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Convergence Analysis for Finite Element Discretizations of the Helmholtz Equation. Part I: the Full Space Problem

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Convergence Analysis for Finite Element Discretizations of the Helmholtz equation.
Part I: The Full Space Problem.

J.M. Melenk* S. Sauter†

Abstract

A rigorous convergence theory for Galerkin methods for a model Helmholtz problem in $\mathbb{R}^d$, $d \in \{1, 2, 3\}$ is presented. General conditions on the approximation properties of the approximation space are stated that ensure quasi-optimality of the method. As an application of the general theory, a full error analysis of the classical hp-version of the finite element method (hp-FEM) is presented where the dependence on the mesh width h, the approximation order p, and the wave number k is given explicitly. In particular, it is shown that quasi-optimality is obtained under the conditions that kh/p is sufficiently small and the polynomial degree p is at least $O(\log k)$.

AMS Subject Classification: 35J05, 65N12, 65N30
Key Words: Helmholtz equation at high wave number, stability, convergence, hp-finite elements

1 Introduction

We consider the numerical solution of the Helmholtz equation by the finite element method or generalizations thereof, which are based on non-standard approximation spaces. Clearly, the derivation of stability and convergence estimates for the classical hp-version of the FEM that are explicit in the wave number, the mesh width, and the approximation order, are of great practical importance. Additionally, such results are also useful for the design and the understanding of generalized finite element methods. Partial results such as sharp estimates for the inf-sup constant of the continuous equations, lower estimates for the convergence rates, one-dimensional analysis by using the discrete Green's function as well as a dispersion analysis for finite element discretizations and generalizations thereof have been derived by many researchers in the past decades (see, e.g., [2, 4, 6, 7, 9–11, 14, 16–18, 21–27, 31, 34, 36, 37, 41, 42]).

The main goal of the present paper is to derive quite general stability and convergence estimates that are:

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- explicit in the wave number, the mesh width, and the polynomial degree of the hp-FEM space;
- valid for problems in d spatial dimensions, $d = 1, 2, 3$;
- only based on approximation properties of the (generalized) finite element space; the rationale behind this requirement is that it is easier to verify such an approximation property than to perform a full-fledged convergence analysis for a given approximation space.

Such types of estimates require the development of new powerful analytical tools and cannot be achieved in one stroke. As a first step, we consider in this paper the Helmholtz equation in a bounded d-dimensional domain with non-reflecting boundary conditions and analyze its finite element discretization. We derive stability and convergence estimates that are explicit in the wave number, the mesh width and the polynomial degree of the finite element space. Forthcoming papers will address more general situations such as the scattering of waves by bounded smooth objects and later the scattering by polygonal/polyhedral domains. The results which we derive in this paper will form the basis for such generalizations.

The outline of this paper is as follows: Section 2 formulates the model problem. Section 3 provides an analysis of the model problem. In particular, the k-dependence of the solution is made explicit (Lemmata 3.7, 3.4). Section 4 analyzes the discrete stability and states explicit conditions on the properties of the approximation space to ensure quasi-optimality of the Galerkin scheme (Theorems 4.2, 4.3). Section 5 applies the results of Section 4 to the hp-version of the FEM. In particular, we show in Corollary 5.5 that quasi-optimality of the hp-FEM can be achieved under the assumption that

$$\frac{kh}{p} + k\left(\frac{kh}{\sigma p}\right)^p \leq C \tag{1.1}$$

where the constants C, $\sigma > 0$ are sufficiently small but independent of h, p, and k. Appendix A provides detailed properties of Bessel functions that are needed in Section 3. Appendix B provides hp-approximation results for functions in the Sobolev spaces H^s which allow simultaneous approximation in the L^2- and H^1-norm. It will turn out that such estimates are essential for the error analysis of Helmholtz problems in the high frequency regime. Appendix C finally provides hp-approximation results for functions that are analytic. These latter approximation results are tailored to regularity properties of solution of Helmholtz-type problems.

2 Formulation of the model Helmholtz problem

The Helmholtz problem in the full space $\mathbb{R}^d$ with Sommerfeld radiation condition is given by: Find $U \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ such that

$$\begin{aligned} (-\Delta - k^2)\, U &= f && \text{in } \mathbb{R}^d, \\ \left|\frac{\partial U}{\partial r} - \mathrm{i}\, kU\right| &= o\left(\|\mathbf{x}\|^{\frac{1-d}{2}}\right) && \|\mathbf{x}\| \to \infty \end{aligned} \tag{2.1}$$

is satisfied in a weak sense (cf. [33]). Here, $\partial/\partial r$ denotes the derivative in radial direction $\mathbf{x}/\left\|\mathbf{x}\right\|$. We assume throughout the paper that the wave number is positive and bounded away from zero, i.e.,

$$k \geq k_0 > 0. \tag{2.2}$$

We assume that f is local in the sense that there exists a bounded, simply connected domain $\Omega \subset \mathbb{R}^d$ that satisfies $\operatorname{supp} f \subset \Omega$. The complement of Ω is denoted by $\Omega^+ := \mathbb{R}^d \backslash \overline{\Omega}$ and the interface by $\Gamma := \overline{\Omega} \cap \overline{\Omega^+}$. Then (2.1) can be formulated in an equivalent way as a transmission problem by seeking functions $u \in H^1(\Omega)$ and $u^+ \in H^1_{\mathrm{loc}}(\Omega^+)$ such that

$$\begin{array}{rl} (-\Delta - k^2)\, u = f & \text{in } \Omega,, \\ (-\Delta - k^2)\, u^+ = 0 & \text{in } \Omega^+, \\ u = u^+ \quad \text{and} \quad \partial u/\partial n = \partial u^+/\partial n & \text{on } \Gamma, \\ \left| \dfrac{\partial u^+}{\partial r} - \mathrm{i}\, k u^+ \right| = o\left(\left\|\mathbf{x}\right\|^{\frac{1-d}{2}} \right) & \left\|\mathbf{x}\right\| \to \infty. \end{array} \tag{2.3}$$

Here, n denote the normal vector pointing into the exterior domain Ω^+.

It can be shown that, for given $g \in H^{1/2}(\Gamma)$, the problem:

$$\text{find } w \in H^1_{\mathrm{loc}}\left(\Omega^+\right) \text{ such that } \left\{ \begin{array}{ll} (-\Delta - k^2)\, w = 0 & \text{in } \Omega^+, \\ w = g & \text{on } \partial\Omega, \\ \left| \dfrac{\partial w}{\partial r} - \mathrm{i}\, k w \right| = o\left(\left\|\mathbf{x}\right\|^{\frac{1-d}{2}} \right) & \left\|\mathbf{x}\right\| \to \infty. \end{array} \right.$$

has a unique weak solution. The mapping $g \mapsto w$ is called the *Steklov-Poincaré operator* and denoted by $S_P : H^{1/2}(\Gamma) \to H^1_{\mathrm{loc}}(\Omega^+)$. The *Dirichlet-to-Neumann map* is given by $T := \gamma_1 S_P : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, where $\gamma_1 := \partial/\partial n$ is the normal trace operator. Hence, problem (2.3) can be reformulated as: Find $u \in H^1(\Omega)$ such that

$$\begin{array}{rl} (-\Delta - k^2)\, u = f & \text{in } \Omega, \\ \partial u/\partial n = Tu & \text{on } \Gamma. \end{array} \tag{2.4}$$

The weak formulation of this equation is given by: Find $u \in H^1(\Omega)$ such that

$$a(u,v) := \int_\Omega \langle \nabla u, \nabla \bar{v} \rangle - k^2 u \bar{v} - \int_\Gamma (Tu)\, \bar{v} = \int_\Omega f \bar{v} \quad \forall v \in H^1(\Omega). \tag{2.5}$$

The exact solution of (2.1) can be written as the acoustic volume potential. Let $G_k : \mathbb{R}^d \backslash \{0\} \to \mathbb{C}$ denote the fundamental solution to the operator $\mathcal{L}_k := -\Delta - k^2$, i.e., $G_k(z) = g_k(\|z\|)$, where

$$g_k(r) := \left\{ \begin{array}{ll} -\frac{\mathrm{e}^{\mathrm{i}\, kr}}{2\,\mathrm{i}\, k} & d = 1, \\ \frac{\mathrm{i}}{4} H_0^{(1)}(kr) & d = 2, \\ \frac{\mathrm{e}^{\mathrm{i}\, kr}}{4\pi r} & d = 3. \end{array} \right.$$

Then, the solution of (2.1) is given by

$$U(x) := (N_k f)(x) := \int_\Omega G_k(x-y)\, f(y)\, dy \qquad \forall x \in \mathbb{R}^d. \tag{2.6}$$

Consequently, the solution of (2.4) and (2.5) is given by

$$u(x) := (N_k f)(x) := \int_\Omega G_k(x-y)\, f(y)\, dy \qquad \forall x \in \Omega.$$

Finally, we recall that a Galerkin method for (2.5) is given as follows: For a (typically finite dimensional) space $S \subset H^1(\Omega)$, the Galerkin approximation $u_S \in S$ to the exact solution u is given by:

$$\text{Find } u_S \in S \text{ s.t.} \quad a(u_S, v) = \int_\Omega f\overline{v} \quad \forall v \in S. \tag{2.7}$$

3 Analysis of the continuous problem

The analysis of the continuous problem is split into three parts. First, we provide some estimates for the Dirichlet-to-Neumann map T. Then, we prove some mapping properties of the solution operator and, finally, state the existence and uniqueness of the continuous problem.

3.1 Estimates for the operator T

We equip the space $H^1(\Omega)$ with the norm

$$\|u\|_{\mathcal{H}} := \left(|u|_{1,\Omega}^2 + k^2 \|u\|_{0,\Omega}^2\right)^{1/2},$$

which is obviously equivalent to the H^1-norm. For $d = 1$, the boundary $\partial\Omega$ consists of the two endpoint of Ω and the $L^2(\partial\Omega)$- and $H^{1/2}(\partial\Omega)$-scalar product and norm is understood as

$$(u, v)_{L^2(\Gamma)} := \sum_{x\in\{-R,R\}} u(x)\,\overline{v(x)} \quad \text{and} \quad \|u\|_{L^2(\Gamma)} = \|u\|_{H^{1/2}(\Gamma)} = \sqrt{\sum_{x\in\{-R,R\}} |u(x)|^2}.$$

For Lipschitz domains, it is well known that a trace estimate holds.

Lemma 3.1 *There exists a constant C_{tr} depending only on Ω and k_0 such that*

$$\forall u \in H^1(\Omega): \quad \|u\|_{H^{1/2}(\Gamma)} \leq C_{\mathrm{tr}} \|u\|_{\mathcal{H}} \tag{3.1a}$$

and

$$\forall u \in H^1(\Omega): \quad \|u\|_{L^2(\Gamma)} \leq C_{\mathrm{tr}} \|u\|_{L^2(\Omega)}^{1/2} \|u\|_{H^1(\Omega)}^{1/2}. \tag{3.1b}$$

Corollary 3.2 *For $u \in H^1(\Omega)$, we have*

$$\sqrt{k}\,\|u\|_{L^2(\Gamma)} \leq \tilde{C}_{\mathrm{tr}} \|u\|_{\mathcal{H}} \quad \textit{with} \quad \tilde{C}_{\mathrm{tr}} := \frac{C_{\mathrm{tr}}}{\sqrt{2}} \frac{\sqrt{1+k_0^2}}{k_0}$$

where k_0 is as in (2.2).

Proof. There holds

$$\begin{aligned} k\|u\|_{L^2(\Gamma)}^2 &\leq C_{\mathrm{tr}}^2 k \|u\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)} \leq \frac{C_{\mathrm{tr}}^2}{2}\left(k^2\|u\|_{L^2(\Omega)}^2 + \|u\|_{H^1(\Omega)}^2\right) \\ &= \frac{C_{\mathrm{tr}}^2}{2}\left(\left(1+k^2\right)\|u\|_{L^2(\Omega)}^2 + |u|_{H^1(\Omega)}^2\right) \leq \tilde{C}_{\mathrm{tr}}^2 \|u\|_{\mathcal{H}}^2. \end{aligned} \tag{3.2}$$

■

Since the right-hand side f in (2.3) has compact support, we may always choose Ω to be a ball $B(0,R)$ of radius R centered at the origin. In the following analysis we will always restrict our attention to this case and assume that

$$R \geq R_0 > 0. \tag{3.3}$$

Lemma 3.3 *Let $\Omega \subset \mathbb{R}^d$ be the ball of radius R centered at the origin and let (3.3) and (2.2) be satisfied. For $d = 2$, we assume, in addition, that $k_0 \geq 1$ holds. Then, there exist constants $c, C > 0$ that only depend on R_0 and k_0 such that the following is true:*

1.
$$\left|(Tu, v)_{L^2(\Gamma)}\right| \leq C \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}} \qquad \forall u, v \in H^1(\Omega). \tag{3.4a}$$

2. *For $d \in \{2, 3\}$ and all $u \in H^{1/2}(\Gamma)$ the real and imaginary parts of (Tu, u) satisfy*

$$-\operatorname{Re}(Tu, u)_{L^2(\Gamma)} \geq c \frac{\|u\|^2_{L^2(\Gamma)}}{R}, \tag{3.4b}$$

$$\operatorname{Im}(Tu, u)_{L^2(\Gamma)} > 0 \quad \text{for } u \neq 0. \tag{3.4c}$$

For $d = 1$, instead of (3.4b), (3.4c), there holds

$$-\operatorname{Re}(Tu, u)_{L^2(\Gamma)} = 0, \tag{3.4d}$$

$$\operatorname{Im}(Tu, u)_{L^2(\Gamma)} \geq k \|u\|^2_{L^2(\Gamma)}. \tag{3.4e}$$

Proof. Case $d = 3$.
The Dirichlet data on $\Gamma = \partial B(0, R)$ can be expanded according to

$$u(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_\ell^m Y_\ell^m(\theta, \phi), \tag{3.5}$$

where (R, θ, ϕ) are the spherical coordinates for $x \in \Gamma$ and the functions Y_ℓ^m are the standard spherical harmonics. The solution to the exterior homogeneous Helmholtz problem with Sommerfeld radiation conditions at infinity and prescribed Dirichlet data at Γ can be expanded in the form

$$u(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_\ell^m Y_\ell^m(\theta, \phi) \frac{h_\ell^{(1)}(kr)}{h_\ell^{(1)}(kR)}, \tag{3.6}$$

where (r, θ, ϕ) are the spherical coordinates of $x \in \mathbb{R}^3 \backslash \overline{\Omega}$. By taking the normal derivative at the boundary we end up with a representation of the Dirichlet-to-Neumann map

$$Tu = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_\ell^m Y_\ell^m(\theta, \phi) \frac{z_\ell(kR)}{R} \tag{3.7}$$

with the functions $z_\ell(r) := r \frac{\left(h_\ell^{(1)}\right)'(r)}{h_\ell^{(1)}(r)}$. These functions have been analyzed in [33, Theorem 2.6.1] where it is shown that

$$1 \leq -\operatorname{Re}(z_\ell(r)) \leq \ell + 1 \quad \text{and} \quad 0 < \operatorname{Im}(z_\ell(r)) \leq r. \tag{3.8}$$

(In [33, Theorem 2.6.1], only $\operatorname{Im} z_\ell(r) \geq 0$ is stated, while the strict positivity follows from the positivity of the function q_ℓ in [33, (2.6.34)].) It follows from (3.7) that

$$\int_\Gamma (Tu)\,\overline{v} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{z_\ell(kR)}{R} u_\ell^m \overline{v_\ell^m}$$

and from (3.8) we conclude that

$$\begin{aligned}
\left|\operatorname{Re}\int_\Gamma (Tu)\,\overline{v}\right| &= \left|\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left\{\frac{\operatorname{Re} z_\ell(kR)}{R}\operatorname{Re}\left(u_\ell^m\overline{v_\ell^m}\right) - \frac{\operatorname{Im} z_\ell(kR)}{R}\operatorname{Im}\left(u_\ell^m\overline{v_\ell^m}\right)\right\}\right| \\
&\leq \frac{1}{R}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left\{\left|\operatorname{Re} z_\ell(kR)\right| + \left|\operatorname{Im} z_\ell(kR)\right|\right\}\left|u_\ell^m\right|\left|v_\ell^m\right| \\
&\leq \frac{1}{R}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left\{|\ell+1| + kR\right\}\left|u_\ell^m\right|\left|v_\ell^m\right| \\
&\leq C\left(R^{-1}\|u\|_{H^{1/2}(\Gamma)}\|v\|_{H^{1/2}(\Gamma)} + k\|u\|_{L^2(\Gamma)}\|v\|_{L^2(\Gamma)}\right).
\end{aligned}$$

Using Corollary 3.2 we get

$$\left|\operatorname{Re}\int_\Gamma (Tu)\,\overline{v}\right| \leq C\tilde{C}_{\mathrm{tr}}^2\left(1+\frac{1}{R_0 k_0}\right)\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}}.$$

By repeating these steps for the imaginary part we get the same upper bound and, hence,

$$\left|\int_\Gamma (Tu)\,\overline{v}\right| \leq C\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}},$$

where C only depends on R_0 and k_0.

The lower estimate of the real part follows from

$$-\operatorname{Re}\int_\Gamma (Tu)\,\overline{u} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{-\operatorname{Re} z_\ell(kR)}{R}\left|u_\ell^m\right|^2 \geq \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{R}\left|u_\ell^m\right|^2 = \frac{\|u\|_{L^2(\Omega)}^2}{R}.$$

The upper estimate for the imaginary part is just a repetition of the previous arguments.

For the lower estimate of the imaginary part, we consider $u \in H^{1/2}(\Gamma)\setminus\{0\}$. Hence, there exists $(m_\star, \ell_\star)$ in the expansion (3.5) so that $u_{\ell_\star}^{m_\star} \neq 0$. This leads to

$$\operatorname{Im}\int_\Gamma (Tu)\,\bar{u} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{\operatorname{Im} z_\ell(kR)}{R}\left|u_\ell^m\right|^2 \geq \left|u_{\ell_\star}^{m_\star}\right|^2 > 0$$

and the lower bound is proved.

Case $d=2$.

We expand the Dirichlet data on Γ in polar coordinates

$$u(x) = \sum_{m\in\mathbb{Z}} u_m\,\mathrm{e}^{\mathrm{i}m\theta},$$

where (R,θ) are the polar coordinates of $x\in\Gamma$. It follows (see, e.g., [12, (2.10)]) that

$$Tu=\sum_{m\in\mathbb{Z}}u_m\frac{w_m\left(kR\right)}{R}\,\mathrm{e}^{\mathrm{i}\,m\theta}\quad\text{with}\quad w_m\left(r\right):=r\frac{\left(H_{|m|}^{(1)}\right)'\left(r\right)}{H_{|m|}^{(1)}\left(r\right)}.\tag{3.9}$$

Obviously, it is sufficient to analyze w_m only for $m\in\mathbb{N}_0$. By decomposing w_m into its real and imaginary part we get

$$w_\ell=r\frac{J_\ell'J_\ell+Y_\ell'Y_\ell+\mathrm{i}\left(Y_\ell'J_\ell-J_\ell'Y_\ell\right)}{J_\ell^2+Y_\ell^2}.$$

For the imaginary part, we obtain

$$Y_\ell'J_\ell-J_\ell'Y_\ell\overset{[1,\,9.1.27]}{=}Y_{\ell-1}J_\ell-J_{\ell-1}Y_\ell\overset{[1,\,9.1.16]}{=}\frac{2}{\pi r}.$$

We set $M_\ell:=\left|H_\ell^{(1)}\right|$ and obtain

$$w_\ell=r\frac{J_\ell'J_\ell+Y_\ell'Y_\ell}{M_\ell^2}+\mathrm{i}\,\frac{2}{\pi M_\ell^2}=\frac{r}{2}\frac{\frac{d}{dr}M_\ell^2}{M_\ell^2}+\mathrm{i}\,\frac{2}{\pi M_\ell^2}.\tag{3.10}$$

In the next step, we derive estimates for the coefficients w_ℓ.

Case $d=2$ and $\ell\in\mathbb{N}_{\geq 2}$.
Let

$$M_{\ell,n}^2\left(r\right):=\frac{2}{\pi r}\sum_{m=0}^{n}\frac{\delta_{\ell,m}}{r^{2m}}\quad\text{with}\quad\delta_{\ell,m}:=\frac{(2m)!\gamma_{\ell,m}}{\left(m!\right)^2 16^m}\text{ and }\gamma_{\ell,m}:=\prod_{k=1}^{m}\left(4\ell^2-\left(2k-1\right)^2\right)\tag{3.11}$$

and define $R_{\ell,n}^M:=M_\ell^2-M_{\ell,n}^2$. Note that

$$\gamma_{\ell,\ell}=\frac{(4\ell)!}{2^{2\ell}\left(2\ell\right)!}\geq 0\quad\text{and}\quad\gamma_{\ell,\ell+1}=-\left(4\ell+1\right)\gamma_{\ell,\ell}<0.\tag{3.12}$$

We conclude from [44, §13.75] that, for the choice $n=\ell-1\geq 0$, there holds $R_{\ell,\ell-1}^M\left(r\right)\geq 0$. Thus,

$$M_\ell^2\left(r\right)\geq M_{\ell,\ell-1}^2\left(r\right)\quad\forall r\geq 0.\tag{3.13}$$

Let K_ν be the modified Bessel function of order ν. From [44, §13.75] we obtain

$$N_\ell^2:=\frac{d}{dr}M_\ell^2=-\frac{16}{\pi^2}\int_0^\infty K_1\left(2r\sinh t\right)\sinh t\cosh\left(2\ell t\right)dt$$

and

$$\frac{\cosh\left(2\ell t\right)}{\cosh t}=\sum_{m=0}^{n}\frac{\gamma_{\ell,m}}{(2m)!}\sinh^{2m}t+\tilde{R}_{\ell,n}^2.$$

If $n>\ell-3/2$, the remainder $\tilde{R}_{\ell,n}$ satisfies

$$\tilde{R}_{\ell,n}^2\in\begin{cases}\left[0,\dfrac{\gamma_{\ell,n+1}}{(2n+2)!}\sinh^{2(n+1)}t\right] & \text{if }\gamma_{\ell,n+1}>0,\\[2ex] \left[\dfrac{\gamma_{\ell,n+1}}{(2n+2)!}\sinh^{2(n+1)}t,0\right] & \text{otherwise.}\end{cases}\tag{3.14}$$

We introduce

$$
\begin{aligned}
N_{\ell,n}^{2} &:= -\frac{16}{\pi^{2}}\sum_{m=0}^{n}\frac{\gamma_{\ell,m}}{(2m)!}\int_{0}^{\infty}K_{1}\left(2r\sinh t\right)\left(\cosh t\right)\left(\sinh^{2m+1}t\right)dt \\
&= -\frac{16}{\pi^{2}}\sum_{m=0}^{n}\frac{\gamma_{\ell,m}}{(2m)!\,(2r)^{2m+2}}\int_{0}^{\infty}K_{1}\left(z\right)z^{2m+1}dz \\
&= -\frac{2}{\pi r^{2}}\sum_{m=0}^{n}\left(2m+1\right)\frac{\delta_{\ell,m}}{r^{2m}} = \frac{d}{dr}M_{\ell,n}^{2}.
\end{aligned}
$$

Note that $M_{\ell}^{2}\left(r\right)$ is monotone decreasing for $r>0$ (cf. [35, §9-7.3]) and hence $N_{\ell}^{2}\left(r\right)<0$ for $r>0$. Thus,

$$
\left|N_{\ell}^{2}\left(r\right)\right| = -N_{\ell,n}^{2}\left(r\right)+R_{\ell,n}^{N} \quad \text{with} \quad R_{\ell,n}^{N} := -N_{\ell}^{2}\left(r\right)+N_{\ell,n}^{2}\left(r\right)
$$

and $R_{\ell,n}^{N}$ has the explicit representation

$$
R_{\ell,n}^{N}\left(r\right) = \frac{16}{\pi^{2}}\int_{0}^{\infty}K_{1}\left(2r\sinh t\right)\left(\sinh t\right)\left(\cosh t\right)\tilde{R}_{\ell,n}^{2}\left(t\right)dt.
$$

Note that sinh, cosh, and K_1 are positive on the positive real axes (cf. [1, 9.6.23]). We choose $n=\ell$ and obtain from (3.12) and (3.14) that $\tilde{R}_{\ell,\ell}\left(t\right)$ is negative for $t>0$ and hence

$$
\left|N_{\ell}^{2}\left(r\right)\right| \leq -N_{\ell,\ell}^{2}\left(r\right) \quad \forall r>0. \tag{3.15}
$$

In summary, we have proved that

$$
\begin{aligned}
\left|\operatorname{Re}w_{\ell}\right| &\leq -\frac{r}{2}\frac{N_{\ell,\ell}^{2}}{M_{\ell,\ell-1}^{2}} = \frac{1}{2}\frac{\sum_{m=0}^{\ell}\left(2m+1\right)\frac{\delta_{\ell,m}}{r^{2m}}}{\sum_{m=0}^{\ell-1}\frac{\delta_{\ell,m}}{r^{2m}}} \leq \frac{2\ell-1}{2}+\frac{2\ell+1}{2}\frac{\frac{\delta_{\ell,\ell}}{r^{2\ell}}}{\frac{\delta_{\ell,\ell-1}}{r^{2\ell-2}}} \\
&= \frac{2\ell-1}{2}+\frac{\left(4\ell-1\right)\left(4\ell^{2}-1\right)}{16\ell r^{2}}.
\end{aligned} \tag{3.16}
$$

Hence, for $\ell\geq 2$ and $r\geq C_{1}\sqrt{\ell}$, the estimate

$$
\left|\operatorname{Re}w_{\ell}\right| \leq \frac{2\ell-1}{2}\left(1+\frac{9}{8C_{1}^{2}}\right)
$$

follows.

It remains to consider the case

$$
r \leq C_{1}\sqrt{\ell}. \tag{3.17}
$$

We derive from (3.10) and [1, 9.1.27]

$$
\left|\frac{r}{2}N_{\ell}^{2}\left(r\right)\right| = -\frac{r}{2}N_{\ell}^{2}\left(r\right) = \ell M_{\ell}^{2}\left(r\right)-r\left(J_{\ell-1}J_{\ell}+Y_{\ell}Y_{\ell-1}\right)
$$

and this leads to

$$
\left|\operatorname{Re}w_{\ell}\right| = \frac{\left|\frac{r}{2}N_{\ell}^{2}\left(r\right)\right|}{M_{\ell}^{2}\left(r\right)} = \ell-\frac{r\left(J_{\ell}J_{\ell-1}+Y_{\ell}Y_{\ell-1}\right)}{M_{\ell}^{2}\left(r\right)}. \tag{3.18}
$$

We deduce from [1, 9.5.2, 9.1.7, 9.1.9] that

$$J_\ell(r) > 0 \quad \text{and} \quad Y_\ell(r) < 0 \quad \forall 0 \le r \le \ell$$

and, thus,

$$J_\ell J_{\ell-1} + Y_\ell Y_{\ell-1} > 0 \quad \forall 0 \le r \le \ell - 1.$$

If $C_1 \le 2^{-1/2}$ there holds $C_1\sqrt{\ell} \le \ell - 1$ for all $\ell \ge 2$ and we have proved $|\operatorname{Re} w_\ell| \le \ell$.

To derive a lower bound of $(-\operatorname{Re} w_\ell)$, we proceed as for (3.16) and obtain, for $r \ge k_0$,

$$-\operatorname{Re} w_\ell(r) \ge -\frac{r}{2}\frac{N^2_{\ell,\ell-1}(r)}{M^2_{\ell,\ell}(r)} = \frac{1}{2}\frac{\sum_{m=0}^{\ell-1}(2m+1)\frac{\delta_{\ell,m}}{r^{2m}}}{\sum_{m=0}^{\ell}\frac{\delta_{\ell,m}}{r^{2m}}} \ge \frac{1}{2}\frac{1}{1+\frac{\frac{\delta_{\ell,\ell}}{r^{2\ell}}}{(2\ell-1)\frac{\delta_{\ell,\ell-1}}{r^{2\ell-2}}}} \tag{3.19}$$

$$= \frac{1}{2}\frac{1}{1+\frac{4\ell-1}{8\ell r^2}} \ge \frac{1}{2}\frac{1}{1+\frac{1}{2k_0^2}}.$$

For the imaginary part we get

$$\operatorname{Im} w_\ell(r) = \frac{2}{\pi M^2_\ell(r)} > 0 \quad \forall \ell \in \mathbb{N}_0 \quad \forall r \ge k_0 \tag{3.20}$$

because M^2_ℓ is non-negative and decreasing for $r > 0$ (cf. [35, §9-7.3]). For the upper bound, we combine [19, 8.479] with the fact that M^2_ℓ is decreasing to obtain for $\ell \in \mathbb{N}_{\ge 1}$

$$M^2_\ell(r) \ge \frac{2}{\pi r} \quad \forall r \ge 1. \tag{3.21a}$$

Hence, the upper bound

$$\operatorname{Im} w_\ell(r) = \frac{2}{\pi M^2_\ell(r)} \le r \tag{3.22}$$

follows.

Case $d = 2$ **and** $\ell = 0, 1$.

For $\ell = 0$, we use [44, §13.75] and get

$$M^2_0(r) \ge M^2_{0,1}(r) = \frac{2}{\pi r}\left(1 - \frac{1}{8r^2}\right).$$

For $d = 2$, there holds $k_0 > 1/2$ by our assumptions and, thus, for $r \ge k_0$ we get

$$M^2_0(r) \ge \frac{1}{\pi r}. \tag{3.21b}$$

The combination of (3.10) and (3.21) implies

$$|\operatorname{Re} w_\ell(r)| \le \frac{\pi r^2}{2}\left|N^2_\ell(r)\right|.$$

We deduce from (3.15) (which is also valid for $\ell = 0, 1$)

$$\left|N^2_\ell(r)\right| \le \left|N^2_{\ell,\ell}(r)\right| \le \frac{2}{\pi r^2}\sum_{m=0}^{\ell}(2m+1)\frac{\delta_{\ell,m}}{r^{2m}} \le \frac{2}{\pi r^2}\begin{cases} 1 & \ell = 0, \\ 1 + \frac{9}{8r^2} & \ell = 1. \end{cases}$$

This implies, for $r \geq k_0$ (cf. (2.2))

$$\left|N_\ell^2(r)\right| \leq C \frac{2}{\pi r^2},$$

where C depends solely on k_0. Thus, for $\ell = 0, 1$,

$$|\operatorname{Re} w_\ell| \leq C \leq C(\ell + 1).$$

Since M_ℓ^2 is monotone decreasing (see [35, §9-7.3]), it follows from (3.9) that $\operatorname{Re} w_\ell(r) < 0$ for all $r > 0$.

In (3.19) we have derived a lower bound for $(-\operatorname{Re} w_\ell)$ provided $\ell \geq 1$. It remains to consider the case $\ell = 0$. The assumption on k_0 implies $r \geq k_0 \geq \frac{1}{2}\sqrt{3}$ so that

$$-\operatorname{Re} w_0(r) \geq -\frac{r}{2}\frac{N_{0,1}^2(r)}{M_{0,0}^2(r)} = \frac{1}{2}\left(1 - \frac{3}{8r^2}\right) \geq \frac{1}{4}.$$

To summarize both cases, we have proved that

$$0 < c \leq -\operatorname{Re} w_\ell(r) \leq C(\ell + 1) \quad \forall r \geq k_0 \quad \forall \ell \in \mathbb{N}_0, \tag{3.23a}$$

where c, C only depends on k_0.

For the imaginary part, it remains (cf. (3.21a), (3.22)) to prove the upper bound for $(\operatorname{Im} w_0)$ and employ (3.10) and (3.21b) to obtain

$$\operatorname{Im} w_0 = \frac{2}{\pi M_0^2} \leq 2r. \tag{3.23b}$$

By proceeding as for $d = 3$ (after (3.8)) the estimates (3.4) follow from (3.23).

Case $d = 1$

For boundary values $\psi : \{-R, R\} \to \mathbb{R}$, the Dirichlet-to-Neumann operator is given by

$$T\psi = \mathrm{i}\, k\psi.$$

The trace theorem (in one dimension) leads to

$$\begin{aligned}
\left|\operatorname{Re}\int_\Gamma (Tu)\,\overline{v}\right| &= \left|\operatorname{Re}\left(\mathrm{i}\, k \sum_{r=\pm R} u(r)\,\overline{v}(r)\right)\right| \\
&\leq k\left|\operatorname{Im}\sum_{r=\pm R} u(r)\,\overline{v}(r)\right| \leq k \sum_{r=\pm R} |u(r)|\,|\overline{v}(r)| \\
&\overset{\text{Cor. 3.2}}{\leq} C\,\|u\|_{\mathcal{H}}\,\|v\|_{\mathcal{H}},
\end{aligned}$$

where C only depends on R_0 and k_0. By the same techniques we can estimate the imaginary part and, thus, obtain (3.4a). The lower bounds (3.4d), (3.4e) follow from

$$\begin{aligned}
-\operatorname{Re}\int_\Gamma (Tu)\,\overline{u} &= -\operatorname{Re}\left(\mathrm{i}\, k \sum_{r=\pm R} |u(r)|^2\right) = 0 \\
\operatorname{Im}\int_\Gamma (Tu)\,\overline{u} &= k \sum_{r=\pm R} |u(r)|^2 \geq k\,\|u\|_{L^2(\Gamma)}^2.
\end{aligned}$$

■

3.2 Local estimates for the solution operator

In this section, we derive some explicit bounds for the solution operator under the assumption that the right-hand side is in $L^2(\Omega)$. These estimates will be the basic tool for proving the discrete stability of the finite element discretization and the convergence. The key step for the analysis of the hp-FEM in Section 5 is the following decomposition result:

Lemma 3.4 (decomposition lemma) *Let Ω be contained in a ball of radius $R > 0$. Then there exist constants C, $\gamma > 0$ depending only R and k_0 such that for $f \in L^2(\Omega)$ the function v given by*

$$v(x) = N_k f(x) = \int_\Omega G_k(x-y) f(y)\, dy, \qquad x \in \Omega$$

satisfies

$$\|v\|_{H^1(\Omega)} + k\|v\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

and can be decomposed as $v = v_{H^2} + v_{\mathcal{A}}$ with

$$\|v_{H^2}\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}, \tag{3.24a}$$

$$\|\nabla^p v_{\mathcal{A}}\|_{L^2(\Omega)} \leq C\left(\gamma|k|\right)^{p-1}\|f\|_{L^2(\Omega)} \qquad \forall p \in \mathbb{N}_0. \tag{3.24b}$$

Here, $\nabla^p v_{\mathcal{A}}$ stands for a sum over all derivatives of order p (see (5.1) for details).

Proof. We start by recalling the definition of the Fourier transform for functions with compact support

$$\hat{u}(\xi) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\langle\xi,x\rangle}\, u(x)\, dx \qquad \forall \xi \in \mathbb{R}^d$$

and the inversion formula

$$u(x) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle x,\xi\rangle}\, \hat{u}(\xi)\, d\xi \qquad \forall x \in \mathbb{R}^d.$$

Let $B_\Omega \subset \mathbb{R}^d$ be a ball of radius R containing Ω. Extend f by zero outside of Ω and denote this extended function again by f. Let $\mu \in C^\infty(\mathbb{R}_{\geq 0})$ be a cutoff function such that

$$\begin{aligned} &\operatorname{supp}\mu \subset [0,4R], && \mu|_{[0,2R]} = 1, \quad |\mu|_{W^{1,\infty}(\mathbb{R}_{\geq 0})} \leq \frac{C}{R}, \\ &\forall x \in \mathbb{R}_{\geq 0} : 0 \leq \mu(x) \leq 1, && \mu|_{[4R,\infty[} = 0, \quad |\mu|_{W^{2,\infty}(\mathbb{R}_{\geq 0})} \leq \frac{C}{R^2}. \end{aligned} \tag{3.25}$$

Define $M(z) := \mu(\|z\|)$ and

$$v_\mu(x) := \int_{B_\Omega} G_k(x-y)\, M(x-y)\, f(y)\, dy \qquad \forall x \in \mathbb{R}^d.$$

The properties of μ guarantee $v_\mu|_{B_\Omega} = v|_{B_\Omega}$ so that we may restrict our attention to the function v_μ. Since $\operatorname{supp} f \subset B_\Omega$ we may write

$$v_\mu = (G_k M) \star f, \tag{3.26}$$

where "$\star$" denotes the convolution in $\mathbb{R}^d$. We will define a decomposition of v_μ (which will determine the decomposition of v on B_Ω) by decomposing its Fourier transform, i.e.,

$$\widehat{v}_\mu = \widehat{v}_{H^2} + \widehat{v}_{\mathcal{A}}. \tag{3.27}$$

In order to define the two terms on the right-hand side of (3.27), we let $B_{3k/2}(0)$ denote the ball of radius $3k/2$ centered at the origin. The characteristic function of $B_{3k/2}(0)$ is denoted by χ_k. The Fourier transform of f is then decomposed as

$$\widehat{f} = \widehat{f}\chi_k + (1-\chi_k)\widehat{f} =: \widehat{f}_k + \widehat{f}_k^c.$$

By the inverse Fourier transformation, this decomposition of $\widehat{f}$ entails a decomposition of f into f_k and f_k^c given by

$$f_k(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle x,\xi\rangle} \chi_k(\xi) \hat{f}(\xi)\, d\xi \quad \text{and} \quad f_k^c(x) := f - f_k. \tag{3.28}$$

Accordingly, we define the decomposition of v_μ in

$$v_{\mu,H^2} := (G_k M) \star f_k^c \quad \text{and} \quad v_{\mu,\mathcal{A}} := (G_k M) \star f_k. \tag{3.29}$$

The functions v_{H^2} and $v_{\mathcal{A}}$ in (3.27) are then obtained by setting $v_{H^2} := v_{\mu,H^2}|_\Omega$ and $v_{\mathcal{A}} := v_{\mu,\mathcal{A}}|_\Omega$. We will obtain the desired estimates by showing the following, stronger estimates:

$$\|v_{\mu,H^2}\|_{H^2(\mathbb{R}^d)} \le C\|f\|_{L^2(\mathbb{R}^d)}, \tag{3.30a}$$

$$\|D^\alpha v_{\mu,\mathcal{A}}\|_{L^2(\mathbb{R}^d)} \le C\gamma^{|\alpha|}|k|^{|\alpha|-1}\|f\|_{L^2(\mathbb{R}^d)}, \qquad \forall \alpha \in \mathbb{N}_0^d. \tag{3.30b}$$

The estimates (3.30) are obtained by Fourier techniques. To that end, we compute the Fourier transform of $G_k M$:

$$\begin{aligned}
\left(\widehat{G_k M}\right)(\xi) &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\,\mathrm{i}\langle \xi,x\rangle} G_k(x) M(x)\, dx \\
&= (2\pi)^{-d/2} \int_0^\infty g_k(r)\,\mu(r)\, r^{d-1} \left(\int_{\mathbb{S}_{d-1}} \mathrm{e}^{-\,\mathrm{i}\, r\langle \xi,\zeta\rangle}\, dS_\zeta \right) dr \\
&= (2\pi)^{-d/2} I(\xi).
\end{aligned}$$

The integral $I(\xi)$ can be evaluated analytically and $I(\xi) = \iota(\|\xi\|)$ with

$$\iota(s) = \begin{cases} 2\displaystyle\int_0^\infty g_k(r)\,\mu(r)\cos(sr)\, dr & d=1, \\ 2\pi\displaystyle\int_0^\infty g_k(r)\,\mu(r)\, r J_0(rs)\, dr & d=2, \\ 4\pi\displaystyle\int_0^\infty g_k(r)\,\mu(r)\, r^2 \frac{\sin(rs)}{(rs)} dr & d=3. \end{cases} \tag{3.32}$$

Applying the Fourier transform to the convolutions (3.29) leads to

$$\begin{aligned}
\widehat{v}_{\mu,H^2} &= (2\pi)^{d/2}\, \widehat{G_k M}\, \widehat{f}_k^c = (2\pi)^{d/2}\, \widehat{G_k M}\, \widehat{f}(1-\chi_k), \\
\widehat{v}_{\mu,\mathcal{A}} &= (2\pi)^{d/2}\, \widehat{G_k M}\, \widehat{f}_k = (2\pi)^{d/2}\, \widehat{G_k M}\, \widehat{f}\chi_k.
\end{aligned}$$

To estimate higher order derivatives of v_{μ,H^2} and $v_{\mu,\mathcal{A}}$ we define for a multi-index $\alpha \in \mathbb{N}_0^d$ the function $P_\alpha : \mathbb{R}^d \to \mathbb{R}^d$ by $P_\alpha(\xi) := \xi^\alpha$ and obtain – by using standard properties of the Fourier transformation and the support properties of χ_k – for all $|\alpha| \leq 2$

$$\begin{aligned} \|\partial^\alpha v_{\mu,H^2}\|_{L^2(\mathbb{R}^d)} &= (2\pi)^{d/2} \left\| P_\alpha \widehat{G_k M} (1-\chi_k) \widehat{f} \right\|_{L^2(\mathbb{R}^d)} \\ &\leq (2\pi)^{d/2} \left(\max_{\xi \in \mathbb{R}^d : |\xi| \geq 3k/2} |P_\alpha I(\xi)| \right) \left\| (1-\chi_k) \widehat{f} \right\|_{L^2(\mathbb{R}^d)} \\ &\leq (2\pi)^{d/2} \left(\max_{s \geq 3k/2} \left| s^{|\alpha|} \iota(s) \right| \right) \|f\|_{L^2(\Omega)} . \end{aligned} \tag{3.33}$$

Completely analogously, we derive for all $\alpha \in \mathbb{N}_0^d$

$$\|\partial^\alpha v_{\mu,\mathcal{A}}\|_{L^2(\mathbb{R}^d)} \leq (2\pi)^{d/2} \left(\max_{s \leq 3k/2} \left| s^{|\alpha|} \iota(s) \right| \right) \|f\|_{L^2(\Omega)} .$$

We can complete the proof of the lemma using the bounds on the function ι given in Lemma 3.5 below. ■

Lemma 3.5 *For the function ι defined in (3.32) the quantity $s^m \iota(s)$ can be estimated*

(i) for $m = 0$ by

$$|\iota(s)| \leq C \frac{R}{k},$$

(ii) for $m = 1$ by

$$s|\iota(s)| \leq CR \begin{cases} 1 + (Rk)^{-1} & d = 1, \\ |\log kR| & d = 2 \quad \text{and} \quad 4Rk \leq 1, \\ 1 & d = 2 \quad \text{and} \quad 4Rk > 1, \\ 1 & d = 3, \end{cases}$$

(iii) and for $m = 2$ by

$$s^2 |\iota(s)| \leq C \begin{cases} Rk + \dfrac{1}{Rk} & d = 1, \\ |\log(kR)| & d = 2 \quad \text{and} \quad 4Rk \leq 1, \\ Rk & d = 2 \quad \text{and} \quad 4Rk > 1, \\ 1 + kR & d = 3. \end{cases}$$

(iv) For fixed R_0, $R_1 > 0$ there exists $C > 0$ such that for $R_0 \leq R \leq R_1$ and any $|s| \geq 3k/2$

$$s^2 |\iota(s)| \leq C.$$

(v) For $m \in \mathbb{N}_0$, $|s| \leq 3k/2$, and $0 < R_0 \leq R \leq R_1$, we have

$$|s|^m |\iota(s)| \leq C \left(\frac{3k}{2} \right)^{m-1} .$$

Proof. In this proof, C denotes a generic constant which may vary from term to term. It suffices to prove the estimates (i)–(iv) because (v) follows directly from (i). We discuss the cases $d=3$, $d=1$, and $d=2$ in turn.

Case 1: $d=3$.

There holds

$$\left|s\iota\left(s\right)\right| = C\left|\int_0^\infty \mathrm{e}^{\mathrm{i}\,kr}\,\mu\left(r\right)\sin\left(rs\right)dr\right| \leq CR.$$

Applying integration by parts we obtain

$$\begin{aligned}\left|\iota\left(s\right)\right| &= \frac{C}{k}\left|\int_0^\infty \mathrm{e}^{\mathrm{i}\,kr}\left(\mu'\left(r\right)\frac{\sin\left(rs\right)}{s}+\mu\left(r\right)\sin\left(rs\right)\right)dr\right|\\ &\leq \frac{C}{k}\int_0^{4R}\left(\frac{C}{R}r+1\right)dr = C\frac{R}{k}.\end{aligned}$$

For the product $s^2\iota\left(s\right)$, we get

$$\begin{aligned}\left|s^2\iota\left(s\right)\right| &= C\left|\int_0^\infty \mathrm{e}^{\mathrm{i}\,kr}\,\mu\left(r\right)s\sin\left(rs\right)dr\right| = C\left|\int_0^\infty \mathrm{e}^{\mathrm{i}\,kr}\,\mu\left(r\right)\partial_r\cos\left(rs\right)dr\right|\\ &\leq C\left(\left|\int_0^\infty \cos\left(rs\right)\partial_r\left(\mathrm{e}^{\mathrm{i}\,kr}\,\mu\left(r\right)\right)dr\right|+1\right)\\ &\leq Ck\left|\int_0^\infty \cos\left(rs\right)\mathrm{e}^{\mathrm{i}\,kr}\,\mu\left(r\right)dr\right| + C\left(\left|\int_0^\infty \cos\left(rs\right)\mathrm{e}^{\mathrm{i}\,kr}\,\mu'\left(r\right)dr\right|+1\right)\\ &=: T^{\mathrm{I}}+T^{\mathrm{II}}.\end{aligned}$$

The estimates $T^{\mathrm{I}}\leq C^{\mathrm{I}}kR$ and $T^{\mathrm{II}}\leq C^{\mathrm{II}}$ follows from the properties of μ (cf. (3.25)). For $|s|\geq 3k/2$, the estimate of T^{I} can be refined by using integration by parts

$$\begin{aligned}T^{\mathrm{I}} &\leq Ck\left|\int_0^\infty \cos\left(rs\right)\mathrm{e}^{\mathrm{i}\,kr}\,\mu\left(r\right)dr\right| = C\frac{k}{2}\left|\int_0^\infty \left(\mathrm{e}^{\mathrm{i}(k+s)r}+\mathrm{e}^{\mathrm{i}(k-s)r}\right)\mu\left(r\right)dr\right|\\ &\leq C'\left(\frac{k^2}{s^2-k^2}+\int_0^\infty \frac{k^2}{s^2-k^2}\left|\mu'\left(r\right)\right|dr\right)\leq C''.\end{aligned}$$

Case 2: $d=1$.

There holds

$$\left|\iota\left(s\right)\right| \leq \frac{1}{k}\int_0^\infty \mu\left(r\right)dr \leq C\frac{R}{k}.$$

To estimate $s\iota\left(s\right)$, we apply integration by parts to obtain

$$\left|s\iota\left(s\right)\right| \leq \left|\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}\,kr}}{k}\mu\left(r\right)\partial_r\sin\left(sr\right)dr\right| = \left|\int_0^\infty \sin\left(sr\right)\partial_r\left(\frac{\mathrm{e}^{\mathrm{i}\,kr}}{k}\mu\left(r\right)\right)dr\right| \leq C\frac{1+Rk}{k}.$$

Similarly, we get by two-fold integration by parts

$$\begin{aligned}
\left|s^{2}\iota\left(s\right)\right| &\leq \left|\int_{0}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\,kr}}{k}\mu\left(r\right)\partial_{r}^{2}\cos\left(sr\right)dr\right| = \left|\int_{0}^{\infty}\left\{\partial_{r}\cos\left(sr\right)\right\}\left\{\partial_{r}\left(\frac{\mathrm{e}^{\mathrm{i}\,kr}}{k}\mu\left(r\right)\right)\right\}dr\right| \\
&\leq \left|\int_{0}^{\infty}\cos\left(sr\right)\left\{\partial_{r}^{2}\left(\frac{\mathrm{e}^{\mathrm{i}\,kr}}{k}\mu\left(r\right)\right)\right\}dr+1\right| \\
&\leq k\left|\int_{0}^{\infty}\cos\left(sr\right)\mathrm{e}^{\mathrm{i}\,kr}\,\mu\left(r\right)dr\right| + \left|\int_{0}^{\infty}\cos\left(sr\right)\left(2\,\mathrm{i}\,\mathrm{e}^{\mathrm{i}\,kr}\,\mu'\left(r\right)+\frac{\mathrm{e}^{\mathrm{i}\,kr}}{k}\mu''\left(r\right)\right)dr+1\right| \\
&=: T^{\mathrm{I}}+T^{\mathrm{III}}.
\end{aligned}$$

The estimate $T^{\mathrm{III}} \leq C\left(1+\frac{1}{kR}\right)$ directly follows from the properties of the cutoff function μ (3.25). The term T^{I} was estimated already in Case 1 so that the proof of the case $d=1$ is complete.

Case 3a: $d=2$ and $r \leq 4R \leq 1/k$.
For brevity, we write

$$h_{k}\left(r\right) := H_{0}^{(1)}\left(kr\right) \quad \text{and} \quad j_{\nu,s}\left(r\right) := J_{\nu}\left(sr\right).$$

Estimate (A.3c) implies

$$\forall 0<r<4R\leq 1/k : \left|h_{k}\left(r\right)\right| \leq C\left(1+\left|\log kr\right|\right) \quad \text{and} \quad \forall r\geq 0 : \left|J_{0}\left(r\right)\right| \overset{[1,\ 9.1.60]}{\leq} 1.$$

Hence,

$$\left|\iota\left(s\right)\right| \leq C\int_{0}^{4R}\left(1+\left|\log kr\right|\right)rdr = CR^{2}\left(1+\left|\log\left(4kR\right)\right|\right).$$

For the estimate of $s^{m}\iota\left(s\right)$, we employ the relations (see [1, 9.1.30], [1, 9.1.1])

$$\left(rj_{1,s}\left(r\right)\right)' = rsj_{0,s}\left(r\right) \qquad \text{and} \qquad \left(rj_{0,s}'\left(r\right)\right)' = -rs^{2}j_{0,s}\left(r\right). \tag{3.34}$$

Integration by parts results in

$$\begin{aligned}
\left|s\iota\left(s\right)\right| &\leq C\left|\int_{0}^{\infty}h_{k}\mu\left(rj_{1,s}\right)'dr\right| = C\left|\int_{0}^{\infty}rj_{1,s}\left(\mu'h_{k}+\mu h_{k}'\right)dr\right| \\
&\overset{\text{(A.8), (A.3c), (A.11)}}{\leq} C\int_{0}^{4kR}r\left\{\frac{\left(1+\left|\log kr\right|\right)}{R}+\frac{1}{r}+k^{2}r\right\}dr \\
&\leq CR\left\{1+\left|\log kR\right|+k^{2}R^{2}\right\} \leq CR\left(1+\left|\log kR\right|\right) \leq CR\left|\log kR\right|.
\end{aligned}$$

Finally, we estimate $s^{2}\iota\left(s\right)$ by two-fold integration by parts

$$\left|s^{2}\iota\left(s\right)\right| = C\left|\int_{0}^{\infty}h_{k}\mu\left(rj_{0,s}'\right)'\right| \leq C\left(\left|\int_{0}^{4R}j_{0,s}\left(r\left(h_{k}\mu\right)'\right)'\right| + \left|\lim_{r\rightarrow 0}\left(rh_{k}'\left(r\right)\right)\right|\right). \tag{3.35}$$

Note that $\lim_{r\rightarrow 0}rh_{k}'\left(r\right) = 2\,\mathrm{i}\,/\pi$. For the first term, we use

$$\left(r\left(h_{k}\mu\right)'\right)' = \mu\left(rh_{k}'\right)' + 2r\mu'h_{k}' + \left(r\mu'\right)'h_{k}. \tag{3.36}$$

We employ (A.12) for the first, (3.25), (A.11) for the second, (3.25), and (A.3c) for the third term on the right-hand side in (3.36) to obtain

$$\left|\left(r\left(h_k\mu\right)'(r)\right)'\right| \leq Ck^2 r\left(1+\left|\log(kr)\right|\right)+\frac{1}{R}+\frac{R+r}{R^2}\left(1+\left|\log(kr)\right|\right).$$

Hence,

$$\begin{aligned}\left|s^2\iota(s)\right| &\leq C\left((kR)^2\left(1+\left|\log(kR)\right|\right)+1+\left|\log(kR)\right|\right)\\ &\leq C\left(1+\left|\log(kR)\right|\right).\end{aligned}$$

Case 3b: $d=2$ and $4Rk>1$.

We define $\varphi_k(r) := h_k(r)\,\mu(r)\,r$ and denote its antiderivative by $\Phi_k(r) := \int_{1/k}^{r}\varphi_k(t)\,dt$. We use the splitting

$$\iota(s)=\frac{\pi\,\mathrm{i}}{2}\int_0^{1/k}\varphi_k j_{0,s}+\frac{\pi\,\mathrm{i}}{2}\int_{1/k}^{4R}\varphi_k j_{0,s} =: \iota_{\mathrm{I}}(s)+\iota_{\mathrm{II}}(s).$$

For $\iota_{\mathrm{I}}(s)$, we employ the estimates as in Case 3a (with $4R$ replaced by $1/k$ therein) to obtain

$$\left|\iota_{\mathrm{I}}(s)\right| \leq \frac{C}{k^2}.$$

It remains to estimate $\iota_{\mathrm{II}}(s)$. Note that $j_{0,s}'=-sj_{1,s}$. There holds

$$\iota_{\mathrm{II}}(s)=\frac{\pi\,\mathrm{i}}{2}\int_{1/k}^{\infty}\varphi_k j_{0,s}=\frac{\pi\,\mathrm{i}}{2}\int_{1/k}^{4R}\Phi_k s j_{1,s}+\frac{\pi\,\mathrm{i}}{2}\Phi_k j_{0,s}\Big|_{r=1/k}^{4R}. \tag{3.37}$$

In the next step, we will estimate Φ_k. Let $\tilde{\varphi}_k(r) := \mathrm{e}^{-\mathrm{i}\,kr}\varphi_k(r)$ so that Φ_k can be written as

$$\begin{aligned}\Phi_k(r) &:= \int_{1/k}^{r}\mathrm{e}^{\mathrm{i}\,kt}\,\tilde{\varphi}_k(t)\,dt = -\int_{1/k}^{r}\frac{\mathrm{e}^{\mathrm{i}\,kt}}{\mathrm{i}\,k}\tilde{\varphi}_k'(t)\,dt+\frac{\varphi_k(t)}{\mathrm{i}\,k}\Big|_{t=1/k}^{r}\\ &= \underbrace{-\int_{1/k}^{r}\frac{\mathrm{e}^{\mathrm{i}\,kt}}{\mathrm{i}\,k}\tilde{\varphi}_k'(t)\,dt}_{=:\Phi_k^{\mathrm{I}}(r)}+\underbrace{\frac{1}{\mathrm{i}\,k}\,th_k\mu\big|_{t=1/k}^{r}}_{=:\Phi_k^{\mathrm{II}}(r)}.\end{aligned}$$

By using (A.6) and

$$\forall t>0: \quad \left|\left(t\mu(t)\right)'\right| \leq C,$$

we obtain

$$\begin{aligned}\left|\Phi_k^{\mathrm{I}}(r)\right| &\leq \frac{1}{k}\int_{1/k}^{r}\left|\tilde{\varphi}_k'\right|dt=\frac{1}{k}\int_{1/k}^{r}\left|t\mu\left(\mathrm{e}^{-\mathrm{i}\,kt}\,h_k\right)'+(t\mu)'\,\mathrm{e}^{-\mathrm{i}\,kt}\,h_k\right|dt\\ &\leq \frac{C}{k}\int_{1/k}^{r}\frac{1}{\sqrt{kt}}dt \leq \frac{C}{k}\sqrt{\frac{r}{k}}.\end{aligned}$$

The function Φ_k^{II} can be estimated by using (A.3a)

$$\left|\Phi_k^{\mathrm{II}}(r)\right| = \left|\frac{1}{\mathrm{i}\,k}\,th_k\mu\big|_{t=1/k}^{r}\right| \leq C\left(\frac{1}{k}\sqrt{\frac{r}{k}}+\frac{1}{k^2}\right) \overset{\sqrt{\frac{1}{k}}\leq\sqrt{r}}{\leq} \frac{C}{k}\sqrt{\frac{r}{k}}.$$

In summary we have proved

$$\left|\Phi_k(r)\right| \leq \frac{C}{k}\sqrt{\frac{r}{k}}.$$

By inserting this estimate and (A.3b) into (3.37) we get

$$\begin{aligned}\left|\iota_{\mathrm{II}}(s)\right| &\leq C\frac{\sqrt{s}}{k^{3/2}}\int_0^{4R}\sqrt{\frac{sr}{1+rs}}dr + \frac{C}{k}\sqrt{\frac{r}{k}}\Big|_{r=1/k}^{4R} \\ &\leq C\left(\frac{4R\sqrt{s}}{k^{3/2}} + \frac{1}{k}\sqrt{\frac{4R}{k}} + \frac{1}{k^2}\right).\end{aligned}$$

This leads to

$$\left|\iota(s)\right| \leq C\frac{R}{k}\sqrt{\frac{s}{k}} + \frac{R}{k}. \tag{3.38}$$

Next, we estimate $s^2\iota(s)$ by two-fold integration by parts

$$\begin{aligned}\left|s^2\iota(s)\right| &= C\left|\int_0^{\infty} h_k\mu\left(rj_{0,s}'\right)'\right| \\ &\leq C\left(\left|\int_0^{4R} j_{0,s}\left(r\left(h_k\mu\right)'\right)'\right| + \underbrace{\left|\lim_{r\to 0} rh_k'(r)\right|}_{=2/\pi}\right).\end{aligned} \tag{3.39}$$

The first summand can be split according to

$$\left|\int_0^{4R} j_{0,s}\left(r\left(h_k\mu\right)'\right)'\right| \leq \underbrace{\left|\int_0^{1/k} j_{0,s}\left(r\left(h_k\mu\right)'\right)'\right|}_{=:I_1} + \underbrace{\left|\int_{1/k}^{4R} j_{0,s}\left(r\left(h_k\mu\right)'\right)'\right|}_{=:I_2}. \tag{3.40}$$

We conclude from Case 3a that $|I_1| \leq C$ holds. For the second integral, we employ (A.3a), (3.25), (A.5), (A.7) to get

$$\begin{aligned}\left|\left(r\left(h_k\mu\right)'\right)'\right| &= \left|h_k\left(\mu' + r\mu''\right) + h_k'\left(\mu + 2r\mu'\right) + rh_k''\mu\right| \leq C\left(\frac{1}{r\sqrt{kr}} + \sqrt{\frac{k}{r}} + rk\sqrt{\frac{k}{r}}\right) \\ &\leq C\left(\frac{1}{r\sqrt{kr}} + k\sqrt{kr}\right).\end{aligned} \tag{3.41}$$

The combination of (3.40), (3.41), and (A.3b) leads to

$$I_2 \leq CkR\sqrt{\frac{Rk}{1+Rs}}.$$

Thus, we have proved

$$\left|s^2\iota(s)\right| \leq CkR\sqrt{\frac{Rk}{1+Rs}}. \tag{3.42}$$

For $0 \leq s \leq k$, we employ (3.38) and for $s > k$ we use (3.42) to obtain for $m = 0, 1, 2$

$$s^m \left|\iota\left(s\right)\right| \leq CRk^{m-1}.$$

For $|s| \geq 3k/2$ and $R \leq C$, the result can further be improved. In view of the assumption $R_0 \leq R \leq R_1$, we may take as our starting point (3.35), which leads immediately to the expression (3.40). The integral I_1 in (3.40) is already seen to be bounded independent of k. Since, by [1, 9.1.1],

$$(rh_k')' = -k^2 rh_k$$

we can write the integral I_2 as

$$I_2 = \left|\int_{1/k}^{4R} j_{0,s}\left(-k^2 rh_k\mu + 2rh_k'\mu' + (r\mu')'h_k\right)\right|.$$

Recalling that $\mu' \equiv 0$ on $(0, 2R)$, we can estimate I_2 by

$$I_2 \leq \underbrace{\left|\int_{1/k}^{4R} j_{0,s}k^2 rh_k\mu\right|}_{=:I_2^I} + CR \sup_{r\in(2R,4R)} \left\{|j_{0,s}h_k'| + |j_{0,s}h_k|\right\}.$$

We conclude from (A.3), (A.5), and (A.1) together with (A.2)

$$CR \sup_{r\in(2R,4R)} \left\{|j_{0,s}h_k'| + |j_{0,s}h_k|\right\} \leq CR\frac{1}{\sqrt{|s|R}}\left(\frac{1}{R\sqrt{Rk}} + \sqrt{\frac{k}{R}} + \frac{1}{\sqrt{Rk}}\right) \leq C,$$

where we used $|s| \geq 3/2k$ and the fact that $k \geq k_0$. It remains to bound I_2^I. Lemma A.1 allows us to write

$$I_2^I = \frac{2k^2}{\pi\sqrt{k|s|}}\left|\int_{1/k}^{4R} g^I(kr)\mu(r)\left\{e^{\mathrm{i}(s+k)r}f^I(sr) + e^{\mathrm{i}(s-k)r}f^{II}(sr)\right\}\right|.$$

Since f^I, f^{II}, g^I are bounded functions by Lemma A.1, an integration by parts leads to

$$\begin{aligned} I_2^I &\leq Ck\left(\frac{1}{|s+k|} + \frac{1}{|s-k|}\right) \\ &+ Ck\left|\int_{1/k}^{4R} \frac{e^{\mathrm{i}(s+k)r}}{s+k}\partial_r\left(f^I(sr)g^I(kr)\mu(r)\right) + \frac{e^{\mathrm{i}(s-k)r}}{s-k}\partial_r\left(f^{II}(sr)g^I(kr)\mu(r)\right)\right|. \end{aligned}$$

Since $|s| \geq 3/2k$, Lemma A.1 provides the estimates

$$\left|\partial_r\left(f^I(sr)g^I(kr)\mu(r)\right)\right| + \left|\partial_r\left(f^{II}(sr)g^I(kr)\mu(r)\right)\right| \leq C, \qquad r \geq 1/k$$

Combining these results, we arrive at the desired $I_2^I \leq C$. ∎

3.3 Existence and uniqueness

Existence, uniqueness, and well-posedness of problem (2.5) has been studied in much more generality (concerning the assumption on the domain Ω) in [12] by using different techniques.

The main goal of the estimates which we have derived in the previous sections is their application to the proof of the discrete stability for the finite element discretization and the convergence rates. However, since existence, uniqueness, and well-posedness for our model problem are simple by-products we state them in passing.

Theorem 3.6 *Let Ω be a ball of radius $R > 0$. Then, there exists a constant $C(\Omega, k) > 0$ such that for all $f \in (H^1(\Omega))'$ the unique solution u of problem (2.5) satisfies*

$$\|u\|_{\mathcal{H}} \leq C(\Omega, k) \|f\|_{H^1(\Omega)'} .$$

Proof. The coercivity of the bilinear form $a(u, v)$ follows from the compact embedding $H^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega)$ and (3.4b), (3.4d):

$$\operatorname{Re} a(u, u) \geq \|u\|_{\mathcal{H}}^2 - 2k^2 \|u\|_{L^2(\Omega)}^2 - \operatorname{Re} \int_\Gamma (Tu)\, \bar{u} \geq \|u\|_{\mathcal{H}}^2 - 2k^2 \|u\|_{L^2(\Omega)}^2 .$$

Next, we show uniqueness of the adjoint problem:

$$\overline{a(v, u)} = 0 \quad \forall v \in H^1(\Omega) \implies u = 0.$$

Let $u \in H^1(\Omega)$ be a solution of the homogeneous adjoint problem. We choose $v = u$ and consider the imaginary part:

$$0 = \operatorname{Im} \overline{a(u, u)} = -\operatorname{Im} \overline{\int_\Gamma (Tu)\, \bar{u}} = \operatorname{Im} \int_\Gamma (Tu)\, \bar{u}.$$

Lemma 3.3 implies $u = 0$ on Γ. Hence, $u \in H_0^1(\Omega)$ and satisfies

$$\int_\Omega \langle \nabla u, \nabla \overline{v} \rangle = k^2 \int_\Omega u \overline{v} \qquad \forall v \in H^1(\Omega) . \tag{3.43}$$

This means in particular that $u \in H_0^1(\Omega)$ is an eigenfunction of $(-\Delta)^{-1}$ with eigenvalue k^{-2}. However, for any $\tilde{\Omega} \supset \Omega$, equation (3.43) implies that the extension

$$\tilde{u}(x) := \begin{cases} u(x) & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$$

satisfies (3.43) with Ω replaced by $\tilde{\Omega}$, i.e., $\tilde{u}$ is also an eigenfunction of $(-\Delta)^{-1}$ with eigenvalue k^{-2} on any domain $\tilde{\Omega} \supset \Omega$. A simple scaling argument shows that this is impossible. ■

Note that the proof of Theorem 3.6 does not provide how the constant $C(\Omega, k)$ depends on the wave number. In [12], this question has been investigated in much more generality and, hence, will not be discussed here. The Fourier analysis which we developed in Section 3.2 give explicit bounds on this constant provided the right-hand side is in $L^2(\Omega)$.

Lemma 3.7 *For any $f \in L^2(\Omega)$ and $v := N_k f$, there holds*

$$\|v\|_{\mathcal{H}} \leq C \|f\|_{L^2(\Omega)} ,$$

where C only depends on k_0 and R_0 (cf. (2.2), (3.3)).

Proof. The radius of the minimal ball that contains Ω is denoted by R_Ω. If $4kR_\Omega > 1$, the estimates

$$\|v\|_{L^2(\Omega)} \leq C \frac{R_\Omega}{k} \|f\|_{L^2(\Omega)}$$

and

$$\|\nabla v\|_{L^2(\Omega)} \leq C \left(\frac{1}{k_0} + R_\Omega \right) \|f\|_{L^2(\Omega)}$$

follow from Lemma 3.5. If $\alpha < 4kR_\Omega \leq 1$, then $|\log kR_\Omega| \leq |\log \alpha|$. Hence, both estimates remain valid (cf. Lemma 3.5), possibly with a different constant C which, in addition, depend on α. ■

Theorem 3.8 *Let Ω be a ball of radius R and boundary Γ. Then, there exists $C_c > 0$ that only depend on k_0 and R_0 (cf. (2.2), (3.3)) such that for all $u, v \in H^1(\Omega)$*

$$|a(u,v)| \leq C_c \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}.$$

Proof. The estimate

$$|a(u,v)| \leq |u|_{H^1(\Omega)} |v|_{H^1(\Omega)} + k^2 \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \left| \int_\Gamma (Tu)\, \bar{v} \right|$$

is obvious. Hence, the assertion follows from Lemma 3.3. ■.

4 Stability and convergence analysis

This section is devoted to the analysis of the discrete problem (2.7) for the finite-dimensional space $S \subset H^1(\Omega)$; we will provide conditions on S under which unique solvability and quasi-optimality of (2.7) can be guaranteed.

We employ the generalization of the theory of [31] that has been developed in [37]. There, a measure of "almost invariance" of the approximation space S under the solution operator of an adjoint Helmholtz problem has been introduced.

Adjoint Problem:

For given $f \in L^2(\Omega)$, find $z \in H^1(\Omega)$ such that

$$a(v,z) = (f,v)_{L^2(\Omega)} \qquad \forall v \in H^1(\Omega).$$

Note that the strong formulation to this problem is: Find z such that

$$\begin{aligned} -\Delta z - k^2 z &= f && \text{in } \Omega, \\ \frac{\partial z}{\partial n} &= T^\star z && \text{on } \Gamma, \end{aligned} \tag{4.1}$$

where $T^\star$ is the adjoint of T. The operator $T^\star$ can be expressed as the normal trace applied to the solution operator to the problem

$$\text{find } w \in H^1_{\text{loc}}\left(\Omega^+\right) \text{ such that } \begin{cases} \left(-\Delta - k^2\right) w = 0 & \text{in } \Omega^+, \\ w = g & \text{on } \Gamma, \\ \left| \dfrac{\partial w}{\partial r} + \mathrm{i}\, k w \right| = o\left(\|\mathbf{x}\|^{\frac{1-d}{2}} \right) & \|\mathbf{x}\| \to \infty. \end{cases}$$

The solution operator $f \mapsto z$ for problem (4.1) is denoted by $N_k^\star : L^2(\Omega) \rightarrow H^1(\Omega)$ and explicitly given by

$$z := N_k^\star f := \int_\Omega \overline{G_k}(x-y) f(y)\, ds_y. \tag{4.2}$$

Remark 4.1 *The stability estimate*

$$\|z\|_{\mathcal{H}} \leq C \|f\|_{L^2(\Omega)}$$

holds as in Lemma 3.7, because $\overline{z} = N_k \overline{f}$ *and* $\|z\|_{\mathcal{H}} = \|\overline{z}\|_{\mathcal{H}}$, $\left\|\overline{f}\right\|_{L^2(\Omega)} = \|f\|_{L^2(\Omega)}$.

For the stability of the discrete problem, the following measure of almost invariance plays a crucial role

$$\eta(S) := \sup_{f \in L^2(\Omega)\setminus\{0\}} \inf_{v \in S} \frac{\|N_k^\star f - v\|_{\mathcal{H}}}{\|f\|_{L^2(\Omega)}}. \tag{4.3}$$

(Note that the quantity $\eta(S)$ was denoted in [37] by $\tilde{\eta}(S)$.)

4.1 Discrete stability

In this section, we will prove the discrete stability in the form of an inf-sup condition.

Theorem 4.2 *Let the assumptions of Lemma 3.3 be satisfied. Assume that the space S is chosen such that*

$$k\eta(S) \leq \frac{1}{4C_c}. \tag{4.4}$$

Then, the discrete inf-sup constant satisfies

$$\inf_{u \in S} \sup_{v \in S\setminus\{0\}} \frac{|a(u,v)|}{\|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}} \geq \frac{1/2}{1 + (2C_c)^{-1} + k}$$

and this ensures existence and uniqueness of the discrete problem (2.7).

Proof. Let $u \in S$ and set $z := 2k^2 N_k^\star u$. Then,

$$\begin{aligned} a(u, u+z) &= \left(\int_\Omega \langle \nabla u, \nabla \overline{u} \rangle + k^2 |u|^2 - \int_\Gamma (Tu)\,\overline{u} \right) + a(u,z) - 2k^2 \int_\Omega |u|^2 \\ &= \int_\Omega \langle \nabla u, \nabla \overline{u} \rangle + k^2 |u|^2 - \int_\Gamma (Tu)\,\overline{u}. \end{aligned}$$

We derive from Lemma 3.3

$$\operatorname{Re} a(u, u+z) \geq \|u\|_{\mathcal{H}}^2.$$

Let $z_S \in S$ denote the best approximation of z with respect to the $\|\cdot\|_{\mathcal{H}}$-norm. Then,

$$\begin{aligned} \operatorname{Re} a(u, u+z_S) &\geq \operatorname{Re} a(u, u+z) - |a(u, z - z_S)| \overset{\text{Thm. 3.8}}{\geq} \|u\|_{\mathcal{H}}^2 - C_c \|u\|_{\mathcal{H}} \|z - z_S\|_{\mathcal{H}} \\ &\geq \|u\|_{\mathcal{H}} \left(\|u\|_{\mathcal{H}} - 2k^2 C_c \eta(S) \|u\|_{L^2(\Omega)} \right) \geq (1 - 2kC_c\eta(S)) \|u\|_{\mathcal{H}}^2. \end{aligned}$$

The stability of the continuous problem (cf. Lemma 3.7) implies

$$\begin{aligned}\|u+z_S\|_{\mathcal{H}} &\leq \|u\|_{\mathcal{H}}+\|z-z_S\|_{\mathcal{H}}+\|z\|_{\mathcal{H}} \leq \|u\|_{\mathcal{H}}+2k^2\eta(S)\|u\|_{L^2(\Omega)}+2k^2\|u\|_{L^2(\Omega)} \\ &\leq (1+2k\eta(S)+k)\|u\|_{\mathcal{H}}\end{aligned}$$

so that

$$\operatorname{Re} a(u,u+z_S) \geq \frac{1-2C_c k\eta(S)}{1+2k\eta(S)+k}\|u\|_{\mathcal{H}}\|u+z_S\|_{\mathcal{H}}.$$

Thus, we have proved

$$\inf_{u\in S}\sup_{v\in S\backslash\{0\}}\frac{|a(u,v)|}{\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}}} \geq \frac{C}{k}.$$

■

4.2 Convergence analysis

The convergence of the finite element discretization is proved by applying the theory as developed in [37] (see also [7,31,38], [8, Sec.5.7]).

Theorem 4.3 *Let the assumptions of Theorem 4.2 be satisfied.*
Then

$$\|e\|_{\mathcal{H}} \leq 2C_c \inf_{v\in S}\|u-v\|_{\mathcal{H}}. \tag{4.5}$$

The L^2*-error satisfies*

$$\|e\|_{L^2(\Omega)} \leq C_c\eta(S)\|e\|_{\mathcal{H}}.$$

Proof. In the first step, we will estimate the L^2-error by the H^1-error and employ the Aubin-Nitsche technique. The Galerkin error is denoted by $e=u-u_S$. We set $\psi := N_k^{\star}e$ (cf. (4.2)) and denote by $\psi_S \in S$ the best approximation of ψ with respect to the $\mathcal{H}$-norm.

The L^2-error can be estimated by

$$\begin{aligned}\|e\|^2_{L^2(\Omega)} &= a(e,\psi) \leq a(e,\psi-\psi_S) \leq C_c\|e\|_{\mathcal{H}}\|\psi-\psi_S\|_{\mathcal{H}} \\ &\leq C_c\eta(S)\|e\|_{\mathcal{H}}\|e\|_{L^2(\Omega)},\end{aligned} \tag{4.6}$$

i.e.,

$$\|e\|_{L^2(\Omega)} \leq C_c\eta(S)\|e\|_{\mathcal{H}}. \tag{4.7}$$

To estimate the $\mathcal{H}$-norm of the error we proceed as follows. Note that (3.4b), (3.4d) imply

$$\operatorname{Re}(Tu,u)_{L^2(\Gamma)} \leq 0. \tag{4.8}$$

Hence, for any $v_S \in S$

$$\begin{aligned}\|e\|^2_{\mathcal{H}} &= \operatorname{Re}(a(e,e)) + \left\{\|e\|^2_{\mathcal{H}} - \operatorname{Re} a(e,e)\right\} \\ &= \operatorname{Re} a(e,u-v_S) + 2k^2\|e\|^2_{L^2(\Omega)} + \operatorname{Re}\int_\Gamma (Te)\,\bar{e} \\ &\overset{(4.7),\,(4.8)}{\leq} C_c\|e\|_{\mathcal{H}}\|u-v_S\|_{\mathcal{H}} + 2(kC_c\eta(S))^2\|e\|^2_{\mathcal{H}}.\end{aligned}$$

Note that (4.4) implies

$$2\left(kC_c\eta\left(S\right)\right)^2 \leq \frac{1}{2} \tag{4.9}$$

so that we arrive at the final estimate

$$\left\|e\right\|_{\mathcal{H}} \leq 2C_c \left\|u - v_S\right\|_{\mathcal{H}}.$$

■

5 Example: *hp*-FEM

Theorem 4.2, 4.3 show quasi-optimality of arbitrary approximation spaces under the assumption (4.4) on the measure of almost invariance $\eta\left(S\right)$. However, for concrete finite element spaces, or generalizations thereof, the verification of condition (4.4) is far from trivial. The purpose of this section is two-fold: firstly, we show that for classical higher order FEM spaces the assumption (4.4) can be met under a relatively mild condition on the local polynomial order of the classical FEM space; in particular, we will demonstrate that for spaces consisting of piecewise polynomials of degree p on quasi-uniform meshes that satisfy the side condition $p \geq c \ln k$, the key condition (4.4) is satisfied. Secondly, we derive conditions on the approximation space that may be easier to ascertain in practice than the condition (4.4).

In view of the fact that the circle (in 2D) and the sphere (in 3D) are relevant geometries for our theory (recall that Thm. 4.2, 4.3 have been shown for circles/spheres), we consider triangulations with curved elements that permit inclusion of these geometries. We adopt the setting of [13]. The triangulation $\mathcal{T}_h$ consists of elements which are the image of the reference triangle (in 2D) or the reference tetrahedron (in 3D). We do not allow hanging nodes and assume – as is standard – that the element maps of elements sharing an edge or a face induce the same parametrization on that edge or face. The maximal mesh width is denoted by $h := \max_{K\in\mathcal{T}_h} \operatorname{diam} K$. Additionally, we make the following assumption on the element maps $F_K : \widehat{K} \to K$ but, first, have to introduce some notation. For a function $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^d$, we write

$$|\nabla^n u(x)|^2 = \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|=n} \frac{n!}{\alpha!}|D^\alpha u(x)|^2. \tag{5.1}$$

For later purposes, we recall the multinomial formula and a simple fact that follows from the Cauchy-Schwarz inequality for sums:

$$\frac{d^n}{n!} = \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|=n} \frac{1}{\alpha!}, \tag{5.2}$$

$$\sum_{\alpha\in\mathbb{N}_0^d:|\alpha|=n} \frac{1}{\alpha!}|D^\alpha u(x)| \leq \frac{1}{n!}d^{n/2}|\nabla^n u(x)|. \tag{5.3}$$

Assumption 5.1 (quasi-uniform regular triangulation) *Each element map F_K can be written as $F_K = R_K \circ A_K$, where A_K is an affine map and the maps R_K and A_K satisfy for constants C_{affine}, C_{metric}, $\gamma > 0$ independent of h:*

$$\|A_K'\|_{L^\infty(\widehat{K})} \leq C_{\text{affine}} h, \qquad \|(A_K')^{-1}\|_{L^\infty(\widehat{K})} \leq C_{\text{affine}} h^{-1}$$
$$\|(R_K')^{-1}\|_{L^\infty(\widetilde{K})} \leq C_{\text{metric}}, \qquad \|\nabla^n R_K\|_{L^\infty(\widetilde{K})} \leq C_{\text{metric}} \gamma^n n! \qquad \forall n \in \mathbb{N}_0.$$

Here, $\widetilde{K} = A_K(\widehat{K})$.

Remark 5.2 *Triangulations satisfying Assumption 5.1 can be obtained by patchwise construction of the mesh: Let* $\mathcal{T}^{macro}$ *be a* fixed *triangulation (with curved elements) with analytic element maps that resolves the geometry. If the triangulation* $\mathcal{T}_h$ *is obtained by quasi-uniform refinements of the reference element* $\widehat{K}$ *and the final mesh is obtained by mapping the subdivisions of the reference element with the macro element maps, then the resulting element maps satisfy the assumptions of Assumption 5.1.*

For meshes $\mathcal{T}_h$ satisfying Assumption 5.1 with element maps F_K we denote the usual space of piecewise (mapped) polynomials by $S^{p,1}(\mathcal{T}_h) := \{u \in H^1(\Omega) \,|\, \forall K \in \mathcal{T}_h\colon u|_K \circ F_K \in \mathcal{P}_p\}$, where $\mathcal{P}_p$ denotes the space of polynomials of degree p. The construction of approximants of a given (sufficiently smooth) function from the space $S^{p,1}(\mathcal{T}_h)$ is most conveniently achieved in an element-by-element fashion. To that end, we introduce the following definition:

Definition 5.3 (element-by-element construction) *Let* $\widehat{K}$ *be the reference simplex in* $\mathbb{R}^d$, $d \in \{2,3\}$. *A polynomial* π *is said to permit* an element-by-element *construction of polynomial degree* p *for* $u \in H^s(\widehat{K})$, $s > d/2$, *if:*

(i) $\pi(V) = u(V)$ *for all* $d+1$ *vertices* V *of* $\widehat{K}$,

(ii) for every edge e *the restriction* $\pi|_e \in \mathcal{P}_p$ *is the unique minimizer of*

$$\pi \mapsto p^{1/2}\|u-\pi\|_{L^2(e)} + \|u-\pi\|_{H^{1/2}_{00}(e)} \tag{5.4}$$

under the constraint that π *satisfies (i); here the Sobolev norm* $H^{1/2}_{00}$ *is defined in (B.1).*

(iii) (for $d=3$*) for every face* f *the restriction* $\pi|_f \in \mathcal{P}_p$ *is the unique minimizer of*

$$\pi \mapsto p\|u-\pi\|_{L^2(f)} + \|u-\pi\|_{H^1(f)} \tag{5.5}$$

under the constraint that π *satisfies (i), (ii) for all vertices and edges of the face* f.

We are now in position to show that the solution $v = N_k^\star f$ can be approximated well by the FEM space $S^{p,1}(\mathcal{T}_h)$ provided that kh/p is sufficiently small and $p \geq c\ln k$.

Theorem 5.4 *Let* $d \in \{1,2,3\}$ *and* $\Omega \subset \mathbb{R}^d$ *be a bounded domain. Then there exist constants* C, $\sigma > 0$ *that depend solely on the constants appearing in Assumption 5.1 such that for every* $f \in L^2(\Omega)$ *the function* $v := N_k^\star f$ *satisfies*

$$\inf_{w\in S^{p,1}(\mathcal{T}_h)} k\|v-w\|_{\mathcal{H}} \leq C\|f\|_{L^2(\Omega)}\left(1+\frac{kh}{p}\right)\left\{\frac{kh}{p} + k\left(\frac{kh}{\sigma p}\right)^p\right\}.$$

Proof. We will only prove the cases $d \in \{2,3\}$. The case $d=1$ follows by similar arguments where the appeal to Theorem B.4 and Lemma C.3 is replaced with that to [39, Thm. 3.17].

By Lemma 3.4, we write $v = v_{H^2} + v_{\mathcal{A}}$ with $v_{H^2} \in H^2(\Omega)$ and $v_{\mathcal{A}}$ analytic; we have the bounds (cf. Remark 4.1)

$$\|v_{H^2}\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}, \qquad \|\nabla^p v_{\mathcal{A}}\|_{L^2(\Omega)} \leq C(\gamma k)^{p-1}\|f\|_{L^2(\Omega)} \qquad \forall p \in \mathbb{N}_0.$$

We approximate v_{H^2} and $v_{\mathcal{A}}$ separately. Theorem B.4 provides an approximant $w_{H^2} \in S^{p,1}(\mathcal{T}_h)$ such that for every $K \in\mathcal{T}_h$ we have, for $q = 0, 1$,

$$\|v_{H^2} - w_{H^2}\|_{H^q(K)} \le C\left(\frac{h}{p}\right)^{2-q} \|v_{H^2}\|_{H^2(K)} \qquad \forall K \in \mathcal{T}_h.$$

Hence, by summation over all elements, we arrive at

$$k\|v_{H^2} - w_{H^2}\|_{\mathcal{H}} \le C\left(\frac{kh}{p} + \left(\frac{kh}{p}\right)^2\right)\|f\|_{L^2(\Omega)}.$$

We now turn to the approximation of $v_{\mathcal{A}}$. Again, we construct the approximation $w_{\mathcal{A}} \in S^{p,1}(\mathcal{T}_h)$ in an element-by-element fashion. We start by defining for each element $K \in\mathcal{T}_h$ the constant C_K by

$$C_K^2 := \sum_{p\in\mathbb{N}_0} \frac{\|\nabla^p v_{\mathcal{A}}\|^2_{L^2(K)}}{(2\gamma k)^{2p}} \tag{5.6}$$

and we note

$$\|\nabla^p v_{\mathcal{A}}\|_{L^2(K)} \le C(2\gamma k)^p C_K \qquad \forall p \in \mathbb{N}_0, \tag{5.7}$$

$$\sum_{K\in\mathcal{T}_h} C_K^2 \le \frac{4}{3}\left(\frac{C}{\gamma k}\right)^2 \|f\|^2_{L^2(\Omega)}. \tag{5.8}$$

Let the element map for K be $F_K = R_K \circ A_K$. Lemma C.1 gives that the function $\tilde{v} := v_{\mathcal{A}}|_K \circ R_K$ satisfies, for suitable constants $\tilde{C}$, C (which depend additionally on the constants describing the analyticity of the element maps R_K)

$$\|\nabla^p \tilde{v}\|_{L^2(\widetilde{K})} \le C\tilde{C}^p \max\{p, k\}^p C_K \qquad \forall p \in \mathbb{N}_0.$$

Since A_K is affine, the function $\hat{v} := v_{\mathcal{A}}|_K \circ F_K = \tilde{v} \circ A_K$ therefore satisfies

$$\|\nabla^p \hat{v}\|_{L^2(\widehat{K})} \le Ch^{-d/2}\tilde{C}^p h^p \max\{p, k\}^p C_K \qquad \forall p \in \mathbb{N}_0.$$

Hence, the assumptions of Lemma C.3 are satisfied, and we get an approximation w on the element K that admits an element-by-element construction and satisfies for $q \in \{0, 1\}$

$$\|v_{\mathcal{A}} - w\|_{H^q(K)} \le Ch^{d/2-q}h^{-d/2}C_K\left\{\left(\frac{h}{h+\sigma}\right)^{p+1} + \left(\frac{kh}{\sigma p}\right)^{p+1}\right\}.$$

Summation over all elements $K \in\mathcal{T}_h$ gives

$$\|v_{\mathcal{A}} - w\|^2_{\mathcal{H}} \le \left[\left(\frac{h}{h+\sigma}\right)^{2p} + k^2\left(\frac{h}{h+\sigma}\right)^{2p+2} + \frac{k^2}{p^2}\left(\frac{kh}{\sigma p}\right)^{2p} + k^2\left(\frac{kh}{\sigma p}\right)^{2p+2}\right]\sum_{K\in\mathcal{T}_h} C_K^2. \tag{5.9}$$

The combination of (5.9) and (5.8) yields

$$k\|v_{\mathcal{A}} - w\|_{\mathcal{H}} \le C\left[\left(\frac{h}{h+\sigma}\right)^p\left(1 + \frac{hk}{h+\sigma}\right) + k\left(\frac{kh}{\sigma p}\right)^p\left(\frac{1}{p} + \frac{kh}{\sigma p}\right)\right]\|f\|_{L^2(\Omega)}.$$

Furthermore, we estimate using $h \leq \operatorname{diam}\Omega$ and $\sigma > 0$ (independent of h)

$$\left(\frac{h}{h+\sigma}\right)^p \left(1 + \frac{kh}{\sigma+h}\right) \leq Ch(1+kh)\left(\frac{h}{\sigma+h}\right)^{p-1} \leq Ch(1+kh)p^{-2} \leq C\frac{h}{p}\left(\frac{1}{p} + \frac{kh}{p}\right).$$

We therefore arrive at

$$k\|v_{\mathcal{A}} - w\|_{\mathcal{H}} \leq C\left(\frac{1}{p} + \frac{kh}{p}\right)\left[\frac{kh}{p} + k\left(\frac{kh}{\sigma p}\right)^p\right]\|f\|_{L^2(\Omega)},$$

which completes the proof of the theorem. ■

Combining Theorems 5.4, 4.3 produces the condition (1.1) for quasi-optimality of the *hp*-FEM announced in the Introduction. We extract from Theorem 5.4 that quasi-optimality of the h-version FEM can be achieved under the side condition that $p \geq C \log k$:

Corollary 5.5 *Let the assumptions of Lemma 3.3 and Theorem 5.4 be satisfied. Then there exist constants c_1, $c_2 > 0$ independent of k, h, and p such that (4.4) is implied by the following condition:*

$$\frac{kh}{p} \leq c_1 \qquad \textit{together with} \qquad p \geq c_2 \ln k. \tag{5.10}$$

Proof. Theorem 5.4 implies

$$k\eta(S) \leq C\left(1 + \frac{kh}{p}\right)\left(\frac{kh}{p} + k\left(\frac{kh}{\sigma p}\right)^p\right).$$

The right-hand side needs to be bounded by $1/C_c$. It is now easy to see that we can select c_1, c_2 such that this can be ensured. ■

Remark 5.6 *To the best of the authors' knowledge, discrete stability in 2D and 3D has only been shown under much more restrictive conditions than (5.10), e.g., the condition $k^2h \lesssim 1$. Even in one dimension, condition (5.10) improves the stability condition $kh \lesssim 1$ that was required in [26].*

Next, we reformulate Theorem 5.4 by deriving the statement under some conditions on abstract approximation spaces that are easier to verify than a direct proof of (4.4).

The key step in Theorem 5.4 is the ability to decompose $v = N_k^\star f$ into an analytic, but oscillatory part and an H^2-regular part and to approximate each part separately. This gives rise to the definition of two types of approximation properties.

Definition 5.7 *For given C^{I}, C^{II}, $\gamma > 0$, and $k \geq k_0 > 0$ let*

$$\mathcal{H}^{\mathrm{osc}}(C^I, \gamma, k) := \left\{ v \in H^1(\Omega) \;\middle|\; \|\nabla^p v\|_{L^2(\Omega)} \leq C^{\mathrm{I}} (\gamma\,|k|)^{p-1} \quad \forall p \in \mathbb{N}_0 \right\},$$

$$\mathcal{H}^{H^2}(C^{\mathrm{II}}) := \left\{ v \in H^2(\Omega) \mid \|v\|_{H^2(\Omega)} \leq C^{\mathrm{II}} \right\}.$$

Let $S \subset H^1(\Omega)$ be the—possibly k-dependent—finite dimensional approximation space for the Galerkin method. The approximation properties for the oscillatory and the H^2-part are then defined as:

$$\eta^{\mathrm{I}}_{\mathrm{apx}}(S,k) := \frac{k}{C^{\mathrm{I}}} \sup_{v \in \mathcal{H}^{\mathcal{A}}(C^I,\gamma,k)} \inf_{w \in S} \|v - w\|_{\mathcal{H}}, \tag{5.11}$$

$$\eta^{\mathrm{II}}_{\mathrm{apx}}(S,k) := \frac{k}{C^{\mathrm{II}}} \sup_{v \in \mathcal{H}^{H^2}(C^{\mathrm{II}})} \inf_{w \in S} \|v - w\|_{\mathcal{H}}.$$

Corollary 5.8 *Let $d \in \{1,2,3\}$ and assume the hypotheses of Lemma 3.4. Let $S \subset H^1(\Omega)$ be a finite dimensional approximation space. Then*

$$k\eta(S) \leq C\left\{\eta_{\mathrm{apx}}^{\mathrm{I}}(S,k) + \eta_{\mathrm{apx}}^{\mathrm{II}}(S,k)\right\}.$$

Proof. We split $v = N_k^\star f$ as in the proof of Theorem 5.4: $v = v_{H^2} + v_{\mathcal{A}}$. Then, $v_{\mathcal{A}} \in \mathcal{H}^{\mathcal{A}}(C^{\mathrm{I}}, \gamma, k)$ if we choose $C^{\mathrm{I}} = C\|f\|_{L^2(\Omega)}$ and C, γ as in (3.24b). Hence, the minimizer $w_{\mathrm{opt}}^{\mathrm{I}} \in S$ of the right-hand side in (5.11) satisfies

$$k\left\|v_{\mathcal{A}} - w_{\mathrm{opt}}^{\mathrm{I}}\right\|_{\mathcal{H}} \leq C\|f\|_{L^2(\Omega)}\,\eta_{\mathrm{apx}}^{\mathrm{I}}(S,k).$$

In a similar fashion, we obtain

$$k\left\|v_{H^2} - w_{\mathrm{opt}}^{\mathrm{II}}\right\|_{\mathcal{H}} \leq C\|f\|_{L^2(\Omega)}\,\eta_{\mathrm{apx}}^{\mathrm{II}}(S,k).$$

By the triangle inequality we get for $w_{\mathrm{opt}} = w_{\mathrm{opt}}^{\mathrm{I}} + w_{\mathrm{opt}}^{\mathrm{II}}$ the estimate

$$k\|v - w_{\mathrm{opt}}\|_{\mathcal{H}} \leq C\left\{\eta_{\mathrm{apx}}^{\mathrm{I}}(S,k) + \eta_{\mathrm{apx}}^{\mathrm{II}}(S,k)\right\}\|f\|_{L^2(\Omega)}.$$

■

A Estimate of Bessel functions

In this appendix we derive some estimates for the Hankel and Bessel functions that are used in Subsection 3.2. First, we will consider the case of large arguments $z > 1$ and then the case $0 < z \leq 1$.

Case 1: $z = kr > 1$.

From [1, 9.2.5-9.2.16], we conclude that the Hankel functions $H_\ell^{(1)}$ and Bessel functions J_ℓ, $\ell \in \mathbb{N}_0$, can be written in the form

$$J_\ell(z) \overset{[1,\ 9.2.5]}{=} \sqrt{\frac{2}{\pi z}}\left(P_\ell(z)\cos\chi - Q_\ell(z)\sin\chi\right), \tag{A.1a}$$

$$H_\ell^{(1)}(z) \overset{[1,\ 9.2.7]}{=} \sqrt{\frac{2}{\pi z}}\left(P_\ell(z) + \mathrm{i}\,Q_\ell(z)\right)\mathrm{e}^{\mathrm{i}\,\chi}, \tag{A.1b}$$

where $\chi := z - \pi/4$. The functions P_ℓ, Q_ℓ have the following property: Upon defining

$$P_{\ell,m}(z) := \sum_{k=0}^{m} \frac{\beta_{\ell,2k}}{z^{2k}} \tag{A.1c}$$

$$Q_{\ell,m}(z) = -\mathrm{i}\sum_{k=0}^{m} \frac{\beta_{\ell,2k+1}}{z^{2k+1}} \tag{A.1d}$$

with

$$\beta_{\ell,k} := \frac{\mathrm{i}^k\,\gamma_{\ell,k}}{2^{3k}k!} \quad \text{and} \quad \gamma_{\ell,m} \text{ as in (3.11)}$$

there holds

$$\forall z > 0 \quad \forall m > \frac{\ell}{2} - \frac{1}{4} \quad \begin{cases} \left|\left(P_\ell - P_{\ell,m-1}\right)(z)\right| \leq \dfrac{\left|\gamma_{\ell,2m}\right|}{2^{6m}\,(2m)!}\dfrac{1}{z^{2m}}, \\[2ex] \left|\left(Q_\ell - Q_{\ell,m-1}\right)(z)\right| \leq \dfrac{\gamma_{\ell,2m+1}}{2^{6m+2}\,(2m+1)!}\dfrac{1}{z^{2m+1}}. \end{cases}$$

Note that in Subsection 3.2 the order ℓ is always small, i.e., $\ell \in \{0, 1\}$ and, hence, we do not analyze the dependence of the constants on ℓ in the following estimates.

We conclude that

$$\forall z \geq 1 : \left|P_\ell(z)\right| \leq \left|P_{\ell,\lceil \frac{\ell}{2} \rceil - 1}(z)\right| + \frac{\left|\gamma_{\ell,2\lceil \frac{\ell}{2} \rceil}\right|}{2^{6\lceil \frac{\ell}{2} \rceil}\left(2\lceil \frac{\ell}{2} \rceil\right)!}\frac{1}{z^{2\lceil \frac{\ell}{2} \rceil}} \leq C. \tag{A.2a}$$

and similarly

$$\forall z \geq 1 : \quad \left|Q_\ell(z)\right| \leq \frac{C}{z}, \quad \left|P_\ell'(z)\right| \leq \frac{C}{z^3}, \quad \left|Q_\ell'(z)\right| \leq \frac{C}{z^2}. \tag{A.2b}$$

Hence, for $f \in \left\{J_\ell, H_\ell^{(1)}\right\}$, $\ell \in \mathbb{N}_0$, there holds

$$\forall z \geq 1 \quad \left|f(z)\right| \leq \frac{C}{\sqrt{z}} \tag{A.3a}$$

and the combination with $\left|J_\ell(z)\right| \overset{[1,\ 9.1.60]}{\leq} C$ for all $z \geq 0$ yields

$$\forall z \geq 0 \quad \left|J_\ell(z)\right| \leq C\sqrt{\frac{1}{1+z}}. \tag{A.3b}$$

We need an estimate of the derivative at the argument $z = kr$ for $z \geq 1$. The derivative of (A.1b) can be written in the form

$$\begin{aligned} \frac{d}{dr}H_0^{(1)}(kr) &\overset{[1,\ 9.2.7]}{=} C\,\mathrm{e}^{\mathrm{i}\,kr}\sqrt{\frac{1}{kr}}\frac{d}{dr}\left(P_0(kr) + \mathrm{i}\,Q_0(kr)\right) \\ &\quad + C\left(P_0(kr) + \mathrm{i}\,Q_0(kr)\right)\frac{d}{dr}\left(\mathrm{e}^{\mathrm{i}\,kr}\sqrt{\frac{1}{kr}}\right). \end{aligned} \tag{A.4}$$

The combination of (A.4) and (A.2) leads to

$$\left|\frac{d}{dr}H_0^{(1)}(kr)\right| \leq C\left(\frac{1}{r\sqrt{kr}} + \sqrt{\frac{k}{r}}\right). \tag{A.5}$$

We also need an estimate of $\partial_r\left(\mathrm{e}^{-\mathrm{i}\,kr}\,H_0^{(1)}(kr)\right)$. Employing (A.1b) we obtain

$$\frac{d}{dr}\left(\mathrm{e}^{-\mathrm{i}\,kr}\,H_0^{(1)}(kr)\right) = C\frac{d}{dr}\left(\sqrt{\frac{1}{kr}}\left(P(0,kr) + \mathrm{i}\,Q(0,kr)\right)\right).$$

Thus, for $kr \geq 1$, we get

$$\left|\partial_r\left(\mathrm{e}^{-\mathrm{i}\,kr}\,H_0^{(1)}\left(kr\right)\right)\right| \leq \frac{C}{r\sqrt{kr}}. \tag{A.6}$$

An estimate of the second derivative of $H_0^{(1)}$ is derived by using [1, 9.1.27, 9.1.28]

$$\left|\frac{d^2}{dr^2}H_0^{(1)}\left(kr\right)\right| = k^2\left|-H_0^{(1)}\left(kr\right) + \frac{H_1^{(1)}\left(kr\right)}{kr}\right| \overset{\text{(A.3a)}}{\leq} Ck\sqrt{\frac{k}{r}}. \tag{A.7}$$

Case 2: $z = kr \in (0,1)$.

To estimate $H_0^{(1)}\left(z\right)$ in the range $(0,1)$ we employ

$$H_0^{(1)}\left(z\right) = J_0\left(z\right) + iY_0\left(z\right)$$

and use for $Y_0\left(z\right)$ the expansion

$$Y_0\left(z\right) = \frac{2}{\pi}\left(\log\frac{z}{2}\right)J_0\left(z\right) - \frac{2}{\pi}\sum_{k=0}^{\infty}\psi\left(k+1\right)\frac{\left(-\frac{z^2}{4}\right)^k}{\left(k!\right)^2},$$

where

$$\psi\left(n\right) := -\gamma + \sum_{k=1}^{n-1}k^{-1} \quad \text{and} \quad \gamma := 0.57721566\ldots \text{ is Euler's constant.}$$

For $0 \leq z \leq 1$, we have

$$\left|Y_0\left(z\right)\right| \leq \frac{2}{\pi}\left|\log\frac{z}{2}\right| + \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{\psi\left(k+1\right)}{4^k\left(k!\right)^2}.$$

Furthermore

$$\left|\psi\left(k+1\right)\right| \leq \gamma + 1 + \sum_{s=2}^{k}\frac{1}{s} \leq \gamma + 1 + \int_1^k\frac{1}{x}dx = \gamma + 1 + \log k =: \gamma' + \log k.$$

Thus, for $0 \leq z \leq 1$, we have

$$\left|Y_0\left(z\right)\right| \leq \frac{2}{\pi}\left|\log\frac{z}{2}\right| + \frac{2}{\pi}\left(\gamma + \sum_{k=1}^{\infty}\frac{\gamma' + \log k}{4^k\left(k!\right)^2}\right)$$

Since $\frac{\gamma'+\log k}{4^k k!} \leq 1$ we get

$$\left|Y_0\left(z\right)\right| \leq \frac{2}{\pi}\left|\log z\right| + C.$$

This leads to

$$\forall z \in \left]0,1\right] \quad \left|H_0^{(1)}\left(z\right)\right| \leq \frac{2}{\pi}\left|\log z\right| + C. \tag{A.3c}$$

The combination with (A.3a) finally results in

$$\left|H_0^{(1)}\left(z\right)\right| \leq \min\left\{\frac{2}{\pi}\left|\log z\right| + C, \frac{C}{\sqrt{z}}\right\}. \tag{A.3d}$$

We will need further estimates of J_1 and $\partial_r H_0^{(1)}$. From [1, 9.1.60], we conclude

$$\forall z \geq 0 \qquad J_1(z) \leq 1/\sqrt{2}. \tag{A.8}$$

For the derivative of Y_0, we obtain (by using $J_0' = -J_1$)

$$Y_0'(z) = \frac{2}{\pi}\left(\frac{J_0(z)}{z} - J_1(z)\log\frac{z}{2}\right) + \frac{z}{\pi}\sum_{k=0}^{\infty}\psi(k+2)\frac{\left(-\frac{z^2}{4}\right)^k}{k!\,(k+1)!}. \tag{A.9}$$

For $0 \leq z \leq 1$, we obtain

$$\frac{z}{\pi}\sum_{k=0}^{\infty}\psi(k+2)\frac{\left(-\frac{z^2}{4}\right)^k}{k!\,(k+1)!} \leq \frac{z}{\pi}\sum_{k=0}^{\infty}\frac{\gamma' + \log(k+1)}{k!4^k(k+1)!} \leq \frac{z}{\pi}\sum_{k=0}^{\infty}\frac{1}{k!} = Cz.$$

Now,

$$|J_1(z)| \overset{[1,\ 9.1.62]}{\leq} z/2 \tag{A.10}$$

and we get

$$Y_0'(z) \leq \frac{2}{\pi}\left(z^{-1} + \frac{z}{2}\log\frac{z}{2} + \frac{\mathrm{e}\,z}{2}\right) \leq \frac{2}{\pi z} + C.$$

Hence, for $0 \leq r \leq 1/k$, we get

$$\left|\partial_r H_0^{(1)}(kr)\right| = k\left(|J_0'(kr)| + |Y_0'(kr)|\right) \leq \frac{2}{\pi r} + Ck + \frac{k^2 r}{2} \leq C\left(\frac{1}{r} + \frac{k^2 r}{2}\right). \tag{A.11}$$

In addition, we need some weighted estimates for second order derivatives of $H_0^{(1)}$. From (A.9) we obtain

$$\partial_r(r\partial_r Y_0(kr)) = \frac{2k}{\pi}\left(-2J_1(kr) - kr\log\frac{kr}{2}J_0(kr) + kr\sum_{k=0}^{\infty}\psi(k+2)\frac{\left(-\frac{(kr)^2}{4}\right)^k}{(k!)^2}\right).$$

This leads to the estimate, for $0 < z \leq 1$,

$$|\partial_r(r\partial_r Y_0(kr))| \leq \frac{2}{\pi}k^2 r\left(C + \left|\log\frac{kr}{2}\right|\right).$$

Note that

$$\partial_r\left(r\partial_r H_0^{(1)}(kr)\right) = -rk^2 J_0(kr) + \mathrm{i}\,\partial_r(r\partial_r Y_0(kr))$$

and, hence,

$$\forall 0 \leq kr \leq 1 \qquad \left|\partial_r\left(r\partial_r H_0^{(1)}(kr)\right)\right| \leq \frac{2}{\pi}k^2 r\left(C + \left|\log\frac{kr}{2}\right|\right). \tag{A.12}$$

We finally state a lemma required for the proof of Lemma 3.5:

Lemma A.1 *Let $|s| \geq 3/2k$ and $k \geq k_0 > 0$. Then*

$$r J_0(sr) H_0^{(1)}(kr) = \frac{2}{\pi\sqrt{k|s|}} \left\{ e^{\mathrm{i}(s+k)r} f^I(sr) + e^{\mathrm{i}(s-k)r} f^{II}(sr) \right\} g^I(kr)$$

where the functions f^I, f^{II}, g^I satisfy for $r \geq 1$ and a $C > 0$ depending solely on k_0:

$$|f^I(r)| + |f^{II}(r)| + |g^I(r)| \leq C,$$
$$r^2 \left(|\frac{d}{dr} f^I(r)| + |\frac{d}{dr} f^{II}(r)| + |\frac{d}{dr} g^I(r)| \right) \leq C.$$

Proof. (A.1a), (A.1b) imply the stated representation with $f^I(sr) = \frac{1}{2}\left(P_0(sr) - Q_0(sr)\right)$, $f^{II}(sr) = \frac{1}{2}\left(P_0(sr) + Q_0(sr)\right)$, and $g^I(kr) = P_0(kr) + \mathrm{i}\, Q_0(kr)$. The estimates for f^I, f^{II}, g^I now follow from the bounds for P_0, Q_0, P_0', Q_0' given in (A.2a), (A.2b). ■

B Approximation by hp-finite elements. Case I: finite regularity

The purpose of the present section is the proof of Theorem B.4, which constructs a polynomial approximation to a function $u \in H^s$, $s > d/2$, in an element-by-element fashion (see Def. 5.3). The novelty of the present construction over existing operators such as those of [3], [32] is that we obtain optimal rates (in p) *simultaneously* in the H^1-norm and the L^2-norm. Closely related results can be found in the recent [20], where similar duality arguments are employed to obtain estimates in L^2.

B.1 Lifting operators

In the p-FEM, globally continuous, piecewise polynomial approximations to a function u are typically constructed in two steps: in a first step, discontinuous approximations are constructed element by element. In a second step, the jumps across the element interfaces are corrected by suitable lifting operators. The construction of these lifting operators is the purpose of the present section; the ensuing Section B.2 is devoted to the analysis of polynomial approximation.

Before proceeding we recall the definition of the Sobolev space $H_{00}^{1/2}(\Omega)$. If Ω is an edge or a face of a triangle or a tetrahedron, then the Sobolev norm $\|\cdot\|_{H_{00}^{1/2}(\Omega)}$ is defined by

$$\|u\|^2_{H_{00}^{1/2}(\Omega)} := \|u\|^2_{H^{1/2}(\Omega)} + \left\| \frac{u}{\sqrt{\operatorname{dist}(\cdot, \partial\Omega)}} \right\|^2_{L^2(\Omega)}. \tag{B.1}$$

Lemma B.1 *Let $\widehat{K}^{2D}$ be the reference triangle in 2D. Vertex and edge lifting operators can be constructed with the following properties:*

1. *For each vertex V of $\widehat{K}^{2D}$ there exists a polynomial $L_{V,p} \in \mathcal{P}_p$ that attains the value 1 at the vertex V and vanishes on the edge of $\widehat{K}^{2D}$ opposite to V. Additionally, for every $s \geq 0$, there exists $C_s > 0$ such that $\|L_{V,p}\|_{H^s(\widehat{K}^{2D})} \leq C_s p^{-1+s}$.*

2. *For every edge e of $\widehat{K}^{2D}$ there exists a bounded linear operator $\pi_e : H^{1/2}_{00}(e) \to H^1(\widehat{K}^{2D})$ with the following properties:*

   (a) $\forall u \in \mathcal{P}_p \cap H^{1/2}_{00}(e): \quad \pi_e u \in \mathcal{P}_p,$

   (b) $\forall u \in H^{1/2}_{00}(e): \quad \pi_e u|_{\partial \widehat{K}^{2D} \setminus e} = 0,$

   (c) $\forall u \in H^{1/2}_{00}(e): \quad p\|\pi_e u\|_{L^2(\widehat{K}^{2D})} + \|\pi_e u\|_{H^1(\widehat{K}^{2D})} \le C\left(\|u\|_{H^{1/2}_{00}(e)} + p^{1/2}\|u\|_{L^2(e)}\right).$

Proof. Let $\widehat{K}^{2D} = \{(x,y)\,|\,0 < x < 1, 0 < y < 1-x\}$. Let $w^{1D}_q(x) = (1-x)^q$. The vertex function $L_{V,p}$ for the vertex $V = (0,0)$ is defined as $L_{V,p}(x,y) = w^{1D}_{\lfloor p/2\rfloor}(x) w^{1D}_{\lfloor p/2\rfloor}(y)$ for $p \ge 2$ and taken as the standard linear hat function $L_{V,1}(x) = (1-x-y)$ that is associated with the vertex V for the case $p = 1$. A simple calculation then shows the result. The functions $L_{V,p}$ for the remaining 2 vertices are obtained by completely analogous constructions.

For the edge lifting, let e be the edge $e = \{(x, 1-x)\,|\,0 < x < 1\}$. By [3] there exists a bounded linear operator $E : H^{1/2}_{00}(e) \to H^1(\widehat{K}^{2D})$ with the following properties: $Eu|_e = u$, $Eu|_{\partial\widehat{K}^{2D}\setminus e} = 0$, and $Eu \in \mathcal{P}_p$ if $u \in \mathcal{P}_p \cap H^{1/2}_{00}(e)$. Introduce the auxiliary operator $(Gu)(x,y) := w^{1D}_p(1-(x+y))(Eu)(x,y)$. By [29, Lemma B.5], we have

$$p\|Gu\|_{L^2(\widehat{K}^{2D})} + \|Gu\|_{H^1(\widehat{K}^{2D})} \le C\left(|Eu|_{H^1(\widehat{K}^{2D})} + p^{1/2}\|u\|_{L^2(e)}\right) \le C\left(\|u\|_{H^{1/2}_{00}(e)} + p^{1/2}\|u\|_{L^2(e)}\right).$$

Denote by $\Pi^{H^1}_p : H^1_0(\widehat{K}^{2D}) \to H^1_0(\widehat{K}^{2D}) \cap \mathcal{P}_p$ the H^1-projection and set $\pi_e u := Eu + \Pi^{H^1}_p(Gu - Eu)$. Then by the stability of $\Pi^{H^1}_p$ and E

$$\|\pi_e u\|_{H^1(\widehat{K}^{2D})} \le \|Gu\|_{H^1(\widehat{K}^{2D})} + 2\|Gu - Eu\|_{H^1(\widehat{K}^{2D})} \le C\left(\|u\|_{H^{1/2}_{00}(e)} + p^{1/2}\|u\|_{L^2(e)}\right),$$

which is the desired H^1-stability result. For the L^2-bound, we use a duality argument as in [20]:

$$\|Gu - Eu - \Pi^{H^1}_p(Gu - Eu)\|_{L^2(\widehat{K}^{2D})} \le Cp^{-1}\|(Gu - Eu) - \Pi^{H^1}_p(Gu - Eu)\|_{H^1(\widehat{K}^{2D})}.$$

The H^1-stability of $\Pi^{H^1}_p$ together with stability properties of E and G produces the desired L^2-bound. ■

Lemma B.2 *Let $\widehat{K}^{3D}$ be the reference tetrahedron in 3D. Vertex, edge, and face lifting operators can be constructed with the following properties:*

(i) *For each vertex V of $\widehat{K}^{3D}$ there exists a polynomial $L_{V,p} \in \mathcal{P}_p$ that attains the value 1 at the vertex V and vanishes on the face opposite V. Additionally, for every $s \ge 0$ there exists $C_s > 0$ such that $\|L_{V,p}\|_{H^s(\widehat{K}^{3D})} \le C_s p^{-3/2+s}$.*

(ii) *For every edge e of $\widehat{K}^{3D}$ there exists a bounded linear operator $\pi_e : H^{1/2}_{00}(e) \to H^1(\widehat{K}^{3D})$ with the following properties:*

   (a) $\pi_e u \in \mathcal{P}_p$ *if* $u \in \mathcal{P}_p \cap H^{1/2}_{00}(e)$

   (b) $(\pi_e u)|_f = 0$ *for the two faces f with* $\overline{f} \cap e = \emptyset$

(c) for the two faces f adjacent to e (i.e., $\overline{f} \cap e = e$)

$$p\|\pi_e u\|_{L^2(f)} + \|\pi_e u\|_{H^1(f)} \leq C\|u\|_{H^{1/2}_{00}(e)} + p^{1/2}\|u\|_{L^2(e)},$$

$$\|\pi_e u\|_{H^{1/2}(\partial\widehat{K}^{3D})} \leq C\left(p^{-1/2}\|u\|_{H^{1/2}_{00}(e)} + \|u\|_{L^2(e)}\right),$$

$$p\|\pi_e u\|_{L^2(\widehat{K}^{3D})} + \|\pi_e u\|_{H^1(\widehat{K}^{3D})} \leq C\left(p^{-1/2}\|u\|_{H^{1/2}_{00}(e)} + \|u\|_{L^2(e)}\right).$$

(iii) For every face f of $\widehat{K}^{3D}$ there exists a bounded linear operator $\pi_f : H^{1/2}_{00}(f) \to H^1(\widehat{K}^{3D})$ with the following properties:

(a) $\pi_f u \in \mathcal{P}_p$ if $u \in \mathcal{P}_p \cap H^{1/2}_{00}(f)$

(b) $(\pi_e u)|_{f'} = 0$ for the faces $f' \neq f$

$$p\|\pi_f u\|_{L^2(\widehat{K}^{3D})} + \|\pi_f u\|_{H^1(\widehat{K}^{3D})} \leq C\left(\|u\|_{H^{1/2}_{00}(f)} + p^{1/2}\|u\|_{L^2(f)}\right).$$

Proof. We take the reference tetrahedron $\widehat{K}^{3D}$ to be $\widehat{K}^{3D} = \{(x,y,z)\,|\,0 < x < 1, 0 < y < 1-x, 0 < z < 1-x-y\}$.

Proof of (i): We construct the function $L_{V,p}$ for the vertex $V = (0,0,0)$, the other cases being handled analogously. For $p \in \{1,2\}$ we take for $L_{V,p}$ the standard linear hat function associated with V. For $p \geq 3$ the function $L_{V,p}(x,y,z) := w^{1D}_{\lfloor p/3\rfloor}(x) w^{1D}_{\lfloor p/3\rfloor}(y) w^{1D}_{\lfloor p/3\rfloor}(z)$ with $w^{1D}_q(x) = (1-x)^q$ has all the desired properties.

Proof of (iii): [32, Lemma 8] exhibits a bounded linear operator $F : H^{1/2}_{00}(f) \to H^1(\widehat{K}^{3D})$ with the additional property that $Fu \in \mathcal{P}_p$ if $u \in \mathcal{P}_p \cap H^{1/2}_{00}(f)$. Let, without loss of generality, $f = \partial\widehat{K}^{3D} \cap \{z = 0\}$. Define the auxiliary operator $(Gu)(x,y,z) := w^{1D}_p(z)(Fu)(x,y,z)$. This operator satisfies (see [29, Lemma B.5] where the analogous arguments are worked out in the 2D setting)

$$p\|Gu\|_{L^2(\widehat{K}^{3D})} + \|Gu\|_{H^1(\widehat{K}^{3D})} \leq C\left(|Fu|_{H^1(\widehat{K}^{3D})} + p^{1/2}\|Fu\|_{L^2(f)}\right) \leq C\|u\|_{H^{1/2}_{00}(f)} + p^{1/2}\|u\|_{L^2(f)}.$$

Letting again $\Pi^{H^1}_p : H^1_0(\widehat{K}^{3D}) \to H^1_0(\widehat{K}^{3D}) \cap \mathcal{P}_p$ be the H^1-projection, we can set $\pi_f u := Fu + \Pi^{H^1}_p(Gu - Fu)$. The desired properties of π_f are then seen in exactly the same way as in the 2D case of Lemma B.1.

Proof of (ii): Set $f_{e,1} = \partial\widehat{K}^{3D} \cap \{z = 0\}$ and $f_{e,2} = \partial\widehat{K}^{3D} \cap \{1-x-y-z = 0\}$. The edge shared by the faces $f_{e,1}$ and $f_{e,2}$ is $e = \{(x, 1-x, 0)\,|\,0 < x < 1\}$. By Lemma B.1 a function $u \in H^{1/2}_{00}(e)$ can be lifted to a function $Eu \in H^1(f_{e,1})$ such that $Eu|_{\partial f_{e,1}\setminus e} = 0$ and

$$p\|Eu\|_{L^2(f_{e,1})} + \|Eu\|_{H^1(f_{e,1})} \leq C\left(\|u\|_{H^{1/2}_{00}(e)} + p^{1/2}\|u\|_{L^2(e)}\right).$$

Additionally, if $u \in \mathcal{P}_p$, then $Eu \in \mathcal{P}_p$. Since the same lifting can be done for the face $f_{e,2}$, we can find a function, again denoted $Eu \in H^1(\partial\widehat{K}^{3D})$, that vanishes on $\partial\widehat{K}^{3D} \setminus (f_{e,1} \cup f_{e,2} \cup e)$, such that $p\|Eu\|_{L^2(\partial\widehat{K}^{3D})} + \|Eu\|_{H^1(\partial\widehat{K}^{3D})} \leq C\left(\|u\|_{H^{1/2}_{00}(e)} + p^{1/2}\|u\|_{L^2(e)}\right)$. Additionally, Eu is a piecewise polynomial of degree p if $u \in \mathcal{P}_p$. An interpolation inequality gives

$$\|Eu\|_{H^{1/2}(\partial\widehat{K}^{3D})} \leq C\|Eu\|^{1/2}_{L^2(\partial\widehat{K}^{3D})}\|Eu\|^{1/2}_{H^1(\partial\widehat{K}^{3D})} \leq Cp^{-1/2}\left(\|u\|_{H^{1/2}_{00}(\partial\widehat{K}^{3D})} + p^{1/2}\|u\|_{L^2(\partial\widehat{K}^{3D})}\right).$$

For this function Eu, [32, Lemma 8] provides a lifting $Fu \in H^1(\widehat{K}^{3D})$ with $\|Fu\|_{H^1(\widehat{K}^{3D})} \leq C\|Eu\|_{H^{1/2}(\widehat{K}^{3D})}$. To get a better L^2-bound, we introduce the distance functions $d_1(\cdot) := \operatorname{dist}(\cdot, f_{e,1})$ and $d_2(\cdot) := \operatorname{dist}(\cdot, f_{e,2})$ as well as $d(\cdot) := \operatorname{dist}(\cdot, f_{e,1} \cup f_{e,2}) = \min\{d_1(\cdot), d_2(\cdot)\}$ and set $w := (1-d)^p$. Define $Gu := wFu$. Then $(Gu)|_{\partial\widehat{K}^{3D}} = (Fu)|_{\partial\widehat{K}^{3D}}$ since $w|_{f_{e,1}\cup f_{e,2}} \equiv 1$ and $Fu|_{\partial\widehat{K}^{3D}\setminus(f_{e,1}\cup f_{e,2})} = 0$. Additionally, $Gu \in H^1(\widehat{K}^{3D})$ since w is Lipschitz continuous. Furthermore, we have

$$p\|Gu\|_{L^2(\widehat{K}^{3D})} + \|Gu\|_{H^1(\widehat{K}^{3D})} \leq C\left(\|Fu\|_{H^1(\widehat{K}^{3D})} + p^{1/2}\|Fu\|_{L^2(f_{e,1}\cup f_{e,2})}\right). \tag{B.2}$$

To see this, we adapt the proof given in [29, Lemma B.5]. We split $\widehat{K}^{3D} = K_1 \cup K_2$ with $K_i = \{(x,y,z) \in \widehat{K}^{3D} \,|\, d(x,y,z) \leq d_i(x,y,z)\}$, $i \in \{1,2\}$. We note that on K_1, we have $d(x,y,z) = d_1(x,y,z) = z$. Hence, by the arguments given in [29, Lemma B.5], we get

$$p\|Gu\|_{L^2(K_1)} + \|Gu\|_{H^1(K_1)} \leq C\left(\|Fu\|_{H^1(K_1)} + p^{1/2}\|Fu\|_{L^2(f_{e,1})}\right).$$

Proceeding completely analogously for K_2 gives us (B.2). Since $Fu|_{\partial K^{3D}}$ coincides with Eu, we conclude that Gu satisfies

$$p\|Gu\|_{L^2(K_1)} + \|Gu\|_{H^1(K_1)} \leq Cp^{-1/2}\left(\|u\|_{H^{1/2}_{00}(e)} + p^{1/2}\|u\|_{L^2(e)}\right). \tag{B.3}$$

We recall that $\Pi_p^{H^1} : H_0^1(\widehat{K}^{3D}) \to H_0^1(\widehat{K}^{3D}) \cap \mathcal{P}_p$ denotes the H^1-projection and define

$$\pi_e u := Fu + \Pi_p^{H^1}(Gu - Fu).$$

If u is a polynomial of degree p, then $\pi_e u$ is a polynomial of degree p. Additionally, $\pi_e u = Fu$ on $\partial\widehat{K}^{3D}$ so that the estimates for π_e on the faces of $\widehat{K}^{3D}$ are satisfied. To see the $H^1(\widehat{K}^{3D})$- and $L^2(\widehat{K}^{3D})$-bounds we note that the stability of $\Pi_p^{H^1}$ together with (B.3) and the stability of F gives us the H^1-bound. The L^2-bound follows as in the proof of Lemma B.1 and in [20] from Nitsche's trick: $\|\pi_e u\|_{L^2(\widehat{K}^{3D})} \leq \|\pi_e u - Gu\|_{L^2(\widehat{K}^{3D})} + \|Gu\|_{L^2(\widehat{K}^{3D})} \leq Cp^{-1}\|Fu - Gu\|_{H^1(\widehat{K}^{3D})} + \|Gu\|_{L^2(\widehat{K}^{3D})}$. ■

B.2 Approximation operators

Lemma B.3 provides polynomial approximation results on triangles and tetrahedra. The lifting operators of the preceding subsection are employed in Theorem B.4 to modify the approximations of Lemma B.3 such that approximations are obtained that permit an element-by-element construction in the sense of Def. 5.3.

Lemma B.3 *Let $\widehat{K}$ be the reference triangle or the reference tetrahedron. Let $s > d/2$. Then there exists for every p a bounded linear operator $\pi_p : H^s(\widehat{K}) \to \mathcal{P}_p$ and for each $t \in [0,s]$ a constant $C > 0$ (depending only on s and t) such that*

$$\|u - \pi_p u\|_{H^t(\widehat{K})} \leq Cp^{-(s-t)}|u|_{H^s(\widehat{K})}, \qquad p \geq s-1. \tag{B.4}$$

Additionally, we have $\|u - \pi_p u\|_{L^\infty(\widehat{K})} \leq Cp^{-(s-d/2)}|u|_{H^s(\widehat{K})}$. For the case $d = 2$ we furthermore have $\|u - \pi_p u\|_{H^t(e)} \leq Cp^{-(s-1/2-t)}|u|_{H^s(\widehat{K})}$ for $0 \leq t \leq s-1/2$ for every edge and for the case $d = 3$ we have $\|u - \pi_p u\|_{H^t(f)} \leq Cp^{-(s-1/2-t)}|u|_{H^s(\widehat{K})}$ for $0 \leq t \leq s-1/2$ for every face f and $\|u - \pi_p u\|_{H^t(e)} \leq Cp^{-(s-1-t)}|u|_{H^k(\widehat{K})}$ for $0 \leq t \leq s-1$ for every edge.

Proof. The construction of π_p with the property (B.4) is fairly classical (see, e.g., [5]). One possible construction is worked out in [29, Appendix A] first for integers s, t and, then, interpolation arguments remove this restriction. Next, we consider the L^∞-bound, for which we need the assumption $s > d/2$: We recall that for a Lipschitz domain $K \subset \mathbb{R}^d$ and $s > d/2$ there exists $C > 0$ such that

$$\|u\|_{L^\infty(K)} \leq C\|u\|_{L^2(K)}^{1-d/(2s)}\|u\|_{H^s(K)}^{d/(2s)} \qquad \forall u \in H^s(K). \tag{B.5}$$

From this, the desired L^∞-bound follows easily. The inequality (B.5) can be seen as follows: First, using an extension operator for K (e.g., the one given in [40, Chap. VI]) it suffices to show this estimate with K replaced with the full space $\mathbb{R}^d$. Next, [43, Thm. 4.6.1] asserts the embedding $B^{d/2}_{2,1}(\mathbb{R}^d) \subset C(\mathbb{R}^d)$. Finally, the Besov space $B^{d/2}_{2,1}(\mathbb{R}^d)$ is recognized as an interpolation space between $L^2(\mathbb{R}^d)$ and $H^s(\mathbb{R}^d)$: $B^{d/2}_{2,1}(\mathbb{R}^d) = (L^2(\mathbb{R}^d), H^s(\mathbb{R}^d))_{d/(2s),1}$. The interpolation inequality then produces the desired result. The remaining estimates on the edges and faces follow from appropriate trace inequalities. Specifically: let $\omega \subset \partial\widehat{K}$ be an edge (for $d = 2$) or a face (for $d = 3$). By [43, Thm. 2.9.3] the trace operator γ is a continuous mapping in the following spaces:

$$\gamma : B^{1/2}_{2,1}(\widehat{K}) \to L^2(\omega), \quad \text{and} \quad \gamma : H^t(\widehat{K}) \to H^{t-1/2}(\omega), \qquad t > 1/2.$$

Together with the observation $B^{1/2}_{2,1}(\widehat{K}) = (L^2(\widehat{K}), H^s(\widehat{K}))_{s,1}$ the desired estimates can be inferred. It remains to see the case of traces on an edge e of the tetrahedron in the case $d = 3$. In this case [43, Thm. 2.9.4] asserts the continuity of the trace operator in the following spaces:

$$\gamma : B^{1}_{2,1}(\widehat{K}) \to L^2(e), \quad \text{and} \quad \gamma : H^t(\widehat{K}) \to H^{t-1}(e), \qquad t > 1.$$

Again, these continuity properties are sufficient to establish the desired error estimates. ■

We conclude this section with the construction of an approximation operator that permits an easy element-by-element construction.

Theorem B.4 *Let $\widehat{K} \subset \mathbb{R}^d$ be the reference triangle or the reference tetrahedron. Let $s > d/2$. Then there exists $C > 0$ (depending only on s and d) and for every p a linear operator $\pi : H^s(\widehat{K}) \to \mathcal{P}_p$ that permits an element-by-element construction in the sense of Definition 5.3 such that*

$$p\|u - \pi u\|_{L^2(\widehat{K})} + \|u - \pi u\|_{H^1(\widehat{K})} \leq Cp^{-(s-1)}|u|_{H^s(\widehat{K})} \qquad \forall p \geq s - 1. \tag{B.6}$$

Proof. We discuss only the case $d = 3$ – the case $d = 2$ is treated very similarly. Also, we will construct πu for a given u—inspection of the construction shows that $u \mapsto \pi u$ is in fact a linear operator.

Let $\pi^1 \in \mathcal{P}_p$ be given by Lemma B.3. Then, for $p \geq s - 1$ there holds

$$\|u - \pi^1\|_{H^t(\widehat{K})} \leq Cp^{-(s-t)}|u|_{H^s(\widehat{K})}, \qquad 0 \leq t \leq s \tag{B.7}$$

$$\|u - \pi^1\|_{H^t(f)} \leq Cp^{-(s-t-1/2)}|u|_{H^s(\widehat{K})}, \qquad \forall \text{ faces } f, \quad 0 \leq t \leq s - 1/2 \tag{B.8}$$

$$\|u - \pi^1\|_{H^t(e)} \leq Cp^{-(s-t-1)}|u|_{H^s(\widehat{K})}, \qquad \forall \text{ edges } e, \quad 0 \leq t \leq s - 1 \tag{B.9}$$

$$\|u - \pi^1\|_{L^\infty(\widehat{K})} \leq Cp^{-(s-3/2)}|u|_{H^s(\widehat{K})}. \tag{B.10}$$

From (B.10) and the vertex-lifting properties given in Lemma B.2, we may adjust π^1 by vertex liftings to obtain a polynomial π^2 satisfying (B.7)–(B.9) and additionally the condition (i) of Definition 5.3. We next adjust the edge values. The polynomial π^2 coincides with u in the vertices and satisfies (B.9). By fixing a $t \in (1/2, s-1)$, we get from an interpolation inequality:

$$\begin{aligned} p^{1/2}\|u-\pi^2\|_{L^2(e)} + \|u-\pi^2\|_{H^{1/2}_{00}(e)} &\le p^{1/2}\|u-\pi^2\|_{L^2(e)} + C\|u-\pi^2\|^{1-1/(2t)}_{L^2(e)}\|u-\pi^2\|^{1/(2t)}_{H^t(e)} \\ &\le Cp^{-(s-3/2)}|u|_{H^s(\widehat{K})}. \end{aligned}$$

Hence, for an edge e, the minimizer π^e of the functional (5.4) satisfies $p^{1/2}\|u-\pi^e\|_{L^2(e)} + \|u-\pi^e\|_{H^{1/2}_{00}(e)} \le Cp^{-(s-3/2)}|u|_{H^k(\widehat{K})}$; the triangle inequality therefore gives that the correction $\pi^e - \pi^2$ needed to obtain condition (ii) of Def. 5.3 likewise satisfies $p^{1/2}\|\pi^e-\pi^2\|_{L^2(e)} + \|\pi_e - \pi^2\|_{H^{1/2}_{00}(e)} \le Cp^{-(s-3/2)}|u|_{H^s(\widehat{K})}$. We conclude that the edge lifting of Lemma B.2 allows us to adjust π^2 to get a polynomial $\pi^3 \in \mathcal{P}_p$ that satisfies the conditions (i) and (ii) of Def. 5.3. Additionally, we have

$$\begin{aligned} p\|u-\pi^3\|_{L^2(\widehat{K})} + \|u-\pi^3\|_{H^1(\widehat{K})} &\le Cp^{-(s-1)}|u|_{H^s(\widehat{K})}, \\ p\|u-\pi^3\|_{L^2(f)} + \|u-\pi^3\|_{H^1(f)} &\le Cp^{-(s-3/2)}|u|_{H^s(\widehat{K})} \qquad \text{for all faces } f. \end{aligned}$$

Since $\pi^3|_e = \pi^e$ for the edges, the minimizer π^f of the functional (5.5) for each face f has to satisfy $p\|u-\pi^f\|_{L^2(f)} + \|u-\pi^f\|_{H^1(f)} \le p\|u-\pi^3\|_{L^2(f)} + \|u-\pi^3\|_{H^1(f)} \le Cp^{-(s-3/2)}|u|_{H^s(\widehat{K})}$. From the triangle inequality, we conclude

$$p\|\pi^3-\pi^f\|_{L^2(f)} + \|\pi^3-\pi^f\|_{H^1(f)} \le Cp^{-(s-3/2)}|u|_{H^k(\widehat{K})}, \qquad \text{together with } \pi^3-\pi^f \in H^1_0(f).$$

Hence, the face lifting of Lemma B.2 allows us to correct the face values to achieve also condition (iii) of Definition 5.3. Lemma B.2 also implies that the correction is such that (B.6) is true. ∎

C Approximation by *hp*-finite elements. Case II: analytic regularity

In this section, we construct a polynomial approximation operator for analytic functions that permits element-by-element construction in the sense of Def. 5.3 and leads to exponential rates of convergence.

Lemma C.1 *Let $d \in \{2,3\}$. Let G_1, $G \subset \mathbb{R}^d$ be bounded open sets. Assume that $g : \overline{G_1} \to \mathbb{R}^d$ satisfies $g(G_1) \subset G$. Assume additionally that g is injective on $\overline{G_1}$, analytic on G_1 and satisfies*

$$\|\nabla^p g\|_{L^\infty(G_1)} \le C_g\gamma_g^p p! \qquad \forall p \in \mathbb{N}_0, \qquad |\det(g')| \ge c_0 > 0 \qquad \text{on } G_1.$$

Let f be analytic on G and satisfy, for some C_f, γ_f, $\kappa > 0$,

$$\|\nabla^p f\|_{L^2(G)} \le C_f\gamma^p \max\{p,\kappa\}^p \qquad \forall p \in \mathbb{N}_0. \tag{C.1}$$

Then, the function $f \circ g$ is analytic on G_1 and there exist constants C, $\gamma_1 > 0$ that depend solely on γ_g, C_g, c_0, and γ_f such that

$$\|\nabla^p(f\circ g)\|_{L^2(G)} \le CC_f\gamma_1^p \max\{p,\kappa\}^p \qquad \forall p \in \mathbb{N}_0.$$

Proof. This is essentially proved in [28, Lemma 4.3.1]. Specifically, [28, Lemma 4.3.1] analyzes the case $d = 2$ and states that C, γ_1 depends on the function g. Inspection of the proof shows that the case $d = 3$ can be handled analogously and shows that the dependence on the function g can be reduced to a dependence on C_g, γ_g, and γ_f. ■

Lemma C.2 *Let $d \in \{1, 2, 3\}$, and let $\widehat{K} \subset \mathbb{R}^d$ be the reference simplex. Let $\overline{\gamma}$, $\widetilde{C} > 0$ be given. Then there exist constants C, $\sigma > 0$ that depend solely on $\overline{\gamma}$ and $\widetilde{C}$ such that for any u that satisfies for some $C_u > 0$, $h \in (0, 1]$, $\kappa \geq 1$ the conditions*

$$\|\nabla^p u\|_{L^2(\widehat{K})} \leq C_u(\overline{\gamma}h)^p \max\{p, \kappa\}^p \qquad \forall p \in \mathbb{N}_0 \tag{C.2}$$

there holds for all $p \in \mathbb{N}_0$ that satisfy

$$\kappa h/p \leq \widetilde{C} \tag{C.3}$$

the bound

$$\inf_{\pi\in\mathcal{P}_p} \|u - \pi\|_{W^{2,\infty}(\widehat{K})} \leq CC_u \left[\left(\frac{h}{\sigma + h} \right)^{p+1} + \left(\frac{\kappa h}{\sigma p} \right)^{p+1} \right]. \tag{C.4}$$

Proof. The Sobolev embedding theorem $H^2(\widehat{K}) \subset C(\overline{\widehat{K}})$ gives us for suitable $C > 0$

$$\begin{aligned}\|\nabla^n u\|_{L^\infty(\widehat{K})} &\leq C_u C \left[(\overline{\gamma}h)^{n+2} \max\{n + 2, \kappa\}^{n+2} + (\overline{\gamma}h)^n \max\{n + 2, \kappa\}^n \right] \\ &\leq CC_u(\overline{\gamma}h)^n \max\{n + 2, \kappa\}^n \left(1 + \max\{(n + 2)h, h\kappa\}^2\right).\end{aligned}$$

In view of $h\kappa \leq \widetilde{C}p$, we may estimate for any $\gamma > 1$ and appropriate $C > 0$:

$$\|\nabla^n u\|_{L^\infty(\widehat{K})} \leq CC_u p^2(\overline{\gamma}\gamma h)^n \max\{n + 2, \kappa\}^n \qquad \forall n \in \mathbb{N}_0. \tag{C.5}$$

Define

$$\mu := \gamma\overline{\gamma}\sqrt{d}\,\mathrm{e}, \tag{C.6}$$

and let $r_0 = \operatorname{diam}(\widehat{K})$ and $b_{\widehat{K}}$ be the barycenter of $\widehat{K}$. The bounds (C.5), (5.2) and Stirling's formula in the form $n! \geq (n/\mathrm{e})^n$ imply that the Taylor series of u about $x \in \widehat{K}$ converges on a (complex) ball $B_{1/(\mu h)}(x) \subset \mathbb{C}^d$ of radius $1/(\mu h)$ and center $x \in \widehat{K}$. For the polynomial approximation of u, we distinguish the cases $\mu h \leq 1/(2r_0)$ and $\mu h > 1/(2r_0)$.

The case $\mu h \leq 1/(2r_0)$: In this case the Taylor series of u about $b_{\widehat{K}}$ converges on an open ball that contains the closure of $\widehat{K}$. We may therefore approximate u by its truncated Taylor series $T_p u$. The error is then given by

$$u(x) - T_p u(x) = \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|\geq p+1} \frac{1}{\alpha!} D^\alpha u(b_{\widehat{K}})(x - b_{\widehat{K}})^\alpha, \qquad x \in B_{1/(\mu h)}(b_{\widehat{K}}) \subset \mathbb{C}^d.$$

Hence (5.2) implies

$$\begin{aligned}\|u - T_p u\|_{L^\infty(B_{r_0}(b_{\widehat{K}}))} &\leq \sum_{|\alpha|\geq p+1} \frac{1}{\alpha!} |D^\alpha u(b_{\widehat{K}})| r_0^{|\alpha|} \leq \sum_{n=p+1}^\infty r_0^n d^{n/2} \frac{1}{n!} \|\nabla^n u\|_{L^\infty(\widehat{K})} \\ &\leq CC_u p^2 \sum_{n=p+1}^\infty \frac{1}{n!} \max\{n + 2, \kappa\}^n d^{n/2} (\gamma\overline{\gamma}h)^n r_0^n =: S.\end{aligned}$$

This last sum S is split further using Stirling's formula $n! \geq (n/\mathrm{e})^n$ and $(1+2/n)^n \leq \mathrm{e}^2$:

$$S = CC_u p^2 \left(\sum_{p+1 \leq n \leq \kappa-2} \frac{1}{n!} (\sqrt{d} r_0 \gamma \overline{\gamma} \kappa h)^n + \sum_{n \geq \max\{p+1, \kappa-2\}} (\gamma\overline{\gamma} h r_0 \sqrt{d})^n \frac{(n+2)^n}{n!} \right)$$
$$\leq CC_u p^2 \mathrm{e}^2 \left(\sum_{n \geq p+1} \frac{1}{n!} \left(\sqrt{d} r_0 \gamma \overline{\gamma} \kappa h \right)^n + \sum_{n \geq p+1} (\underbrace{\mathrm{e}\, \gamma \overline{\gamma} \sqrt{d}}_{=\mu} r_0 h)^n \right) =: S_1 + S_2.$$

We estimate these two sums separately. For S_2, we use the assumption $\mu r_0 h \leq 1/2$, which allows us to estimate

$$S_2 \leq CC_u p^2 \mathrm{e}^2 (\mu r_0 h)^{p+1} = CC_u p^2 \mathrm{e}^2 \left(\frac{h}{\frac{1}{2\mu r_0} + \frac{1}{2\mu r_0}} \right)^{p+1} \leq CC_u p^2 \mathrm{e}^2 \left(\frac{h}{\frac{1}{2\mu r_0} + h} \right)^{p+1}.$$

For S_1, we recall that Taylor's formula gives, for $x > 0$,

$$\sum_{n \geq p+1} \frac{1}{n!} x^n = \mathrm{e}^x - \sum_{n=0}^{p} \frac{1}{n!} x^n = \frac{1}{p!} \int_0^x (x-t)^p \mathrm{e}^t \, dt \leq \frac{x^{p+1}}{p!} \mathrm{e}^x.$$

Hence, we can estimate S_1 by (recall $\overline{\gamma}\gamma\sqrt{d} = \mu/\mathrm{e}$),

$$S_2 \leq CC_u p^2 \frac{((\mu/\mathrm{e}) r_0 \kappa h)^{p+1}}{p!} \mathrm{e}^{(\mu/\mathrm{e}) r_0 \kappa h} \leq CC_u p^3 \left(\frac{\mathrm{e}^\theta \mu r_0 \kappa h}{p+1} \right)^{p+1},$$

where, in the second inequality, we have used the assumption $h\kappa/p \leq \widetilde{C}$ and Stirling's formula $n! \geq (n/\mathrm{e})^n$ and have abbreviated $\theta := \widetilde{C}\mu/\mathrm{e} r_0$. Combining the estimates for S_1 and S_2 we arrive at the following estimate for suitable $\sigma > 0$ (depending only on μ,r_0, and $\widetilde{C}$):

$$\|u - T_p u\|_{L^\infty(B_{r_0}(b_{\widetilde{K}}))} \leq S \leq CC_u \left(\left(\frac{\kappa h}{\sigma p} \right)^{p+1} + \left(\frac{h}{\sigma + h} \right)^{p+1} \right).$$

Since $\operatorname{dist}(\widehat{K}, \partial B_{r_0}(b_{\widetilde{K}})) > 0$, the Cauchy integral formula for derivatives then implies

$$\|u - T_p u\|_{W^{2,\infty}(\widehat{K})} \leq CC_u \left(\left(\frac{\kappa h}{\sigma p} \right)^{p+1} + \left(\frac{h}{\sigma + h} \right)^{p+1} \right).$$

The case $\mu h > 1/(2r_0)$: In this case, h is bounded away from 0, namely, $h_0 := 1/(2r_0\mu) \leq h \leq 1$. We recall that for every $x \in \widehat{K}$ the Taylor series of u about x converges on the (complex) ball $B_{1/(\mu h)}(x) \subset \mathbb{C}^d$. Setting $r_1 := 1/(2\mu)$, we infer from this that u is analytic on $B_{2r_1} := \cup_{x \in \widehat{K}} B_{2r_1}(x) \subset \mathbb{C}^d$. The estimate (C.5) and a calculation analogous to the above reveals that on $B_{r_1} := \cup_{x \in \widehat{K}} B_{r_1}(x)$ we have (note that $h \geq h_0$)

$$\|u\|_{L^\infty(B_{r_1})} \leq CC_u p^2 \mathrm{e}^{\vartheta \kappa h}.$$

for a constant $\vartheta > 0$ independent of p, κ, h. Approximation results for analytic functions on triangles/tetrahedra (see [28, Prop. 3.2.16] for the case $d = 2$ and [15, Thm. 1] for the case $d = 3$) imply the existence of C, $b > 0$ that depend solely on r_1 such that

$$\inf_{\pi \in \mathcal{P}_{p+1}} \|u - \pi\|_{W^{2,\infty}(\widehat{K})} \leq CC_u p^2 \, \mathrm{e}^{\vartheta \kappa h} \, \mathrm{e}^{-bp} \qquad \forall p \in \mathbb{N}_0.$$

We finally distinguish two further cases: If $\vartheta\kappa h < pb/2$, then we can estimate

$$p^2 \mathrm{e}^{\vartheta\kappa h p} \mathrm{e}^{-bp} \leq p^2 \mathrm{e}^{-b/2p} \leq C \left(\frac{h_0}{\sigma + h_0} \right)^{p+1},$$

for suitable constants C, $\sigma > 0$ depending only on b and h_0. Since $h \geq h_0$ and the function $h \mapsto h/(\sigma + h)$ is monotone increasing, we have reached the desired bound. If, on the other hand, $\vartheta\kappa h \geq pb/2$, then

$$p^2 \mathrm{e}^{\vartheta\kappa h} \mathrm{e}^{-bp} \leq C \mathrm{e}^{\vartheta\kappa h} \leq C \mathrm{e}^{\vartheta\widetilde{C}p} = C \left(\mathrm{e}^{\vartheta\widetilde{C}} \right)^p \leq C \left(\frac{\kappa h}{p} \frac{2}{b} \mathrm{e}^{\vartheta\widetilde{C}} \right)^p;$$

we recognize this bound to have the desired form. ■

Lemma C.3 *Assume the hypotheses of Lemma C.2. Then one can find a polynomial $\pi \in \mathcal{P}_p$ that satisfies*

$$\|u - \pi\|_{W^{1,\infty}(\widehat{K})} \leq C C_u \left[\left(\frac{h}{\sigma + h} \right)^{p+1} + \left(\frac{\kappa h}{\sigma p} \right)^{p+1} \right] \tag{C.7}$$

and additionally admits an element-by-element construction as defined in Definition 5.3.

Proof. The construction follows standard lines. We will only outline the arguments for the case $d = 3$. In order to keep the notation compact, we introduce the expression

$$E(C, \sigma) := C C_u \left[\left(\frac{h}{\sigma + h} \right)^{p+1} + \left(\frac{\kappa h}{\sigma p} \right)^{p+1} \right].$$

In what follows, the constants C_i, $\sigma_i > 0$ $(i = 1, 2, \ldots)$ will be independent of C_u, h, p, and κ. Let $\pi \in \mathcal{P}_p$ be the polynomial given by Lemma C.2. It satisfies $\|u - \pi\|_{W^{2,\infty}(\widehat{K})} \leq E(C, \sigma)$. Therefore, we may correct π by a linear polynomial without sacrificing the approximation rate to ensure $u(V) - \pi(V)$ for all vertices $V \in \mathcal{V}$. This corrected polynomial, denoted π^2, vanishes in the vertices and still satisfies $\|u - \pi^2\|_{W^{2,\infty}(\widehat{K})} \leq E(C_2, \sigma_2)$. Next, we correct the edges. We illustrate the procedure only for one edge. Without loss of generality, we assume that $\widehat{K} = \{(x, y, z) \,|\, 0 < x, y, z < 1, x + y < 1 - z\}$ and that the edge e considered is $e = \{(0, 0, z) \,|\, z \in (0, 1)\}$. Let the univariate polynomial $\pi^e \in \mathcal{P}_p$ be the minimizer of the functional (5.4). From $\|u - \pi^2\|_{W^{2,\infty}(e)} \leq \|u - \pi^2\|_{W^{2,\infty}(\widehat{K})} \leq E(C_2, \sigma_2)$ we can conclude that $p^{1/2}\|u - \pi^e\|_{L^2(e)} + \|u - \pi^e\|_{H^{1/2}_{00}(e)} \leq C p^{1/2} E(C_2, \sigma_2)$. Hence, for the required correction $\pi^c := \pi^2|_e - \pi^e$, which vanishes in the two endpoints of e, we get from a triangle inequality and standard polynomial inverse estimates $\|\frac{1}{1-z}\pi^c\|_{L^\infty(e)} + \|\pi^c\|_{L^\infty(e)} \leq E(C_3, \sigma_3)$. We may lift this univariate function to $\widehat{K}$ by

$$\widetilde{\pi}^e(x, y, z) := \frac{1 - x - y - z}{1 - z} \pi^c(z)$$

This is a polynomial of degree $\leq p$ that vanishes on all edges but the edge e; clearly, $\|\widetilde{\pi}^e\|_{L^\infty(\widehat{K})} \leq E(C_3, \sigma_3)$. The polynomial inverse estimate $\|\widetilde{\pi}^e\|_{W^{1,\infty}(\widehat{K})} \leq C p^2 \|\widetilde{\pi}^e\|_{L^\infty(\widehat{K})}$ shows that $\|\widetilde{\pi}^e\|_{W^{2,\infty}(\widehat{K})} \leq E(C_4, \sigma_4)$. Proceeding in this fashion for all edges, we arrive at a polynomial π^3 with the desired behavior on all edges of $\widehat{K}$ and satisfies $\|u - \pi^3\|_{W^{2,\infty}(\widehat{K})} \leq E(C_5, \sigma_5)$.

It remains to construct a correction for the faces. To that end, the key issue is again that of a lifting from a face f. Without loss of generality, this face is $f := \{(x,y,0) \,|\, 0 < x, y, x+y < 1\}$. For a polynomial π^c defined on f that additionally vanishes on ∂f, we define the lifting $\widetilde{\pi}^f$ by

$$\widetilde{\pi}^f(x,y,z) = \frac{xy(1-x-y-z)}{xy(1-x-y)} \pi^c(x,y).$$

This is a polynomial that vanishes on all faces of $\widehat{K}$ except on f. Additionally, it is a lifting, i.e., $\widetilde{\pi}^f|_f = \pi^c$. As in the case of the lifting from the edge we see that if π^c is exponentially small on f, then the lifting is likewise exponentially small. To see that the required correction π^c is exponentially small, let π^f be the minimizer of the functional (5.5). Since π^3 has the desired behavior on the edges of f, we have $\pi^3|_{\partial f} = \pi^f|_{\partial f}$ and therefore $\|u - \pi^3\|_{W^{2,\infty}(\widehat{K})} \leq E(C_5, \sigma_5)$ allows us to conclude $\|\pi^3 - \pi^f\|_{H^1(f)} \leq CE(C_5, \sigma_5)$. Polynomial inverse estimates then imply for the lifting $\widetilde{\pi}^f$ that $\|\widetilde{\pi}^f\|_{W^{1,\infty}(\widehat{K})} \leq E(C_6, \sigma_6)$. ∎

D comments on the proof of Lemma B.3

We have heavily used "non standard" Besov spaces in the proof of Lemma B.3. The following two lemmas show these spaces, being intermediary in the proof anyway, can be avoided.

Lemma D.1 *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let $s > d/2$. Then there exists $C_s > 0$ such that*

$$\|u\|_{L^\infty(\Omega)} \leq C_s \|u\|_{L^2(\Omega)}^{1-d/(2s)} \|u\|_{H^s(\Omega)}^{d/(2s)}$$

Proof. A short proof is as follows: Let $E : L^2(\Omega) \to L^2(\mathbb{R}^d)$ be the Stein extension operator. Then $\|Eu\|_{L^2(\mathbb{R}^d)} \leq C\|u\|_{L^2(\Omega)}$ and $\|Eu\|_{H^s(\mathbb{R}^d)} \leq C\|u\|_{H^s(\Omega)}$. By [43, Thm.4.6.1], we have the embedding estimate $B^{d/2}_{2,1}(\mathbb{R}^d) \subset C(\mathbb{R}^d)$; in particular, $\|u\|_{L^\infty(\mathbb{R}^d)} \leq C\|u\|_{B^{d/s}_{2,1}(\mathbb{R}^d)}$. Next, we recognize that $B^{d/2}_{2,1}(\mathbb{R}^d)$ is obtained by interpolation between $L^2(\mathbb{R}^d)$ and $H^s(\mathbb{R}^d)$ via the K-method; specifically, $B^{d/2}_{2,1}(\mathbb{R}^d) = (L^2(\mathbb{R}^d), H^s(\mathbb{R}^d))_{\theta,1}$ with $\theta = d/(2s)$. Hence,

$$\|u\|_{B^{d/2}_{2,1}(\mathbb{R}^d)} \leq C\|u\|_{L^2(\mathbb{R}^d)}^{1-d/(2s)} \|u\|_{H^s(\mathbb{R}^d)}^{d/(2s)}.$$

We conclude

$$\|u\|_{L^\infty(\Omega)} \leq C\|Eu\|_{L^\infty(\mathbb{R}^d)} \leq C\|Eu\|_{L^2(\mathbb{R}^d)}^{1-d/(2s)} \|Eu\|_{H^s(\mathbb{R}^d)}^{d/(2s)} \leq C\|u\|_{L^2(\Omega)}^{1-d/(2s)} \|u\|_{H^s(\Omega)}^{d/(2s)}$$

An alternative proof that avoids the Besov space $B^{d/2}_{2,1}(\mathbb{R}^d)$ is as follows: We assume that s is not an integer (the case of s being an integer is shown analogously). For the unit cube $Q = (0,1)^d$, the Sobolev embedding theorem asserts

$$\|u\|_{L^\infty(Q)} \leq C\|u\|_{H^s(Q)} \qquad \forall u \in H^s(Q).$$

(This can be seen by expanding u in a Fourier series). For the norm $\|u\|_{H^s(Q)}$, we now use the equivalent norm (the Aronstein-Slobodeckij norm)

$$\|u\|^2_{H^s(Q)} := \|u\|^2_{L^2(Q)} + |u|^2_{H^s(Q)}, \qquad \text{where} \qquad |u|^2_{H^s(Q)} := \sum_{|\alpha| = \lfloor s \rfloor} \int_Q \int_Q \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{2(s-\lfloor s \rfloor)}} \, dx \, dy$$

We use the analogous expression for $|u|_{H^s(\mathbb{R}^d)}$. By covering $\mathbb{R}^d$ with translates of the unit cube Q, we can infer

$$\|u\|_{L^\infty(\mathbb{R}^d)} \leq C\left[\|u\|_{L^2(\mathbb{R}^d)} + |u|_{H^s(\mathbb{R}^d)}\right] \qquad \forall u \in C_0^\infty(\mathbb{R}^d). \tag{D.1}$$

We next proceed in the standard way to infer from this a multiplicative estimate. For $u \in C_0^\infty(\mathbb{R}^d)$ we define, for $R > 0$ to be chosen below, the function $u_R(x) := u(Rx)$. Then

$$\|u\|_{L^\infty(\mathbb{R}^d)} = \|u_R\|_{L^\infty(\mathbb{R}^d)} \leq C\left[\|u_R\|_{L^2(\mathbb{R}^d)} + |u_R|_{H^s(\mathbb{R}^d)}\right] = C\left[R^{d/2}\|u\|_{L^2(\mathbb{R}^d)} + R^{d/2-s}|u|_{H^s(\mathbb{R}^d)}\right].$$

This estimate holds for every $R > 0$ with $C > 0$ independent of R and u. Selecting

$$R = \left(\frac{|u|_{H^s(\mathbb{R}^d)}}{\|u\|_{L^2(\mathbb{R}^d)}}\right)^{1/s}$$

produces

$$\|u\|_{L^\infty(\mathbb{R}^d)} \leq |u|^{d/(2s)}_{H^s(\mathbb{R}^d)}\|u\|^{1-d/(2s)}_{L^2(\mathbb{R}^d)} \qquad \forall v \in C_0^\infty(\mathbb{R}^d).$$

From this estimate, the desired bound on Ω follows easily. ∎

Lemma D.2 *Let $K = \mathbb{R}^d$ and $\omega = \mathbb{R}^{d-1} \times \{0\}$ be a hyperplane and $s > 1/2$. Then there exists $C > 0$ (depending s) such that*

$$\|u\|_{L^2(\omega)} \leq C\|u\|^{1-1/(2s)}_{L^2(K)}\|u\|^{1/(2s)}_{H^s(K)} \qquad \forall u \in H^s(K).$$

Proof. A proof based on the Besov space $B^{1/2}_{2,1}(K)$ can be found in [30, Thm. A.2]. An "elementary" proof based on the continuity of the trace operator $H^s(K) \to H^{s-1/2}(\omega)$ can be shown using the same techniques as in the proof of Lemma D.1—see [30, Exercise A.1] for details. ∎

Lemma D.3 *Let $d \geq 3$, $K = \mathbb{R}^d$, $\omega = \mathbb{R}^{d-2} \times \{0\} \times \{0\}$ be a hyperplane of co-dimension 2. Let $s > 1$. Then*

$$\|u\|_{L^2(\omega)} \leq C\|u\|^{(s-1)/s}_{L^2(K)}\|u\|^{1/s}_{H^s(K)} \qquad \forall u \in H^s(K).$$

Proof. The proof consists in iterating Lemma D.2. Let $\omega' = \mathbb{R}^{d-1}\times\{0\}$. Applying Lemma D.2 with $s' = s - 1/2 > 1/2$, we get in view of $1/(2s') = \frac{1}{2s-1}$ and $1 - 1/(2s') = \frac{2s-s}{2s-1}$

$$\|u\|_{L^2(\omega)} \leq C\|u\|^{(2s-2)/(2s-1)}_{L^2(\omega')}\|u\|^{1/(2s-1)}_{H^{s-1/2}(\omega')}.$$

Applying again Lemma D.2 and the trace theorem we arrive at

$$\|u\|_{L^2(\omega)} \leq C\|u\|^{(1-\frac{1}{2s})\frac{2s-2}{2s-1}}_{L^2(K)}\|u\|^{\frac{1}{2s}\frac{2s-2}{2s-1}}_{H^s(K)}\|u\|^{\frac{1}{2s-1}}_{H^s(K)};$$

elementary manipulations of the exponents produce the desired form. ∎

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