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Optimal additive Schwarz methods for the hp-BEM: the hypersingular integral operator in 3D on locally refined meshes

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We propose and analyze an overlapping Schwarz preconditioner for the p and hp boundary element method for the hypersingular integral equation in 3D. We consider surface triangulations consisting of triangles. We prove a bound for the condition number that is independent of the mesh size h and polynomial order p. Additionally, we provide an extension to adaptive meshes based on a local multilevel preconditioner for the lowest order space. Numerical experiments on different geometries support our theoretical findings.

1 Introduction

Many elliptic boundary value problems that are treated in practice are linear and have constant (or at least piecewise constant) coefficients. In this setting, the boundary element method (BEM, [HW08, SS11, Ste08a, McL00]) has established itself as an effective alternative to the finite element method (FEM). Just as in the FEM applied to this particular problem class, high order methods are highly attractive since they can produce rapidly convergent schemes on suitably chosen adaptive meshes. The discretization leads to large systems of equations, and a use of iterative solvers brings the question of preconditioning to the fore.

In the present work, we study high order Galerkin discretizations of the hypersingular operator. This is an operator of order 1, and we therefore have to expect the condition number of the system matrix to increase as the mesh size h decreases and the approximation order p increases. We present an additive overlapping Schwarz preconditioner that offsets this degradation and results in condition numbers that are bounded independent of the mesh size and the approximation order. This is achieved by combining the recent $H^{1/2}$-stable decomposition of spaces of piecewise polynomials of degree p of [KMR] and the multilevel diagonal scaling preconditioner of [FFPS13, Füh14] for the hypersingular operator discretized by piecewise linears.

The first p-robust additive overlapping Schwarz preconditioner for the hypersingular operator on a 2-dimensional surface $\Gamma \subset \mathbb{R}^3$ is [Heu99]. The difference between [Heu99] on the one side and [KMR] and the present article on the other side is that [Heu99] required the triangulations of the surface Γ to consists of quadrilaterals whereas [KMR] and we study simplicial triangulations. While this seems a minor difference, the underlying $H^{1/2}$-stable space decomposition is achieved with completely different tools. Historically earlier is the development in hp-FEM, where [Pav94] presented, for quadrilateral meshes, a decomposition that is stable in H^1 and, in fact, also in L^2. Essential in [Pav94] are stability properties of the 1D Gauß-Lobatto interpolation operator; tensor product arguments then allow for the treatment of quadrilaterals and hexahedra. These techniques were generalized in [Heu99] to obtain $H^{1/2}$-stable space decompositions. Since [Pav94] relied on tensor product elements (quadrilaterals, hexahedra), the corresponding H^1-stable decomposition of *hp*-FEM spaces on simplicial triangulations needed new techniques. These weren't developed until [SMPZ08]. Reference [KMR] then employs methods similar to those of [SMPZ08] and derived $H^{1/2}$-stable decompositions of high order approximation spaces. Non-overlapping additive Schwarz preconditioners for the hypersingular operator are also available, [AG00]; as it is typical of this class of preconditioners, the condition number grows polylogarithmically in p.

Our preconditioner removes the mesh dependence of the condition number with the aid of $H^{1/2}$-stable decompositions of the space of piecewise linears. For sequences of uniformly refined meshes, the first multilevel space decomposition appears to be [TS96] (see also [Osw99]). For locally refined meshes multilevel spaces decompositions have recently been made available in [FFPS13, Füh14] for sequences of meshes that are obtained in typical h-adaptive environments with the aid of *newest vertex bisection* (NVB).

The paper is organized as follows: In Section 2 we introduce the hypersingular equation and the discretization by high order piecewise polynomial spaces. Section 3 collects properties of the fractional Sobolev spaces including the scaling properties. Section 4 studies in detail the p-dependence of the condition number of the unpreconditioned system. The polynomial basis on the reference triangle chosen by us is a hierarchical basis of the form first proposed by Karniadakis & Sherwin, [KS99, Appendix D.1.1.2]; the precise form is the one from [Zag06, Section 5.2.3]. We prove bounds for the condition number of the stiffness matrix not only in the $H^{1/2}$-norm but also in the norms of L^2 and H^1. This is also of interest for *hp*-FEM and could not be found in the literature. Section 5 develops several preconditioners. The first one (Theorem 5.3) is based on decomposing the high order approximation space into the global space of piecewise linears and local high order spaces of functions associated with the vertex patches. The second one (Theorem 5.4) is based on a further multilevel decomposition of the global space of piecewise linears. The third one (Theorem 5.10) exploits the observation that topologically, only a finite number of vertex patches can occur. Hence, significant memory savings for the preconditioner are possible if the exact bilinear forms for the vertex patches are replaced with scaled versions of simpler ones defined on a finite number of reference configurations. Numerical experiments in Section 6 illustrate that the proposed preconditioners are indeed robust with respect to both h and p.

We close with a remark on notation: The expression $a \sim b$ always means that there exist two constants c_1, $c_2 > 0$ such that $c_1\, a \leq b \leq c_2\, a$. The constants c_1, c_2 do not depend on the mesh size h and the approximation order p, but may depend on the geometry and the shape regularity of the triangulation. We will also sometimes use the notation $a \lesssim b$ to mean $a \leq Cb$.

2 hp-discretization of the hypersingular integral equation

2.1 Hypersingular integral equation

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz polyhedron and $\Gamma \subseteq \partial\Omega$ an open, connected subset of $\partial\Omega$. If $\Gamma \neq \partial\Omega$, we assume it to be a Lipschitz hypograph, [McL00]; the key property needed is that Γ is such that the ellipticity condition (2.2) holds. Furthermore, we will use affine, shape regular triangulations of Γ below, which further imposes conditions on Γ. In this work, we are concerned with preconditioning high order discretizations of the hypersingular integral operator, which is given by

$$(Du)(x) := -\partial_{n_x}^{int} \int_\Gamma \partial_{n_y}^{int} G(x,y) u(y)\, ds_y \quad \text{for } x \in \Gamma, \tag{2.1}$$

where $G(x,y) := \frac{1}{4\pi}\frac{1}{|x-y|}$ is the fundamental solution of the 3D-Laplacian and $\partial_{n_y}^{int}$ denotes the (interior) normal derivative with respect to $y \in \Gamma$.

We will need some results from the theory of Sobolev and interpolation spaces, see [McL00, Appendix B]. For an open subset $\omega \subset \partial\Omega$, let $L^2(\omega)$ and $H^1(\omega)$ denote the usual Sobolev spaces and set $\widetilde{H}^1(\omega) := \overline{C_0^\infty(\omega)}^{\|\cdot\|_{H^1}}$. On $\widetilde{H}^1(\omega)$, we consider the equivalent norm $\|u\|_{\widetilde{H}^1(\omega)}^2 := \|\nabla_\Gamma u\|_{L^2(\omega)}^2$.

We will define broken Sobolev norms by interpolation. The following Proposition 2.1 collects key properties of interpolation spaces that we will need; we refer to [Tar07, Tri95] for a comprehensive treatment. For two Banach spaces $(X_0, \|\cdot\|_0)$ and $(X_1, \|\cdot\|_1)$, with continuous inclusion $X_1 \subseteq X_0$ and a parameter $s \in (0,1)$ the interpolation norm is defined as

$$\|u\|_{[X_0,X_1]_s}^2 := \int_{t=0}^\infty t^{-2s} \left(\inf_{v \in X_1} \|u-v\|_0 + t\|v\|_1 \right)^2 \frac{dt}{t}.$$

The interpolation space is given by $[X_0, X_1]_s := \left\{ u \in X_0 : \|u\|_{[X_0,X_1]_s} < \infty \right\}$.

An important result, which we use in this paper, is the following interpolation theorem:

Proposition 2.1. *Let X_i, Y_i, $i \in \{0,1\}$ be two pairs of Banach spaces with continuous inclusions $X_1 \subseteq X_0$ and $Y_1 \subseteq Y_0$ and let $s \in (0,1)$.*

(i) If a linear operator T is bounded as an operator $X_0 \to Y_0$ and $X_1 \to Y_1$ then it is also bounded as an operator $[X_0,X_1]_s \to [Y_0,Y_1]_s$ with

$$\|T\|_{[X_0,X_1]_s \to [Y_0,Y_1]_s} \le \|T\|_{X_0 \to Y_0}^{1-s} \|T\|_{X_1 \to Y_1}^s.$$

(ii) There exists a constant $C > 0$ such that for all $x \in X_1$:

$$\|x\|_{[X_0,X_1]_s} \le C \|x\|_{X_0}^{1-s} \|x\|_{X_1}^s. \qquad \square$$

We define the fractional Sobolev spaces by interpolation. For $s \in (0,1)$, we set:

$$H^s(\omega) := \left[L^2(\omega), H^1(\omega)\right]_s, \quad \widetilde{H}^s(\omega) := \left[L^2(\omega), \widetilde{H}^1(\omega)\right]_s.$$

Here, we will only consider the case $s = 1/2$. We define $H^{-1/2}(\Gamma)$ as the dual space of $\widetilde{H}^{1/2}(\Gamma)$ where duality is understood with respect to the (continuously) extended $L^2(\Gamma)$-scalar product and denoted by $\langle\cdot,\cdot\rangle_\Gamma$. An equivalent norm on $H^{1/2}(\Gamma)$ is given by $\|u\|_{H^{1/2}(\Gamma)}^2 \sim \|u\|_{L^2(\Gamma)}^2 + |u|_{H^{1/2}(\Gamma)}^2$, where $|\cdot|_{H^{1/2}(\Gamma)}$ is given by the Sobolev-Slobodeckij seminorm (see [SS11] for the exact definition).

We now state some important properties of D, see, e.g., [SS11, McL00, HW08, Ste08a]. First, the operator $D : \widetilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is a bounded linear operator.

For open surfaces $\Gamma \subsetneqq \partial\Omega$ the operator is elliptic

$$\langle Du, u\rangle_\Gamma \ge c_{ell} \|u\|_{\widetilde{H}^{1/2}(\Gamma)}^2 \tag{2.2}$$

with some constant $c_{ell} > 0$ that only depends on Γ. In the case of a closed surface, i.e. $\Gamma = \partial\Omega$ we note that $\widetilde{H}^{1/2}(\Gamma) = H^{1/2}(\Gamma)$ and the operator D is still semi-elliptic, i.e.

$$\langle Du, u\rangle_\Gamma \geq c_{ell} \, |u|^2_{H^{1/2}(\Gamma)} .$$

If Γ is simply connected, then the kernel of D consists of only the constant functions, $\ker(D) = \operatorname{span}(1)$.

To get unique solvability and strong ellipticity for the case of a closed surface it is customary to introduce a stabilized bilinear form by

$$\left\langle \widetilde{D}u, v \right\rangle_\Gamma := \langle Du, v\rangle_\Gamma + \alpha^2 \langle u, 1\rangle_\Gamma \langle v, 1\rangle_\Gamma , \quad \alpha > 0. \tag{2.3}$$

In order to avoid having to distinguish the two cases $\Gamma = \partial\Omega$ and $\Gamma \subsetneq \partial\Omega$, we will only work with the stabilized form and just set $\alpha = 0$ in the case of $\Gamma \subsetneq \partial\Omega$.

The basic integral equation involving the hypersingular operator D then reads: For given $g \in H^{-1/2}(\Gamma)$, find $u \in \widetilde{H}^{1/2}(\Gamma)$ such that

$$\left\langle \widetilde{D}u, v \right\rangle_\Gamma = \langle g, v\rangle_\Gamma \quad \forall v \in \widetilde{H}^{1/2}(\Gamma). \tag{2.4}$$

We note that in the case of the closed surface, the solution of the stabilized system above is equivalent to the solution of $\langle Du, v\rangle_\Gamma = \langle g, v\rangle_\Gamma$ under the side constraint $\langle u, 1\rangle_\Gamma = \frac{\langle g,1\rangle_\Gamma}{\alpha^2\,|\Gamma|}$.

Moreover, it is well known that $\left\langle \widetilde{D}\cdot, \cdot \right\rangle_\Gamma$ is symmetric, elliptic and induces an equivalent norm on $\widetilde{H}^{1/2}(\Gamma)$, i.e.,

$$\left\langle \widetilde{D}u, u \right\rangle_\Gamma \sim \|u\|^2_{\widetilde{H}^{1/2}(\Gamma)} .$$

2.2 Discretization

Let $\mathcal{T} = \{K_1, \ldots, K_N\}$ denote a regular (in the sense of Ciarlet) triangulation of the two-dimensional manifold $\Gamma \subseteq \partial\Omega$ into compact, non-degenerate planar surface triangles. We say a triangulation is γ shape regular, if there exists a constant $\gamma > 0$ such that

$$\max_{K\in\mathcal{T}} \frac{\operatorname{diam}(K)^2}{|K|} \leq \gamma.$$

For each element K we denote the affine mapping from the reference triangle $\widehat{K} := \operatorname{conv}\{(0,0),(1,0),(0,1)\}$ by $F_K : \widehat{K} \to K$. We will write $P^p(\widehat{K})$ for the space of polynomials of degree p on $\widehat{K}$. The space of piecewise polynomials on $\mathcal{T}$ is given by

$$P^p(\mathcal{T}) := \left\{ u \in L^2(\Gamma) \,:\, u \circ F_K \in P^p(\widehat{K}) \text{ for all } K \in \mathcal{T} \right\}. \tag{2.5}$$

The elementwise constant mesh width function $h := h_\mathcal{T} \in P^0(\mathcal{T})$ is defined by $(h_\mathcal{T})|_K := \operatorname{diam}(K)$ for all $K \in \mathcal{T}$.

Let $\mathcal{V} = \{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_M\}$ denote the set of all vertices of the triangulation $\mathcal{T}$ that are not on the boundary of Γ. We define the (vertex) patch $\omega_{\boldsymbol{z}}$ for a vertex $\boldsymbol{z} \in \mathcal{V}$ by

$$\omega_{\boldsymbol{z}} := \operatorname{interior}\left(\bigcup_{\{K\in\mathcal{T}:\, \boldsymbol{z}\in K\}} K \right), \tag{2.6}$$

where the interior is understood with respect to the topology of Γ. For $p \geq 1$, define

$$\widetilde{S}^p(\mathcal{T}) := P^p(\mathcal{T}) \cap \widetilde{H}^{1/2}(\Gamma). \tag{2.7}$$

Then, the Galerkin discretization of (2.4) consists in replacing $\widetilde{H}^{1/2}(\Gamma)$ with the discrete subspace $\widetilde{S}^p(\mathcal{T})$, i.e., we seek a solution of the variational formulation: Find $u_h \in \widetilde{S}^p(\mathcal{T})$ such that

$$\left\langle \widetilde{D}u_h, v_h \right\rangle_\Gamma = \langle g, v_h \rangle_\Gamma \quad \text{for all } v_h \in \widetilde{S}^p(\mathcal{T}). \tag{2.8}$$

Remark 2.2. We require a uniform polynomial degree across all elements. This is not essential and done for simplicity of presentation. For details on the more general case, see [KMR].

After choosing a basis of $\widetilde{S}^p(\mathcal{T})$ the problem (2.8) can be written as a linear system of equations, and we write $\widetilde{D}_h^p$ for the resulting system matrix. Our goal is to construct a preconditioner for $\widetilde{D}_h^p$. It is well-known that the condition number of $\widetilde{D}_h^p$ depends on the choice of the basis of $\widetilde{S}^p(\mathcal{T})$. Here, we will work with the basis determined by the basis of Definition 2.4 below. We remark in passing that the preconditioned system of Section 5.2 will no longer depend on the basis.

2.3 Polynomial basis on the reference element

For the matrix representation of the Galerkin formulation (2.8) we have to specify a polynomial basis on the reference triangle $\widehat{K}$. We use a basis that is based on a collapsed tensor product representation of the triangle and is given in [Zag06, Section 5.2.3]. This kind of basis was first proposed for the *hp*-FEM by Karniadakis & Sherwin, [KS99, Appendix D.1.1.2]; closely related, earlier works on polynomial basis that rely on a collapsed tensor product representation of the triangle are [Koo75, Dub91].

We introduce the following notations:

Definition 2.3 (polynomial basis on the reference triangle)**.** For coefficients α, $\beta > -1$ the family of *Jacobi polynomials* on the interval $(-1,1)$ is denoted by $P_n^{(\alpha,\beta)}(s)$, $n \in \mathbb{N}_0$. They are orthogonal with respect to the L^2 inner product on $(-1,1)$ with weight $(1-x)^\alpha(1+x)^\beta$. (See for example [KS99, Appendix A] or [Zag06, Appendix A.3] for the exact definitions and a list of important properties). The *Legendre polynomials* are a special case of the Jacobi polynomials for $\alpha = \beta = 0$ and denoted by $\ell_n(s) := P_n^{(0,0)}(s)$. The *integrated Legendre polynomials* L_n and the *scaled polynomials* are defined by

$$\begin{aligned} L_n(s) &:= \int_{-1}^{s} \ell_{n-1}(t)dt \quad \text{for } n \in \mathbb{N}, \\ P_n^{\mathcal{S},(\alpha,\beta)}(s,t) &:= t^n P_n^{(\alpha,\beta)}(s/t), \\ L_n^{\mathcal{S}}(s,t) &:= t^n L_n(s/t). \end{aligned}$$

On the reference triangle, our basis reads as follows:

Definition 2.4. Let $p > 0$ and let $\lambda_1, \lambda_2, \lambda_3$ be the barycentric coordinates on the reference triangle $\widehat{K}$. Then the basis functions for the reference triangle are given by:

(a) for $i = 1, 2, 3$ the vertex functions:

$$\varphi_i^{\mathcal{V}} := \lambda_i;$$

(b) for $m=1,2,3$ and an edge $\mathcal{E}_m$ with edge vertices e_1, e_2, the edge functions are given by:

$$\varphi_i^{\mathcal{E}_m} := \sqrt{\frac{2i+3}{2}}\, L_{i+2}^{\mathcal{S}}(\lambda_{e_2}-\lambda_{e_1}, \lambda_{e_1}+\lambda_{e_2}), \qquad 0 \le i \le p-2;$$

(c) for $0 \le i+j \le p-3$ we have the cell based functions:

$$\varphi_{(i,j)}^{\mathcal{I}} := c_{ij}\lambda_1\lambda_2\lambda_3 P_i^{\mathcal{S},(2,2)}(\lambda_1-\lambda_2, \lambda_1+\lambda_2)\, P_j^{(2i+5,2)}(2\lambda_3-1).$$

with c_{ij} such that $\left\|\varphi_{(i.j)}^{\mathcal{I}}\right\|_{L^2(\widehat{K})} = 1$.

Remark 2.5. In order to get a basis of $\widetilde{S}^p(\mathcal{T})$ we take the composition with the element mappings $\varphi \circ F_K$. To ensure continuity along edges we take an arbitrary orientation of the edges and observe that the edge basis functions $\varphi_i^{\mathcal{E}}$ are symmetric under permutation of λ_{e_1} and λ_{e_2} up to a sign change $(-1)^i$.

3 Properties of $\widetilde{H}^{1/2}(\Gamma)$

3.1 Quasi-interpolation in $\widetilde{H}^{1/2}(\Gamma)$

Several results of this paper depend on results in [KMR]. Therefore we present a short summary of the main results of the paper in this section. In [KMR] the authors propose an $H^{1/2}$-stable space decomposition on triangles. It is based on quasi-interpolation operators constructed by local averaging on elements.

We introduce the following spaces:

$$X_0 := \prod_{\boldsymbol{z}\in\mathcal{V}} L^2(\omega_{\boldsymbol{z}}), \qquad X_1 := \prod_{\boldsymbol{z}\in\mathcal{V}} \widetilde{H}^1(\omega_{\boldsymbol{z}})$$

The spaces $L^2(\omega_{\boldsymbol{z}})$ and $\widetilde{H}^1(\omega_{\boldsymbol{z}})$ are endowed with the L^2- and $\widetilde{H}^1$-norm, respectively.

Proposition 3.1 (localization, [KMR])**.** *There exists an operator* $J : L^2(\Gamma) \to \left(\mathcal{S}^1(\mathcal{T}), \|\cdot\|_{L^2(\Gamma)}\right) \times X_0$ *with the following properties:*

(i) J *is linear and bounded.*

(ii) $J|_{\widetilde{H}^1(\Gamma)}$ *is also bounded as an operator* $\widetilde{H}^1(\Gamma) \to \left(\mathcal{S}^1(\mathcal{T}), \|\cdot\|_{\widetilde{H}^1(\Gamma)}\right) \times X_1$.

(iii) If $u \in \widetilde{S}^p(\mathcal{T})$ *then each component of* Ju *is in* $\widetilde{S}^p(\mathcal{T})$.

(iv) For all $u \in L^2(\Gamma)$*, if we write* $Ju =: (u_1, U)$*, and* $U_{\boldsymbol{z}}$ *for the component of* U *in* X_0 *corresponding to the space* $L^2(\omega_{\boldsymbol{z}})$*, then* Ju *represents a decomposition of* u*, i.e.,*

$$u = u_1 + \sum_{z\in\boldsymbol{z}} U_{\boldsymbol{z}}, \tag{3.1}$$

and the projection is $\widetilde{H}^{1/2}(\Gamma)$ *stable:*

$$\|u_1\|^2_{\widetilde{H}^{1/2}(\Gamma)} + \sum_{z\in\boldsymbol{z}} \|U_{\boldsymbol{z}}\|^2_{\widetilde{H}^{1/2}(\omega_{\boldsymbol{z}})} \le C\,\|u\|^2_{\widetilde{H}^{1/2}(\Gamma)}.$$

The norms of J in points (i)—(iii) and the constant $C > 0$ in (iv) depend only on Γ and the shape regularity constant γ.

Sketch of proof: The first component of J (i.e., the mapping $u \mapsto u_1$) consists of the Scott-Zhang operator, as modified in [AFF$^+$14, Section 3.2]. The local components (i.e., the functions $U_{\boldsymbol{z}}$, $z \in \mathcal{V}$) then are based on a successive decomposition into vertex, edge and interior parts, similar to what is done in [SMPZ08]. We give a flavor of the procedure. In order to define the vertex parts for a vertex $\boldsymbol{z}$, we select an element $K \subset \omega_{\boldsymbol{z}}$ of the patch $\omega_{\boldsymbol{z}}$ and do a suitable local averaging on that element. In order to extend this function, which is defined in terms of $u|_K$, to the patch $\omega_{\boldsymbol{z}}$ and thus obtain the function $u_{\boldsymbol{z}}$, we define $u_{\boldsymbol{z}}$ by "rotating" it around the vertex $\boldsymbol{z}$. Provided the averaging is done appropriately, it can be shown that for continuous functions u we have $u(\boldsymbol{z}) = u_{\boldsymbol{z}}(\boldsymbol{z})$ and appropriate stability properties in L^2 and H^1. The edge contributions are contructed from the function $\tilde{u} := u - \sum_{\boldsymbol{z}\in\mathcal{V}} u_{\boldsymbol{z}}$. For an edge E one selects an element (of which E is an edge), averages there, and extends the obtained averaged function to the edge patch by symmetry across the edge E. It again holds for sufficiently smooth u that $\tilde{u}(x) = \tilde{u}_E(x)\ \forall x \in E$. The interiors are then just given by restricting the remainder. For $u \in \widetilde{H}^1(\Gamma)$, since $u - \sum_{E\in\mathcal{E}} \tilde{u}_E - \sum_{\boldsymbol{z}\in\mathcal{V}} u_{\boldsymbol{z}}$ vanishes on all edges, the resulting functions are again in $\widetilde{H}^1(K)\ \forall K \in \mathcal{T}$. The $\widetilde{H}^{1/2}$ stability then is a direct consequence of the L^2 and H^1 stability and interpolation properties given in Proposition 2.1, (i). $\square$

Remark 3.2. Independently, a decomposition analogous to Proposition 3.1 was presented in [FW13].

The construction in Proposition 3.1 can be modified and used to relate the $\widetilde{H}^{1/2}(\omega_{\boldsymbol{z}})$-norm to the $\widetilde{H}^{1/2}(\mathcal{O})$-norm, if $\mathcal{O} \supset \omega_{\boldsymbol{z}}$:

Corollary 3.3. *Let $\boldsymbol{z} \in \mathcal{V}$ and $\mathcal{O}$ be some set of triangles of $\mathcal{T}$ with $\omega_{\boldsymbol{z}} \subseteq \mathcal{O} \subseteq \Gamma$. Then there exist constants c_1, c_2 that only depend on $\mathcal{O}$, Γ, and the shape regularity of $\mathcal{T}$ such that, for all $u \in \widetilde{H}^{1/2}(\mathcal{O})$ with $\operatorname{supp}(u) \subseteq \overline{\omega_{\boldsymbol{z}}}$, we can estimate:*

$$c_1\ \|u\|_{\widetilde{H}^{1/2}(\omega_{\boldsymbol{z}})} \leq \|u\|_{\widetilde{H}^{1/2}(\mathcal{O})} \leq c_2\, \|u\|_{\widetilde{H}^{1/2}(\omega \boldsymbol{z})}\,.$$

Proof. To see the second inequality, consider the extension operator E that extends the function u by 0 outside of $\omega_{\boldsymbol{z}}$. This operator is continuous $L^2(\omega_{\boldsymbol{z}}) \to L^2(\mathcal{O})$ and $\widetilde{H}^1(\omega_{\boldsymbol{z}}) \to \widetilde{H}^1(\mathcal{O})$, both with constant 1. Applying Proposition 2.1, (i) to this extension operator E gives the second inequality. The first inequality is more involved. We start by noting that the stability assertion of Proposition 3.1 gives $u = u_1 + \sum_{\boldsymbol{z}'\in\mathcal{V}} U_{\boldsymbol{z}'}$ and

$$\|u_1\|^2_{\widetilde{H}^{1/2}(\mathcal{O})} + \sum_{\boldsymbol{z}'\in\mathcal{V}} \|U_{\boldsymbol{z}'}\|^2_{\widetilde{H}^{1/2}(\omega_{\boldsymbol{z}'})} \leq C\, \|u\|^2_{\widetilde{H}^{1/2}(\mathcal{O})}\,. \tag{3.2}$$

The constant C depends only on the set $\mathcal{O}$ and the shape regularity of the triangulation. The decomposition in Proposition 3.1 is not unique, and we will now exploit this by requiring more. Specifically, we assert that the operator J, which effects the decomposition, can be chosen such that, for given $\omega_{\boldsymbol{z}}$, we have $\operatorname{supp} u_1 \subset \overline{\omega_{\boldsymbol{z}}}$ and $U_{\boldsymbol{z}'} = 0$ for $\boldsymbol{z}' \neq \boldsymbol{z}$. If this can be achieved, then $u = u_1 + U_{\boldsymbol{z}}$ and (3.2) gives

$$\|u_1\|_{\widetilde{H}^{1/2}(\mathcal{O})} + \|U_{\boldsymbol{z}}\|_{\widetilde{H}^{1/2}(\omega_{\boldsymbol{z}})} \leq C\, \|u\|_{\widetilde{H}^{1/2}(\prime)}\,.$$

Using the assumption that $\operatorname{supp} u_1 \subset \overline{\omega_{\boldsymbol{z}}}$, we can exploit $\|u_1\|_{\widetilde{H}^{1/2}(\omega_{\boldsymbol{z}})} \leq c_2\, \|u_1\|_{\widetilde{H}^{1/2}(\mathcal{O})}$; the triangle inequality $\|u\|_{\widetilde{H}^{1/2}(\omega_{\boldsymbol{z}})} \leq \|u_1\|_{\widetilde{H}^{1/2}(\omega_{\boldsymbol{z}})} + \|U_{\boldsymbol{z}}\|_{\widetilde{H}^{1/2}(\omega_{\boldsymbol{z}})}$ then concludes the proof.

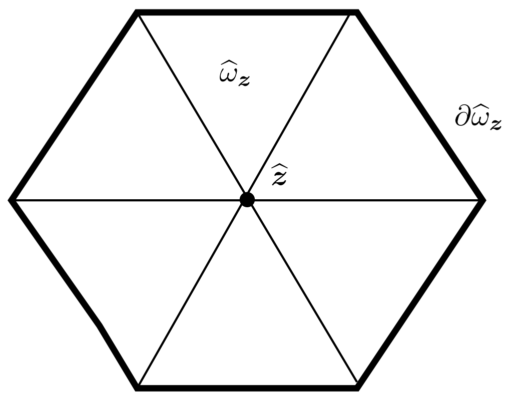

Figure 3.1: Example of an interior reference patch $\widehat{\omega}_{\boldsymbol{z}}$

It therefore remains to see that we can construct the operator J of Proposition 3.1 with the additional property that $u = u_1 + U_{\boldsymbol{z}}$ if u is such that $\operatorname{supp} u \subset \overline{\omega_{\boldsymbol{z}}}$. This follows by carefully selecting the elements on which the local averaging is done, namely, whenever one has to choose an element on which to average, one selects an element that is *not* contained in $\omega_{\boldsymbol{z}}$. For example for vertex contributions $\boldsymbol{z'} \in \partial\omega_{\boldsymbol{z}}$ we make sure to use elements K' which are not in $\omega_{\boldsymbol{z}}$. This implies that for $\operatorname{supp}(u) \subseteq \overline{\omega_{\boldsymbol{z}}}$ we get $u_{\boldsymbol{z'}} = 0$. The same choice is made for the Scott-Zhang operator and the edge contributions. $\square$

The stable space decomposition of Proposition 3.1 is one of several ingredients of the proof that the interpolation space obtained by interpolating the $\widetilde{S}^p(\mathcal{T})$ endowed with the L^2-norm and the H^1-norm yields the space $\widetilde{S}^p(\mathcal{T})$ endowed with the appropriate fractional Sobolev norm:

Proposition 3.4 ([KMR])**.** *Let $s \in (0,1)$ and let $\mathcal{T}$ be a shape regular triangulation of Γ. Let $p \geq 1$. Then:*

$$\left[\left(\widetilde{S}^p(\mathcal{T}), \|\cdot\|_{L^2(\Gamma)}\right), \left(\widetilde{S}^p(\mathcal{T}), \|\cdot\|_{L^2(\Gamma)}\right)\right]_s = \left(\widetilde{S}^p(\mathcal{T}), \|\cdot\|_{\widetilde{H}^s(\Gamma)}\right).$$

The constants implied in the norm equivalence depend only on Γ, s, and the shape regularity constant of the mesh $\mathcal{T}$.

3.2 Geometry of vertex patches

We recall that $\mathcal{V}$ is the set of all inner vertices, i.e., $\mathcal{V} \ni \boldsymbol{z} \notin \partial\Gamma$. We define the patch size for a vertex $z \in \mathcal{V}$ as $h_{\boldsymbol{z}} := \operatorname{diam}(\omega_{\boldsymbol{z}})$ and stress that γ-shape regularity implies $h_{\boldsymbol{z}} \sim h|_K$ for all elements $T \subseteq \omega_{\boldsymbol{z}}$.

Due to shape regularity, the number of elements meeting at a vertex is bounded. The following definition allows us to transform the vertex patches $\omega_{\boldsymbol{z}}$ to a finite number of reference configurations.

Definition 3.5 (reference patch)**.** Let $\omega_{\boldsymbol{z}}$ be an interior patch consisting of n triangles. We may define a Lipschitz continuous map $F_{\boldsymbol{z}} : \widehat{\omega}_{\boldsymbol{z}} \to \omega_{\boldsymbol{z}}$, where $\widehat{\omega}_{\boldsymbol{z}} \subseteq \mathbb{R}^2$ is a regular polygon with n edges, see Fig. 3.1. The map $F_{\boldsymbol{z}}$ is piecewise defined as a concatenation of affine maps from triangles comprising the regular polygon to the reference element $\widehat{K}$ with the element maps F_K. We note that $F_{\boldsymbol{z}}(\partial\widehat{\omega}_{\boldsymbol{z}}) = \partial\omega_{\boldsymbol{z}}$.

The following lemma tells us how the hypersingular integral operator behaves under the patch transformation.

Lemma 3.6. *Let* $u \in \widetilde{H}^{1/2}(\Gamma)$ *with* $\operatorname{supp}(u) \subseteq \overline{\omega_{\boldsymbol{z}}}$. *Define* $\widehat{u} := u \circ F_{\boldsymbol{z}}$ *and the integral operator* $\widehat{D}$ *as* $\widehat{D}\widehat{u}\,(x) := -\partial_{n_x}^{int} \int_{\widehat{\omega_{\boldsymbol{z}}}} \partial_{n_y}^{int} G(x,y)\widehat{u}(y)\, ds(y)$ *for* $x \in \widehat{\omega}_{\boldsymbol{z}}$, *where we treat* $\widehat{\omega_{\boldsymbol{z}}} \subseteq \mathbb{R}^2$ *as a screen embedded in* $\mathbb{R}^3$ *and* ∂_{n_x} *is the derivative in direction of the vector* $(0,0,1)$. *Then the hypersingular operator scales like*

$$\langle Du, u\rangle_{\omega_{\boldsymbol{z}}} \sim h_{\boldsymbol{z}} \left\langle \widehat{D}\widehat{u}, \widehat{u} \right\rangle_{\widehat{\omega}_{\boldsymbol{z}}},$$

where the implied constants depend only on the γ*-shape regularity* (5.2) *and the geometry of* Γ.

Proof. We will prove this in three steps:

1. $\langle Du, u\rangle_\Gamma \sim \|u\|^2_{\widetilde{H}^{1/2}(\omega_{\boldsymbol{z}})}$,
2. $\|u\|^2_{\widetilde{H}^{1/2}(\omega_{\boldsymbol{z}})} \sim h_{\boldsymbol{z}} \|\widehat{u}\|^2_{\widetilde{H}^{1/2}(\widehat{\omega}_{\boldsymbol{z}})}$,
3. $\|\widehat{u}\|^2_{\widetilde{H}^{1/2}(\widehat{\omega}_{\boldsymbol{z}})} \sim \left\langle \widehat{D}\widehat{u}, \widehat{u} \right\rangle_{\widehat{\omega}_{\boldsymbol{z}}}$.

Proof of 1: It is well-known, that D is continuous and elliptic on $\widetilde{H}^{1/2}(\omega_{\boldsymbol{z}})$. We note that in our case, the ellipticity constants can be chosen independently of the patch $\omega_{\boldsymbol{z}}$ and instead only depend on Γ. To do so we embed the spaces $\widetilde{H}^{1/2}(\omega_{\boldsymbol{z}})$ into finitely many larger spaces $\widetilde{H}^{1/2}(\mathcal{O})$ for some open sub-surfaces $\mathcal{O}$ such that $\omega_{\boldsymbol{z}} \subseteq \mathcal{O} \subseteq \Gamma$ (for the screen problem we may use $\mathcal{O} := \Gamma$, for the case of closed surfaces we can for example use $\mathcal{O}_j := \Gamma \setminus F_j$, where F_j is the j-th face of the polyhedron Ω such that $\omega_{\boldsymbol{z}} \cap F_j = \emptyset$). It is important that these surfaces are open, since for closed surfaces Γ we don't have full ellipticity of D but only for $\widetilde{D}$, and the stabilization term has a different scaling behavior. Since u vanishes outside of ω_z we can use the ellipticity on $\widetilde{H}^{1/2}(\mathcal{O})$ and see $\langle Du, u\rangle_\Gamma \sim \|u\|^2_{\widetilde{H}^{1/2}(\mathcal{O})}$. By Corollary 3.3 the norms on $\omega_{\boldsymbol{z}}$ and $\mathcal{O}$ are equivalent, which implies the statement 1.

Proof of 2: The scaling of the L^2-norm and H^1-seminorm (we can use the seminorm, since we are working on $\widetilde{H}^{1/2}$ of an open surface) is well-known as

$$\|u\|_{L^2(\omega_{\boldsymbol{z}})} \sim h_{\boldsymbol{z}} \|\hat{u}\|_{L^2(\widehat{\omega}_{\boldsymbol{z}})}, \quad \|\nabla u\|_{L^2(\omega_{\boldsymbol{z}})} \sim \|\nabla \hat{u}\|_{L^2(\widehat{\omega}_{\boldsymbol{z}})}.$$

The interpolation theorem (Proposition 2.1, (i)) then proves part 2.

Proof of 3: We again use ellipticity and continuity of $\hat{D}$. Since there are only finitely many reference patches, the constants can be chosen independently of the individual patches. □

4 Condition number of the hp-Galerkin Matrix

In this section we investigate the condition number of the unpreconditioned Galerkin matrix to motivate the need for good preconditioning. We will work on the reference triangle $\widehat{K} := \operatorname{conv}\{(0,0),(1,0),(0,1)\}$, and will need the following well-known inverse inequalities for polynomials on $\widehat{K}$:

Proposition 4.1 (Inverse inequalities, [Sch98, Theorem 4.76])**.** *Let* $\widehat{K}$ *denote the reference triangle and let* E *be one of its edges. There exists a generic constant* C *such that for all* $p \in \mathbb{N}$

and for all $v \in P^p(\widehat{K})$ the following estimates hold:

$$\|v\|_{L^\infty(\widehat{K})} \leq Cp^2 \|v\|_{L^2(\widehat{K})}, \tag{4.1}$$

$$\|v\|_{L^\infty(\widehat{K})} \leq C\sqrt{\log(p+1)}\, \|v\|_{H^1(\widehat{K})}, \tag{4.2}$$

$$\|v\|_{H^1(\widehat{K})} \leq Cp^2 \|v\|_{L^2(\widehat{K})}, \tag{4.3}$$

$$\|v\|_{L^2(E)} \leq Cp \|v\|_{L^2(\widehat{K})}. \tag{4.4}$$

□

First we investigate the L^2 and H^1 conditioning of our basis on the reference triangle.

Lemma 4.2. *Let $u \in P^p(\widehat{K})$ and let $\alpha_j^{\mathcal{V}}$, $\alpha_j^{\mathcal{E}_m}$, $\alpha_{(ij)}^{\mathcal{I}}$ be the coefficients with respect to the basis in Definition 2.4. We decompose $u = u_{\mathcal{V}} + u_{\mathcal{E}_1} + u_{\mathcal{E}_2} + u_{\mathcal{E}_3} + u_{\mathcal{I}}$ with*

$$u_{\mathcal{V}} := \sum_{j=1}^{3} \alpha_j^{\mathcal{V}} \varphi_j^{\mathcal{V}}, \quad u_{\mathcal{E}_m} := \sum_{j=0}^{p-2} \alpha_j^{\mathcal{E}_m} \varphi_j^{\mathcal{E}_m}, \quad u_{\mathcal{I}} := \sum_{i+j \leq p-3} \alpha_{(ij)}^{\mathcal{I}} \varphi_{(ij)}^{\mathcal{I}}. \tag{4.5}$$

Then we can estimate with generic constants that do not depend on u or p:

$$\|u_{\mathcal{V}}\|_{L^2(\widehat{K})}^2 \leq C \sum_{j=1}^{3} \left|\alpha_j^{\mathcal{V}}\right|^2, \tag{4.6}$$

$$\|u_{\mathcal{E}_m}\|_{L^2(\widehat{K})}^2 \leq C \sum_{j=0}^{p-2} \left|\alpha_j^{\mathcal{E}_m}\right|^2, \tag{4.7}$$

$$\|u_{\mathcal{I}}\|_{L^2(\widehat{K})}^2 = \sum_{i+j \leq p-3} \left|\alpha_{(ij)}^{\mathcal{I}}\right|^2. \tag{4.8}$$

Combined this gives:

$$\|u\|_{L^2(\widehat{K})}^2 \leq C \left(\sum_{j=1}^{3} \left|\alpha_j^{\mathcal{V}}\right|^2 + \sum_{m=1}^{3} \sum_{j=0}^{p-2} \left|\alpha_j^{\mathcal{E}_m}\right|^2 + \sum_{i,j \leq p-3} \left|\alpha_{(ij)}^{\mathcal{I}}\right|^2 \right). \tag{4.9}$$

Proof. We estimate each term individually. For $u_{\mathcal{V}}$ the estimate is clear. For the edge contributions, we restrict ourselves to the edge $(0,0)-(1,0)$, i.e., $m=1$ and drop the index m in the notation. The other edges can be estimated by symmetry.

$$\begin{aligned}
\|u_{\mathcal{E}}\|_{L^2(\widehat{K})}^2 &= \left\| \sum_{j=0}^{p-2} \alpha_j^{\mathcal{E}} L_{i+2}^{\mathcal{S}}(\lambda_1 - \lambda_2, \lambda_1 + \lambda_2) \right\|_{L^2(\widehat{K})}^2 \\
&= \sum_{i,j=0}^{p-2} \alpha_j^{\mathcal{E}} \alpha_i^{\mathcal{E}} \int_{\widehat{K}} L_{i+2}^{\mathcal{S}}(\lambda_1 - \lambda_2, \lambda_1 + \lambda_2) L_{j+2}^{\mathcal{S}}(\lambda_1 - \lambda_2, \lambda_1 + \lambda_2) dx \\
&= \frac{1}{4} \sum_{i,j=0}^{p-2} \alpha_j^{\mathcal{E}} \alpha_i^{\mathcal{E}} \int_{-1}^{1} \int_{-1}^{1} L_{i+2}(\xi) L_{j+2}(\xi) \left(\frac{1-\eta}{2}\right)^{i+j+5} d\xi d\eta.
\end{aligned}$$

where in the last step we transformed the reference triangle to $(-1,1) \times (-1,1)$ via the map $(\xi, \eta) \mapsto (\frac{1}{4}(1+\xi)(1-\eta), \frac{1}{2}(1+\eta))$.

It is well-known (see [Sch98, Page 65]), that the mass matrix of the integrated Legendre polynomials is pentadiagonal, and the non-zero entries satisfy $\left|M_{(ij)}\right| \sim \frac{1}{(i+1)\,(j+1)}$. It is easy to check that $\left|\int_{-1}^{1}\left(\frac{1-\eta}{2}\right)^{i+j+5} d\eta\right| \leq C(i+j+6)^{-1}$. We can estimate:

$$\|u_{\mathcal{E}}\|^2_{L^2(\widehat{K})} \leq C\sum_{j=0}^{p-2}\left|\alpha_j^{\mathcal{E}}\right|^2 \frac{1}{(j+1)^3} \leq C\sum_{j=0}^{p-2}\left|\alpha_j^{\mathcal{E}}\right|^2.$$

The bubble basis functions are chosen orthogonal. Thus we can write

$$\|u_{\mathcal{I}}\|^2_{L^2(\widehat{K})} = \sum_{i+j\leq p-3}\left|\alpha^{\mathcal{I}}_{(ij)}\right|^2 \|\varphi_{(ij)}\|^2_{L^2(\widehat{K})} = \sum_{i+j\leq p-3}\left|\alpha^{\mathcal{I}}_{(ij)}\right|^2$$

by our choice of scaling of the bubble functions.

Finally, we split the function u into vertex, edge and inner components and apply the triangle inequality, which gives

$$\|u\|^2_{L^2(\widehat{K})} \leq 5\,\|u_{\mathcal{V}}\|^2_{L^2(\widehat{K})} + 5\sum_{m=1}^{3}\left\|u_{\mathcal{E}_j}\right\|^2_{L^2(\widehat{K})} + 5\,\|u_{\mathcal{I}}\|^2_{L^2(\widehat{K})}$$

and thus (4.9). □

More interesting are the inverse estimates.

Lemma 4.3. *Let again $u \in P^p(\widehat{K})$. Then, there holds*

$$\sum_{j=1}^{3}\left|\alpha_j^{\mathcal{V}}\right|^2 \leq C_1 p^4\,\|u\|^2_{L^2(\widehat{K})} \qquad \textit{as well as} \qquad \sum_{j=1}^{3}\left|\alpha_j^{\mathcal{V}}\right|^2 \leq C_1\log(p+1)\,\|u\|^2_{H^1(\widehat{K})}.$$

For the edge parts we have

$$\sum_{j=0}^{p-2}\left|\alpha_j^{\mathcal{E}_m}\right|^2 \leq C_2 p^6\,\|u\|^2_{L^2(\widehat{K})} \qquad \textit{and} \qquad \sum_{j=0}^{p-2}\left|\alpha_j^{\mathcal{E}_m}\right|^2 \leq C_2 p^2\,\|u\|^2_{H^1(\widehat{K})}.$$

Moreover, if u vanishes on ∂T, then

$$\sum_{i+j\leq p-3}\left|\alpha^{\mathcal{I}}_{(ij)}\right|^2 = \|u\|^2_{L^2(\widehat{K})}. \tag{4.10}$$

Proof. Since $\alpha_j^{\mathcal{V}} = u(\boldsymbol{z}_j)$ where $\boldsymbol{z}_j$ denotes the $j-th$ vertex, we can use the L^∞ inverse estimates (4.1) and (4.2) to get estimates for the vertex part.

For the edge parts, we again only consider the bottom edge, $\mathcal{E} = \mathcal{E}_m$ with $m = 1$. First we assume that u vanishes in all vertices. If we consider the restriction of u to the edge $\mathcal{E}$ we only have contributions by the edge basis, i.e., we can write

$$u\,(x,0) = \sum_{i=0}^{p-2}\alpha_i^{\mathcal{E}} L_{i+2}(2x-1), \quad x \in (0,1),$$
$$\frac{\partial}{\partial x}u(x,0) = \sum_{i=0}^{p-2}\alpha_i^{\mathcal{E}}\, 2\,\sqrt{\frac{2i+3}{2}}\,\ell_{i+1}(2x-1), \quad x \in (0,1).$$

The factor was chosen to get an L^2-normed basis, since we have $\|\ell_{i+1}\|^2_{L^2(-1,1)} = \frac{2}{2i+3}$. The Legendre polynomials are orthogonal on $(-1,1)$, and therefore simple calculations show

$$\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(\mathcal{E})} = 2\sum_{i=0}^{p-2} \left|\alpha_i^{\mathcal{E}}\right|^2.$$

If we consider a general $u \in P^p(\widehat{K})$, we apply the previous estimate to $u_2 := u - I^1 u$ where I^1 denotes the nodal interpolation operator to the linears. Then we get from the triangle inequality

$$\sum_{i=0}^{p-2} \left|\alpha_i^{\mathcal{E}}\right|^2 \le 2\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(\mathcal{E})} + 2\left\|\frac{\partial I^1 u}{\partial x}\right\|^2_{L^2(\mathcal{E})}.$$

We apply the trace estimate (4.4) to the first part, and the trace and norm equivalence for the second and obtain:

$$\sum_{i=0}^{p-2} \left|\alpha_i^{\mathcal{E}}\right|^2 \lesssim p^2 \left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(\widehat{K})} + \left\|I^1 u\right\|^2_{L^2(\widehat{K})}.$$

The H^1 estimate then follows from the L^∞ estimate for the nodal interpolant (4.2). For the L^2 estimate we then simply use the inverse estimate (4.3).

For the proof of (4.10) we just note that if u vanishes on ∂T we have $u = u_{\mathcal{I}}$ and thus we can use the equality (4.8). $\square$

Lemma 4.4. *There exist generic constants c_0, C_0, c_1, $C_1 > 0$ independent of p such that, for all $u \in P^p(\hat{K})$, the following estimates hold:*

$$c_0 \|u\|^2_{L^2(\widehat{K})} \le \sum_{j=1}^{3} \left|\alpha_j^{\mathcal{V}}\right|^2 + \sum_{m=1}^{3}\sum_{j=0}^{p-2} \left|\alpha_j^{\mathcal{E}_m}\right|^2 + \sum_{i,j\le p-3} \left|\alpha_{(ij)}^{\mathcal{I}}\right|^2 \le C_0 p^6 \|u\|^2_{L^2(\widehat{K})}, \tag{4.11}$$

$$c_1 p^{-4} \|u\|^2_{H^1(\widehat{K})} \le \sum_{j=1}^{3} \left|\alpha_j^{\mathcal{V}}\right|^2 + \sum_{m=1}^{3}\sum_{j=0}^{p-2} \left|\alpha_j^{\mathcal{E}_m}\right|^2 + \sum_{i,j\le p-3} \left|\alpha_{(ij)}^{\mathcal{I}}\right|^2 \le C_1 p^2 \|u\|^2_{H^1(\widehat{K})}. \tag{4.12}$$

Proof of (4.11)*:* The lower bound was already shown in Lemma 4.2. For the upper bound we apply the preceding Lemma 4.3 to u and see $\sum_{j=1}^3 \left|\alpha_j^{\mathcal{V}}\right|^2 \le C_1 p^4 \|u\|^2_{L^2(\widehat{K})}$. For edge $\mathcal{E}_m$ we get

$$\sum_{j=0}^{p-2} \left|\alpha_j^{\mathcal{E}_m}\right|^2 \le C p^6 \|u\|^2_{L^2(\widehat{K})}.$$

Next we set $u_2 := u - u_V - u_{\mathcal{E}}$, where $u_{\mathcal{E}}$ is the sum of the edge contributions $u_{\mathcal{E}_m}$. This function vanishes on the boundary and we can apply Lemma 4.3 to get:

$$\sum_{i+j\le p-3} \left|\alpha_{(ij)}^{\mathcal{I}}\right|^2 \le C_3 \|u_2\|^2_{L^2(\widehat{K})}.$$

Since we can always estimate the L^2 norms by the ℓ^2 norms of the coefficients (Lemma 4.2), we obtain

$$\begin{aligned}\|u_2\|^2_{L^2(\widehat{K})} &\le 3\|u\|^2_{L^2(\widehat{K})} + 3\|u_V\|^2_{L^2(\widehat{K})} + 3\|u_{\mathcal{E}}\|^2_{L^2(\widehat{K})} \\ &\le C\left(\|u\|^2_{L^2(\widehat{K})} + p^6 \|u\|^2_{L^2(\widehat{K})} + p^4 \|u\|^2_{L^2(\widehat{K})}\right).\end{aligned}$$

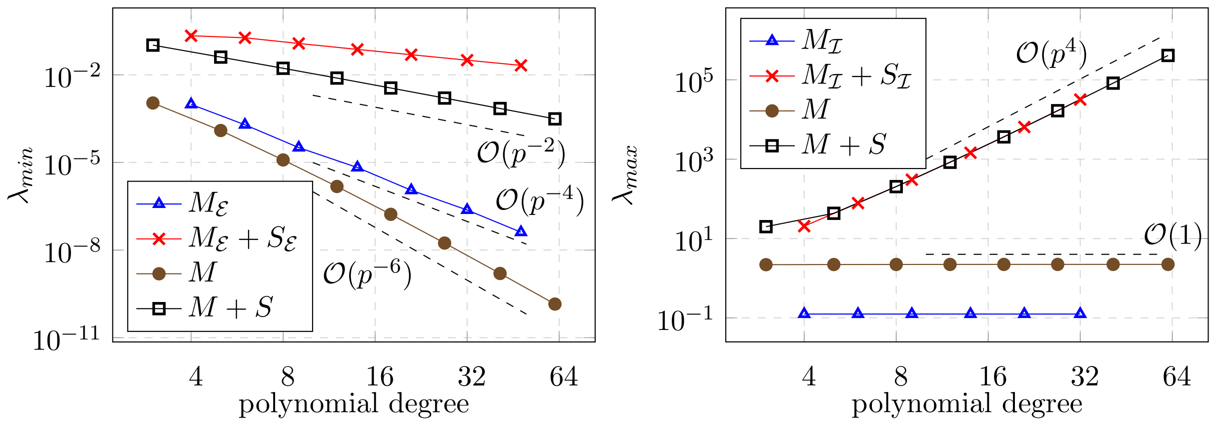

Figure 4.1: Numerical computation of the extremal eigenvalues of the mass matrices and the sum of mass and H^1-stiffness matrix $M + S$ for the full system and different subblocks.

Proof of (4.12)*:* The proof for the upper H^1 estimate works along the same lines, but using the sharper H^1-estimates from Lemma 4.3 for the vertex and edge parts. For the lower estimate, we just make use of the inverse estimate (4.3) and the fact that the L^2 norm is uniformly bounded by the coefficients, to conclude the proof. $\square$

Example 4.5. In Figure 4.1 we compare our theoretical bounds on the reference element from Theorem 4.4 with a numerical experiment that studies the maximal and minimal eigenvalues of the mass matrix M and the stiffness matrix S (corresponding to the bilinear form $(\nabla\cdot, \nabla\cdot)_{L^2(\widehat{K})}$). We focus on the full system and the subblocks that contributed the highest order in our theoretical investigations, i.e. the edge blocks $M_\mathcal{E}$, $S_\mathcal{E}$, and the block of inner basis functions $M_\mathcal{I}$, $S_\mathcal{I}$. We see that the estimates on the full condition numbers are not overly pessimistic: the numerics show a behavior of the minimal eigenvalue of $\mathcal{O}(p^{5.5})$ instead of $\mathcal{O}(p^6)$. If we focus solely on the edge contributions, we see that the bound we used for the lower eigenvalue is not sharp there. This can partly be explained by the fact that if no inner basis functions are present it is possible to improve the estimate (4.4) by a factor of p. But since we also need to encompass coupling of inner and edge basis functions this improvement in order is lost again when looking at the full systems.

The estimates on the reference triangle can now be transferred to the global space $\widetilde{S}^p(\mathcal{T})$ on quasiuniform meshes.

Theorem 4.6. *Let $\mathcal{T}$ be a quasiuniform triangulation with mesh size h. With the polynomial basis on the reference triangle $\widehat{K}$ given by Definition 2.4 let $\{\varphi_i \,|\, i = 1, \dots, N\}$ be the basis of $\widetilde{S}^p(\mathcal{T})$.*

Then there exist constants c_0, $c_{1/2}$, c_1, C_0, $C_{1/2}$, $C_1 > 0$ that only depend on Γ and the shape regularity constant γ such that, for every $\mathfrak{u} \in \mathbb{R}^N$ and $u = \sum_{j=0}^N \mathfrak{u}_j \varphi_j \in \widetilde{S}^p(\mathcal{T})$:

$$c_0 \frac{1}{h^2} \|u\|^2_{L^2(\Gamma)} \leq \|\mathfrak{u}\|^2_{\ell^2} \leq C_0 \frac{p^6}{h^2} \|u\|^2_{L^2(\Gamma)} \tag{4.13}$$

$$c_1 p^{-4} \|u\|^2_{H^1(\Gamma)} \leq \|\mathfrak{u}\|^2_{\ell^2} \leq C_1 \left(p^2 + h^{-2}\right) \|u\|^2_{H^1(\Gamma)} \tag{4.14}$$

$$c_{1/2} h^{-1} p^{-2} \|u\|^2_{\widetilde{H}^{1/2}(\Gamma)} \leq \|\mathfrak{u}\|^2_{\ell^2} \leq C_{1/2} \left(\frac{p^4}{h} + h^{-2}\right) \|u\|^2_{\widetilde{H}^{1/2}(\Gamma)}. \tag{4.15}$$

Proof. The L^2 estimate (4.13). can be shown easily by transforming to the reference element and applying (4.11).

To prove the other estimates (4.14), (4.15), we need the Scott-Zhang interpolation operator $J_h : L^2(\Gamma) \to \widetilde{S}^1(\mathcal{T})$ as modified in [AFF+14]. It has the following important properties:

1. J_h is a bounded linear operator from $L^2(\Gamma)$ to $\widetilde{S}^1(\mathcal{T})$.
2. For every $s \in [0,1]$ there holds $\|J_h v\|_{\widetilde{H}^s(\Gamma)} \leq C_{stab}(s)\, \|v\|_{\widetilde{H}^s(\Gamma)}\ \forall v \in \widetilde{H}^s(\Gamma)$.
3. For every $K \in \mathcal{T}$ let $\omega_K := \bigcup \{K' \in \mathcal{T} : K \cap K' \neq \emptyset\}$ denote the element patch, i.e., the union of all elements that touch K. Then, for for all $v \in \widetilde{H}^1(\Gamma)$

$$\|(1 - J_h)\, v\|_{L^2(K)} \leq C_{sz} h_K \, \|\nabla v\|_{L^2(\omega_K)}\,, \tag{4.16}$$

$$\|\nabla\, (1 - J_h)\, v\|_{L^2(K)} \leq C_{sz}\, \|\nabla v\|_{L^2(\omega_K)}\,. \tag{4.17}$$

The constant C_{sz} depends only on the γ-shape regularity of $\mathcal{T}$, and $C_{stab}(s)$ additionally depends on Γ and s.

We will use the following notation: For a function $u \in \widetilde{S}^p(\mathcal{T})$ we will write $\mathfrak{u} \in \mathbb{R}^N$ for its representation in the basis $\{\varphi_i \,|\, i = 1, \ldots, N\}$. For an element $K \in \mathcal{T}$ we write $\mathfrak{u}|_K$ for the part of the coefficient vector that belongs to basis functions whose support intersects the interior of K. In addition to the function $u \in \widetilde{S}^p(\mathcal{T})$, we will employ the function $\tilde{u} := u - J_h u$. Its vector representation will be denoted $\tilde{\mathfrak{u}} \in \mathbb{R}^N$. Finally, the vector representation of $J_h u$ (again $u \in \widetilde{S}^p(\mathcal{T})$) will be $\mathfrak{J}_h \mathfrak{u} \in \mathbb{R}^N$.

1. step: We claim the following stability estimates:

$$\|\mathfrak{J}_h \mathfrak{u}\|_{\ell^2}^2 \lesssim h^{-2}\, \|J_h u\|_{L^2(\Gamma)}^2 \lesssim h^{-2}\, \|u\|_{L^2(\Gamma)}^2 \lesssim \|\mathfrak{u}\|_{\ell^2}^2\,, \tag{4.18}$$

$$\|J_h u\|_{H^{1/2}(\Gamma)}^2 \lesssim\ h^{-1}\, \|J_h u\|_{L^2}^2\,, \tag{4.19}$$

$$\|J_h u\|_{H^{1/2}(\Gamma)}^2 \lesssim h\, \|\mathfrak{u}\|_{\ell^2}^2\,. \tag{4.20}$$

The inequalities (4.18) are just a simple scaling argument combined with the L^2 stability of the Scott-Zhang projection. The inequality (4.19) follows from the inverse inequality (note that $J_h u$ has degree 1). Finally, (4.20) follows from combining (4.19) and (4.18).

2. step: Next, we investigate the function $\tilde{u} = u - J_h u$. We claim the following estimates:

$$\|\tilde{u}\|_{L^2(\Gamma)}^2 \lesssim h^2\, \|\mathfrak{u}\|_{\ell^2}^2\,, \tag{4.21}$$

$$\|\tilde{u}\|_{H^1(\Gamma)}^2 \lesssim p^4\, \|\mathfrak{u}\|_{\ell^2}^2\,, \tag{4.22}$$

$$\|\tilde{u}\|_{\widetilde{H}^{1/2}(\Gamma)}^2 \lesssim p^2\, h\, \|\mathfrak{u}\|_{\ell^2}^2\,. \tag{4.23}$$

The estimate (4.21) is a simple consequence of (4.13) and the L^2-stability of the Scott-Zhang operator J_h. For the proof of (4.22) we combine a simple scaling argument, together with (4.12) and the stability estimate (4.18) to get

$$\|\tilde{u}\|_{H^1(\Gamma)}^2 \lesssim p^4\, \|\tilde{\mathfrak{u}}\|_{\ell^2}^2 \lesssim p^4\, \|\mathfrak{u}\|_{\ell^2}^2\,.$$

The bound (4.23) follows from the interpolation estimate of Proposition 2.1, (ii) and the estimates (4.21) and (4.22).

3. step: We assert:

$$\|\tilde{\mathfrak{u}}\|_{\ell^2}^2 \lesssim \frac{p^6}{h^2}\,\|u\|_{L^2(\Gamma)}^2\,, \tag{4.24}$$

$$\|\tilde{\mathfrak{u}}\|_{\ell^2}^2 \lesssim p^2\,\|u\|_{H^1(\Gamma)}^2\,, \tag{4.25}$$

$$\|\tilde{\mathfrak{u}}\|_{\ell^2}^2 \lesssim \frac{p^4}{h}\,\|u\|_{\widetilde{H}^{1/2}(\Gamma)}^2\,. \tag{4.26}$$

Again, (4.24) is a simple consequence of (4.13) and the L^2-stability of the Scott-Zhang operator J_h. For the bound (4.25) we calculate

$$\|\tilde{\mathfrak{u}}\|_{\ell^2}^2 \leq \sum_{K\in\mathcal{T}} \|\tilde{\mathfrak{u}}|_K\|_{\ell^2}^2 \lesssim p^2 \sum_{K\in\mathcal{T}} h^{-2}\,\|\tilde{u}\|_{L^2(K)}^2 + |\tilde{u}|_{H^1(K)}^2$$

By applying the local L^2 interpolation estimate (4.16) and H^1 stability (4.17) we get

$$\|\tilde{\mathfrak{u}}\|_{\ell^2}^2 \lesssim p^2 \sum_{K\in\mathcal{T}} \|u\|_{H^1(\omega_K)}^2 \lesssim p^2\,\|u\|_{H^1(\Gamma)}^2\,,$$

where in the last step we used the fact for shape-regular meshes each element is contained in at most M different patches, where M depends solely on the shape regularity.

We next prove (4.26). We apply Proposition 2.1, (i) to the map $\ell^2 \to \widetilde{S}^p(\mathcal{T}) :\ \mathfrak{u} \mapsto u$, where the space $\widetilde{S}^p(\mathcal{T})$ is once equipped with the L^2- and once with the H^1-norm. By Proposition 3.4 interpolating between (4.21) and (4.22) yields $\|\tilde{\mathfrak{u}}\|_{\ell^2}^2 \lesssim \left(p^6 h^{-2} p^2\right)^{1/2} \|u\|_{\widetilde{H}^{1/2}(\Gamma)}^2 \lesssim p^4 h^{-1}\,\|u\|_{\widetilde{H}^{1/2}(\Gamma)}^2$.

4. step: The above steps allow us to obtain the H^1 and $H^{1/2}$ estimates (4.14), (4.15) of the theorem. We decopose $u = \tilde{u} + J_h u$ and correspondingly $\mathfrak{u} = \tilde{\mathfrak{u}} + \mathfrak{J}_h\mathfrak{u}$. Then:

$$\|\mathfrak{u}\|_{\ell^2}^2 \lesssim \|\tilde{\mathfrak{u}}\|_{\ell^2}^2 + \|\mathfrak{J}_h\mathfrak{u}\|_{\ell^2}^2 \overset{(4.26),(4.18)}{\lesssim} \frac{p^4}{h}\,\|u\|_{\widetilde{H}^{1/2}(\Gamma)}^2 + \frac{1}{h^2}\,\|u\|_{L^2(\Gamma)}^2 \lesssim \frac{p^4\,h+1}{h^2}\,\|u\|_{\widetilde{H}^{1/2}(\Gamma)}^2$$

$$\|u\|_{\widetilde{H}^{1/2}(\Gamma)}^2 \lesssim \|\tilde{u}\|_{\widetilde{H}^{1/2}(\Gamma)}^2 + \|J_h u\|_{\widetilde{H}^{1/2}(\Gamma)}^2 \overset{(4.23),(4.20)}{\lesssim} h\,p^2\,\|\mathfrak{u}\|_{\ell^2}^2 + h\,\|\mathfrak{u}\|_{\ell^2}^2 \lesssim h\,p^2\,\|\mathfrak{u}\|_{\ell^2}^2$$

This shows (4.15). The H^1 estimate (4.14) follows along the same lines: An elementwise inverse estimate gives

$$\|u\|_{H^1(\Gamma)}^2 \lesssim \frac{p^4}{h^2}\,\|u\|_{L^2(\Gamma)}^2 \lesssim p^4\|\mathfrak{u}\|_{\ell^2}^2,$$

and the splitting $\mathfrak{u} = \tilde{\mathfrak{u}} + \mathfrak{J}_h\mathfrak{u}$ produces

$$\|\mathfrak{u}\|_{\ell^2}^2 \lesssim \|\tilde{\mathfrak{u}}\|_{\ell^2}^2 + \|\mathfrak{J}_h\mathfrak{u}\|_{\ell^2}^2 \overset{(4.25),(4.18)}{\lesssim} \left(p^2 + h^{-2}\right)\|u\|_{H^1(\Gamma)}^2\,.$$

□

Corollary 4.7. *The condition number of the unpreconditioned Galerkin matrix $\widetilde{D}_h^p$ can be bounded by*

$$\kappa\left(\widetilde{D}_h^p\right) \leq C\left(\frac{p^2}{h} + p^6\right)$$

with a constant $C > 0$ that depends only on Γ and the shape regularity constant γ.

Proof. The bilinear form induced by the stabilized hypersingular operator is elliptic and continuous with respect to the $H^{1/2}$ norm. By applying the estimates (4.15) to the Rayleigh quotients we get the stated result. $\square$

Remark 4.8. In this section we did not consider the effect of diagonal scaling. The numerical results in Section 6 suggest that it improves the p-dependence of the condition number significantly.

5 hp-preconditioning

5.1 Abstract additive Schwarz methods

Additive Schwarz preconditioners are based on decompositions of a vector space $\mathbb{V}$ into smaller subspaces $\mathbb{V}_i$, $i = 0, \ldots, J$, on which a local problem is solved. We recall some of the basic definitions and important results. Details can be found in [TW05, chapter 2].

Let $a(\cdot, \cdot) : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ be a symmetric, positive definite bilinear form on the finite dimensional vector space $\mathbb{V}$. For a given $f \in \mathbb{V}'$ consider the problem of finding $u \in \mathbb{V}$ such that

$$a(u, v) = f(v) \quad \forall v \in \mathbb{V}.$$

We will write A for the corresponding Galerkin matrix.

Let $\mathbb{V}_i \subset \mathbb{V}$, $i = 0, \ldots, J$ be finite dimensional vector spaces with corresponding prolongation operators $R_i^T : \mathbb{V}_i \to \mathbb{V}$. We will commit a slight abuse of notation and also denote the matrix representation of the operator as R_i^T and R_i as its transposed matrix. We also assume that $\mathbb{V}$ permits a (in general not direct) decomposition into

$$\mathbb{V} = R_0^T \mathbb{V}_0 + \sum_{i=1}^{J} R_i^T \mathbb{V}_i.$$

We assume that for each subspace $\mathbb{V}_i$ a symmetric, positive definite bilinear form

$$\widetilde{a}_i(\cdot, \cdot) : \mathbb{V}_i \times \mathbb{V}_i \to \mathbb{R}, \quad i = 0, \ldots, J,$$

is given. We write $\widetilde{A}_i$ for the matrix representation of $\widetilde{a}_i(\cdot, \cdot)$. Sometimes these bilinear forms are referred to as the "local solvers," in the simplest case of "exact local solvers" they are just restrictions of $a(\cdot, \cdot)$, i.e., $\widetilde{a}_i(u_i, v_i) := a\left(R_i^T u_i, R_i^T v_i\right)$ for all $u_i, v_i \in \mathbb{V}_i$.

Then, the corresponding additive Schwarz preconditioner is given by

$$B^{-1} := \sum_{i=0}^{J} R_i^T \widetilde{A}_i^{-1} R_i.$$

The following proposition allows to bound the condition number of the preconditioned system $B^{-1}A$. The first part is often referred to as the Lemma of Lions (see [Zha92, Lio88, MN85]).

Proposition 5.1. *(a) Assume that there exists a constant $C_0 > 0$ such that every $u \in \mathbb{V}$ admits a decomposition $u = \sum_{i=0}^{J} R_i^T u_i$ with $u_i \in \mathbb{V}_i$ such that*

$$\sum_{i=0}^{J} \widetilde{a}_i(u_i, u_i) \leq C_0 \, a(u, u).$$

Then, the minimal eigenvalue of $B^{-1}A$ can be bounded by $\lambda_{min}\left(B^{-1}A\right) \geq C_0^{-1}$.

(b) Assume that there exists $C_1 > 0$ such that for every decomposition $u = \sum_{i=0}^{J} R_i^T v_i$ with $v_i \in \mathbb{V}_i$ the following estimate holds:

$$a(u,u) \leq C_1 \sum_{i=0}^{J} \widetilde{a}_i(v_i, v_i).$$

Then, the maximal eigenvalue of $B^{-1}A$ can be bounded by $\lambda_{max}\left(B^{-1}A\right) \leq C_1$.

These two estimates together give an estimate for the condition number of the preconditioned linear system:

$$\kappa\left(B^{-1}A\right) := \frac{\lambda_{max}}{\lambda_{min}} \leq C_0 C_1. \qquad \square$$

5.2 An hp-stable preconditioner

In order to define an additive Schwarz preconditioner, we decompose the boundary element space $\mathbb{V} := \widetilde{S}^p(\mathcal{T})$ into several overlapping subspaces. We define $\mathbb{V}_h^1 := \widetilde{S}^1(\mathcal{T})$ as the space of globally continuous and piecewise linear functions on $\mathcal{T}$ that vanish on $\partial\Gamma$ and denote the corresponding canonical embedding operator by $R_h^T : \mathbb{V}_h^1 \to \mathbb{V}$. We also define for each vertex $\boldsymbol{z} \in \mathcal{V}$ the local space

$$\mathbb{V}_{\boldsymbol{z}}^p := \{u \in \widetilde{S}^p(\mathcal{T}) |\, \operatorname{supp}(u) \subset \overline{\omega_{\boldsymbol{z}}}\}$$

and denote the canonical embedding operators by $R_{\boldsymbol{z}}^T : \mathbb{V}_{\boldsymbol{z}}^p \to \mathbb{V}$.

The space decomposition then reads

$$\mathbb{V} = \mathbb{V}_h^1 + \sum_{\boldsymbol{z}\in\mathcal{V}} \mathbb{V}_{\boldsymbol{z}}^p. \tag{5.1}$$

We will denote the restriction of the Galerkin matrix $\widetilde{D}_h^p$ to the subspaces $\mathbb{V}_h^1$ and $\mathbb{V}_{\boldsymbol{z}}^p$ as $\widetilde{D}_h^1$ and $\widetilde{D}_{h,\boldsymbol{z}}^p$, respectively.

Lemma 5.2. *There exist constants $c_1, c_2 > 0$, which depend only on Γ and the γ-shape regularity of $\mathcal{T}$, such that the following holds:*

- *For every $u \in \widetilde{S}^p(\mathcal{T})$ there exists a decomposition $u = u_1 + \sum_{\boldsymbol{z}\in\mathcal{V}} u_{\boldsymbol{z}}$ with $u_1 \in \mathbb{V}_h^1$ and $u_{\boldsymbol{z}} \in \mathbb{V}_{\boldsymbol{z}}^p$ and*

$$\|u_1\|_{\widetilde{H}^{1/2}(\Gamma)}^2 + \sum_{\boldsymbol{z}\in\mathcal{V}} \|u_{\boldsymbol{z}}\|_{\widetilde{H}^{1/2}(\Gamma)}^2 \leq c_1 \; \|u\|_{\widetilde{H}^{1/2}(\Gamma)}^2 .$$

- *Any decomposition $u = v_1 + \sum_{\boldsymbol{z}\in\mathcal{V}} v_{\boldsymbol{z}}$ with $v_1 \in \mathbb{V}_h^1$ and $v_{\boldsymbol{z}} \in \mathbb{V}_{\boldsymbol{z}}^p$ satisfies*

$$\|u\|_{\widetilde{H}^{1/2}(\Gamma)}^2 \leq c_2 \left(\|v_1\|_{\widetilde{H}^{1/2}(\Gamma)}^2 + \sum_{\boldsymbol{z}\in\mathcal{V}} \|v_{\boldsymbol{z}}\|_{\widetilde{H}^{1/2}(\Gamma)}^2 \right).$$

Proof. The first estimate is the assertion of Proposition 3.1. The second estimate can be shown by a so-called coloring argument, along the same lines as in [Heu99, Lemma 2]. It is based on the following estimate (see [SS11, Lemma 4.1.49] or [vP89, Lemma 3.2]):

Let w_j, $j = 1, \ldots, n$ be functions in $\widetilde{H}^s(\Gamma)$ for $s \geq 0$ with pairwise disjoint support. Then it holds

$$\left\| \sum_{i=1}^{n} w_i \right\|^2_{\widetilde{H}^s(\Gamma)} \leq C \sum_{i=1}^{n} \|w_i\|^2_{\widetilde{H}^s(\Gamma)},$$

where $C > 0$ depends only on Γ. By γ-shape regularity, the number of elements in any vertex patch, and therefore also the number of vertices in a patch is uniformly bounded by some constant N_c which is uniformly bounded in terms of γ. Thus, we can sort the vertices into sets $J_1, \ldots, J_{N_c}$, such that $\bigcup_{i=1}^{N_c} J_i = \mathcal{V}$ and $|\omega_{\boldsymbol{z}} \cap \omega_{\boldsymbol{z'}}| = 0$ for all $\boldsymbol{z}, \boldsymbol{z'}$ in the same index set J_i. Repeated application of the triangle inequality together with the previous inequality then gives:

$$\begin{aligned}
\|u\|^2_{\widetilde{H}^{1/2}(\Gamma)} &\leq 2\,\|v_1\|^2_{\widetilde{H}^{1/2}(\Gamma)} + 2 \left\| \sum_{\boldsymbol{z} \in \mathcal{V}} v_{\boldsymbol{z}} \right\|^2_{\widetilde{H}^{1/2}(\Gamma)} \\
&\leq 2\,\|v_1\|^2_{\widetilde{H}^{1/2}(\Gamma)} + 2N_c \sum_{i=1}^{N_c} \left\| \sum_{\boldsymbol{z} \in J_i} v_{\boldsymbol{z}} \right\|^2_{\widetilde{H}^{1/2}(\Gamma)} \\
&\leq 2\,\|v_1\|^2_{\widetilde{H}^{1/2}(\Gamma)} + 2\,N_c\,C \sum_{\boldsymbol{z} \in \mathcal{V}} \|v_{\boldsymbol{z}}\|^2_{\widetilde{H}^{1/2}(\Gamma)}.
\end{aligned}$$

This proves the second inequality. □

The previous lemma only made statements about the $\widetilde{H}^{1/2}(\Gamma)$-norm.

Theorem 5.3. *Let $\mathcal{T}$ be a γ-shape regular triangulation of Γ.*
Based on the space decomposition (5.1) define the preconditioner as

$$B^{-1} := R_h^T \left(\widetilde{D}_h^1\right)^{-1} R_h \; + \; \sum_{\boldsymbol{z} \in \mathcal{V}} R_{\boldsymbol{z}}^T \left(\widetilde{D}_{h,\boldsymbol{z}}^p\right)^{-1} R_{\boldsymbol{z}}.$$

Then, the preconditioned system has a condition number that is bounded by a constant that depends only on Γ and the γ-shape regularity of $\mathcal{T}$:

$$\kappa(B^{-1}\widetilde{D}_h^p) \leq C.$$

Proof. The bilinear form $\left\langle \widetilde{D}\cdot, \cdot \right\rangle_\Gamma$ is equivalent to $\|\cdot\|^2_{\widetilde{H}^{1/2}(\Gamma)}$. Hence, the combination of Lemma 5.2 and Proposition 5.1 give the boundedness of the condition number. □

5.3 Adaptive meshes

The preconditioner of Theorem 5.3 relies on the space decomposition (5.1). In this section, we discuss how the space $\widetilde{S}^1(\mathcal{T})$ of piecewise linear function can be further decomposed in a multilevel fashion. Our setting will be one where $\mathcal{T}$ is the finest mesh of a sequence $(\mathcal{T}_\ell)_{\ell=0}^L$ of nested meshes that are generated by Newest Vertex Bisection. This situation is encountered, for example, in adaptive environments.

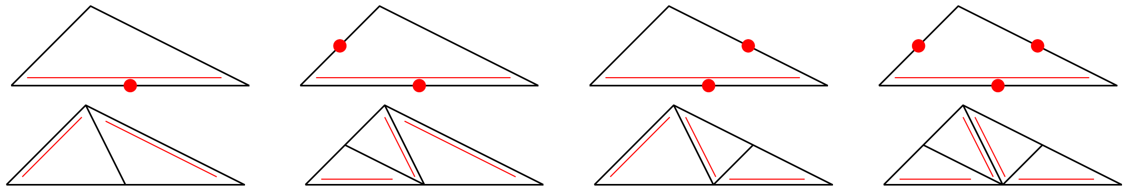

Figure 5.1: For each element $K \in \mathcal{T}_\ell$ there exists exactly one *reference edge* indicated by the red line (upper left plot). The element K is refined by bisecting its reference edge. This leads to a new node (red dot) and two son elements. The reference edges of the son elements are opposite to the *newest vertex* (lower left plot). Hanging nodes are avoided as follows: Assume that some of the edges of the triangle, but at least the reference edge, are marked for refinement (upper plots). The triangle will be split into two, three or four son elements by iterative application of the *newest vertex bisection (NVB)*.

5.3.1 Mesh refinement

In this section we consider the mesh refinement that is often used in adaptive algorithms: Starting with an *initial* triangulation $\mathcal{T}_0$, the mesh $\mathcal{T}_\ell$ is obtained from $\mathcal{T}_{\ell-1}$ by refining at least the marked elements $\mathcal{M}_{\ell-1} \subseteq \mathcal{T}_{\ell-1}$. Usually, the following assumptions on the mesh refinement are made:

- $\mathcal{T}_\ell$ is regular for all $\ell \in \mathbb{N}_0$, i.e., there exist no hanging nodes;
- The meshes $\mathcal{T}_0, \mathcal{T}_1, \ldots$ are γ-shape-regular, i.e.,

$$\sup_{\ell \in \mathbb{N}_0} \max_{K \in \mathcal{T}_\ell} \frac{\operatorname{diam}(K)^2}{|K|} \leq \gamma. \tag{5.2}$$

Here, $|K|$ denotes the surface area of an element $K \in \mathcal{T}_\ell$ and $\operatorname{diam}(K)$ is the Euclidean diameter.

We will restrict ourselves to the *newest vertex bisection (NVB)*, see Figure 5.1 for a description. NVB has many good properties, in particular it preserves shape regularity, i.e., ensures (5.2) uniformly in ℓ. Furthermore, it is often employed in h-adaptivity and the proofs of optimality of adaptive FEM [CKNS08, FFP14] and BEM [FFK$^+$14, FFK$^+$13, Gan13] strongly rely on the use of NVB. Further details on this mesh refinement can be found in [Ste08b, KPP13]. In particular, we note that the reference edges in the initial triangulation $\mathcal{T}_0$ can be chosen arbitrarily, see [KPP13].

We call a triangulation $\widehat{\mathcal{T}}$ the uniform refinement of $\mathcal{T}$, if all elements $K \in \mathcal{T}$ are refined, i.e., each element $K \in \mathcal{T}$ is split into four son elements $K_1, \ldots, K_4 \in \widehat{\mathcal{T}}$ with $|K_i| = |K|/4$ (see the last column of Figure 5.1).

We stress that the preconditioner described in Section 5.2 is independent of the chosen refinement as long as it satisfies the assumptions above, whereas the results for the local multilevel preconditioner presented in Section 5.3 depend on the NVB refinement.

For the remainder of the work, we consider a sequence of triangulations $\mathcal{T}_0, \ldots, \mathcal{T}_L$ with corresponding sets of vertices $\mathcal{V}_0, \ldots, \mathcal{V}_L$. For a vertex $\boldsymbol{z} \in \mathcal{V}_\ell$, the associated patch is denoted by $\omega_{\ell,\boldsymbol{z}}$.

In the construction of the p-preconditioner in Section 5.2 we only considered a single mesh $\mathcal{T}$. For the remainder of the paper, the p part will always constructed with respect to the finest mesh $\mathcal{T}_L$. For a simpler presentation we set $\mathcal{T} := \mathcal{T}_L$ and $\mathcal{V} := \mathcal{V}_L$.

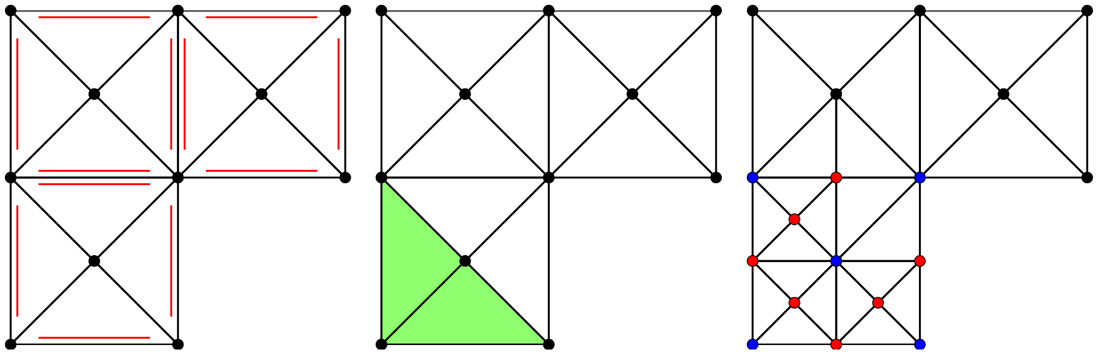

Figure 5.2: Visualization of the definition (5.3) of the local subsets $\widetilde{\mathcal{V}}_\ell$: Starting with an triangulation $\mathcal{T}_{\ell-1}$ (left), we mark two elements, indicated by green triangles (middle), for refinement. Using iterated NVB refinement, see Section 5.3.1, we obtain the mesh $\mathcal{T}_\ell$ (right). The set $\widetilde{\mathcal{V}}_\ell$ consists of the new vertices (red) and neighboring vertices, where the corresponding vertex patches have changed (blue).

5.3.2 A refined splitting for adaptive meshes

The space decomposition from (5.1) involves the global lowest-order space $\mathbb{V}_h^1 = \widetilde{S}^1(\mathcal{T}_L)$. Therefore, the computation of the corresponding additive Schwarz operator needs the inversion of a global problem, which in practice becomes very costly, and is often not feasible. To overcome this disadvantage, we consider a refined splitting of the space $\mathbb{V}_h^1$ that relies on the hierarchy of the adaptively refined meshes $\mathcal{T}_0, \ldots, \mathcal{T}_L$. The corresponding local multilevel preconditioner was introduced and analyzed in [FFPS13, Füh14]. See also [HWZ12, WC06, XCH10, XCN09] for local multilevel preconditioners for (adaptive) FEM and [TS96, HM12] for (uniform) BEM.

Set $\widetilde{\mathcal{V}}_0 := \mathcal{V}_0$ and define the local subsets

$$\widetilde{\mathcal{V}}_\ell := \mathcal{V}_\ell \backslash \mathcal{V}_{\ell-1} \cup \{\boldsymbol{z} \in \mathcal{V}_{\ell-1} \,:\, \omega_{\ell,\boldsymbol{z}} \subsetneq \omega_{\ell-1,\boldsymbol{z}}\} \quad \text{for } \ell \geq 1 \tag{5.3}$$

of newly created vertices plus some of their neighbors, see Figure 5.2 for a visualization. Based on these sets, we consider the space decomposition

$$\mathbb{V}_h^1 = \sum_{\ell=0}^{L} \sum_{\boldsymbol{z} \in \widetilde{\mathcal{V}}_\ell} \mathbb{V}_{\ell,\boldsymbol{z}}^1 \quad \text{with} \quad \mathbb{V}_{\ell,\boldsymbol{z}}^1 := \operatorname{span}\{\varphi_{\ell,\boldsymbol{z}}\}, \tag{5.4}$$

where $\varphi_{\ell,\boldsymbol{z}} \in \widetilde{S}^1(\mathcal{T}_\ell)$ is the nodal hat function with $\varphi_{\ell,\boldsymbol{z}}(\boldsymbol{z}) = 1$ and $\varphi_{\ell,\boldsymbol{z}}(\boldsymbol{z}') = 0$ for all $\boldsymbol{z}' \in \mathcal{V}_\ell \backslash \{\boldsymbol{z}\}$. The basic idea of this splitting is that we do a diagonal scaling only in the regions where the meshes have been refined. We will use local exact solvers, i.e.,

$$\widetilde{a}_{\ell,\boldsymbol{z}}(u_{\ell,\boldsymbol{z}}, v_{\ell,\boldsymbol{z}}) := \left\langle \widetilde{D}(R_{\ell,\boldsymbol{z}})^T u_{\ell,\boldsymbol{z}}, (R_{\ell,\boldsymbol{z}})^T v_{\ell,\boldsymbol{z}} \right\rangle_\Gamma \quad \text{for all } u_{\ell,\boldsymbol{z}}, v_{\ell,\boldsymbol{z}} \in \mathbb{V}_{\ell,\boldsymbol{z}}^1,$$

where $(R_{\ell,\boldsymbol{z}})^T : \mathbb{V}_{\ell,\boldsymbol{z}}^1 \to \mathbb{V}_h^1$ denotes the canonical embedding operator. Let $\widetilde{D}_h^1$ denote the Galerkin matrix of $\widetilde{D}$ with respect to the basis $(\varphi_{L,\boldsymbol{z}})_{\boldsymbol{z} \in \mathcal{V}_L}$ of $\mathbb{V}_h^1$ and define $\widetilde{D}_{\ell,\boldsymbol{z}}^1 := \widetilde{a}_{\ell,\boldsymbol{z}}(\varphi_{\ell,\boldsymbol{z}}, \varphi_{\ell,\boldsymbol{z}})$. Then, the *local multilevel diagonal (LMLD)* preconditioner associated to the splitting (5.4) reads

$$(B_h^1)^{-1} := \sum_{\ell=0}^{L} \sum_{\boldsymbol{z} \in \widetilde{\mathcal{V}}_\ell} (R_{\ell,\boldsymbol{z}})^T \left(\widetilde{D}_{\ell,\boldsymbol{z}}^1\right)^{-1} R_{\ell,\boldsymbol{z}}. \tag{5.5}$$

We stress that this preconditioner corresponds to a diagonal scaling with respect to the local subset of vertices $\widetilde{\mathcal{V}}_\ell$ on each level $\ell = 0, \dots, L$. Further details and the proof of the following result are found in [FFPS13, Füh14].

Proposition 5.4. *The splitting* (5.4) *together with* $\widetilde{a}_{\ell,\boldsymbol{z}}(\cdot,\cdot)$ *and the operators* $R^T_{\ell,\boldsymbol{z}}$ *satisfies the requirements of Proposition 5.1 with constants depending only on* Γ *and the initial triangulation* $\mathcal{T}_0$. *For the additive Schwarz operator* $P^1_h := (B^1_h)^{-1}\widetilde{D}^1_h$ *there holds in particular*

$$c\left\langle \widetilde{D}u_h, u_h\right\rangle_\Gamma \le \left\langle \widetilde{D}P^1_h u_h, u_h\right\rangle_\Gamma \le C\left\langle \widetilde{D}u_h, u_h\right\rangle_\Gamma \quad \textit{for all } u_h \in \mathbb{V}^1_h. \tag{5.6}$$

The constants c, $C > 0$ *depend only on* Γ, *the initial triangulation* $\mathcal{T}_0$, *and the use of NVB for refinement, i.e.,* $\mathcal{T}_{\ell+1} = \mathrm{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$ *with arbitrary* $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$. □

We replace the space $\mathbb{V}^1_h$ in (5.1) by the refined splitting (5.4) and end up with the space decomposition

$$\mathbb{V} = \sum_{\ell=0}^{L}\sum_{\boldsymbol{z}\in\widetilde{\mathcal{V}}_\ell} \mathbb{V}^1_{\ell,\boldsymbol{z}} + \sum_{\boldsymbol{z}\in\mathcal{V}_L} \mathbb{V}^p_{L,\boldsymbol{z}}. \tag{5.7}$$

The following Lemma 5.5 shows that the preconditioner resulting from the decomposition (5.7) is *hp*-stable. The result formalizes the observation that the combination of stable subspace decompositions leads again to a stable subspace decomposition. It is a simple consequence of the well-known theory for additive Schwarz methods, see Section 5.1. Therefore, details are left to the reader.

Lemma 5.5. *Let* $\mathbb{V}$ *be a finite dimensional vector space, and* $\mathbb{V}_j$, $R^T_{\mathbb{V},j}$, *and* $\widetilde{a}_{\mathbb{V},j}(\cdot,\cdot)$ *for* $j = 0, \dots, J$ *be a decomposition of* $\mathbb{V}$ *in the sense of Section 5.1 that satisfies the assumptions of Proposition 5.1 with constants* $C_{0,\mathbb{V}}$ *and* $C_{1,\mathbb{V}}$. *Consider an additional decomposition* $\mathbb{W}_l$, $R^T_{\mathbb{W},l}$ *and* $\widetilde{a}_{\mathbb{W},\ell}(\cdot,\cdot)$ *with* $\ell = 0, \dots, L$ *of* $\mathbb{V}_0$, *that also satisfies the requirements of Preposition 5.1 for the bilinear form* $\widetilde{a}_{\mathbb{V},0}(\cdot,\cdot)$ *with constants* $C_{0,\mathbb{W}}$ *and* $C_{1,\mathbb{W}}$. *Define a new additive Schwarz preconditioner as:*

$$\widetilde{B}^{-1} := R^T_{\mathbb{V},0}\left(\sum_{\ell=0}^{L} R^T_{\mathbb{W},\ell}\widetilde{A}^{-1}_{\mathbb{W},\ell}R_{\mathbb{W},\ell}\right)R_{\mathbb{V},0} \;+\; \sum_{j=1}^{J} R^T_{\mathbb{V},j}\widetilde{A}^{-1}_{\mathbb{V},j}R_{\mathbb{V},j}.$$

This new preconditioner satisfies the assumptions of Proposition 5.1 with $C_0 = \max(1, C_{0,\mathbb{W}})\, C_{0,\mathbb{V}}$ *and* $C_1 = \max(1, C_{1,\mathbb{W}})\, C_{1,\mathbb{V}}$. □

Theorem 5.6. *Assume* $\mathcal{T}$ *is generated from a regular and shape-regular initial triangulation* $\mathcal{T}_0$ *by successive application of Newest Vertex Bisection. Based on the space decomposition* (5.7) *define the preconditioner*

$$B_2^{-1} := R_h^T (B^1_h)^{-1} R_h \;+\; \sum_{\boldsymbol{z}\in\mathcal{V}_L} R^T_{\boldsymbol{z}}\left(\widetilde{D}^p_{h,\boldsymbol{z}}\right)^{-1} R_{\boldsymbol{z}},$$

Then, for constants c, $C > 0$ *that depend solely on* Γ *and the initial triangulation* $\mathcal{T}_0$, *the extremal eigenvalues of* $B_2^{-1}\widetilde{D}^p_h$ *satisfy*

$$c \le \lambda_{\min}(B_2^{-1}\widetilde{D}^p_h) \le \lambda_{\max}(B_2^{-1}\widetilde{D}^p_h) \le C.$$

In particular, the condition number $\kappa(B_2^{-1}\widetilde{D}^p_h)$ *is bounded independently of* h *and* p.

Proof. The proof follows from Lemma 5.2, Proposition 5.4, and Lemma 5.5. □

5.4 Spectrally equivalent local solvers

For each vertex patch, we need to store the dense matrix $\left(\widetilde{D}^p_{h,\boldsymbol{z}}\right)^{-1}$. For higher polynomial orders, storing these blocks is a significant part of the memory consumption of the preconditioner. To reduce these costs, we can make use of the fact that the abstract additive Schwarz theory allows us to replace the local bilinear forms $a(R_i^T u_i, R_i^T v_i)$ with spectrally equivalent forms, as long as they satisfy the conditions stated in Proposition 5.1. This is for example guaranteed, if the decomposition is stable for the exact local solvers and there exist some constants $c_1, c_2 > 0$ such that

$$c_1\, \widetilde{a}_i(u_i, u_i) \leq a(R_i^T u_i, R_i^T u_i) \leq c_2 \widetilde{a}_i(u_i, u_i) \quad \forall u_i \in \mathbb{V}_i.$$

The new preconditioner will be based on a finite number of reference patches, for which the Galerkin matrix has to be inverted.

First we prove the simple fact that we can drop the stabilization term from (2.4) when assembling the local bilinear forms:

Lemma 5.7. *There exists a constant $c_1 > 0$ that does not depend on p or h such that for any vertex patch $\omega_{\boldsymbol{z}}$ the following estimates hold:*

$$\langle Du, u\rangle_\Gamma \leq \left\langle \widetilde{D}u, u\right\rangle_\Gamma \leq c_1 \langle Du, u\rangle_\Gamma \qquad \forall u \in \mathbb{V}^p_{\boldsymbol{z}}.$$

Proof. The first estimate is trivial, as $\widetilde{D}$ only adds an additional non-negative term. For the second inequality, we note that the functions in $\mathbb{V}^p_{\boldsymbol{z}}$ all vanish outside of $\omega_{\boldsymbol{z}}$ and therefore $\mathbb{V}^p_{\boldsymbol{z}} \cap \ker(D) = \emptyset$. We transform to the reference patch, use the fact that $\hat{D}$ is elliptic on $\widetilde{H}^{1/2}(\hat{\omega}_{\boldsymbol{z}})$, and transform back by applying Lemma 3.6:

$$\begin{aligned}\|u\|^2_{L^2(\omega_{\boldsymbol{z}})} &\leq C h_{\boldsymbol{z}}^2 \|\hat{u}\|^2_{L^2(\hat{\omega}_{\boldsymbol{z}})} \leq C h_{\boldsymbol{z}}^2 \|\hat{u}\|^2_{\widetilde{H}^{1/2}(\hat{\omega}_{\boldsymbol{z}})} \\ &\leq C h_{\boldsymbol{z}}^2 \left\langle \hat{D}\hat{u}, \hat{u}\right\rangle_{\hat{\omega}\boldsymbol{z}} \leq C h_{\boldsymbol{z}} \langle Du, u\rangle_{\omega \boldsymbol{z}}.\end{aligned}$$

Thus, we can simply estimate the stabilization:

$$\begin{aligned}\alpha^2 \langle u, \mathbb{1}\rangle^2_{\omega_{\boldsymbol{z}}} &\leq \alpha^2 \|u\|^2_{L^2(\omega_{\boldsymbol{z}})} \|\mathbb{1}\|^2_{L^2(\omega_{\boldsymbol{z}})} \\ &\leq \alpha^2 \|\mathbb{1}\|^2_{L^2(\omega_{\boldsymbol{z}})} C h_{\boldsymbol{z}} \langle Du, u\rangle_\Gamma.\end{aligned}$$

This gives the full estimate with the constant

$$\begin{aligned}c_1 &:= \max\left(1, \alpha^2 \|\mathbb{1}\|^2_{L^2(\omega_{\boldsymbol{z}})} C h_{\boldsymbol{z}}\right) \\ &\leq \max\left(1, C\alpha^2 h_{\boldsymbol{z}}^3\right).\end{aligned}$$

□

Remark 5.8. The proof of the previous lemma shows that this modification does not significantly affect the stability of the preconditioner and its effect will even vanish with h-refinement.

We are now able to define the new local bilinear forms as:

Definition 5.9. Take $\boldsymbol{z} \in \mathcal{V}$, and let $F_{\boldsymbol{z}} : \widehat{\omega}_{\boldsymbol{z}} \to \omega_{\boldsymbol{z}}$ be the pullback mapping to the reference patch as in Definition 3.5. Set

$$\widetilde{a}_{\boldsymbol{z}}(u, v) := h_{\boldsymbol{z}} \left\langle \widehat{D}\left(\, u \circ F_{\boldsymbol{z}}\right), v \circ F_{\boldsymbol{z}}\right\rangle_{\widehat{\omega}_{\boldsymbol{z}}} \qquad \forall u, v \in \mathbb{V}^p_{\boldsymbol{z}}.$$

(see Lemma 3.6 for the definition of $\widehat{D}$). We denote the Galerkin matrix corresponding the bilinear form $\widetilde{a}_{\boldsymbol{z}}$ on the reference patch by $\widehat{D}^p_{h,\mathrm{ref}(\boldsymbol{z})}$.

The above definition only needs to evaluate $\left\langle \widehat{D}\hat{u}, \hat{v} \right\rangle_\Gamma$ on the reference patch. Since the reference patch depends only on the number of elements belonging to the patch, the number of blocks that need to be stored depends only on the shape regularity and is independent of the number of vertices in the triangulation $\mathcal{T}$.

Theorem 5.10. *Assume $\mathcal{T}$ is generated from a regular and shape-regular initial triangulation $\mathcal{T}_0$ by successive application of Newest Vertex Bisection. The preconditioner using the local solvers from Definition 5.9 is optimal, i.e., for*

$$B_3^{-1} := R_h^T (B_h^1)^{-1} R_h \; + \sum_{\boldsymbol{z} \in \mathcal{V}_L} h_{\boldsymbol{z}}^{-1} R_{\boldsymbol{z}}^T \left(\widehat{D}^p_{h,\mathrm{ref}(\boldsymbol{z})} \right)^{-1} R_{\boldsymbol{z}},$$

the condition number of the preconditioned system satisfies

$$\kappa(B_3^{-1} \tilde{D}_h^p) \leq C,$$

where $C > 0$ only depends on Γ and $\mathcal{T}_0$. It is in particular independent of h and p.

Proof. The scaling properties of $\langle Du, u \rangle_\Gamma$ were stated in Lemma 3.6. Therefore, we can conclude the argument by using the standard additive Schwarz theory. $\square$

5.4.1 Numerical Realization

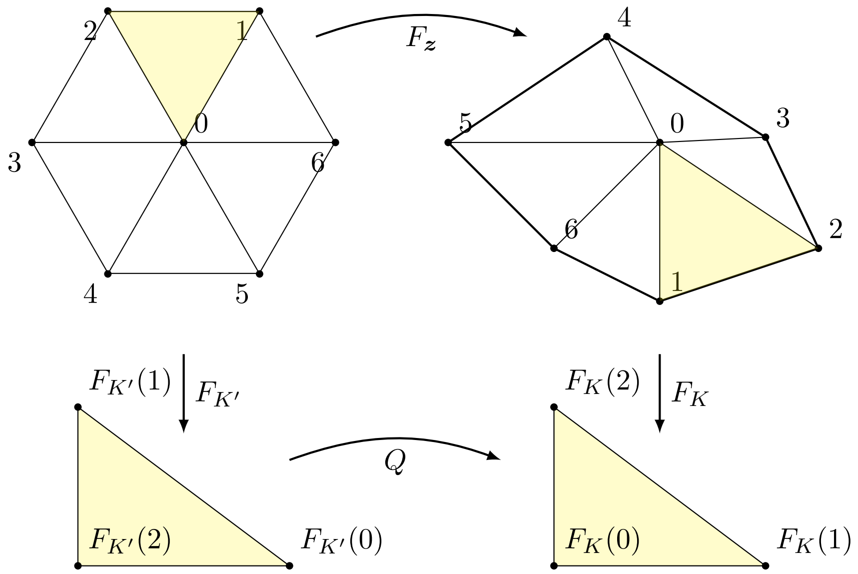

Figure 5.3: The mapping to the reference patch described as a combination of element maps.

When implementing the preconditioner as defined above, it is important to note that for a basis function φ_i on $\omega_{\boldsymbol{z}}$ the transformed function $\varphi_i \circ F_{\boldsymbol{z}}^{-1}$ does not necessarily correspond to the i-th basis function on $\widehat{\omega}_{\boldsymbol{z}}$. Depending on the chosen basis we may run into orientation problems. This can be fixed in the following way:

Let $\boldsymbol{z} \in \mathcal{V}$ be fixed. Choose a numbering for the vertices $\boldsymbol{z}_i$ and elements K_i of $\omega_{\boldsymbol{z}}$ such that adjacent elements have adjacent numbers (for example enumerate clockwise or counter-clockwise). We also choose a similar enumeration on the reference patch and denote it as $\widehat{\boldsymbol{z}}_i$

and $\widehat{K}_i$. The enumeration is such that the reference map $F_{\boldsymbol{z}}$ maps $\boldsymbol{z}_i$ to $\widehat{\boldsymbol{z}}_i$ and K_i to $\widehat{K}_i$. Let $N_{\boldsymbol{z}}$ be the number of vertices in the patch.

For elements $K \subset \omega_{\boldsymbol{z}}$ and $K' \subset \widehat{\omega}_{\boldsymbol{z}}$, the bases on $\omega_{\boldsymbol{z}}$ and on $\widehat{\omega}_{\boldsymbol{z}}$ are locally defined by the pullback of polynomials on the reference triangle $\widehat{K}$. We denote the element maps as $F_K : \widehat{K} \to K$ and $F'_K : \widehat{K} \to K'$ respectively. The basis functions are then given as $\varphi_j := \widehat{\varphi}_j \circ F_K$ on $\omega_{\boldsymbol{z}}$ and $\psi_j := \widehat{\psi}_j \circ F_{K'}$ on $\widehat{\omega}_{\boldsymbol{z}}$.

Due to the fact that the local element maps do not necessarily map the same points in the local ordering, we need to introduce another map $Q : \widehat{K} \to \widehat{K}$ that represents a vertex permutation.

Then, we can write the patch-pullback restricted to K' as $F_{\boldsymbol{z}}|_{K'} = F_K^{-1} \circ Q \circ F_{K'}$ (see Figure 5.3). We observe:

i) For the hat function the mapping is trivial: $\varphi_{\boldsymbol{z}} \circ F_{\boldsymbol{z}} = \psi_{\widehat{\boldsymbol{z}}}$.

ii) For the edge basis, permuting the vertices on the reference element only changes the sign of the corresponding edge functions. Thus, we have $\varphi_j^{E_m} \circ F_{\boldsymbol{z}} = (-1)^j \psi_j^{E_m}$, if the orientation of the edge in the global triangulation does not match the orientation of the reference patch.

iii) The inner basis functions transformation under Q is not so simple. Since the basis functions all have support on a single element we can restrict our consideration to this element and assemble the necessary basis transformations for all 5 permutations of vertices on the reference triangle without losing the memory advantage of using the reference patch.

Remark 5.11. One could also exploit the symmetry (up to a sign change) of the permutation of λ_1 and λ_2 in the definition of the inner basis functions to reduce the number of basis transformation matrices needed to 2 instead of 5.

6 Numerical Results

The following numerical experiments confirm that the proposed preconditioners (Theorem 5.3, Theorem 5.4 and Theorem 5.10) do indeed yield a system with a condition number that is bounded uniformly in h and p, whereas the unpreconditioned system grows in p with a rate slightly smaller than predicted in Corollary 4.7 (The condition seems to behave like $\kappa \sim \mathcal{O}(p^{5.5})$). Diagonal preconditioning appears to reduce the condition number to $\mathcal{O}(h^{-1}p^{2.5})$ for uniform refinement. All of the following experiments were done using the BEM++ software library ([BAP+13]; `www.bempp.org/`) which internally uses the AHMED software library for $\mathcal{H}$-matrix compression ([Beb08], [Beb14]). We used the basis described in Section 2.2.

Example 6.1 (unpreconditioned p-dependence)**.** We consider a quadratic screen in $\mathbb{R}^3$ (see Figure 6.1, right). In order to confirm our bounds from Corollary 4.7 we look at the condition number of the unpreconditioned system on different uniformly refined meshes and compare the growth behavior with respect to p. In Figure 6.2 we see, in accordance to the estimates of Corollary 4.7, that depending on the mesh size h we get a preasymptotic phase, in which the $\mathcal{O}(h^{-1}p^2)$ term is dominating, but then end up with an h-independent asymptotic behavior of $\mathcal{O}(p^{5.5})$, which is slightly better than the prediction of $\mathcal{O}(p^6)$ of Corollary 4.7.

Example 6.2 (Fichera's cube)**.** We compare the preconditioner that uses the local multilevel preconditioner for the h-part and the inexact local solvers based on the reference patches to the unpreconditioned system and to a simple diagonal scaling. We consider the problem on a closed geometry, the surface of the Fichera cube with side length 1, and employ a stabilization parameter $\alpha = 0.2$. To generate the adaptive meshes, we used Newest Vertex Bisection, where

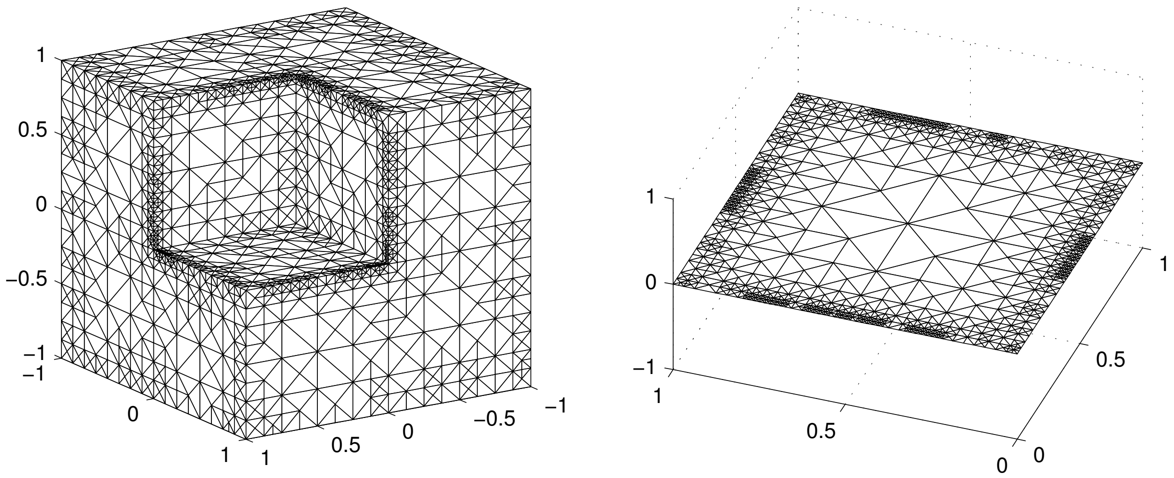

Figure 6.1: Adaptive meshes on the Fichera cube and for a screen problem.

in each step, the set of marked elements originated from a lowest order adaptive algorithm with a ZZ-type error estimator (as described in [AFF$^+$14]). The left part of Figure 6.1 shows an example of one of the meshes used.

Figure 6.3 confirms that the condition of the preconditioned system does not depend on the polynomial degree of the discretization and Figure 6.4 confirms the independence of the preconditioned system from the series of adaptively refined meshes. The unpreconditioned and the diagonally preconditioned system do not show the bad behavior with respect to h, probably due to the already large condition number for $p > 1$.

Example 6.3 (screen problem)**.** As a second test case for our preconditioner, we again look at the screen problem in $\mathbb{R}^3$ with a quadratic screen of side length 1 (see Figure 6.1), which represents the case $\Gamma \neq \partial\Omega$ and $\alpha = 0$ in (2.3), and perform the same experiments as we did for Fichera's cube in Example 6.2. In Figure 6.5 we again observe that the condition number is independent of the polynomial degree. Figures 6.6–6.9 demonstrate the independence of the mesh size h.

Example 6.4 (inexact local solvers)**.** In the next example we compare the different preconditioners proposed in this paper. While the numerical experiments all show that the preconditioner is indeed robust in h and p, the constant differs if we use the different simplifications described in the Sections 5.3 and 5.4 to the preconditioner. In Figures 6.10 and 6.11 we can observe the different constants for the geometry given by Fichera's cube of Example 6.2.

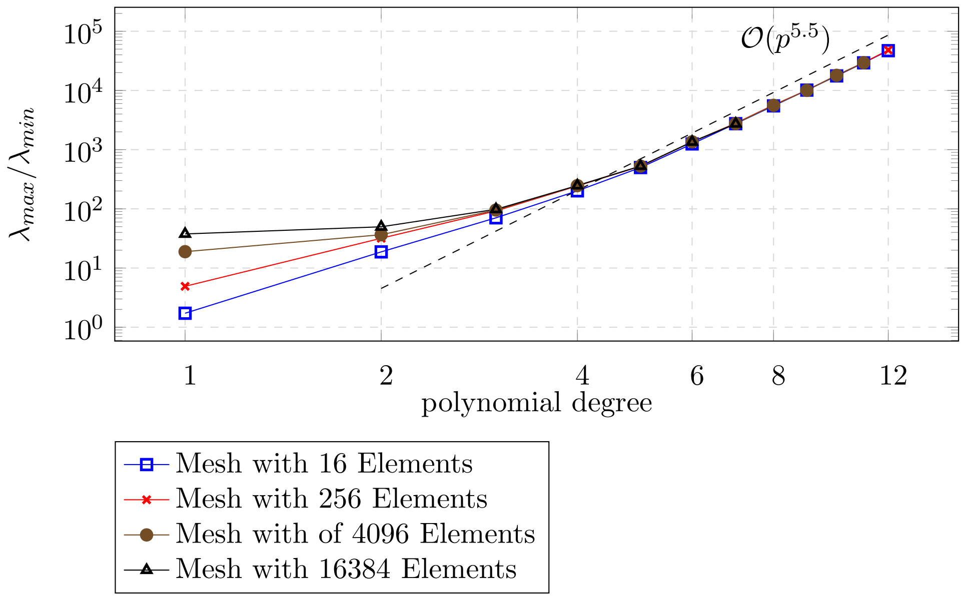

Figure 6.2: Comparison of the condition number of $\widetilde{D}_h^p$ for the screen problem on different uniform meshes (Example 6.1)

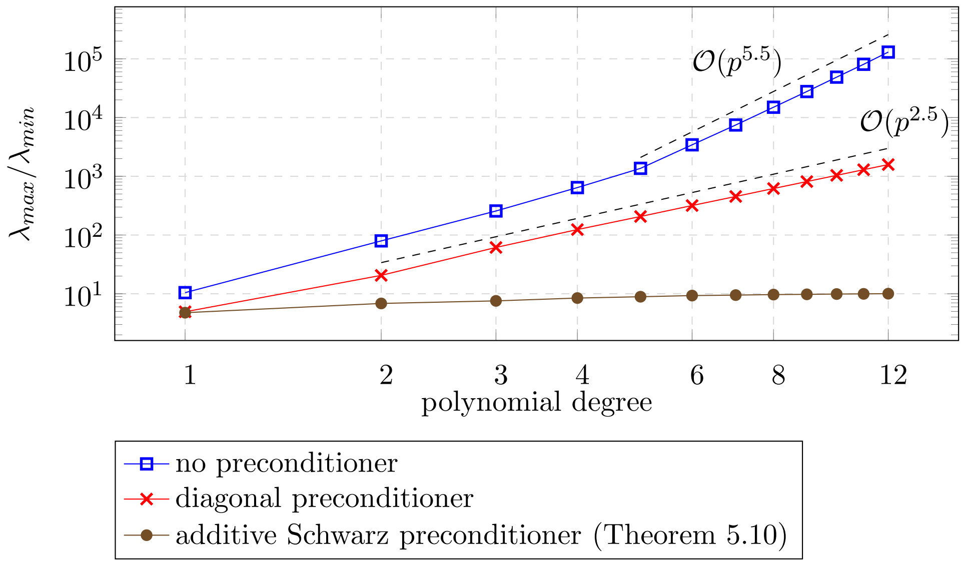

Figure 6.3: Fichera cube, condition numbers for fixed uniform mesh with 70 elements (Example 6.2).

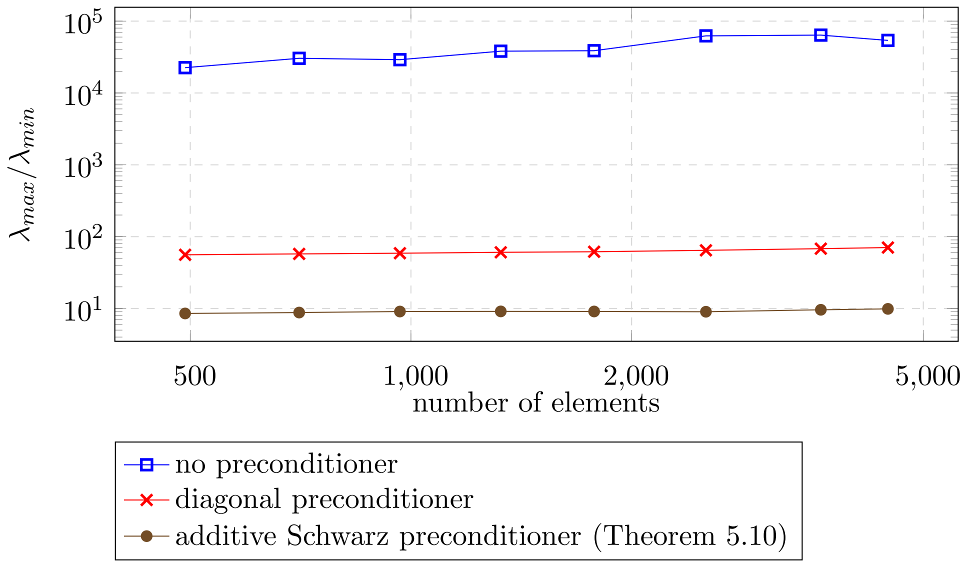

Figure 6.4: Fichera cube, adaptive h-refinement for $p = 3$ (Example 6.2).

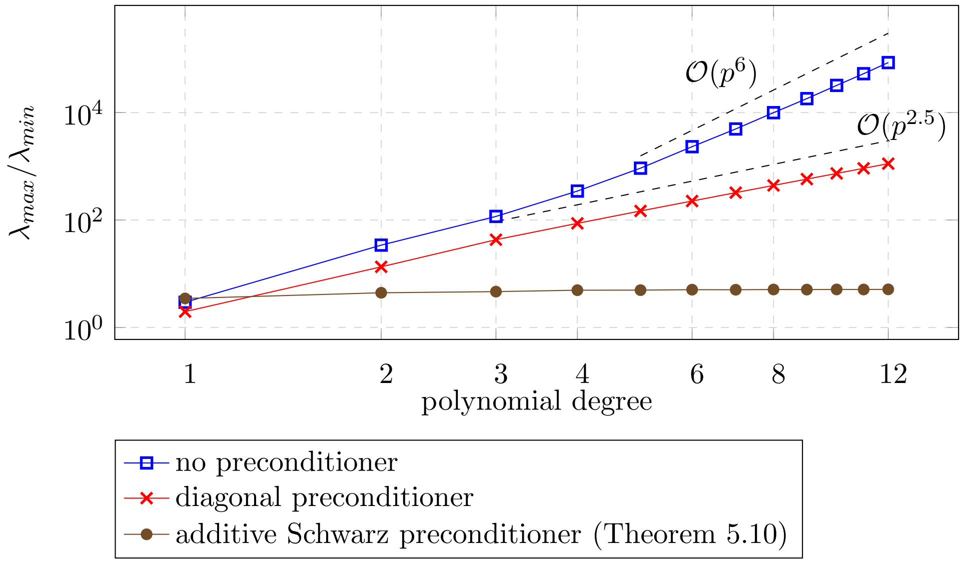

Figure 6.5: Screen problem, condition numbers for uniform mesh with 45 elements (Example 6.3).

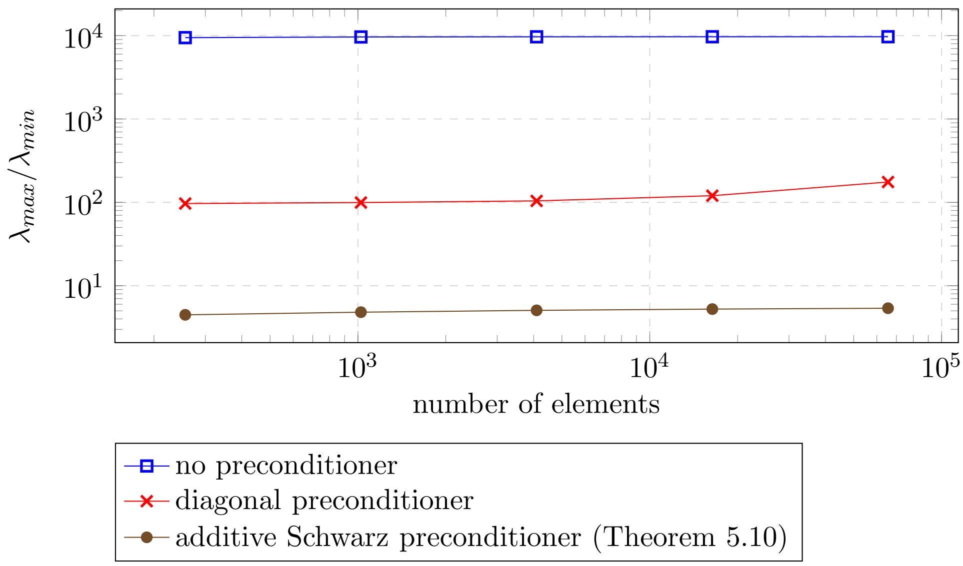

Figure 6.6: Screen problem, uniform h-refinement for $p = 4$ (Example 6.3).

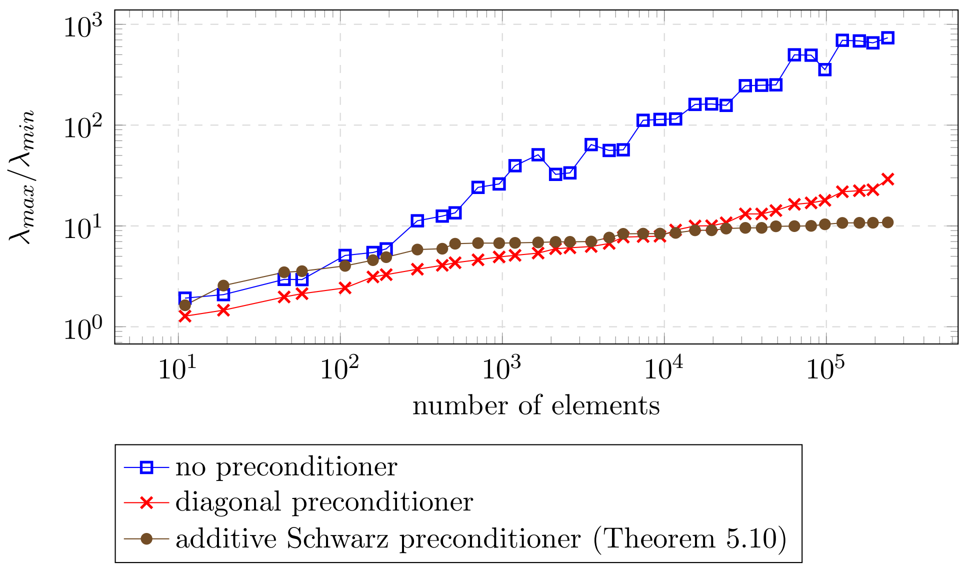

Figure 6.7: Screen problem, adaptive h-refinement for $p = 1$ (Example 6.3).

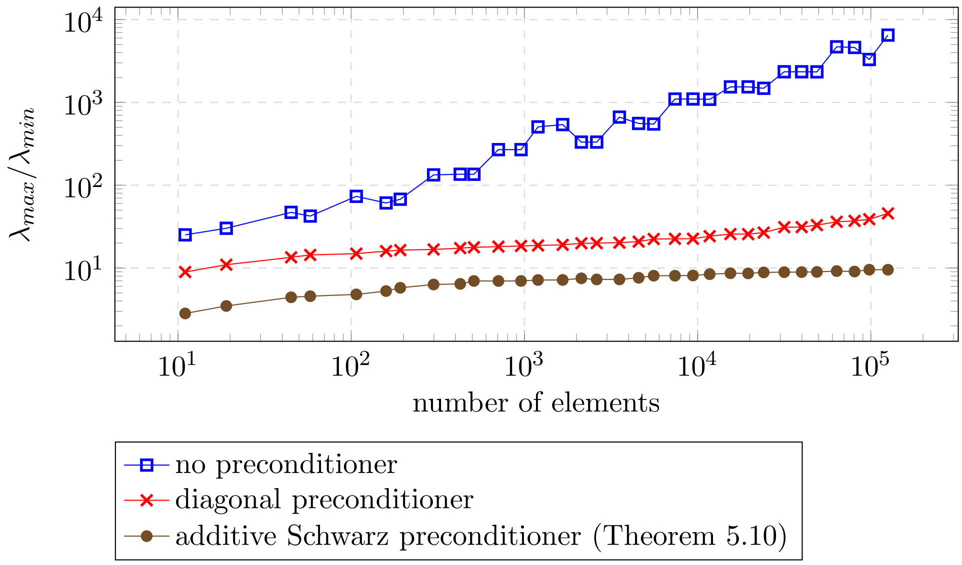

Figure 6.8: Screen problem, adaptive h-refinement for $p = 2$ (Example 6.3).

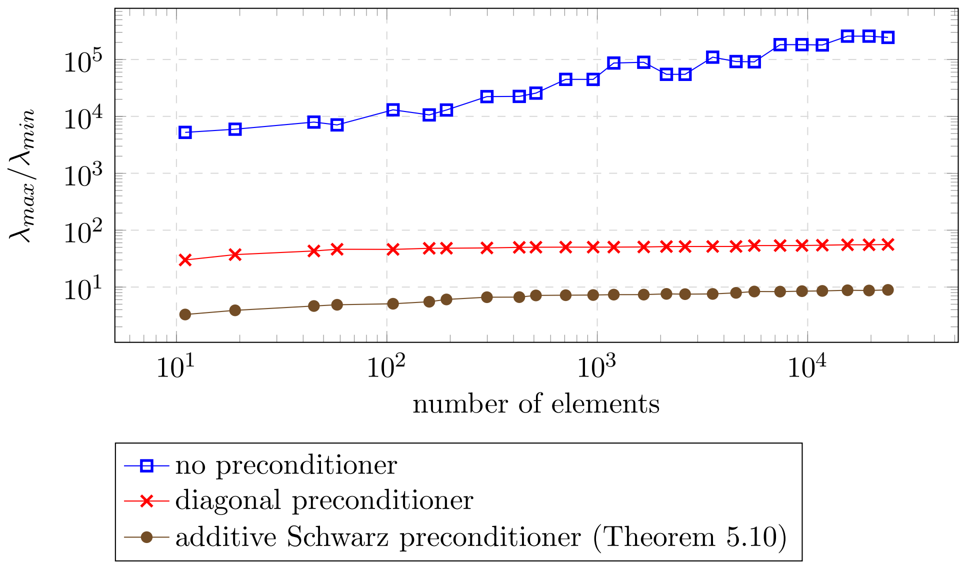

Figure 6.9: Screen problem, adaptive h-refinement for $p = 3$ (Example 6.3).

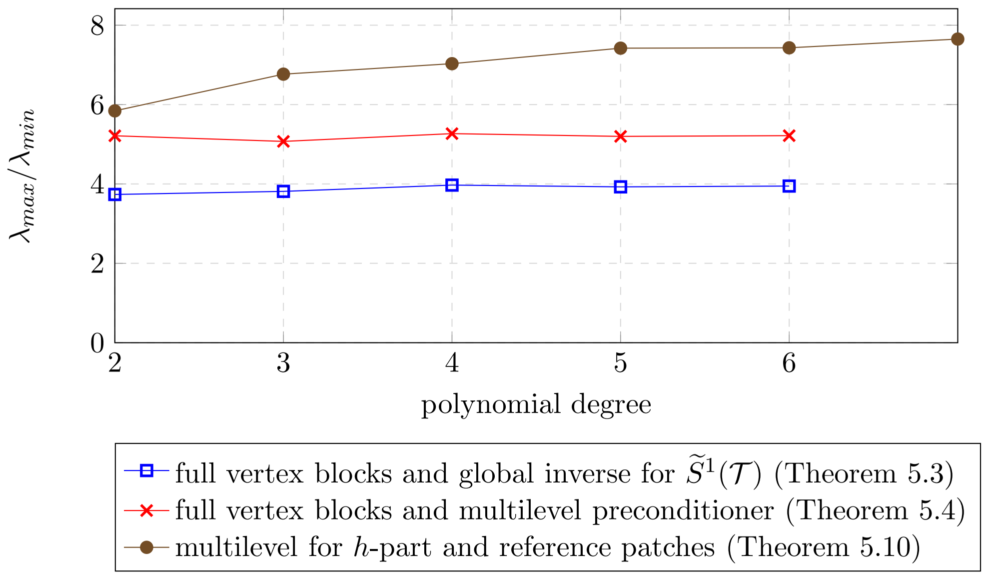

Figure 6.10: Comparison of the different proposed preconditioners for a fixed uniform mesh with 70 elements on the Fichera cube (Example 6.4).

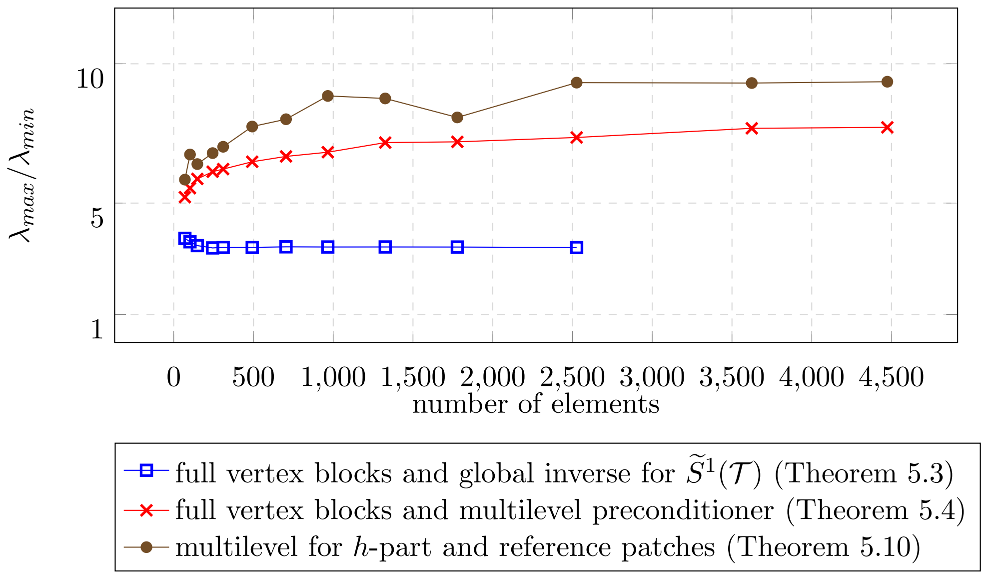

Figure 6.11: Comparison of the different proposed preconditioners for adaptive mesh refinement on the Fichera cube with $p = 2$.

Example 6.5 (inexact local solvers)**.** We continue with the geometry of Example 6.4. We motivated Section 5.4 by stating the large memory requirement of the preconditioner when storing the dense local block inverses. That the reference patch based preconditioner does indeed solve this issue, can be seen in Table 6.1, where we compare the memory requirements, not considering the lowest order space $\mathbb{V}_h^1$. For comparison, we included the storage requirements for the full matrix $\widetilde{D}_h^p$ and the $\mathcal{H}$-matrix approximation with accuracy 10^{-8} which is denoted as $D_h^{p,\mathcal{H}}$. While we still get linear growth in the number of elements, due to some bookkeeping requirements, such as element orientation etc., which could theoretically also be avoided, we observe a much reduced storage cost. For $p = 3$ and $55,298$ degrees of freedom, the memory requirement is less than 2.5% of the full block storage. For $p = 4$ and $393,218$ degrees of freedom the memory requirement is just 0.6% and for higher polynomial orders, this ratio would become even smaller. Comparing only the number of blocks that need to be stored, we see that in this particular geometry we only need to store the inverse for 6 reference blocks.

p	N_{dof}	$\text{mem}\left(\widetilde{D}_h^p\right)/N_{dof}$ [KB]	$\text{mem}\left(\widetilde{D}_h^{p,\mathcal{H}}\right)/N_{dof}$ [KB]	$\text{mem}\left(B^{-1}\right)/N_{dof}$ [KB]	$\text{mem}\left(B_3^{-1}\right)/N_{dof}$ [KB]
2	98	0.765 62	0.769 45	0.094 547	0.039 222
2	298	2.3281	2.3334	0.098 259	0.027 318
2	986	7.7031	7.1907	0.100 25	0.020 522
2	3558	27.797	15.006	0.100 68	0.018 394
2	8950	69.922	19.817	0.100 78	0.017 902
3	218	1.7031	1.7103	0.313 14	0.083 787
3	668	5.2188	5.1523	0.325 08	0.041 261
3	2216	17.312	10.921	0.332	0.017 895
3	5969	46.633	16.452	0.3343	0.011 556
3	16 310	127.42	22.738	0.332 74	0.009 182 4
3	20 135	157.3	24.04	0.333 72	0.008 922 2
4	386	3.0156	3.0197	0.670 86	0.169 73
4	1954	15.266	10.879	0.702 33	0.048 29
4	5634	44.016	17.39	0.710 85	0.019 619
4	14 226	111.14	21.209	0.713 64	0.010 424
4	35 794	279.64	27.596	0.714 28	0.006 790 8
5	602	4.7031	4.7083	1.1694	0.293 24
5	1852	14.469	14.476	1.2122	0.129 45
5	8802	68.766	68.772	1.2384	0.029 459
5	16 577	129.51	129.4	1.2468	0.016 961
5	45 302	353.92	353.79	1.2404	0.007 989 8
5	55 927	436.93	436.78	1.2443	0.007 006 2

Table 6.1: Comparison of the memory requirement relative to the number of degrees of freedom N_{dof} between storing the full block structure and the reference block based preconditioner from Section 5.4 (Example 6.5).

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