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On the stability of the polynomial L^2-projection on triangles and tetrahedra

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Abstract

For the reference triangle or tetrahedron $\mathcal{T}$, we study the stability properties of the $L^2(\mathcal{T})$-projection Π_N onto the space of polynomials of degree N. We show $\|\Pi_N u\|^2_{L^2(\partial\mathcal{T})} \leq C\|u\|_{L^2(\mathcal{T})}\|u\|_{H^1(\mathcal{T})}$ and $\|\Pi_N u\|_{H^1(\mathcal{T})} \leq C(N+1)^{1/2}\|u\|_{H^1(\mathcal{T})}$. This implies optimal convergence rates for the approximation error $\|u - \Pi_N uu\|_{L^2(\partial\mathcal{T})}$ for all $u \in H^k(\mathcal{T})$, $k > 1/2$.

1 Introduction and main results

The study of polynomials and their properties as the polynomial degree tends to infinity has a very long history in numerical mathematics. Concerning approximation and stability properties of various high order approximation operators, the univariate case is reasonably well understood (in the way of examples, we mention the monographs [26] for orthogonal polynomials and [8] for issues concerning approximation); by tensor product arguments, also for the case of polynomial approximation on d-dimensional hyper cubes a significant number of results is available. The situation is less developed for simplices, and it is the purpose of this note to contribute in the area by studying the stability properties of the polynomial L^2-projection on triangles or tetrahedra. Our main results are Theorems 1.1 and 1.3 below. These two theorems generalize known results for tensor product domains: Theorem 1.1 is the analog of [12, Lemma 4.2] (and correspondingly, Cor. 1.2 is the analog of [12, Lemma 4.4] and [15, Lemma 3.5]) and Theorem 1.3 corresponds to [4, Thm. 2.2]. Independently, closely related results have recently been obtained in [5]. The novelty of the present work over [5] is twofold: Firstly, in the language of Corollary 1.2 below, we extend the approximation result of [5] from $s \geq 1$ to $s > 1/2$. Secondly, we study the H^1-stability of the L^2 projection.
Although the results of the present note are of independent interest, the stability result of Theorem 1.1 has applications in the analysis of the hp-version of discontinuous Galerkin methods (hp-DGFEM) as demonstrated in [24]. More generally, simplicial elements are, due to their greater geometric flexibility as compared to tensor product elements, commonly used in high order finite element codes so that an understanding of stability and approximation properties of polynomial operators defined on simplices could be useful in other applications of high order finite element methods (hp-FEM) as well. We refer to [6, 7, 16, 21, 23] for various aspects of hp-FEM.

To fix the notation, we introduce the reference triangle $\mathcal{T}^2$, the reference tetrahedron $\mathcal{T}^3$ as well

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as the reference cube by

$$\mathcal{T}^2 := \{(x,y) \in \mathbb{R}^2 : -1 < x < 1 \ , \ -1 < y < -x\}, \tag{1.1a}$$
$$\mathcal{T}^3 := \{(x,y,z) \in \mathbb{R}^3 : -1 < x,y,z \ , \ x+y+z < -1\}, \tag{1.1b}$$
$$\mathcal{S}^d := (-1,1)^d, \quad d \in \{1,2,3\}. \tag{1.1c}$$

Throughout, we will denote by $\mathcal{P}_N$ the space of polynomials of (total) degree N. We then have:

Theorem 1.1. *Let $\mathcal{T}$ be the reference triangle or tetrahedron and denote by $\Pi_N : L^2(\mathcal{T}) \to \mathcal{P}_N$ the $L^2(\mathcal{T})$-projection onto the space of polynomials of degree N. Then there exists a constant $C > 0$ independent of N such that*

$$\|\Pi_N u\|^2_{L^2(\partial\mathcal{T})} \leq C\|u\|_{L^2(\mathcal{T})}\|u\|_{H^1(\mathcal{T})} \qquad \forall u \in H^1(\mathcal{T}). \tag{1.2}$$

In particular, therefore,

$$\|\Pi_N u\|_{L^2(\partial\mathcal{T})} \leq C\|u\|_{B^{1/2}_{2,1}(\mathcal{T})} \qquad \forall u \in B^{1/2}_{2,1}(\mathcal{T}), \tag{1.3}$$

where the Besov space $B^{1/2}_{2,1}(\mathcal{T})$ is defined by $B^{1/2}_{2,1}(\mathcal{T}) = (L^2(\mathcal{T}), H^1(\mathcal{T}))_{1/2,1}$ and we used the real method of interpolation (see, e.g., [27, 28] for details).

Corollary 1.2. *Let $\mathcal{T}$ be the reference triangle or tetrahedron. Then for every $s > 1/2$ there exists a constant $C_s > 0$ such that*

$$\|u - \Pi_N u\|_{L^2(\partial\mathcal{T})} \leq C_s(N+1)^{-(s-1/2)}\|u\|_{H^s(\mathcal{T})} \qquad \forall u \in H^s(\mathcal{T}). \tag{1.4}$$

Theorem 1.3. *Let $\mathcal{T}$ be the reference triangle or tetrahedron. Then there exists a constant $C > 0$ such that for all $N \in \mathbb{N}$*

$$\|\Pi_N u\|_{H^1(\mathcal{T})} \leq C(N+1)^{1/2}\|u\|_{H^1(\mathcal{T})} \qquad \forall u \in H^1(\mathcal{T}). \tag{1.5}$$

We will only show the proofs for the 3D case in Theorem 5.3 (this theorem combines Theorems 1.1 and Corollary 1.2) and in Theorem 6.2, which formulates Theorem 1.3. The 2D case is treated with similar ideas. Details of the proof of Theorem 1.1 in 2D can be found in the Bachelor Thesis [29].

2 Numerical results

In this section, we illustrate the sharpness of Theorems 1.1, 1.3 for the 1D and the 2D case. We present the best constants in the following 1D and 2D situations:

$$|(\Pi_N u)(1)|^2 \leq C^{1D}_{mult}\|u\|_{L^2(I)}\|u\|_{H^1(I)}, \qquad \|\Pi_N u\|_{H^1(I)} \leq C^{1D}_{H^1}\sqrt{N+1}\|u\|_{H^1(I)} \quad \forall u \in \mathcal{P}_{2N} \tag{2.1}$$
$$\|(\Pi_N u)\|^2_{L^2(\Gamma)} \leq C^{2D}_{mult}\|u\|_{L^2(\mathcal{T}^2)}\|u\|_{H^1(\mathcal{T}^2)}, \qquad \|\Pi_N u\|_{H^1(\mathcal{T}^2)} \leq C^{2D}_{H^1}\sqrt{N+1}\|u\|_{H^1(\mathcal{T}^2)} \quad \forall u \in \mathcal{P}_{2N} \tag{2.2}$$

where $I = (-1,1)$ and $\Gamma = (-1,1) \times \{-1\} \subset \partial\mathcal{T}^2$. The best constants C^{1D}_{mult}, C^{2D}_{mult} are solutions of constrained maximization problem. For example,

$$C^{2D}_{mult} = \max\{\|\Pi_N u\|^2_{L^2(\Gamma)} \,|\, \|u\|^2_{L^2(\mathcal{T}^2)}\|u\|^2_{H^1(\mathcal{T}^2)} = 1, \quad u \in \mathcal{P}_{2N}\},$$

which can be solved using the technique of Lagrange multipliers. The constant $C^{2D}_{H^1}$ is more readily accessible as the solution of an eigenvalue problem since

$$C^{2D}_{H^1} = \sup_{u \in \mathcal{P}_{2N}} \frac{\|u\|^2_{L^2(\Gamma)}}{\|u\|^2_{H^1(\mathcal{T}^2)}}.$$

The result of the 1D situation are presented in Table 1 whereas the outcome of the 2D calculations are shown in Table 2. The 2D calculations are in agreement with the results of Theorem 1.1, 1.3 whereas the 1D results illustrate [12, Lemma 4.1] and [4, Thm. 2.2].

N	$\sup_{u\in\mathcal{P}_{2N}} \frac{\lvert(\Pi_N u)(1)\rvert^2}{\lVert u\rVert_{L^2(I)}\lVert u\rVert_{H^1(I)}}$	$\sup_{u\in\mathcal{P}_{2N}} \frac{\lvert(\Pi_N u)(1)\rvert^2}{\lVert u\rVert^2_{H^1(I)}}$	N	$\sup_{u\in\mathcal{P}_{2N}} \frac{\lvert(\Pi_N u)(1)\rvert^2}{\lVert u\rVert_{L^2(I)}\lVert u\rVert_{H^1(I)}}$	$\sup_{u\in\mathcal{P}_{2N}} \frac{\lvert(\Pi_N u)(1)\rvert^2}{\lVert u\rVert^2_{H^1(I)}}$
1	1.18184916854199	0.87500000000000	55	2.96809547801154	1.04624257963827
2	1.82982112979637	1.14361283167718	60	2.97190920471158	1.04551089085115
3	2.15270769390416	1.15072048261852	65	2.97503926131730	1.04489004319026
4	2.34106259718609	1.13538600864567	70	2.97764417211453	1.04435662477469
5	2.45948991256407	1.11992788388317	75	2.97983833781387	1.04389338318215
10	2.72197668882430	1.08267283507986	80	2.98170613800160	1.04348732490767
15	2.82210388053635	1.06853381605478	85	2.98331100621813	1.04312847760604
20	2.87406416223951	1.06111064764886	90	2.98470143645119	1.04280906028005
25	2.90512455645115	1.05653834380496	95	2.98591505801301	1.04252291272595
30	2.92540310256400	1.05343954290018	100	2.98698146107879	1.04226509441461
35	2.93948150346734	1.05120092371494	105	2.98792419448666	1.04203159687688
40	2.94971129296239	1.04950802550192	110	2.98876220322068	1.04181913380901
45	2.95741139670201	1.04818299976127	115	2.98951087919110	1.04162498545355
50	2.96337279789140	1.04711769879646	120	2.99018284042270	1.04144688155757

Table 1: 1D maximization problems

N	$\sup_{u\in\mathcal{P}_{2N}} \frac{\lVert\Pi_N u\rVert^2_{L^2(\Gamma)}}{\lVert u\rVert_{L^2(\mathcal{T}^2)}\lVert u\rVert_{H^1(\mathcal{T}^2)}}$	$\sup_{u\in\mathcal{P}_{2N}} \frac{\lVert\Pi_N u\rVert^2_{L^2(\Gamma)}}{\lVert u\rVert^2_{H^1(\mathcal{T}^2)}}$	$\sup_{u\in\mathcal{P}_{2N}} \frac{\lVert\Pi_N u\rVert^2_{H^1(\mathcal{T}^2)}}{\lVert u\rVert^2_{H^1(\mathcal{T}^2)}(N+1)}$
1	1.841709179979923	1.471718130438879	0.630765682656827
2	2.482008261007026	1.705122181047698	0.534605787746493
3	2.840187661660685	1.715716151553367	0.502563892227422
4	3.069499343400879	1.698817794545474	0.469837538241653
5	3.221409644442824	1.681456018543639	0.451490548981791
6	3.328227344860761	1.668376974400849	0.442841777476419
7	3.407909278234681	1.658518924126468	0.439947515744027
8	3.470131098161663	1.650883702558015	0.438357597812212
9	3.520381327094282	1.644802661753009	0.437324308925344
10	3.561974205023130	1.639847070385123	0.436649057543612
15	3.695751965776597	1.624493814472113	0.435028483384906
20	3.768125200720072	1.616550757911536	0.434212531866692
25	3.812940340675182	1.611706243752921	0.433709285223336
30	3.843021842931179	1.608446746919096	0.433462666731238
35	3.864360185593495	1.606105243942980	0.433441925213746
40	3.880125733031547	1.604342473993514	0.433618345964902
45	3.892146634658522	1.602967814218030	0.433964716705375
50	3.901546220032895	1.601865982255488	0.434455030379973
55	3.909049652499364	1.600963192345851	0.435064216731618

Table 2: 2D maximization problems

3 One-dimensional results

In the tensor-product setting of squares and hexahedra, the arguments leading to Theorems 1.1, 1.3 can be reduced to a one-dimensional setting. Most of this reduction to a one-dimensional setting is also possible in the present case of simplices, and the present section provides the necessary one-dimensional results. Our basic tool for this dimension reduction is the so-called Duffy transformation (see (4.1) below), which maps the simplex into a hyper cube. More importantly, as noted already by [9, 16, 17] orthogonal polynomials on the simplex can be defined in the transformed variables through products of univariate Jacobi polynomials, which expressed the desired reduction to one-dimensional settings. Although the situation is technically more complicated than the tensor-product setting since Jacobi polynomials arise—the tensor-product setting of squares and hexahedra ultimately leads to the study of the more common Gegenbauer/ultraspherical polynomials (see, e.g., [21, Sec. 3.3.1], [1, 13, 22])—the Jacobi polynomials are classical orthogonal polynomials and one can draw on a plethora of known properties for the purpose of both analysis and design of algorithms. This underlies, for example, the works [2, 5, 16, 18] and is also at the heart of the present analysis.

We denote by $P_n^{(\alpha,\beta)}$, α, $\beta > -1$, $n \in \mathbb{N}$, the Jacobi polynomials, [26]. From [26, (4.3.3)] we have the following orthogonality relation for Jacobi polynomials and p, $q \in \mathbb{N}_0$:

$$\int_{-1}^{1} (1-x)^\alpha (1-x)^\beta P_p^{(\alpha,\beta)}(x) P_q^{(\alpha,\beta)}(x) dx = \gamma_p^{(\alpha,\beta)} \delta_{p,q} \tag{3.1}$$

here, $\delta_{p,q}$ represent the Kronecker symbol and

$$\gamma_p^{(\alpha,\beta)} := \frac{2^{\alpha+\beta+1}}{2p+\alpha+\beta+1} \frac{\Gamma(p+\alpha+1)\Gamma(p+\beta+1)}{p!\Gamma(p+\alpha+\beta+1)}. \tag{3.2}$$

Furthermore, we abbreviate factors that will appear naturally in our computations:

$$\begin{aligned} h_1(q,\alpha) &:= -\frac{2(q+1)}{(2q+\alpha+1)(2q+\alpha+2)}, & g_1(q,\alpha) &:= \frac{2q+2\alpha}{(2q+\alpha-1)(2q+\alpha)}, \\ h_2(q,\alpha) &:= \frac{2\alpha}{(2q+\alpha+2)(2q+\alpha)}, & g_2(q,\alpha) &:= \frac{2\alpha}{(2q+\alpha-2)(2q+\alpha)}, \\ h_3(q,\alpha) &:= \frac{2(q+\alpha)}{(2q+\alpha+1)(2q+\alpha)}, & g_3(q,\alpha) &:= -\frac{2q-2}{(2q+\alpha-1)(2q+\alpha-2)}. \end{aligned} \tag{3.3}$$

By direct calculation we can establish relations between h_i and g_i.

Lemma 3.1. *Let h_1, h_2, h_3 and g_1, g_2, g_3 be defined in (3.3). Then there holds for any $q \geq 1$ and $\alpha \in \mathbb{N}_0$*

$$\frac{g_1(q+1,\alpha)}{\gamma_q^{(\alpha,0)}} = \frac{h_3(q+1,\alpha)}{\gamma_{q+1}^{(\alpha,0)}}, \qquad \frac{g_2(q+1,\alpha)}{\gamma_q^{(\alpha,0)}} = \frac{h_2(q,\alpha)}{\gamma_q^{(\alpha,0)}}, \qquad \frac{g_3(q+1,\alpha)}{\gamma_q^{(\alpha,0)}} = \frac{h_1(q-1,\alpha)}{\gamma_{q-1}^{(\alpha,0)}}, \tag{3.4}$$

$$(-1)^q \frac{1}{\gamma_q^{(\alpha,0)}} h_1(q,\alpha) + (-1)^{q+1} \frac{1}{\gamma_{q+1}^{(\alpha,0)}} h_2(q+1,\alpha) + (-1)^{q+2} \frac{1}{\gamma_{q+2}^{(\alpha,0)}} h_3(q+2,\alpha) = 0. \tag{3.5}$$

Furthermore, for any $q \geq 0$

$$h_2(q,\alpha) - h_1(q,\alpha) = h_3(q,\alpha). \tag{3.6}$$

Proof. This follows directly by simple calculation and the definition of the terms. Details can be found in Appendix B. □

We will denote by $\widehat{P}_q^{(\alpha,0)}$ the antiderivative of $P_{q-1}^{(\alpha,0)}$, i.e.,

$$\widehat{P}_q^{(\alpha,0)}(x) := \int_{-1}^{x} P_{q-1}^{(\alpha,0)}(t)\,dt. \tag{3.7}$$

The following lemma states important relations between Jacobi polynomials, their derivatives, and their antiderivatives.

Lemma 3.2. *Let $\alpha \in \mathbb{N}_0$ and h_i, g_i, $i \in \{1,2,3\}$ be given by (3.3) and $\gamma_p^{(\alpha,\beta)}$ by (3.2). Then we have*

(i) for $q \geq 1$

$$\int_{-1}^{x} (1-t)^\alpha P_q^{(\alpha,0)}(t)\,dt = -(1-x)^\alpha \Big(h_1(q,\alpha) P_{q+1}^{(\alpha,0)}(x) + h_2(q,\alpha) P_q^{(\alpha,0)}(x) + h_3(q,\alpha) P_{q-1}^{(\alpha,0)}(x) \Big),$$

(ii) for $q \geq 2$

$$\widehat{P}_q^{(\alpha,0)}(x) = g_1(q,\alpha) P_q^{(\alpha,0)}(x) + g_2(q,\alpha) P_{q-1}^{(\alpha,0)}(x) + g_3(q,\alpha) P_{q-2}^{(\alpha,0)}(x),$$

(iii) for $q \geq 1$

$$\frac{1}{\gamma_q^{(\alpha,0)}} P_q^{(\alpha,0)}(x) = \frac{h_1(q-1,\alpha)}{\gamma_{q-1}^{(\alpha,0)}} \big(P_{q-1}^{(\alpha,0)}\big)'(x) + \frac{h_2(q,\alpha)}{\gamma_q^{(\alpha,0)}} \big(P_q^{(\alpha,0)}\big)'(x) + \frac{h_3(q+1,\alpha)}{\gamma_{q+1}^{(\alpha,0)}} \big(P_{q+1}^{(\alpha,0)}\big)'(x).$$

Proof. The proof of (i) relies on relations satisfied by Jacobi polynomials; see Appendix B for details. (ii) is taken from [2]; (iii) is obtained by differentiating (ii) and using Lemma 3.1. □

The next lemma will be used in the proof of the ensuing Lemma 3.5, which is the 1D version of Theorem 1.3.

Lemma 3.3. *There exists $K > 0$ such that for all α, $q \in \mathbb{N}_0$*

$$\int_{-1}^{1} (1-x)^\alpha \left| \big(P_q^{(\alpha,0)}\big)'(x) \right|^2 dx \leq K q (q+1+\alpha)^2 \gamma_q^{(\alpha,0)}.$$

Proof. The assertion in case of $q = 0$ is trivial. For $\alpha = 0$ see [1, (5.3)]. A direct calculation shows

$$I_0^2 = 0, \qquad I_1^2 = \frac{(\alpha+2)^2}{4} \frac{2^{\alpha+1}}{\alpha+1}, \qquad I_2^2 = \frac{(3+\alpha)^2(\alpha+2)}{\alpha^2+4\alpha+3} 2^{\alpha+4};$$

for suitable K, the assertion of the lemma is therefore true for $q \in \{0,1,2\}$ and all α. Thus, we may assume $\alpha \geq 1$, $q \geq 2$.

We abbreviate $P_q := P_q^{(\alpha,0)}$ and $I_q^2 := \int_{-1}^{1} (1-x)^\alpha |P_q'(x)|^2\,dx$. From Lemma 3.2 (ii) with $q+1$ and q there we get

$$P_{q+1}' = \frac{1}{g_1(q+1,\alpha)} P_q - \frac{g_2(q+1,\alpha)}{g_1(q+1,\alpha) g_1(q,\alpha)} P_{q-1} + (1-\varepsilon_q) P_{q-1}' - \varepsilon_q P_{q-2}', \tag{3.8}$$

where

$$\varepsilon_q := -\frac{g_2(q+1,\alpha) g_3(q,\alpha)}{g_1(q+1,\alpha) g_1(q,\alpha)} = \frac{\alpha(2q+1+\alpha)(q-1)}{(q+1+\alpha)(2q+\alpha-2)(q+\alpha)}.$$

We note that $0 \leq \varepsilon_q \leq 1$. Furthermore, we calculate

$$\big((1-\varepsilon_q) P_{q-1}' - \varepsilon_q P_{q-2}'\big)^2 = (1-\varepsilon_q)^2 (P_{q-1}')^2 + \varepsilon_q^2 (P_{q-2}')^2 - 2\varepsilon_q (1-\varepsilon_q) P_{q-1}' P_{q-2}'$$

so that by integration, Cauchy-Schwarz, and $0 \le \varepsilon_q \le 1$:

$$\int_{-1}^{1} (1-x)^\alpha \left((1-\varepsilon_q)P'_{q-1}(x) - \varepsilon_q P'_{q-2}(x)\right)^2 \, dx \le \left((1-\varepsilon_q)I_{q-1} + \varepsilon_q I_{q-2}\right)^2 .$$

By the orthogonality properties of the Jacobi polynomials, we conclude in view of (3.8)

$$\begin{aligned}
I_{q+1}^2 &= \left(\frac{1}{g_1(q+1,\alpha)}\right)^2 \gamma_q^{(\alpha,0)} + \left(\frac{g_2(q+1,\alpha)}{g_1(q+1,\alpha)g_1(q,\alpha)}\right)^2 \gamma_{q-1}^{(\alpha,0)} \\
&\quad + \int_{-1}^{1} (1-x)^\alpha \left((1-\varepsilon_q)P'_{q-1}(x) - \varepsilon_q P'_{q-2}(x)\right)^2 \, dx \\
&\le \left(\frac{1}{g_1(q+1,\alpha)}\right)^2 \gamma_q^{(\alpha,0)} + \left(\frac{g_2(q+1,\alpha)}{g_1(q+1,\alpha)g_1(q,\alpha)}\right)^2 \gamma_{q-1}^{(\alpha,0)} + \left((1-\varepsilon_q)I_{q-1} + \varepsilon_q I_{q-2}\right)^2 .
\end{aligned}$$

We proceed now by an induction argument. We note that for any K the claimed formula is monotone increasing in q. Hence, we can estimate

$$\left((1-\varepsilon_q)I_{q-1} + \varepsilon_q I_{q-2}\right)^2 \le K(2(q-1)+\alpha+1)^2(q-1)\gamma_{q-1}^{(\alpha,0)}$$

and obtain by some tedious estimates for the other terms (see Appendix B) :

$$\begin{aligned}
I_{q+1}^2 &\le \left(\frac{1}{g_1(q+1,\alpha)}\right)^2 \gamma_q^{(\alpha,0)} + \left(\frac{g_2(q+1,\alpha)}{g_1(q+1,\alpha)g_1(q,\alpha)}\right)^2 \gamma_{q-1}^{(\alpha,0)} + K(2q+\alpha-1)^2(q-1)\gamma_{q-1}^{(\alpha,0)} \\
&\le K(2(q+1)+1+\alpha)^2(q+1)\gamma_{q+1}^{(\alpha,0)} \left[\frac{1}{K(q+1)} + \frac{1}{K(q+1)} + \frac{(2q+\alpha-1)^2(q-1)\gamma_{q-1}^{(\alpha,0)}}{(2q+1+\alpha)^2(q+1)\gamma_{q+1}^{(\alpha,0)}}\right] \\
&\le K(2(q+1)+1+\alpha)^2 q\gamma_{q+1}^{(\alpha,0)} \left[\frac{1}{K(q+1)} + \frac{1}{K(q+1)} + \frac{(2q+\alpha-1)^2(q-1)(2q+\alpha+3)}{(2q+3+\alpha)^2(q+1)(2q+\alpha-1)}\right] \\
&\le K(2(q+1)+1+\alpha)^2 q\gamma_{q+1}^{(\alpha,0)} \left[\frac{1}{K(q+1)} + \frac{1}{K(q+1)} + 1 - \frac{1}{(q+1)}\right].
\end{aligned}$$

The proof is complete by ensuring that $K \ge 2$ so that the expression in brackets is bounded by 1. □

The essential ingredient of the one-dimensional analysis in [4, Thm. 2.2], [15, Lemma 3.5], [12, Lemma 4.1] is the ability to relate the expansion coefficients $(u_n)_{n=0}^\infty$ of the Legendre expansion $u = \sum_n u_n P_n^{(0,0)}$ to the expansion coefficients $(b_n)_{n=0}^\infty$ of the Legendre expansion $u' = \sum_n b_n P_n^{(0,0)}$. This relation generalizes to the case of expansions in Jacobi polynomials. A first result in this direction is (see also [5, Lemma 2.1] and [2, Lemma 2.2]):

Lemma 3.4. *Let $\alpha \in \mathbb{N}_0$. Let $U \in C^1(-1,1)$ and let $(1-x)^\alpha U(x)$ as well as $(1-x)^{\alpha+1}U'(x)$ be integrable. Furthermore, assume $\lim_{x\to 1}(1-x)^{1+\alpha}U(x) = 0$ and $\lim_{x\to -1}(1+x)U(x) = 0$. Then the expansion coefficients*

$$\begin{aligned}
u_q &:= \int_{-1}^{1} (1-x)^\alpha U(x) P_q^{(\alpha,0)}(x)dx, \\
b_q &:= \int_{-1}^{1} (1-x)^\alpha U'(x) P_q^{(\alpha,0)}(x)dx.
\end{aligned}$$

satisfy the following connection formula for $q \ge 1$:

$$u_q = h_1(q,\alpha)b_{q+1} + h_2(q,\alpha)b_q + h_3(q,\alpha)b_{q-1}.$$

Proof. Follows from an integration by parts and the representation of antiderivatives of Jacobi polynomials in terms of Jacobi polynomials given in Lemma 3.2 (i). We refer to Appendix B for details. □

The following result is the generalization of [4, Thm. 2.2] on the H^1-stability of the L^2-projection. While [4, Thm. 2.2] studied the H^1-stability of the truncated Legendre expansion, we study here the effect of truncating a Jacobi expansion.

Lemma 3.5. *Let $\alpha \in \mathbb{N}_0$. Let u_q and b_q be defined as in Lemma 3.4. Then there exists a constant $C > 0$ independent of α and N such that for every $N \in \mathbb{N}$ we have*

$$\int_{-1}^{1} (1-x)^\alpha \left| \sum_{q=0}^{N} \frac{1}{\gamma_q^{(\alpha,0)}} u_q \big(P_q^{(\alpha,0)}\big)'(x) \right|^2 dx \leq CN \sum_{q=0}^{\infty} \frac{1}{\gamma_q^{(\alpha,0)}} |b_q|^2.$$

Proof. We abbreviate $P_q := P_q^{(\alpha,0)}$. We compute

$$\begin{aligned}
\sum_{q\geq N+1} \frac{1}{\gamma_q^{(\alpha,0)}} u_q P_q' &= \sum_{q\geq N+1} \frac{1}{\gamma_q^{(\alpha,0)}} \left[h_1(q,\alpha) b_{q+1} + h_2(q,\alpha) b_q + h_3(q,\alpha) b_{q-1}\right] P_q' \\
&= \sum_{q\geq N} b_q \frac{1}{\gamma_{q+1}^{(\alpha,0)}} h_3(q+1,\alpha) P_{q+1}' + \sum_{q\geq N+1} b_q \frac{1}{\gamma_q^{(\alpha,0)}} h_2(q,\alpha) P_q' \\
&\quad + \sum_{q\geq N+2} b_q \frac{1}{\gamma_{q-1}^{(\alpha,0)}} h_1(q-1,\alpha) P_{q-1}' \\
&= \sum_{q\geq N+2} b_q \left[\frac{1}{\gamma_{q-1}^{(\alpha,0)}} h_1(q-1,\alpha) P_{q-1}' + \frac{1}{\gamma_q^{(\alpha,0)}} h_2(q,\alpha) P_q' + \frac{1}{\gamma_{q+1}^{(\alpha,0)}} h_3(q+1,\alpha) P_{q+1}' \right] \\
&\quad + \frac{1}{\gamma_{N+1}^{(\alpha,0)}} h_3(N+1,\alpha) P_{N+1}' b_N + \sum_{q=N}^{N+1} \frac{1}{\gamma_q^{(\alpha,0)}} h_2(q,\alpha) P_q' b_q.
\end{aligned}$$

With Lemma 3.2 (iii) we therefore conclude

$$\sum_{q\geq N+1} \frac{1}{\gamma_q^{(\alpha,0)}} u_q P_q' = \sum_{q\geq N+2} b_q \frac{1}{\gamma_q^{(\alpha,0)}} P_q + \frac{1}{\gamma_{N+1}^{(\alpha,0)}} h_3(N+1,\alpha) P_{N+1}' b_N + \sum_{q=N}^{N+1} \frac{1}{\gamma_q^{(\alpha,0)}} h_2(q,\alpha) P_q' b_q.$$

Inserting the result of Lemma 3.3 gives

$$\begin{aligned}
\int_{-1}^{1} (1-x)^\alpha \left| \sum_{q\geq N+1} \frac{1}{\gamma_q^{(\alpha,0)}} u_q P_q' \right|^2 dx &\leq \sum_{q\geq N+2} \frac{3}{\gamma_q^{(\alpha,0)}} b_q^2 + CN \frac{h_3(N+1,\alpha)^2}{\gamma_N^{(\alpha,0)}} (N+\alpha)^2 b_N^2 \\
&\quad + CN \sum_{q=N}^{N+1} \frac{1}{\gamma_q^{(\alpha,0)}} (N+\alpha)^2 h_2(q,\alpha)^2 b_q^2 \\
&\leq CN \sum_{q\geq N} \frac{1}{\gamma_q^{(\alpha,0)}} b_q^2.
\end{aligned}$$

This allows us to conclude the argument since $\big(P_0^{(\alpha,0)}\big)'(x) = 0$.

□

Lemma 3.6. *For $\beta > -1$ and $U \in C^1(0,1) \cap C((0,1])$ there holds*

$$\int_0^1 x^\beta \,|U(x)|^2 \, dx \leq \left(\frac{2}{\beta+1}\right)^2 \int_0^1 x^{\beta+2} \,|U'(x)|^2 \, dx + \frac{1}{\beta+1} |U(1)|^2.$$

Proof. This variant of the Hardy inequality can be shown using [14, Thm. 330]. See Appendix B for details. □

Lemma 3.7. *Let $U \in C^1(-1,1)$ and assume*

$$\int_{-1}^{1} |U(x)|^2(1-x)^\alpha dx < \infty, \qquad \int_{-1}^{1} |U'(x)|^2(1-x)^\alpha dx < \infty.$$

Let u_q and b_q be defined as in Lemma 3.4. Then there exist constants C_1, $C_2 > 0$ independent of α and U such that

$$\sum_{q=1}^{\infty} \frac{1}{\gamma_q^{(\alpha,0)}}(q+\alpha)^2|u_q|^2 \le C_1 \sum_{q=0}^{\infty} \frac{1}{\gamma_q^{(\alpha,0)}}|b_q|^2 \le C_2 \int_{-1}^{1} \left|U'(x)\right|^2 (1-x)^\alpha dx.$$

Proof. The result follows from the relation between u_q and b_q given in Lemma 3.4 and from bounds for h_1, h_2, h_3. □

Next, we show a short lemma that will be useful in the proof of Lemma 3.9.

Lemma 3.8. *Let $\alpha \in \mathbb{N}_0$ and $q \ge 1$. Then there exists a constant $C > 0$ independent of q and α such that*

$$\alpha \sum_{j=q+\alpha}^{N} \frac{1}{j^2} \le 2\frac{\alpha}{q+\alpha} \qquad \forall N \ge q+\alpha.$$

Proof. The proof follows by the standard argument of majorizing the sum by an integral. □

While Lemma 3.4 shows that the coefficients u_q can be expressed as a short linear combination of the coefficients b_q (a maximum of 3 coefficients suffices), the converse is not so easy. The following lemma may be regarded as a weak converse of Lemma 3.4 since it allows us to bound the coefficients b_q in terms of the coefficients u_q and weighted sums of the coefficients b_q. This is the main result of this section and the key ingredient of the proof of Theorem 1.1 as it is responsible for the multiplicative structure of the bound in Theorem 1.1.

Lemma 3.9. *Assume the hypotheses of Lemma 3.4. Let $\alpha \in \mathbb{N}_0$. Let u_q and b_q be defined as in Lemma 3.4. Then for $q \ge 1$ there exists a constant $C > 0$ independent of q and α such that*

$$|b_{q-1}|^2 + |b_q|^2 \le C 2^{\alpha+1} \left(\sum_{j \ge q} \frac{1}{\gamma_j^{(\alpha,0)}} u_j^2 \right)^{1/2} \left(\sum_{j \ge q-1} \frac{1}{\gamma_j^{(\alpha,0)}} b_j^2 \right)^{1/2}.$$

Proof. We may assume that the right-hand side of the estimate in the lemma is finite. In view of the sign properties of h_1, h_2, h_3 and (3.6) we have

$$|h_1(q,\alpha)| + |h_2(q,\alpha)| = |h_3(q,\alpha)|. \tag{3.9}$$

We introduce the abbreviations

$$\alpha_q := \frac{h_2(q,\alpha)}{h_3(q,\alpha)} = \frac{\alpha(2q+\alpha+1)}{(2q+\alpha+2)(q+\alpha)},$$
$$\varepsilon_q := \alpha_q(1-\alpha_{q+1}) = \frac{\alpha(q+2)(2q+\alpha+1)}{(2q+\alpha+4)(q+1+\alpha)(q+\alpha)}.$$

By rearranging terms in Lemma 3.4 and using the triangle inequality we get

$$|h_3(q,\alpha)|\,|b_{q-1}| \le |u_q| + |h_2(q,\alpha)|\,|b_q| + |h_1(q,\alpha)|\,|b_{q+1}|.$$

We set

$$z_q := \frac{|u_q|}{|h_3(q,\alpha)|}$$

and by applying (3.9) we arrive at

$$|b_{q-1}| \le z_q + \alpha_q|b_q| + (1-\alpha_q)|b_{q+1}|. \tag{3.10}$$

Iterating (3.10) once gives

$$\begin{aligned}|b_{q-1}| &\le z_q + \alpha_q\big(z_{q+1} + \alpha_{q+1}|b_{q+1}| + (1-\alpha_{q+1})|b_{q+2}|\big) + (1-\alpha_q)|b_{q+1}| \\ &\le z_q + \alpha_q z_{q+1} + \big(1-\alpha_q(1-\alpha_{q+1})\big)|b_{q+1}| + \alpha_q(1-\alpha_{q+1})|b_{q+2}| \\ &= z_q + \alpha_q z_{q+1} + (1-\varepsilon_q)|b_{q+1}| + \varepsilon_q|b_{q+2}|.\end{aligned}$$

Squaring and Cauchy-Schwarz yields

$$\begin{aligned}b_{q-1}^2 \le{}& (z_q+\alpha_q z_{q+1})^2 + 2(z_q+\alpha_q z_{q+1})\big((1-\varepsilon_q)|b_{q+1}| + \varepsilon_q|b_{q+2}|\big) \\ &+ (1-\varepsilon_q)^2 b_{q+1}^2 + \varepsilon_q^2 b_{q+2}^2 + 2\varepsilon_q(1-\varepsilon_q)|b_{q+1}|\,|b_{q+2}| \\ \le{}& (z_q+\alpha_q z_{q+1})^2 + 2(z_q+\alpha_q z_{q+1})\big((1-\varepsilon_q)|b_{q+1}| + \varepsilon_q|b_{q+2}|\big) \\ &+ \big((1-\varepsilon_q)^2 + \varepsilon_q(1-\varepsilon_q)\big)b_{q+1}^2 + \big(\varepsilon_q^2 + \varepsilon_q(1-\varepsilon_q)\big)b_{q+2}^2.\end{aligned}$$

If we abbreviate for the first two addends

$$f_q := (z_q+\alpha_q z_{q+1})^2 + 2(z_q+\alpha_q z_{q+1})\big((1-\varepsilon_q)|b_{q+1}| + \varepsilon_q|b_{q+2}|\big) \tag{3.11}$$

we obtain

$$b_{q-1}^2 \le f_q + (1-\varepsilon_q)b_{q+1}^2 + \varepsilon_q b_{q+2}^2,$$

which we rewrite as

$$b_{q-1}^2 - b_{q+1}^2 \le f_q + \varepsilon_q\left(b_{q+2}^2 - b_{q+1}^2\right).$$

Next, we want to employ a telescoping sum. Since we assume that the sums in the right side of the statement of this lemma are finite, i.e.

$$\sum_j \frac{1}{\gamma_j^{(\alpha,0)}} u_j^2 < \infty, \qquad \sum_j \frac{1}{\gamma_j^{(\alpha,0)}} b_j^2 < \infty, \tag{3.12}$$

and since $\frac{1}{\gamma_j^{(\alpha,0)}} \lesssim (j+\alpha)2^{-\alpha}$ we have $\sqrt{q}\,|b_q| \to 0$ for $q\to\infty$. Hence, we can write

$$\begin{aligned}b_{q-1}^2 + b_q^2 &= \sum_{j=0}^{\infty} b_{q-1+2j}^2 - b_{q-1+2j+2}^2 + b_{q+2j}^2 - b_{q+2j+2}^2 \\ &\le \sum_{j=0}^{\infty} f_{q+2j} + \varepsilon_{q+2j}\left(b_{q+2+2j}^2 - b_{q+1+2j}^2\right) + f_{q+1+2j} + \varepsilon_{q+1+2j}\left(b_{q+3+2j}^2 - b_{q+2+2j}^2\right) \\ &= \sum_{j=0}^{\infty} f_{q+j} - \sum_{j=0}^{\infty}\varepsilon_{q+2j}b_{q+1+2j}^2 + \sum_{j=0}^{\infty}\left(\varepsilon_{q+2j} - \varepsilon_{q+2j+1}\right)b_{q+2+2j}^2 + \sum_{j=0}^{\infty}\varepsilon_{q+1+2j}b_{q+3+2j}^2 \\ &= \sum_{j=0}^{\infty} f_{q+j} - \varepsilon_q b_{q+1}^2 - \sum_{j=0}^{\infty}\varepsilon_{q+2+2j}b_{q+3+2j}^2 + \sum_{j=0}^{\infty}\left(\varepsilon_{q+2j} - \varepsilon_{q+2j+1}\right)b_{q+2+2j}^2 + \sum_{j=0}^{\infty}\varepsilon_{q+1+2j}b_{q+3+2j}^2 \\ &= \sum_{j=0}^{\infty} f_{q+j} - \varepsilon_q b_{q+1}^2 + \sum_{j=0}^{\infty}\left(\varepsilon_{q+1+2j} - \varepsilon_{q+2+2j}\right)b_{q+3+2j}^2 + \sum_{j=0}^{\infty}\left(\varepsilon_{q+2j} - \varepsilon_{q+2j+1}\right)b_{q+2+2j}^2 \\ &= \sum_{j=0}^{\infty} f_{q+j} - \varepsilon_q b_{q+1}^2 + \sum_{j=0}^{\infty}\left(\varepsilon_{q+j} - \varepsilon_{q+j+1}\right)b_{q+2+j}^2.\end{aligned}$$

We conclude, noting that $\varepsilon_q \geq 0$,

$$b_{q-1}^2 + b_q^2 \leq b_{q-1}^2 + b_q^2 + \varepsilon_q b_{q+1}^2 \leq F_q + S_{q+2}, \tag{3.13}$$

where

$$F_q := \sum_{j \geq q} f_j, \tag{3.14}$$

$$S_q := \sum_{j \geq q} \varepsilon'_j b_j^2 \quad \text{with} \quad \varepsilon'_j := |\varepsilon_{j-2} - \varepsilon_{j-1}|. \tag{3.15}$$

By positivity of ε'_j and f_j we have $S_{q+1} \leq S_q$ as well as $F_{q+1} \leq F_q$. Therefore, we get from (3.13) and the definition of S_q

$$\begin{aligned} S_q &= \varepsilon'_q b_q^2 + \varepsilon'_{q+1} b_{q+1}^2 + S_{q+2} \\ &\leq S_{q+2} + \max\{\varepsilon'_q, \varepsilon'_{q+1}\} S_{q+3} + \max\{\varepsilon'_q, \varepsilon'_{q+1}\} F_{q+1} \\ &\leq (1 + \max\{\varepsilon'_q, \varepsilon'_{q+1}\}) S_{q+2} + \max\{\varepsilon'_q, \varepsilon'_{q+1}\} F_q. \end{aligned}$$

Applying the notation

$$\varepsilon''_q := \max\{\varepsilon'_q, \varepsilon'_{q+1}\}$$

we have

$$S_q \leq (1 + \varepsilon''_q) S_{q+2} + \varepsilon''_q F_q. \tag{3.16}$$

Iterating (3.16) N times leads to

$$S_q \leq S_{q+2N+2} \prod_{j=0}^{N} (1 + \varepsilon''_{q+2j}) + \sum_{j=0}^{N} \varepsilon''_{q+2j} F_{q+2j} \prod_{i=0}^{j-1} (1 + \varepsilon''_{q+2i}). \tag{3.17}$$

A calculation shows

$$\varepsilon'_j \lesssim \frac{\alpha(\alpha + j)^3}{(\alpha + j)^5} = \frac{\alpha}{(\alpha + j)^2}. \tag{3.18}$$

From the definition of S_q in (3.15), (3.12), and (3.18) it follows that $\lim_{q \to \infty} S_q = 0$. Furthermore, we can bound the product uniformly in N:

$$\prod_{j=0}^{N} (1 + \varepsilon''_{q+2j}) = \exp\left(\sum_{j=0}^{N} \ln(1 + \varepsilon''_{q+2j}) \right) \leq \exp\left(\sum_{j=0}^{N} \varepsilon''_{q+2j} \right), \tag{3.19}$$

where in the last estimate we used the fact that $\ln(1 + x) \leq x$ for $x \geq 0$. From (3.18) we get

$$\sum_{j=0}^{N} \varepsilon''_{q+2j} \lesssim \sum_{j=0}^{N} \frac{\alpha}{(\alpha + q + 2j)^2} \lesssim \alpha \sum_{j=q}^{N} \frac{1}{(\alpha + j)^2} \lesssim \frac{\alpha}{\alpha + q} \quad \forall N \geq q, \tag{3.20}$$

where we have used Lemma 3.8 in the last step. Since $\frac{\alpha}{\alpha+q} < 1$, inserting (3.20) in (3.19) gives

$$\prod_{j=0}^{N} (1 + \varepsilon''_{q+2j}) \leq C. \tag{3.21}$$

Now, by passing to the limit $N \to \infty$ in (3.17), we obtain a closed form bound for S_q:

$$S_q \leq \sum_{j=0}^{\infty} \varepsilon''_{q+2j} F_{q+2j} \prod_{i=0}^{j-1} (1 + \varepsilon''_{q+2i}).$$

Applying (3.20), (3.21), (3.18) and the definition of F_q we can simplify

$$S_q \lesssim \sum_{j=0}^{\infty} \varepsilon''_{q+2j} F_{q+2j} \lesssim \sum_{j\geq q} \sum_{i \geq j} f_i \frac{\alpha}{(\alpha+j)^2} = \sum_{i \geq q} f_i \sum_{j=q}^{i} \frac{\alpha}{(\alpha+j)^2} \lesssim \frac{\alpha}{\alpha+q} F_q.$$

Inserting this estimate in (3.13) and using $\frac{\alpha}{\alpha+q+2} < 1$, we arrive at

$$b_{q-1}^2 + b_q^2 \lesssim F_q + \frac{\alpha}{\alpha+q+2} F_{q+2} \lesssim F_q + F_{q+2} \lesssim F_q. \tag{3.22}$$

We are left with estimating F_q. By the definition of F_q in (3.14) and the definition of f_q in (3.11) we have

$$F_q = \sum_{j \geq q} (z_j + \alpha_j z_{j+1})^2 + 2 \sum_{j \geq q} (z_j + \alpha_j z_{j+1}) \big((1-\varepsilon_j)|b_{j+1}| + \varepsilon_j |b_{j+2}| \big). \tag{3.23}$$

Now we estimate both sums separately starting with the first one:

$$\sum_{j \geq q} (z_j + \alpha_j z_{j+1})^2 \lesssim \sum_{j \geq q} z_j^2 + \underbrace{\alpha_j^2}_{\leq 1} z_{j+1}^2 \lesssim \sum_{j \geq q} z_j^2. \tag{3.24}$$

Next, we use the relation between u_q and b_q from Lemma 3.4. Furthermore, we note that $h_3(q,\alpha) \gtrsim 2^{-(\alpha+1)} \gamma_q^{(\alpha,0)}$. Hence, we obtain

$$\begin{aligned} z_q^2 = \frac{|u_q|^2}{|h_3(q,\alpha)|^2} &\lesssim 2^{\alpha+1} \frac{|u_q|}{\gamma_q^{(\alpha,0)}} \frac{|u_q|}{h_3(q,\alpha)} \\ &= 2^{\alpha+1} \frac{|u_q|}{\gamma_q^{(\alpha,0)}} \frac{1}{h_3(q,\alpha)} \, |h_1(q,\alpha) b_{q+1} + h_2(q,\alpha) b_q + h_3(q,\alpha) b_{q-1}| \\ &\lesssim 2^{\alpha+1} \frac{|u_q|}{\gamma_q^{(\alpha,0)}} \big((1-\alpha_q)|b_{q+1}| + \alpha_q |b_q| + |b_{q-1}| \big). \end{aligned}$$

Inserting this in the bound (3.24), we get by applying the Cauchy-Schwarz inequality for sums

$$\begin{aligned} \sum_{j \geq q} (z_j + \alpha_j z_{j+1})^2 &\lesssim 2^{\alpha+1} \sum_{j \geq q} \frac{1}{\gamma_j^{(\alpha,0)}} |u_j| \big((1-\alpha_j)|b_{j+1}| + \alpha_j |b_j| + |b_{j-1}| \big) \\ &\lesssim 2^{\alpha+1} \left(\sum_{j \geq q} \frac{1}{\gamma_j^{(\alpha,0)}} |u_j|^2 \right)^{1/2} \left(\sum_{j \geq q-1} \frac{1}{\gamma_j^{(\alpha,0)}} |b_j|^2 \right)^{1/2}. \end{aligned}$$

We continue by estimating the second sum in (3.23). Using again $z_q^2 \lesssim 2^{\alpha+1}|u_q|/\gamma_q^{(\alpha,0)}$ we get

$$
\begin{aligned}
&\sum_{j\geq q}(z_j+\alpha_j z_{j+1})\big((1-\varepsilon_j)|b_{j+1}|+\varepsilon_j|b_{j+2}|\big)\\
&\quad\lesssim 2^{\alpha+1}\sum_{j\geq q}\frac{1}{\gamma_j^{(\alpha,0)}}\big(|u_j|+\underbrace{\alpha_j}_{\leq 1}|u_{j+1}|\big)\big(\underbrace{(1-\varepsilon_j)}_{\leq 1}|b_{j+1}|+\underbrace{\varepsilon_j}_{\leq 1}|b_{j+2}|\big)\\
&\quad\lesssim 2^{\alpha+1}\sum_{j\geq q}\frac{1}{\gamma_j^{(\alpha,0)}}\left(|u_j|+|u_{j+1}|\right)\left(|b_{j+1}|+|b_{j+2}|\right)\\
&\quad\lesssim 2^{\alpha+1}\left(\sum_{j\geq q}\frac{1}{\gamma_j^{(\alpha,0)}}|u_j|^2\right)^{1/2}\left(\sum_{j\geq q}\frac{1}{\gamma_{j+1}^{(\alpha,0)}}|b_{j+1}|^2\right)^{1/2}\\
&\quad\lesssim 2^{\alpha+1}\left(\sum_{j\geq q}\frac{1}{\gamma_j^{(\alpha,0)}}|u_j|^2\right)^{1/2}\left(\sum_{j\geq q-1}\frac{1}{\gamma_j^{(\alpha,0)}}|b_j|^2\right)^{1/2}.
\end{aligned}
$$

In view of (3.22) the last two estimates conclude the proof. □

4 Expansions

To save space we will sometimes denote points in $\mathbb{R}^3$ by just one letter, i.e. $\xi = (\xi_1, \xi_2, \xi_3)$ for points in $\mathcal{T}^3$ and $\eta = (\eta_1, \eta_2, \eta_3)$ for points in $\mathcal{S}^3$.

4.1 Duffy-Transformation

We recall the definition of the reference triangle, tetrahedron, and the d-dimensional hyper cube in (1.1). The 3D-Duffy transformation $D : \mathcal{S}^3 \to \mathcal{T}^3$, [10], is given by

$$
D(\eta_1,\eta_2,\eta_3) := (\xi_1,\xi_2,\xi_3) = \left(\frac{(1+\eta_1)(1-\eta_2)(1-\eta_3)}{4}-1, \frac{(1+\eta_2)(1-\eta_3)}{2}-1, \eta_3\right) \tag{4.1}
$$

with inverse

$$
D^{-1}(\xi_1,\xi_2,\xi_3) = (\eta_1,\eta_2,\eta_3) = \left(-2\,\frac{1+\xi_1}{\xi_2+\xi_3}-1, 2\,\frac{1+\xi_2}{1-\xi_3}-1, \xi_3\right).
$$

Lemma 4.1. *The Duffy transformation is a bijection between the (open) cube $\mathcal{S}^3$ and the (open) tetrahedron $\mathcal{T}^3$. Additionally,*

$$
D'(\eta) := \left[\frac{\partial \xi_i}{\partial \eta_j}\right]_{i,j=1}^3 = \begin{bmatrix} \frac{1}{4}(1-\eta_2)(1-\eta_3) & 0 & 0 \\ -\frac{1}{4}(1+\eta_1)(1-\eta_3) & \frac{1}{2}(1-\eta_3) & 0 \\ -\frac{1}{4}(1+\eta_1)(1-\eta_2) & -\frac{1}{2}(1+\eta_2) & 1 \end{bmatrix}^\top,
$$

$$
\big(D'(\eta)\big)^{-1} = \frac{1}{(1-\eta_2)(1-\eta_3)}\begin{bmatrix} 4 & 2(1+\eta_1) & 2(1+\eta_1) \\ 0 & 2(1-\eta_2) & 1-\eta_2^2 \\ 0 & 0 & (1-\eta_2)(1-\eta_3) \end{bmatrix},
$$

$$
\det D' = \left(\frac{1-\eta_2}{2}\right)\left(\frac{1-\eta_3}{2}\right)^2.
$$

Proof. See, for example, [16]. □

Lemma 4.2. *Let D be the Duffy transformation and $\Gamma := \mathcal{T}^2 \times \{-1\}$. Then $D(\Gamma) = \Gamma$ and D is an isometric isomorphism with respect to the $L^2(\Gamma)$-norm, i.e., for sufficiently smooth functions u, we have for $\tilde{u} = u \circ D$ the relation $\|u\|_{L^2(\Gamma)} = \|\tilde{u}\|_{L^2(\Gamma)}$.*

Proof. Follows by inspection. □

4.2 Orthogonal polynomials on tetrahedra

In terms of Jacobi polynomials $P_n^{(\alpha,\beta)}$ we introduce orthogonal polynomials on the reference tetrahedron $\mathcal{T}^3$ often associated with the names of Dubiner or Koornwinder, [9, 16, 17].

Lemma 4.3. *Let p, q, $r \in \mathbb{N}_0$ and set $\psi_{p,q,r} := \tilde{\psi}_{p,q,r} \circ D^{-1}$, where $\tilde{\psi}_{p,q,r}$ is defined by*

$$\tilde{\psi}_{p,q,r}(\eta) := P_p^{(0,0)}(\eta_1) P_q^{(2p+1,0)}(\eta_2) P_r^{(2p+2q+2,0)}(\eta_3) \left(\frac{1-\eta_2}{2}\right)^p \left(\frac{1-\eta_3}{2}\right)^{p+q}.$$

Then the functions $\psi_{p,q,r}$ are $L^2(\mathcal{T}^3)$ orthogonal and satisfy $\psi_{p,q,r} \in \mathcal{P}_{p+q+r}(\mathcal{T}^3)$ and

$$\begin{aligned}\int_{\mathcal{T}^3} \psi_{p,q,r}(\xi)\psi_{p',q',r'}(\xi)\, d\xi &= \delta_{p,p'}\delta_{q,q'}\delta_{r,r'} \frac{2}{2p+1}\frac{2}{2p+2q+2}\frac{2}{2p+2q+2r+3} \\ &= \delta_{p,p'}\delta_{q,q'}\delta_{r,r'} \gamma_p^{(0,0)} \frac{\gamma_q^{(2p+1,0)}}{2^{2p+1}} \frac{\gamma_r^{(2p+2q+2,0)}}{2^{2p+2q+2}}.\end{aligned}$$

Proof. The proof can be found in Appendix B. □

4.3 Expansion in terms of $\psi_{p,q,r}$

Since $(\psi_{p,q,r})_{p,q,r\in\mathbb{N}_0}$ form a set of orthogonal polynomials we have that any $u \in L^2(\mathcal{T}^3)$ can be expanded as

$$u = \sum_{p,q,r=0}^{\infty} \frac{\langle \psi_{p,q,r}, u\rangle_{L^2(\mathcal{T}^3)}}{\|\psi_{p,q,r}\|^2_{L^2(\mathcal{T}^3)}} \psi_{p,q,r} = \sum_{p,q,r=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{2^{2p+1}}{\gamma_q^{(2p+1,0)}} \frac{2^{2p+2q+2}}{\gamma_r^{(2p+2q+2,0)}} u_{p,q,r}\psi_{p,q,r}, \tag{4.2}$$

where

$$u_{p,q,r} := \langle \psi_{p,q,r}, u\rangle_{L^2(\mathcal{T}^3)}. \tag{4.3}$$

A basic ingredient of the proofs of Theorems 1.1 and 1.3 is the reduction of the analysis to one-dimensional settings, for which we have provided the necessary results in Section 3. The variable that will play a special role is η_3 as will become apparent in Definition 4.4 below. For a function u defined in $\mathcal{T}^3$ we introduce the transformed function $\tilde{u} := u \circ D$ and get

$$\begin{aligned} u_{p,q,r} &= \int_{\mathcal{T}^3} u(\xi)\psi_{p,q,r}(\xi)\, d\xi = \int_{\mathcal{S}^3} \tilde{u}(\eta)\tilde{\psi}_{p,q,r}(\eta) \left(\frac{1-\eta_2}{2}\right)\left(\frac{1-\eta_3}{2}\right)^2 d\eta \\ &= \int_{\mathcal{S}^3} \tilde{u}(\eta) P_p^{(0,0)}(\eta_1) P_q^{(2p+1,0)}(\eta_2) P_r^{(2p+2q+2,0)}(\eta_3) \left(\frac{1-\eta_2}{2}\right)^{p+1}\left(\frac{1-\eta_3}{2}\right)^{p+q+2} d\eta. \end{aligned} \tag{4.4}$$

Definition 4.4. *Let $p, q \in \mathbb{N}_0$ and $u \in L^2(\mathcal{T}^3)$. We define the functions $U_{p,q} : \mathbb{R} \to \mathbb{R}$ and $\widetilde{U}_{p,q} : \mathbb{R} \to \mathbb{R}$ as well as the coefficients $\tilde{u}_{p,q,r}$ and $\tilde{u}'_{p,q,r}$ by*

$$U_{p,q}(\eta_3) := \int_{-1}^{1}\int_{-1}^{1} \tilde{u}(\eta) P_p^{(0,0)}(\eta_1) P_q^{(2p+1,0)}(\eta_2) \left(\frac{1-\eta_2}{2}\right)^{p+1} d\eta_1 d\eta_2, \tag{4.5}$$

$$\widetilde{U}_{p,q}(\eta_3) := \frac{U_{p,q}(\eta_3)}{(1-\eta_3)^{p+q}}, \tag{4.6}$$

$$\tilde{u}_{p,q,r} := \int_{-1}^{1} (1-\eta_3)^{2p+2q+2} \widetilde{U}_{p,q}(\eta_3) P_r^{(2p+2q+2,0)}(\eta_3)\, d\eta_3, \tag{4.7}$$

$$\tilde{u}'_{p,q,r} := \int_{-1}^{1} (1-\eta_3)^{2p+2q+2} \widetilde{U}'_{p,q}(\eta_3) P_r^{(2p+2q+2,0)}(\eta_3)\, d\eta_3. \tag{4.8}$$

With this notation, we have by comparing (4.4) with (4.7)

$$u_{p,q,r} = \frac{1}{2^{p+q+2}} \int_{-1}^{1} (1-\eta_3)^{2p+2q+2} \widetilde{U}_{p,q}(\eta_3) P_r^{(2p+2q+2,0)}(\eta_3)\, d\eta_3 = \frac{1}{2^{p+q+2}} \tilde{u}_{p,q,r}. \tag{4.9}$$

Since for sufficiently smooth functions u the transformed function $\tilde{u}$ is constant on $\eta_3 = 1$, orthogonalities of the Jacobi polynomials give us

$$U_{p,q}(1) = 0 \qquad \text{for } (p,q) \neq (0,0) \tag{4.10}$$

4.4 Properties of the univariate functions $U_{p,q}$ and $\widetilde{U}_{p,q}$

We start with some preliminary considerations regarding estimates for partial derivatives of the transformed function $\tilde{u}$. We have

$$\partial_{\eta_1} \tilde{u}(\eta) = \frac{(1-\eta_2)(1-\eta_3)}{4} (\partial_1 u) \circ D(\eta), \tag{4.11}$$

$$\partial_{\eta_2} \tilde{u}(\eta) = -\frac{(1+\eta_1)(1-\eta_3)}{4} (\partial_1 u) \circ D(\eta) + \frac{(1-\eta_3)}{2} (\partial_2 u) \circ D(\eta), \tag{4.12}$$

$$\partial_{\eta_3} \tilde{u}(\eta) = -\frac{(1+\eta_1)(1-\eta_2)}{4} (\partial_1 u) \circ D(\eta) - \frac{(1+\eta_2)}{2} (\partial_2 u) \circ D(\eta) + (\partial_3 u) \circ D(\eta), \tag{4.13}$$

where ∂_i is the partial derivative with respect to the i-th argument. In particular, we get

$$\begin{aligned}\int_{\mathcal{S}^3} |\partial_{\eta_1} \tilde{u}(\eta)|^2 \left(\frac{2}{1-\eta_2}\right) d\eta &= \int_{\mathcal{S}^3} |(\partial_1 u) \circ D(\eta)|^2 \left(\frac{1-\eta_2}{2}\right)\left(\frac{1-\eta_3}{2}\right)^2 d\eta \\ &= \|\partial_{\xi_1} u\|^2_{L^2(\mathcal{T}^3)} \leq \|\nabla u\|^2_{L^2(\mathcal{T}^3)},\end{aligned} \tag{4.14}$$

$$\begin{aligned}\int_{\mathcal{S}^3} |\partial_{\eta_2} \tilde{u}(\eta)|^2 \left(\frac{1-\eta_2}{2}\right) d\eta &\lesssim \int_{\mathcal{S}^3} |(\partial_1 u) \circ D(\eta)|^2 \left(\frac{1+\eta_1}{2}\right)^2 \left(\frac{1-\eta_2}{2}\right)\left(\frac{1-\eta_3}{2}\right)^2 d\eta + \|\partial_{\xi_2} u\|^2_{L^2(\mathcal{T}^3)} \\ &\lesssim \|\partial_{\xi_1} u\|^2_{L^2(\mathcal{T}^3)} + \|\partial_{\xi_2} u\|^2_{L^2(\mathcal{T}^3)} \lesssim \|\nabla u\|^2_{L^2(\mathcal{T}^3)}.\end{aligned} \tag{4.15}$$

These estimates will be useful to prove the following lemmas.

Lemma 4.5 (properties of $U_{p,q}$)**.** *Let $u \in H^1(T)$ and $U_{p,q}$ be defined in Definition 4.4. Then there exists a constant $C > 0$ independent of u such that*

$$\sum_{p,q=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{1}{\gamma_q^{(2p+1,0)}} \int_{-1}^{1} |U_{p,q}(\eta_3)|^2 \left(\frac{1-\eta_3}{2}\right)^2 d\eta_3 = \|u\|^2_{L^2(\mathcal{T}^3)}, \tag{4.16}$$

$$\sum_{p,q=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{1}{\gamma_q^{(2p+1,0)}} \int_{-1}^{1} |U'_{p,q}(\eta_3)|^2 \left(\frac{1-\eta_3}{2}\right)^2 d\eta_3 \leq C\|\nabla u\|^2_{L^2(\mathcal{T}^3)}, \tag{4.17}$$

$$\sum_{p,q\geq 1}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{1}{\gamma_q^{(2p+1,0)}} (p+q)^2 \int_{-1}^{1} |U_{p,q}(\eta_3)|^2\, d\eta_3 \leq C\|\nabla u\|^2_{L^2(\mathcal{T}^3)}. \tag{4.18}$$

Furthermore, we have for $\Gamma = \mathcal{T}^2 \times \{-1\}$

$$\sum_{p=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{1}{\gamma_q^{(2p+1,0)}} |U_{p,q}(-1)|^2 = \|u\|^2_{L^2(\Gamma)}. \tag{4.19}$$

Proof. We first prove (4.16) and (4.17). The fact that $P_p^{(0,0)}(\eta_1)P_q^{(2p+1,0)}(\eta_2)\left(\frac{1-\eta_2}{2}\right)^p$ are orthogonal polynomials in a weighted L^2-space on $\mathcal{S}^2$ and the definition of $U_{p,q}$ imply (for fixed η_3) the representation

$$\tilde{u}(\eta) = \sum_{p,q=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{2^{2p+1}}{\gamma_q^{(2p+1,0)}} U_{p,q}(\eta_3) P_p^{(0,0)}(\eta_1) P_q^{(2p+1,0)}(\eta_2) \left(\frac{1-\eta_2}{2}\right)^p,$$

which in turn gives

$$\int_{\mathcal{S}^2} |\tilde{u}(\eta)|^2 \left(\frac{1-\eta_2}{2}\right) d\eta_1 d\eta_2 = \sum_{p,q=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{1}{\gamma_q^{(2p+1,0)}} \left|U_{p,q}(\eta_3)\right|^2. \tag{4.20}$$

Since $\det D' = \left(\frac{1-\eta_2}{2}\right)\left(\frac{1-\eta_3}{2}\right)^2$, multiplication with $\left(\frac{1-\eta_3}{2}\right)^2$ and integration in η_3 gives (4.16).

Similar to the representation of $\tilde{u}$ above, we get for $\partial_{\eta_3}\tilde{u}$

$$\partial_{\eta_3}\tilde{u}(\eta) = \sum_{p,q=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{2^{2p+1}}{\gamma_q^{(2p+1,0)}} U'_{p,q}(\eta_3) P_p^{(0,0)}(\eta_1) P_q^{(2p+1,0)}(\eta_2) \left(\frac{1-\eta_2}{2}\right)^p.$$

Reasoning as in the case of (4.16) yields

$$\sum_{p,q=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{1}{\gamma_q^{(2p+1,0)}} \int_{-1}^{1} \left|U'_{p,q}(\eta_3)\right|^2 \left(\frac{1-\eta_3}{2}\right)^2 d\eta_3 \lesssim \|\nabla u\|^2_{L^2(\mathcal{T}^3)},$$

which immediately leads to (4.17).

We now turn to the proof of (4.18). By definition, we have

$$U_{p,q}(\eta_3) = \frac{1}{2^{p+1}} \int_{\mathcal{S}^2} \tilde{u}(\eta) P_p^{(0,0)}(\eta_1) P_q^{(2p+1,0)}(\eta_2)(1-\eta_2)^{p+1} d\eta_1 d\eta_2. \tag{4.21}$$

We consider the integration in η_2. Integration by parts then yields

$$\begin{aligned}
&\int_{-1}^{1} \tilde{u}(\eta) P_q^{(2p+1,0)}(\eta_2)(1-\eta_2)^{p+1} d\eta_2 \\
&\quad = \left(\tilde{u}(\eta)(1-\eta_2)^{p+1}\widehat{P}_{q+1}^{(2p+1,0)}(\eta_2)\right)\Big|_{-1}^{1} - \int_{-1}^{1} \partial_{\eta_2}\left(\tilde{u}(\eta)(1-\eta_2)^{p+1}\right)\widehat{P}_{q+1}^{(2p+1,0)}(\eta_2) d\eta_2 \\
&\quad = -\int_{-1}^{1} \partial_{\eta_2}\tilde{u}(\eta)(1-\eta_2)^{p+1}\widehat{P}_{q+1}^{(2p+1,0)}(\eta_2) + (p+1)\tilde{u}(\eta)(1-\eta_2)^p \widehat{P}_{q+1}^{(2p+1,0)}(\eta_2) d\eta_2.
\end{aligned}$$

Hence, we obtain by inserting into (4.21)

$$\begin{aligned}
U_{p,q}(\eta_3) = &-\frac{1}{2^{p+1}} \int_{\mathcal{S}^2} \partial_{\eta_2}\tilde{u}(\eta) P_p^{(0,0)}(\eta_1)(1-\eta_2)^{p+1}\widehat{P}_{q+1}^{(2p+1,0)}(\eta_2) d\eta_1 d\eta_2 \\
&- \frac{p+1}{2^{p+1}} \int_{\mathcal{S}^2} \tilde{u}(\eta) P_p^{(0,0)}(\eta_1)(1-\eta_2)^p \widehat{P}_{q+1}^{(2p+1,0)}(\eta_2) d\eta_1 d\eta_2 \\
= &-\frac{1}{2^{p+1}} \int_{\mathcal{S}^2} \partial_{\eta_2}\tilde{u}(\eta) P_p^{(0,0)}(\eta_1)(1-\eta_2)^{p+1}\widehat{P}_{q+1}^{(2p+1,0)}(\eta_2) d\eta_1 d\eta_2 \\
&+ \frac{p+1}{2^{p+1}} \int_{\mathcal{S}^2} (\partial_{\eta_1}\tilde{u})(\eta) \widehat{P}_{p+1}^{(0,0)}(\eta_1)(1-\eta_2)^p \widehat{P}_{q+1}^{(2p+1,0)}(\eta_2) d\eta_1 d\eta_2,
\end{aligned}$$

where in the last equation we used integration by parts in η_1 and the fact that $\int_{-1}^{1} P_p^{(0,0)}(t)dt = 0$ for $p \geq 1$. With the abbreviation $g_i := g_i(q+1, 2p+1)$, $i = 1, 2, 3$ we have by Lemma 3.2 (ii) for $p, q \geq 1$ the following relationships:

$$\widehat{P}_q^{(2p+1,0)}(\eta_2) = g_1 P_{q+1}^{(2p+1,0)}(\eta_2) + g_2 P_q^{(2p+1,0)}(\eta_2) + g_3 P_{q-1}^{(2p+1,0)}(\eta_2), \tag{4.22}$$

$$\widehat{P}_p^{(0,0)}(\eta_1) = \frac{1}{2p+1}\left(P_{p+1}^{(0,0)}(\eta_1) - P_{p-1}^{(0,0)}(\eta_1)\right). \tag{4.23}$$

Furthermore, we introduce two abbreviations

$$z_{p,q}(\eta_3) := \int_{\mathcal{S}^2} (\partial_{\eta_2}\tilde{u})(\eta) P_p^{(0,0)}(\eta_1)\left(\frac{1-\eta_2}{2}\right)^{p+1} P_q^{(2p+1,0)}(\eta_2) d\eta_1 d\eta_2 \tag{4.24}$$

$$\tilde{z}_{p,q}(\eta_3) := \int_{\mathcal{S}^2} ((\partial_1 u)\circ D)(\eta) P_p^{(0,0)}(\eta_1)\left(\frac{1-\eta_2}{2}\right)^{p+1} P_q^{(2p+1,0)}(\eta_2) d\eta_1 d\eta_2. \tag{4.25}$$

Since we have (4.11), using (4.22), (4.23), (4.24) and (4.25) we get

$$\begin{aligned} U_{p,q}(\eta_3) = &-\big(g_1 z_{p,q+1}(\eta_3) + g_2 z_{p,q}(\eta_3) + g_3 z_{p,q-1}(\eta_3)\big) \\ &+ \left(\frac{p+1}{2p+1}\right)\left(\frac{1-\eta_3}{4}\right)\Big[g_1\big(\tilde{z}_{p+1,q+1}(\eta_3) - \tilde{z}_{p-1,q+1}(\eta_3)\big) \\ &+ g_2\big(\tilde{z}_{p+1,q}(\eta_3) - \tilde{z}_{p-1,q}(\eta_3)\big) + g_3\big(\tilde{z}_{p+1,q-1}(\eta_3) - \tilde{z}_{p-1,q-1}(\eta_3)\big)\Big]. \end{aligned}$$

We use $g_1, g_2, g_3 \lesssim \frac{1}{p+q}$ to arrive at

$$(p+q)|U_{p,q}(\eta_3)| \lesssim \sum_{j=0}^{2}\left(\frac{1-\eta_3}{2}\right)\Big(|\tilde{z}_{p+1,q+1-j}(\eta_3)| + |\tilde{z}_{p-1,q+1-j}(\eta_3)|\Big) + |z_{p,q+1-j}(\eta_3)|. \tag{4.26}$$

To estimate the terms on the right side we note that the abbreviations $z_{p,q}$ and $\tilde{z}_{p,q}$ lead us to the representations

$$\partial_{\eta_2}\tilde{u}(\eta) = \sum_{p,q=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{2^{2p+1}}{\gamma_q^{(2p+1,0)}} z_{p,q}(\eta_3) P_p^{(0,0)}(\eta_1)\left(\frac{1-\eta_2}{2}\right)^p P_q^{(2p+1,0)}(\eta_2)$$

$$((\partial_1 u)\circ D)(\eta) = \sum_{p,q=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{2^{2p+1}}{\gamma_q^{(2p+1,0)}} \tilde{z}_{p,q}(\eta_3) P_p^{(0,0)}(\eta_1)\left(\frac{1-\eta_2}{2}\right)^p P_q^{(2p+1,0)}(\eta_2).$$

Since the polynomials $P_p^{(0,0)}(\eta_1)\left(\frac{1-\eta_2}{2}\right)^p P_q^{(2p+1,0)}(\eta_2)$ are orthogonal polynomials on the reference triangle $\mathcal{T}^2$ we have

$$\int_{\mathcal{S}^2} |\partial_{\eta_2}\tilde{u}(\eta)|^2 \left(\frac{1-\eta_2}{2}\right) d\eta_1 d\eta_2 = \sum_{p,q=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{1}{\gamma_q^{(2p+1,0)}} |z_{p,q}(\eta_3)|^2 \tag{4.27}$$

$$\int_{\mathcal{S}^2} |((\partial_1\tilde{u})\circ D)(\eta)|^2 \left(\frac{1-\eta_2}{2}\right) d\eta_1 d\eta_2 = \sum_{p,q=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{1}{\gamma_q^{(2p+1,0)}} |\tilde{z}_{p,q}(\eta_3)|^2. \tag{4.28}$$

Formula (4.27) together with an integration in η_3 and an application of (4.15) gives

$$\sum_{p,q=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{1}{\gamma_q^{(2p+1,0)}} \int_{-1}^{1} |z_{p,q}(\eta_3)|^2 d\eta_3 = \int_{\mathcal{S}^3} |\partial_{\eta_2}\tilde{u}(\eta)|^2 \left(\frac{1-\eta_2}{2}\right) d\eta \lesssim \|\nabla u\|^2_{L^2(\mathcal{T}^3)}.$$

From the representation (4.28) we get by a multiplication with $\left(\frac{1-\eta_3}{2}\right)^2$, an integration in η_3 and the use of (4.14) that

$$\sum_{p,q=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{1}{\gamma_q^{(2p+1,0)}} \int_{-1}^{1} |\tilde{z}_{p,q}(\eta_3)|^2 \left(\frac{1-\eta_3}{2}\right)^2 d\eta_3$$

$$= \int_{\mathcal{S}^3} |((\partial_1 u) \circ D)(\eta)|^2 \left(\frac{1-\eta_2}{2}\right) \left(\frac{1-\eta_3}{2}\right)^2 d\eta \lesssim \|\nabla u\|^2_{L^2(\mathcal{T}^3)}.$$

The last two results in combination with (4.26) conclude the argument.

We finally prove (4.19). For the estimate (4.19), we use (4.20) with $\eta_3 = -1$. Noting Lemma 4.2 we have

$$\|u\|^2_{L^2(\Gamma)} = \|\tilde{u}\|^2_{L^2(\Gamma)} = \int_{\mathcal{S}^2} |\tilde{u}(\eta_1, \eta_2, -1)|^2 \, d\eta_1 d\eta_2$$

and therefore the result follows. □

The estimate (4.18) does not provide a bound for the special cases $p = 0$ or $q = 0$. The treatment of these two special cases is the purpose of the following lemma.

Lemma 4.6. *Assume the same hypotheses as in Lemma 4.5. Then there exists a constant $C > 0$ independent of u such that*

$$\sum_{q=0}^{\infty} \frac{1}{\gamma_0^{(0,0)}} \frac{1}{\gamma_q^{(1,0)}} q^2 \int_{-1}^{1} |U_{0,q}(\eta_3)|^2 \, d\eta_3 \le C \|\nabla u\|^2_{L^2(\mathcal{T}^3)} \tag{4.29}$$

$$\sum_{p=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{1}{\gamma_0^{(2p+1,0)}} p^2 \int_{-1}^{1} |U_{p,0}(\eta_3)|^2 \, d\eta_3 \le C \|\nabla u\|^2_{L^2(\mathcal{T}^3)}. \tag{4.30}$$

Proof. The proof of (4.29) start with the observation

$$U_{0,q} = \int_{-1}^{1} \left(\int_{-1}^{1} \tilde{u}(\eta) d\eta_1 \right) P_q^{(1,0)}(\eta_2) \left(\frac{1-\eta_2}{2}\right) d\eta_2.$$

We expand for fixed η_3 the function $\eta_2 \mapsto \int_{-1}^{1} \tilde{u}(\eta_1, \eta_2, \eta_3) d\eta_1$ as well as its derivative in terms of $P_q^{(1,0)}$. Lemma 3.7 then yields

$$\sum_{q=0}^{\infty} \frac{1}{\gamma_0^{(0,0)}} \frac{1}{\gamma_q^{(1,0)}} q^2 \int_{-1}^{1} |U_{0,q}(\eta_3)|^2 \, d\eta_3 \le 2 \sum_{q=1}^{\infty} \frac{1}{\gamma_q^{(1,0)}} (q+1)^2 \int_{-1}^{1} |U_{0,q}(\eta_3)|^2 \, d\eta_3$$

$$\lesssim \int_{-1}^{1} \int_{-1}^{1} \left| \int_{-1}^{1} \partial_{\eta_2} \tilde{u}(\eta) d\eta_1 \right|^2 (1-\eta_2) d\eta_2 d\eta_3 \lesssim \int_{\mathcal{S}^3} |\partial_{\eta_2} \tilde{u}(\eta)|^2 \left(\frac{1-\eta_2}{2}\right) d\eta \lesssim \|\nabla u\|^2_{L^2(\mathcal{T}^3)},$$

where we appealed to (4.15) in the last estimate.

Analogously we deal with (4.30), where we expand the map $\eta_1 \mapsto \int_{-1}^{1} \tilde{u}(\eta_1, \eta_2, \eta_3) \left(\frac{1-\eta_2}{2}\right)^{p+1} d\eta_2$ again for fixed η_3 in terms of $P_p^{(0,0)}$ and conclude with Lemma 3.7 and (4.14). □

The estimates for the functions $U_{p,q}$ imply easily corresponding bounds for the functions $\widetilde{U}_{p,q}$:

Lemma 4.7 (properties of $\widetilde{U}_{p,q}$)**.** *Let $u \in H^1(T)$ and $\widetilde{U}_{p,q}$ be defined in Definition 4.4. Then there exists a constant $C > 0$ independent of u such that*

$$\frac{1}{4} \sum_{p,q=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{1}{\gamma_q^{(2p+1,0)}} \int_{-1}^{1} (1-\eta_3)^{2p+2q+2} \left| \widetilde{U}_{p,q}(\eta_3) \right|^2 d\eta_3 = \|u\|^2_{L^2(\mathcal{T}^3)}, \tag{4.31}$$

$$\frac{1}{4} \sum_{p,q=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{1}{\gamma_q^{(2p+1,0)}} \int_{-1}^{1} (1-\eta_3)^{2p+2q+2} \left| \widetilde{U}'_{p,q}(\eta_3) \right|^2 d\eta_3 \le C \|\nabla u\|^2_{L^2(\mathcal{T}^3)}. \tag{4.32}$$

Proof. We have $\widetilde{U}_{p,q}(\eta_3) = (1-\eta_3)^{-(p+q)} U_{p,q}(\eta_3)$ and therefore $\widetilde{U}'_{p,q}(\eta_3) = (1-\eta_3)^{-(p+q)} U'_{p,q}(\eta_3) + (p+q)(1-\eta_3)^{-(p+q+1)} U_{p,q}(\eta_3)$. Hence,

$$\frac{1}{4}\int_{-1}^{1} (1-\eta_3)^{2p+2q+2} \left|\widetilde{U}_{p,q}(\eta_3)\right|^2 d\eta_3 = \int_{-1}^{1} \left(\frac{1-\eta_3}{2}\right)^2 |U_{p,q}(\eta_3)|^2 \, d\eta_3$$
$$\frac{1}{4}\int_{-1}^{1} (1-\eta_3)^{2p+2q+2} \left|\widetilde{U}'_{p,q}(\eta_3)\right|^2 d\eta_3 \lesssim \int_{-1}^{1} \left(\frac{1-\eta_3}{2}\right)^2 \left|U'_{p,q}(\eta_3)\right|^2 d\eta_3 + (p+q)^2 \int_{-1}^{1} |U_{p,q}(\eta_3)|^2 \, d\eta_3.$$

Using the results of Lemma 4.5 concludes the argument. □

These results allow us to get bounds for weighted sums of the coefficients $\tilde{u}_{p,q,r}$ and $\tilde{u}'_{p,q,r}$ given in Definition 4.4:

Corollary 4.8. *Assume the hypotheses of Lemma 4.7 and let $\tilde{u}_{p,q,r}$, $\tilde{u}'_{p,q,r}$ be given by Definition 4.4. Then there exist constants C independent of u such that*

$$\sum_{p,q,r=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{1}{\gamma_q^{(2p+1,0)}} \frac{1}{\gamma_r^{(2p+2q+2,0)}} |\tilde{u}_{p,q,r}|^2 \le C\|u\|^2_{L^2(\mathcal{T}^3)}, \tag{4.33}$$
$$\sum_{p,q,r=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{1}{\gamma_q^{(2p+1,0)}} \frac{1}{\gamma_r^{(2p+2q+2,0)}} |\tilde{u}'_{p,q,r}|^2 \le C\|\nabla u\|^2_{L^2(\mathcal{T}^3)}. \tag{4.34}$$

Proof. Since the polynomials $P_r^{(2p+2q+2,0)}$ are orthogonal polynomials in a weighted L^2-space, expanding $\widetilde{U}_{p,q}$ yields the representation

$$\widetilde{U}_{p,q}(\eta_3) = \sum_{r=0}^{\infty} \frac{1}{\gamma_r^{(2p+2q+2,0)}} \tilde{u}_{p,q,r} P_r^{(2p+2q+2,0)}(\eta_3).$$

Furthermore, we have

$$\int_{-1}^{1} (1-\eta_3)^{2p+2q+2} |\widetilde{U}_{p,q}(\eta_3)|^2 d\eta_3 = \sum_{r=0}^{\infty} \frac{1}{\gamma_r^{(2p+2q+2,0)}} |\tilde{u}_{p,q,r}|^2.$$

The statement (4.33) now follows directly from (4.31) of Lemma 4.7. Analogously, we deal with (4.34), where we expand $\widetilde{U}'_{p,q}$ and conclude with (4.32) of Lemma 4.7. □

4.5 Connections between $\tilde{u}_{p,q,r}$ and $\tilde{u}'_{p,q,r}$

To obtain the multiplicative structure in the estimation of Theorem 1.1 one key ingredient will be connections between $\tilde{u}_{p,q,r}$ and $\tilde{u}'_{p,q,r}$. This connection is essentially a one-dimensional effect and follows from Lemma 3.4:

Corollary 4.9. *Let $\tilde{u}_{p,q,r}$ and $\tilde{u}'_{p,q,r}$ be as in Definition 4.4 and h_1, h_2 and h_3 be given by (3.3). Then for $r \ge 1$ and p, $q \ge 0$ there holds*

$$\tilde{u}_{p,q,r} = h_1(r, 2p+2q+2)\tilde{u}'_{p,q,r+1} + h_2(r, 2p+2q+2)\tilde{u}'_{p,q,r} + h_3(r, 2p+2q+2)\tilde{u}'_{p,q,r-1}.$$

Proof. By density we may assume that $u \in C^\infty(\overline{\mathcal{T}^3})$. Lemma 3.4 then implies the result. For a detailed version of this proof see Appendix B. □

5 Trace stability

First, we will establish a representation for the transformed function $\tilde{u} = u \circ D$ on the face $\Gamma := \mathcal{T}^2 \times \{-1\}$. Using (A.4) and (A.3) we get

$$P_r^{(2p+2q+2,0)}(-1) = (-1)^r P_r^{(0,2p+2q+2)}(1) = (-1)^r.$$

Therefore we have for $\xi \in \Gamma$

$$\psi_{p,q,r}(\xi) = \tilde{\psi}_{p,q,r}\left(D^{-1}(\xi_1,\xi_2,-1)\right) = (-1)^r \left(\frac{1-\eta_2}{2}\right)^p P_p^{(0,0)}(\eta_1) P_q^{(2p+1,0)}(\eta_2).$$

Applying the expansion of u in (4.2) we arrive at

$$\begin{aligned}\tilde{u}(\eta_1,\eta_2,-1) &= u(\xi_1,\xi_2,-1)\\ &= \sum_{p,q,r=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{2^{2p+1}}{\gamma_q^{(2p+1,0)}} P_p^{(0,0)}(\eta_1) \left(\frac{1-\eta_2}{2}\right)^p P_q^{(2p+1,0)}(\eta_2)(-1)^r u_{p,q,r} \frac{2^{2p+2q+2}}{\gamma_r^{(2p+2q+2,0)}}. \quad (5.1)\end{aligned}$$

In view of Lemma 4.2 as well as the expansion (4.2), we obtain for the $L^2(\Gamma)$-norm

$$\|u\|^2_{L^2(\Gamma)} = \|\tilde{u}\|^2_{L^2(\Gamma)} = \sum_{p,q=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{1}{\gamma_q^{(2p+1,0)}} \left| \sum_{r=0}^{\infty} (-1)^r u_{p,q,r} \frac{2^{2p+2q+2}}{\gamma_r^{(2p+2q+2,0)}} \right|^2. \quad (5.2)$$

The next lemma will show that the infinite sum over r in (5.2) can be expressed as a finite sum.

Lemma 5.1. *Let $N \geq 1$. Then there holds*

$$\begin{aligned}\sum_{r=N}^{\infty} (-1)^r \frac{2^{2p+2q+2}}{\gamma_r^{(2p+2q+2,0)}} u_{p,q,r} &= (-1)^N h_2(N, 2p+2q+2) \frac{2^{p+q}}{\gamma_N^{(2p+2q+2,0)}} \tilde{u}'_{p,q,N}\\ &\quad + \sum_{r=N-1}^{N} (-1)^{r+1} h_3(r+1, 2p+2q+2) \frac{2^{p+q}}{\gamma_{r+1}^{(2p+2q+2,0)}} \tilde{u}'_{p,q,r}.\end{aligned}$$

Proof. We abbreviate $n_{pq} := 2p+2q+2$. In view of Corollary 4.9 we have

$$\begin{aligned}\sum_{r=N}^{\infty} (-1)^r \frac{2^{n_{pq}}}{\gamma_r^{(n_{pq},0)}} u_{p,q,r} &\overset{(4.9)}{=} \sum_{r=N}^{\infty} (-1)^r \frac{2^{p+q}}{\gamma_r^{(n_{pq},0)}} \tilde{u}_{p,q,r}\\ &= \sum_{r=N}^{\infty} (-1)^r \frac{2^{p+q}}{\gamma_r^{(n_{pq},0)}} \Big(h_1(r,n_{pq}) \tilde{u}'_{p,q,r+1} + h_2(r,n_{pq}) \tilde{u}'_{p,q,r} + h_3(r,n_{pq}) \tilde{u}'_{p,q,r-1} \Big)\\ &= \sum_{r=N+1}^{\infty} (-1)^{r-1} h_1(r-1,n_{pq}) \frac{2^{p+q}}{\gamma_{r-1}^{(n_{pq},0)}} \tilde{u}'_{p,q,r}\\ &\quad + \sum_{r=N}^{\infty} (-1)^r h_2(r,n_{pq}) \frac{2^{p+q}}{\gamma_r^{(n_{pq},0)}} \tilde{u}'_{p,q,r} + \sum_{r=N-1}^{\infty} (-1)^{r+1} h_3(r+1,n_{pq}) \frac{2^{p+q}}{\gamma_{r+1}^{(n_{pq},0)}} \tilde{u}'_{p,q,r}\\ &= \sum_{r=N+1}^{\infty} (-1)^r \tilde{u}'_{p,q,r} 2^{p+q} \left[-\frac{h_1(r-1,n_{pq})}{\gamma_{r-1}^{(n_{pq},0)}} + \frac{h_2(r,n_{pq})}{\gamma_r^{(n_{pq},0)}} - \frac{h_3(r+1,n_{pq})}{\gamma_{r+1}^{(n_{pq},0)}} \right]\\ &\quad + (-1)^N h_2(N,n_{pq}) \frac{2^{p+q}}{\gamma_N^{(n_{pq},0)}} \tilde{u}'_{p,q,N} + \sum_{r=N-1}^{N} (-1)^{r+1} h_3(r+1,n_{pq}) \frac{2^{p+q}}{\gamma_{r+1}^{(n_{pq},0)}} \tilde{u}'_{p,q,r}.\end{aligned}$$

By (3.5), the expression in brackets vanishes and that concludes the proof. □

Since Lemma 5.1 assumes $N \geq 1$, the terms corresponding to $r = 0$ in (5.2) are not included. We study this case now in Lemma 5.2:

Lemma 5.2. *Let $u \in H^1(\mathcal{T}^3)$ and consider the representation of the norms in (5.2). For $r = 0$ there exists a constant $C > 0$ independent of p, q and u such that*

$$\sum_{p,q=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{1}{\gamma_q^{(2p+1,0)}} \left| u_{p,q,0} \frac{2^{2p+2q+2}}{\gamma_0^{(2p+2q+2,0)}} \right|^2 \leq C \|u\|_{L^2(\mathcal{T}^3)} \|u\|_{H^1(\mathcal{T}^3)}.$$

Proof. We first note that the coefficient $|u_{0,0,0}| \leq \|u\|_{L^2(\mathcal{T}^3)}$ so that we may focus on the sum with $(p,q) \neq (0,0)$.

Since $\gamma_p^{(0,0)} = \frac{2}{2p+1}$, $\gamma_q^{(2p+1,0)} = \frac{2^{2p+1}}{q+p+1}$ and $\gamma_0^{(2p+2q+2,0)} = \frac{2^{2p+2q+3}}{2p+2q+3}$, we get

$$\sum_{p,q=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{1}{\gamma_q^{(2p+1,0)}} \left| u_{p,q,0} \frac{2^{2p+2q+2}}{\gamma_0^{(2p+2q+2,0)}} \right|^2 \lesssim \sum_{p,q=0}^{\infty} (p+q+1)^4 \left| u_{p,q,0} \right|^2.$$

To bound the sum on the right-hand side, we note that an integration by parts and (4.10) give (for $(p,q) \neq (0,0)$)

$$2^{p+q+2} u_{p,q,0} = \int_{-1}^{1} (1-\eta_3)^{p+q+2} U_{p,q}(\eta_3) d\eta_3 \tag{5.3}$$

$$= \frac{1}{p+q+3} \left(2^{p+q+3} U_{p,q}(-1) + \int_{-1}^{1} (1-\eta_3)^{p+q+3} U'_{p,q}(\eta_3) d\eta_3 \right), \tag{5.4}$$

These two equations yield two estimates for $u_{p,q,0}$. From (5.3), we get with the Cauchy-Schwarz inequality

$$\begin{aligned} |2^{p+q+2} u_{p,q,0}|^2 &= \left| \int_{-1}^{1} (1-\eta_3)^{p+q+2} U_{p,q}(\eta_3) d\eta_3 \right|^2 \\ &\leq \left(\int_{-1}^{1} \left((1-\eta_3)^{p+q+\frac{3}{2}} \right)^2 d\eta_3 \right) \left(\int_{-1}^{1} (1-\eta_3) |U_{p,q}(\eta_3)|^2 d\eta_3 \right) \\ &= \frac{2^{2p+2q+4}}{p+q+3} \int_{-1}^{1} \frac{1-\eta_3}{2} |U_{p,q}(\eta_3)|^2 d\eta_3 \end{aligned} \tag{5.5}$$

From (5.4), we obtain a second estimate as follows:

$$|2^{p+q+2} u_{p,q,0}|^2 \leq \frac{2}{(p+q+3)^2} \left(2^{2p+2q+6} |U_{p,q}(-1)|^2 + \left| \int_{-1}^{1} (1-\eta_3)^{p+q+3} U'_{p,q}(\eta_3) d\eta_3 \right|^2 \right),$$

where

$$\begin{aligned} \left| \int_{-1}^{1} (1-\eta_3)^{p+q+3} U'_{p,q}(\eta_3) d\eta_3 \right|^2 &\leq \left(\int_{-1}^{1} \left((1-\eta_3)^{p+q+\frac{5}{2}} \right)^2 d\eta_3 \right) \left(\int_{-1}^{1} (1-\eta_3) |U'_{p,q}(\eta_3)|^2 d\eta_3 \right) \\ &= \frac{2^{2p+2q+6}}{p+q+3} \int_{-1}^{1} \frac{1-\eta_3}{2} |U'_{p,q}(\eta_3)|^2. \end{aligned}$$

Inserting this in the bound before yields

$$|2^{p+q+2} u_{p,q,0}|^2 \leq 2 \frac{2^{2p+2q+6}}{(p+q+1)^2} \left(|U_{p,q}(-1)|^2 + \frac{1}{p+q+1} \int_{-1}^{1} \frac{1-\eta_3}{2} |U'_{p,q}(\eta_3)|^2 d\eta_3 \right). \tag{5.6}$$

Next, we abbreviate

$$\sigma_{p,q}^2 := \int_{-1}^{1} \frac{1-\eta_3}{2} |U_{p,q}(\eta_3)|^2 d\eta_3, \qquad \tau_{p,q}^2 := \int_{-1}^{1} \frac{1-\eta_3}{2} |U'_{p,q}(\eta_3)|^2 d\eta_3.$$

Hence, applying (5.5) and (5.6) we have

$$\begin{aligned} |2^{p+q+2} u_{p,q,0}|^2 &\le \min\left\{ \frac{2^{2p+2q+4}}{p+q+1} \sigma_{p,q}^2 \,,\, 2\frac{2^{2p+2q+6}}{(p+q+1)^2} \left(|U_{p,q}(-1)|^2 + \frac{1}{p+q+1} \tau_{p,q}^2 \right) \right\} \\ &\le 2\frac{2^{2p+2q+6}}{(p+q+1)^2} |U_{p,q}(-1)|^2 + 2\min\left\{ \frac{2^{2p+2q+4}}{p+q+1} \sigma_{p,q}^2 \,,\, \frac{2^{2p+2q+6}}{(p+q+1)^3} \tau_{p,q}^2 \right\} \\ &\le 2\frac{2^{2p+2q+6}}{(p+q+1)^2} |U_{p,q}(-1)|^2 + 2\frac{2^{2p+2q+5}}{(p+q+1)^2} \sigma_{p,q}\tau_{p,q}, \end{aligned}$$

where in the last step we used that $\min\{a^2, b^2\} \le |a||b|$. This leads us to

$$|u_{p,q,0}|^2 \lesssim \frac{1}{(p+q+1)^2} |U_{p,q}(-1)|^2 + \frac{1}{(p+q+1)^2} \sigma_{p,q}\tau_{p,q}. \tag{5.7}$$

Hence, we conclude

$$\begin{aligned} &\sum_{p,q=0}^{\infty} (p+q+1)^4 |u_{p,q,0}|^2 \lesssim \sum_{p,q=0}^{\infty} (p+q+1)^2 |U_{p,q}(-1)|^2 + \sum_{p,q=0}^{\infty} (p+q+1)^2 \sigma_{p,q}\tau_{p,q} \\ &\lesssim \sum_{p,q=0}^{\infty} (p+q+1)^2 |U_{p,q}(-1)|^2 + \left(\sum_{p,q=0}^{\infty} (p+q+1)^2 \sigma_{p,q}^2 \right)^{1/2} \left(\sum_{p,q=0}^{\infty} (p+q+1)^2 \tau_{p,q}^2 \right)^{1/2} \\ &\lesssim \|u\|_{L^2(\Gamma)}^2 + \|u\|_{L^2(\mathcal{T}^3)} \|\nabla u\|_{L^2(\mathcal{T}^3)} \end{aligned}$$

where, in the last inequality, we appealed to Lemma 4.5. The statement of the lemma now from the multiplicative trace inequality $\|u\|_{L^2(\Gamma)}^2 \lesssim \|u\|_{L^2(\mathcal{T}^3)} \|u\|_{H^1(\mathcal{T}^3)}$ of, e.g., [3, Thm. 1.6.6] and the trivial bound. ☐

We are now in position to prove Theorem 1.1 as well as Corollary 1.2:

Theorem 5.3 (trace stability of L^2-projection, approximation properties)**.** *For $N \in \mathbb{N}_0$ denote by Π_N the $L^2(\mathcal{T}^3)$-projection onto $\mathcal{P}_N(\mathcal{T}^3)$. There exists a constant $C > 0$ independent of N and u such that*

$$\|\Pi_N u\|_{L^2(\Gamma)}^2 \le C \|u\|_{L^2(\mathcal{T}^3)} \|u\|_{H^1(\mathcal{T}^3)} \qquad \forall u \in H^1(\mathcal{T}^3). \tag{5.8}$$

Furthermore,

$$\|\Pi_N u\|_{L^2(\Gamma)} \le C \|u\|_{B_{2,1}^{1/2}(\mathcal{T}^3)} \qquad \forall u \in B_{2,1}^{1/2}(\mathcal{T}^3). \tag{5.9}$$

Additionally, for each $s > 1/2$ there is $C_s > 0$ such that

$$\|u - \Pi_N u\|_{L^2(\Gamma)} \le C_s (N+1)^{-(s-1/2)} \|u\|_{H^s(\mathcal{T}^3)} \qquad \forall u \in H^s(\mathcal{T}). \tag{5.10}$$

Proof. In view of the multiplicative trace inequality $\|u\|_{L^2(\Gamma)}^2 \lesssim \|u\|_{L^2(\mathcal{T}^3)} \|u\|_{H^1(\mathcal{T}^3)}$ (see, e.g., [3, Thm. 1.6.6]) we will only show the statement $\|u - \Pi_N u\|_{L^2(\Gamma)}^2 \lesssim \|u\|_{L^2(\mathcal{T}^3)} \|u\|_{H^1(\mathcal{T}^3)}$. We abbreviate $n_{pq} := 2p + 2q + 2$ and $c_{pq} := \frac{1}{\gamma_p^{(0,0)}} \frac{1}{\gamma_q^{(2p+1,0)}}$. By (5.2), we have to bound

$$\|u - \Pi_N u\|_{L^2(\Gamma)}^2 = \sum_{p,q=0}^{\infty} c_{pq} \left| \sum_{r=\max\{0,N+1-p-q\}}^{\infty} (-1)^r \frac{2^{n_{pq}}}{\gamma_r^{(n_{pq},0)}} u_{p,q,r} \right|^2$$

$$
\begin{aligned}
&= \sum_{p+q\le N} c_{pq}\left|\sum_{r=N+1-p-q}^{\infty}(-1)^r\frac{2^{n_{pq}}}{\gamma_r^{(n_{pq},0)}}u_{p,q,r}\right|^2 + \sum_{p+q\ge N+1} c_{pq}\left|\sum_{r=0}^{\infty}(-1)^r\frac{2^{n_{pq}}}{\gamma_r^{(n_{pq},0)}}u_{p,q,r}\right|^2\\
&= \sum_{p=0}^{N}\sum_{q=0}^{N-p} c_{pq}\left|\sum_{r=N+1-p-q}^{\infty}(-1)^r\frac{2^{n_{pq}}}{\gamma_r^{(n_{pq},0)}}u_{p,q,r}\right|^2 + \sum_{p=0}^{N}\sum_{q=N+1-p}^{\infty} c_{pq}\left|\sum_{r=0}^{\infty}(-1)^r\frac{2^{n_{pq}}}{\gamma_r^{(n_{pq},0)}}u_{p,q,r}\right|^2\\
&\quad + \sum_{p=N+1}^{\infty}\sum_{q=0}^{\infty} c_{pq}\left|\sum_{r=0}^{\infty}(-1)^r\frac{2^{n_{pq}}}{\gamma_r^{(n_{pq},0)}}u_{p,q,r}\right|^2\\
&\lesssim \underbrace{\sum_{p=0}^{N}\sum_{q=0}^{N-p} c_{pq}\left|\sum_{r=N+1-p-q}^{\infty}(-1)^r\frac{2^{n_{pq}}}{\gamma_r^{(n_{pq},0)}}u_{p,q,r}\right|^2}_{=:\,S_1} + \underbrace{\sum_{p=0}^{N}\sum_{q=N+1-p}^{\infty} c_{pq}\left|\sum_{r=1}^{\infty}(-1)^r\frac{2^{n_{pq}}}{\gamma_r^{(n_{pq},0)}}u_{p,q,r}\right|^2}_{=:\,S_2}\\
&\quad + \underbrace{\sum_{p=N+1}^{\infty}\sum_{q=0}^{\infty} c_{pq}\left|\sum_{r=1}^{\infty}(-1)^r\frac{2^{n_{pq}}}{\gamma_r^{(n_{pq},0)}}u_{p,q,r}\right|^2}_{=:\,S_3} + \underbrace{\sum_{p=0}^{N}\sum_{q=N+1-p}^{\infty} c_{pq}\left|\frac{2^{n_{pq}}}{\gamma_0^{(n_{pq},0)}}u_{p,q,0}\right|^2}_{=:\,S_4}\\
&\quad + \underbrace{\sum_{p=N+1}^{\infty}\sum_{q=0}^{\infty} c_{pq}\left|\frac{2^{n_{pq}}}{\gamma_0^{(n_{pq},0)}}u_{p,q,0}\right|^2}_{=:\,S_5}.
\end{aligned}
$$

Lemma 5.2 immediately gives $S_4+S_5 \lesssim \|u\|_{L^2(\mathcal{T}^3)}\|u\|_{H^1(\mathcal{T}^3)}$. From Lemma 5.1 as well as the estimates

$$
\begin{aligned}
&h_2(N+1-p-q, n_{pq}) \lesssim \frac{p+q}{N^2} \lesssim \frac{1}{N}, \qquad p+q = 0,\dots,N\\
&h_3(N+1-p-q, n_{pq}),\ h_3(N+2-p-q, n_{pq}) \lesssim \frac{N+p+q}{N^2} \lesssim \frac{1}{N}, \qquad p+q = 0,\dots,N
\end{aligned}
$$

and

$$
\frac{1}{\gamma_{N+1-p-q}^{(n_{pq},0)}} = \frac{2N+5}{2^{2p+2q+3}}, \qquad \frac{1}{\gamma_{N+2-p-q}^{(n_{pq},0)}} = \frac{2N+7}{2^{2p+2q+3}}
$$

we obtain for S_1

$$
S_1 \lesssim \sum_{p=0}^{N}\sum_{q=0}^{N-p} c_{pq}\left(\left|2^{-(p+q+3)}\tilde{u}'_{p,q,N+1-p-q}\right|^2 + \left|2^{-(p+q+3)}\tilde{u}'_{p,q,N-p-q}\right|^2\right).
$$

Analogously, we get for S_2 and S_3

$$
\begin{aligned}
S_2 &\lesssim \sum_{p=0}^{N}\sum_{q=N+1-p}^{\infty} c_{pq}\left(\left|2^{-(p+q+3)}\tilde{u}'_{p,q,1}\right|^2 + \left|2^{-(p+q+3)}\tilde{u}'_{p,q,0}\right|^2\right)\\
S_3 &\lesssim \sum_{p=N+1}^{\infty}\sum_{q=0}^{\infty} c_{pq}\left(\left|2^{-(p+q+3)}\tilde{u}'_{p,q,1}\right|^2 + \left|2^{-(p+q+3)}\tilde{u}'_{p,q,0}\right|^2\right).
\end{aligned}
$$

Applying Lemma 3.9 the powers of two in the estimates above and in the lemma annihilate each other.

Hence, $S_1+S_2+S_3$ gives

$$
\begin{aligned}
S_1+S_2+S_3 \lesssim & \sum_{p=0}^{N}\sum_{q=0}^{N-p} c_{pq}\left(\sum_{r\geq N+1-p-q}\frac{1}{\gamma_r^{(n_{pq},0)}}|\tilde{u}_{p,q,r}|^2\right)^{1/2}\left(\sum_{r\geq N-p-q}\frac{1}{\gamma_r^{(n_{pq},0)}}|\tilde{u}'_{p,q,r}|^2\right)^{1/2}\\
&+\sum_{p=0}^{N}\sum_{q=N+1-p}^{\infty} c_{pq}\left(\sum_{r\geq 1}\frac{1}{\gamma_r^{(n_{pq},0)}}|\tilde{u}_{p,q,r}|^2\right)^{1/2}\left(\sum_{r\geq 0}\frac{1}{\gamma_r^{(n_{pq},0)}}|\tilde{u}'_{p,q,r}|^2\right)^{1/2}\\
&+\sum_{p=N+1}^{\infty}\sum_{q=0}^{\infty} c_{pq}\left(\sum_{r\geq 1}\frac{1}{\gamma_r^{(n_{pq},0)}}|\tilde{u}_{p,q,r}|^2\right)^{1/2}\left(\sum_{r\geq 0}\frac{1}{\gamma_r^{(n_{pq},0)}}|\tilde{u}'_{p,q,r}|^2\right)^{1/2}\\
\lesssim & \|u\|_{L^2(\mathcal{T}^3)}\|\nabla u\|_{L^2(\mathcal{T}^3)},
\end{aligned}
$$

where in the last estimate, we have used the Cauchy-Schwarz inequality for sums and Corollary 4.8. Since $\|\nabla u\|_{L^2(\mathcal{T}^3)} \leq \|u\|_{H^1(\mathcal{T}^3)}$ this concludes the proof of (5.8).
The estimate (5.9) follows directly from (5.8) in view of [27, Lemma 25.3].
Finally (5.10) is shown in the standard way. For arbitrary $q \in \mathcal{P}_N$ we have by the projection property of Π_N as well as the continuity of the trace operator $\gamma_0 : B^{1/2}_{1/2,1}(\mathcal{T}) \to L^2(\partial\mathcal{T}^3)$ (cf., e.g., [27, Sec. 32])

$$
\|u-\Pi_N u\|_{L^2(\partial\mathcal{T})} \leq \|u-q\|_{L^2(\partial\mathcal{T})} + \|\Pi_N(u-q)\|_{L^2(\partial\mathcal{T})} \leq C\|u-q\|_{B^{1/2}_{2,1}(\mathcal{T})}.
$$

Hence, $\|u-\Pi_N u\|_{L^2(\partial\mathcal{T})} \leq C \inf_{q\in\mathcal{P}_N}\|u-q\|_{B^{1/2}_{2,1}(\mathcal{T})}$. Fix $s > 1/2$. Let $\widetilde{\Pi}_N : L^2(\mathcal{T}) \to \mathcal{P}_N$ be an approximation operator with the properties (for example, one can combine the approximation results of [19, Appendix A] for hyper cubes with the well-known extension operator of Stein, [25, Chap. VII])

$$
\|u-\widetilde{\Pi}_N u\|_{L^2(\mathcal{T})} \leq CN^{-s}\|u\|_{H^s(\mathcal{T})}, \qquad \|u-\widetilde{\Pi}_N u\|_{H^s(\mathcal{T})} \leq C\|u\|_{H^s(\mathcal{T})} \qquad \forall u \in H^s(\mathcal{T}).
$$

By interpolation theory, we then have that $\operatorname{Id}-\widetilde{\Pi}_N$ is a bounded linear operator $H^s(\mathcal{T}) \to B^{1/2}_{2,1}(\mathcal{T}) = (L^2(\mathcal{T}), H^s(\mathcal{T}))_{(s-1/2)/s,1}$ with norm $\|\operatorname{Id}-\widetilde{\Pi}_N\|_{B^{1/2}_{2,1}(\mathcal{T})\leftarrow H^s(\mathcal{T})} \leq CN^{-s(s-1/2)/s}$. □

6 H^1-stability

Our procedure to study the H^1-stability of the L^2-projection Π_N is to study the directional derivative that corresponds to the derivative ∂_{η_3} in the transformed variable. The full gradient can be obtained from this directional derivative and appropriate affine transformations of the reference tetrahedron $\mathcal{T}^3$.

The key step is therefore to control $\partial_{\eta_3}\widetilde{\Pi}_N u$, where we denote $\widetilde{\Pi}_N u := (\Pi_N u)\circ D$. This is the purpose of the ensuing lemma.

Lemma 6.1. *There exists a constant $C > 0$ independent of N such that*

$$
\int_{\mathcal{S}^3}(1-\eta_2)(1-\eta_3)^2\left|\partial_{\eta_3}\widetilde{\Pi}_N u(\eta)\right|^2 d\eta \leq CN\|\nabla u\|^2_{L^2(\mathcal{T}^3)} \qquad \forall u \in H^1(\mathcal{T}^3).
$$

Proof. We abbreviate $n_{pq} := 2p+2q+2$. We see that

$$
\widetilde{\Pi}_N u(\eta) = \sum_{p=0}^{N}\sum_{q=0}^{N-p}\sum_{r=0}^{N-p-q}\frac{1}{\gamma_p^{(0,0)}}\frac{2^{2p+1}}{\gamma_q^{(2p+1,0)}}\frac{2^{p+q}}{\gamma_r^{(n_{pq},0)}}\tilde{u}_{p,q,r}\tilde{\psi}_{p,q,r}(\eta)
$$

by recalling the relation between $u_{p,q,r}$ and $\tilde{u}_{p,q,r}$. Differentiating with respect to η_3 shows us that we have to estimate the two terms

$$I_1 := \int_{\mathcal{S}^3} (1-\eta_2)(1-\eta_3)^2 \left| \sum_{p=0}^{N} \sum_{q=0}^{N-p} \sum_{r=0}^{N-p-q} \frac{1}{\gamma_p^{(0,0)}} \frac{2^{2p+1}}{\gamma_q^{(2p+1,0)}} \frac{1}{\gamma_r^{(n_{pq},0)}} \tilde{u}_{p,q,r} \right.$$

$$\left. \times P_p^{(0,0)}(\eta_1) P_q^{(2p+1,0)}(\eta_2) \left(\frac{1-\eta_2}{2}\right)^p (1-\eta_3)^{p+q} \big(P_r^{(n_{pq},0)}\big)'(\eta_3) \right|^2 d\eta \tag{6.1}$$

$$I_2 := \int_{\mathcal{S}^3} (1-\eta_2)(1-\eta_3)^2 \left| \sum_{p=0}^{N} \sum_{q=0}^{N-p} \sum_{r=0}^{N-p-q} \frac{1}{\gamma_p^{(0,0)}} \frac{2^{2p+1}}{\gamma_q^{(2p+1,0)}} \frac{p+q}{\gamma_r^{(n_{pq},0)}} \tilde{u}_{p,q,r} \right.$$

$$\left. \times P_p^{(0,0)}(\eta_1) P_q^{(2p+1,0)}(\eta_2) \left(\frac{1-\eta_2}{2}\right)^p (1-\eta_3)^{p+q-1} P_r^{(n_{pq},0)}(\eta_3) \right|^2 d\eta \tag{6.2}$$

First, we consider (6.1). From Lemma 3.5 with $\alpha = n_{pq}$ we get

$$\begin{aligned} I_1 &= \sum_{p=0}^{N} \sum_{q=0}^{N-p} \frac{1}{\gamma_p^{(0,0)}} \frac{1}{\gamma_q^{(2p+1,0)}} \int_{-1}^{1} (1-\eta_3)^{n_{pq}} \left| \sum_{r=0}^{N-p-q} \frac{1}{\gamma_r^{(n_{pq},0)}} \tilde{u}_{p,q,r} \big(P_r^{(n_{pq},0)}\big)'(\eta_3) \right|^2 d\eta_3 \\ &\lesssim \sum_{p=0}^{N} \sum_{q=0}^{N-p} \frac{1}{\gamma_p^{(0,0)}} \frac{1}{\gamma_q^{(2p+1,0)}} (N-p-q) \sum_{r=0}^{\infty} \frac{1}{\gamma_r^{(n_{pq},0)}} |\tilde{u}'_{p,q,r}|^2 \lesssim N \|\nabla u\|^2_{L^2(\mathcal{T}^3)}, \end{aligned}$$

where in the last step, we appealed to Corollary 4.8. Thus, we arrive at the desired bound for I_1. Next, we consider (6.2). We have

$$I_2 = \sum_{p=0}^{N} \sum_{q=0}^{N-p} \frac{1}{\gamma_p^{(0,0)}} \frac{(p+q)^2}{\gamma_q^{(2p+1,0)}} \int_{-1}^{1} (1-\eta_3)^{2p+2q} \left| \sum_{r=0}^{N-p-q} \frac{1}{\gamma_r^{(n_{pq},0)}} \tilde{u}_{p,q,r} P_r^{(n_{pq},0)}(\eta_3) \right|^2 d\eta_3$$

Lemma 3.6 with $\beta = 2p+2q$ and the normalization convention for Jacobi polynomials (A.3) now yield

$$\begin{aligned} I_2 &\le \sum_{p=0}^{N} \sum_{q=0}^{N-p} \frac{1}{\gamma_p^{(0,0)}} \frac{(p+q)^2}{\gamma_q^{(2p+1,0)}} \left(\frac{1}{(p+q)^2} \int_{-1}^{1} (1-\eta_3)^{n_{pq}} \left| \sum_{r=0}^{N-p-q} \frac{1}{\gamma_r^{(n_{pq},0)}} \tilde{u}_{p,q,r} \big(P_r^{(n_{pq},0)}\big)'(\eta_3) \right|^2 d\eta_3 \right. \\ &\qquad \left. + \frac{1}{p+q} \left| \sum_{r=0}^{N-p-q} \frac{1}{\gamma_r^{(n_{pq},0)}} \tilde{u}_{p,q,r} P_r^{(n_{pq},0)}(-1) \right|^2 \right) \\ &\le I_1 + \sum_{p=0}^{N} \sum_{q=0}^{N-p} \frac{1}{\gamma_p^{(0,0)}} \frac{p+q}{\gamma_q^{(2p+1,0)}} \left| \sum_{r=0}^{N-p-q} \frac{1}{\gamma_r^{(n_{pq},0)}} \tilde{u}_{p,q,r} P_r^{(n_{pq},0)}(-1) \right|^2 \\ &= I_1 + \sum_{p=0}^{N} \sum_{q=0}^{N-p} \frac{1}{\gamma_p^{(0,0)}} \frac{p+q}{\gamma_q^{(2p+1,0)}} \left| \sum_{r=0}^{N-p-q} \frac{1}{\gamma_r^{(n_{pq},0)}} \tilde{u}_{p,q,r} (-1)^r \right|^2 \\ &\le I_1 + 2N \sum_{p=0}^{N} \sum_{q=0}^{N-p} \frac{1}{\gamma_p^{(0,0)}} \frac{1}{\gamma_q^{(2p+1,0)}} \left(\left| \sum_{r=0}^{\infty} \frac{(-1)^r}{\gamma_r^{(n_{pq},0)}} \tilde{u}_{p,q,r} \right|^2 + \left| \sum_{r=N-p-q+1}^{\infty} \frac{(-1)^r}{\gamma_r^{(n_{pq},0)}} \tilde{u}_{p,q,r} \right|^2 \right) \\ &\lesssim N \|\nabla u\|^2_{L^2(\mathcal{T}^3)} + N \|u\|^2_{L^2(\{\eta \in \mathcal{T}^3 : \eta_3 = -1\})} \\ &\lesssim N \|\nabla u\|^2_{L^2(\mathcal{T}^3)} + N \|u\|_{L^2(\mathcal{T}^3)} \|u\|_{H^1(\mathcal{T}^3)} \lesssim N \|\nabla u\|^2_{L^2(\mathcal{T}^3)} \end{aligned}$$

where we used the fact that I_1 is bounded above and the multiplicative trace inequality [3, Thm. 1.6.6] as in Theorem 5.3 but now applied to the bottom face $\{\eta \in \mathcal{T}^3 : \eta_3 = -1\}$. □

We can now prove Theorem 1.3:

Theorem 6.2 (H^1-stability of L^2-projection)**.** *There exists a constant $C > 0$ independent of N such that*

$$\|\nabla \Pi_N u\|_{L^2(\mathcal{T}^3)} \le C\sqrt{N}\|\nabla u\|_{L^2(\mathcal{T}^3)} \qquad \forall u \in H^1(\mathcal{T}^3).$$

Proof. For a function v and the transformed function $\tilde{v} = v \circ D$, the formula (4.13) provides a relation between $\partial_{\eta_3}\tilde{v}$ and ∇v. Rearranging terms yields

$$(\partial_{\eta_3}\tilde{v}) \circ D^{-1}(\xi) = -\frac{1+\xi_1}{1-\xi_3}\partial_1 v(\xi) - \frac{1+\xi_2}{1-\xi_3}\partial_2 v(\xi) + \partial_3 v(\xi).$$

Therefore, when transforming to $\mathcal{T}^3$ in Lemma 6.1 we get

$$\int_{\mathcal{T}^3} \left| -\frac{1+\xi_1}{1-\xi_3}\partial_1 \Pi_N u(\xi) - \frac{1+\xi_2}{1-\xi_3}\partial_2 \Pi_N u(\xi) + \partial_3 \Pi_N u(\xi)\right|^2 d\xi \lesssim N\|\nabla u\|^2_{L^2(\mathcal{T}^3)}. \tag{6.3}$$

By the symmetry properties of $\mathcal{T}^3$, we see that also the following two other permutations of indices are valid estimates:

$$\int_{\mathcal{T}^3} \left| -\frac{1+\xi_2}{1-\xi_1}\partial_2 \Pi_N u(\xi) - \frac{1+\xi_3}{1-\xi_1}\partial_3 \Pi_N u(\xi) + \partial_1 \Pi_N u(\xi)\right|^2 d\xi \lesssim N\|\nabla u\|^2_{L^2(\mathcal{T}^3)}, \tag{6.4}$$

$$\int_{\mathcal{T}^3} \left| -\frac{1+\xi_3}{1-\xi_2}\partial_3 \Pi_N u(\xi) - \frac{1+\xi_1}{1-\xi_2}\partial_1 \Pi_N u(\xi) + \partial_2 \Pi_N u(\xi)\right|^2 d\xi \lesssim N\|\nabla u\|^2_{L^2(\mathcal{T}^3)}. \tag{6.5}$$

We abbreviate $a(x,y) := -\frac{1+x}{1-y}$, $a_{ij} := a(\xi_i, \xi_j)$ and

$$A(\xi_1,\xi_2,\xi_3) := \begin{pmatrix} 1 + a_{13}^2 + a_{12}^2 & a_{13}a_{23} + a_{21} + a_{12} & a_{13} + a_{31} + a_{32}a_{12} \\ sym & 1 + a_{23}^2 + a_{21}^2 & a_{23} + a_{21}a_{31} + a_{32} \\ sym & sym & 1 + a_{31}^2 + a_{32}^2 \end{pmatrix}.$$

Hence, we see that by adding (6.3), (6.4), and (6.5) we arrive at

$$\int_{\mathcal{T}^3} (\nabla \Pi_N u)^\top A(\xi) \nabla \Pi_N u \, d\xi \lesssim N\|\nabla u\|^2_{L^2(\mathcal{T}^3)}$$

Next, we observe that near the top vertex $(-1,-1,1)$, we have

$$\left|\frac{1+\xi_1}{1-\xi_3}\right| \le 1 \quad \text{and} \quad \left|\frac{1+\xi_2}{1-\xi_3}\right| \le 1.$$

This implies that the functions a_{13} and a_{23} are uniformly bounded on $\mathcal{T}^3$. Analogously, we get bounds for a_{12}, a_{32} and a_{21}, a_{31} by studying the vertices $(1,-1,-1)$ and $(-1,1,-1)$. Therefore, we have

$$\sup_{\xi \in \mathcal{T}^3} \|A(\xi)\|_{L^\infty(\mathcal{T}^3)} < \infty.$$

By construction, the matrix $A(\xi)$ is (pointwise) symmetric positive semidefinite. Our goal is to show that $A(\xi)$ is in fact positive definite on the set that stays away from the face F opposite the vertex $(-1,-1,-1)$. This can be done with techniques as in [11] by establishing lower bounds for the eigenvalues of $A(\xi)$. A direct calculation reveals

$$\begin{aligned} \det A(\xi) &= 16\frac{\xi_1^2 + 2\xi_1\xi_2 + 2\xi_1\xi_3 + 2\xi_1 + \xi_2^2 + 1 + 2\xi_3 + 2\xi_3\xi_2 + 2\xi_2 + \xi_3^2}{(-1+\xi_1)^2(-1+\xi_3)^2(-1+\xi_2)^2} \\ &= 16\frac{(\xi_1+\xi_2+\xi_3)^2 + 2(\xi_1+\xi_2+\xi_3) + 1}{(-1+\xi_1)^2(-1+\xi_3)^2(-1+\xi_2)^2} = 16\frac{(1+\xi_1+\xi_2+\xi_3)^2}{(-1+\xi_1)^2(-1+\xi_3)^2(-1+\xi_2)^2}. \end{aligned}$$

The face opposite the vertex $(-1,-1,-1)$ contains the vertices $(-1-1,1)$, $(-1,1,-1)$, $(1,-1,-1)$ and is given by the equation $\xi_1+\xi_2+\xi_3+1=0$. Furthermore, we conclude that the signed distance of an arbitrary point ξ from this face F is given by

$$\operatorname{dist}(\xi,F)=\frac{1}{\sqrt{3}}(\xi_1+\xi_2+\xi_3+1).$$

Let, for arbitrary $\delta>0$,

$$T_\delta := \{\xi\in\mathcal{T}^3 \mid \operatorname{dist}(\xi,F)<-\delta\}.$$

Then, since we stay away from the face F, it is clear that there exists $C_\delta>0$ such that

$$\det A(\xi)\geq C_\delta \quad \forall\xi\in\overline{T}_\delta.$$

Combining the above findings, we have that on T_δ the matrix $A(\xi)$ is in fact symmetric positive definite. Since the entries of $A(\xi)$ are uniformly bounded in ξ, Gershgorin's circle theorem provides a constant C_{upper} such that all eigenvalues of $A(\xi)$ are bounded by C_{upper}.

A lower bound for the eigenvalues is obtained as follows: Denoting for fixed $\xi\in T_\delta$ the eigenvalues $0<\lambda_1\leq\lambda_2\leq\lambda_3$, we get from $\det A=\lambda_1\lambda_2\lambda_3$

$$C_\delta\leq\det A=\lambda_1\lambda_2\lambda_3\leq\lambda_1 C_{upper}^2.$$

This provides the desired lower bound for λ_1. Thus, we conclude that for every $\delta>0$ we can find $c_\delta>0$ such that $A(\xi)\geq c_\delta I$ on T_δ. Hence,

$$c_\delta\int_{T_\delta}|\nabla\Pi_N u|^2\,d\xi\leq\int_{\mathcal{T}^3}(\nabla\Pi_N u)^\top A(\xi)\nabla\Pi_N u\;d\xi\lesssim N\|\nabla u\|^2_{L^2(\mathcal{T}^3)}.$$

Affine transformations allow us to get analogous estimates for the sets that stay away from the other faces of $\mathcal{T}^3$. We therefore get the desired result. □

A Properties of Jacobi polynomials

We have the following useful formulas (see [16, p. 350 f], [26]):

Recursion Relations

$$a_n^1 P_{n+1}^{(\alpha,\beta)}(x)=(a_n^2+a_n^3x)P_n^{(\alpha,\beta)}(x)-a_n^4P_{n-1}^{(\alpha,\beta)}(x) \tag{A.1}$$

with

$$\begin{aligned}
a_n^1 &:= 2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)\\
a_n^2 &:= (2n+\alpha+\beta+1)(\alpha^2-\beta^2)\\
a_n^3 &:= (2n+\alpha+\beta)(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)\\
a_n^4 &:= 2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)
\end{aligned}$$

$$b_n^1(x)\frac{d}{dx}P_n^{(\alpha,\beta)}(x)=b_n^2(x)P_n^{(\alpha,\beta)}(x)+b_n^3(x)P_{n-1}^{(\alpha,\beta)}(x) \tag{A.2}$$

with

$$\begin{aligned}
b_n^1(x) &:= (2n+\alpha+\beta)(1-x^2)\\
b_n^2(x) &:= n\,(\alpha-\beta-(2n+\alpha+\beta)x)\\
b_n^3(x) &:= 2(n+\alpha)(n+\beta)
\end{aligned}$$

Special Values

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} \tag{A.3}$$

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x) \tag{A.4}$$

Special Cases For the Legendre Polynomial $L_n(x)$ there holds

$$L_n(x) = P_n^{(0,0)}(x) \tag{A.5}$$

Miscellaneous Relations

$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(\alpha+\beta+n+1)P_{n-1}^{(\alpha+1,\beta+1)}(x) \tag{A.6}$$

$$2n\int_{-1}^{x}(1-t)^{\alpha}(1+t)^{\beta}P_n^{(\alpha,\beta)}(t)\,dt = -(1-x)^{\alpha+1}(1+x)^{\beta+1}P_{n-1}^{(\alpha+1,\beta+1)}(x) \tag{A.7}$$

B Selected proofs

B.1 Selected proofs for Section 3

Proof of Lemma 3.2 (i). Using rearranged versions of (A.2), (A.6) and (A.7) we obtain

$$\begin{aligned}
\int_{-1}^{x}(1-t)^{\alpha}P_q^{(\alpha,0)}(t)dt &\overset{(A.7)}{=} -\frac{1}{2q}(1+x)(1-x)^{\alpha+1}P_{q-1}^{(\alpha+1,1)}(x) \\
&= -\frac{1}{2q}(1-x^2)(1-x)^{\alpha}P_{q-1}^{(\alpha+1,1)}(x) \\
&\overset{(A.6)}{=} -(1-x)^{\alpha}\frac{1}{2q}(1-x^2)\frac{2}{q+\alpha+1}\frac{d}{dx}P_q^{(\alpha,0)}(x) \\
&\overset{(A.2)}{=} -(1-x)^{\alpha}\frac{1}{q}\frac{1}{q+\alpha+1}\frac{q(\alpha-(2q+\alpha)x)P_q^{(\alpha,0)}(x)+2q(q+\alpha)P_{q-1}^{(\alpha,0)}(x)}{2q+\alpha} \\
&= -(1-x)^{\alpha}\frac{\alpha P_q^{(\alpha,0)}(x)+2(q+\alpha)P_{q-1}^{(\alpha,0)}(x)-(2q+\alpha)xP_q^{(\alpha,0)}(x)}{(q+\alpha+1)(2q+\alpha)}
\end{aligned}$$

(A.1) allows us now to replace the term $xP_q^{(\alpha,0)}(x)$ by terms involving $P_{q+1}^{(\alpha,0)}(x)$, $P_q^{(\alpha,0)}(x)$ and $P_{q-1}^{(\alpha,0)}(x)$. Hence, we get

$$\begin{aligned}
\int_{-1}^{x}(1-t)^{\alpha}P_q^{(\alpha,0)}(t)dt = -(1-x)^{\alpha}\frac{1}{(q+\alpha+1)(2q+\alpha)}\Bigg\{\alpha P_q^{(\alpha,0)}(x)+2(q+\alpha)P_{q-1}^{(\alpha,0)}(x) \\
-\frac{1}{(2q+\alpha+1)(2q+\alpha+2)}\Big(2(q+1)(q+\alpha+1)(2q+\alpha)P_{q+1}^{(\alpha,0)}(x) \\
+2q(q+\alpha)(2q+\alpha+2)P_{q-1}^{(\alpha,0)}(x)-(2q+\alpha+1)\alpha^2P_q^{(\alpha,0)}(x)\Big)\Bigg\}
\end{aligned}$$

Rearranging terms gives

$$\int_{-1}^{x}(1-t)^{\alpha}P_q^{(\alpha,0)}(t)dt = -(1-x)^{\alpha}\frac{1}{(q+\alpha+1)(2q+\alpha)}\Bigg\{-\frac{2(q+1)(q+\alpha+1)(2q+\alpha)}{(2q+\alpha+1)(2q+\alpha+2)}P_{q+1}^{(\alpha,0)}(x) + \alpha\frac{2q+2\alpha+2}{2q+\alpha+2}P_q^{(\alpha,0)}(x) + 2(q+\alpha)\frac{q+\alpha+1}{2q+\alpha+1}P_{q-1}^{(\alpha,0)}(x)\Bigg\}$$

$$= -(1-x)^{\alpha}\Bigg\{\underbrace{-\frac{2(q+1)}{(2q+\alpha+1)(2q+\alpha+2)}}_{h_1(q,\alpha)}P_{q+1}^{(\alpha,0)}(x) + \underbrace{\frac{2\alpha}{(2q+\alpha+2)(2q+\alpha)}}_{h_2(q,\alpha)}P_q^{(\alpha,0)}(x) + \underbrace{\frac{2(q+\alpha)}{(2q+\alpha+1)(2q+\alpha)}}_{h_3(q,\alpha)}P_{q-1}^{(\alpha,0)}(x)\Bigg\}$$

□

Proof of (3.5). By definition of $\gamma_p^{(\alpha,\beta)}$ we obtain in particular

$$\gamma_q^{(\alpha,0)} = \frac{2^{\alpha+1}}{2q+\alpha+1},$$

which leads in combination with the definition of h_1, h_2 and h_3 to

$$(-1)^q\frac{1}{\gamma_q^{(\alpha,0)}}h_1(q,\alpha) + (-1)^{q+1}\frac{1}{\gamma_{q+1}^{(\alpha,0)}}h_2(q+1,\alpha) + (-1)^{q+2}\frac{1}{\gamma_{q+2}^{(\alpha,0)}}h_3(q+2,\alpha)$$
$$= (-1)^{q+1}\frac{2q+\alpha+1}{2^{\alpha+1}}\frac{2q+2}{(2q+\alpha+1)(2q+\alpha+2)} + (-1)^{q+1}\frac{2q+\alpha+3}{2^{\alpha+1}}\frac{2\alpha}{(2q+\alpha+4)(2q+\alpha+2)} + (-1)^{q+2}\frac{2q+\alpha+5}{2^{\alpha+1}}\frac{2(q+\alpha+2)}{(2q+\alpha+5)(2q+\alpha+4)}$$
$$= \frac{(-1)^{q+1}}{2^{\alpha}}\left(\frac{q+1}{2q+\alpha+2} + \frac{(2q+\alpha+3)\alpha}{(2q+\alpha+4)(2q+\alpha+2)} - \frac{q+\alpha+2}{2q+\alpha+4}\right)$$
$$= \frac{(-1)^{q+1}}{2^{\alpha}}\left(\frac{(q+1)(2q+\alpha+4)+(2q+\alpha+3)\alpha-(q+\alpha+2)(2q+\alpha+2)}{(2q+\alpha+4)(2q+\alpha+2)}\right)$$

Simply multiplying out the numerator concludes the proof regarding the first equation.

Inserting the definition of h_1, h_2 and h_3 also leads in the case of the second equation to the conclusion

$$h_2(q,\alpha) - h_1(q,\alpha) = \frac{2\alpha(2q+\alpha+1)+2(q+1)(2q+\alpha)}{(2q+\alpha)(2q+\alpha+1)(2q+\alpha+2)}$$
$$= \frac{4q^2+4q+6q\alpha+2\alpha^2+4\alpha}{(2q+\alpha)(2q+\alpha+1)(2q+\alpha+2)}$$
$$= \frac{(2q+\alpha+2)(2q+2\alpha)}{(2q+\alpha)(2q+\alpha+1)(2q+\alpha+2)} = h_3(q,\alpha).$$

□

Lemma B.1 (details of the proof of Lemma 3.3)**.** *Define for α, $q \in \mathbb{N}_0$*

$$\varepsilon_q := -\frac{g_2(q+1,\alpha)g_3(q,\alpha)}{g_1(q+1,\alpha)g_1(q,\alpha)} = \frac{\alpha(2q+1+\alpha)(q-1)}{(q+1+\alpha)(2q+\alpha-2)(q+\alpha)}.$$

Then, for α, $q \geq 1$ we have $0 \leq \varepsilon_q \leq 1$.

Proof. Clearly, $\varepsilon_q \geq 0$. To see the estimate $\varepsilon_q \leq 1$, we have to show

$$\begin{aligned}
\alpha(2q+1+\alpha)(q-1) &\overset{?}{\leq} (q+\alpha)(q+1+\alpha)(2q+\alpha-2) \\
\iff \quad \alpha(q-1)(2q+\alpha-2)+3\alpha(q-1) &\overset{?}{\leq} (2q+\alpha-2)(q+\alpha)(q+\alpha+1) \\
\iff \quad \underbrace{(2q+\alpha-2)}_{\geq 1}(\alpha(q-1)-(q+\alpha+1)(q+\alpha))+3\alpha(q-1) &\overset{?}{\leq} 0
\end{aligned}$$

This last inequality is certainly true if

$$\begin{aligned}
\alpha(q-1)-(q+\alpha+1)(q+\alpha)+3\alpha(q-1) &\overset{?}{\leq} 0 \\
\iff \quad 4\alpha(q-1)-(q+\alpha+1)(q+\alpha) &\overset{?}{\leq} 0 \\
\iff \quad 4\alpha(q-1)-(q+\alpha)^2-(q+\alpha) &\overset{?}{\leq} 0 \\
\iff \quad 4\alpha(q-1)-q^2-2\alpha q-\alpha^2-(q+\alpha) &\overset{?}{\leq} 0 \\
\iff \quad 2\alpha q-4\alpha-q^2-\alpha^2-(q+\alpha) &\overset{?}{\leq} 0 \\
\iff \quad -4\alpha-(q-\alpha)^2-(q+\alpha) &\overset{?}{\leq} 0
\end{aligned}$$

□

Lemma B.2 (details of the proof of Lemma 3.3)**.** *For $\alpha \in \mathbb{N}$, $q \geq 2$ we have*

$$\left(\frac{1}{g_1(q+1,\alpha)}\right)^2 \gamma_q^{(\alpha,0)} \leq (2(q+1)+\alpha+1)^2 \gamma_{q+1}^{(\alpha,0)}(q+1)\frac{1}{q+1} \tag{B.1}$$

$$\left(\frac{g_2(q+1,\alpha)}{g_1(q+1,\alpha)g_1(q,\alpha)}\right)^2 \gamma_{q-1}^{(\alpha,0)} \leq (2(q+1)+\alpha+1)\gamma_{q+1}^{(\alpha,0)}(q+1)\frac{1}{q+1} \tag{B.2}$$

Proof. We start by observing that

$$\gamma_p^{(\alpha,0)} = \frac{2^{\alpha+1}}{2p+\alpha+1}, \qquad p \geq 0 \tag{B.3}$$

so that we obtain

$$\begin{aligned}
\frac{\gamma_q^{(\alpha,0)}}{\gamma_{q+1}^{(\alpha,0)}} &= \frac{2q+\alpha+3}{2q+\alpha+1} \\
\frac{\gamma_{q-1}^{(\alpha,0)}}{\gamma_{q+1}^{(\alpha,0)}} &= \frac{2q+\alpha+3}{2q+\alpha-1}
\end{aligned}$$

We start with the bound (B.1). Then

$$\begin{aligned}
\frac{1}{(g_1(q+1,\alpha))^2}\gamma_q^{(\alpha,0)} &= \frac{1}{(g_1(q+1,\alpha))^2}\frac{\gamma_q^{(\alpha,0)}}{\gamma_{q+1}^{(\alpha,0)}}\gamma_{q+1}^{(\alpha,0)} = \frac{(2q+\alpha+1)^2(2q+\alpha+2)^2}{(2q+2\alpha+2)^2}\frac{2q+\alpha+3}{2q+\alpha+1}\gamma_{q+1}^{(\alpha,0)} \\
&\leq (2q+\alpha+1)(2q+\alpha+3)\gamma_{q+1}^{(\alpha,0)} \leq (2(q+1)+\alpha+1)^2(q+1)\frac{1}{q+1}\gamma_{q+1}^{(\alpha,0)}
\end{aligned}$$

where, in the last step, we used $q \geq 2$, $\alpha \geq 1$.

We now turn to the bound (B.2). Then

$$\begin{aligned}
&\left(\frac{g_2(q+1,\alpha)}{g_1(q+1,\alpha)g_1(q,\alpha)}\right)^2 \gamma_{q-1}^{(\alpha,0)} \\
&= \left(\frac{2\alpha}{(2q+\alpha)(2q+\alpha+2)} \frac{(2q+\alpha+1)(2q+\alpha+2)}{2q+2\alpha+2} \frac{(2q+\alpha-1)(2q+\alpha)}{2q+2\alpha}\right)^2 \gamma_{q-1}^{(\alpha,0)} \\
&= \left(\frac{2\alpha}{(2q+\alpha)(2q+\alpha+2)} \frac{(2q+\alpha+1)(2q+\alpha+2)}{2q+2\alpha+2} \frac{(2q+\alpha-1)(2q+\alpha)}{2q+2\alpha}\right)^2 \frac{\gamma_{q-1}^{(\alpha,0)}}{\gamma_{q+1}^{(\alpha,0)}} \gamma_{q+1}^{(\alpha,0)} \\
&= \left(\frac{2\alpha}{(2q+\alpha)(2q+\alpha+2)} \frac{(2q+\alpha+1)(2q+\alpha+2)}{2q+2\alpha+2} \frac{(2q+\alpha-1)(2q+\alpha)}{2q+2\alpha}\right)^2 \frac{2q+\alpha+3}{2q+\alpha-1} \gamma_{q+1}^{(\alpha,0)} \\
&\leq \gamma_{q+1}^{(\alpha,0)} \left(\frac{(2q+\alpha+1)2\alpha}{(2q+\alpha)(2q+\alpha+2)} \frac{(2q+\alpha+2)}{2q+2\alpha+2} \frac{(2q+\alpha-1)(2q+\alpha)}{2q+2\alpha}\right)^2 \frac{2q+\alpha+3}{2q+\alpha-1} \\
&\leq \gamma_{q+1}^{(\alpha,0)} \left(\frac{2p+\alpha+1}{2q+\alpha+2} \frac{2\alpha}{2q+2\alpha} \frac{2q+\alpha+2}{2q+2\alpha+2}\right)^2 (2q+\alpha+3)(2q+\alpha-1) \\
&\leq (2q+\alpha+3)^2 \gamma_{q+1}^{(\alpha,0)}
\end{aligned}$$

□

Lemma B.3. *Let $U \in C^1(-1,1)$ and let $(1-x)^\alpha U(x)$ as well as $(1-x)^{\alpha+1}U'$ be integrable. Furthermore, let*

$$\lim_{x\to 1}(1-x)^{1+\alpha}U(x) = 0 \quad \text{and} \quad \lim_{x\to -1}(1+x)U(x) = 0.$$

Consider h_1, h_2 and h_3 from (3.3). We define

$$\begin{aligned}
u_q &:= \int_{-1}^{1} (1-x)^\alpha U(x) P_q^{(\alpha,0)}(x)dx, \\
b_q &:= \int_{-1}^{1} (1-x)^\alpha U'(x) P_q^{(\alpha,0)}(x)dx.
\end{aligned}$$

Then for $q \geq 1$ and $\alpha \in \mathbb{N}_0$ the following relationship holds:

$$u_q = h_1(q,\alpha)b_{q+1} + h_2(q,\alpha)b_q + h_3(q,\alpha)b_{q-1}.$$

Proof. From (A.7) we have for $x \to -1$

$$\int_{-1}^{x} (1-t)^\alpha P_q^{(\alpha,0)}(t)dt = O(1+x)$$

and for $x \to 1$

$$\int_{-1}^{x} (1-t)^\alpha P_q^{(\alpha,0)}(t)dt = O\left((1-x)^{\alpha+1}\right).$$

Hence, using the stipulated behavior of U at the endpoints, the following integration by parts can be justified:

$$\begin{aligned}
u_q &= \int_{-1}^{1} (1-x)^\alpha U(x) P_q^{(\alpha,0)}(x)dx \\
&= \left(U(x) \int_{-1}^{x} (1-t)^\alpha P_q^{(\alpha,0)}(x)\right)\Big|_{-1}^{1} - \int_{-1}^{1} U'(x) \int_{-1}^{x} (1-t)^\alpha P_q^{(\alpha,0)}(t)dt\, dx.
\end{aligned}$$

In particular, we note that b_q is well-defined. Furthermore,

$$\begin{aligned} u_q &= -\int_{-1}^{1} U'(x) \int_{-1}^{x} (1-t)^{\alpha} P_q^{(\alpha,0)}(t)dt\,dx \\ &= \int_{-1}^{1} (1-x)^{\alpha} U'(x) \left(h_1(q,\alpha) P_{q+1}^{(\alpha,0)}(x) + h_2(q,\alpha) P_q^{(\alpha,0)}(x) + h_3(q,\alpha) P_{q-1}^{(\alpha,0)}(x) \right) dx \\ &= h_1(q,\alpha) b_{q+1} + h_2(q,\alpha) b_q + h_3(q,\alpha) b_{q-1}, \end{aligned}$$

where in the second equation we appealed to Lemma 3.2 (i). □

Lemma B.4. *For* $\beta > -1$ *and* $U \in C^1(0,1) \cap C((0,1])$ *there holds*

$$\int_0^1 x^{\beta} |U(x)|^2 \, dx \le \left(\frac{2}{\beta+1}\right)^2 \int_0^1 x^{\beta+2} \left|U'(x)\right|^2 dx + \frac{1}{\beta+1} |U(1)|^2.$$

Proof. We define according to the notation of [14, Thm. 330]

$$f(x) := \begin{cases} U'(x) & 0 < x < 1 \\ 0 & x > 1 \end{cases} \quad \text{and} \quad F(x) := \int_x^{\infty} f(t)\,dt = \begin{cases} 0 & x > 1 \\ U(1) - U(x) & 0 < x < 1. \end{cases}$$

For $\beta > -1$, [14, Thm. 330] states

$$\int_0^{\infty} x^{\beta} |F(x)|^2 dx \le \left(\frac{2}{\beta+1}\right)^2 \int_0^{\infty} x^{\beta+2} |f(x)|^2 dx$$

Hence,

$$\int_0^1 x^{\beta} (U(x) - U(1))^2 dx \le \left(\frac{2}{\beta+1}\right)^2 \int_0^1 x^{\beta+2} |U'(x)|^2 dx.$$

By rearranging terms, we get

$$\begin{aligned} \int_0^1 x^{\beta} |U(x)|^2 dx &\le \left(\frac{2}{\beta+1}\right)^2 \int_0^1 x^{\beta+2} |U'(x)|^2 dx + |U(1)|^2 \int_0^1 x^{\beta} dx \\ &\le \left(\frac{2}{\beta+1}\right)^2 \int_0^1 x^{2+\beta} |U'(x)|^2 \, dx + \frac{1}{\beta+1} |U(1)|^2. \end{aligned}$$

□

B.2 Selected proofs for Section 4

Proof of Lemma 4.3. With the definition of D^{-1} and the abbreviation $n_{pq} := 2p + 2q + 2$ we get

$$\begin{aligned} &\psi_{p,q,r}(\xi_1, \xi_2, \xi_3) = \tilde{\psi}_{p,q,r}\left(-2\,\frac{1+\xi_1}{\xi_2+\xi_3} - 1, 2\,\frac{1+\xi_2}{1-\xi_3} - 1, \xi_3\right) \\ &= P_p^{(0,0)}\left(-2\,\frac{1+\xi_1}{\xi_2+\xi_3} - 1\right) P_q^{(2p+1,0)}\left(2\,\frac{1+\xi_2}{1-\xi_3} - 1\right) P_r^{(n_{pq},0)}(\xi_3) \left(\frac{1 - 2\,\frac{1+\xi_2}{1-\xi_3} - 1}{2}\right)^p \left(\frac{1-\xi_3}{2}\right)^{p+q}. \end{aligned}$$

Expanding $P_p^{(0,0)}(x-1) P_q^{(2p+1,0)}(y-1) = \sum_{k=0}^{p} \sum_{l=0}^{q} c_{kl} x^k y^l$ leads to

$$\begin{aligned} \psi_{p,q,r}(\xi_1, \xi_2, \xi_3) &= \sum_{k=0}^{p} \sum_{l=0}^{q} c_{kl}\; 2^{k+l} \left(\frac{1+\xi_1}{\xi_2+\xi_3}\right)^k \left(\frac{1+\xi_2}{1-\xi_3}\right)^l P_r^{(n_{pq},0)}(\xi_3) \left(1 - \frac{1+\xi_2}{1-\xi_3}\right)^p \left(\frac{1-\xi_3}{2}\right)^{p+q} \\ &= \sum_{k=0}^{p} \sum_{l=0}^{q} c_{kl}\; \frac{2^{k+l}}{2^{p+q}} (1+\xi_1)^k (1+\xi_2)^l \frac{(1-\xi_3)^{q-l} (\xi_2 - \xi_3)^p}{(\xi_2+\xi_3)^k} P_r^{(n_{pq},0)}(\xi_3). \end{aligned}$$

Since $P_r^{(n_{pq},0)}$ is a polynomial of degree r, we see by the last 2 terms in the sum above that $\psi_{p,q,r} \in \mathcal{P}_{p+q+r}(\mathcal{T}^3)$.

To see the orthogonality property, we transform to the cube S^3 and make use of (3.1) three times

$$\begin{aligned}
\int_{\mathcal{T}^3} \psi_{p,q,r}(\xi)\psi_{p',q',r'}(\xi)d\xi &= \int_{\mathcal{S}^3} \tilde{\psi}_{p,q,r}(\eta)\tilde{\psi}_{p',q',r'}(\eta)\left(\frac{1-\eta_2}{2}\right)\left(\frac{1-\eta_3}{2}\right)^2 d\eta \\
&= \int_{-1}^1\int_{-1}^1\int_{-1}^1 P_p^{(0,0)}(\eta_1)P_{p'}^{(0,0)}(\eta_1)\left(\frac{1-\eta_2}{2}\right)^{p+p'+1} P_q^{(2p+1,0)}(\eta_2)P_{q'}^{(2p'+1,0)}(\eta_2) \\
&\quad \times \left(\frac{1-\eta_3}{2}\right)^{p+q+p'+q'+2} P_r^{(n_{pq},0)}(\eta_3)P_{r'}^{(n_{p'q'},0)}(\eta_3)\; d\eta_1 d\eta_2 d\eta_3 \\
&= \frac{2}{2p+1}\delta_{pp'}2^{-(2p+1)}\int_{-1}^1\int_{-1}^1 (1-\eta_2)^{2p+1}P_q^{(2p+1,0)}(\eta_2)P_{q'}^{(2p+1,0)}(\eta_2) \\
&\quad \times \left(\frac{1-\eta_3}{2}\right)^{2p+q+q'+2} P_r^{(n_{pq},0)}(\eta_3)P_{r'}^{(n_{pq'},0)}(\eta_3)\; d\eta_2 d\eta_3 \\
&= \frac{2}{2p+1}\delta_{pp'}\frac{2}{2p+2q+2}\delta_{qq'}2^{-n_{pq}}\int_{-1}^1 (1-\eta_3)^{n_{pq}}P_r^{(n_{pq},0)}(\eta_3)P_{r'}^{(n_{pq},0)}(\eta_3)\; d\eta_3 \\
&= \frac{2}{2p+1}\delta_{pp'}\frac{2}{2p+2q+2}\delta_{qq'}\frac{2}{2p+2q+2r+3}\delta_{rr'}.
\end{aligned}$$

□

Lemma B.5 (details of Lemma 4.2)**.** *Let D be the Duffy transformation and $\Gamma := \mathcal{T}^2 \times \{-1\}$. Then $D(\Gamma) = \Gamma$ and D is an isometric isomorphism with respect to the $L^2(\Gamma)$-norm.*

Proof. Obviously D is an isomorphism and by definition $D(\Gamma) = \Gamma$, so we will only show the isometry property.

Let u be a quadratic integrable function on $\mathcal{T}^3$ and consider the transformed function $\tilde{u} = u \circ D$. We have

$$\begin{aligned}
\|\tilde{u}\|^2_{L^2(\Gamma)} &= \int_\Gamma |\tilde{u}(\eta_1,\eta_2,\eta_3)|^2\, d\eta_1 d\eta_2 d\eta_3 = \int_{-1}^1\int_{-1}^1 \left|u\big(D(\eta_1,\eta_2,-1)\big)\right|^2 d\eta_1 d\eta_2 \\
&= \int_{-1}^1\int_{-1}^1 \left|u\left(\frac{(1+\eta_1)(1+\eta_2)}{2}-1,\eta_2,-1\right)\right|^2 d\eta_1 d\eta_2 \\
&= \int_{\mathcal{T}^2} |u(\xi_1,\xi_2,-1)|^2\, d\xi_1\xi_2 = \|u\|^2_{L^2(\Gamma)}.
\end{aligned}$$

□

Proof of Corollary 4.9. To prove this corollary we want to make use of Lemma 3.4. Therefore, we have to clarify that the conditions in the lemma are satisfied. We proceed in two steps. First, we require $u \in C^\infty(\overline{\mathcal{T}^3})$ and show the statement in this case and then we argue by density to achieve results in $H^1(\mathcal{T}^3)$.

Step 1: By assuming that $u \in C^\infty(\overline{\mathcal{T}^3})$ we get $\tilde{u} \in C^1(\mathcal{S}^3)$. Hence, for fixed p and q, if we recall the definition of $U_{p,q}$, we see that the map $\eta_3 \mapsto U_{p,q}(\eta_3)$ is smooth on $[-1,1]$. Considering the definition of $\widetilde{U}_{p,q}$

$$\widetilde{U}_{p,q}(\eta_3) = \frac{U_{p,q}(\eta_3)}{(1-\eta_3)^{p+q}},$$

we see that $\widetilde{U}_{p,q} \in C^1([-1,1))$ and that $\widetilde{U}_{p,q}$ has at most one pole of maximal order $p+q$ at the point $\eta_3 = 1$. In view of these preliminary considerations we conclude that the following limits exist and that the conditions in Lemma 3.4 are satisfied:

$$\lim_{\eta_3\to 1}(1-\eta_3)^{2p+2q+3}\widetilde{U}_{p,q}(\eta_3) = \lim_{\eta_3\to 1}(1-\eta_3)^{p+q+3}U_{p,q}(\eta_3) = 0.$$

and

$$\lim_{\eta_3\to -1}(1+\eta_3)\widetilde{U}_{p,q}(\eta_3) = 0.$$

Now the statement follows directly from Lemma 3.4 when looking at the definition of $\tilde{u}_{p,q,r}$ and $\tilde{u}'_{p,q,r}$ and consequently replacing U with $\widetilde{U}_{p,q}$ and α with $2p+2q+1$.

Step 2: Let $u \in H^1(\mathcal{T}^3)$. Since $C^\infty(\overline{\mathcal{T}^3})$ is dense in $H^1(\mathcal{T}^3)$, there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset C^\infty(\overline{\mathcal{T}^3})$ such that $u_n \to u$ in $H^1(\mathcal{T}^3)$ for $n\to\infty$. Because we have already proved that u_n, $n\in\mathbb{N}$ satisfies our statement, ensuring that the sequences of coefficients $\tilde{u}^n_{p,q,r}$ and $\tilde{u}'^n_{p,q,r}$ corresponding to u_n converge for fixed p,q and r will conclude the proof:

We have

$$\tilde{u}^n_{p,q,r} = 2^{p+q+2}u^n_{p,q,r} = 2^{p+q+2}\int_{\mathcal{T}^3} u_n(\xi_1,\xi_2,\xi_3)\psi_{p,q,r}(\xi_1,\xi_2,\xi_3)d\xi_1 d\xi_2 d\xi_3.$$

Since $(\psi_{p,q,r})_{p,q,r\in\mathbb{N}_0}$ forms an orthogonal basis for $L^2(\mathcal{T}^3)$ and since $H^1(\mathcal{T}^3) \subset L^2(\mathcal{T}^3)$, the maps $F: u \mapsto \tilde{u}_{p,q,r}$ are continuous linear functionals on $H^1(\mathcal{T}^3)$ and thus $\lim_{n\to\infty} F(u_n) = F(u)$.
In case of $\tilde{u}'_{p,q,r}$ we study the functionals $\widetilde{F}: u \mapsto \tilde{u}'_{p,q,r}$ that map $C^\infty(\overline{\mathcal{T}^3})$ into $\mathbb{R}$. Since $\widetilde{F}$ is a linear functional that is continuous with respect to the $H^1(\mathcal{T}^3)$-norm, we see by density of $C^\infty(\overline{\mathcal{T}^3})$ in $H^1(\mathcal{T}^3)$ that it is indeed a well-defined continuous linear functional on $H^1(\mathcal{T}^3)$ and thus again $\lim_{n\to\infty}\widetilde{F}(u_n) = \widetilde{F}(u)$. □

C Extended versions of the tables of Section 2

N	$\sup_{u\in\mathcal{P}_{2N}} \frac{\|(\Pi_N u)(1)\|^2}{\|u\|_{L^2}\|u\|_{H^1}}$	$\sup_{u\in\mathcal{P}_{2N}} \frac{\|(\Pi_N u)(1)\|^2}{\|u\|_{H^1}^2}$
1	1.18184916854199	0.87500000000000
2	1.82982112979637	1.14361283167718
3	2.15270769390416	1.15072048261852
4	2.34106259718609	1.13538600864567
5	2.45948991256407	1.11992788388317
10	2.72197668882430	1.08267283507986
15	2.82210388053635	1.06853381605478
20	2.87406416223951	1.06111064764886
25	2.90512455645115	1.05653834380496
30	2.92540310256400	1.05343954290018
35	2.93948150346734	1.05120092371494
40	2.94971129296239	1.04950802550192
45	2.95741139670201	1.04818299976127
50	2.96337279789140	1.04711769879646
55	2.96809547801154	1.04624257963827
60	2.97190920471158	1.04551089085115
65	2.97503926131730	1.04489004319026
70	2.97764417211453	1.04435662477469
75	2.97983833781387	1.04389338318215
80	2.98170613800160	1.04348732490767
85	2.98331100621813	1.04312847760604
90	2.98470143645119	1.04280906028005
95	2.98591505801301	1.04252291272595
100	2.98698146107879	1.04226509441461
105	2.98792419448666	1.04203159687688
110	2.98876220322068	1.04181913380901
115	2.98951087919110	1.04162498545355
120	2.99018284042270	1.04144688155757

Table 3: Computed constants C for 1D maximization problems

N	$\sup_{u\in\mathcal{P}_{2N}} \frac{\|\Pi_N u\|^2_{L^2(\Gamma)}}{\|u\|_{L^2(\mathcal{T}^2)}\|u\|_{H^1(\mathcal{T}^2)}}$	$\sup_{u\in\mathcal{P}_{2N}} \frac{\|\Pi_N u\|^2_{L^2(\Gamma)}}{\|u\|^2_{H^1(\mathcal{T}^2)}}$	$\sup_{u\in\mathcal{P}_{2N}} \frac{\|\Pi_N u\|^2_{H^1(\mathcal{T}^2)}}{\|u\|^2_{H^1(\mathcal{T}^2)}(N+1)}$
1	1.841709179979923	1.471718130438879	0.630765682656827
2	2.482008261007026	1.705122181047698	0.534605787746493
3	2.840187661660685	1.715716151553367	0.502563892227422
4	3.069499343400879	1.698817794545474	0.469837538241653
5	3.221409644442824	1.681456018543639	0.451490548981791
6	3.328227344860761	1.668376974400849	0.442841777476419
7	3.407909278234681	1.658518924126468	0.439947515744027
8	3.470131098161663	1.650883702558015	0.438357597812212
9	3.520381327094282	1.644802661753009	0.437324308925344
10	3.561974205023130	1.639847070385123	0.436649057543612
11	3.597047215946491	1.635732061317178	0.436162962404347
12	3.627055403122174	1.632261308353419	0.435795222777319
13	3.653032616328430	1.629295122853766	0.435497902586582
14	3.675739731999510	1.626731407637615	0.435247417526015
15	3.695751965776597	1.624493814472113	0.435028483384906
16	3.713514379154265	1.622524114095057	0.434833152302013
17	3.729377544445405	1.620777127516557	0.434656212926501
18	3.743622134346961	1.619217268307188	0.434494915639700
19	3.756475749181842	1.617816128453177	0.434347360750190
20	3.768125200720072	1.616550757911536	0.434212531866692
21	3.778725310656065	1.615402415968316	
22	3.788405606490951	1.614355650089980	
23	3.797275282737163	1.613397606293759	
24	3.805427098445950	1.612517505912479	
25	3.812940340675182	1.611706243752921	0.433709285223336
26	3.819883212382239	1.610956076034725	
27	3.826314685175567	1.610260375564773	
28	3.832286023912474	1.609613437845351	
29	3.837841988875210	1.609010326178230	
30	3.843021842931179	1.608446746919096	0.433462666731238
31	3.847860156122867	1.607918948256749	
32	3.852387490412042	1.607423637503739	
33	3.856630952540156	1.606957913069210	
34	3.860614671188471	1.606519208163681	
35	3.864360185593495	1.606105243942980	0.433441925213746
36	3.867886785255372	1.605713990297966	
37	3.871211788573052	1.605343632872021	
38	3.874350789295843	1.604992545187163	
39	3.877317859826744	1.604659264974558	
40	3.880125733031547	1.604342473993514	0.433618345964902
41	3.882785952906251	1.604040980750689	
42	3.885309010715330	1.603753705645651	
43	3.887704458222425	1.603479668157049	
44	3.889981011011230	1.603217975745874	
45	3.892146634658522	1.602967814218030	0.433964716705375
46	3.894208624113123	1.602728439323389	
47	3.896173670029407	1.602499169413209	
48	3.898047920431772	1.602279379001790	
49	3.899837032314585	1.602068493106013	
50	3.901546220032895	1.601865982255488	0.434455030379973
51	3.903180295826488	1.601671358082506	
52	3.904743708152147	1.601484169413596	
53	3.906240573788223	1.601303998798579	
54	3.907674708463756	1.601130459422183	
55	3.909049652499364	1.600963192345851	0.435064216731618

Table 4: Extended version of Table 2

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