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Adaptive *hp*-FEM for the contact problem with Tresca friction in linear elasticity: The primal formulation

P. Dörsek and J.M. Melenk

Abstract We present an *a priori* analysis of the *hp*-version of the finite element method for the primal formulation of frictional contact in linear elasticity. We introduce a new limiting case estimate for the interpolation error at Gauss and Gauss-Lobatto quadrature points. An *hp*-adaptive strategy is presented; numerical results shows that this strategy can lead to exponential convergence.

1 Introduction

We study the *hp*-version of the finite element method (*hp*-FEM) applied to a contact problem with Tresca friction in two-dimensional linear elasticity. In contrast to the more realistic Coulomb friction model, Tresca friction leads to a convex minimisation problem, which is simpler from a mathematical point of view. Nevertheless, the efficient numerical treatment of Tresca friction problems is important since solvers for such problems are building blocks for solvers for Coulomb friction problems (see [20, Section 2.5.4]).

The mathematical formulation of the frictional contact problem as a minimisation problem is provided in [12] and can be shown to be equivalent to a variational inequality of the second kind. First order h-version approximations have been available since the 1980s, see [16, 15], where the approximations can actually be chosen to be conforming and the nondifferentiable functional can be evaluated exactly. When moving to higher order discretisations, it is highly impractical to retain these properties. For the closely related variational inequalities of the first kind stemming from non-frictional obstacle and contact problems, Maischak and Stephan analysed *hp*-boundary element methods in [24, 25], and obtained convergence rates under certain regularity

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assumptions on the exact solution; they also presented an adaptive strategy based on a multilevel estimator. Results for the frictional contact problem in the *hp*-boundary element method were next provided in [6]; however, the variational crimes associated with approximating the nondifferentiable friction functional j, which is clearly necessary in a high order context, were not addressed. In [18], this discretisation error was analysed.

In the present article, we focus on two issues: Firstly, we provide an *a priori* analysis for the errors arising from a discretisation of the non-differentiable friction functional j. We proceed in a different way than it was done in [18] and base our analysis on a new limiting case interpolation error estimate for functions in the Besov space $\mathrm{B}^{1/2}_{2,1}(a,b)$. Secondly, we show numerically for a two-dimensional model problem from [19] that *hp*-adaptivity can yield exponential convergence.

2 Problem formulation

Let $\Omega \subseteq \mathbb{R}^2$ be a polygonal domain. We decompose its boundary Γ with normal vector $\boldsymbol{\nu}$ into three relatively open, disjoint parts Γ_D, Γ_N and Γ_C. On Γ_D with $|\Gamma_\mathrm{D}| > 0$ we prescribe homogeneous Dirichlet conditions, on Γ_N Neumann conditions with given traction $\mathbf{t}$, and on Γ_C contact conditions with Tresca friction, where the friction coefficient g is assumed to be constant. The volume forces are denoted by $\mathbf{F}$. Furthermore, we assume that contact holds on the entirety of Γ_C. For simplicity of exposition, we will assume that Γ_C a single edge of Ω.

We denote by $\mathrm{H}^s(\Omega)$ the usual Sobolev spaces on Ω, and similarly on the boundary parts, with norms defined through the Slobodeckij seminorms (see [29]). $\mathrm{H}^{-s}_{00}(\Gamma_\mathrm{C})$ denotes the dual space of $\mathrm{H}^s(\Gamma_\mathrm{C})$. The Besov spaces $\mathrm{B}^s_{2,q}(\Omega)$, $s \in (k, k+1)$, $k \in \mathbb{N}_0$, $q \in [1,\infty]$, are defined as the interpolation spaces $(\mathrm{H}^k(\Omega), \mathrm{H}^{k+1}(\Omega))_{s-k,q}$ (note that the J- and the K-method of interpolation generate the same spaces with equivalent norms, see e.g. [30, Lemma 24.3]). For $q = 2$, the Besov space $\mathrm{B}^s_{2,2}(\Omega)$ and the Sobolev space $\mathrm{H}^s(\Omega)$ coincide with equivalent norms, which yields that fractional order Sobolev spaces can be defined by interpolation.

Applying standard notation of linear elasticity, $\boldsymbol{\varepsilon}_{ij}(\mathbf{v}) := \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$ denotes the small strain tensor and $\boldsymbol{\sigma}(\mathbf{v}) := \mathsf{C}\boldsymbol{\varepsilon}(\mathbf{v})$ the stress tensor. Here, C is the Hooke tensor, which is assumed to be uniformly positive definite. For a vector field $\boldsymbol{\mu}$ on Γ_C, $\boldsymbol{\mu}_n := \boldsymbol{\mu}\cdot\boldsymbol{\nu}$ is its normal component and $\boldsymbol{\mu}_t := \boldsymbol{\mu} - (\boldsymbol{\mu}\cdot\boldsymbol{\nu})\boldsymbol{\nu}$ its tangential component. With the trace operator $\gamma_{0,\Gamma_\mathrm{D}} \colon (\mathrm{H}^1(\Omega))^2 \to (\mathrm{H}^{1/2}(\Gamma_\mathrm{D}))^2$, we set

$$V := \left\{\mathbf{v} \in \left(\mathrm{H}^1(\Omega)\right)^2 : \gamma_{0,\Gamma_\mathrm{D}}(\mathbf{v}) = 0\right\}. \quad (1)$$

and define the bilinear form $a\colon V\times V\to\mathbb{R}$, the linear form $L:V\to\mathbb{R}$ and the convex, nondifferentiable functional $j:V\to\mathbb{R}$ by

$$a(\mathbf{v},\mathbf{w}) := \int_\Omega \boldsymbol{\sigma}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{w})\mathrm{d}\mathbf{x}, \tag{2}$$

$$L(\mathbf{v}) := \int_\Omega \mathbf{F}\cdot\mathbf{v}\mathrm{d}\mathbf{x} + \int_{\Gamma_\mathrm{N}} \mathbf{t}\cdot\mathbf{v}\mathrm{d}s_\mathbf{x}, \qquad j(\mathbf{v}) := \int_{\Gamma_\mathrm{C}} g|\mathbf{v}_t|\mathrm{d}s_\mathbf{x}. \tag{3}$$

The primal version of the continuous version of the linearly elastic contact problem with Tresca friction then reads:

$$\text{Find the minimiser } \mathbf{u}\in V \text{ of } J(\mathbf{v}) := \frac{1}{2}a(\mathbf{v},\mathbf{v}) - L(\mathbf{v}) + j(\mathbf{v}). \tag{4}$$

As is well-known, this minimiser can also be characterised by (see [12])

$$a(\mathbf{u},\mathbf{v}-\mathbf{u}) + j(\mathbf{v}) - j(\mathbf{u}) \geq L(\mathbf{v}-\mathbf{u}) \qquad \forall\mathbf{v}\in V \tag{5}$$

The unique solvability of (4) follows by standard arguments since the Hooke tensor C is uniformly positive definite and Γ_D has positive measure, see [20, 19, 22].

Choosing a discrete finite-dimensional subspace $V_N\subseteq V$ and a discretisation $j_N\colon V_N\to\mathbb{R}$ of j, we obtain the discrete primal formulation:

$$\text{Find the minimiser } \mathbf{u}_N\in V_N \text{ of } J_N(\mathbf{v}) := \frac{1}{2}a(\mathbf{v},\mathbf{v}) - L(\mathbf{v}) + j_N(\mathbf{v}). \tag{6}$$

Let $\mathcal{T}_N$ be regular shape regular triangulations of Ω consisting of affine triangles or quadrilateral elements $K\in\mathcal{T}_N$ with diameter $h_{N,K}$. Assume that the boundary parts Γ_C, Γ_D and Γ_N are resolved by the mesh. For each $K\in\mathcal{T}_N$, let $p_{N,K}\in\mathbb{N}$ be a polynomial degree. We assume that neighboring elements have comparable polynomial degrees, i.e.,

$$p_{N,K}\sim p_{N,K'} \qquad \forall K,K'\in\mathcal{T}_N \text{ with } \overline{K}\cap\overline{K'}\neq\emptyset. \tag{7}$$

Set

$$V_N := \left\{\mathbf{v}_N\in V\colon \mathbf{v}_N|_K\in\Pi^{p_{N,K}}(K) \text{ for all } K\in\mathcal{T}_N\right\}, \tag{8}$$

where $\Pi^p(K)$ is the space of polynomials of (total) degree p if K is a triangle and $\Pi^p(K)$ is the tensor product space $\mathcal{Q}^p$ of polynomials of degree p in each variable if K is a quadrilateral.

We denote the set of edges on the contact boundary by $\mathcal{E}_{\mathrm{C},N}$, that is,

$$\mathcal{E}_{\mathrm{C},N} := \left\{E\colon E\subset\Gamma_\mathrm{C} \text{ is an edge of } \mathcal{T}_N\right\}. \tag{9}$$

We see that for every $E\in\mathcal{E}_{\mathrm{C},N}$, there exists a unique $K_E\in\mathcal{T}_N$ such that E is an edge of K_E.

3 A priori error estimates

We obtain the discretisations j_N of the functionals j by a quadrature formula: Given an element $E \in \mathcal{E}_{\mathrm{C},N}$, let $\widetilde{\mathrm{G}}_{E,q}$ be the the points of either the Gauss or Gauss-Lobatto quadrature on E with $q+1$ points, together with the corresponding weights $\omega_{E,q,\mathbf{x}}$ for $\mathbf{x} \in E$, obtained by applying an affine transformation from the reference edge $\hat{E}$ to E. Then, for $\mathbf{v}_N \in V_N$, and choosing a vector $(q_{N,E})_{E \in \mathcal{E}_{\mathrm{C},N}}$, we define

$$j_N(\mathbf{v}_N) := \sum_{E \in \mathcal{E}_{\mathrm{C},N}} j_{N,E}(\mathbf{v}_N), \qquad \text{where} \tag{10}$$

$$j_{N,E}(\mathbf{v}_N) := \sum_{\mathbf{x} \in \widetilde{\mathrm{G}}_{E,q_{N,E}}} g|\mathbf{v}_{N,t}(\mathbf{x})| \omega_{E,q_{N,E},\mathbf{x}}. \tag{11}$$

Note, in particular, that j_N is well-defined, as $\mathbf{v}_N$ is continuous on $\overline{\Omega}$, and thus also on $\overline{\Gamma_{\mathrm{C}}}$. We shall assume that there exists a constant $C > 0$ independent of N and E such that

$$C^{-1} p_{N,K_E} \le q_{N,E} \le C p_{N,K_E}. \tag{12}$$

The main result of this section is:

Theorem 3.1. *Let* $\mathbf{u} \in \mathrm{H}^{3/2}(\Omega)$ *be the solution of* (4) *and* $\mathbf{u}_N \in V_N$ *be the solution of* (6) *where* j_N *is chosen as in* (10), (11). *Assuming* (12), *we have the a priori error estimate*

$$\|\mathbf{u} - \mathbf{u}_N\|_{\mathrm{H}^1(\Omega)} \le C_{\mathbf{u}} \max_{K \in \mathcal{T}_N} h_{N,K}^{1/4} p_{N,K}^{-1/4} (1 + \sqrt[4]{\ln p_{N,K}}). \tag{13}$$

3.1 An interpolation error estimate for $\mathrm{B}^{1/2}_{2,1}$-functions

In [3], error estimates for the one-dimensional Gauss-Lobatto i_N and Gauss interpolation operators j_N are proved, namely, for $u \in \mathrm{H}^{1/2+\varepsilon}(\hat{E})$,

$$\|u - i_N u\|_{\mathrm{L}^2(\hat{E})} + \|u - j_N u\|_{\mathrm{L}^2(\hat{E})} \le C_\varepsilon N^{-1/2-\varepsilon} |u|_{\mathrm{H}^{1/2+\varepsilon}(\hat{E})} \tag{14}$$

where $\hat{E} := (-1, +1)$ is the reference element and $\varepsilon > 0$ arbitrary. As functions in $\mathrm{H}^{1/2}(\hat{E})$ are not necessarily continuous, the choice $\varepsilon = 0$ is not admissible. Thus, we consider the Besov space $\mathrm{B}^{1/2}_{2,1}(\hat{E}) = \big(\mathrm{L}^2(\hat{E}), \mathrm{H}^1(\hat{E})\big)_{1/2,1}$, which is defined as the J-method interpolation space of $\mathrm{L}^2(\hat{E})$ and $\mathrm{H}^1(\hat{E})$ with parameters $\theta = 1/2$ and $q = 1$, and consists of continuous functions.

The main result is:

Theorem 3.2. *For all* $u \in \mathrm{B}^{1/2}_{2,1}(\hat{E})$ *we have*

$$\|u - i_N u\|_{\mathrm{L}^2(\hat{E})} + \|u - j_N u\|_{\mathrm{L}^2(\hat{E})} \lesssim N^{-1/2} \|u\|_{\mathrm{B}^{1/2}_{2,1}(\hat{E})}. \tag{15}$$

We shall only provide proofs for the case of Gauss-Lobatto interpolation; for Gauss interpolation, one proceeds analogously.

The following result is a multiplicative variant of [2, Lemme III.1.4] obtained by applying the Gagliardo-Nirenberg-Sobolev inequality instead of the Sobolev imbedding theorem.

Lemma 3.3. *Let* $\psi \in \mathrm{H}^1(a,b)$. *Then,*

$$\|\psi\|^2_{\mathrm{L}^\infty(a,b)} \lesssim \frac{1}{b-a} \|\psi\|^2_{\mathrm{L}^2(a,b)} + \|\psi\|_{\mathrm{L}^2(a,b)} \|\psi'\|_{\mathrm{L}^2(a,b)}. \tag{16}$$

Let $\eta_{N,i} = \cos(\xi_{N,i})$ and $\rho_{N,i}$, $i = 0, \ldots, N$, be the nodes and weights of Gauss-Lobatto quadrature with $N+1$ points. Define the Gauss-Lobatto interpolation operator $i_N \colon \mathrm{C}([-1,+1]) \to \mathcal{P}^N$ by

$$i_N u := \sum_{j=0}^{N} u(\eta_{N,j}) L_{N,j}, \tag{17}$$

where $L_{N,j}(t) := \prod_{k \neq j} \frac{t - \eta_{N,k}}{\eta_{N,j} - \eta_{N,k}}$ is the Lagrange interpolation polynomial at $\eta_{N,j}$.

By applying the sharper estimate given in Lemma 3.3 in the proof of [2, Théorème III.1.15], we obtain the following multiplicative result.

Proposition 3.4. *For every* $u \in \mathrm{H}^1(\hat{E})$,

$$\begin{aligned} \|i_N u\|^2_{\mathrm{L}^2(\hat{E})} \lesssim & N^{-2} \left(|u(-1)|^2 + |u(1)|^2 \right) + \|u\|^2_{\mathrm{L}^2(\hat{E})} \\ & + N^{-1} \|u\|_{\mathrm{L}^2(\hat{E})} \|u' \sqrt{1-x^2}\|_{\mathrm{L}^2(\hat{E})}. \end{aligned} \tag{18}$$

Remark 3.5. Proposition 3.4 is a special case of the following, more general result. Let $\mathrm{H}^{k,\alpha}(\hat{E})$ be the space of all functions with

$$\|v\|^2_{\mathrm{H}^{k,\alpha}(\hat{E})} := \sum_{\ell=0}^{k} \int_{-1}^{+1} |u^{(\ell)}(x)|^2 (1-x^2)^{\alpha+\ell} \mathrm{d}x < \infty, \tag{19}$$

and set $\mathrm{L}^{2,\alpha}(\hat{E}) := \mathrm{H}^{0,\alpha}(\hat{E})$. These spaces were also considered in [17, Section 3]. One can show (see Appendix 5)

$$\begin{aligned} \|i^\alpha_N u\|^2_{\mathrm{L}^{2,\alpha}(\hat{E})} \lesssim & \|u\|^2_{\mathrm{L}^{2,\alpha}(\hat{E})} + N^{-1} \|u\|_{\mathrm{L}^{2,\alpha}(\hat{E})} \|u'\|_{\mathrm{L}^{2,\alpha+1}(\hat{E})} \\ & + N^{-2-2\alpha} \left(u(-1)^2 + u(+1)^2 \right) \end{aligned} \tag{20}$$

for all $u \in \mathrm{H}^{1,\alpha}(-1,+1) \cap \mathrm{C}([-1,+1])$ and all $\alpha > -1$, where i_N^α is the Gauss-Jacobi-Lobatto interpolant. Additionally, Appendix 5 shows that the Chebyshev-Lobatto interpolation operator $i_N^{-1/2}$ is stable on $\mathrm{H}^{1,-1/2}(-1,+1)$ as well as the interpolation space $\big(\mathrm{L}^{2,-1/2}(\hat{E}), \mathrm{H}^{1,-1/2}(\hat{E})\big)_{1/2,1}$.

Combining Lemma 3.3 with Proposition 3.4 yields:

Corollary 3.6. *For* $u \in \mathrm{H}^1(\hat{E})$,

$$\|i_N u\|_{\mathrm{L}^2(\hat{E})} \lesssim \|u\|_{\mathrm{L}^2(\hat{E})} + N^{-1/2} \|u\|_{\mathrm{L}^2(\hat{E})}^{1/2} \|u\|_{\mathrm{H}^1(\hat{E})}^{1/2}. \tag{21}$$

A key step towards the proof of the main result of this section, Theorem 3.2, is the following theorem:

Theorem 3.7. *Let* $T_N \colon \mathrm{C}([-1,+1]) \to \mathcal{P}^N$, $N \in \mathbb{N}$, *be continuous linear operators satisfying*

$$T_N p = p \quad \textit{for } p \in \mathcal{P}^N \quad \textit{and} \tag{22}$$

$$\|T_N u\|_{\mathrm{L}^2(\hat{E})} \lesssim \|u\|_{\mathrm{L}^2(\hat{E})} + N^{-1/2} \|u\|_{\mathrm{L}^2(\hat{E})}^{1/2} \|u\|_{\mathrm{H}^1(\hat{E})}^{1/2} \quad \textit{for } u \in \mathrm{H}^1(\hat{E}). \tag{23}$$

Then,

$$\|u - T_N u\|_{\mathrm{L}^2(\hat{E})} \lesssim N^{-1/2} \|u\|_{\mathrm{B}^{1/2}_{2,1}(\hat{E})} \quad \textit{for all } u \in \mathrm{B}^{1/2}_{2,1}(\hat{E}). \tag{24}$$

Note that $T_N \colon \mathrm{B}^{1/2}_{2,1}(\hat{E}) \to \mathcal{P}^N$ is well-defined and continuous as we have the continuous injection $\mathrm{B}^{1/2}_{2,1}(\hat{E}) \hookrightarrow \mathrm{C}([-1,+1])$ (see [30]).

Proof. We shall first prove the multiplicative error estimate

$$\|u - T_N u\|_{\mathrm{L}^2(\hat{E})} \lesssim N^{-1/2} \|u\|_{\mathrm{L}^2(\hat{E})}^{1/2} \|u\|_{\mathrm{H}^1(\hat{E})}^{1/2} : \tag{25}$$

By [26, Proposition A.2], there exists a sequence of operators $\pi_N \colon \mathrm{L}^2(\hat{E}) \to \mathcal{P}^N$ with

$$\|\pi_N u\|_{\mathrm{L}^2(\hat{E})} \lesssim \|u\|_{\mathrm{L}^2(\hat{E})} \qquad \text{for all } u \in \mathrm{L}^2(\hat{E}), \tag{26}$$

$$\|u - \pi_N u\|_{\mathrm{L}^2(\hat{E})} \lesssim N^{-1} \|u\|_{\mathrm{H}^1(\hat{E})} \qquad \text{for all } u \in \mathrm{H}^1(\hat{E}), \tag{27}$$

$$\text{and} \quad \|\pi_N u\|_{\mathrm{H}^1(\hat{E})} \lesssim \|u\|_{\mathrm{H}^1(\hat{E})} \qquad \text{for all } u \in \mathrm{H}^1(\hat{E}). \tag{28}$$

As $T_N \circ \pi_N = \pi_N$, we see by (23) that

$$\begin{aligned}
\|u - T_N u\|_{\mathrm{L}^2(\hat{E})} &\le \|u - \pi_N u\|_{\mathrm{L}^2(\hat{E})} + \|T_N(u - \pi_N u)\|_{\mathrm{L}^2(\hat{E})} \\
&\lesssim \|u - \pi_N u\|_{\mathrm{L}^2(\hat{E})} + N^{-1/2} \|u - \pi_N u\|_{\mathrm{L}^2(\hat{E})}^{1/2} \|u - \pi_N u\|_{\mathrm{H}^1(\hat{E})}^{1/2} \\
&\lesssim N^{-1/2} \|u\|_{\mathrm{L}^2(\hat{E})}^{1/2} \|u\|_{\mathrm{H}^1(\hat{E})}^{1/2}.
\end{aligned} \tag{29}$$

A careful analysis of the proof of [30, Theorem 25.3] shows that this yields

$$\|u - T_N u\|_{\mathrm{L}^2(\hat{E})} \lesssim N^{-1/2} \|u\|_{\mathrm{B}^{1/2}_{2,1}(\hat{E})}, \tag{30}$$

that is, the claimed estimate. □

Theorem 3.2 follows by combining Corollary 3.6 and Theorem 3.7.

3.2 A polynomial inverse estimate

We need the following inverse estimate:

Lemma 3.8 (Generalised $\mathrm{B}^{1/2}_{2,1}$-$\mathrm{H}^{1/2}$ p-version inverse inequality). *There exists a constant $C >$ such that for all polynomials $q \in \mathcal{P}^p$ and all $\kappa \in \mathbb{R}$,*

$$\big\| |q| - \kappa \big\|_{\mathrm{B}^{1/2}_{2,1}(\hat{E})} \leq C(1 + \sqrt{\ln p}) \left(|q|_{\mathrm{H}^{1/2}(\hat{E})} + |\kappa - \bar{q}| \right), \tag{31}$$

where $\bar{q} := \frac{1}{2} \int_{-1}^{+1} |q(x)| \mathrm{d}x$ is the integral mean of $|q|$.

The particular choices $\kappa = \bar{q}$ and $\kappa = 0$ lead to

$$\big\| |q| - \bar{q} \big\|_{\mathrm{B}^{1/2}_{2,1}(\hat{E})} \leq C(1 + \sqrt{\ln p}) \left(|q|_{\mathrm{H}^{1/2}(\hat{E})} \right),$$
$$\big\| |q| \big\|_{\mathrm{B}^{1/2}_{2,1}(\hat{E})} \leq C(1 + \sqrt{\ln p}) \left(|q|_{\mathrm{H}^{1/2}(\hat{E})} + |\bar{q}| \right) \leq C(1 + \sqrt{\ln p}) \|q\|_{\mathrm{H}^{1/2}(\hat{E})}.$$

Proof. We use the K-method of interpolation (see [29, 30]). Let

$$K(t,u) := \inf_{v \in \mathrm{H}^1(\hat{E})} \left[\|u - v\|^2_{\mathrm{L}^2(\hat{E})} + t^2 \|v\|^2_{\mathrm{H}^1(\hat{E})} \right]^{1/2}. \tag{32}$$

By [7, p.193, equation (7.4)], we see that for arbitrary $\varepsilon \in (0,1]$,

$$\begin{aligned} \big\| |q| - \kappa \big\|_{\mathrm{B}^{1/2}_{2,1}(\hat{E})} &\sim \int_0^1 t^{-1/2} K(t, |q| - \kappa) \frac{\mathrm{d}t}{t} \\ &= \int_0^\varepsilon t^{-1/2} K(t, |q| - \kappa) \frac{\mathrm{d}t}{t} + \int_\varepsilon^1 t^{-1/2} K(t, |q - \kappa|) \frac{\mathrm{d}t}{t}. \end{aligned} \tag{33}$$

For the first integral, choose $v = |q| - \kappa$ in K so that, by the Deny-Lions Lemma (see [11, Theorem 6.1] for the version used here),

$$\int_0^\varepsilon t^{-1/2} K(t, |q| - \kappa) \frac{\mathrm{d}t}{t} \le \int_0^\varepsilon t^{1/2} \big\| |q| - \kappa \big\|_{\mathrm{H}^1(\hat{E})} \frac{\mathrm{d}t}{t}$$
$$= 2\sqrt{\varepsilon} \big\| |q| - \kappa \big\|_{\mathrm{H}^1(\hat{E})} \le 2\sqrt{\varepsilon} \left(\big\| |q| - \bar{q} \big\|_{\mathrm{H}^1(\hat{E})} + 2|\kappa - \bar{q}| \right)$$
$$\le 2\sqrt{\varepsilon} \left(\big| |q| \big|_{\mathrm{H}^1(\hat{E})} + 2|\kappa - \bar{q}| \right). \tag{34}$$

Note that $\big| |q| \big|_{\mathrm{H}^1(\hat{E})} = |q|_{\mathrm{H}^1(\hat{E})}$. The inverse inequality in [1, p. 100, Theorem III.4.2] implies

$$\sqrt{\varepsilon} |q|_{\mathrm{H}^1(\hat{E})} \lesssim \sqrt{\varepsilon} p |q|_{\mathrm{H}^{1/2}(\hat{E})}. \tag{35}$$

For the second integral, we see that, by applying the Cauchy-Schwarz inequality for the measure $\frac{\mathrm{d}t}{t}$ to the functions $t \mapsto 1$ and $t \mapsto t^{-1/2} K(t, |q| - \kappa)$, and the definition of $\big\| |q| - \kappa \big\|_{\mathrm{H}^{1/2}(\hat{E})}$ by the K-method,

$$\int_\varepsilon^1 t^{-1/2} K(t, |q| - \kappa) \frac{\mathrm{d}t}{t} \le \sqrt{\int_\varepsilon^1 \frac{\mathrm{d}t}{t}} \sqrt{\int_\varepsilon^1 \left(t^{-1/2} K(t, |q| - \kappa) \right)^2 \frac{\mathrm{d}t}{t}}$$
$$\lesssim \sqrt{-\ln \varepsilon} \big\| |q| - \kappa \big\|_{\mathrm{H}^{1/2}(\hat{E})} \lesssim \sqrt{-\ln \varepsilon} \left(\big\| |q| - \bar{q} \big\|_{\mathrm{H}^{1/2}(\hat{E})} + |\kappa - \bar{q}| \right)$$
$$\lesssim \sqrt{-\ln \varepsilon} \left(\big| |q| \big|_{\mathrm{H}^{1/2}(\hat{E})} + |\kappa - \bar{q}| \right), \tag{36}$$

where the last step again follows by the Deny-Lions Lemma. Additionally, by the definition of the $\mathrm{H}^{1/2}$-seminorm, we see easily that $\big| |q| \big|_{\mathrm{H}^{1/2}(\hat{E})} \le |q|_{\mathrm{H}^{1/2}(\hat{E})}$, which yields

$$\int_\varepsilon^1 t^{-1/2} K(t, |q| - \kappa) \frac{\mathrm{d}t}{t} \lesssim \sqrt{-\ln \varepsilon} \left(|q|_{\mathrm{H}^{1/2}(\hat{E})} + |\kappa - \bar{q}| \right), \tag{37}$$

We set $\varepsilon := \frac{1}{p^2}$ and obtain

$$\big\| |q| - \kappa \big\|_{\mathrm{B}^{1/2}_{2,1}(\hat{E})} \lesssim (1 + \sqrt{\ln p}) |q|_{\mathrm{H}^{1/2}(\hat{E})} + (p^{-1} + \sqrt{\ln p}) |\kappa - \bar{q}|$$
$$\le (1 + \sqrt{\ln p}) \left(|q|_{\mathrm{H}^{1/2}(\hat{E})} + |\kappa - \bar{q}| \right). \quad \square \tag{38}$$

3.3 Convergence rates: Proof of Theorem 3.1

We now prove a convergence rate result for the primal formulation of the friction problem. We follow in style the article [5]. A similar estimate was derived in [18, Lemma 4.1] using different techniques.

In the following, $\mathbf{u}$ and $\mathbf{u}_N$ denote the solutions of (4) and (6).

Proposition 3.9. *Define* $S_{\mathbf{u}}(\mathbf{v}) := a(\mathbf{u}, \mathbf{v}) - L(\mathbf{v})$. *Then, for all* $\mathbf{v}_N \in V_N$,

$$a(\mathbf{u}-\mathbf{u}_N,\mathbf{u}-\mathbf{u}_N) \le a(\mathbf{u}-\mathbf{u}_N,\mathbf{u}-\mathbf{v}_N) + S_{\mathbf{u}}(\mathbf{v}_N-\mathbf{u}) \\ + j_N(\mathbf{v}_N) - j(\mathbf{v}_N) + j(\mathbf{u}_N) - j_N(\mathbf{u}_N) + j(\mathbf{v}_N-\mathbf{u}). \tag{39}$$

Proof. It follows from (6) that

$$\begin{aligned}
&a(\mathbf{u}-\mathbf{u}_N,\mathbf{u}-\mathbf{u}_N) = \\
&\ a(\mathbf{u}-\mathbf{u}_N,\mathbf{u}-\mathbf{v}_N) + a(\mathbf{u},\mathbf{v}_N-\mathbf{u}_N) - a(\mathbf{u}_N,\mathbf{v}_N-\mathbf{u}_N) \\
&\ \le a(\mathbf{u}-\mathbf{u}_N,\mathbf{u}-\mathbf{v}_N) + a(\mathbf{u},\mathbf{v}_N-\mathbf{u}_N) - L(\mathbf{v}_N-\mathbf{u}_N) + j_N(\mathbf{v}_N) - j_N(\mathbf{u}_N) \\
&\ \le a(\mathbf{u}-\mathbf{u}_N,\mathbf{u}-\mathbf{v}_N) + S_{\mathbf{u}}(\mathbf{v}_N-\mathbf{u}_N) + j_N(\mathbf{v}_N) - j_N(\mathbf{u}_N).
\end{aligned}$$

Since for all $\mathbf{v} \in V$ we have

$$\begin{aligned}
S_{\mathbf{u}}(\mathbf{v}_N-\mathbf{u}_N) &= S_{\mathbf{u}}(\mathbf{v}_N-\mathbf{u}) + S_{\mathbf{u}}(\mathbf{u}-\mathbf{v}) + S_{\mathbf{u}}(\mathbf{v}-\mathbf{u}_N) \\
&\le S_{\mathbf{u}}(\mathbf{v}_N-\mathbf{u}) + j(\mathbf{v}) - j(\mathbf{u}) + S_{\mathbf{u}}(\mathbf{v}-\mathbf{u}_N),
\end{aligned} \tag{40}$$

we obtain

$$\begin{aligned}
a(\mathbf{u}-\mathbf{u}_N,\mathbf{u}-\mathbf{u}_N) \le &a(\mathbf{u}-\mathbf{u}_N,\mathbf{u}-\mathbf{v}_N) + S_{\mathbf{u}}(\mathbf{v}_N-\mathbf{u}) + S_{\mathbf{u}}(\mathbf{v}-\mathbf{u}_N) \\
&+ j_N(\mathbf{v}_N) - j(\mathbf{u}) + j(\mathbf{v}) - j_N(\mathbf{u}_N).
\end{aligned} \tag{41}$$

Choose now $\mathbf{v} = \mathbf{u}_N$ and note that $j(\mathbf{v}_N) \le j(\mathbf{v}_N-\mathbf{u}) + j(\mathbf{u})$, and thus $-j(\mathbf{u}) \le -j(\mathbf{v}_N) + j(\mathbf{u}-\mathbf{v}_N)$. The claim then follows. □

Proposision 3.9 shows that the main task is to estimate the error introduced by approximating j by j_N on V_N. This will be done now.

Theorem 3.10. *Let $E \in \mathcal{E}_{\mathrm{C},N}$ and $K_E \in \mathcal{T}_N$ be such that E is an edge of K_E. Let $\mathbf{w}_N \in V_N$, and set*

$$j_E(\mathbf{w}_N) := \int_E g|\mathbf{w}_N| \mathrm{d}s_{\mathbf{x}}. \tag{42}$$

Then, we have the estimate

$$\begin{aligned}
|j_E(\mathbf{w}_N) - j_{N,E}(\mathbf{w}_N)| &\lesssim h_{N,K_E}(1+\sqrt{\ln p_{N,K_E}})q_{N,E}^{-1/2}|\mathbf{w}_N|_{\mathrm{H}^1(K_E)} \\
&\lesssim h_{N,K_E}(1+\sqrt{\ln p_{N,K_E}})p_{N,K_E}^{-1/2}|\mathbf{w}_N|_{\mathrm{H}^1(K_E)}.
\end{aligned} \tag{43}$$

Proof. It is clear that

$$j_{N,E}(\mathbf{w}_N) = g\int_E i_{E,q_{N,E}}|\mathbf{w}_N| \mathrm{d}s_{\mathbf{x}}, \tag{44}$$

where $i_{E,q_{N,E}}$ denotes the local interpolation operator on E at the $q_{N,E}+1$ Gauss-Lobatto points. Theorem 3.2 provides a constant $C>0$ such that

$$\big\||\mathbf{w}| - i_q|\mathbf{w}|\big\|_{\mathrm{L}^2(\hat{E})} \le Cq^{-1/2}\|\mathbf{w}\|_{\mathrm{B}^{1/2}_{2,1}(\hat{E})} \qquad \forall q \in \mathbb{N}. \tag{45}$$

Apply now a scaling argument: Let $\Phi_E\colon \hat{E} \to E$ be an invertible, affine mapping. As $i_{E,q_{N,E}}$ reproduces constant functions, we have for any $\kappa \in \mathbb{R}$,

$$\begin{aligned}\big\||\mathbf{w}_N| - i_{E,q_{N,E}}|\mathbf{w}_N|\big\|_{\mathrm{L}^2(E)} &= \|(|\mathbf{w}_N| - \kappa) - i_{E,q_{N,E}}(|\mathbf{w}_N| - \kappa)\|_{\mathrm{L}^2(E)} \\ &= \frac{h_{N,E}^{1/2}}{2}\big\|(|\mathbf{w}_N \circ \Phi_E| - \kappa) - i_{q_{N,E}}(|\mathbf{w}_N \circ \Phi_E| - \kappa)\big\|_{\mathrm{L}^2(\hat{E})} \\ &\le C h_{N,E}^{1/2} q_{N,E}^{-1/2}\big\||\mathbf{w}_N \circ \Phi_E| - \kappa\big\|_{\mathrm{B}^{1/2}_{2,1}(\hat{E})}.\end{aligned}$$

With the choice $\kappa := \frac{1}{2}\int_{\hat{E}}|\mathbf{w}_N \circ \Phi_E|\mathrm{d}x$, Lemma 3.8 gives

$$\big\||\mathbf{w}_N \circ \Phi_E| - \kappa\big\|_{\mathrm{B}^{1/2}_{2,1}(\hat{E})} \lesssim \left(1 + \sqrt{\ln p_{N,E}}\right)|\mathbf{w}_N \Phi_E|_{\mathrm{H}^{1/2}(\hat{E})}.$$

Thus, again by scaling,

$$\||\mathbf{w}_N| - i_{E,q_{N,E}}|\mathbf{w}_N|\|_{\mathrm{L}^2(E)} \lesssim C h_{N,E}^{1/2}(1 + \sqrt{\ln p_{N,E}}) q_{N,E}^{-1/2}|\mathbf{w}_N|_{\mathrm{H}^{1/2}(E)}.$$

We obtain by the trace theorem and (12) that

$$\begin{aligned}|j_E(\mathbf{w}_N) - j_{N,E}(\mathbf{w}_N)| &\lesssim h_{N,E}(1 + \sqrt{\ln p_{N,E}}) q_{N,E}^{-1/2}|\mathbf{w}_N|_{\mathrm{H}^{1/2}(E)} \\ &\lesssim h_{N,K_E}(1 + \sqrt{\ln p_{N,K_E}}) q_{N,E}^{-1/2}|\mathbf{w}_N|_{\mathrm{H}^1(K_E)} \\ &\lesssim h_{N,K_E}(1 + \sqrt{\ln p_{N,K_E}}) p_{N,K_E}^{-1/2}|\mathbf{w}_N|_{\mathrm{H}^1(K_E)}. \quad \square\end{aligned}$$

Let h_N, p_N and q_N be the local mesh width, polynomial degree and quadrature order and introduce the local approximation quantification

$$\omega_N := h_N^{1/2} p_N^{-1/2}(1 + \sqrt{\ln p_N}). \tag{46}$$

Corollary 3.11. *Set* $\mathcal{S}_N := \bigcup_{E \in \mathcal{E}_{\mathrm{C},N}} K_E$. *Let* ω_N *be given by (46). Then: For every* $\mathbf{w}_N \in V_N$,

$$|j(\mathbf{w}_N) - j_N(\mathbf{w}_N)| \lesssim \|\omega_N \nabla \mathbf{w}_N\|_{\mathrm{L}^2(\mathcal{S}_N)} \le \|\omega_N \nabla \mathbf{w}_N\|_{\mathrm{L}^2(\Omega)}. \tag{47}$$

Proof. Applying Theorem 3.10 to $\mathbf{w}_N$ and summing over $E \in \mathcal{E}_{\mathrm{C},N}$, we obtain by the discrete Cauchy-Schwarz inequality and the trace theorem that

$$\begin{aligned}|j_N(\mathbf{w}_N) - j(\mathbf{w}_N)| &\le \sum_{E \in \mathcal{E}_{\mathrm{C},N}} |j_E(\mathbf{w}_N) - j(\mathbf{w}_N)| \\ &\lesssim \sum_{E \in \mathcal{E}_{\mathrm{C},N}} h_{N,E}^{1/2} h_{N,E}^{1/2}(1 + \sqrt{\ln p_{N,K_E}}) q_{N,E}^{-1/2}|\mathbf{w}_N|_{\mathrm{H}^1(K_E)} \\ &\le \left(\sum_{E \in \mathcal{E}_{\mathrm{C},N}} h_{N,E}\right)^{1/2} \left(\sum_{E \in \mathcal{E}_{\mathrm{C},N}} h_{N,E}(1 + \ln p_{N,K_E}) q_{N,K_E}^{-1}|\mathbf{w}_N|^2_{\mathrm{H}^1(K_E)}\right)^{1/2}\end{aligned}$$

$$= |\Gamma_{\mathrm{C}}|^{1/2} \left(\sum_{E \in \mathcal{E}_{\mathrm{C},N}} h_{N,E}(1 + \ln p_{N,K_E}) q_{N,K_E}^{-1} |\mathbf{w}_N|^2_{\mathrm{H}^1(K_E)} \right)^{1/2}$$
$$\lesssim \|h_N^{1/2}(1 + \sqrt{\ln p_N}) q_N^{-1/2} \nabla \mathbf{w}_N\|_{\mathrm{L}^2(\mathcal{S}_N)} \quad \square$$

Theorem 3.12. *Set $\mathcal{S}_N := \bigcup_{E \in \mathcal{E}_{\mathrm{C},N}} K_E$, and let $\mathbf{u}_N$ and $\mathbf{u}$ be the solutions of* (6) *and* (4)*, respectively. Let ω_N be given by (46). Then:*

$$\begin{aligned}
&\|\mathbf{u} - \mathbf{u}_N\|_{\mathrm{H}^1(\Omega)} \\
&\lesssim \inf_{\mathbf{v}_N \in V_N} \Big(\|\omega_N \nabla \mathbf{u}_N\|_{\mathrm{L}^2(\mathcal{S}_N)} + \|\omega_N \nabla \mathbf{v}_N\|_{\mathrm{L}^2(\mathcal{S}_N)} \\
&\qquad + \|\mathbf{u} - \mathbf{v}_N\|_{\mathrm{H}^1(\Omega)} + \|\mathbf{u} - \mathbf{v}_N\|^2_{\mathrm{H}^1(\Omega)} + |S_{\mathbf{u}}(\mathbf{u} - \mathbf{v}_N)| \Big)^{1/2} \\
&\lesssim \inf_{\mathbf{v}_N \in V_N} \Big(\|\omega_N \nabla \mathbf{u}_N\|_{\mathrm{L}^2(\Omega)} + \|\omega_N \nabla \mathbf{v}_N\|_{\mathrm{L}^2(\Omega)} \\
&\qquad + \|\mathbf{u} - \mathbf{v}_N\|_{\mathrm{H}^1(\Omega)} + \|\mathbf{u} - \mathbf{v}_N\|^2_{\mathrm{H}^1(\Omega)} + |S_{\mathbf{u}}(\mathbf{u} - \mathbf{v}_N)| \Big)^{1/2}.
\end{aligned}$$

Before proving Theorem 3.12, we remark that estimating the $\mathrm{L}^2(\mathcal{S}_N)$-norm by the $\mathrm{L}^2(\Omega)$-norm is typically very pessimistic. Heuristically, the strip is of area $O(h)$, so we expect to obtain another power of h in this estimate.

Proof. We employ Proposition 3.9. By the V-boundedness and V-ellipticity of a and the V-boundedness of j, we see that

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_N\|^2_{\mathrm{H}^1(\Omega)} \lesssim{}& \|\mathbf{u} - \mathbf{u}_N\|_{\mathrm{H}^1(\Omega)} \|\mathbf{u} - \mathbf{v}_N\|_{\mathrm{H}^1(\Omega)} + \|\mathbf{v}_N - \mathbf{u}\|_{\mathrm{H}^1(\Omega)} \\
&+ j_N(\mathbf{v}_N) - j(\mathbf{v}_N) + j(\mathbf{u}_N) - j_N(\mathbf{u}_N) + S_{\mathbf{u}}(\mathbf{v}_N - \mathbf{u}),
\end{aligned}$$

from which it follows by applying the ε-trick that

$$\begin{aligned}
&\|\mathbf{u} - \mathbf{u}_N\|^2_{\mathrm{H}^1(\Omega)} \lesssim \|\mathbf{u} - \mathbf{v}_N\|^2_{\mathrm{H}^1(\Omega)} + \|\mathbf{u} - \mathbf{v}_N\|_{\mathrm{H}^1(\Omega)} \\
&\qquad + |j_N(\mathbf{v}_N) - j(\mathbf{v}_N)| + |j_N(\mathbf{u}_N) - j(\mathbf{u}_N)| + S_{\mathbf{u}}(\mathbf{v}_N - \mathbf{u}).
\end{aligned}$$

Applying Corollary 3.11 to $\mathbf{u}_N$ and $\mathbf{v}_N$, the result now follows by the local equivalence of p_N and q_N. $\square$

Clearly, choosing $\mathbf{v}_N \in V_N$ to be the best approximation of $\mathbf{u}$ with respect to the H^1-norm proves that $\|h_N^{1/2}(1 + \sqrt{\ln p_N}) p_N^{-1/2} \mathbf{v}_N\|_{\mathrm{H}^1(\Omega)}$ stays bounded and converges with a rate of $h_N^{1/2}(1 + \sqrt{\ln p_N}) p_N^{-1/2}$, and $\|\mathbf{u} - \mathbf{v}_N\|_{\mathrm{H}^1(\Omega)} \to 0$ if $h_N/p_N \to 0$. It still remains to show that $\|\mathbf{u}_N\|_{\mathrm{H}^1(\Omega)}$ stays bounded.

Lemma 3.13. *The norms in $\mathrm{H}^1(\Omega)$ of the solutions $\mathbf{u}_N$ of* (6) *stay bounded for $N \to \infty$.*

Proof. Choose $\mathbf{v}_N = 0$. Then, as $j_N(\mathbf{w}_N) \geq 0$ for all $\mathbf{w}_N \in V_N$,

$$a(\mathbf{u}_N, \mathbf{u}_N) \leq L(\mathbf{u}_N) - j_N(\mathbf{u}_N) \leq L(\mathbf{u}_N). \tag{48}$$

The result now follows by the coercitivity of a and the boundedness of L. □

Remark 3.14. Lemma 3.13 shows that the primal method converges if $\bigcup_N V_N$ is dense in V. This can also be shown similarly as in [18] using Glowinski's theorem.

Finally, Theorem 3.1 now easily follows from Theorem 3.12 together with the interpolation operators in [26].

4 Numerical experiments

4.1 A posteriori error estimation

One way to realise numerically the minimisation problem (6) is by dualisation. Specifically, we assume that the quadrature points $\tilde{G}_{E,q_{N,E}}$ are the Gauss points and that

$$q_{N,E} \geq p_{N,K_E} - 1 \qquad \forall E \in \mathcal{E}_{\mathrm{C},N}. \tag{49}$$

We introduce the bilinear forms b and b_N by

$$b(\mathbf{u},\lambda) := g\int_{\Gamma_{\mathrm{C}}} \mathbf{u}_t\lambda, \qquad b_N(\mathbf{u},\lambda) := g\sum_{E\in\mathcal{E}_{\mathrm{C},N}}\sum_{\mathbf{x}\in\tilde{G}_{E,q_{N,E}}} \omega_{E,q_N,E,\mathbf{x}}\mathbf{u}_t(\mathbf{x})\lambda(\mathbf{x}),$$

$$W_N := \{\lambda \in L^2(\Gamma_{\mathrm{C}}) :\ \lambda|_E \in \mathcal{P}^{q_{N,E}} \quad \forall E \in \mathcal{E}_{\mathrm{C},N}\},$$
$$\Lambda_N := \{\lambda \in W_N :\ |\lambda(\mathbf{x})| \leq 1 \quad \forall \mathbf{x} \in \tilde{G}_{E,q_{N,E}} \quad \forall E \in \mathcal{E}_{\mathrm{C},N}\},$$

where, in the present 2D setting, we view the tangential component $\mathbf{u}_t$ of $\mathbf{u}$ as a scalar function in the definition of b and b_N. It is easy to see that $j_N(\mathbf{u}) = \sup_{\lambda\in\Lambda_N} b_N(\mathbf{u},\lambda)$. Hence, the minimisation problem (6) can be reformulated as a saddle point problem of finding $(\mathbf{u}_N,\lambda_N) \in V_N \times \Lambda_N$ such that

$$a(\mathbf{u}_N,\mathbf{v}) \qquad +b_N(\mathbf{v},\lambda_N) = L(\mathbf{v}) \qquad \forall v \in V_N, \tag{50a}$$
$$b_N(\mathbf{u}_N,\mu-\lambda_N) \qquad \leq 0 \qquad \forall \mu \in \Lambda_N. \tag{50b}$$

(50) has solutions; the component $\mathbf{u}_N$ is the unique solution of (6), which justifies our using the same symbol. Any Lagrange multiplier λ_N can be used for *a posteriori* error estimation. Indeed, exploiting the fact that $b(\mathbf{v},\lambda) = b_N(\mathbf{v},\lambda)$ for all $\mathbf{v} \in V_N$ and $\lambda \in W_N$, one can proceed as in [8, Sec. 4] to show the following result (see [9, Appendix] for the details):

Theorem 4.1. *Assume (49). Let $\mathbf{u}$, $\mathbf{u}_N$ solve (4), (6), and let λ_N be a Langrange multiplier satisfying (50). Then $\|\mathbf{u}-\mathbf{u}_N\|^2_{H^1(\Omega)} \leq C\eta_N^2$, where the* error indicator

$$\eta_N^2 := \sum_{K\in\mathcal{T}_N} \eta_{N,K}^2 \tag{51}$$

is defined in terms of element error indicators

$$\eta_{N,K}^2 := h_{N,K}^2 p_{N,K}^{-2} \|\mathbf{r}_K\|_{\mathrm{L}^2(K)}^2 + h_{N,K} p_{N,K}^{-1} \sum_{E \subseteq \partial K} \|\mathbf{R}_E\|_{\mathrm{L}^2(E)}^2 \\ + j_{\partial K \cap \Gamma_{\mathrm{C}}}(\mathbf{u}_N) - b_{\partial K \cap \Gamma_{\mathrm{C}}}(\mathbf{u}_N, \tilde{\lambda}_N) + g^2 \|\lambda_N - \tilde{\lambda}_N\|_{\mathrm{H}_{00}^{-1/2}(\partial K \cap \Gamma_{\mathrm{C}})}^2, \tag{52}$$

where the element residuals $\mathbf{r}_K$ *and the edge jumps* $\mathbf{R}_E$ *are given by*

$$\mathbf{r}_K := -\operatorname{div} \boldsymbol{\sigma}(\mathbf{u}_N) - \mathbf{F}, \qquad \mathbf{R}_E := \begin{cases} \frac{1}{2} [\boldsymbol{\sigma}(\mathbf{u}_N) \cdot \boldsymbol{\nu}]_E & \text{if } E \subset \Omega, \\ (\boldsymbol{\sigma}(\mathbf{u}_N) \cdot \boldsymbol{\nu})_t + g\lambda_N & \text{if } E \subset \Gamma_{\mathrm{C}} \\ \boldsymbol{\sigma}(\mathbf{u}_N) \cdot \boldsymbol{\nu} - \mathbf{t} & \text{if } E \subset \Gamma_{\mathrm{N}}, \\ 0 & \text{if } E \subset \Gamma_{\mathrm{D}}. \end{cases}$$

Finally, $\tilde{\lambda}_N$ *is the* $L^2(\Gamma_{\mathrm{C}})$*-projection of* λ_N *onto* Λ_N*.*

Remark 4.2. In our numerical experiments, we estimate the error indicator η_N further by replacing the $H_{00}^{-1/2}$-norm by the L^2-norm and estimating rather generously the contributions of $j_{\partial K \cap \Gamma_{\mathrm{C}}}(\mathbf{u}_N) - b_{\partial K \cap \Gamma_{\mathrm{C}}}(\mathbf{u}_N, \tilde{\lambda}_N)$ for those edges $E \subset \Gamma_{\mathrm{C}}$ where $\lambda_N|_E \neq \tilde{\lambda}_N|_E$. We refer to [8, Remark 4.3] for details.

4.2 Numerical examples

We consider the two-dimensional numerical problem of [19, Example 6.12]. Let $\Omega = (0,4) \times (0,4)$, assume homogeneous Dirichlet conditions on $\Gamma_{\mathrm{D}} := \{4\} \times (0,4)$, frictional contact on $\Gamma_{\mathrm{C}} := (0,4) \times \{0\}$, and Neumann conditions on $\Gamma_{\mathrm{N}} := (\{0\} \times (0,4)) \cup ((0,4) \times \{4\})$, where $\mathbf{t}(0,s) = (150(5-s), -75)\mathrm{daN/mm}^2$ for $s \in (0,4)$ and $\mathbf{t} = 0$ on $(0,4) \times \{4\}$. The elasticity parameters are chosen to be $E = 1500\mathrm{daN/mm}^2$ and $\nu = 0.4$, the friction coefficient is $g = 450\mathrm{daN/mm}^2$. We assume plane stress conditions.

We perform 6 numerical experiments: *h*-uniform and *h*-adaptive methods with polynomial degrees 2 and 3; a *p*-uniform method starting with polynomial degree 2; and an *hp*-adaptive method starting with polynomial degree 3. The initial meshes are uniform and consist of 16 squares.

Quadrilateral meshes with hanging nodes are used. We require the "one hanging node rule" and that all irregular nodes be one-irregular. Differing polynomial degrees on neighbouring elements are resolved by using the minimum rule on the edge. For the discretisation of j, we choose Gaussian quadrature and $q_{N,E} = p_{N,K_E} - 1$ for $E \in \mathcal{E}_{\mathrm{C},N}$, i.e., we use p_{N,K_E} quadrature points in (11). As described in Section 4.1, the minimisation problem (6) is recast in primal-dual form and the resulting first kind variational inequality is solved with the MPRGP algorithm (see [10]). As a by-product, we obtain a Lagrange multiplier $\lambda_N \in W_N$, which is used to define error indicators of (51). These are plotted in Figure 1. All calculations were done using `maiprogs` ([23]). For the static condensation of the internal degrees of freedom, `pardiso` was used ([27, 28, 21]).

In the *hp*-adaptive scheme, each adaptive step refines those 20% of the elements that have the largest error indicators (52). The decision of whether to perform an h-refinement or a p-enrichnement is based on [8, Alg. 5.1] with $\delta = 1$. The essential idea of that algorithm is similar to Strategy II of [13]: A p-enrichment for an element K can only be done if two conditions are met: (i) the coefficients of the Legendre expansion of the displacement field decay sufficiently rapidly and (ii), if K has an edge E on the contact boundary Γ_{C}, then λ_N satisfies $\|\lambda_N\|_{L^\infty(E)} \leq 1$. This last condition $\|\lambda_N\|_{\mathrm{L}^\infty(E)} \leq 1$ is strictly enforced by ensuring that an upper bound for $\|\lambda_N\|_{\mathrm{L}^\infty(E)}$ is bounded by 1. This upper bound is obtained by expanding the polynomial $\lambda_N|_E$ into a Legendre series, computing the extrema of the leading quadratic part explicitly and estimating the remainder with the triangle inequality; we refer to [8] for details, where a similar strategy is employed in the context of a primal-dual formulation.

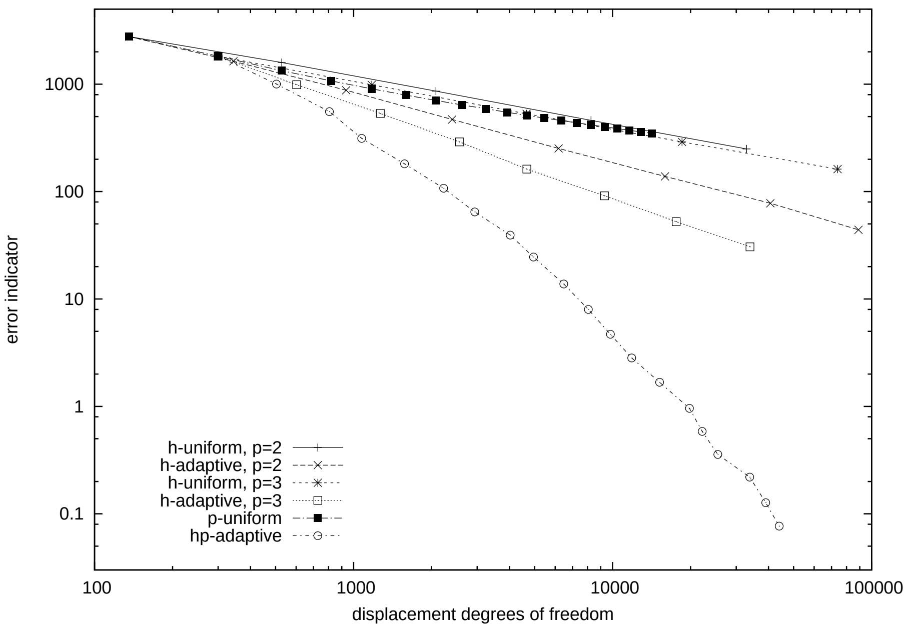

Fig. 1 Error indicator η_N vs. problem size

Figure 1 shows the error indicators for the two uniform and adaptive h-methods, the uniform p-method and the *hp*-adaptive method. Assuming that the error behaves like $\|\mathbf{u} - \mathbf{u}_N\|_{\mathrm{H}^1(\Omega)} = CN^{-\alpha}$ in the uniform h- and p-

versions and the adaptive h-versions, we obtain by a least squares fit rates of about $\alpha = 0.44$ for the h-uniform and $\alpha = 0.33$ for the p-uniform methods and about $\alpha = 0.64$ and $\alpha = 0.87$ for the adaptive schemes with polynomial degrees 2 and 3, respectively. For the hp-adaptivity, we obtain $\gamma = 0.35$, assuming an error behaviour of the form $\|\mathbf{u} - \mathbf{u}_N\|_{H^1(\Omega)} = C\exp(-\gamma N^{1/3})$.

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5 Appendix A: Interpolation in weighted norms

Let $\xi^\alpha_{N,j} = \cos\theta^\alpha_{N,N-j}$, $j = 0,\dots,N$, be the Gauss-Lobatto-Jacobi quadrature points, and $\rho^\alpha_{N,j}$, $j = 0,\dots,N$ the corresponding weights. Recall that the $(\xi^\alpha_{N,j})_{j=0,\dots,N}$ are symmetric with respect to 0.

The following results are well-known (see [4, (IV.19.24) and (IV.19.28)]).

Lemma 5.1. *For* $1 \le j \le \lfloor \frac{N}{2} \rfloor$ *and* $\alpha > -1$,

$$\frac{(2j-1)\pi}{2N-2} \le \theta^\alpha_{N,j} \le \frac{(2j+1)\pi}{2N-2} \tag{53}$$

and for $\alpha > -1$ *and* $1 \le j \le N-1$,

$$\rho_{N,j}^{\alpha} \lesssim N^{-1} \sin^{2\alpha+1} \theta_{N,j}^{\alpha}. \tag{54}$$

For $j = 0$ *and* $j = N$,

$$\rho_{N,0}^{\alpha} = \rho_{N,N}^{\alpha} \lesssim N^{-2-2\alpha}. \tag{55}$$

For the quadrature of Jacobi polynomials, we have (cf. [2, Lemme III.1.12]):

Lemma 5.2. *For* $\alpha > -1$ *and* $N \ge 1$,

$$\sum_{j=0}^{N} J_N^{\alpha}(\xi_j^{\alpha})^2 \rho_j^{\alpha} = \left(2 + \frac{2\alpha+1}{N}\right) \|J_N^{\alpha}\|^2_{\mathrm{L}^{2,\alpha}(-1,+1)}. \tag{56}$$

Proof. By [4, (IV.19.30)],

$$J_N^{\alpha}(\xi_j^{\alpha})^2 \rho_j^{\alpha} = \frac{2^{2\alpha+1}\Gamma(N+\alpha+1)^2}{NN!\Gamma(N+2\alpha+2)} \tag{57}$$

and by [4, (IV.19.1) and (IV.19.31)],

$$J_N^{\alpha}(-1)^2 \rho_0^{\alpha} = J_N^{\alpha}(+1)^2 \rho_N^{\alpha} = \frac{2^{2\alpha+1}\Gamma(N+\alpha+1)^2}{NN!\Gamma(N+2\alpha+2)}(\alpha+1). \tag{58}$$

As, by [4, (IV.19.8)],

$$\begin{aligned} \|(J_N^{\alpha})^2\|_{\mathrm{L}^{2,\alpha}(-1,+1)} &= \frac{2^{2\alpha+1}\Gamma(N+\alpha+1)^2}{(2N+2\alpha+1)N!\Gamma(N+2\alpha+1)} \\ &= \frac{N(N+2\alpha+1)}{2N+2\alpha+1} \frac{2^{2\alpha+1}\Gamma(N+\alpha+1)^2}{NN!\Gamma(N+2\alpha+2)}, \end{aligned} \tag{59}$$

we see that

$$\begin{aligned} \sum_{j=0}^{N} J_N^{\alpha}(\xi_j^{\alpha})^2 \rho_j^{\alpha} &= (N+2\alpha+1) \frac{2N+2\alpha+1}{N(2N+2\alpha+1)} \|J_N^{\alpha}\|^2_{\mathrm{L}^{2,\alpha}(-1,+1)} \\ &= \frac{2N+2\alpha+1}{N} \|J_N^{\alpha}\|^2_{\mathrm{L}^{2,\alpha}(-1,+1)}. \quad \square \end{aligned} \tag{60}$$

As the $(J_N^{\alpha})_N$ are orthogonal in $\mathrm{L}^{2,\alpha}(-1,+1)$, we obtain similarly as in [2, Corollaire III.1.13]:

Lemma 5.3. *There exists a constant* $C > 0$ *such that for every* $\varphi_N \in \mathcal{P}_N$,

$$\|\varphi_N\|^2_{\mathrm{L}^{2,\alpha}(-1,+1)} \le \sum_{j=0}^{N} \varphi_N(\xi_j^{\alpha})^2 \rho_j^{\alpha} \le C \|\varphi_N\|^2_{\mathrm{L}^{2,\alpha}(-1,+1)}. \tag{61}$$

With the Gauss-Lobatto-Jacobi interpolation operator at $(\xi_{N,j}^{\alpha})_{j=0,\dots,N}$ denoted by i_N^{α}, we obtain:

Theorem 5.4. *Let $\alpha \geq -1/2$. For every $u \in \mathrm{H}^{1,\alpha}(-1,+1) \cap \mathrm{C}([-1,+1])$,*

$$\|i_N^{\alpha} u\|^2_{\mathrm{L}^{2,\alpha}(-1,+1)} \lesssim \|u\|^2_{\mathrm{L}^{2,\alpha}(-1,+1)} + N^{-1}\|u\|_{\mathrm{L}^{2,\alpha}(-1,+1)}\|u'\|_{\mathrm{L}^{2,\alpha+1}(-1,+1)} + N^{-2-2\alpha}\left(u(-1)^2 + u(+1)^2\right). \quad (62)$$

Note that by [17, Lemma 3.1], $\mathrm{H}^{1,-1/2}(-1,+1) \subset \mathrm{C}([-1,+1])$.

Proof. Set $\hat{u}(\theta) := u(\cos\theta)$. Define the open intervals $\hat{I}_{N,j}$, $j = 1,\dots,N-1$, by $\hat{I}_{N,j} := \left(\frac{2j-1}{2N-2}\pi, \frac{2j+1}{2N-2}\pi\right)$ for $j \leq \lfloor \frac{N}{2} \rfloor$, and by $\hat{I}_{N,j} := \pi - \hat{I}_{N,N-j}$ otherwise. It follows by Lemma 5.1 that $\theta_{N,j}^{\alpha} \in \hat{I}_{N,j}$, and that there exists a constant $C > 0$ independent of N or j such that $C^{-1}\sin\theta_{N,j}^{\alpha} \leq \sin\theta \leq C\sin\theta_{N,j}^{\alpha}$ for $\theta \in \hat{I}_{N,j}$. Furthermore, $|\hat{I}_{N,j}| \sim N^{-1}$, and the intervals overlap at most near $\frac{\pi}{2}$.

Set $I_{N,j} := \cos\hat{I}_{N,j}$. By Lemma 5.3,

$$\begin{aligned}\|i_N^{\alpha} u\|^2_{\mathrm{L}^{2,\alpha}(-1,+1)} &\leq \sum_{j=0}^{N} u(\xi_{N,j}^{\alpha})^2 \rho_{N,j}^{\alpha} \\ &= u(-1)^2\rho_{N,0}^{\alpha} + u(+1)^2\rho_{N,N}^{\alpha} + \sum_{j=1}^{N-1} \hat{u}(\theta_{N,j}^{\alpha})^2 \rho_{N,j}^{\alpha}. \end{aligned} \quad (63)$$

Lemma 3.3 yields

$$\hat{u}(\theta_{N,j}^{\alpha})^2 \leq \|\hat{u}\|^2_{\mathrm{L}^{\infty}(\hat{I}_{N,j})} \lesssim \frac{1}{|\hat{I}_{N,j}|}\|\hat{u}\|^2_{\mathrm{L}^2(\hat{I}_{N,j})} + \|\hat{u}\|_{\mathrm{L}^2(\hat{I}_{N,j})}\|\hat{u}'\|_{\mathrm{L}^2(\hat{I}_{N,j})}. \quad (64)$$

Now, by $\sin\theta_{N,j}^{\alpha} \leq \sin\theta \leq C\sin\theta_{N,j}^{\alpha}$,

$$\begin{aligned}\|\hat{u}\|^2_{\mathrm{L}^2(\hat{I}_{N,j})}\rho_{N,j}^{\alpha} &\lesssim N^{-1}\int_{\hat{I}_{N,j}} \hat{u}(\theta)^2 \sin^{2\alpha+1}(\theta)\mathrm{d}\theta \\ &= N^{-1}\int_{I_{N,j}} u(x)^2(1-x^2)^{\alpha}\mathrm{d}x \end{aligned} \quad (65)$$

and

$$\begin{aligned}\|\hat{u}'\|^2_{\mathrm{L}^2(\hat{I}_{N,j})}\rho_{N,j}^{\alpha} &\lesssim N^{-1}\int_{\hat{I}_{N,j}} \hat{u}'(\theta)^2 \sin^{2\alpha+1}(\theta)\mathrm{d}\theta \\ &= N^{-1}\int_{\hat{I}_{N,j}} u'(\cos\theta)^2 \sin^{2\alpha+3}(\theta)\mathrm{d}\theta \\ &= N^{-1}\int_{I_{N,j}} u'(x)^2(1-x^2)^{\alpha+1}\mathrm{d}x. \end{aligned} \quad (66)$$

As $\|\hat{I}_{N,j}\| \sim N^{-1}$,

$$\hat{u}(\theta^\alpha_{N,j})^2\rho^\alpha_{N,j} \le \|u\|^2_{\mathrm{L}^{2,\alpha}(I_{N,j})} + N^{-1}\|u\|_{\mathrm{L}^{2,\alpha}(I_{N,j})}\|u'\|_{\mathrm{L}^{2,\alpha+1}(I_{N,j})}, \tag{67}$$

from which the result follows.

We single out the special case $\alpha = -1/2$, for which we can show stability of $i_N^{-1/2}$ in $(\mathrm{L}^{2,-1/2}(-1,+1), \mathrm{H}^{1,-1/2}(-1,+1))_{1/2,1}$:

Corollary 5.5. *The operator* $i_N^{-1/2}$ *is uniformly (in* N*) stable on* $\mathrm{H}^{1,-1/2}(-1,+1)$ *and the interpolation space* $(\mathrm{L}^{2,-1/2}(-1,+1), \mathrm{H}^{1,-1/2}(-1,+1))_{1/2,1}$.

Proof. From [17, Thm. 3.1] we have the existence of an operator Π_N : $\mathrm{H}^{1,-1/2}(-1,+1) \to \mathcal{P}^N$ with

$$\|u - \Pi_N u\|_{\mathrm{L}^{2,-1/2}(-1,1)} \lesssim N^{-1}\|u\|_{\mathrm{H}^{1,-1/2}(-1,1)}, \tag{68}$$

$$\|\Pi_N u\|_{\mathrm{L}^{2,-1/2}(-1,1)} \le \|u\|_{\mathrm{L}^{2,-1/2}(-1,1)}, \tag{69}$$

$$\|\Pi_N u\|_{\mathrm{H}^{1,-1/2}(-1,1)} \le \|u\|_{\mathrm{H}^{1,-1/2}(-1,1)}. \tag{70}$$

Furthermore, we have the inverse estimate (see [1, Prop. 6.1])

$$\|q_N'\|_{\mathrm{L}^{2,+1/2}(-1,1)} \lesssim N\|q_N\|_{\mathrm{L}^{2,-1/2}(-1,1)} \quad \text{for all } q_N \in \mathcal{P}^N,$$

which readily implies

$$\|q_N\|_{\mathrm{H}^{1,-1/2}(-1,1)} \lesssim N\|q_N\|_{\mathrm{L}^{2,-1/2}(-1,1)} \quad \text{for all } q_N \in \mathcal{P}^N. \tag{71}$$

We are now ready to show the stability of $i_N^{-1/2}$ on $\mathrm{H}^{1,-1/2}(-1,1)$: Combining the stability result Lemma 5.6 below with (68)–(71) gives

$$\begin{aligned}\|i_N^{-1/2}u\|_{\mathrm{H}^{1,-1/2}} &\le \|\Pi_N u\|_{\mathrm{H}^{1,-1/2}} + \|i_N^{-1/2}(u - \Pi_N u)\|_{\mathrm{H}^{1,-1/2}} \\ &\lesssim \|u\|_{\mathrm{H}^{1,-1/2}} + N\|i_N^{-1/2}(u - \Pi_N u)\|_{\mathrm{L}^{2,-1/2}} \\ &\lesssim \|u\|_{\mathrm{H}^{1,-1/2}} + N\|(u - \Pi_N u)\|_{\mathrm{L}^{2,-1/2}} \\ &\qquad + N^{1/2}\|u - \Pi_N u\|^{1/2}_{\mathrm{L}^{2,-1/2}}\|u - \Pi_N u\|^{1/2}_{\mathrm{H}^{1,-1/2}} \\ &\lesssim \|u\|_{\mathrm{H}^{1,-1/2}}.\end{aligned} \tag{72}$$

In order to see the stability of $i_N^{-1/2}$ on $(\mathrm{L}^{2,-1/2}(-1,+1), \mathrm{H}^{1,-1/2}(-1,+1))_{1/2,1}$, we first consider the operator $i_N^{-1/2} - \Pi_N$. We have for the interpolation norm $\|\cdot\|_{1/2,1}$ associated with the space $(\mathrm{L}^{2,-1/2}(-1,+1), \mathrm{H}^{1,-1/2}(-1,+1))_{1/2,1}$ by using the inverse estimate (71) and Lemma 5.6 below

$$
\begin{aligned}
&\|i_N^{-1/2}u - \varPi_N u\|_{1/2,1}^2 \le \|i_N^{-1/2}u - \varPi_N u\|_{\mathrm{L}^{2,-1/2}} \|i_N^{-1/2}u - \varPi_N u\|_{\mathrm{H}^{1,-1/2}} \\
&= \|i_N^{-1/2}(u - \varPi_N u)\|_{\mathrm{L}^{2,-1/2}} \|i_N^{-1/2}(u - \varPi_N u)\|_{\mathrm{H}^{1,-1/2}} \\
&\lesssim N \|i_N^{-1/2}(u - \varPi_N u)\|_{\mathrm{L}^{2,-1/2}}^2 \\
&\lesssim N \|u - \varPi_N u\|_{\mathrm{L}^{2,-1/2}}^2 + \|u - \varPi_N u\|_{\mathrm{L}^{2,-1/2}} \|u - \varPi_N u\|_{\mathrm{H}^{1,-1/2}} \\
&\lesssim \|u - \varPi_N u\|_{\mathrm{L}^{2,-1/2}} \|u\|_{\mathrm{H}^{1,-1/2}} \lesssim \|u\|_{\mathrm{L}^{2,-1/2}} \|u\|_{\mathrm{H}^{1,-1/2}};
\end{aligned}
$$

here we used the approximation property (68) and the stability estimate (70) in the penultimate step and the stability estimate (68) in the final step. We conclude with the triangle inequality and the multiplicative estimate $\|\varPi_N u\|_{1/2,1}^2 \le \|\varPi_N u\|_{\mathrm{L}^{2,-1/2}} \|\varPi_N u\|_{\mathrm{H}^{1,-1/2}}$ that

$$
\|i_N^{-1/2}u\|_{1/2,1}^2 \lesssim \|u\|_{\mathrm{L}^{2,-1/2}} \|u\|_{\mathrm{H}^{1,-1/2}} \qquad \forall\, u \in \mathrm{H}^{1,-1/2}.
$$

An application of [30, Lemma 25.3] allows us to conclude $\|i_N^{-1/2}u\|_{1/2,1} \lesssim \|u\|_{1/2,1}$ for all $u \in (\mathrm{L}^{2,-1/2}, \mathrm{H}^{1,-1/2})_{1/2,1}$.

Lemma 5.6. *For every* $u \in \mathrm{H}^{1,-1/2}(-1,+1)$

$$
\begin{aligned}
\|u\|_{\mathrm{L}^\infty(-1,+1)}^2 &\lesssim \|u\|_{\mathrm{L}^{2,-1/2}(-1,+1)} \|u\|_{\mathrm{H}^{1,-1/2}(-1,+1)}, \\
\|i_N^{-1/2}u\|_{\mathrm{L}^{2,-1/2}(-1,+1)}^2 &\lesssim \|u\|_{\mathrm{L}^{2,-1/2}(-1,+1)}^2 + N^{-1} \|u\|_{\mathrm{L}^{2,-1/2}(-1,+1)} \|u\|_{\mathrm{H}^{1,-1/2}(-1,+1)}.
\end{aligned}
$$

Proof. We start with the L^∞-bound. Let $u \in C^\infty([-1,1])$. Then for arbitrary x, $y \in (-1,1)$:

$$
\begin{aligned}
u^2(x) - u^2(y) &= \int_y^x (u^2)'(t)\mathrm{d}t = 2\int_y^x u u' \mathrm{d}t \\
&\le 2\sqrt{\int_{-1}^1 u^2(t)(1-t^2)^{-1/2}\mathrm{d}t} \sqrt{\int_{-1}^1 |u'(t)|^2 (1-t^2)^{1/2}\mathrm{d}t} \\
&= 2\|u\|_{\mathrm{L}^{2,-1/2}(-1,1)} \|u'\|_{\mathrm{L}^{2,+1/2}(-1,1)}.
\end{aligned}
$$

Multiplying by $(1-y^2)^{-1/2}$, integrating in y, and observing $\int_{-1}^1 (1-y^2)^{-1/2}\mathrm{d}y = \pi$ produces

$$
\pi u^2(x) \le \|u\|_{\mathrm{L}^{2,-1/2}(-1,1)}^2 + 2\pi \|u\|_{\mathrm{L}^{2,-1/2}(-1,1)} \|u'\|_{\mathrm{L}^{2,+1/2}(-1,1)},
$$

which implies the desired L^∞-bound.

The estimate for $i_N^{-1/2}u$ now follows from Theorem 5.4.

6 Appendix B: A posteriori error estimation

6.1 Preliminaries

We state a well-known lemma:

Lemma 6.1. *Let $\mathcal{K} \subset \mathcal{V}$ and $\Lambda \subset \mathcal{W}$ be a closed convex subsets of the Hilbert spaces $\mathcal{V}$ and $\mathcal{W}$. Let $a : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ and $b : \mathcal{V} \times \mathcal{W} \to \mathbb{R}$ be continuous bilinear forms; let $L : \mathcal{V} \to \mathbb{R}$ and $G : \mathcal{W} \to \mathbb{R}$ be continuous linear forms. Let a be symmetric positive definite. Assume additionally that Λ is bounded.*

1. *The functional J given by*

$$J(u) := \sup_{\lambda \in \Lambda} \mathcal{L}(u, \lambda), \qquad \mathcal{L}(u, \lambda) := \frac{1}{2} a(u, u) - L(u) + b(u, \lambda) - G(\lambda)$$

   is continuous and strictly convex on $\mathcal{K}$. Hence, there exists a unique minimizer $u \in \mathcal{K}$ of J.
2. *There exists a Lagrange multiplier $\lambda \in \Lambda$ such that $(u, \lambda) \in \mathcal{K} \times \Lambda$ solves the saddle point problem*

$$a(u, v) + b(v, \lambda) \geq L(v - u) \qquad \forall v \in \mathcal{K}, \tag{73a}$$

$$b(u, \mu - \lambda) \leq G(\mu - \lambda) \qquad \forall \mu \in \Lambda. \tag{73b}$$

Proof.
Proof of Part 1:
1. step: It is not difficult to see that $z : u \mapsto \sup_{\lambda \in \Lambda} b(u, \lambda)$ is convex. Hence, J is convex. Furthermore, since $u \mapsto a(u, u)$ is strictly convex, we get that J is strictly convex.
2. step: We claim that J is continuous. To that end, it suffices to see that the function z is continuous. However, since Λ is bounded, it is clear that $|z(u)| \leq C\|u\|_{\mathcal{V}}$ for all $u \in \mathcal{V}$ and a suitable $C > 0$. Since z is convex, the continuity of z follows (see, e.g., [14, Chap. 1, Lemma 2.1]).
3. step: Since $|z(u)| \leq C\|u\|_{\mathcal{V}}$ and a is symmetric positive definite, we infer that J is coercive. By general results of convex analysis, J has a minimizer. By strict convexity of J, this minimizer is unique.

Proof of Part 2: Define the Lagrangian

$$\mathcal{L}(u, \lambda) := \frac{1}{2} a(u, u) - L(u) + b(u, \lambda) - G(\lambda)$$

Then [14, Prop. VI.2.4] provides the existence of a saddle point (u, λ) of $\mathcal{L}$. This saddle point satisfies (73), and the component u is the minimiser of J.

6.2 A posteriori error estimation

Introduce

$$b(\mathbf{v},\boldsymbol{\mu}) = \int_{\Gamma_C} g\mathbf{v}_t \cdot \boldsymbol{\mu},$$
$$\Lambda = \{\boldsymbol{\lambda} \in (L^\infty(\Gamma_C))^2 \mid \|\boldsymbol{\lambda}\|_{L^\infty(\Gamma_C)} \le 1\}.$$

We then note that the friction functional j can be written as

$$j(\mathbf{u}) = \sup_{\lambda\in\Lambda} b(\mathbf{u},\boldsymbol{\lambda}).$$

Lemma 6.1 asserts that the saddle point problem

$$a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},\boldsymbol{\lambda}) = L(\mathbf{v}) \qquad \forall \mathbf{v}\in V, \tag{74a}$$
$$b(\mathbf{u},\boldsymbol{\mu}-\boldsymbol{\lambda}) \le 0 \qquad \forall \boldsymbol{\mu}\in\Lambda, \tag{74b}$$

is solvable and that the component $\mathbf{u}$ is the minimiser of the minimisation problem (4). In fact, as is shown in [8], the Lagrange multiplier $\boldsymbol{\lambda}$ is unique.

Let j_N be realised by a Gaussian quadrature with $q_{N,E}+1$ points on each edge $E \in \mathcal{E}_{\mathrm{C},N}$ and assume

$$q_{N,E} \ge p_{N,E} - 1 \qquad \forall E \in \mathcal{E}_{\mathrm{C},N}. \tag{75}$$

Introduce

$$W_N := \{\boldsymbol{\mu} \in (L^2(\Gamma_\mathrm{C}))^2 \mid \boldsymbol{\mu}|_E \in \mathcal{P}^{q_{N,E}} \quad \forall E \in \mathcal{E}_{\mathrm{C},N}\},$$
$$b_N(\mathbf{v},\boldsymbol{\mu}) := g \sum_{E\in\mathcal{E}_{\mathrm{C},N}} \sum_{\mathbf{x}\in\widetilde{G}_{E,q_{N,E}}} \omega_{e,q_{N,E},\mathbf{x}} \mathbf{v}_t(\mathbf{x})\boldsymbol{\mu}(\mathbf{x}),$$
$$\Lambda_N := \{\boldsymbol{\mu} \in W_N : |\boldsymbol{\mu}(\mathbf{x})| \le 1 \qquad \forall E \in \mathcal{E}_{\mathrm{C},N}\ \forall \mathbf{x} \in \widetilde{G}_{E,q_{N,E}}\}.$$

We notice

$$\sup_{\boldsymbol{\mu}\in\Lambda_N} b_N(\mathbf{v},\boldsymbol{\mu}) = j_N(\mathbf{v}) \qquad \forall \mathbf{v}\in V_N, \tag{76}$$
$$b_N(\mathbf{v},\boldsymbol{\mu}) = b(\mathbf{v},\boldsymbol{\mu}) \qquad \forall \mathbf{v}\in V_N, \quad \boldsymbol{\mu}\in W_N. \tag{77}$$

Again, by Lemma 6.1, there exists a solution $(\mathbf{u}_N, \boldsymbol{\lambda}_N)$ of the saddle point problem saddle point problem

$$a(\mathbf{u}_N,\mathbf{v}) + b_N(\mathbf{v},\boldsymbol{\lambda}_N) = L(\mathbf{v}) \qquad \forall \mathbf{v}\in V_N, \tag{78a}$$
$$b_N(\mathbf{u}_N,\boldsymbol{\mu}-\boldsymbol{\lambda}_N) \le 0 \qquad \forall \boldsymbol{\mu}\in\Lambda_N. \tag{78b}$$

The solution $\mathbf{u}_N$ is the minimiser of the discrete problem (6).

For $\boldsymbol{\lambda}_N \in W_N$ let $\tilde{\boldsymbol{\lambda}}_N \in \Lambda_N$ be the $L^2(\Gamma_C)$-projection onto Λ. For $\mathbf{v} \in V$ let $I\mathbf{v} \in V_N$ be arbitrary. Then (74) and (78) yield:

$$\begin{aligned} &-a(\mathbf{u}_N, \mathbf{v}) + L(\mathbf{v}) - b(\mathbf{v}, \tilde{\boldsymbol{\lambda}}_N) \\ &= -a(\mathbf{u}_N, \mathbf{v} - I\mathbf{v}) + L(\mathbf{v} - I\mathbf{v}) - a(\mathbf{u}_N, I\mathbf{v}) + L(I\mathbf{v}) - b(\mathbf{v}, \tilde{\boldsymbol{\lambda}}_N) \\ &= -a(\mathbf{u}_N, \mathbf{v} - I\mathbf{v}) + L(\mathbf{v} - I\mathbf{v}) + b_N(I\mathbf{v}, \boldsymbol{\lambda}_N) - b(\mathbf{v}, \tilde{\boldsymbol{\lambda}}_N) \\ &= -a(\mathbf{u}_N, \mathbf{v} - I\mathbf{v}) + L(\mathbf{v} - I\mathbf{v}) - b(\mathbf{v} - I\mathbf{v}, \boldsymbol{\lambda}_N) + b(\mathbf{v}, \boldsymbol{\lambda}_N - \tilde{\boldsymbol{\lambda}}_N), \quad (79) \end{aligned}$$

where, in the last step we additionally used (77).

We are now in position to provide *a posteriori* error estimates. The starting point is [8, Thm. 4.1], which directly gives

$$\|\mathbf{u} - \mathbf{u}_N\|^2_{\mathrm{H}^1(\Omega)} \leq C\Big\{ \sup_{\mathbf{v}\in V} \|\mathbf{v}\|^{-2}_{\mathrm{H}^1(\Omega)} \Big(-a(\mathbf{u}_N, \mathbf{v}) + L(\mathbf{v}) - b(\mathbf{v}, \tilde{\boldsymbol{\lambda}}_N)\Big)^2 + j(\mathbf{u}_N) - b(\mathbf{u}_N, \tilde{\boldsymbol{\lambda}}_N)\Big\}.$$

Inserting (79) yields

$$\|\mathbf{u} - \mathbf{u}_N\|^2_{\mathrm{H}^1(\Omega)} \leq C\Big\{ \sup_{\mathbf{v}\in V} \|\mathbf{v}\|^{-2}_{\mathrm{H}^1(\Omega)} \left(-a(\mathbf{u}_N, \mathbf{v} - I\mathbf{v}) + L(\mathbf{v} - I\mathbf{v}) - b(\mathbf{v} - I\mathbf{v}, \boldsymbol{\lambda}_N)\right)^2 + (b(\mathbf{v}, \boldsymbol{\lambda}_N - \tilde{\boldsymbol{\lambda}}_N))^2 + j(\mathbf{u}_N) - b(\mathbf{u}_N, \tilde{\boldsymbol{\lambda}}_N)\Big\}.$$

We recognise that this expression is exactly the same as in [8]. Hence, we may proceed in exactly the same way as there. For $K \in \mathcal{T}_N$, we define the interior residuals by

$$\mathbf{r}_K := -\operatorname{Div} \boldsymbol{\sigma}(\mathbf{u}_N) - \mathbf{F} \quad (80)$$

and for $E \in \mathcal{E}_N$ the boundary residuals by

$$\mathbf{R}_E := \begin{cases} \frac{1}{2} \left[\boldsymbol{\sigma}(\mathbf{u}_N) \cdot \boldsymbol{\nu}\right]_E & \text{if } E \subset \Omega, \\ (\boldsymbol{\sigma}(\mathbf{u}_N) \cdot \boldsymbol{\nu})_t + g(\boldsymbol{\lambda}_N)_t & \text{i f } E \subset \Gamma_{\mathrm{C}} \\ \boldsymbol{\sigma}(\mathbf{u}_N) \cdot \boldsymbol{\nu} - \mathbf{t} & \text{if } E \subset \Gamma_{\mathrm{N}}, \\ 0 & \text{if } E \subset \Gamma_{\mathrm{D}}, \end{cases} \quad (81)$$

where

$$\left[\boldsymbol{\sigma}(\mathbf{u}_N) \cdot \boldsymbol{\nu}\right]_E := \boldsymbol{\sigma}(\mathbf{u}_N)|_{K_{E,1}} \cdot \boldsymbol{\nu}_{K_{E,1}} + \boldsymbol{\sigma}(\mathbf{u}_N)|_{K_{E,2}} \cdot \boldsymbol{\nu}_{K_{E,2}}$$

is the boundary jump with E the common edge of $K_{E,1}$ and $K_{E,2}$ and $\boldsymbol{\nu}_{K_{E,1}}$ pointing from $K_{E,1}$ to $K_{E,2}$, and $\boldsymbol{\nu}_{K_{E,2}} = -\boldsymbol{\nu}_{K_{E,1}}$.

We define *local error indicators* by

$$\eta_{N,K}^2 := h_{N,K}^2 p_{N,K}^{-2} \|\mathbf{r}_K\|_{\mathrm{L}^2(K)}^2 + h_{N,K} p_{N,K}^{-1} \sum_{E\subseteq\partial K} \|\mathbf{R}_E\|_{\mathrm{L}^2(E)}^2 + j_{\partial K\cap\Gamma_{\mathrm{C}}}(\mathbf{u}_N) - b_{\partial K\cap\Gamma_{\mathrm{C}}}(\mathbf{u}_N, \tilde{\boldsymbol{\lambda}}_N) + g^2 \|\boldsymbol{\lambda}_N - \tilde{\boldsymbol{\lambda}}_N\|_{\mathrm{H}_{00}^{-1/2}(\partial K\cap\Gamma_{\mathrm{C}})}^2, \tag{82}$$

where the last three terms vanish if $\partial K \cap \Gamma_{\mathrm{C}} = \emptyset$, and the *global error indicator* by

$$\eta_N^2 := \sum_{K\in\mathcal{T}_N} \eta_{N,K}^2. \tag{83}$$

Theorem 6.2 (Reliability). *Assume that the meshes $\mathcal{T}_N$ are affine and shape regular. Assume (7).*

Assume (75). Then there exists a constant $C > 0$ such that the following is true: For Lagrange multiplier $\boldsymbol{\lambda}_N$ such that $(\mathbf{u}_N, \boldsymbol{\lambda}_N)$ solves (78) the residual error indicator given by (82), (83) satisfies

$$\|\mathbf{u} - \mathbf{u}_N\|_{\mathrm{H}^1(\Omega)} \le C\eta_N \quad \text{for all } N. \tag{84}$$

The efficiency result of [8] is also true as can be checked by inspection:

Theorem 6.3 (Efficiency). *Let $(\mathbf{u}, \boldsymbol{\lambda}) \in V \times \Lambda$ be the solution of (74). Assume (7). For each K let $\bar{\mathbf{r}}_K \in \Pi^{p_{N,K}}(K)$ be a polynomial approximation of $\mathbf{r}_K$. For each edge E, let $\bar{\mathbf{R}}_E$ be a polynomial approximation of $\mathbf{R}_E$ of degree p_E, where $p_E = \min\{p_K : E \text{ is edge of } K \in \mathcal{T}_N\}$.*

For $K \in \mathcal{T}_N$ denote by K_{patch} the union of elements of $\mathcal{T}_N$ that share an edge with K. Let $\mathbf{r}_{patch}$ and $\bar{\mathbf{r}}_{patch}$ be defined on K_{patch} in an elementwise fashion by $\mathbf{r}_{patch}|_{K'} = r_{K'}$ and $\bar{\mathbf{r}}_{patch}|_{K'} = \bar{\mathbf{r}}_{K'}$ for all $K' \subset K_{patch}$.

Let $\beta \in (1/2, 1]$. Then there exists a constant $C > 0$ such that the residual error indicator satisfies

$$\eta_{N,K}^2 \lesssim p_{N,K}^{2\beta}\Big(p_{N,K}\|\mathbf{u} - \mathbf{u}_N\|_{\mathrm{H}^1(K_{patch})}^2 + h_{N,K}^2 p_{N,K}^{-3+2\beta} \|\bar{\mathbf{r}}_{K_{patch}} - \mathbf{r}_{K_{patch}}\|_{\mathrm{L}^2(K_{patch})} + h_{N,K} p_K^{-1} \sum_{E\subseteq\partial K} \|\bar{\mathbf{R}}_E - \mathbf{R}_E\|_{\mathrm{L}^2(E)}^2 + g^2 h_{N,K} p_K^{-1} \|\boldsymbol{\lambda}_N - \boldsymbol{\lambda}\|_{\mathrm{L}^2(\partial K\cap\Gamma_{\mathrm{C}})}^2\Big) + g h_{N,\partial K\cap\Gamma_{\mathrm{C}}}^{1/2} \|\mathbf{u}_N - \mathbf{u}\|_{\mathrm{L}^2(\partial K\cap\Gamma_{\mathrm{C}})} + g\|\mathbf{u}\|_{\mathrm{L}^2(\partial K\cap\Gamma_{\mathrm{C}})} \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_N\|_{\mathrm{L}^2(\partial K\cap\Gamma_{\mathrm{C}})} + g^2\|\boldsymbol{\lambda}_N - \boldsymbol{\lambda}\|_{\mathrm{L}^2(\partial K\cap\Gamma_{\mathrm{C}})}^2 \tag{85}$$

for all N and all $K \in \mathcal{T}_N$.

Remark 6.4. It is worth pointing out that that the Lagrange multiplier $\boldsymbol{\lambda}_N$ may not be unique. In the special case considered in Section 4.2, it is, however, unique: The special choice $q_{N,E} = p_{N,E} - 1$ and the boundary conditions (Dirichlet boundary conditions at one end point) ensure that a $\boldsymbol{\lambda} \in W_N$ that satisfies

$$b_N(\mathbf{u}, \boldsymbol{\lambda}) = 0 \quad \forall \mathbf{u} \in V_N$$

has to vanish identically. Thus, the Lagrange multiplier $\boldsymbol{\lambda}_N$ is unique in this case.