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General DG-Methods for Highly Indefinite Helmholtz Problems (with Appendix C)

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Abstract We develop a stability and convergence theory for a Discontinuous Galerkin formulation (DG) of a highly indefinite Helmholtz problem in $\mathbb{R}^d$, $d \in \{1,2,3\}$. The theory covers conforming as well as non-conforming generalized finite element methods. In contrast to conventional Galerkin methods where a minimal resolution condition is necessary to guarantee the unique solvability, it is proved that the DG-method admits a unique solution under much weaker conditions. As an application we present the error analysis for the hp-version of the finite element method explicitly in terms of the mesh width h, polynomial degree p and wavenumber k. It is shown that the optimal convergence order estimate is obtained under the conditions that $kh/\sqrt{p}$ is sufficiently small and the polynomial degree p is at least $O(\log k)$. On regular meshes, the first condition is improved to the requirement that kh/p be sufficiently small.
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1 Introduction

In this paper we analyze a discontinuous Galerkin method applied to the following model Helmholtz problem:

$$-\Delta u - k^2 u = f \quad \text{in } \Omega, \tag{1.1}$$

$$\frac{\partial u}{\partial \mathbf{n}} + \mathrm{i} k u = g \quad \text{on } \partial\Omega. \tag{1.2}$$

Here, Ω is a bounded Lipschitz domain in $\mathbb{R}^d$, $d \in \{2,3\}$, and $k \geq k_0 > 0$ is the real and positive wavenumber bounded away from zero. The outer normal vector to $\partial\Omega$ is denoted $\mathbf{n}$, and we write $\mathrm{i} = \sqrt{-1}$ for the imaginary unit. We assume $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$. By

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$H^s(\Omega)$ we denote the usual Sobolev space with norm $\|\cdot\|_{H^s(\Omega)}$, [1]. The seminorm which contains only the derivatives of order s is denoted by $|\cdot|_{H^s(\Omega)}$.

The weak formulation for (1.1) is given by: Find $u \in V := H^1(\Omega)$ such that

$$a(u,v) = F(v) \qquad \forall v \in H^1(\Omega), \tag{1.3}$$

where

$$a(u,v) := \int_\Omega \left(\nabla u \nabla \bar{v} - k^2 u \bar{v}\right) + \mathrm{i}\,k \int_{\partial\Omega} u\bar{v}, \tag{1.4}$$

$$F(v) := \int_\Omega f\bar{v} + \int_{\partial\Omega} g\bar{v}. \tag{1.5}$$

Existence and uniqueness for the continuous problem were proved in [36] for bounded Lipschitz domains.

Problems in high-frequency scattering of acoustic or electro-magnetic waves are highly indefinite, and the design of discretization methods that behave robustly with respect to the amount of indefiniteness is of great importance. For our model problem, the highly indefinite case arises for high wavenumbers k, and the solution u is highly oscillatory. It is well-known for such problems that low order finite elements suffer from the *pollution effect*, which mandates very fine meshes, [32]. For example, the classical analysis for lowest order $\mathbb{P}_1$-finite element spaces (see, e.g., [44], [32, Sec. 4]) guarantees unique solvability and quasi-optimality only under the condition that the number of degrees of freedom N satisfies $N \gtrsim k^{2d}$, where d is the spatial dimension. We hasten to add that the conditions on the mesh size are less stringent for higher order FEM. A particular example is the analysis of [39,40], which shows for high order methods that linking the polynomial degree p logarithmically to the wavenumber can lead to a stable method with few degrees of freedom per wavelength. We mention that on regular meshes the pollution error can also be understood by a dispersion analysis that quantifies the phase difference between the exact solution and the numerical solution, [2–4, 6, 15, 18, 32–34].

While the existence of discrete solutions for classical, conforming finite element discretizations is understood, it is worth stressing that a minimal resolution condition is required to ensure their existence. This observation motivates the quest for stabilized variational formulations that always guarantee the discrete stability of the method (existence and uniqueness of the discrete solution). Prominent examples of these types of methods are those incorporating least squares ideas, [19, 28, 29, 41] and Discontinuous Galerkin (DG) methods. Several variants of DG methods based on standard piecewise polynomial spaces are analyzed, for example, in [21–23, 47, 48]. They feature unique solvability of the discrete systems without any resolution conditions; yet, it is worth pointing out that reduced or no convergence takes place in the preasymptotic regime.

The *Ultra Weak Variational Formulation* (UWVF) of Cessenat and Després [10, 11, 16] can be understood as a DG method that permits using non-standard, discontinuous local discretization spaces such as plane waves (see [8,25,30,31]). In present paper we follow the idea of [25], where a DG method was derived from the UWVF for the Helmholtz problem. For plane waves as local ansatz spaces in this DG method, [25] shows linear convergence of the method under appropriate resolution conditions. By specializing to homogeneous Helmholtz problems [30] establishes quasi-optimal convergence (in a norm dictated by the method) without any resolution condition.

The goal of our work is to develop a theory for the same DG formulation as in [25] that allows us to infer the convergence behavior of abstract conforming and non-conforming generalized finite element spaces from certain local approximation properties and local inverse estimates, which may be easy to check, possibly even at run-time.

This paper is structured as follows: In Section 2, we recall from [25] a DG method for the Helmholtz problem (1.1). Section 3 is devoted to discrete stability and convergence. The unified theory presented there covers two popular choices of approximation spaces, namely, spaces consisting of piecewise plane waves and conforming as well as non-conforming polynomial *hp*-finite element spaces on affine simplicial meshes. Nevertheless, we also derive an abstract approximation criterion for general finite element spaces that implies existence and uniqueness of the discrete solution. Based on these results, we obtain quasi-optimal convergence in the DG-norm for general finite element spaces.

In Section 4 we apply the results of Section 3 to the *hp*-version of the polynomial FEM. We obtain a convergence theory that is explicit in the wavenumber k as well as the mesh width h and the polynomial degree p. These results may be viewed as an extension of the results [39, 40] for classical H^1-conforming discretizations to the DG-setting. In these papers, a scale resolution condition of the form

$$\frac{kh}{p} \leq c_1 \qquad \text{and} \qquad p \geq c_2 \log k \tag{1.6}$$

(for suitable c_1, c_2) is sufficient to guarantee quasi-optimality. For the *hp*-version of the DG-FEM on *regular* meshes, or, more generally, meshes that permit sufficiently rich H^1-conforming subspaces of the non-conforming DG-space, the same condition yields quasi-optimality. In the general case, the slightly stronger condition (4.16) is a sufficient condition for quasi-optimality. In particular, we show, for the first time for a DG method on regular meshes, that quasi-optimality can be obtained for a fixed number of degrees of freedom per wavelength. Two appendices conclude the article. Appendix A gives details for the regularity result Theorem 4.5. Appendix B is concerned with elementwise defined *hp*-approximations that are optimal in the broken H^2-norm; this result is required for the proof of Theorem 4.11.

2 Discontinuous Galerkin Method

2.1 Meshes and Spaces

To formulate the DG method we first introduce some notation. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, denote a polygonal ($d = 2$) or polyhedral ($d = 3$) Lipschitz domain[1]. The DG problem is based on a partition $\mathscr{T}$ of Ω into non-overlapping curvilinear polygonal/polyhedral subdomains ("finite elements") K; hanging nodes are allowed. The local and global mesh width is denoted by

$$h_K := \operatorname{diam} K \quad \text{and} \quad h := \max_{K \in \mathscr{T}} h_K. \tag{2.1}$$

In the case $d = 3$, the boundary of K can be split into *faces* and for $d = 2$ into *edges*. For ease of notation we use the terminology "faces" in both cases. For $K \in \mathscr{T}$, we denote the set of faces by $\mathscr{E}(K)$. The subset of interior faces, i.e., the set of faces of K which are not lying on $\partial\Omega$, is denoted by $\mathscr{E}^{\mathscr{I}}(K)$. For instance the number $\sharp\mathscr{E}(K) = d+1$ if K is a simplex. As a convention we consider the finite elements $K \in \mathscr{T}$ always as open sets and the faces $e \in \mathscr{E}(K)$ as relatively open sets.

[1] The DG method can also be formulated for geometries with curved boundaries.

The interior skeleton $\mathfrak{S}_{\mathscr{T}}^{\mathscr{I}}$ and the boundary skeleton $\mathfrak{S}_{\mathscr{T}}^{\mathscr{B}}$ are given by

$$\mathfrak{S}_{\mathscr{T}}^{\mathscr{I}} := \bigcup_{K\in\mathscr{T}} \bigcup_{e\in\mathscr{E}^{\mathscr{I}}(K)} e, \qquad \mathfrak{S}_{\mathscr{T}}^{\mathscr{B}} := \bigcup_{K\in\mathscr{T}} \bigcup_{\substack{e\in\mathscr{E}(K)\\ e\subset\partial\Omega}} e.$$

Note that $\mathfrak{S}_{\mathscr{T}}^{\mathscr{I}}$, $\mathfrak{S}_{\mathscr{T}}^{\mathscr{B}}$ are the union of the relative interior of the faces and, consequently, for any point $x\in\mathfrak{S}_{\mathscr{T}}^{\mathscr{I}}$, there exist exactly two elements in $\mathscr{T}$ (denoted by K_x^+, K_x^-) with $x\in\overline{K_x^+}\cap\overline{K_x^-}$.

Also define $\nabla_{\mathscr{T}}$ and $\Delta_{\mathscr{T}}$ as elementwise applications of the operators ∇ and Δ, respectively. The one-sided restrictions of some $\mathscr{T}$-piecewise smooth function v for $x\in\mathfrak{S}_{\mathscr{T}}^{\mathscr{I}}$ are denoted by

$$v^+(x) := \lim_{\substack{y\in K_x^+\\ y\to x}} v(y) \quad \text{and} \quad v^-(x) := \lim_{\substack{y\in K_x^-\\ y\to x}} v(y).$$

We use the same notation for vector-valued functions.

We define the averages and jumps for $\mathscr{T}$-piecewise smooth scalar-valued functions v and vector-valued functions σ_S on $\mathfrak{S}_{\mathscr{T}}^{\mathscr{I}}$ by

$$\text{the averages: } \{v\} := \frac{1}{2}\left(v^+ + v^-\right), \quad \{\sigma_S\} := \frac{1}{2}\left(\sigma_S^+ + \sigma_S^-\right),$$

$$\text{the jumps: } \quad [\![v]\!]_N := v^+\mathbf{n}^+ + v^-\mathbf{n}^-, \; [\![\sigma_S]\!]_N := \sigma_S^+\cdot\mathbf{n}^+ + \sigma_S^-\cdot\mathbf{n}^-.$$

where $\mathbf{n}^+(x)$, $\mathbf{n}^-(x)$ denote the (outer) normal vectors of elements K_x^+, K_x^-.

Based on the partition $\mathscr{T}$ we can introduce *broken Sobolev spaces* in the standard way: For $s\geq 0$, we set

$$H^s_{\mathrm{pw}}(\Omega) := L^2(\Omega)\cap\prod_{K\in\mathscr{T}} H^s(K). \tag{2.2}$$

2.2 Discrete Formulation

We approximate the solution of (1.3) from an *abstract* finite-dimensional space $S\subset H^2_{\mathrm{pw}}(\Omega)$, i.e., only the following two conditions are imposed:

$$S\subset L^2(\Omega) \quad \text{and} \quad S\subset\prod_{K\in\mathscr{T}} H^2(K). \tag{2.3}$$

We briefly recall the derivation of the Discontinuous Galerkin formulation from the UWVF as in [25]. We denote by $(\cdot,\cdot)$ the L^2 inner product on Ω, i.e., $(u,v)=\int_\Omega u\bar{v}dV$. Let S be a discrete space as in (2.3). Let $\alpha\in L^\infty\left(\mathfrak{S}_{\mathscr{T}}^{\mathscr{I}}\right)$, $\beta\in L^\infty\left(\mathfrak{S}_{\mathscr{T}}^{\mathscr{I}}\right)$, and $\delta\in L^\infty\left(\mathfrak{S}_{\mathscr{T}}^{\mathscr{B}}\right)$ be some positive and bounded functions on the mesh skeletons. (It will turn out that these functions can be chosen to be piecewise constant on a certain partition of the skeleton as elaborated in Remark 2.2.) Then, the DG formulation can be written in the following form, [25,30]:

Find $u_S\in S$ such that, for all $v\in S$,

$$a_{\mathscr{T}}(u_S,v) - k^2(u_S,v) = (f,v) - \int_{\mathfrak{S}_{\mathscr{T}}^{\mathscr{B}}} \delta\frac{1}{\mathrm{i}k} g\overline{\nabla_{\mathscr{T}} v\cdot\mathbf{n}}dS + \int_{\mathfrak{S}_{\mathscr{T}}^{\mathscr{B}}} (1-\delta)g\bar{v}dS =: F_{\mathscr{T}}(v), \tag{2.4}$$

where $a_{\mathcal{T}}(\cdot,\cdot)$ is the DG-bilinear form on $S \times S$ defined by

$$\begin{aligned}a_{\mathcal{T}}(u,v) := &(\nabla_{\mathcal{T}} u, \nabla_{\mathcal{T}} v) - \int_{\mathfrak{S}_{\mathcal{T}}^{\mathcal{I}}} [\![u]\!]_N \cdot \{\overline{\nabla_{\mathcal{T}} v}\} dS - \int_{\mathfrak{S}_{\mathcal{T}}^{\mathcal{I}}} \{\nabla_{\mathcal{T}} u\} \cdot [\![\overline{v}]\!]_N dS \\ &- \int_{\mathfrak{S}_{\mathcal{T}}^{\mathcal{B}}} \delta u \overline{\nabla_{\mathcal{T}} v \cdot \mathbf{n}} dS - \int_{\mathfrak{S}_{\mathcal{T}}^{\mathcal{B}}} \delta \nabla_{\mathcal{T}} u \cdot \mathbf{n} \overline{v} dS \\ &- \frac{1}{\mathrm{i}k} \int_{\mathfrak{S}_{\mathcal{T}}^{\mathcal{I}}} \beta [\![\nabla_{\mathcal{T}} u]\!]_N [\![\overline{\nabla_{\mathcal{T}} v}]\!]_N dS - \frac{1}{\mathrm{i}k} \int_{\mathfrak{S}_{\mathcal{T}}^{\mathcal{B}}} \delta \nabla_{\mathcal{T}} u \cdot \mathbf{n} \overline{\nabla_{\mathcal{T}} v \cdot \mathbf{n}} dS \\ &+ \mathrm{i}k \int_{\mathfrak{S}_{\mathcal{T}}^{\mathcal{I}}} \alpha [\![u]\!]_N [\![\overline{v}]\!]_N dS + \mathrm{i}k \int_{\mathfrak{S}_{\mathcal{T}}^{\mathcal{B}}} (1-\delta) u \overline{v} dS. \end{aligned} \tag{2.5}$$

Note that $a_{\mathcal{T}}(\cdot,\cdot)$ can be extended to a sesquilinear form on $H_{\mathrm{pw}}^{3/2+\varepsilon}(\Omega) \times H_{\mathrm{pw}}^{3/2+\varepsilon}(\Omega)$ for any $\varepsilon > 0$. So far, the functions α, β, δ are arbitrary, positive L^∞ functions. Our analysis will rely on certain properties of α that depend on some trace inverse estimates for the space S. We therefore introduce:

Definition 2.1 (inverse trace inequality) For each element K, the constant $C_{\mathrm{trace}}(S,K)$ is the smallest constant such that

$$\|\nabla (v|_K)\|_{L^2(\partial K)} \leq C_{\mathrm{trace}}(S,K) \|\nabla v\|_{L^2(K)} \qquad \forall v \in S. \tag{2.6}$$

Remark 2.2 The analysis of the continuity and coercivity will lead to the condition

$$\alpha(x) \geq \frac{4}{3k} \max_{K \in \{K_x^+, K_x^-\}} C_{\mathrm{trace}}^2(S,K) \quad \forall x \in \mathfrak{S}_{\mathcal{T}}^{\mathcal{I}}. \tag{2.7}$$

For the special case that S is a conforming/non-conforming polynomial hp-finite element space, the estimate of the approximation property of S with respect to the $\|\cdot\|_{DG}$ and $\|\cdot\|_{DG^+}$ norms, (cf. Section 4.2 ahead) leads to the choices

$$\alpha(x) = \mathfrak{a} \max_{K \in \{K_x^+, K_x^-\}} \frac{p^2}{kh_K}, \quad \beta = \mathfrak{b}\frac{kh}{p}, \quad \delta = \quad \mathfrak{d}\frac{kh}{p}, \tag{2.8}$$

where the parameter $\mathfrak{a}$ is selected fixed but sufficiently large; the parameters $\mathfrak{b}$, $\mathfrak{d}$ are selected to be of size $O(1)$. ■

Remark 2.3 It is easy to see that $x \mapsto \alpha(x)$ can be chosen piecewise constant with respect to a sub-partition $\mathcal{E}$ of the set of all faces. More precisely, we define a subdivision of the set of inner faces by

$$\mathcal{E}^{\mathcal{I}} := \left\{ \overset{\circ}{\partial} K \cap \overset{\circ}{\partial} K' \cap \Omega \mid K \in \mathcal{T}, \quad K' \in \mathcal{T} \backslash \{K\} \right\},$$

where $\overset{\circ}{\partial} K := \bigcup_{e \in \mathcal{E}(K)} e$. For any $e' \in \mathcal{E}^{\mathcal{I}}$, the maximum in (2.7) over $x \in e'$ can always be chosen as *one* fixed element K so that the value of α is constant along e'. Hence, without loss of generality we may assume in the following that α is chosen as an $\mathcal{E}$-piecewise constant function. Note that the assumption "α is positive" then implies for each $K \in \mathcal{T}$

$$\alpha_{\partial K}^{\min} := \inf_{x \in \partial K} \alpha(x) = \alpha(X) \tag{2.9}$$

for some $X \in \overset{\circ}{\partial} K \cap \Omega$. ■

In the rest of this section we will show that the discretization given by the sesquilinear form $a_{\mathscr{T}}$ is consistent as well as adjoint consistent. The latter property will prove particularly useful to obtain error estimates.

Lemma 2.4 (consistency) *Let the exact solution u of (1.2) be in $H^{3/2+\varepsilon}(\Omega)$ for some $\varepsilon > 0$. Then u satisfies, with the right-hand side $F_{\mathscr{T}}$ given in (2.4), the consistency condition*

$$a_{\mathscr{T}}(u,v) - k^2(u,v) = F_{\mathscr{T}}(v) \qquad \forall v \in S.$$

Proof From the $H^{3/2+\varepsilon}$-regularity of u it follows that u and ∇u have well-defined traces on ∂K for each $K \in \mathscr{T}$ and

$$[\![u]\!]_N = 0, \quad [\![\nabla u]\!]_N = 0, \quad \{\nabla u\} = \nabla u \quad \text{on} \quad \mathfrak{S}_{\mathscr{T}}^{\mathscr{I}}.$$

We multiply both sides of equation (1.1) by a test function $v \in S$, integrate elementwise, sum over all elements, and integrate by parts to get

$$\sum_{K \in \mathscr{T}} \left(\int_{\partial K} (-\nabla u \cdot \mathbf{n}) \bar{v} + \int_K \nabla u \cdot \nabla \bar{v} \right) - \int_\Omega k^2 u \bar{v} = \int_\Omega f \bar{v}. \tag{2.10}$$

From the definition of the jumps on the inner faces and the boundary condition (1.2), we get

$$\begin{aligned} -\sum_{K \in \mathscr{T}} \int_{\partial K} (\nabla u \cdot \mathbf{n}) \bar{v} dS = &- \int_{\mathfrak{S}_{\mathscr{T}}^{\mathscr{B}}} \delta \nabla u \cdot \mathbf{n} \bar{v} dS - \int_{\mathfrak{S}_{\mathscr{T}}^{\mathscr{B}}} (1-\delta) g \bar{v} dS \\ &+ \int_{\mathfrak{S}_{\mathscr{T}}^{\mathscr{B}}} \mathrm{i} k (1-\delta) u \bar{v} dS - \int_{\mathfrak{S}_{\mathscr{T}}^{\mathscr{I}}} \nabla u \cdot [\![\bar{v}]\!]_N dS. \end{aligned}$$

The boundary condition (1.2) gives us

$$\begin{aligned} -\sum_{K \in \mathscr{T}} \int_{\partial K} (\nabla u \cdot \mathbf{n}) \bar{v} dS = &- \int_{\mathfrak{S}_{\mathscr{T}}^{\mathscr{B}}} \delta \nabla u \cdot \mathbf{n} \bar{v} dS - \int_{\mathfrak{S}_{\mathscr{T}}^{\mathscr{B}}} (1-\delta) g \bar{v} dS + \int_{\mathfrak{S}_{\mathscr{T}}^{\mathscr{B}}} \mathrm{i} k (1-\delta) u \bar{v} dS \\ &- \int_{\mathfrak{S}_{\mathscr{T}}^{\mathscr{I}}} \nabla u \cdot [\![\bar{v}]\!]_N dS + \frac{1}{\mathrm{i}k} \int_{\mathfrak{S}_{\mathscr{T}}^{\mathscr{B}}} \delta g \overline{\nabla_{\mathscr{T}} v \cdot \mathbf{n}} dS - \frac{1}{\mathrm{i}k} \int_{\mathfrak{S}_{\mathscr{T}}^{\mathscr{B}}} \delta \nabla u \cdot \mathbf{n} \overline{\nabla_{\mathscr{T}} v \cdot \mathbf{n}} dS \\ &- \int_{\mathfrak{S}_{\mathscr{T}}^{\mathscr{B}}} \delta u \overline{\nabla_{\mathscr{T}} v \cdot \mathbf{n}} dS. \end{aligned}$$

Inserting this result into equation (2.10) leads to

$$a_{\mathscr{T}}(u,v) - k^2(u,v) = (f,v) - \int_{\mathfrak{S}_{\mathscr{T}}^{\mathscr{B}}} \delta \frac{1}{\mathrm{i}k} g \overline{\nabla_{\mathscr{T}} v \cdot \mathbf{n}} dS + \int_{\mathfrak{S}_{\mathscr{T}}^{\mathscr{B}}} (1-\delta) g \bar{v} dS, \quad \forall v \in S,$$

which (2.4) as desired. □

Lemma 2.7 below will establish the consistency with respect to the following adjoint problem.

Definition 2.5 (adjoint solution operator N_k^*) The *adjoint Helmholtz problem* is given by:

$$\text{For } w \in L^2(\Omega) \text{ find } \phi \in H^1(\Omega) \text{ such that } a(v,\phi) = (v,w) \quad \forall v \in H^1(\Omega). \tag{2.11}$$

The *solution operator* $N_k^* : L^2(\Omega) \to H^1(\Omega)$ is characterized by the condition

$$a(v, N_k^*(w)) = (v,w). \tag{2.12}$$

We say that Problem (2.11) has $H^s(\Omega)$*-regularity* for some $s > 1$ if for any given right-hand side $w \in L^2(\Omega)$ the solution ϕ of (2.11) is in $H^s(\Omega)$ and satisfies

$$\|\phi\|_{H^s(\Omega)} \leq C_{\text{reg}} \|w\|_{L^2(\Omega)}$$

for some positive constant C_{reg} that is independent of w.

Remark 2.6 The adjoint problem (2.11) is a well-posed problem, for which even k-explicit regularity is available. For example, if Ω convex (or smooth and star-shaped), then $\phi \in H^2(\Omega)$ and

$$\begin{aligned} k\|\phi\|_{L^2(\Omega)} + \|\nabla\phi\|_{L^2(\Omega)} &\leq C_1(\Omega)\|w\|_{L^2(\Omega)}, \\ \|\nabla^2\phi\|_{L^2(\Omega)} &\leq C_2(\Omega)(1+k)\|w\|_{L^2(\Omega)}, \end{aligned}$$

with $C_1(\Omega), C_2(\Omega) > 0$ independent of $k \geq k_0 > 0$ (k_0 is arbitrary but fixed), [36, Prop. 8.1.4] for $d = 2$ and [12] for $d = 3$. For general Lipschitz domains, we have by [17, Thm. 2.4]

$$k\|\phi\|_{L^2(\Omega)} + \|\nabla\phi\|_{L^2(\Omega)} \leq C_3(\Omega)k^{5/2}\|w\|_{L^2(\Omega)}$$

for a constant $C_3(\Omega)$ independent of $k \geq k_0$. For polygonal/polyhedral Lipschitz domains Ω the classical elliptic regularity theory provides $\phi \in H^{3/2+\varepsilon}(\Omega)$ for some $\varepsilon > 0$, which depends on the geometry of Ω. ■

Lemma 2.7 (adjoint consistency) *Let the adjoint Helmholtz problem be* $H^{3/2+\varepsilon}(\Omega)$*-regular for some* $\varepsilon > 0$*. Then for any* $w \in L^2(\Omega)$*, the solution* $\phi := N_k^*(w)$ *of the adjoint problem (2.11) satisfies*

$$a_{\mathscr{T}}(v,\phi) - k^2(v,\phi) = (v,w) \qquad \forall v \in H^{3/2+\varepsilon}_{\text{pw}}(\Omega). \tag{2.13}$$

Proof From the $H^{3/2+\varepsilon}(\Omega)$-regularity of ϕ it follows that ϕ and $\nabla\phi$ have well-defined traces on ∂K for each $K \in \mathscr{T}$ and

$$[\![\phi]\!]_N = 0, \quad [\![\nabla\phi]\!]_N = 0, \quad \{\nabla\phi\} = \nabla\phi \quad \text{on} \quad \mathfrak{S}^{\mathscr{I}}_{\mathscr{T}}.$$

The rest of the proof is just a repetition of the arguments in the proof of Lemma 2.4 by taking into account the zero Robin boundary conditions for the adjoint problem. □

On $H^{3/2+\varepsilon}_{\text{pw}}(\Omega)$ for $\varepsilon > 0$ we will use the mesh-dependent norms $\|\cdot\|_{DG}$ and $\|\cdot\|_{DG^+}$ that were introduced in [25]:

$$\begin{aligned} \|v\|^2_{DG} :=& \|\nabla_{\mathscr{T}} v\|^2_{L^2(\Omega)} + k^{-1}\|\beta^{1/2}[\![\nabla_{\mathscr{T}} v]\!]_N\|^2_{L^2(\mathfrak{S}^{\mathscr{I}}_{\mathscr{T}})} + k\|\alpha^{1/2}[\![v]\!]_N\|^2_{L^2(\mathfrak{S}^{\mathscr{I}}_{\mathscr{T}})} \\ &+ k^{-1}\|\delta^{1/2}\nabla_{\mathscr{T}} v \cdot \mathbf{n}\|^2_{L^2(\mathfrak{S}^{\mathscr{B}}_{\mathscr{T}})} + k\|(1-\delta)^{1/2}v\|^2_{L^2(\mathfrak{S}^{\mathscr{B}}_{\mathscr{T}})} + k^2\|v\|^2_{L^2(\Omega)}, \\ \|v\|^2_{DG^+} :=& \|v\|^2_{DG} + k^{-1}\|\alpha^{-1/2}\{\nabla_{\mathscr{T}} v\}\|^2_{L^2(\mathfrak{S}^{\mathscr{I}}_{\mathscr{T}})}. \end{aligned}$$

3 Discrete Stability and Convergence Analysis

This section is devoted to the analysis of the discrete problem for the finite dimensional space S satisfying the condition (2.3).

3.1 Continuity and Coercivity

Proposition 3.1 *Define $b_{\mathcal{T}}(u,v) := a_{\mathcal{T}}(u,v) + k^2(u,v)$. For any $0 < \delta < \frac{1}{3}$ and α satisfying (2.7), there exist constants c_{coer}, $C_{\text{c}} > 0$ independent of h, k, α, β, δ, and $C_{\text{trace}}(S,K)$ such that the following two statements are true:*

a) The sesquilinear form $b_{\mathcal{T}}(\cdot,\cdot)$ is coercive:

$$|b_{\mathcal{T}}(v,v)| \geq c_{\text{coer}} \|v\|_{DG}^2 \qquad \forall v \in S.$$

b) For any $\varepsilon > 0$, the sesquilinear form $b_{\mathcal{T}}(\cdot,\cdot)$ satisfies the following continuity estimates

$$|b_{\mathcal{T}}(v,w_S)| \leq C_{\text{c}} \|v\|_{DG^+} \|w\|_{DG^+} \qquad \forall v,w \in H_{\text{pw}}^{3/2+\varepsilon}(\Omega), \tag{3.1}$$

$$|b_{\mathcal{T}}(v,w_S)| \leq C_{\text{c}} \|v\|_{DG^+} \|w_S\|_{DG} \qquad \forall v \in H_{\text{pw}}^{3/2+\varepsilon}(\Omega), \quad \forall w_S \in S, \tag{3.2}$$

$$|b_{\mathcal{T}}(w_S,v)| \leq C_{\text{c}} \|v\|_{DG^+} \|w_S\|_{DG} \qquad \forall v \in H_{\text{pw}}^{3/2+\varepsilon}(\Omega), \quad \forall w_S \in S. \tag{3.3}$$

Proof The proof uses the same argument as [25, Props. 4.2, 4.4]; we trace the dependence on our abstract framework and work out constants explicitly.
a) The definition of $b_{\mathcal{T}}(.,.)$ leads to

$$\begin{aligned} b_{\mathcal{T}}(v,v) = {} & \|\nabla_{\mathcal{T}} v\|_{L^2(\Omega)}^2 - 2\operatorname{Re}\left(\int_{\mathfrak{S}_{\mathcal{T}}^{\mathcal{I}}} [\![v]\!]_N \cdot \{\overline{\nabla_{\mathcal{T}} v}\} dS\right) - 2\operatorname{Re}\left(\int_{\mathfrak{S}_{\mathcal{T}}^{\mathcal{B}}} \delta v \overline{\nabla_{\mathcal{T}} v \cdot \mathbf{n}} dS\right) \\ & + \mathrm{i} k^{-1} \|\beta^{1/2} [\![\nabla_{\mathcal{T}} v]\!]_N\|_{L^2(\mathfrak{S}_{\mathcal{T}}^{\mathcal{I}})}^2 + \mathrm{i} k^{-1} \|\delta^{1/2} \nabla_{\mathcal{T}} v \cdot \mathbf{n}\|_{L^2(\mathfrak{S}_{\mathcal{T}}^{\mathcal{B}})}^2 \\ & + \mathrm{i} k \|\alpha^{1/2} [\![v]\!]_N\|_{L^2(\mathfrak{S}_{\mathcal{T}}^{\mathcal{I}})}^2 + \mathrm{i} k \|(1-\delta)^{1/2} v\|_{0,\mathfrak{S}_{\mathcal{T}}^{\mathcal{B}}}^2 + k^2 \|v\|_{L^2(\Omega)}^2. \end{aligned}$$

By using Young's inequality for some positive function $s \in L^\infty\left(\overline{\mathfrak{S}_{\mathcal{T}}^{\mathcal{I}}}\right)$ we get for the second term in the representation of $b_{\mathcal{T}}(\cdot,\cdot)$

$$\left|2\operatorname{Re}\int_{\mathfrak{S}_{\mathcal{T}}^{\mathcal{I}}} [\![v]\!]_N \cdot \{\overline{\nabla_{\mathcal{T}} v}\} dS\right| \leq k \left\|\sqrt{\frac{s}{\alpha}} \alpha^{1/2} [\![v]\!]_N\right\|_{L^2(\mathfrak{S}_{\mathcal{T}}^{\mathcal{I}})}^2 + \frac{1}{k} \left\|\frac{1}{\sqrt{s}} \nabla\left(v|_K\right)\right\|_{L^2(\mathfrak{S}_{\mathcal{T}}^{\mathcal{I}})}^2.$$

We choose $s := 4\alpha/5$. By using (2.7) we get

$$\left|2\operatorname{Re}\int_{\mathfrak{S}_{\mathcal{T}}^{\mathcal{I}}} [\![v]\!]_N \cdot \{\overline{\nabla_{\mathcal{T}} v}\} dS\right| \leq \frac{4}{5} k \|\alpha^{1/2} [\![v]\!]_N\|_{L^2(\mathfrak{S}_{\mathcal{T}}^{\mathcal{I}})}^2 + \sum_{K \in \mathcal{T}} \frac{5}{4k} \left\|\frac{1}{\alpha^{1/2}} \nabla\left(v|_K\right)\right\|_{L^2(\Omega \cap \partial K)}^2.$$

For the second summand, we get with $\alpha_{\partial K}^{\min}$ as in (2.9)

$$\sum_{K \in \mathcal{T}} \frac{5}{4k} \left\|\frac{1}{\alpha^{1/2}} \nabla\left(v|_K\right)\right\|_{L^2(\Omega \cap \partial K)}^2 \leq \sum_{K \in \mathcal{T}} \frac{5}{4k} \frac{C_{\text{trace}}^2(S,K)}{\alpha_{\partial K}^{\min}} \|\nabla v\|_{L^2(K)}^2.$$

Let $X \in \overset{\circ}{\partial K} \cap \Omega$ be defined as in Remark 2.3. Since $K \in \left\{K_X^+, K_X^-\right\}$, the condition on α (cf. (2.6)) implies

$$\alpha_{\partial K}^{\min} = \alpha(X) \geq \frac{4}{3k} \max_{K' \in \left\{K_X^+, K_X^-\right\}} C_{\text{trace}}^2\left(S,K'\right) \geq \frac{4}{3k} C_{\text{trace}}^2(S,K). \tag{3.4}$$

Hence,

$$\sum_{K\in\mathcal{T}} \frac{5}{4k} \left\| \frac{1}{\alpha^{1/2}} \nabla \left(v|_K \right) \right\|^2_{L^2(\Omega\cap\partial K)} \leq \frac{15}{16} \|\nabla_{\mathcal{T}} v\|^2_{L^2(\Omega)}.$$

All in all we have derived

$$\left| 2\,\mathrm{Re} \int_{\mathfrak{S}^{\mathcal{I}}_{\mathcal{T}}} [\![v]\!]_N \cdot \{\overline{\nabla_{\mathcal{T}} v}\} dS \right| \leq \frac{4k}{5} \|\alpha^{1/2} [\![v]\!]_N\|^2_{L^2(\mathfrak{S}^{\mathcal{I}}_{\mathcal{T}})} + \frac{15}{16} \|\nabla_{\mathcal{T}} v\|^2_{L^2(\Omega)}.$$

The third term in $b_{\mathcal{T}}(\cdot,\cdot)$ can be estimated in a similar fashion for any $t>0$ by

$$\left| 2\,\mathrm{Re} \int_{\mathfrak{S}^{\mathcal{B}}_{\mathcal{T}}} \delta v \overline{\nabla_{\mathcal{T}} v} \cdot \mathbf{n} dS \right| \leq tk \frac{\delta}{1-\delta} \|(1-\delta)^{1/2} v\|^2_{L^2(\mathfrak{S}^{\mathcal{B}}_{\mathcal{T}})} + \frac{1}{tk} \|\delta^{1/2} \nabla_{\mathcal{T}} v \cdot \mathbf{n}\|^2_{L^2(\mathfrak{S}^{\mathcal{B}}_{\mathcal{T}})}.$$

By choosing $0<\delta<\frac{1}{3}$ as well as $t=3/2$ we obtain

$$\begin{aligned}
|b_{\mathcal{T}}(v,v)| &\geq \frac{1}{\sqrt{2}} \left(|\mathrm{Re}(b_{\mathcal{T}}(v,v))| + |\mathrm{Im}(b_{\mathcal{T}}(v,v))| \right) \\
&\geq \frac{1}{\sqrt{2}} \Big(\frac{1}{16} \|\nabla_{\mathcal{T}} v\|^2_{L^2(\Omega)} + \frac{k}{5} \|\alpha^{1/2} [\![v]\!]_N\|^2_{L^2(\mathfrak{S}^{\mathcal{I}}_{\mathcal{T}})} + \frac{k}{4} \|(1-\delta)^{1/2} v\|^2_{L^2(\mathfrak{S}^{\mathcal{B}}_{\mathcal{T}})} \\
&\quad + \frac{1}{3k} \|\delta^{1/2} \nabla_{\mathcal{T}} v \cdot \mathbf{n}\|^2_{0,\mathfrak{S}^{\mathcal{B}}_{\mathcal{T}}} + k^{-1} \|\beta^{1/2} [\![\nabla_{\mathcal{T}} v]\!]_N\|^2_{L^2(\mathfrak{S}^{\mathcal{I}}_{\mathcal{T}})} + k^2 \|v\|^2_{L^2(\Omega)} \Big) \\
&\geq c_{\mathrm{coer}} \|v\|^2_{DG}.
\end{aligned} \tag{3.5}$$

b) Using Young's inequality we get

$$\begin{aligned}
&|b_{\mathcal{T}}(v,w)| \leq \qquad\qquad (3.6) \\
&\quad |(\nabla_{\mathcal{T}} v, \nabla_{\mathcal{T}} w)| + k^2 |(v,w)| + \left| \int_{\mathfrak{S}^{\mathcal{I}}_{\mathcal{T}}} [\![v]\!]_N \cdot \{\overline{\nabla_{\mathcal{T}} w}\} dS \right| + \left| \int_{\mathfrak{S}^{\mathcal{I}}_{\mathcal{T}}} \{\nabla_{\mathcal{T}} v\} \cdot [\![\overline{w}]\!]_N dS \right| \\
&+ \left| \int_{\mathfrak{S}^{\mathcal{B}}_{\mathcal{T}}} \delta v \overline{\nabla_{\mathcal{T}} w \cdot \mathbf{n}} dS \right| + \left| \int_{\mathfrak{S}^{\mathcal{B}}_{\mathcal{T}}} \delta \nabla_{\mathcal{T}} v \cdot \mathbf{n} \overline{w} dS \right| + \frac{1}{k} \left| \int_{\mathfrak{S}^{\mathcal{I}}_{\mathcal{T}}} \left(\beta [\![\nabla_{\mathcal{T}} v]\!]_N [\![\overline{\nabla_{\mathcal{T}} w}]\!]_N \right) dS \right| \\
&+ \frac{1}{k} \left| \int_{\mathfrak{S}^{\mathcal{B}}_{\mathcal{T}}} \left(\delta \nabla_{\mathcal{T}} v \cdot \mathbf{n} \overline{\nabla_{\mathcal{T}} w \cdot \mathbf{n}} \right) dS \right| + \left| \int_{\mathfrak{S}^{\mathcal{I}}_{\mathcal{T}}} (k\alpha [\![v]\!]_N [\![\overline{w}]\!]_N) dS \right| + k \left| \int_{\mathfrak{S}^{\mathcal{B}}_{\mathcal{T}}} (1-\delta) v \overline{w} dS \right|.
\end{aligned}$$

For $0<\delta<1/3$ and for any v, $w \in H^{3/2+\varepsilon}_{\mathrm{pw}}(\Omega)$ we finally obtain

$$|b_{\mathcal{T}}(v,w)| \leq C_{\mathrm{c}} \|v\|_{DG^+} \|w\|_{DG^+}.$$

Estimates in weaker norms are possible if one of these two functions is from the discrete space S, e.g., $w \in S$. A careful inspection of equation (3.6) shows that the only term which requires the DG^+-norm instead of DG-norm for w in the continuity estimate is $\int_{\mathfrak{S}^{\mathcal{I}}_{\mathcal{T}}} [\![v]\!]_N \cdot \{\overline{\nabla_{\mathcal{T}} w}\} dS$. Using Young's inequality we get

$$\left| \int_{\mathfrak{S}^{\mathcal{I}}_{\mathcal{T}}} [\![v]\!]_N \cdot \{\overline{\nabla_{\mathcal{T}} w}\} dS \right| \leq \sum_{K\in\mathcal{T}} \left\{ \|[\![v]\!]_N\|_{L^2(\Omega\cap\partial K)} \|\nabla \left(w|_K \right)\|_{L^2(\Omega\cap\partial K)} \right\}.$$

We apply the trace inequality in (2.6) and also (2.7) to obtain

$$\left|\int_{\mathfrak{S}_{\mathcal{T}}^{\mathcal{I}}} [\![v]\!]_N \cdot \{\overline{\nabla_{\mathcal{T}} w}\} dS\right| \leq \sum_{K\in\mathcal{T}} \left\{ \frac{1}{\sqrt{\alpha_{\partial K}^{\min}}} \left\| \alpha^{\frac{1}{2}} [\![v]\!]_N \right\|_{L^2(\Omega\cap\partial K)} C_{\text{trace}}(S,K) \|\nabla_{\mathcal{T}} w\|_{L^2(K)} \right\}$$

$$\overset{(3.4)}{\leq} \sqrt{\frac{3k}{4}} \sum_{K\in\mathcal{T}} \left\{ \left\| \alpha^{\frac{1}{2}} [\![v]\!]_N \right\|_{L^2(\Omega\cap\partial K)} \|\nabla_{\mathcal{T}} w\|_{L^2(K)} \right\}$$

$$\leq \sqrt{\frac{3k}{2}} \left\| \alpha^{\frac{1}{2}} [\![v]\!]_N \right\|_{L^2\left(\mathfrak{S}_{\mathcal{T}}^{\mathcal{I}}\right)} \|\nabla_{\mathcal{T}} w\|_{L^2(\Omega)}.$$

Hence, we finally obtain (3.2). The estimate (3.3) can be shown using the same techniques or derived from (3.2) by observing that for v, $w \in H_{\text{pw}}^{3/2+\varepsilon}(\Omega)$ we have

$$b_{\mathcal{T},k}(v,w) = \overline{b_{\mathcal{T},-k}(w,v)},$$

where we have added the subscript k (or $-k$) to emphasize how the parameter k enters the definition. □

Remark 3.2 The restriction $0 < \delta < 1/3$ in Proposition 3.1 was made to simplify the proof and may be relaxed to $0 < \delta < 1/2$. Then, the coercivity constant is bounded from below but degenerates to zero as $\delta \to 1/2$. This can be shown by assuming $0 < \delta \leq 1/2 - \varepsilon$ and $t = 1/(1-2\varepsilon)$ with $0 < \varepsilon < 1/2$. Following similar steps as in (3.5), one can show

$$C_{\text{coer}} = \frac{1}{\sqrt{2}} \min\left\{\frac{1}{16}, \frac{2\varepsilon}{1+2\varepsilon}\right\}. \qquad ■$$

As a corollary of (3.3) we have the following continuity assertion, which will be useful for certain adjoint problems:

Corollary 3.3 *For any $\varepsilon > 0$, it holds*

$$|a_{\mathcal{T}}(v,u) - k^2(v,u)| \leq C_{\text{c}} \|u\|_{DG+} \|v\|_{DG} \qquad \forall u \in H_{\text{pw}}^{3/2+\varepsilon}(\Omega) \quad \forall v \in S. \tag{3.7}$$

3.2 Quasi-Optimality

We start with a definition: We say that a pair $(u, u_S) \in H_{\text{pw}}^{3/2+\varepsilon}(\Omega) \times S$ of functions satisfies the Galerkin orthogonality if

$$a_{\mathcal{T}}(u - u_S, v) = 0 \qquad \forall v \in S. \tag{3.8}$$

Our starting point for the analysis of our DG problem is a quasi-optimality result which is proved under the *assumption* that the above Galerkin orthogonality is valid. The existence and uniqueness of a solution u_S of the discrete problem (2.4) is then shown in a second step based on the quasi-optimality result.

Proposition 3.4 *There exists a constant $\widetilde{C} > 0$ depending solely on the constants C_c, c_{coer} of Proposition 3.1 such that the following is true: Any pair $(u, u_S) \in H_{\text{pw}}^{3/2+\varepsilon}(\Omega) \times S$ meeting the orthogonality condition (3.8) satisfies*

$$\|u - u_S\|_{DG} \leq \widetilde{C} \left(\inf_{v\in S} \|u - v\|_{DG^+} + \sup_{0\neq w_S\in S} \frac{k|(u - u_S, w_S)|}{\|w_S\|_{L^2(\Omega)}} \right).$$

Proof For the reader's convenience, we include the proof taken from [25, Proposition 4.4]. We start with a triangle inequality

$$\|u-u_S\|_{DG} \le \|u-v\|_{DG} + \|v-u_S\|_{DG} \qquad \forall v \in S \tag{3.9}$$

and employ the coercivity of $b_{\mathscr{T}}(\cdot,\cdot)$

$$\begin{aligned}\|v-u_S\|_{DG}^2 &\le \frac{1}{c_{\text{coer}}}|b_{\mathscr{T}}(v-u_S, v-u_S)| \\ &\le \frac{1}{c_{\text{coer}}}|b_{\mathscr{T}}(v-u, v-u_S)| + \frac{1}{c_{\text{coer}}}|b_{\mathscr{T}}(u-u_S, v-u_S)| \\ &= \frac{1}{c_{\text{coer}}}|b_{\mathscr{T}}(v-u, v-u_S)| + \frac{2k^2}{c_{\text{coer}}}|(u-u_S, v-u_S)|,\end{aligned} \tag{3.10}$$

where in the last inequality we employed the orthogonality condition (3.8). The continuity of $b_{\mathscr{T}}(\cdot,\cdot)$ expressed in (3.1) together with (3.10) implies

$$\|v-u_S\|_{DG}^2 \le \frac{C_{\text{c}}}{c_{\text{coer}}}\|v-u\|_{DG^+}\|v-u_S\|_{DG} + \frac{2k^2}{c_{\text{coer}}}|(u-u_S, v-u_S)|.$$

We combine this result with (3.9) and obtain

$$\|u-u_S\|_{DG} \le \|u-v\|_{DG} + \frac{C_{\text{c}}}{c_{\text{coer}}}\|v-u\|_{DG^+} + \frac{2k}{c_{\text{coer}}} \sup_{0\neq w_S\in S}\frac{|(u-u_S, w_S)|}{\|w_S\|_{L^2(\Omega)}}. \qquad \square$$

Next, we will use the adjoint problem to gauge the contribution $\sup_{w_S\in S}\frac{k|(u-u_S,w_S)|}{\|w_S\|_{L^2(\Omega)}}$ in Proposition 3.4.

Proposition 3.5 *Assume that the adjoint Helmholtz problem is $H^{3/2+\varepsilon}(\Omega)$ regular for some $\varepsilon > 0$. Let the coefficients in the definition of $a_{\mathscr{T}}(\cdot,\cdot)$ satisfy $0<\delta<1/3$ and (2.7). Then the following is true: For any pair $(u,u_S) \in H^{3/2+\varepsilon}_{\text{pw}}(\Omega)\times S$ that satisfies (3.8) we have*

$$\sup_{0\neq w_S\in S}\frac{k|(u-u_S,w_S)_{L^2(\Omega)}|}{\|w_S\|_{L^2(\Omega)}} \le (1+3C_{\text{c}})\,\eta_k(S)\left(\inf_{v\in S}\|u-v\|_{DG^+} + \|u-u_S\|_{DG}\right),$$

where the adjoint approximation property *is defined by*

$$\eta_k(S) := \sup_{f\in L^2(\Omega)\backslash\{0\}}\inf_{\psi_S\in S}\frac{k\|N_k^*(f)-\psi_S\|_{DG^+}}{\|f\|_{L^2(\Omega)}}. \tag{3.11}$$

Proof Write $\phi = N_k^*(w_S)$ for the solution of (2.12) with right-hand side $w_S \in S \subset L^2(\Omega)$. Our regularity assumption implies $\phi \in H^{3/2+\varepsilon}(\Omega)$ for some $\varepsilon > 0$ (cf. Remark 2.6). The adjoint consistency of the method stated in Lemma 2.7 then provides

$$(u-u_S, w_S) = a_{\mathscr{T}}(u-u_S,\phi) - k^2(u-u_S,\phi).$$

Using the definition of the sesquilinear form $a_{\mathscr{T}}$ and the Galerkin orthogonality, we get for any $v \in S$

$$\begin{aligned}|(u-u_S,w_S)| &\le |a_{\mathscr{T}}(u-v,\phi-\psi_S)| + |a_{\mathscr{T}}(v-u_S,\phi-\psi_S)| + k^2|(u-u_S,\phi-\psi_S)| \\ &\le (C_{\text{c}}\|u-v\|_{DG^+} + C_{\text{c}}\|v-u_S\|_{DG} + \|u-u_S\|_{DG})\,\|\phi-\psi_S\|_{DG^+} \\ &\le (2C_{\text{c}}\|u-v\|_{DG^+} + (1+C_{\text{c}})\|u-u_S\|_{DG})\,\|\phi-\psi_S\|_{DG^+}.\end{aligned}$$

Since v, $\psi_S \in S$ are arbitrary, the statement follows. $\square$

The combination of the previous results leads to the following wavenumber-explicit error estimate (still under the assumption of existence of a discrete solution).

Theorem 3.6 (quasi-optimal convergence) *Assume that the adjoint Helmholtz problem is $H^{3/2+\varepsilon}(\Omega)$ regular for some $\varepsilon > 0$. Let the coefficients in the definition of $a_{\mathscr{T}}(\cdot,\cdot)$ satisfy $0 < \delta < 1/3$ and (2.7). If the condition*

$$\eta_k(S) < \frac{c_{\mathrm{coer}}}{4(1+C_c)}$$

holds, then for any pair $(u, u_S) \in H^{3/2+\varepsilon}_{\mathrm{pw}}(\Omega) \times S$ that satisfies (3.8) we have

$$\|u - u_S\|_{DG} \leq C \inf_{v \in S} \|u - v\|_{DG^+}, \tag{3.12}$$

where C depends solely on C_c and c_{coer}.

Proof By combining the results of Propositions 3.4 and 3.5, we get the following:

$$\|u - u_S\|_{DG} \leq \left(1 + \frac{C_c}{c_{\mathrm{coer}}} + \frac{4C_c}{c_{\mathrm{coer}}}\eta_k(S)\right) \inf_{v \in S} \|u - v\|_{DG^+} + \frac{2(1+C_c)}{c_{\mathrm{coer}}}\eta_k(S)\|u - u_S\|_{DG}.$$

The condition $\frac{2(1+C_c)}{c_{\mathrm{coer}}}\eta_k(S) < 1/2$ allows us to absorb the error term on the right-hand side in the left-hand side. □

3.3 Discrete Stability

The preceding section provides an error analysis under the assumption of existence of the discrete solution $u_S \in S$ of (2.4). Extra conditions have to be imposed for existence as the following Example 3.7 shows. That is, the discontinuous Galerkin method for the Helmholtz problem is not necessarily stable for an arbitrary discrete space S that only satisfies the minimal condition (2.3).

Example 3.7 Let $\Omega := \operatorname{conv}\{(0,0)^\intercal, (1,0)^\intercal, (0,1)^\intercal\}$ and let the mesh $\mathscr{T}$ consists of the single element $\{\Omega\}$. A (one-dimensional) space S that satisfies condition (2.3) is defined by the span of the squared cubic bubble function, $S = \operatorname{span}\{(27\lambda_1\lambda_2\lambda_3)^2\}$, where $\lambda_1 = \xi_1$, $\lambda_2 = \xi_2$, $\lambda_3 = 1 - \xi_1 - \xi_2$ and $0 \leq \xi_1 \leq 1$, $0 \leq \xi_2 \leq 1 - \xi_1$. In this case, equation (3.16) reduces to

$$(\nabla w_S, \nabla v_S) - k^2(w_S, v_S) = 0 \qquad \forall v_S \in S. \tag{3.13}$$

As S is a one-dimensional space we get the following 1×1 system $(A - k^2B)w = 0$, where $A = \int_{\widehat{K}} \nabla b_1 \cdot \nabla b_1 = 5.1125$, $B = \int_{\widehat{K}} b_1^2 = 0.0843$ and $b_1 = (27\lambda_1\lambda_2\lambda_3)^2$. Obviously, the value of $k = \sqrt{\frac{A}{B}}$ is a critical wavenumber where the system matrix becomes singular. ■

In this section, we will study conditions under which the DG problem admits a unique solution in the discrete space S. One possible condition (3.14) is formulated in Theorem 3.8 and it is shown that this condition is always satisfied for plane waves methods as well as for conforming and non-conforming polynomial *hp*-finite element spaces on affine simplicial meshes (cf. Remark 3.9). Thus, Theorem 3.8 presents a unified stability theory for these types of methods and shows that a unique numerical solution always exists for these important choices of spaces. This is in contrast to conventional Galerkin methods applied to (1.3), where a minimal resolution condition on the finite element space, e.g., on the mesh width, has to be imposed in order to guarantee unique solvability of the discrete equations.

Alternatively, as in the classical Galerkin discretization, a condition on the adjoint approximation property on the abstract space can be employed to prove existence, uniqueness, and quasi-optimality of the discretization. This is proved in Theorem 3.10.

Theorem 3.8 *Let the discrete space S satisfy (2.3). Let $\beta \geq 0$, $0 < \delta < 1/3$, and choose α such that (2.7) is satisfied. Then, the DG problem (2.4) has a unique solution $u_S \in S$ if*

$$C_S < \frac{k}{2(1+C_c)} \quad \textit{with} \quad C_S := \sup_{w_S \in S \cap H_0^2(\Omega) \setminus \{0\}} \inf_{v_S \in S} \frac{\| \langle x, \nabla w_S \rangle - v_S \|_{DG^+}}{\| w_S \|_{L^2(\Omega)}}. \tag{3.14}$$

Furthermore, let the exact solution of (1.3) satisfy $u \in H^{3/2+\varepsilon}(\Omega)$, and let the adjoint Helmholtz problem be $H^{3/2+\varepsilon}(\Omega)$ regular for some $\varepsilon > 0$. Assume the adjoint approximation condition

$$\eta_k(S) < \frac{c_{\mathrm{coer}}}{4(1+C_c)}.$$

Then, the quasi-optimal error estimate

$$\| u - u_S \|_{DG} \leq C \inf_{v \in S} \| u - v \|_{DG^+}$$

holds, where C is independent of k and the space S.

Proof If the discrete solution $u_S \in S$ of (2.4) exists, then the consistency statement Lemma 2.4 implies the orthogonality condition (3.8) so that the quasi-optimality assertion follows from Theorem 3.6. It therefore remains to assert existence of $u_S \in S$. By dimension arguments, existence of a solution $u_S \in S$ of (2.4) follows, if we can verify the following uniqueness assertion:

$$\forall w_S \in S \setminus \{0\} \quad \exists v_S \in S \quad \text{s.t.} \quad |a_{\mathscr{T}}(w_S, v_S) - k^2(w_S, v_S)| > 0. \tag{3.15}$$

We prove (3.15) indirectly, by showing the equivalent implication:
For any $w_S \in S$ it holds:

$$\big(\forall v_S \in S \quad a_{\mathscr{T}}(w_S, v_S) - k^2(w_S, v_S) = 0\big) \Rightarrow w_S = 0. \tag{3.16}$$

Our assumption in (3.16) implies for any $w_S \in S$

$$\operatorname{Im}\big(a_{\mathscr{T}}(w_S, v_S) - k^2(w_S, v_S)\big) = 0 \quad \text{and} \quad \operatorname{Re}\big(a_{\mathscr{T}}(w_S, v_S) - k^2(w_S, v_S)\big) = 0. \tag{3.17}$$

First we choose $v_S = w_S$ in (3.17). From the equation for the imaginary part we obtain

$$\begin{aligned} [\![\nabla_{\mathscr{T}} w_S]\!]_N &= 0 \quad \text{on} \quad \mathfrak{S}_{\mathscr{T}}^{\mathscr{I}}, \\ \nabla_{\mathscr{T}} w_S \cdot \mathbf{n} &= 0 \quad \text{on} \quad \mathfrak{S}_{\mathscr{T}}^{\mathscr{B}}, \\ [\![w_S]\!]_N &= 0 \quad \text{on} \quad \mathfrak{S}_{\mathscr{T}}^{\mathscr{I}}, \\ w_S &= 0 \quad \text{on} \quad \mathfrak{S}_{\mathscr{T}}^{\mathscr{B}}, \end{aligned}$$

and this implies $w_S \in H_0^2(\Omega) \cap S$ (in particular, it implies $\nabla_{\mathscr{T}} w_S = \nabla w_S$). Hence, the real part of equation (3.17) gives us

$$\| \nabla w_S \|_{L^2(\Omega)}^2 - k^2 \| w_S \|_{L^2(\Omega)}^2 = 0. \tag{3.18}$$

Define $v_S^*(x) = \langle x, \nabla w_S \rangle$. From the real part of equation (3.17) it follows

$$
\begin{aligned}
0 &= \operatorname{Re}\left(a_{\mathscr{T}}(w_S, v_S^*) - k^2(w_S, v_S^*)\right) + \operatorname{Re}\left(a_{\mathscr{T}}(w_S, v_S - v_S^*) - k^2(w_S, v_S - v_S^*)\right) \\
&\geq \operatorname{Re}\left(a_{\mathscr{T}}(w_S, v_S^*) - k^2(w_S, v_S^*)\right) - |a_{\mathscr{T}}(w_S, v_S^* - v_S)| - |k^2(w_S, v_S^* - v_S)|.
\end{aligned}
$$

By using $2\operatorname{Re}(w_S \nabla \overline{w_S}) = \nabla(|w_S|^2)$ for the first term, and continuity of $a_{\mathscr{T}}$, and applying Cauchy-Schwarz inequality we get (see also [36], [21])

$$
\begin{aligned}
0 \geq & (2-d)\|\nabla w_S\|^2_{L^2(\Omega)} + dk^2 \|w_S\|^2_{L^2(\Omega)} - 2C_c \|w_S\|_{DG} \|v_S^* - v_S\|_{DG^+} \\
& - 2k^2 \|w_S\|_{L^2(\Omega)} \|v_S^* - v_S\|_{L^2(\Omega)} \\
\geq & (2-d)\|\nabla w_S\|^2_{L^2(\Omega)} + dk^2 \|w_S\|^2_{L^2(\Omega)} - 2C_c C_S \|w_S\|_{DG} \|w_S\|_{L^2(\Omega)} \\
& - 2C_S k \|w_S\|^2_{L^2(\Omega)}. \qquad (3.19)
\end{aligned}
$$

Using the definition of DG-norm and taking into account that $w_S \in H_0^2(\Omega) \cap S$ we get $\|w_S\|_{DG} = \|w_S\|_{\mathscr{H}}$, where $\|w_S\|^2_{\mathscr{H}} := \|\nabla w_S\|^2_{L^2(\Omega)} + k^2 \|w_S\|^2_{L^2(\Omega)}$. For $d = 1$, we get

$$
0 \geq \|w_S\|^2_{\mathscr{H}} - 2C_c C_S \|w_S\|_{\mathscr{H}} \|w_S\|_{L^2(\Omega)} - 2C_S k \|w_S\|^2_{L^2(\Omega)} \geq \left(1 - \frac{2C_c C_S}{k} - \frac{2C_S}{k}\right) \|w_S\|^2_{\mathscr{H}}.
$$

If $C_S < \frac{k}{2(1+C_c)}$ then it follows that $w_S = 0 \operatorname{in} \Omega$. For $d = 2, 3$ we add (3.18) to the equation (3.19) and then proceed with the same argument as in 1d. □

Remark 3.9 For general finite-dimensional spaces S, condition (3.14) could be interpreted as a condition on the scale resolution. However, the condition (3.14) is always satisfied in the following two important cases:

- In [25] the variational formulation (2.4) was derived for the discretization by locally (discontinuous) plane waves. In that setting, condition (3.14) is not imposed since it is trivially satisfied as then $S \cap H_0^2(\Omega) = \{0\}$ (this equality follows from the unique continuation principle for elliptic PDEs—see, e.g., the discussion in [17, Sec. 6.3] for details).
- DG-methods based on classical piecewise polynomials on affine triangulations (consisting of simplices) satisfy (3.14) automatically as $\langle x, \nabla_{\mathscr{T}} w_S \rangle \in S$. The proof is closely related to the arguments presented in [21–23]. Indeed, the key observation in these references is that, for given $u \in S$, elementwise defined test functions of the form u and $x \cdot \nabla u$ or, more generally, $\alpha(x - x_\Omega) \cdot \nabla u + \beta u$ (for constants α, β and a chosen point x_Ω) are useful to provide stability and error estimates. ■

For new generalized finite element spaces, it might be complicated to verify condition (3.14). In the following theorem, we present a different criterion which also implies discrete stability.

Theorem 3.10 *Let the exact solution of (1.3) satisfy $u \in H^{3/2+\varepsilon}(\Omega)$ and let the adjoint Helmholtz problem be $H^{3/2+\varepsilon}(\Omega)$ regular for some $\varepsilon > 0$. Assume that the coefficients in the definition of $a_{\mathscr{T}}(\cdot,\cdot)$ satisfy $0 < \delta < 1/3$ and (2.7). If the condition*

$$
\eta_k(S) < \frac{c_{\text{coer}}}{4(1 + C_c)}
$$

holds, then the DG problem (2.4) has a unique solution $u_S \in S$ and satisfies the quasi-optimality property (3.12).

Proof The proof follows the lines in [35, Thm. 3.9]. We merely have to show existence of u_S. Since the (2.4) corresponds to a linear system of equations, it suffices to show uniqueness. Therefore, let $u_S \in S$ be in the kernel of the discrete operator, i.e., $a_{\mathscr{T}}(u_S, v) - k^2(u_S, v) = 0$ for all $v \in S$. Then the pair $(0, u_S) \in H^{3/2+\varepsilon}(\Omega) \times S$ satisfies the orthogonality condition (3.8). Hence, Theorem 3.6 implies $\|0 - u_S\|_{DG} \leq C \inf_{v \in S} \|0 - v\|_{DG+} = 0$, which shows $u_S = 0$. Again, the quasi-optimality follows as a combination of Theorem 3.6 and Lemma 2.4. □

4 Application to Polynomial hp-Finite Elements

Theorem 3.6 provides a quasi-optimal error estimate for abstract approximation spaces S that satisfy the conditions (2.3) and (3.14). The concrete choice of the space S enters the analysis via a) the constant $C_{\text{trace}}(S, K)$, b) the estimate of the approximation error $\inf_{v \in S} \|u - v\|_{DG^+}$, c) the adjoint approximation property $\eta_k(S)$, and d) the constant C_S in (3.14). As explained in Remark 3.9 the condition on C_S is "automatically" satisfied for polynomial hp-finite element spaces if *affine* meshes are considered. The focus in the present section is on non-affine meshes so that the stability of the DG method will be inferred from the condition on the adjoint approximability as discussed in Theorem 3.8. Our primary reason for considering curved elements is that our regularity theory for Helmholtz problems (see Theorems 4.5) is done for smooth (more precisely: analytic) geometries. In this setting, we derive explicit estimates for these quantities in the context of polynomial hp-finite element space which are explicit with respect to the polynomial degree p, and the mesh size h.

4.1 Preliminaries

We consider a partition of the domain Ω into "simplicial" elements. That is, the finite element mesh $\mathscr{T}$ consists of elements K that are the images of the reference element $\widehat{K}$, i.e., the reference triangle (in 2D) or the reference tetrahedron (in 3D), under the element map $F_K : \widehat{K} \to K$. The mesh width is denoted by $h := \max_{K \in \mathscr{T}} \operatorname{diam} K$ (cf. (2.1)).

We use the symbol ∇^n to denote derivatives of order n; more precisely, for a function $u : \Omega \to \mathbb{R}, \Omega \subset \mathbb{R}^d$, we set

$$|\nabla^n u(x)|^2 = \sum_{\alpha \in \mathbb{N}_0^d : |\alpha| = n} \frac{n!}{\alpha!} |D^\alpha u(x)|^2.$$

We will need some conditions on the element maps F_K of the triangulations in order to capture the approximation properties of the polynomial hp-FEM spaces. The following assumption will make this more precise. We emphasize that, in contrast to the case of $H^1(\Omega)$-conforming subspaces, we do not require in the present context of DG-methods a "compatibility" condition for element maps of neighboring elements.

Assumption 4.1 *("simplicial" finite element mesh). Each element map F_K can be written as $F_K = R_K \circ B_K$, where B_K is an affine map (containing the scaling by h_K) and R_K is analytic. Let $\widetilde{K} := B_K(K)$. The maps R_K and B_K satisfy for* shape regularity *constants $C_{\text{affine}}, C_{\text{metric}}, \gamma > 0$ independent of h:*

$$\|B_K'\|_{L^\infty(\widehat{K})} \leq C_{\text{affine}} h_K, \qquad \|(B_K')^{-1}\|_{L^\infty(\widehat{K})} \leq C_{\text{affine}} h_K^{-1}$$
$$\|(R_K')^{-1}\|_{L^\infty(\widetilde{K})} \leq C_{\text{metric}}, \qquad \|\nabla^n R_K\|_{L^\infty(\widetilde{K})} \leq C_{\text{metric}} \gamma^n n! \quad \forall n \in \mathbb{N}_0.$$

Remark 4.2 If the mapping R_K in Assumption 4.1 are *affine* we say that $\mathscr{T}$ is an *affine* triangulation.

The constants C in the estimates below may depend on the shape regularity constants in a continuous way and, possibly, increase with increasing values of C_{affine}, C_{metric}, and γ. ■

In this paper we are allowed to consider non-conforming meshes with general interfaces, i.e., one mesh can be a submesh of the other one, or meshes can have entirely unmatched interfaces.

For meshes $\mathscr{T}$ satisfying Assumption 4.1 we define the following non-conforming space of piecewise (mapped) polynomials by

$$S^{p,0}(\mathscr{T}) := \{u \in L^2(\Omega) | \quad \forall K \in \mathscr{T} : u|_K \circ F_K \in \mathscr{P}_p\},$$

where $\mathscr{P}_p$ denotes the space of polynomials of degree p. The *mesh size function* $h_{\mathscr{T}}$ is defined by $h_{\mathscr{T}}|_K := \operatorname{diam} K$ for all $K \in \mathscr{T}$. The estimate of $C_{\text{trace}}(S,K)$ in these cases is a local trace estimate for multivariate polynomials:

Lemma 4.3 *Let $\mathscr{T}$ satisfy Assumption 4.1. Then there exists $c_{\text{inv}} > 0$ independent of $K \in \mathscr{T}$ and p such that for the polynomial hp-finite element space $S^{p,0}(\mathscr{T})$ we have (cf. (2.6))*

$$C_{\text{trace}}(S,K) \le \frac{c_{\text{inv}} p}{\sqrt{h_K}}$$

Furthermore, for

$$\mathfrak{a} > \frac{4}{3} c_{\text{inv}}^2, \tag{4.1}$$

which is independent of K, p, and k, the choice of α given in (2.8) implies the condition (2.7).

Proof We merely prove the inverse estimate. On the reference element $\widehat{K}$, we have with the multiplicative trace inequality and a standard polynomial inverse estimate (see, e.g., [45, Thm. 4.76], where the case $d = 2$ is covered) for any $v \in \mathscr{P}_p$

$$\|v\|^2_{L^2(\partial\widehat{K})} \le C\|v\|_{L^2(\widehat{K})}\|v\|_{H^1(\widehat{K})} \le Cp^2\|v\|^2_{L^2(\widehat{K})}.$$

The assumptions on the element maps F_K are such that the same h-dependence as in classical scaling argument are obtained, i.e., for $v \in S^{p,0}(\mathscr{T})$ we get for each $K \in \mathscr{T}$

$$\|v\|_{L^2(\partial K)} \le Cph^{-1/2}\|v\|_{L^2(K)}. \tag{4.2}$$

For the actual estimate of interest, we let $v \in S^{p,0}(\mathscr{T})$, fix K, and set $\widehat{v} := v|_K \circ F_K$. We note $\nabla v = (\nabla\widehat{v}) \circ F_K \circ (F_K')^{-1}$ with, by the assumptions on the properties of B_K and R_K,

$$\|(F_K')^{-1}\|_{L^\infty(\widehat{K})} \le Ch_K^{-1}, \qquad \|(F_K')\|_{L^\infty(\widehat{K})} \le Ch_K. \tag{4.3}$$

Applying the estimate (4.2) to the components of $\nabla\widehat{v} \circ F_K$ and observing (4.3), one can show the desired result. □

The trace inequality of Lemma 4.3 shows that the constant $\mathfrak{a}$ in (2.8) can be selected such that (2.7) is satisfied. This observation implies the following result:

Theorem 4.4 *Let α, β, and δ be chosen according to (2.8) with $\mathfrak{a}$ sufficiently large. Let $S = S^{p,0}(\mathscr{T})$ be the polynomial hp-finite element space based on a mesh $\mathscr{T}$ that satisfies Assumption 4.1.*

- *If C_S satisfies condition (3.14) then the DG problem has a unique solution in S.*
- *If $\mathscr{T}$ is an affine triangulation of Ω and satisfies Assumption 4.1, then the DG problem has a unique solution in S.*

4.2 Convergence Analysis

In this section we will show that the solution u of the model boundary value problem (1.1), (1.2) can be approximated from the finite element space $S^{p,0}(\mathscr{T})$ provided that $kh/\sqrt{p}$ is small enough and $p \geq c\log k$ (with c sufficiently large independent of h, k, p). Under more stringent conditions on the mesh, we will show that this condition can be relaxed to the condition that kh/p be small enough and $p \geq c\log k$.

The proof of this approximation property is based on the following decomposition lemma, which is a generalization of [40, Theorem 4.10], where the special case $s = 0$ is covered:

Theorem 4.5 (Decomposition Lemma) *Let $\Omega \in \mathbb{R}^d$, $d \in \{2,3\}$ be a bounded Lipschitz domain. Assume additionally that Ω has an analytic boundary. Assume furthermore that the solution operator $(f,g) \mapsto u := S_k(f,g)$ for the Helmholtz boundary value problem (1.1), (1.2) satisfies*

$$\|u\|_{\mathscr{H},\Omega} \leq C_{\mathrm{stab}} k^{\vartheta} \left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \right) \tag{4.4}$$

for some C_{stab} and $\vartheta \geq 0$ independent of k. Fix $s \in \mathbb{N}_0$. Then there exist constants C, $\lambda > 0$ independent of $k \geq k_0$ such that for every $f \in H^s(\Omega)$ and $g \in H^{s+1/2}(\partial\Omega)$ the solution $u = S_k(f,g)$ of the Helmholtz problem (1.3) can be written as $u = u_{H^{s+2}} + u_{\mathscr{A}}$, where, for all $n \in \mathbb{N}_0$

$$\|u_{\mathscr{A}}\|_{\mathscr{H},\Omega} \leq Ck^{\vartheta} \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)} \right), \tag{4.5}$$

$$\|\nabla^{n+2} u_{\mathscr{A}}\|_{L^2(\Omega)} \leq C\lambda^n k^{\vartheta-1} \max\{n,k\}^{n+2} \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)} \right), \tag{4.6}$$

$$\|u_{H^{s+2}}\|_{H^{s+2}(\Omega)} + k^{s+2} \|u_{H^{s+2}}\|_{L^2(\Omega)} \leq C \left(\|f\|_{H^s(\Omega)} + \|g\|_{H^{s+1/2}(\partial\Omega)} \right). \tag{4.7}$$

Proof The proof follows the lines of [40, Theorem 4.10]. The key modifications are collected in Appendix A. □

Remark 4.6 For the present model problem (1.1), (1.2) the assumption (4.4) holds with $\vartheta = 5/2$ by [17, Thm. 2.4]. For star-shaped domains, $\vartheta = 0$ is possible as shown in [36, Prop. 8.1.4] for $d = 2$ and subsequently for $d = 3$ in [12]. ■

4.2.1 Convergence analysis for general non-conforming polynomial hp-finite elements

In this section we consider general non-conforming polynomial hp-finite elements, where no interelement compatibility conditions are imposed on the element maps F_K that relate element maps of neighboring elements to each other. Hence, the conforming subspace $S \cap H^1(\Omega) \subset S$ may be small. As we will discuss in more detail in Section 5 below, better results can be expected if the conforming subspace $S \cap H^1(\Omega) \subset S$ is sufficiently rich.

We start with a lemma that takes the role of the standard scaling argument:

Lemma 4.7 *Let $\mathscr{T}$ be a shape-regular mesh in the sense of Assumption 4.1. Fix $s \in \mathbb{N}_0$. Then for each $K \in \mathscr{T}$ and every sufficiently smooth v the following relations between v and*

$\widehat{v} := v|_K \circ F_K$ are true:

$$\begin{aligned}
\|v\|_{L^2(K)} &\sim h^{d/2}\|\widehat{v}\|_{L^2(\widehat{K})},\\
\|\nabla v\|_{L^2(K)} &\sim h^{d/2-1}\|\nabla\widehat{v}\|_{L^2(\widehat{K})},\\
\|\nabla^{s+2}\widehat{v}\|_{L^2(\widehat{K})} &\leq Ch^{s+2-d/2}\|v\|_{H^{s+2}(K)},\\
\|v\|_{L^2(\partial K)} &\sim h^{(d-1)/2}\|\widehat{v}\|_{L^2(\partial\widehat{K})},\\
\|\nabla v\|_{L^2(\partial K)} &\sim h^{(d-1)/2-1}\|\nabla\widehat{v}\|_{L^2(\partial\widehat{K})},
\end{aligned}$$

where C and the implied constants depend solely on the constants appearing in Assumption 4.1.

Proof We will only consider the case of the $(s+2)$nd derivatives. We note the form $F_K = R_K \circ A_K$, where A_K is affine. This implies the estimates

$$\|F_K'\|_{L^\infty(\widehat{K})} \leq Ch_K, \qquad \sum_{\alpha\in\mathbb{N}_0^2:|\alpha|=s+2} \|D^\alpha F_K\|_{L^\infty(\widehat{K})} \leq Ch_K^{s+2},$$

where the constants depend only on the constants appearing in Assumption 4.1. The chain rule then implies the estimates for $\|\nabla^{s+2}\widehat{v}\|_{L^2(\widehat{K})}$. □

For shape-regular triangulations (cf. Assumption 4.1) we have the following result:

Theorem 4.8 *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ be a bounded Lipschitz domain with analytic boundary. Let the mesh $\mathscr{T}$ be shape-regular in the sense of Assumption 4.1. Fix $s \in \mathbb{N}_0$. Let α, β, δ be chosen according to (2.8). Fix $\overline{C} > 0$ and assume $p \geq s+1$ as well as $kh/p \leq \overline{C}$. Then there exist constants C, $\sigma > 0$ independent of h, p, and p such that, for every $f \in H^s(\Omega)$ and $g \in H^{s+1/2}(\partial\Omega)$, there holds*

$$\inf_{v\in S} k\|u-v\|_{DG^+} \leq C_{f,g}\left(\left(\frac{h}{p}\right)^s \frac{kh}{\sqrt{p}} + k^\vartheta\left\{\left(\frac{h}{h+\sigma}\right)^p + k\left(\frac{kh}{\sigma p}\right)^p\right\}\right), \tag{4.8}$$

where $C_{f,g} := \|f\|_{H^s(\Omega)} + \|g\|_{H^{s+1/2}(\partial\Omega)}$ and $\vartheta \geq 0$ is given by (4.4) (note also Remark 4.6).

Proof We employ the splitting $u = u_{H^{s+2}} + u_{\mathscr{A}}$ of Theorem 4.5 with $u_{H^{s+2}} \in H^{s+2}(\Omega)$ and the analytic part $u_{\mathscr{A}}$.

Following [39, Thm. 5.5], we approximate $u_{H^{s+2}}$ and $v_{\mathscr{A}}$ separately in the ensuing steps 1 and 2.

1. step: From, e.g., [39, Lemma B.3], we know that for every $s' > d/2$ and every $p \geq s'-1$ there exists a bounded linear operator $\pi_p : H^{s'}(\widehat{K}) \to \mathscr{P}_p$ such that

$$\|u-\pi_p u\|_{H^t(\widehat{K})} \leq Cp^{-(s'-t)}|u|_{H^{s'}(\widehat{K})} \qquad \text{for} \quad 0 \leq t \leq s', \tag{4.9}$$

$$\|u-\pi_p u\|_{H^t(\widehat{e})} \leq Cp^{-(s'-1/2-t)}|u|_{H^{s'}(\widehat{K})} \quad \text{for} \quad 0 \leq t \leq s'-1/2. \tag{4.10}$$

Here, the constant $C > 0$ depends only on s'. By $\widehat{K}$ we denote the reference element and by $\widehat{e}$ one of its edges (in 2D) or faces (in 3D). We apply this approximation result with $s' = s+2$. The elementwise application of the operator π_p to $u_{H^{s+2}}$ (pulled back to the reference element $\widehat{K}$) defines an approximation $w_{H^{s+2}} \in S^{p,0}(\mathscr{T})$. By a scaling argument (cf. Lemma 4.7) and summation over all elements, the bound (4.9) with $s' = s+2$ implies that $w_{H^{s+2}}$ satisfies

$$k\left(k\|u_{H^{s+2}} - w_{H^{s+2}}\|_{L^2(\Omega)} + \|\nabla_{\mathscr{T}}(u_{H^{s+2}} - w_{H^{s+2}})\|_{L^2(\Omega)}\right) \leq \\ C\left(k\left(\frac{h}{p}\right)^{s+1} + k^2\left(\frac{h}{p}\right)^{s+2}\right)\left(\|f\|_{H^s(\Omega)} + \|g\|_{H^{s+1/2}(\partial\Omega)}\right).$$

In order to estimate the terms of the DG^+-norm associated with the skeleton, we employ the choice of the parameters α, β, δ given in (2.8), viz.,

$$\alpha(x) = \frac{4}{3}\max_{K\in\{K_x^+, K_x^-\}} \frac{p^2}{kh_K} \qquad \forall x \in \mathfrak{S}_{\mathscr{T}}^{\mathscr{I}} \quad \text{and} \quad \beta = O(\frac{kh}{p}), \quad \delta = O(\frac{kh}{p}). \tag{4.11}$$

Recall the definition of $\alpha_{\partial K}^{\min}$ as in Remark 2.3 and estimate (3.4). On the inner skeleton $\mathfrak{S}_{\mathscr{T}}^{I}$ we get

$$k\|\alpha^{-1/2}\{\nabla_{\mathscr{T}}(u_{H^{s+2}} - w_{H^{s+2}})\}\|^2_{L^2(\mathfrak{S}_{\mathscr{T}}^{I})} \leq \sum_{K\in\mathscr{T}} \frac{k}{\alpha_{\partial K}^{\min}} \|\{\nabla_{\mathscr{T}}(u_{H^{s+2}} - w_{H^{s+2}})\}\|^2_{L^2(\Omega\cap\partial K)}.$$

Let X denote the minimizer as in (3.4). Then, with the definition (4.11) we get

$$\alpha_{\partial K}^{\min} = \alpha(X) = \frac{4}{3}\max_{K'\in\{K_X^+, K_X^-\}} \frac{p^2}{kh_{K'}} \geq \frac{4}{3}\frac{p^2}{kh_K} \tag{4.12}$$

so that

$$k\|\alpha^{-1/2}\{\nabla_{\mathscr{T}}(u_{H^{s+2}} - w_{H^{s+2}})\}\|^2_{L^2(\mathfrak{S}_{\mathscr{T}}^{I})} \\ \leq \sum_{K\in\mathscr{T}} \frac{3k^2 h_K}{4p^2}\|\nabla((u_{H^{s+2}} - w_{H^{s+2}})|_K)\|^2_{L^2(\Omega\cap\partial K)}.$$

Thus, we get by scaling (4.9), (4.10) to the mesh $\mathscr{T}$

$$k\|\alpha^{-1/2}\{\nabla_{\mathscr{T}}(u_{H^{s+2}} - w_{H^{s+2}})\}\|^2_{L^2(\mathfrak{S}_{\mathscr{T}}^{I})} \leq C\sum_{K\in\mathscr{T}} \frac{k^2 h}{p^2}\left(\frac{h_K}{p}\right)^{2s+1} \|u_{H^{s+2}}\|^2_{H^{s+2}(K)} \\ \leq C\frac{k^2}{p}\left(\frac{h}{p}\right)^{2s+2}\|u_{H^{s+2}}\|^2_{H^{s+2}(\Omega)} \leq C\frac{k^2}{p}\left(\frac{h}{p}\right)^{2s+2}\left(\|f\|^2_{H^s(\Omega)} + \|g\|^2_{H^{s+1/2}(\partial\Omega)}\right).$$

The following estimates can be obtained by similar arguments:

$$k^{1/2}\|\beta^{1/2}[\![\nabla_{\mathscr{T}}(u_{H^{s+2}} - w_{H^{s+2}})]\!]_N\|_{L^2(\mathfrak{S}_{\mathscr{T}}^{\mathscr{I}})} \leq Ck\left(\frac{h}{p}\right)^{s+1}\left(\|f\|_{H^s(\Omega)} + \|g\|_{H^{s+1/2}(\partial\Omega)}\right),$$

$$k^{3/2}\|\alpha^{1/2}[\![u_{H^{s+2}} - w_{H^{s+2}}]\!]_N\|_{L^2(\mathfrak{S}_{\mathscr{T}}^{\mathscr{I}})} \leq Ck\sqrt{p}\left(\frac{h}{p}\right)^{s+1}\left(\|f\|_{H^s(\Omega)} + \|g\|_{H^{s+1/2}(\partial\Omega)}\right),$$

$$k^{1/2}\|\delta^{1/2}\nabla_{\mathscr{T}}(u_{H^{s+2}} - w_{H^{s+2}})\cdot\mathbf{n}\|_{H^s(\mathfrak{S}_{\mathscr{T}}^{\mathscr{B}})} \leq Ck\left(\frac{h}{p}\right)^{s+1}\left(\|f\|_{H^s(\Omega)} + \|g\|_{H^{s+1/2}(\partial\Omega)}\right),$$

$$k^{3/2}\|(1-\delta)^{1/2}(u_{H^2} - w_{H^2})\|_{L^2(\mathfrak{S}_{\mathscr{T}}^{\mathscr{B}})} \leq Ck^{3/2}\left(\frac{h}{p}\right)^{s+3/2}\left(\|f\|_{H^s(\Omega)} + \|g\|_{H^{s+1/2}(\partial\Omega)}\right).$$

In total, we get the following approximation property for the H^{s+2}-part:

$$k\|u_{H^{s+2}}-w_{H^{s+2}}\|_{DG^+}\leq C\left(\frac{h}{p}\right)^s\left(\frac{kh}{\sqrt{p}}+\left(\frac{kh}{p}\right)^{3/2}+\left(\frac{kh}{p}\right)^2\right)\left(\|f\|_{H^s(\Omega)}+\|g\|_{H^{s+1/2}(\partial\Omega)}\right).$$

Using the assumption $kh/p\leq\overline{C}$, this can be simplified to

$$k\|u_{H^{s+2}}-w_{H^{s+2}}\|_{DG^+}\leq C\left(\frac{h}{p}\right)^s\frac{kh}{\sqrt{p}}\left(\|f\|_{H^s(\Omega)}+\|g\|_{H^{s+1/2}(\partial\Omega)}\right).$$

2. step: For the approximation of the analytic part $u_{\mathscr{A}}$, we construct an element $w_{\mathscr{A}}\in S^{p,0}(\mathscr{T})$ as follows. For each $K\in\mathscr{T}$, let the constant C_K by defined by

$$C_K^2:=\sum_{n\in\mathbb{N}_0}\frac{\|\nabla^n u_{\mathscr{A}}\|^2_{L^2(K)}}{(2\lambda\max\{n,k\})^{2n}}.$$

Then, we have

$$\begin{aligned}\|\nabla^n u_{\mathscr{A}}\|_{L^2(K)}&\leq(2\lambda\max\{n,k\})^n C_K\quad\forall n\in\mathbb{N}_0,\\ \sum_{K\in\mathscr{T}}C_K^2&\leq C\left(\frac{1}{\lambda k}\right)^2k^{2\vartheta}\left(\|f\|^2_{L^2(\Omega)}+\|g\|^2_{H^{1/2}(\partial\Omega)}\right).\end{aligned}\tag{4.13}$$

For $q\in\{0,1,2\}$ we get the following estimate (see [39, Proof of Theorem 5.5]) for suitable $\sigma>0$:

$$\|u_{\mathscr{A}}-w_{\mathscr{A}}\|_{H^q(K)}\leq Ch_K^{-q}C_K\left\{\left(\frac{h_K}{h_K+\sigma}\right)^{p+1}+\left(\frac{kh_K}{\sigma p}\right)^{p+1}\right\}.\tag{4.14}$$

It is convenient to define the abbreviations:

$$\begin{aligned}E(\sigma)&:=\left(\frac{h}{h+\sigma}\right)^p+k\left(\frac{kh}{\sigma p}\right)^p,\\ M&:=k^{\vartheta}\left(\|f\|_{L^2(\Omega)}+\|g\|_{H^{1/2}(\partial\Omega)}\right).\end{aligned}$$

By summing over all elements, it follows as in [39] by suitably adjusting the constant σ

$$k\|u_{\mathscr{A}}-w_{\mathscr{A}}\|_{\mathscr{H}}\leq C\left(\frac{1}{p}+\frac{kh}{p}\right)E(\sigma)M.\tag{4.15}$$

In order to treat the terms associated with the skeleton $\mathfrak{S}^{\mathscr{I}}_{\mathscr{T}}\cup\mathfrak{S}^{\mathscr{B}}_{\mathscr{T}}$ we use the multiplicative trace inequality (on $\widehat{K}$ and Lemma 4.7)

$$\|v\|^2_{L^2(\partial K)}\leq C\left(\|v\|_{L^2(K)}|v|_{H^1(K)}+h_K^{-1}\|v\|^2_{L^2(K)}\right)$$

to obtain

$$k\|\alpha^{-1/2}\{\nabla_{\mathscr{T}}(u_{\mathscr{A}}-w_{\mathscr{A}})\}\|^2_{L^2(\mathfrak{S}^I_{\mathscr{T}})}\leq\sum_{K\in\mathscr{T}}\frac{k}{\alpha^{\min}_{\partial k}}\|\nabla_{\mathscr{T}}((u_{\mathscr{A}}-w_{\mathscr{A}})|_K)\|^2_{L^2(\Omega\cap\partial K)}.$$

By using the estimate (4.12) we obtain

$$\begin{aligned}
& k\left\|\alpha^{-1/2}\{\nabla_{\mathscr{T}}(u_{\mathscr{A}}-w_{\mathscr{A}})\}\right\|^2_{L^2(\mathfrak{S}^I_{\mathscr{T}})} \\
& \leq \sum_{K\in\mathscr{T}} \frac{3k^2h_K}{4p^2}\,\|\nabla(\,(u_{\mathscr{A}}-w_{\mathscr{A}})|_K)\|^2_{L^2(\Omega\cap\partial K)} \\
& \leq \sum_{K\in\mathscr{T}} \frac{3}{4}\left(\frac{k^2h_K}{p^2}\right)\left(\|\nabla(u_{\mathscr{A}}-w_{\mathscr{A}})\|_{L^2(K)}\,|\nabla(u_{\mathscr{A}}-w_{\mathscr{A}})|_{H^1(K)}+h_K^{-1}\,\|\nabla(u_{\mathscr{A}}-w_{\mathscr{A}})\|^2_{L^2(K)}\right).
\end{aligned}$$

By using the estimates in equation (4.14) we get

$$k\|\alpha^{-1/2}\{\nabla_{\mathscr{T}}(u_{\mathscr{A}}-w_{\mathscr{A}})\}\|^2_{L^2(\mathfrak{S}^I_{\mathscr{T}})} \leq \sum_{K\in\mathscr{T}} \frac{3Ck^2}{4p^2}\left\{h_K\left(\frac{h_K}{h_K+\sigma}\right)^{p-1}+\frac{k}{p}\left(\frac{kh_K}{\sigma p}\right)^p\right\}^2 C_K^2.$$

Finally equation (4.13) gives us after suitably adjusting the constant σ

$$k^{1/2}\|\alpha^{-1/2}\{\nabla_{\mathscr{T}}(u_{\mathscr{A}}-w_{\mathscr{A}})\}\|_{L^2\left(\mathfrak{S}^{\mathscr{I}}_{\mathscr{T}}\right)} \leq C\frac{1}{p^2}E(\sigma)M.$$

By the similar arguments we obtain the following estimates

$$\begin{aligned}
& k^{1/2}\|\beta^{1/2}[\![\nabla_{\mathscr{T}}(u_{\mathscr{A}}-w_{\mathscr{A}})]\!]_N\|_{L^2\left(\mathfrak{S}^{\mathscr{I}}_{\mathscr{T}}\right)} \leq C\frac{1}{p^{3/2}}E(\sigma)M, \\
& k^{3/2}\|\alpha^{1/2}[\![u_{\mathscr{A}}-w_{\mathscr{A}}]\!]_N\|_{L^2\left(\mathfrak{S}^{\mathscr{I}}_{\mathscr{T}}\right)} \leq CE(\sigma)M, \\
& k^{1/2}\|\delta^{1/2}\nabla_{\mathscr{T}}(u_{\mathscr{A}}-w_{\mathscr{A}})\cdot\mathbf{n}\|_{L^2\left(\mathfrak{S}^{\mathscr{B}}_{\mathscr{T}}\right)} \leq C\frac{1}{p^{3/2}}E(\sigma)M, \\
& k^{3/2}\|(1-\delta)^{1/2}(u_{\mathscr{A}}-w_{\mathscr{A}})\|_{L^2\left(\mathfrak{S}^{\mathscr{B}}_{\mathscr{T}}\right)} \leq C\frac{(kh)^{1/2}}{p}E(\sigma)M.
\end{aligned}$$

The approximation property for the analytic part $u_{\mathscr{A}}$ with respect to the DG^+ norm is then

$$k\|u_{\mathscr{A}}-w_{\mathscr{A}}\|_{DG^+} \leq C\left(1+\frac{1}{p}+\frac{kh}{p}+\frac{\sqrt{kh}}{p}\right)E(\sigma)M \leq CE(\sigma)M,$$

where, in the last estimate we used the assumption $kh/p \leq \overline{C}$. The combination of the estimates of steps 1 and 2 leads to the assertion. □

The approximation result Theorem 4.8 permits us to estimate the adjoint approximation property $\eta(S)$ of (3.11):

Corollary 4.9 *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded Lipschitz domain with analytic boundary. Let the mesh $\mathscr{T}$ be shape-regular in the sense of Assumption 4.1. Let α, β, δ be chosen according to (2.8). Fix $\overline{C} > 0$ and assume $kh/p \leq \overline{C}$. Then there exist constants C, $\sigma > 0$ such that $\eta_k(S)$ defined in (3.11) satisfies*

$$\eta_k(S) \leq C\left[\frac{kh}{\sqrt{p}}+k^{\vartheta}\left(\left(\frac{h}{h+\sigma}\right)^p+k\left(\frac{kh}{\sigma p}\right)^p\right)\right].$$

Proof We apply Theorem 4.8 with $s=0$ and $g=0$. Given $f \in L^2(\Omega)$ let $v = N_k^*(f) = \overline{N_k(\overline{f})}$. Hence, the regularity estimates of Theorem 4.5 (with $g=0$) are applicable. The assumption $kh/p \le \overline{C}$ allows us to estimate $(kh/p)^2 \le Ckh/\sqrt{p}$. □

Finally, the convergence estimate for polynomial hp-FEM can be stated in the following theorem:

Theorem 4.10 (Convergence Estimate) *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded Lipschitz domain with analytic boundary. Let the mesh $\mathscr{T}$ be shape-regular in the sense of Assumption 4.1. Fix $s \in \mathbb{N}_0$. Let α, β, δ be chosen according to (2.8) with $\mathfrak{a}$ sufficiently large. Moreover, let $0 < \delta < 1/3$. Then, there exist constants c_1, c_2, $C > 0$ independent of k, h, and p such that under the assumptions*

$$\frac{kh}{\sqrt{p}} \le c_1 \quad \textit{together with} \quad p \ge c_2 \log(k) \quad \textit{as well as} \quad p \ge s+1 \tag{4.16}$$

there holds for $f \in H^s(\Omega)$ and $g \in H^{s+1/2}(\partial\Omega)$ the a priori *estimate*

$$\|u-u_S\|_{DG} \le C\left[\sqrt{p}\left(\frac{h}{p}\right)^{s+1} + k^{\vartheta-1}\left\{\left(\frac{h}{h+\sigma}\right)^p + k\left(\frac{kh}{\sigma p}\right)^p\right\}\right]\left[\|f\|_{H^s(\Omega)} + \|g\|_{H^{s+1/2}(\partial\Omega)}\right].$$

In particular, under the additional assumption that $\mathfrak{b}$ and $\mathfrak{d}$ satisfy $\mathfrak{b}$, $\mathfrak{d} \ge c_0 > 0$, there holds

$$\|\nabla_{\mathscr{T}}(u-u_S)\|_{L^2(\Omega)} + \sqrt{\frac{h}{p}}\|[\![\nabla_{\mathscr{T}}(u-u_S)]\!]_N\|_{L^2\left(\mathfrak{S}_{\mathscr{T}}^{\mathscr{I}}\right)} + \frac{p}{\sqrt{h}}\|[\![u-u_S]\!]_N\|_{L^2\left(\mathfrak{S}_{\mathscr{T}}^{\mathscr{I}}\right)} \le C\|u-u_S\|_{DG}.$$

Proof By taking the constant $\mathfrak{a}$ in (2.8) sufficiently large, we can ensure by Lemma 4.3 the condition (2.7). Hence the assertion is a combination of Theorems 3.10, 4.8 and Corollary 4.9. □

4.2.2 Convergence analysis for hp-FEM on regular meshes

When contrasting the estimate for the adjoint approximation property $\eta_k(S)$ given in Corollary 4.9 and the final convergence result Theorem 4.10 with the corresponding ones for the classical conforming hp-FEM presented in [39, 40] one observes the suboptimality in p by half an order. This suboptimality is typical of p-explicit DG-methods and in general sharp, [24]. It can be removed if the hp-approximation space S is such that it contains an $H^1(\Omega)$-conforming subspace that is sufficiently rich. The essential point of the argument is that the approximant $w_{H^{s+2}}$ in the proof of Theorem 4.8 can be chosen to be in $H^1(\Omega)$ so that the following skeleton term vanishes:

$$k^{3/2}\|\alpha^{1/2}[\![u_{H^{s+2}} - w_{H^{s+2}}]\!]_N\|_{L^2\left(\mathfrak{S}_{\mathscr{T}}^{\mathscr{I}}\right)} = 0. \tag{4.17}$$

We illustrate this procedure for a specific setting, namely, that of a regular mesh $\mathscr{T}$ whose element maps satisfy the standard compatibility conditions for an $H^1(\Omega)$-conforming discretization. Specifically, we require the mesh to be H^1*-regular* by which we mean: first, the partition has no hanging nodes or edges and, second, in addition to the conditions of Assumption 4.1 we require the element maps F_K and $F_{K'}$ of two elements K, K' that share an edge or face to induce the same parametrization on this edge or face. One of way of constructing such a mesh is to start from a fixed coarse macro triangulation on Ω into "patches" using curved elements (e.g., constructed with "transfinite blending" [26, 27] and [14, Chap. 5]) and then construct the actual triangulation with elements of size h by transporting refinements of the reference elements to physical space with the patch maps of the coarse triangulation. More details for such a procedure are given in [39, Example 5.1]. On such regular meshes, the standard $H^1(\Omega)$-conforming hp-FEM spaces given as $S^{p,1}(\mathscr{T}) := \{u \in H^1(\Omega) \,|\, \forall K \in \mathscr{T} : u|_K \circ F_k \in \mathscr{P}_p\}$ have good approximation properties, which results in the following improvement over Theorem 4.8:

Theorem 4.11 *Assume the hypotheses of Theorem 4.8. Assume additionally that the mesh $\mathscr{T}$ is H^1-regular in the above sense. Then for $S = S^{p,1}(\mathscr{T})$:*

$$\inf_{v \in S} k\|u - v\|_{DG^+} \leq C_{f,g} \left(\left(\frac{h}{p}\right)^s \frac{kh}{p} + k^{\vartheta} \left\{ \left(\frac{h}{h+\sigma}\right)^p + k \left(\frac{kh}{\sigma p}\right)^p \right\} \right). \tag{4.18}$$

Proof As in the proof of Theorem 4.8, we decompose $u = u_{H^{s+2}} + u_{\mathscr{A}}$. We will not discuss the approximation of $u_{\mathscr{A}}$ since its approximation follows the lines of [39, Thm. 5.5]. We construct an $H^1(\Omega)$-conforming approximation $w_{H^{s+2}} \in S$ to $u_{H^{s+2}}$. This ensures the desired property (4.17). It remains to guarantee that $w_{H^{s+2}}$ is constructed such that the optimal rate of convergence is achieved in the broken H^1-norm and L^2-norm and also for the trace of the gradient on the skeleton. Recall $p \geq s+1$. In Appendix B (Cor. B.4) we construct, for $t > 5/2$ (for $d = 2$) and $t > 5$ (for $d = 3$) a linear operator $I : H^t(\Omega) \to S \cap H^1(\Omega)$ with the following approximation properties:

$$\left(\frac{h_K}{p}\right)^2 \|\nabla^2(u - Iu)\|_{L^2(K)} + \left(\frac{h_K}{p}\right) \|\nabla(u - Iu)\|_{L^2(K)} + \|u - Iu\|_{L^2(K)} \leq C \left(\frac{h_K}{p}\right)^t \|u\|_{H^t(K)}.$$

Set $t^* = 5/2$ for $d = 2$ and $t^* = 5$ for $d = 3$. If $s + 2 > t^*$, we obtain the desired estimate for $\|u - Iu\|_{DG+}$ from this by summation over all elements. If $s + 2 \leq t^*$, then we employ the following interpolation argument due to [7]: Fix $\sigma > t^*$. The Sobolev space $H^{s+2}(\Omega)$ can be characterized by interpolation (using the so-called "K-method" as described, for example, in [46]), and we have $H^{s+2}(\Omega) = (L^2(\Omega), H^{\sigma}(\Omega))_{\theta,2}$ with $\theta = (s+2)/\sigma$. Hence, we can find, for any $t > 0$, a function $v_t \in H^{\sigma}(\Omega)$ such that

$$\|u - v_t\|_{L^2(\Omega)} + t\|v_t\|_{H^{\sigma}(\Omega)} =: K(u,t) \leq Ct^{\theta}\|u\|_{H^{s+2}(\Omega)}.$$

Then [7, Lemma] gives the stability estimate $\|u - v_t\|_{H^{s+2}(\Omega)} \leq C\|u\|_{H^{s+2}(\Omega)}$. Using interpolation estimates, we therefore arrive at

$$\begin{aligned}
\|v_t\|_{H^{\sigma}(\Omega)} &\leq Ct^{\theta-1}\|u\|_{H^{s+2}(\Omega)}, \\
\|u - v_t\|_{L^2(\Omega)} &\leq Ct^{\theta}\|u\|_{H^{s+2}(\Omega)}, \\
\|u - v_t\|_{H^1(\Omega)} &\leq C\|u - v_t\|_{L^2(\Omega)}^{(s+1)/(s+2)} \|u - v_t\|_{H^{s+2}(\Omega)}^{1/(s+2)} \leq Ct^{\theta(s+1)/(s+2)}\|u\|_{H^{s+2}(\Omega)}, \\
\|u - v_t\|_{H^2(\Omega)} &\leq C\|u - v_t\|_{L^2(\Omega)}^{s/(s+2)} \|u - v_t\|_{H^{s+2}(\Omega)}^{2/(s+2)} \leq Ct^{\theta s/(s+2)}\|u\|_{H^{s+2}(\Omega)}.
\end{aligned}$$

We select $t = (h/p)^{\sigma}$. Then, the above estimates take the following form:

$$\begin{aligned}
\|v_t\|_{H^{\sigma}(\Omega)} &\leq (h/p)^{s+2-\sigma}\|u\|_{H^{s+2}(\Omega)}, \\
\|u - v_t\|_{L^2(\Omega)} &\leq C(h/p)^{s+2}\|u\|_{H^{s+2}(\Omega)}, \\
\|u - v_t\|_{H^1(\Omega)} &\leq C(h/p)^{s+1}\|u\|_{H^{s+2}(\Omega)}, \\
\|u - v_t\|_{H^2(\Omega)} &\leq C(h/p)^{s}\|u\|_{H^{s+2}(\Omega)}.
\end{aligned}$$

Using elementwise appropriate multiplicative trace inequalities yields

$$\|u-v_t\|_{DG,+} \le C\left[k(h/p)^{s+2}+(h/p)^{s+1}+k^{1/2}(h/p)^{s+3/2}\right]\|u\|_{H^{s+2}(\Omega)}.$$

Finally, v_t is sufficiently smooth to allow us to apply the approximation operator I of Appendix B and bound $\|v_t - Iv_t\|_{DG+}$ with the aid of Corollary B.4. □

Remark 4.12 For H^1-regular meshes (in the above sense) the approximation result for the adjoint approximation property $\eta_k(S)$ in Corollary 4.9 can be improved to

$$\eta_k(S) \le C\left[\frac{kh}{p}+k^{\vartheta}\left(\left(\frac{h}{h+\sigma}\right)^p + k\left(\frac{kh}{\sigma p}\right)^p\right)\right].$$

In turn, this results in an improvement of Theorem 4.10: the resolution condition (4.16) can be relaxed to

$$\frac{kh}{p} \le c_1 \quad \text{together with} \quad p \ge c_2 \log(k) \quad \text{as well as} \quad p \ge s+1 \tag{4.19}$$

and the approximation result also improves to

$$\|u-u_S\|_{DG} \le C\left[\left(\frac{h}{p}\right)^{s+1}+k^{\vartheta-1}\left\{\left(\frac{h}{h+\sigma}\right)^p+k\left(\frac{kh}{\sigma p}\right)^p\right\}\right]\left[\|f\|_{H^s(\Omega)}+\|g\|_{H^{s+1/2}(\partial\Omega)}\right].$$

■

5 Conclusions

In this paper, we have formulated the discontinuous Galerkin method for abstract finite dimensional test and trial spaces (conforming and non-conforming ones). The concrete choice of this space S enters the stability and convergence analysis via the following four quantities.

(a) *Trace constant* $C_{\text{trace}}(S,K)$. Due to the formulation as a discontinuous Galerkin method, which contains integral jump terms across element faces, it is quite natural that local trace estimates for the space S are required for the error analysis.
(b) *Approximation property* $\inf_{v\in S}\|u-v\|_{DG^+}$. In order to derive quantitative error estimates it is obvious that approximation results for S for functions with higher Sobolev regularity are required. The trace estimate (cf. (a)) allows us to "transfer" the local approximation results for the elements $K \in \mathscr{T}$ to the skeleton norm.
(c) *Adjoint approximation property* $\eta_k(S)$. The decomposition lemma formulated as Theorem 4.5 provides a regularity theory for Helmholtz problems that splits the solution into several contributions, each of which can be approximated by piecewise polynomials with error estimates that are explicit in h, k, and p.
(d) *The constant C_S of (3.14)*. This condition ensures unique solvability of the discrete system (2.4) (see Theorem 3.8). For the important cases of polynomial hp-finite elements on affine, simplicial triangulations or plane wave approximation spaces, the condition (3.14) is automatically satisfied. If the adjoint approximation property can be controlled, then Theorem 3.10 provides an alternative way to ensure unique solvability for (2.4).

As an application of our abstract theory we considered the polynomial hp-finite elements, and we derived sharp stability and convergence estimates for non-conforming polynomial hp-finite element spaces. The *a priori* estimate in Theorem 4.10 is optimal in h (note that $f \in H^s(\Omega)$ with $g \in H^{s+1/2}(\partial\Omega)$ implies $u \in H^{s+2}(\Omega)$ by the assumed smoothness of $\partial\Omega$) but suboptimal in p by half an order. This is typical in p-explicit DG methods. This suboptimality in p can be removed (in both the scale resolution condition (4.16) as well as the *a priori* estimate of Corollary 4.9) by assuming that the approximation space contains an $H^1(\Omega)$-conforming subspace that is sufficiently rich. As an example, we considered the special case of meshes that are H^1-regular in Theorem 4.11 and the ensuing Remark 4.12. These results are formulated for meshes without handing nodes but we believe that similar results hold also for certain meshes with hanging nodes; the essential tool is the existence of an $H^1(\Omega)$-conforming interpolant with appropriate approximation properties. Such a situation arises, e.g., if a conforming hp-finite element mesh is further refined locally in a controlled way by introducing hanging nodes.

We restricted the convergence analysis for polynomial hp-finite element spaces in Section 4 to Lipschitz domains with analytic boundaries in order not to further increase the technicalities in this paper. In [40], the case of polygonal domains for the standard variational formulation of the Helmholtz equation with conforming polynomial hp-finite element spaces was considered and regularity estimates in weighted Sobolev spaces were derived. We expect that the generalization of our theory for the DG method for non-conforming finite element spaces to polygonal domains is possible along those lines.

A Details for the proof of Theorem 4.5

We start with an extension of [40, Lemma 4.6] for the modified Helmholtz equation.

Lemma A.1 *Let Ω be a bounded Lipschitz domain with a smooth boundary. Let S_k^Δ be the solution operator for the boundary value problem*

$$-\Delta u + k^2 u = 0 \quad \text{in } \Omega, \qquad \partial_n u + \mathrm{i} k u = g \quad \text{on } \partial\Omega.$$

Then, for every $s \in \mathbb{N}_0$ there exists $C > 0$ independent of $k \geq k_0$ such that

$$\|S_k^\Delta(g)\|_{H^{s+2}(\Omega)} \leq C\left[\|g\|_{H^{s+1/2}(\partial\Omega)} + k^{s+1/2}\|g\|_{L^2(\partial\Omega)}\right], \tag{A.1}$$

$$\|S_k^\Delta(g)\|_{H^1(\Omega)} + k\|S^\Delta(g)\|_{L^2(\Omega)} \leq Ck^{-1/2}\|g\|_{L^2(\partial\Omega)}. \tag{A.2}$$

Proof The case $s = 0$ in (A.1) as well as the estimate (A.2) is given in [40, Lemma 4.6]. For $s \geq 1$, we employ induction and the standard shift theorem for the Laplacian: Since u solves

$$-\Delta u = -k^2 u \quad \text{in } \Omega, \qquad \partial_n u = g - \mathrm{i} k u \quad \text{on } \partial\Omega,$$

we have

$$\begin{aligned}\|u\|_{H^{s+2}(\Omega)} &\leq C\left[k^2\|u\|_{H^s(\Omega)} + \|g\|_{H^{s+1/2}(\partial\Omega)} + k\|u\|_{H^{s+1/2}(\partial\Omega)}\right]\\ &\leq C\left[k^2\|u\|_{H^s(\Omega)} + \|g\|_{H^{s+1/2}(\partial\Omega)} + k\|u\|_{H^{s+1}(\Omega)}\right],\end{aligned}$$

where we used a trace inequality. Using the induction hypothesis then leads to an estimate that involves norms of g other than $\|g\|_{H^{s+3/2}(\partial\Omega)}$ and $\|g\|_{L^2(\partial\Omega)}$. These can be removed by an interpolation inequality (see, e.g., [20, Thm. 1.4.3.3]) and an appropriate use of the Young inequality. □

The analog of [40, Lemma 4.7] is the following (we use the operator $H^N_{\partial\Omega}$ defined in [40, (4.1c)]):

Lemma A.2 *Let Ω be a bounded Lipschitz domain with a smooth boundary. Fix $q \in (0,1)$ and $s \in \mathbb{N}_0$. Then, the operator $H^N_{\partial\Omega}$ can be selected such that the operator $S^\Delta_k \circ H^H_{\partial\Omega}$ satisfies for some $C > 0$ independent of k*

$$k^{s+2}\|S^\Delta_k(H^N_{\partial\Omega}g)\|_{L^2(\Omega)} + k^2\|S^\Delta_k(H^N_{\partial\Omega}g)\|_{H^s(\Omega)} \le q\|g\|_{H^{s+1/2}(\partial\Omega)}, \tag{A.3}$$

$$\|S^\Delta_k(H^N_{\partial\Omega}g)\|_{H^{s+2}(\Omega)} \le C\|g\|_{H^{s+1/2}(\partial\Omega)}. \tag{A.4}$$

Proof Estimates (A.3) and (A.4) are shown in [40, Lemma 4.7] for the special case $s = 0$. For $s \ge 1$, these estimates are derived as in [40, Lemma 4.7] by combining Lemma A.1 with [40, Lemma 4.2]. We illustrate the procedure for the second term of the left-hand side of (A.3) for the case $s \ge 2$: Lemma A.1 yields

$$\begin{aligned}\|S^\Delta_k(H^N_{\partial\Omega})\|_{H^s(\Omega)} &\le C\left[\|H^N_{\partial\Omega}g\|_{H^{s-3/2}(\partial\Omega)} + k^{s-3/2}\|H^N_{\partial\Omega}g\|_{L^2(\Omega)}\right]\\ &\le C\left[(q/k)^2\|g\|_{H^{s+1/2}(\partial\Omega)} + k^{s-3/2}(q/k)^{s+1/2}\|g\|_{H^{s+1/2}(\partial\Omega)}\right],\end{aligned}$$

where we used [40, Lemma 4.2]. Rearranging terms yields the result. □

We also need properties of the Newton potential N_k, which generalize [40, Lemma 4.5]:

Lemma A.3 *Let Ω be a bounded Lipschitz domain. Fix $s \in \mathbb{N}_0$ and $q \in (0,1)$. Then the operator H_Ω of [40, (4.1b)] can be selected such that for $0 \le s' \le s+2$*

$$\|N_k(H_\Omega f)\|_{H^{s'}(\Omega)} \le C(q/k)^{s+2-s'}\|f\|_{H^s(\Omega)}. \tag{A.5}$$

Proof Follows from the procedure in [40]; see also [38, Lemma 4.2]. The essential point is that [39, (3.35)] can be generalized (by using the notation therein) to

$$\left\|\partial^\alpha v_{\mu,H^2}\right\|_{L^2(\mathbb{R}^d)} = (2\pi)^{d/2}\left\|P_{\alpha-\beta}\widehat{G_k M}(1-\chi_{\lambda k})\widehat{\partial^\beta f}\right\|_{L^2(\mathbb{R}^d)}$$

for all $\alpha \in \mathbb{N}^d_0$ and $\beta \in \mathbb{N}^d_0$. By selecting $|\alpha| = s'$ and $|\beta| = s'-2$, we see that $|\alpha-\beta| = 2$ and this case is considered in [39, (3.35)]. By performing the same estimates as in [39, after (3.35)], we derive for $|\alpha-\beta| = 2$ the estimate

$$\|\partial^\alpha N_k(H_\Omega f)\|_{L^2(\Omega)} \le C\left\|\partial^\beta H_\Omega f\right\|_{L^2(\Omega)}$$

so that

$$\|N_k(H_\Omega f)\|_{H^{s'}(\Omega)} \le C\|H_\Omega f\|_{H^{s'-2}(\Omega)}$$

follows. The combination with [40, Lemma 4.2] leads to the assertion (A.5). □

The next lemma generalizes [40, Lemma 4.15] (note that the boundary condition (1.2) differs from that in [40] by a sign):

Lemma A.4 *Let Ω be a bounded Lipschitz domain with a smooth boundary. Fix $s \in \mathbb{N}_0$. Assume that the solution operator $(f,g) \mapsto S_k(f,g)$ for (1.1), (1.2) satisfies (4.4). Then S_k admits the following decomposition: $u = S_k(f,0) = u_{\mathscr{A}} + u_{H^{s+2}} + \widetilde{u}$, where*

$$\begin{aligned}\|u_{\mathscr{A}}\|_{H^1(\Omega)} + k\|u_{\mathscr{A}}\|_{L^2(\Omega)} &\le Ck^\vartheta\|f\|_{L^2(\Omega)},\\ \|\nabla^{n+2}u_{\mathscr{A}}\|_{L^2(\Omega)} &\le Ck^{\vartheta-1}\gamma^n\max\{k,n\}^{n+2}\|f\|_{L^2(\Omega)} \qquad \forall n \in \mathbb{N}_0,\\ k^{s+2}\|u_{H^{s+2}}\|_{L^2(\Omega)} + \|u_{H^{s+2}}\|_{H^{s+2}(\Omega)} &\le C\|f\|_{H^s(\Omega)}\end{aligned}$$

for constants C, $\gamma > 0$ independent of k and n, and the remainder $\widetilde{u} = S_k(\widetilde{f},0)$ satisfies the boundary value problem

$$-\Delta\widetilde{u} - k^2\widetilde{u} = \widetilde{f} \quad \text{in } \Omega, \qquad \partial_n\widetilde{u} + \mathrm{i}k\widetilde{u} = 0 \quad \text{on } \partial\Omega$$

for a right-hand side $\widetilde{f} \in H^s(\Omega)$ with

$$\|\widetilde{f}\|_{H^s(\Omega)} \le q\|f\|_{H^s(\Omega)}, \qquad \|\widetilde{f}\|_{L^2(\Omega)} \le q\|f\|_{L^2(\Omega)}.$$

Proof The proof follows that of [40, Lemma 4.15]. We flag that the boundary condition (1.2) studied in the present paper differs from that in [40], which accounts for sign differences between the procedure here and in [40, Lemma 4.15]. We only need to show the additional bound $\|u_{H^{s+2}}\|_{H^{s+2}(\Omega)} \leq C\|f\|_{H^s(\Omega)}$. To that end, we have to consider, in the notation of [40, Lemma 4.15], the terms

$$u^I_{H^2} = N_k(H_\Omega f), \tag{A.6}$$

$$u^{II}_{H^2} = S^\Delta_k(H^N_{\partial\Omega}(-\mathrm{i}ku^I_{H^2} - \partial_n u^I_{H^2})). \tag{A.7}$$

For (A.6), we use Lemma A.3 to get

$$k^{s+2}\|N_k(H_\Omega f)\|_{L^2(\Omega)} + k\|N_k(H_\Omega f)\|_{H^{s+1}(\Omega)} + \|N_k(H_\Omega f)\|_{H^{s+2}(\Omega)} \leq C\|f\|_{H^s(\Omega)},$$
$$\|N_k(H_\Omega f)\|_{H^s(\Omega)} \leq C(q/k)^2\|f\|_{H^s(\Omega)}.$$

This implies in particular with a trace inequality that

$$\|-\mathrm{i}ku^I_{H^2} - \partial_n u^I_{H^2}\|_{H^{s+1/2}(\partial\Omega)} \leq Ck\|u^I_{H^2}\|_{H^{s+1}(\Omega)} + C\|u^I_{H^2}\|_{H^{s+2}(\Omega)} \leq C\|f\|_{H^s(\Omega)},$$

so that also for (A.7), we can obtain, with the aid of Lemma A.2, the bounds

$$\|S^\Delta_k(H^N_{\partial\Omega}(-\mathrm{i}ku^I_{H^2} - \partial_n u^I_{H^2}))\|_{H^{s+2}(\Omega)} \leq C\|f\|_{H^s(\Omega)},$$
$$k^{s+2}\|S^\Delta_k(H^N_{\partial\Omega}(-\mathrm{i}ku^I_{H^2} - \partial_n u^I_{H^2}))\|_{L^2(\Omega)} + k^2\|S^\Delta_k(H^N_{\partial\Omega}(-\mathrm{i}ku^I_{H^2} - \partial_n u^I_{H^2}))\|_{H^s(\Omega)} \leq q\|f\|_{H^s(\Omega)}.$$

From the above estimates follows the bound for $\|u_{H^{s+2}}\|_{H^{s+2}(\Omega)}$. The estimate for $\widetilde{f}$ follows also from the above observations by noting that we have to set $\widetilde{f} := 2k^2u^{II}_{H^2}$ and then suitably adjust q as in the proof [40, Lemma 4.15]. □

Finally, we formulate the analog of [40, Lemma 4.16]:

Lemma A.5 *Assume the hypotheses of Lemma A.4. Fix $q \in (0,1)$ and $s \in \mathbb{N}_0$. Then the solution $u = S_k(0,g)$ can be written as $u = u_{\mathscr{A}} + u_{H^{s+2}} + \widetilde{u}$, where*

$$\|u_{\mathscr{A}}\|_{H^1(\Omega)} + k\|u_{\mathscr{A}}\|_{L^2(\Omega)} \leq Ck^\vartheta\|g\|_{H^{1/2}(\partial\Omega)}, \tag{A.8}$$

$$\|\nabla^{n+2}u_{\mathscr{A}}\|_{L^2(\Omega)} \leq Ck^{\vartheta-1}\gamma^n\max\{n,k\}^{n+2}\|g\|_{H^{1/2}(\partial\Omega)} \qquad \forall n \in \mathbb{N}_0, \tag{A.9}$$

$$k^{s+2}\|u_{H^{s+2}}\|_{L^2(\Omega)} + \|u_{H^{s+2}}\|_{H^{s+2}(\Omega)} \leq C\|g\|_{H^{s+1/2}(\partial\Omega)}, \tag{A.10}$$

where the constants C, $\gamma > 0$ are independent of k and n. The remainder $\widetilde{u}$ satisfies the boundary value problem

$$-\Delta\widetilde{u} - k^2\widetilde{u} = 0 \quad \text{in } \Omega, \qquad \partial_n\widetilde{u} + \mathrm{i}k\widetilde{u} = \widetilde{g} \quad \text{on } \partial\Omega$$

for data $\widetilde{g} \in H^{s+1/2}(\partial\Omega)$ with

$$\|\widetilde{g}\|_{H^{s+1/2}(\partial\Omega)} \leq q\|g\|_{H^{s+1/2}(\partial\Omega)}.$$

Proof The proof follows [40, Lemma 4.16], and we will only discuss (A.10). Again, we mention the sign difference between the boundary condition (1.2) and that studied in [40]. We have to consider, in the notation of [40, Lemma 4.16], the terms

$$u^I_{H^2} = S^\Delta_k(H^N_{\partial\Omega}g), \tag{A.11}$$

$$u^{II}_{H^2} = N_k(H_\Omega(2k^2u^I_{H^2})). \tag{A.12}$$

For the term in (A.11), we use Lemma A.2 to get

$$k^{s+2}\|u^I_{H^2}\|_{L^2(\Omega)} + \|u^I_{H^2}\|_{H^{s+2}(\Omega)} \leq C\|g\|_{H^{s+1/2}(\partial\Omega)},$$
$$k^2\|u^I_{H^2}\|_{H^s(\Omega)} \leq q\|g\|_{H^{s+1/2}(\partial\Omega)}.$$

For the term in (A.12), we use Lemma A.3 to arrive at

$$k\|u^{II}_{H^2}\|_{H^{s+1}(\Omega)} + k^{s+2}\|u^{II}_{H^2}\|_{L^2(\Omega)} + \|u^{II}_{H^2}\|_{H^{s+2}(\Omega)} \leq Ck^2\|u^I_{H^2}\|_{H^s(\Omega)} \leq Cq\|g\|_{H^{s+1/2}(\partial\Omega)}.$$

As in the proof of [40, Lemma 4.16], we then set $\widetilde{g} := -\mathrm{i}ku^{II}_{H^2} - \partial_n u^{II}_{H^2}$ and use the above estimates to get with the trace inequality

$$\|\widetilde{g}\|_{H^{s+1/2}(\partial\Omega)} \leq C\left[k\|u^{II}_{H^2}\|_{H^{s+1}(\Omega)} + \|u^{II}_{H^2}\|_{H^{s+2}(\Omega)}\right] \leq Cq\|g\|_{H^{s+1/2}(\partial\Omega)}.$$

Suitably adjusting the constant q yields the result. □

B H^1-conforming approximation

In this appendix we construct an H^1-conforming approximation operator that features optimal rates of convergence not only in L^2 and H^1 but also for the trace and the normal derivative on the element boundaries. This operator can be constructed in an element-by-element fashion. That is, its value at the geometric entities (vertices, edge, faces, elements) is only determined by the function values at these entities. Our construction is closely related to the projection-based interpolation of [13] and the construction in [39, Appendix B]. In contrast to [39, Appendix B], where optimal rates in L^2 and H^1 were sought, we ensure that the optimal rate of convergence for the trace of the gradient is also achieved. We stress that our construction is done with a view to simplicity rather than minimal regularity assumptions.

Definition B.1 (element-by-element construction in 2D) Let $\widehat{K}$ be the reference triangle. Let $s > 5/2$. A polynomial π is said to permit *an element-by-element* construction of boundary polynomial degree $p \geq 7$ for $u \in H^s(\widehat{K})$ if

(i) $\pi(V) = u(V)$ for all $d+1$ vertices V of $\widehat{K}$.
(ii) For every edge e of $\widehat{K}$, the restriction $\pi|_e \in \mathscr{P}_p$ is the unique minimizer of

$$\pi \mapsto p^2 \|u-\pi\|_{L^2(e)} + p|u-\pi|_{H^1(e)} + |u-\pi|_{H^2(e)} \tag{B.1}$$

under two constraints: first, π satisfies (i) and second, the derivative (along e) of $u-\pi$ vanishes in the endpoints of e (i.e., $(u-\pi)|_e \in H_0^2(e)$).

Definition B.2 (element-by-element construction in 3D) Let $\widehat{K}$ be the reference tetrahedron. Let $s > 5$. A polynomial π is said to permit *an element-by-element* construction of edge polynomial degree $p \geq 10$ and face polynomial degree $2p$ for $u \in H^s(\widehat{K})$ if

(i) $\pi(V) = u(V)$ for all $d+1$ vertices V of $\widehat{K}$.
(ii) For every edge e of $\widehat{K}$, the restriction $\pi|_e \in \mathscr{P}_p$ is the unique minimizer of

$$\pi \mapsto p^4 \sum_{j=0}^{4} p^{-j} |u-\pi|_{H^j(e)} \tag{B.2}$$

under two constraints: first, π satisfies (i) and second, the tangential derivatives (along e) up to order 3 vanish in the endpoints of e (i.e., $(u-\pi)|_e \in H_0^4(e)$).
(iii) For every face f of $\widehat{K}$, the restriction $\pi|_f \in \mathscr{P}_{2p}$ is the unique minimizer of

$$\pi \mapsto p^4 \sum_{j=0}^{4} p^{-j} |u-\pi|_{H^j(f)} \tag{B.3}$$

under two constraints: first, π satisfies (i), (ii) for all vertices and edges of f and second, the mixed derivatives of $u-\pi$ vanish in the vertices, i.e., $\partial_{e_1}\partial_{e_2}(u-\pi)(V) = 0$ for each vertex V of f, where e_1, e_2 are two tangential vectors associated with the edges e_1, e_2 of the face f that meet in V.

Theorem B.3 *Let $\widehat{K}$ be the reference triangle or the reference tetrahedron. Set $V_p := \{v \in \mathscr{P}_{2p} \,|\, v|_e \in \mathscr{P}_p$ for all edges $e\}$ if $d = 2$ and $V_p := \{v \in \mathscr{P}_{4p+1} \,|\, v|_f \in \mathscr{P}_{2p}$ for all faces $f, v|_e \in \mathscr{P}_p$ for all edges $e\}$ if $d = 3$. Assume $s > 5/2$ if $d = 2$ and $s > 5$ for $d = 3$. Then, for $p \geq \max\{10, s-1\}$ for $d = 3$ and $p \geq \max\{7, s-1\}$ for $d = 2$, there exists a linear operator $\pi : H^s(\widehat{K}) \to V_p$ that permits an element-by-element construction in the sense of Definition B.1 (for $d = 2$) or Definition B.2 (for $d = 3$) such that*

$$p^2 \|u-\pi(u)\|_{L^2(\widehat{K})} + p|u-\pi(u)|_{H^1(\widehat{K})} + |u-\pi(u)|_{H^2(\widehat{K})} \leq C p^{-(s-2)} |u|_{H^s(\widehat{K})}. \tag{B.4}$$

The constant $C > 0$ depends only on s.

Proof We will only present the arguments for the case $d = 3$. We construct $\pi(u)$ directly—inspection of the proof shows that $u \mapsto \pi(u)$ is a linear operator. To begin with, we mention that the condition $p \geq 10$ ensures that an element-by-element construction in the form of Definition B.2 is feasible: Taking in Lemma B.13 $i = 3$ (and the parameter p there as $p = i+1 = 4$) one can find a polynomial of degree $p' = 2i + p = 10$ that coincides with u and all its derivatives up to order $i = 3$ in all vertices.

Before actually embarking on the proof, we note a trace estimate that will be required frequently, namely, for any edge e of the tetrahedron $\widehat{K} = \widehat{K}^{3D}$, we have for arbitrary but fixed $t > 1$

$$\|v\|_{L^2(e)} \leq C_t \|v\|_{L^2(\widehat{K})}^{(t-1)/t} \|v\|_{H^t(\widehat{K})}^{1/t} \qquad \forall v \in H^t(\widehat{K}); \tag{B.5}$$

this embedding can be shown with appropriate trace estimates $e \to f \to \widehat{K}$ or by combining the continuity assertion for the trace mapping of [46, Thm. 2.9.3] with interpolation inequalities (cf. also the proof of [39, Lemma B.3] where a similar argument is employed).

From [39, Lemma B.3] we have an approximation $\pi^0 \in \mathscr{P}_p$ with

$$|u - \pi^0|_{H^t(\widehat{K})} \leq C p^{-(s-t)} \|u\|_{H^s(\widehat{K})}, \qquad t \in [0,s]. \tag{B.6}$$

Also, [39, Lemma B.3] gives the following L^∞-estimate and, by a similar reasoning, also an L^∞-estimates for the derivatives up to order 3:[2]

$$\sum_{j=0}^{3} p^{-j} \|\nabla^j (u - \pi^0)\|_{L^\infty(\widehat{K})} \leq C p^{-(s-d/2)} \|u\|_{H^s(\widehat{K})}. \tag{B.7}$$

Vertex Correction. With the vertex liftings of Lemma B.13 we can construct a polynomial $\pi^1 \in \mathscr{P}_p$ with the following properties:

$$|u - \pi^1|_{H^t(\widehat{K})} \leq C p^{-(s-t)} \|u\|_{H^s(\widehat{K})}, \qquad t \in [0,s], \tag{B.8}$$

$$D^\beta (u - \pi^1)(V) = 0, \qquad 0 \leq |\beta| \leq 3. \tag{B.9}$$

To see this, we employ the vertex liftings E_V^{3D} of Remark B.14. Specifically, we fix a vertex V and take in Remark B.14 the parameter $q = 3$ and the parameter p there as $p - 6$ to obtain the polynomial

$$\widetilde{\pi}_1 := \pi_0 + E_V^{3D}(u - \pi_0) \in \mathscr{P}_p.$$

By construction in Remark B.14, the polynomial $\widetilde{\pi}_1$ satisfies $D^\beta (u - \widetilde{\pi}_1)(V) = 0$ for $|\beta| \leq 3$ and, by (B.37),

$$\|\widetilde{\pi}_1 - u\|_{H^t(\widehat{K})} \leq C \sum_{|\alpha| \leq 3} \|D^\alpha (u - \pi_0)\|_{L^\infty(\widehat{K})} p^{-d/2+t} p^{-|\alpha|} + \|u - \pi_0\|_{H^t(\widehat{K})} \overset{(B.6),(B.7)}{\leq} C p^{-(s-t)} \|u\|_{H^s(\widehat{K})}.$$

Proceeding in this way for all vertices yields a polynomial $\pi^1 \in \mathscr{P}_p$ with the properties (B.8), (B.9).

The condition (B.9) implies in particular that $u - \pi^1 \in H_0^4(e)$ and $\nabla(u - \pi^1) \in H_0^3(e)$ for each edge e. With the trace estimates (B.5) we get from (B.8) the following estimates on edges:

$$p^4 \|u - \pi^1\|_{L^2(e)} + \sum_{j=0}^{3} p^{3-j} |\nabla(u - \pi^1)|_{H^j(e)} \leq C p^{-(s-5)} \|u\|_{H^s(\widehat{K})} \qquad \forall \text{ edges } e \text{ of } \widehat{K}. \tag{B.10}$$

Edge Correction I. Fix an edge e. Since π^1 satisfies both side constraints in Definition B.2.(ii), the minimizer π_e of (B.2) satisfies by (B.10)

$$p^4 \sum_{j=0}^{4} p^{-j} |u - \pi_e|_{H^j(e)} \leq p^4 \sum_{j=0}^{4} p^{-j} |u - \pi^1|_{H^j(e)} \leq C p^{-(s-5)} \|u\|_{H^s(\widehat{K})}.$$

We note that the difference $\pi_e - \pi^1|_e$ is a polynomial of degree p and $\partial_e^j(\pi_e - \pi^1)$ vanishes at the endpoints of e for $j \in \{0,1,2,3\}$, i.e., $\pi_e - \pi^1 \in H_0^4(e) \cap \mathscr{P}_p$. By writing $\pi^1 - \pi_e = \left(\pi^1 - u\right) + (u - \pi_e)$ we obtain with the triangle inequality

$$p^4 \sum_{j=0}^{4} p^{-j} |\pi^1 - \pi_e|_{H^j(e)} \leq C p^{-(s-5)} \|u\|_{H^s(\widehat{K})}. \tag{B.11}$$

[2] To see this, e.g., for $j = 3$, we employ the interpolation inequality [39, (B.5)] to $\nabla^3 u$ to obtain

$$\left\|\nabla^3 u\right\|_{L^\infty(\widehat{K})} \leq C \left\|\nabla^3 u\right\|_{L^2(\widehat{K})}^{1-d/(2(s-3))} \left\|\nabla^3 u\right\|_{H^{s-3}(\widehat{K})}^{d/(2(s-3))} \qquad \forall u \in H^s(\widehat{K})$$

since $s > 3 + d/2$. The combination with (B.6) yields the desired bound in (B.7).

With the aid of Lemma B.15 we can find an edge lifting $L_e := E^{3D}_{1,e}\left(\pi^1 - \pi_e\right) \in \mathscr{P}_{2p}$ (take as the parameter p in the statement of Lemma B.15 the value $p-1$) to correct the discrepancy $\pi^1 - \pi_e$ with the following properties[3]:

$$
\begin{aligned}
p^4 \sum_{j=0}^{4} p^{-j} \|L_e\|_{H^j(\widehat{K})} &\overset{\text{Lem. B.15.(vi), (vii) and (B.11)}}{\leq} C p^{-(s-4)} \|u\|_{H^s(\widehat{K})}, \\
p^4 \sum_{j=0}^{4} p^{-j} \|L_e\|_{H^j(f)} &\overset{\text{Lem. B.15.(viii) and (B.11)}}{\leq} C p^{-(s-4-1/2)} \|u\|_{H^s(\widehat{K})} \qquad \text{for all faces } f, \\
L_e &= (\pi^1 - \pi_e) \quad \text{on } e, \\
L_e &= 0 \quad \text{on all other edges of } \widehat{K}, \\
(L_e)|_f &= 0 \qquad \text{on all faces } f \text{ that have not } e \text{ as an edge}, \\
(\partial_{n_f} L_e)|_{\partial f} &= 0 \quad \text{for each face } f.
\end{aligned}
$$

With the aid of such a lifting for each edge e, we can find a polynomial $\pi^2 \in \mathscr{P}_{2p}$ with

$$
p^4 \sum_{j=0}^{4} p^{-j} \|u - \pi^2\|_{H^j(\widehat{K})} \leq C p^{4-s} \|u\|_{H^s(\widehat{K})}, \tag{B.12}
$$

$$
p^4 \sum_{j=0}^{4} p^{-j} \|u - \pi^2\|_{H^j(f)} \leq C p^{1/2} p^{4-s} \|u\|_{H^s(\widehat{K})} \qquad \text{for all faces } f \tag{B.13}
$$

and the following two additional properties:

$$
\pi^2|_e = \pi_e \quad \text{for all edges } e \qquad \text{and} \qquad \partial_{n_f} \pi^2|_{\partial f} = \partial_{n_f} \pi^1|_{\partial f} \quad \text{for each face } f. \tag{B.14}
$$

In other words, π^2 satisfies conditions (i), (ii) of Definition B.2.

Relation to face minimizer π_f. For a face f, we denote by π_f the polynomial that is obtained by the minimizing procedure (B.3). We claim that

$$
p^4 \sum_{j=0}^{4} p^{-j} \|u - \pi_f\|_{H^j(f)} \leq C p^{-(s-9/2)} \|u\|_{H^s(\widehat{K})}. \tag{B.15}
$$

To see this, we estimate the error of a modification of π^2. An interpolation inequality and estimate (B.12) imply

$$
\begin{aligned}
\|\nabla^2 (u - \pi^2)\|_{L^\infty(f)} \leq \|\nabla^2 (u - \pi^2)\|_{L^\infty(\widehat{K})} &\leq C \|u - \pi^2\|^{1-3/4}_{H^2(\widehat{K})} \|u - \pi^2\|^{3/4}_{H^4(\widehat{K})} \\
&\leq C p^{-1/2+4-s} \|u\|_{H^s(\widehat{K})}.
\end{aligned} \tag{B.16}
$$

We note that the polynomial π^2 coincides with π_e for each edge e of ∂f. The second order mixed derivatives of $(u - \pi^2)|_f$ may not vanish at the vertices. This can be corrected with a lifting of Lemma B.13. Specifically, for each vertex V Lemma B.13 provides a lifting $L_V \in \mathscr{P}_p$ (take the parameter p in Lemma B.13 as $p-4$) that vanishes on ∂f such that the mixed derivative at V equals 1 and

$$
p^4 \sum_{j=0}^{4} p^{-j} \|L_V\|_{H^j(f)} \leq C p^{-1+4-2},
$$

where we used appropriate trace theorems again. Combining this with (B.16), we can construct a function $\widetilde{\pi}_f$ that satisfies all the desired constraints on ∂f and at the vertices of f and additionally the estimate

$$
p^4 \sum_{j=0}^{4} p^{-j} \|u - \widetilde{\pi}_f\|_{H^j(f)} \leq C p^4 \sum_{j=0}^{4} p^{-j} \|u - \pi^2\|_{H^j(f)} + C p^{-1/2+4-s} p^{-1+4-2} \|u\|_{H^s(\widehat{K})} \leq C p^{1/2+4-s} \|u\|_{H^s(\widehat{K})},
$$

[3] For a face f, the face normal $n_f : \partial f \to \mathbb{S}_2$ is defined to have length 1, lies in the plane of f, and points to the exterior of f. The face normal derivative on ∂f is then given by $\partial_{n_f} := \langle n_f, \nabla \cdot \rangle$.

where we used (B.12) to control $u-\pi^2$. We conclude for the minimizer π_f that (B.15) holds.

Edge Correction II The minimizer π_f satisfies

$$D_f^\beta\left(\pi_f-u\right)(V)=0 \quad \text{for all } 0\le|\beta|\le 2 \text{ at all vertices } V, \tag{B.17}$$

where the subscript f in D_f^β indicates that differentiation is taken in the plane given by f. The observations (B.14), (B.17), and (B.9) ensure that for each face f and each edge e of f, we have $\partial_{n_f}(\pi_f-\pi^2)|_e=\partial_{n_f}(\pi_f-\pi^1)|_e=\partial_{n_f}(\pi_f-u)|_e+\partial_{n_f}(u-\pi^1)|_e\in H_0^2(e)$. With the trace estimate (B.5) we get from (B.15) and (B.12)

$$\begin{aligned}p^2\sum_{j=0}^{2}p^{-j}|\partial_{n_f}(\pi^2-\pi_f)|_{H^j(e)}&\le p^2\sum_{j=0}^{2}p^{-j}|\partial_{n_f}(\pi^2-u)|_{H^j(e)}+p^2\sum_{j=0}^{2}p^{-j}|\partial_{n_f}(u-\pi_f)|_{H^j(e)}\\&\le Cp^{4-s}\|u\|_{H^s(\widehat{K})}.\end{aligned} \tag{B.18}$$

We are now in position to construct for each face a lifting $L_f\in\mathscr{P}_{3p+1}$ (which is composed of liftings $E_{2,e}^{3D}\left(\partial_{n_f}(\pi^2-\pi_f)\Big|_e\right)$ with the lifting operator $E_{2,e}^{3D}$ of Lemma B.16) with the following properties:

$$p^2\sum_{j=0}^{2}p^{-j}\|L_f\|_{H^j(\widehat{K})}\le Cp^{2-s}\|u\|_{H^s(\widehat{K})}, \tag{B.19}$$

$$L_f=0 \quad \text{on all faces except } f, \tag{B.20}$$

$$\partial_{n_f}L_f|_{\partial f}=\partial_{n_f}(\pi^2-\pi_f)|_{\partial f}. \tag{B.21}$$

With these liftings, we may adjust π^2 to produce a polynomial $\pi^3\in\mathscr{P}_{3p+1}$ with

$$p^2\sum_{j=0}^{2}p^{-j}\|u-\pi^3\|_{H^j(\widehat{K})}\le Cp^{2-s}\|u\|_{H^s(\widehat{K})}, \tag{B.22}$$

$$\pi^3-\pi_f\in H_0^2(f) \qquad \text{on all faces } f. \tag{B.23}$$

Face Correction. In view of $\pi^3-\pi_f\in H_0^2(f)$ we may use the final face lifting of Lemma B.17 to produce a polynomial $\pi^4\in\mathscr{P}_{4p+1}$ to enforce the desired behavior on the faces. Since $\pi^4\big|_f=\pi_f$ for all faces, it satisfies the conditions of Definition B.2 and additionally

$$p^2\sum_{j=0}^{2}p^{-j}\|u-\pi^4\|_{H^j(\widehat{K})}\le Cp^{2-s}\|u\|_{H^s(\widehat{K})}. \tag{B.24}$$

Volume correction. As a final step, we replace $\|u\|_{H^s(\widehat{K})}$ on the right-hand side of (B.24) by the semi-norm $|u|_{H^s(\widehat{K})}$ with the classical compactness argument due to Deny-Lions. Specifically, we take $\pi(u)\in\mathscr{P}_{4p+1}$ as the minimizer of

$$v\mapsto p^2\sum_{j=0}^{2}p^{-j}\|u-v\|_{H^j(\widehat{K})}$$

under the constraint that $v|_{\partial\widehat{K}}=\pi^4|_{\partial\widehat{K}}$. Then $u\mapsto\pi(u)$ is a projection on the space V_p (as defined in the theorem) and the full norm $\|u\|_{H^s(\widehat{K})}$ can be replaced with $|u|_{H^s(\widehat{K})}$ for $p\ge s-1$. $\square$

Corollary B.4 *Let $\mathscr{T}$ be an H^1-regular mesh in the sense of the beginning of Section 4.2.2 and $S=S^{p,1}(\mathscr{T})$ be the space of piecewise mapped polynomials of degree p on $\mathscr{T}$. Let $s>5/2$ for $d=2$ and $s>5$ for $d=3$. Then, for every $p\ge s-1$ there exists a linear operator $I:H^s(\Omega)\to S\cap H^1(\Omega)$ such that for all $K\in\mathscr{T}$*

$$\left(\frac{h_K}{p}\right)^2\|\nabla^2(u-Iu)\|_{L^2(K)}+\left(\frac{h_K}{p}\right)\|\nabla(u-Iu)\|_{L^2(K)}+\|u-Iu\|_{L^2(K)}\le C\left(\frac{h_K}{p}\right)^s\|u\|_{H^s(K)}.$$

Proof For large p, we use the operator constructed in Theorem B.3. For example, for $d=3$ and $p'\ge\max\{10,s-1\}$ with $p':=\lfloor(p-1)/4\rfloor$, we can define Iu on the reference element $\widehat{K}$ by taking the operator constructed in Theorem B.3 (with p' taking the role of p there); this yields the desired estimates in p and the appropriate powers of h_K arise from scaling arguments (cf. Lemma 4.7). If $p'<\max\{10,s-1\}$, this corresponds to finitely many possible values of p and the p-dependence in the desired estimate is irrelevant. We take Iu as any standard Lagrange interpolation operator and obtain the required h_K-dependence again by the scaling arguments of Lemma 4.7. $\square$

B.1 Lifting operators

B.1.1 Preliminaries

We start with a convenient definition of the reference triangle $\widehat{K}^{2D}$ and the reference tetrahedron $\widehat{K}^{3D}$:

$$\widehat{K}^{2D} := \{(x,y) \mid -1 < x < 1, \quad 0 < y < 1-|x|\}, \tag{B.25}$$

$$\widehat{K}^{3D} := \{(x,y,z) \mid -1 < x < 1, \quad 0 < y, \quad 0 < z, \quad 0 < y+z < 1-|x|\}. \tag{B.26}$$

Below, we will frequently require the following asymptotics of the Beta function B for $\alpha > -1$ and $p \geq 0$ (cf., e.g., [42, Secs. 1.6, 5.1]):

$$\int_0^1 x^\alpha (1-x)^p \, dx = B(\alpha+1, p+1) = \frac{\Gamma(\alpha+1)\Gamma(p+1)}{\Gamma(\alpha+p+2)} \leq C_\alpha (p+1)^{-1-\alpha}. \tag{B.27}$$

We need a preliminary result that will prove useful for the construction of various vertex liftings:

Lemma B.5 *For $q \in \mathbb{N}$ define on $(0,1)$ the function $L_q(r) := (1-r)^q$. Fix $i \in \mathbb{N}_0$. Then there exists a polynomial $\pi_i \in \mathscr{P}_i$ of the form*

$$\pi_i(r) = \sum_{j=0}^{i} \alpha_j (qr)^j$$

and a constant C_i (which depends solely on i) with the following properties:

$$|\alpha_j| \leq C_i, \qquad j = 0, \dots, i,$$

$$(\pi_i L_q)^{(j)}(0) = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } 0 < j \leq i. \end{cases}$$

Furthermore, the polynomial $\pi_i L_q$ satisfies, for every $a \in [0,1]$, $\alpha \geq 0$, and every $s \in \mathbb{N}_0$

$$\left| \int_0^{1-a} |r^\alpha (\pi_i L_q)^{(s)}(a+r)|^2 \, dr \right| \leq C_{s,i,\alpha} (1-a)^{2(q-s+\alpha)+1} q^{-1+2s-2\alpha} \sum_{j=0}^{i} (qa)^{2j}. \tag{B.28}$$

The constant $C_{s,i,\alpha}$ depends only on s, α, and i.

Proof The polynomials π_i can be defined inductively. We take $\pi_0 \equiv 1$. For π_{i+1} we make the ansatz $\pi_{i+1}(r) = \pi_i(r) + \alpha_{i+1} r^{i+1}$. This implies for $0 \leq m \leq i$ that $\left(\pi_{i+1} L_q\right)^{(m)}(0) = \left(\pi_i L_q\right)^{(m)}(0)$. The unknown coefficient α_{i+1} is then determined by the condition

$$0 \stackrel{!}{=} (\pi_{i+1} L_q)^{(i+1)}(0) = \sum_{j=0}^{i+1} \binom{i+1}{j} \pi_i^{(j)}(0) L_q^{(i+1-j)}(0) + \alpha_{i+1} q^{i+1} (i+1)! L_q(0).$$

Since $L_q^{(j)}(0) = (-1)^j \binom{q}{j} j!$ we get $|L_q^{(j)}(0)| \leq C q^j$ for a constant $C > 0$ independent of $q \in \mathbb{N}$. In view of $\pi_i^{(j)}(0) = \alpha_j q^j j!$, the claimed estimate follows for α_{i+1} by induction and (B.27). We finally show (B.28). For simplicity of notation, let $q \geq s+1$. Since $r^\alpha (\pi_i L_q)^{(s)}$ consists of terms of the form $\left((1-r)^q (qr)^j\right)^{(s)} r^\alpha$, the product rule shows that it consists of terms of the form $((1-r)^q)^{(s-k)} \left((qr)^j\right)^{(k)} r^\alpha$ which can be estimated from above by

$$r^\alpha q^{s-k} q^j r^{j-k} (1-r)^{q-(s-k)}, \qquad 0 \leq j \leq i, \quad 0 \leq k \leq \min\{s, j\}.$$

With these constraints on j and k, we estimate with the change of variables $r = (1-a)\rho$

$$\begin{aligned}
I_{j,k} &:= q^{2s} \int_{r=0}^{1-a} r^{2\alpha} (q(a+r))^{2(j-k)} (1-(a+r))^{2(q-(s-k))} \, dr \\
&= q^{2s+2(j-k)} (1-a)^{2(q-(s-k))+1+2\alpha} \int_{\rho=0}^{1} \rho^{2\alpha} (a+(1-a)\rho)^{2(j-k)} (1-\rho)^{2(q-(s-k))} \, d\rho \\
&\lesssim q^{2s+2(j-k)} (1-a)^{2(q-(s-k))+1+2\alpha} \int_{\rho=0}^{1} \rho^{2\alpha} (a^{2(j-k)} + \rho^{2(j-k)}) (1-\rho)^{2(q-(s-k))} \, d\rho \\
&\stackrel{(B.27)}{\lesssim} q^{2s+2(j-k)-1} (1-a)^{2(q-(s-k))+1+2\alpha} \left(q^{-2\alpha} a^{2(j-k)} + q^{-2(j-k)-2\alpha}\right) \\
&\lesssim q^{2s-1-2\alpha} (1-a)^{2(q-(s-k))+1+2\alpha} \left((qa)^{2(j-k)} + 1\right).
\end{aligned}$$

Summation over all relevant j, k gives the stated estimate. $\square$

We need a working lemma for the edge liftings in 2D and 3D:

Lemma B.6 *Consider $\widehat{K}^{2D}$ and its edge $e = (-1,1) \times \{0\}$. Let $j \in \{0,1,2,3,4\}$. Let $\mathscr{V}$ be the set of vertices of $\widehat{K}^{2D}$ and $d_{\mathscr{V}} := \operatorname{dist}(\cdot, \mathscr{V})$ be the distance from the vertices. Let $w \in C^\infty(\mathbb{R}^4)$. Let $\alpha \in \mathbb{N}_0$. Then there is $C > 0$ such that for every $p \geq 0$ the map $E_{1,e} : H^j_0(e) \to H^j(\widehat{K}^{2D})$ given by*

$$(E_{1,e}u)(x,y) := y^\alpha w\left(x, y, \frac{y}{1-x}, \frac{y}{1+x}\right)(1-y)^p u(x)$$

satisfies:

$$|E_{1,e}u|_{H^j(\widehat{K}^{2D})} \leq C(p+1)^{-\alpha-1/2}\left[p^j\|u\|_{L^2(e)} + p^{j-1}|u|_{H^1(e)} + \cdots + p^0|u|_{H^j(e)}\right]. \tag{B.29}$$

Furthermore, if $0 \leq \alpha \leq j$ and $0 \leq i \leq j$ and additionally[4] *$p \geq j$*

$$\|d_{\mathscr{V}}^{-(j-i)}\nabla^i E_{1,e}u\|_{L^2(e')} \leq C(p+1)^{-\alpha}\left[|u|_{H^j(e)} + p|u|_{H^{j-1}(e)} + \cdots + p^j\|u\|_{L^2(e)}\right] \tag{B.30}$$

for any simplex edge e'. In particular, therefore, $E_{1,e}u \in H^j_0(e')$ for every edge if $p \geq j$.

Proof We start with the proof of (B.29). Without explicitly stating it below, we will assume that p is sufficiently large (specifically, $p \geq 2$). For the case $j = 0$, (B.29) follows from the observation that w is a bounded function on $\widehat{K}^{2D}$ since $y/(1-|x|) \leq 1$ on $\widehat{K}^{2D}$ and the estimate (B.27). For the cases $j \geq 1$, we have to control the derivatives. We use $0 < y < 1-|x|$ and the smoothness of w to estimate

$$|D^\beta w| \leq C(1-|x|)^{-|\beta|}, \qquad (x,y) \in \widehat{K}^{2D}, \tag{B.31}$$

$$|D^\beta(y^\alpha(1-y)^p)| \leq Cp^{|\beta|-\alpha}(1-y)^{p-|\beta|}(1+(yp)^\alpha), \qquad (x,y) \in \widehat{K}^{2D}, \tag{B.32}$$

for arbitrary multiindices $\beta \in \mathbb{N}^2_0$ and $p \geq |\beta|$. Recall that $i \mapsto a^i$ is convex for $i \in \mathbb{N}_0$ and $a > 0$. From the product rule, we therefore infer for fixed $\beta \in \mathbb{N}^2_0$ and $p \geq |\beta|$

$$\begin{aligned}|D^\beta(y^\alpha w(1-y)^p)| &\leq C\left[((1-|x|)^{-|\beta|}(1-y)^{|\beta|} + p^\beta)p^{-\alpha}(1+(py)^\alpha)\right](1-y)^{p-|\beta|}\\ &\leq C\left[((1-|x|)^{-|\beta|} + p^\beta)(p^{-\alpha} + y^\alpha)\right](1-y)^{p-|\beta|}.\end{aligned} \tag{B.33}$$

(B.33) thus allows us to control the derivatives of the function W defined as

$$W(x,y) := y^\alpha w(1-y)^p.$$

We now consider the case $j = 1$ and $|\beta| = 1$. Then Lemma B.8 gives

$$\begin{aligned}&\int_{x=-1}^1\int_{y=0}^{1-|x|}\left(u(x)D^\beta W\right)^2 + (\partial_x u(x)W)^2\,dy\,dx\\ &\leq C\left(p^{-2\alpha-1}\|\frac{1}{1-x}u\|^2_{L^2(e)} + p^2p^{-2\alpha-1}\|u\|^2_{L^2(e)} + p^{-2\alpha-1}\|\partial_x u\|^2_{L^2(e)}\right)\\ &\leq Cp^{-2\alpha-1}\|\partial_x u\|^2_{L^2(e)} + p^2p^{-2\alpha-1}\|u\|^2_{L^2(e)},\end{aligned}$$

where, in the last step, we employed the Hardy inequality of Lemma B.7 (with $\beta = -2$ there). We now consider $j = 2$. Then, we have to bound $\|uD^\beta W\|_{L^2(\widehat{K}^{2D})}$ for $|\beta| = 2$, $\|\partial_x uD^\beta W\|_{L^2(\widehat{K}^{2D})}$ for $|\beta| = 1$ and $\|\partial_x^2 uW\|_{L^2(\widehat{K}^{2D})}$. Writing D^2W and D^1W for the sum of all derivatives of order 2 and 1, respectively, we estimate

$$\begin{aligned}\|\partial_x^2 uW\|^2_{L^2(\widehat{K}^{2D})} &\leq Cp^{-2\alpha-1}|u|^2_{H^2(e)},\\ \|\partial_x uD^1W\|^2_{L^2(\widehat{K}^{2D})} &\leq C\left(p^{-2\alpha-1}\|\frac{1}{1-x}\partial_x u\|^2_{L^2(e)} + p^2p^{-2\alpha-1}\|\partial_x u\|^2_{L^2(e)}\right)\\ &\leq C\left(p^{-2\alpha-1}\|\partial_x^2 u\|^2_{L^2(e)} + p^2p^{-2\alpha-1}\|\partial_x u\|^2_{L^2(e)}\right),\end{aligned}$$

[4] The condition $p \geq j$ can be dropped if $E_{1,e}u$ vanishes to higher order at the vertex $(0,1)$ due to appropriate assumptions on the function w.

where, in the last step, we used again the Hardy inequality of Lemma B.7 with the assumption $\partial_x u(1) = 0$. Estimating uD^2W requires us to control

$$\int_{x=-1}^{1}\int_{y=0}^{1-|x|} |u(x)|^2(1-|x|)^{-4}\left(p^{-\alpha}+y^{\alpha}\right)^2(1-y)^{2p-4}\,dy\,dx \qquad \text{and}$$
$$\int_{x=-1}^{1}\int_{y=0}^{1-|x|} |u(x)|^2 p^4\left(p^{-\alpha}+y^{\alpha}\right)^2(1-y)^{2p-4}\,dy\,dx.$$

The second term is readily bounded by $p^{4-2\alpha-1}\|u\|^2_{L^2(e)}$. For the first term, an application of Lemma B.8 yields

$$p^{-2\alpha-1}\|\frac{1}{(1-|x|)^2}u\|^2_{L^2(e)}.$$

A two-fold application of the Hardy inequality Lemma B.7 yields $\|1/(1-|x|)^2u\|_{L^2(e)} \leq C\|\partial_x^2 u\|^2_{L^2(e)}$, which is the desired estimate. The cases $j = 3$, 4 are shown with similar arguments.

For the estimate (B.30), we argue in a similar way. We focus on the case $e' \neq e$, the case $e' = e$ being slightly simpler. The assumption $p \geq j$ implies that $E_{1,e}u$ vanishes to higher order at the vertex $(0,1)$. We may therefore concentrate on the behavior of $E_{1,e}u$ at the vertices $(-1,0)$ and $(1,0)$. For example, for $i = 0$ we have to estimate terms of the following form (the contribution $(1-y)^p$ is generously estimated by 1) in view of (B.33):

$$\int_{x=-1}^{1} u^2(x)\,(1-|x|)^{-2j}\left(p^{-\alpha}+(1-|x|)^{\alpha}\right)^2 dx, \tag{B.34}$$

where we observed that the factor y^{α} arising in (B.33) is changed into $(1-|x|)$ due to the parametrization of e'. The integral (B.34) can then be treated with the Hardy inequality of Lemma B.7. □

From [46, Rem. 1, Sec. 3.2.6] we have the following variant of Hardy's inequality:

Lemma B.7 (Hardy inequality) *For $\beta < -1$ and $\varphi \in C_0^{\infty}(0,1)$*

$$\int_{x=0}^{1} |\varphi(x)|^2 x^{\beta}\,dx \leq \left(\frac{2}{|\beta+1|}\right)^2 \int_{x=0}^{1} x^{\beta+2}\left|\varphi'(x)\right|^2 dx.$$

Lemma B.8 *Fix $\alpha \geq 0$ and $\beta \in \mathbb{R}$ with $\alpha+\beta \geq 0$. There is some $C > 0$ independent of $x \in (0,1)$ and $p \geq 0$ such that*

$$\int_{y=0}^{1-x}\left(\left(\frac{y}{1-x}\right)^{\alpha} y^{\beta}(1-y)^p\right)^2 dy \leq C\left(\min\{1-x,p^{-1}\}\right)^{1+2\beta}, \tag{B.35}$$

$$\int_{y=0}^{1-x}\left(\frac{y^{\alpha}}{(1-x)^{\alpha+1/2}}(1-y)^p\right)^2 dy \leq C. \tag{B.36}$$

Proof We may assume $p \geq 2$. Both estimates follow by distinguishing between the cases $x < 1-1/p$ and $1-1/p < x < 1$; in the latter case, we use additionally (B.27). □

Lemma B.9 *Let $f \in L^1(\widehat{K}^{2D})$. Then*

$$\int_{\widehat{K}^{3D}} f(x,y+z)\,dx\,dy\,dz = \int_{\widehat{K}^{2D}} y f(x,y)\,dy\,dx,$$
$$\int_{\widehat{K}^{3D}} y f(x,y+z)\,dx\,dy\,dz = \int_{\widehat{K}^{2D}} \frac{1}{2}y^2 f(x,y)\,dy\,dx.$$

Proof Follows from an appropriate application of Fubini's theorem. □

B.1.2 Liftings for the 2D case

We start with vertex liftings that allow us to match the Taylor expansion in the vertices to any desired order.

Lemma B.10 (vertex liftings in 2D) *Fix $i \in \mathbb{N}_0$ and a vertex V of $\widehat{K}^{2D}$. Denote by e_1, e_2 the two edges meeting at V and by ∂_{e_1}, ∂_{e_2} differentiation along e_1, e_2. Fix $(i_1,i_2) \in \mathbb{N}_0^2$ with $i_1+i_2 \le i$. Then for $p \ge i+1$ one can find polynomials $L_{V,(i_1,i_2),p} \in \mathscr{P}_{p+2i}$ with*

$$\partial_{e_1}^{j_1}\partial_{e_2}^{j_2} L_{V,(i_1,i_2),p}(V) = \delta_{i_1,j_1}\delta_{i_2,j_2} \qquad \forall (j_1,j_2) \in \mathbb{N}_0^2 \text{ with } j_1+j_2 \le i,$$
$$\nabla^j L_{V,(i_1,i_2),p}(V') = 0 \qquad \forall 0 \le j \le i, \quad \forall \text{ vertices } V' \ne V.$$

Furthermore $L_{V,(i_1,i_2),p}$ vanishes on the edge opposite V and for every $s \ge 0$, one has for a constant $C_s > 0$ independent of p (but depending on s and i)

$$\|L_{V,(i_1,i_2),p}\|_{H^s(\widehat{K}^{2D})} \le C_s p^{-1+s-(i_1+i_2)}.$$

Proof It is convenient to work with the reference triangle

$$\widetilde{K}^{2D} := \{(x,y)\,|\,0<x<1, 0<y<1-x\}.$$

Let $L_{1,p} \in \mathscr{P}_{p+i}$ be the univariate polynomial given by Lemma B.5 with the property $L_{1,p}^{(j)}(0) = \delta_{j,0}$ for $j = 0,\dots,i$ and $L_{1,p}^{(j)}(1) = 0$ for $j = 0,\dots,p-1$. Set

$$L_{V,(i_1,i_2),p}(x,y) := \frac{1}{i_1!}\frac{1}{i_2!}x^{i_1}y^{i_2}L_{1,p}(x+y) \in \mathscr{P}_{p+i_1+i_2+i}.$$

Since $L_{1,p}(0) = 1$ and $L_{1,p}^{(j)}(0) = 0$ for $j = 1,\dots,i$ and $L_{1,p}^{(j)}(1) = 0$ for $j = 0,\dots,p-1 \ge i$, we see that $L_{V,p}$ has the desired properties in the vertices of $\widetilde{K}^{2D}$. To see the norm bounds, we consider $(s_1,s_2) \in \mathbb{N}_0^2$ with $s_1+s_2 = s$. Then, by the product rule, $D^{(s_1,s_2)}L_{V,(i_1,i_2),p}$ consist of terms of the form

$$x^{i_1-k_1}y^{i_2-k_2}L_{1,p}^{(s_1+s_2-k_1-k_2)}(x+y), \qquad 0 \le k_1 \le \min\{i_1,s_1\}, \quad 0 \le k_2 \le \min\{i_2,s_2\}.$$

Hence, we have to bound

$$I_{k_1,k_2} := \int_{x=0}^1 x^{2(i_1-k_1)}\int_{y=0}^{1-x} y^{2(i_2-k_2)}|L_{1,p}^{(s_1+s_2-k_1-k_2)}(x+y)|^2\,dy\,dx.$$

With the aid of Lemma B.5 in the first step and (B.27) in the second one, we get

$$\begin{aligned} I_{k_1,k_2} &\lesssim \sum_{j=0}^{i}\int_{x=0}^1 x^{2(i_1-k_1)}p^{-1-2(i_2-k_2)+2(s_1+s_2-k_1-k_2)}(1-x)^{2(p-(s_1+s_2-k_1-k_2+i_2-k_2))+1}(xp)^{2j} \\ &\lesssim \sum_{j=0}^{i} p^{-2(i_1-k_1)-1}p^{-1-2i_2+2s_1+2s_2-2k_1} \lesssim p^{2(s_1-i_1+s_2-i_2)-2} = p^{2(s-i_1-i_2)-2}, \end{aligned}$$

which implies the desired estimate. □

Lemma B.11 (edge liftings in 2D) *For every edge of $\widehat{K}^{2D}$ and $j \ge 1$ and $p \in \mathbb{N}$ there is a bounded linear operator $E_{1,e}^{2D} : L^2(e) \to L^2(\widehat{K}^{2D})$ with the following properties with a $C > 0$ independent of p and u:*

(i) $\|E_{1,e}^{2D}u\|_{L^2(\widehat{K}^{2D})} \le Cp^{-1/2}\|u\|_{L^2(e)}$.

(ii) $|E_{1,e}^{2D}u|_{H^k(\widehat{K}^{2D})} \le Cp^{-1/2}\left[p^k\|u\|_{L^2(e)} + p^{k-1}\|\nabla_e u\|_{L^2(e)} + \cdots + |u|_{H^k(e)}\right]$ *if additionally* $u \in H_0^k(e)$.

Additionally, $E_{1,e}^{2D}u$ has a trace on $\partial\widehat{K}^{2D}$ and

(iii) $(E_{1,e}^{2D}u)|_e = u$.

(iv) $(E_{1,e}^{2D}u)|_{\partial\widehat{K}^{2D}\setminus e} = 0$.

Furthermore, if $u \in H_0^j(e)$, then

(v) $\forall u \in \mathscr{P}_q \cap H_0^j(e)$: $E_{1,e}^{2D}u \in \mathscr{P}_{p+q}$.

(vi) $(\nabla^k E_{1,e}^{2D}u)|_{\partial\widehat{K}^{2D}\setminus e} = 0, \qquad k = 0,\dots,j-1.$

Proof We consider the edge $e = \{(x,y)\,|\,y=0\}$. The edge lifting for e is taken to be

$$(E^{2D}_{1,e}u)(x,y) := u(x)\frac{1}{(1-x^2)^j}(1-x-y)^j(1+x-y)^j(1-y)^p = u(x)\left(1-\frac{y}{1-x}\right)^j\left(1-\frac{y}{1+x}\right)^j(1-y)^p.$$

Lemma B.6 implies the norm bounds stated in (i), (ii), since $E^{2D}_{1,e}u$ has the form studied there. The properties concerning the traces and derivatives on $\partial\widehat{K}^{2D}$ given in (iii)—(vi) follow by inspection (and $j>0$)). □

The following result is a variation of Lemma B.11 and will be required for the 3D situation.

Lemma B.12 *Let $\mathscr{V}$ be the vertices of $\widehat{K}^{2D}$ and $d_{\mathscr{V}} := \operatorname{dist}(\cdot,\mathscr{V})$. Then, for every edge e of $\widehat{K}^{2D}$ and $p\in\mathbb{N}$ there is a bounded linear operator $E_{1,e}: L^2(e)\to L^2(\widehat{K}^{2D})$ with the following properties:*

(i) $\|E_{1,e}u\|_{L^2(\widehat{K}^{2D})} \le Cp^{-1/2}\|u\|_{L^2(e)}$.
(ii) $|E_{1,e}u|_{H^j(\widehat{K}^{2D})} \le Cp^{-1/2}p^j\sum_{\ell=0}^{j}p^{-\ell}\|u\|_{H^\ell(e)}$ *if* $u\in H^j_0(e)$, $j\ge 0$.
(iii) If $u\in H^j_0(e)$ for a $j\ge 1$, then $E_{1,e}u|_{e'}\in H^j_0(e')$ for every edge e' of $\widehat{K}^{2D}$ and in fact, for $0\le i\le j\le p$,

$$\|d_{\mathscr{V}}^{-(j-i)}\nabla^i E_{1,e}u\|_{L^2(e')} \le Cp^j\sum_{k=0}^{j}p^{-k}|u|_{H^k(e)}.$$

In the above estimates, the constant $C>0$ is independent of u and p. Additionally, if $u\in H^3_0(e)$, then

(iv) $\forall u\in\mathscr{P}_q\cap H^3_0(e)$: $E_{1,e}u\in\mathscr{P}_{p+q+1}$.
(v) $(E_{1,e}u)|_{\partial\widehat{K}^{2D}\setminus e}=0$.
(vi) $(\nabla E_{1,e}u)|_{\partial\widehat{K}^{2D}\setminus e}=0$.
(vii) $(E_{1,e}u)|_e = u$.
(viii) $(\partial_n E_{1,e}u)|_e = 0$.

Proof We modify the operator $E^{2D}_{1,e}$ of Lemma B.11 slightly and set

$$\begin{aligned}(E_{1,e}u)(x,y) &:= u(x)\frac{1}{(1-x^2)^2}(1-x-y)^2(1+x-y)^2\left(1+py+y\frac{4}{1-x^2}\right)(1-y)^p\\ &= u(x)\left(1-\frac{y}{1-x}\right)^2\left(1-\frac{y}{1+x}\right)^2\left(1+py+y\frac{4}{1-x^2}\right)(1-y)^p.\end{aligned}$$

The control of $|E_{1,e}u|_{H^j(\widehat{K}^{2D})}$ stated in (i), (ii) follows from Lemma B.6 by observing that $2y/(1-x^2) = y/(1-x)+y/(1+x)$ so that $E_{1,e}u = W_1u + pyW_2u$ with functions W_1, W_2 of the form studied in Lemma B.6. Likewise, the bounds given in (iii) on edges e' follow from Lemma B.6 and the special form $E_{1,e}u = W_1u + pyW_2u$. (In fact, the condition $p\ge j$ on the degree p is not completely sharp.) The properties (iv)–(vii) result from the factor $(1-y/(1-x))^2(1-y/(1+x))^2$. The property $(\partial_n E_{1,e}u)|_e = 0$ is a consequence of the factor $1+py+4y/(1-x^2)$. □

B.1.3 Liftings for the 3D case

We start with the vertex liftings:

Lemma B.13 (vertex liftings in 3D) *Fix $i\in\mathbb{N}_0$ and a vertex V of $\widehat{K}^{3D}$. Denote by e_1, e_2, e_3 the three edges meeting at V and by ∂_{e_k}, differentiation along e_k. Fix $(i_1,i_2,i_3)\in\mathbb{N}_0^3$ with $i_1+i_2+i_3\le i$. Then one can find, for every $p\ge i+1$, a polynomial $L_{V,(i_1,i_2,i_3),p}\in\mathscr{P}_{p+2i}$ with*

$$\begin{aligned}\partial_{e_1}^{j_1}\partial_{e_2}^{j_2}\partial_{e_3}^{j_3}L_{V,(i_1,i_2,i_3),p}(V) &= \delta_{i_1,j_1}\delta_{i_2,j_2}\delta_{i_3,j_3} \qquad \forall (j_1,j_2,j_3)\in\mathbb{N}_0^3 \text{ with } j_1+j_2+j_3\le i,\\ \nabla^j L_{V,(i_1,i_2,i_3),p}(V') &= 0 \qquad \forall 0\le j\le i, \quad \forall \text{ vertices } V'\ne V.\end{aligned}$$

Furthermore, $L_{V,(i_1,i_2,i_3),p}$ vanishes on the face opposite V. Additionally, for every $s\ge 0$, one has for a constant $C_s>0$ independent of p (but depending on s and i)

$$\|L_{V,(i_1,i_2,i_3),p}\|_{H^s(\widehat{K}^{3D})} \le C_s p^{-3/2+s-(i_1+i_2+i_3)}.$$

Proof The proof parallels that of the 2D-version detailed in Lemma B.10. It is convenient to work with the reference tetrahedron

$$\widetilde{K}^{3D} := \{(x,y,z)\,|\,0<x<1, 0<y<1-x, 0<z<1-x-y\}.$$

Let $L_{1,p} \in \mathscr{P}_{p+i}$ be the univariate polynomial given by Lemma B.5 with $L^{(j)}_{1,p}(0) = \delta_{j,0}$, $j = 0,\dots,i$ and $L^{(j)}_{1,p}(1) = 0$ for $j = 0,\dots,p-1$. Set

$$L_{V,(i_1,i_2,i_3),p}(x,y,z) := \frac{1}{i_1!}\frac{1}{i_2!}\frac{1}{i_3!}x^{i_1}y^{i_2}z^{i_3}L_{1,p}(x+y+z) \in \mathscr{P}_{p+i_1+i_2+i_3}.$$

Since $L_{1,p}(0) = 1$ and $L^{(j)}_{1,p}(0) = 0$ for $j = 1,\dots,i$ and $L^{(j)}_{1,p}(1) = 0$ for $j = 0,\dots,p-1 \geq i$, we see that $L_{V,p}$ has the desired properties in the vertices of $\widetilde{K}^{3D}$. To see the norm bounds, we consider a $(s_1,s_2,s_3) \in \mathbb{N}_0^3$ with $s_1+s_2+s_3 = s$. Then, by the product rule, $D^{(s_1,s_2,s_3)}L_{V,(i_1,i_2,i_3),p}$ consist of terms of the form

$$x^{i_1-k_1}y^{i_2-k_2}z^{i_3-k_3}L^{(s_1+s_2+s_3-k_1-k_2-k_3)}_{1,p}(x+y+z),$$

where $(k_1,k_2,k_3) \in \mathbb{N}_0^3$ is constrained to satisfy $0 \leq k_1 \leq \min\{i_1,s_1\}$, $0 \leq k_2 \leq \min\{i_2,s_2\}$, $0 \leq k_3 \leq \min\{i_3,s_3\}$. Hence, we have to bound

$$I_{k_1,k_2,k_3} := \int_{x=0}^{1} x^{2(i_1-k_1)}\int_{y=0}^{1-x} y^{2(i_2-k_2)}\int_{z=0}^{1-x-y} z^{2(i_3-k_3)}|L^{(s_1+s_2+s_3-k_1-k_2-k_3)}_{1,p}(x+y+z)|^2\,dz\,dy\,dx.$$

Abbreviating $s = s_1+s_2+s_3$ and $k = k_1+k_2+k_3$ we get with the aid of Lemma B.5

$$I_{k_1,k_2,k_3} \lesssim \sum_{j=0}^{i} p^{-1-2(i_3-k_3)+2(s-k)}\int_{x=0}^{1} x^{2(i_1-k_1)}\int_{y=0}^{1-x} y^{2(i_2-k_2)}(1-(x+y))^{2(p-(s-k+i_3-k_3))+1}\left((x+y)p\right)^{2j}.$$

For the innermost integral, we use the change of variables $y = (1-x)\eta$ and get in view of (B.27)

$$\begin{aligned}\int_{y=0}^{1-x} &= (1-x)^{2+2(i_2-k_2)+2(p-(s-k+i_3-k_3))}\int_{\eta=0}^{1}\eta^{2(i_2-k_2)}(1-\eta)^{2(p-(s-k+i_3-k_3))+1}\left((x+(1-x)\eta)p\right)^{2j}\\ &\lesssim (1-x)^{2+2(i_2-k_2)+2(p-(s-k+i_3-k_3))}p^{-2(i_2-k_2)-1}p^{2j}\left[x^{2j}+p^{-2j}\right].\end{aligned}$$

Thus, we get

$$\begin{aligned}I_{k_1,k_2,k_3} &\lesssim \sum_{j=0}^{i} p^{-1-2(i_3-k_3)+2(s-k)}p^{-1-2(i_2-k_2)}\int_{x=0}^{1} x^{2(i_1-k_1)}(1-x)^{2+2(i_2-k_2)+2(p-(s-k+i_3-k_3))}(px)^{2j}\\ &\lesssim \sum_{j=0}^{i} p^{-1-2(i_3-k_3)+2(s-k)}p^{-1-2(i_2-k_2)}p^{-1-2(i_1-k_1)} \lesssim p^{-3-2(i_1+i_2+i_3)+2s},\end{aligned}$$

which is the claimed estimate. □

Remark B.14 (vertex liftings matching to finite order) Let $s > 0$ and $q \in \mathbb{N}_0$ such that the embedding theorem $H^s(\widehat{K}^{3D}) \subset C^q(\overline{\widehat{K}})$ is valid. Define, for $p \geq q+1$, with the aid of the functions of $L_{V,(i_1,i_2,i_3),p}$ of Lemma B.13 the operator

$$E_V^{3D}u := \sum_{\alpha\in\mathbb{N}_0^3:|\alpha|\leq q}\frac{1}{\alpha!}D^\alpha u(V)L_{V,\alpha,p}.$$

Then $E_V^{3D}u \in \mathscr{P}_{p+2q}$. Furthermore $D^\beta(u-E_V^{3D}u)(V) = 0$ for all $|\beta| \leq q$ and $(D^\beta E_V^{3D}u)(V') = 0$ for all $|\beta| \leq q$ and vertices $V' \neq V$, and $E_V^{3D}u$ vanishes on the face opposite V. Additionally, for $t \geq 0$, we have

$$\|E_V^{3D}u\|_{H^t(\widehat{K}^{3D})} \leq C_t\sum_{|\alpha|\leq q}|D^\alpha u(V)|p^{-|\alpha|}p^{-3/2+t}. \tag{B.37}$$

For the following lemmas, we recall our notion of face normal derivative operator ∂_{n_f}: For a face f of $\widehat{K}^{3D}$ with boundary ∂f, we denote by $\partial_{n_f}v = n_f\cdot\nabla v$, where n_f is the vector of length 1 normal to ∂f in the plane spanned by f.

Lemma B.15 (edge trace lifting) *For each edge e of $\widehat{K}^{3D}$ denote by $f_{1,e}$, $f_{2,e}$ the two faces sharing e. There is a lifting operator $E^{3D}_{1,e} : H^3_0(e) \to H^3(\widehat{K}^{3D})$ with the following lifting properties:*

(i) $(E^{3D}_{1,e}u)|_e = u$.
(ii) $E^{3D}_{1,e}u$ *vanishes on all faces that do not have e as an edge.*
(iii) $E^{3D}_{1,e}u$ *as well as* $\nabla E^{3D}_{1,e}u$ *vanish on all edges except e.*
(iv) *For each of the two faces $f_{1,e}$, $f_{2,e}$, the face normal derivative of $E^{3D}_{1,e}u$ vanish on e, i.e.,* $(\partial_{n_{f_{i,e}}} E^{3D}_{1,e}u)|_e = 0$ *for* $i = 1, 2$.
(v) *If* $u \in \mathscr{P}_q \cap H^3_0(e)$*, then* $E^{3D}_{1,e}u \in \mathscr{P}_{q+p+1}$.

For each fixed $j \geq 0$, the following stability bounds are valid:

(vi) $\|E^{3D}_{1,e}u\|_{L^2(\widehat{K}^{3D})} \leq Cp^{-1}\|u\|_{L^2(e)}$.
(vii) *If* $u \in H^j_0(e)$*, then* $|E^{3D}_{1,e}u|_{H^j(\widehat{K}^{3D})} \leq Cp^{-1}\left[p^j\|u\|_{L^2(e)} + p^{j-1}|u|_{H^1(e)} + \cdots + |u|_{H^j(e)}\right]$.
(viii) *If* $u \in H^j_0(e)$*, then for the faces $f_{i,e}$, $i \in \{1,2\}$,*

$$|E^{3D}_{1,e}u|_{L^2(f_{i,e})} \leq Cp^{-1/2}\|u\|_{L^2(e)},$$
$$|E^{3D}_{1,e}u|_{H^j(f_{i,e})} \leq Cp^{-1/2}\left[p^j\|u\|_{L^2(e)} + p^{j-1}|u|_{H^1(e)} + \cdots + |u|_{H^j(e)}\right].$$

Proof Let $e = (-1,1) \times \{0\} \times \{0\}$. With the operator $E_{1,e}$ of Lemma B.12 define $E^{3D}_{1,e}$ by the formula

$$(E^{3D}_{1,e}u)(x,y,z) := (E_{1,e}u)(x,y+z).$$

The statements (i)–(iv), about where $E^{3D}_{1,e}u$ vanishes follows from the definition. The estimates (viii), follow from Lemma B.12.(ii) and the simple observation that $y = 0$ or $z = 0$ for the faces $f_{1,e}$, $f_{2,e}$. For the volume bounds (vi), (vii), we employ Lemma B.9 and arguments similar to those of the 2D case in Lemma B.6. □

Lemma B.16 (edge normal derivative lifting) *For each edge e of $\widehat{K}^{3D}$ denote by $f_{1,e}$ and $f_{2,e}$ the two faces that share the edge e. There is a lifting operator $E^{3D}_{2,e} : H^2_0(e) \to H^2(\widehat{K}^{3D})$ with the following properties:*

(i) $E^{3D}_{2,e}u$ *vanishes on* $\partial\widehat{K}^{3D} \setminus f_{1,e}$.
(ii) *The face normal derivative* $\partial_{n_{f_{1,e}}} E^{3D}_{2,e}u$ *satisfies* $\partial_{n_{f_{1,e}}}(E^{3D}_{2,e}u)|_e = u$ *and* $\partial_{n_{f_{1,e}}}(E^{3D}_{2,e}u)|_{\partial f_{1,e}\setminus e} = 0$.
(iii) $\|E^{3D}_{2,e}u\|_{L^2(\widehat{K}^{3D})} \leq Cp^{-2}\|u\|_{L^2(e)}$.
(iv) $|E^{3D}_{2,e}u|_{H^2(\widehat{K}^{3D})} \leq Cp^{-2}\left[p^2\|u\|_{L^2(e)} + p|u|_{H^1(e)} + |u|_{H^2(e)}\right]$.
(v) *For the face $f_{1,e}$, we have*

$$|E^{3D}_{2,e}u|_{L^2(f_{1,e})} \leq Cp^{-2+1/2}\|u\|_{L^2(e)},$$
$$|E^{3D}_{2,e}u|_{H^2(f_{1,e})} \leq Cp^{-2+1/2}\left[p^2\|u\|_{L^2(e)} + p|u|_{H^1(e)} + |u|_{H^2(e)}\right].$$

(vi) *If* $u \in \mathscr{P}_q \cap H^2_0(e)$*, then* $E_{2,e}u \in \mathscr{P}_{q+p+1}$.

Proof Let $e = (-1,1) \times \{0\} \times \{0\}$ and let $f_{1,e} = \{(x,y,z) \,|\, \partial\widehat{K}^{3D} \cap \{y = 0\}\}$. With the operator $E_{1,e}$ of Lemma B.11 define $E^{3D}_{2,e}$ by the formula

$$(E^{3D}_{2,e}u)(x,y,z) := y(E_{1,e}u)(x,y+z).$$

The statements (i), (ii) about where $E^{3D}_{2,e}u$ vanishes follows from the definition. The estimates (v) follow by reasoning as in the proof of Lemma B.11. In view of Lemma B.9, we see that we can proceed with analogous arguments as in the 2D case to get the volume bounds of (iii), (iv). □

We finally need a lifting from faces.

Lemma B.17 (face lifting) *For each face f of $\widehat{K}^{3D}$ there is a lifting operator $E^{3D}_f : H^2_0(f) \to H^2(\widehat{K}^{3D})$ with the following properties:*

(i) $(E_f^{3D}u)|_{\partial\widehat{K}^{3D}\setminus f} = 0$.
(ii) $(E_f^{3D}u)|_f = u$.
(iii) $\|E_f^{3D}u\|_{L^2(\widehat{K}^{3D})} \leq Cp^{-1/2}\|u\|_{L^2(f)}$.
(iv) $|E_f^{3D}u|_{H^2(\widehat{K}^{3D})} \leq Cp^{-1/2}\left[p^2\|u\|_{L^2(f)} + p|u|_{H^1(f)} + |u|_{H^2(f)}\right]$.
(v) *If* $u \in \mathscr{P}_q \cap H_0^2(f)$*, then* $E_f^{3D}u \in \mathscr{P}_{p+q}$.

Proof Let $f = \widehat{K}^{2D} \times \{0\}$. Define E_f^{3D} by

$$\begin{aligned}(E_f^{3D}u)(x,y,z) &:= \frac{u(x,y)}{(1-x-y)(1+x-y)}(1-x-y-z)(1+x-y-z)(1-z)^p \\ &= u(x,y)\left(1-\frac{z}{1-x-y}\right)\left(1-\frac{z}{1+x-y}\right)(1-z)^p.\end{aligned}$$

We focus on the bounds for the second derivatives of $E_f^{3D}u$. We note that E_f^{3D} has the form

$$(E_f^{3D}u)(x,y,z) = u(x,y)w(x,y,z/(1-x-y),z/(1+x-y),z)(1-z)^p \tag{B.38}$$

for a smooth function w. Arguing as in the proof of Lemma B.6, we see that for multiindices $\beta \in \mathbb{N}_0^3$, $|\beta| \leq 2$ we have by the smoothness of w and that fact that $|z/(1-x-y)| \leq 1$ as well as $|z/(1+x-y)| \leq 1$ on $\widehat{K}^{3D}$

$$|D^\beta w(x,y,z/(1-x-y),z/(1+x-y),z)| \leq C\left[\left(\frac{1}{1-x-y}\right)^{|\beta|} + \left(\frac{1}{1+x-y}\right)^{|\beta|}\right]. \tag{B.39}$$

With the product rule we get with the abbreviation $d(x,y) := \operatorname{dist}((x,y),\partial\widehat{K}^{2D})$

$$|D^\beta(w(1-z)^p)| \leq C\left(\frac{1}{d}+p\right)^{|\beta|}(1-z)^{p-|\beta|}. \tag{B.40}$$

As in the proof of Lemma B.6, we abbreviate D^1u and D^2u for the sum of all derivatives of order 1 and 2. From (B.38) we obtain with the product rule for differentiation and (B.40)

$$\begin{aligned}\left|E_f^{3D}u\right|_{H^2(\widehat{K}^{3D})} &\leq C\sum_{\ell=0}^{2}\left\|\left(D^{2-\ell}u\right)\left(\frac{1}{d}+p\right)^{\ell}(1-z)^{p-\ell}\right\|_{L^2(\widehat{K}^{3D})} \\ &\leq C\sum_{\ell=0}^{2}\left(\left\|d^{-\ell}\left(D^{2-\ell}u\right)(1-z)^{p-\ell}\right\|_{L^2(\widehat{K}^{3D})} + p^{\ell}\left\|\left(D^{2-\ell}u\right)(1-z)^{p-\ell}\right\|_{L^2(\widehat{K}^{3D})}\right) \\ &\leq Cp^{-1/2}\sum_{\ell=0}^{2}\left(\left\|d^{-\ell}D^{2-\ell}u\right\|_{L^2(\widehat{K}^{2D})} + p^{\ell}|u|_{H^{2-\ell}(\widehat{K}^{2D})}\right) \\ &\overset{([20,\text{ Thm. } 1.4.4.4])}{\leq} Cp^{-1/2}\sum_{\ell=0}^{2}p^{2-\ell}|u|_{H^{\ell}(\widehat{K}^{2D})},\end{aligned}$$

which concludes the proof. □

C 2D Proof

Theorem C.1 *Let $\widehat{K}^{2D} \subset \mathbb{R}^2$ be the reference triangle. Let $s > 5/2$. Then there exists $C > 0$ (depending only on s and d) and for every p a linear operator $\pi : H^s(\widehat{K}) \to \mathscr{P}_{2p}$ that permits an element-by-element construction in the sense of Definition B.1 such that*

$$p^2\|u-\pi(u)\|_{L^2(\widehat{K})} + p\|u-\pi(u)\|_{H^1(\widehat{K})} + \|u-\pi(u)\|_{H^2(\widehat{K})} \leq Cp^{-(s-2)}|u|_{H^s(\widehat{K})} \qquad \forall p \geq s-1. \tag{C.1}$$

Furthermore, the following trace estimates are valid:

$$p\|u-\pi(u)\|_{L^2(\partial\widehat{K})} + \|\nabla(u-\pi(u))\|_{L^2(\partial\widehat{K})} \leq Cp^{-(s-3/2)}|u|_{H^s(\widehat{K})} \qquad p \geq s-1.$$

Proof We will construct $\pi(u)$ for a given u—inspection of the construction shows that $u \mapsto \pi(u)$ is in fact a linear operator.

Before constructing $\pi(u)$, we comment on the requirement $p \geq 7$. This condition is sufficient to ensure that the feasible set for the minimization procedure in Definition B.1 is not empty. Indeed, we have to find polynomials that coincide with u in the vertices to order 2. This can be ensured with the vertex liftings of Lemma B.10 with $i = 2$ and $p \geq i+1 = 3$ there, which results in the polynomial degree $p+2i = 7$.

We start by observing that $u|_e \in H^{s-1/2}(e)$ for each edge. It can then be shown that the polynomial $\pi_e \in \mathscr{P}_p$ defined by the minimization process of Definition B.1 satisfies

$$p^2\|u-\pi_e\|_{L^2(e)} + p\|u-\pi_e\|_{H^1(e)} + |u-\pi_e|_{H^2(e)} \leq Cp^2 p^{s-1/2}\|u\|_{H^s(\widehat{K})}. \tag{C.2}$$

(See the proof of [39, Thm. B.4] for similar arguments.) By [39, Lemma B.3] (and a simple modification) we have a $\pi^1 \in \mathscr{P}_p$ such that for $p \geq s-1$ there holds

$$\|u-\pi^1\|_{H^t(\widehat{K})} \leq Cp^{-(s-t)}|u|_{H^s(\widehat{K})}, \qquad 0 \leq t \leq s, \tag{C.3}$$

$$\|u-\pi^1\|_{H^t(e)} \leq Cp^{-(s-t-1/2)}|u|_{H^s(\widehat{K})}, \qquad \forall \text{ edges } e, \quad 0 \leq t \leq s-1/2, \tag{C.4}$$

$$\|u-\pi^1\|_{L^\infty(\widehat{K})} \leq Cp^{-(s-1)}|u|_{H^s(\widehat{K})}, \tag{C.5}$$

$$\|\nabla(u-\pi^1)\|_{L^\infty(\widehat{K})} \leq Cp^{-(s-2)}|u|_{H^s(\widehat{K})}. \tag{C.6}$$

From (C.5), (C.6) and the vertex-lifting properties given in Lemma B.10, we may adjust π^1 by vertex liftings to obtain a polynomial π^2 satisfying (C.3)–(C.6) and additionally that $u-\pi^1$ as well as $\nabla(u-\pi^1)$ vanishes in the vertices of $\widehat{K}$. The resulting polynomial π^2 satisfies the constraints in (ii) of Definition B.1 and has the following approximation properties:

$$p^2\|u-\pi^2\|_{L^2(\widehat{K})} + p\|u-\pi^2\|_{H^1(\widehat{K})} + \|u-\pi^2\|_{H^2(\widehat{K})} \leq Cp^{-(s-2)}\|u\|_{H^s(\widehat{K})}, \tag{C.7}$$

$$p^2\|u-\pi^2\|_{L^2(e)} + p\|u-\pi^2\|_{H^1(e)} + \|u-\pi^2\|_{H^2(e)} \leq Cp^{-(s-2-1/2)}\|u\|_{H^s(\widehat{K})} \quad \forall e. \tag{C.8}$$

We next adjust π^2 so as to match the minimizer π_e for edge e. We note that $u-\pi_e \in H^2_0(e)$ as well as $u-\pi^2 \in H^2_0(e)$ so that $\pi_e - \pi^2 \in H^2_0(e)$. The triangle inequality and (C.2), (C.7) yield

$$p^2\|\pi^2-\pi_e\|_{L^2(e)} + p\|\pi^2-\pi_e\|_{H^1(e)} + \|\pi^2-\pi_e\|_{H^2(e)} \leq Cp^{-(s-2-1/2)}\|u\|_{H^s(\widehat{K})} \tag{C.9}$$

The edge lifting of Lemma B.11 (with $j = 2$ there) yields an approximation $E_e(\pi^2-\pi_e) \in \mathscr{P}_{2p}$ that vanishes on $\partial\widehat{K} \setminus e$, coincides with $\pi^2 - \pi_e$ on e, and satisfies

$$p^2\|E_e(\pi^2-\pi_e)\|_{L^2(\widehat{K})} + p\|E_e(\pi^2-\pi_e)\|_{H^1(\widehat{K})} + \|E_e(\pi^2-\pi_e)\|_{H^2(\widehat{K})} \leq Cp^{-(s-2)}\|u\|_{H^s(\widehat{K})}.$$

Using such a correction for all edges, we arrive at an approximation $\widetilde{\pi}u$ of u that satisfies the conditions for element-by-element construction stipulated in Definition B.1 and

$$p^2\|u-\widetilde{\pi}u\|_{L^2(\widehat{K})} + p\|u-\widetilde{\pi}u\|_{H^1(\widehat{K})} + \|u-\widetilde{\pi}u\|_{H^2(\widehat{K})} \leq Cp^{-(s-2)}\|u\|_{H^s(\widehat{K})}.$$

This operator π maps into the polynomial space $\widetilde{P}_p := \{u \in \mathscr{P}_{2p} \,|\, u|_e \in \mathscr{P}_p \text{ for each edge } e\}$. We take πu of the form $\pi u = \widetilde{\pi}u + z$, where $z \in \widetilde{\mathscr{P}_p} \cap H^1_0(\widehat{K})$ minimizes

$$p^2\|(u-\widetilde{\pi}u)-z\|_{L^2(\widehat{K})} + \|(u-\widetilde{\pi}u)-z\|_{H^2(\widehat{K})},$$

which determines π. The construction shows that π is a linear operator that is furthermore a projection onto $\widetilde{\mathscr{P}}_p$. This property allows us to replace the full norm $\|u\|_{H^s(\widehat{K})}$ by the seminorm $|u|_{H^s(\widehat{K})}$ by a classical compactness argument (known as Deny-Lions lemma). Finally, estimates for the error on the boundary follow directly from trace estimates. $\square$

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