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An analysis of discretizations of the Helmholtz equation in L^2 and in negative norms (extended version)

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An analysis of discretizations of the Helmholtz equation in L^2 and in negative norms (extended version)

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August 14, 2012

Abstract

For a model Helmholtz problem at high wavenumber we present a wavenumber-explicit error analysis in weak norms such as L^2, H^{-1}. In 1D, we analyze the convergence behavior of the lowest order optimally blended spectral-finite element scheme of [Ainsworth & Wajid, SIAM J. Numer. Anal. (2010)].

1 Introduction

The Helmholtz equation is a basic equation when treating wave propagation problems in a time-harmonic setting. Typical applications include acoustic and electromagnetic scattering problems as well as laser physics. A particular case of interest is that of high wavenumbers k. Then the conditions on the discretization are stringent due to the requirement to resolve the oscillatory nature of the solution. More subtle is that classical discretizations of the Helmholtz equation suffer strongly from dispersion errors. The stability and dispersive properties of discretizations of the Helmholtz equation are by now understood: for regular, translation-invariant grids we refer to [1–3, 6, 9, 11, 13]—see also the discussion in [10]. One of the outcomes of these analyses on regular grids is in particular that high order methods are significantly less susceptible to dispersion errors ("pollution errors") than low order methods. The works [10, 16–18] rigorously establish this observation also for discretizations on unstructured meshes. The present note concentrates on two aspects. The starting point for the first aspect is that the analysis of [10, 16–18] is performed in the H^1-like norm $\|\cdot\|_{\mathcal{H}}$ of (2.3). Here, we focus on weaker norms, namely, the convergence in the L^2-norm, the H^{-1}-norm as well as the convergence of linear functionals that are generated by smooth weighted volume integrals. While the asymptotic convergence rates are, of course, the ones to be expected from the underlying duality arguments, the novel aspect of the present paper is that we are able to extract for our model problem good estimates in the wavenumber k; numerical examples show that in favorable situations our estimates are indeed sharp in k. The second aspect covered in this paper is related to the above mentioned dispersion analysis on translation invariant meshes. Such regular meshes permit the use of powerful tools such as Fourier techniques to understand and analyze discretizations and to design new schemes with good dispersion properties. The recent proposal of [4] shows in particular that a suitable combination of the Galerkin FEM and the spectral element method (SEM) can lead to new methods with significantly reduced dispersion errors. This analysis is done on regular, translation-invariant grids and suggests greatly reduced actual errors. For a 1D model problem on regular grids, we provide an actual error analysis for the lowest order discretization and show that the greatly reduced dispersion error leads to a gain in accuracy by a factor k as compared with the Galerkin FEM.

2 The Helmholtz model problem

We consider a specific model problem on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ with Robin boundary conditions:

$$\begin{aligned} -\Delta u - k^2 u = f \quad &\text{in} \quad \Omega \\ \partial_n u - iku = g \quad &\text{on} \quad \partial\Omega \end{aligned} \tag{2.1}$$

where $k \geq k_0 > 0$.

The weak formulation of our model has the form:

$$\text{Find } u \in H^1(\Omega) \text{ s.t.} \quad B(u,v) = l(v) \quad \forall v \in H^1(\Omega) \tag{2.2}$$

$$B(u,v) := \int_\Omega \nabla u \cdot \nabla \bar{v} - k^2 u\bar{v} - ik \int_{\partial\Omega} u\bar{v}, \qquad l(v) := \int_\Omega f\bar{v} + \int_{\partial\Omega} g\bar{v}.$$

An important role is played by the norm

$$\|u\|_{\mathcal{H}}^2 := \|\nabla u\|_{L^2(\Omega)}^2 + k^2 \|u\|_{L^2(\Omega)}^2. \tag{2.3}$$

In this norm, the bilinear form $B(\cdot,\cdot)$ is continuous uniformly in $k \geq k_0 > 0$, i.e., there exists $C_c > 0$ independent of k such that

$$|B(u,v)| \leq C_c \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}} \qquad \forall u,v \in H^1(\Omega)$$

(see, e.g., [16, Cor. 3.4]). The bilinear form B also satisfies the Gårding inequality

$$\operatorname{Re} B(u,u) = \|\nabla u\|_{L^2(\Omega)}^2 - k^2 \|u\|_{L^2(\Omega)}^2 \qquad \forall u \in H^1(\Omega).$$

The abstract conforming discretization is as follows: given a closed space $V_N \subset H^1(\Omega)$, the finite element approximation $u_N \in V_N$ is given by the condition

$$B(u_N, v) = l(v) \qquad \forall v \in V_N \tag{2.4}$$

With these observations in hand, [17, Thm. 3.2] formulated the following result:

Proposition 2.1 ([17, Thm. 3.2]). *Define the adjoint solution operator $S^* : L^2(\Omega) \to H^1(\Omega)$ by the condition*

$$B(v, S^* f) = \int_\Omega v\overline{f} \qquad \forall v \in H^1(\Omega)$$

and the adjoint approximation property η_N by

$$\eta_N^{L^2} := \sup_{0 \neq f \in L^2(\Omega)} \inf_{v \in V_N} \frac{\|S^* f - v\|_{\mathcal{H}}}{\|f\|_{L^2(\Omega)}}. \tag{2.5}$$

If the condition

$$2C_c k \eta_N^{L^2} \leq 1 \tag{2.6}$$

is fulfilled, then the Galerkin approximation $u_N \in V_N$ defined by (2.4) exists and is unique. Furthermore, the following two a priori estimates are valid:

$$\|u - u_N\|_{\mathcal{H}} \leq 2C_c \inf_{v \in V_N} \|u - v\|_{\mathcal{H}}, \tag{2.7}$$

$$\|u - u_N\|_{L^2(\Omega)} \leq C_c \eta_N^{L^2} \inf_{v \in V_N} \|u - v\|_{\mathcal{H}}. \tag{2.8}$$

Proposition 2.1 gives abstract conditions and estimates for the Galerkin error in the $\|\cdot\|_{\mathcal{H}}$-norm and the $L^2(\Omega)$; we will make these estimates more specific in the context of the *hp*-FEM below. We will also give estimates in the $(H^1(\Omega))'$-norm and estimates for the evaluation of linear functionals. For *a priori* estimates in the $(H^1(\Omega))'$-norm and $H^{-1}(\Omega)$-norm (we set $H^{-1}(\Omega) = (H_0^1(\Omega))'$), we have the following abstract result, which we will quantify below:

Lemma 2.2. *Define*

$$\eta_N^{H^1} := \sup_{f \in H^1(\Omega)} \inf_{v \in V_N} \frac{\|S^* f - v\|_{\mathcal{H}}}{\|f\|_{H^1(\Omega)}}, \qquad \eta_N^{H_0^1} := \sup_{f \in H_0^1(\Omega)} \inf_{v \in V_N} \frac{\|S^* f - v\|_{\mathcal{H}}}{\|f\|_{H^1(\Omega)}}, \tag{2.9}$$

If the solvability condition (2.6) is satisfied then the Galerkin error $u - u_N$ satisfies

$$\begin{aligned} \|u - u_N\|_{(H^1(\Omega))'} &\leq 2C_c^2 \eta_N^{H^1} \inf_{v \in V_N} \|u - v\|_{\mathcal{H}} \\ \|u - u_N\|_{H^{-1}(\Omega)} &\leq 2C_c^2 \eta_N^{H_0^1} \inf_{v \in V_N} \|u - v\|_{\mathcal{H}}. \end{aligned}$$

Proof. We will just prove the estimate for $\|u - u_N\|_{(H^1(\Omega))'}$ using a duality argument and Galerkin orthogonality: For arbitrary $v \in H^1(\Omega)$ and $w_N \in V_N$ we have

$$\left|(u - u_N, v)_{L^2(\Omega)}\right| = |B(u - u_N, S^* v)| = |B(u - u_N, S^* v - w_N)| \leq C_c \|u - u_N\|_{\mathcal{H}} \|S^* v - w_N\|_{\mathcal{H}}.$$

Since w_N is arbitrary, we can conclude

$$\left|(u - u_N, v)_{L^2(\Omega)}\right| \leq C_c \|u - u_N\|_{\mathcal{H}} \|v\|_{H^1(\Omega)} \eta_N^{H^1}$$

Dividing by $\|v\|_{H^1(\Omega)}$, taking the supremum over $v \in H^1(\Omega)$, and inserting the best approximation result (2.7) yields the result. □

We will also be interested in the error in linear functionals. Specifically, we will consider linear functionals of the form

$$v \mapsto L(v) := \int_\Omega \overline{z} v \tag{2.10}$$

where the function $z \in L^2(\Omega)$ (or even smoother). The *a priori* analysis for the error $L(u - u_N)$ is done, as usual, by duality arguments:

Lemma 2.3. *Let $z \in L^2(\Omega)$ and L be given by (2.10). Assume that (2.6) holds. Then*

$$|L(u) - L(u_N)| = |L(u - u_N)| \leq C_c \left(\inf_{v \in V_N} \|u - v\|_{\mathcal{H}} \right) \left(\inf_{w \in V_N} \|S^* z - w\|_{\mathcal{H}} \right).$$

Proof. Follows from arguments very similar to those of the proof Lemma 2.2 □

3 Regularity

The above considerations show that we have to quantify the adjoint approximation properties $\eta_N^{L^2}$, $\eta_N^{H^1}$. This leads to the study of the regularity of the solution operator and the adjoint operator. In this connection, it is worth noting that $S^* f$ is also the solution of a Helmholtz problem; in fact, $S^* f = \overline{S(\overline{f}, 0)}$, where $(f, g) \mapsto S(f, g)$ denotes the solution operator for (2.1). This shows that the regularity properties of S^* can be inferred from those of S.

3.1 Prelude: the 1D situation

Several of the regularity issues for (2.1) can be already be seen in 1D. An important motivation for us to discuss the 1D situation in some detail is that we will study numerically the 1D situation below and will therefore need the regularity assertion given here. As the 1D model problem, we consider the following situation studied already in [11–13]:

$$-u'' - k^2 u = f \quad \text{in } I = (0,1), \qquad u(0) = 0, \qquad u'(1) - \mathbf{i}ku(1) = g \in \mathbb{C} \tag{3.1}$$

The Green's function is known explicitly, namely,

$$G(x,y) = \frac{1}{k}\begin{cases} \sin(kx)e^{\mathbf{i}ky} & 0 \le x \le y \le 1 \\ \sin(ky)e^{\mathbf{i}kx} & 0 \le y \le x \le 1 \end{cases} \tag{3.2}$$

so that the solution can be written as

$$u(x) = \int_0^1 G(x,y)f(y)\,dy + g\frac{\sin kx}{k(\cos k - \mathbf{i}\sin k)} \tag{3.3}$$

One has the stability estimate (see, e.g., [11, Thm. 4.4])

$$\|u\|_{\mathcal{H}} \le C\left[\|f\|_{L^2(I)} + |g|\right]. \tag{3.4}$$

For smooth f, the solution formula (3.3) is an oscillatory integral (for large k) so that integration by parts is expected to give an additional power of k^{-1}. The following lemma asserts the validity of this expectation. Instead of working with the solution formula, we prove it using arguments that will also be used in the multi-d case:

Lemma 3.1. *The solution u of (3.1) satisfies, for a constant C independent of k, f, and g,*

$$\|u\|_{\mathcal{H}} \le C\left[k^{-1}\|f\|_{H^1(I)} + |g|\right]. \tag{3.5}$$

Proof. We may restrict our attention to the case $g = 0$. Define the function $u_0(x) := -k^{-2}f(x) + k^{-2}f(0)\cos kx$. Then $\|u_0\|_{\mathcal{H}} \le Ck^{-1}\|f\|_{H^1(I)}$. The difference $\delta := u - u_0$ satisfies

$$-\delta'' - k^2\delta = -k^{-2}f'',$$
$$\delta(0) = 0, \qquad \delta'(1) - \mathbf{i}k\delta(1) = -(-k^{-2}f'(1) - k^{-1}f(0)\sin k) + \mathbf{i}k(-k^{-2}f(1) + k^{-2}f(0)\cos k).$$

Applying now the stability estimate (3.4) and the Sobolev embedding theorem gives

$$\|\delta\|_{\mathcal{H}} \le C\left[k^{-2}\|f''\|_{L^2(I)} + k^{-2}|f'(1)| + k^{-1}|f(1)| + k^{-1}|f(0)|\right] \le C\left[k^{-2}\|f''\|_{L^2(I)} + k^{-1}\|f\|_{H^1(I)}\right].$$

Hence, we have obtained

$$\|u\|_{\mathcal{H}} \le \|u_0\|_{\mathcal{H}} + \|\delta\|_{\mathcal{H}} \le C\left[k^{-2}\|f\|_{H^2(I)} + k^{-1}\|f\|_{H^1(I)}\right].$$

The term $k^{-2}\|f\|_{H^2(I)}$ can be reduced to a term of the form $k^{-1}\|f\|_{H^1(I)}$ by interpolation arguments as worked out in the proof of Lemma 3.4 below. ☐

Remark 3.2. We note that the term involving $|g|$ in (3.5) is not improved by a factor k^{-1} as compared with (3.4). Inspection of the solution formula (3.3) shows that its k-dependence is sharp. Thus, better estimates (with respect to k) can only be expected for the case of homogeneous boundary conditions. See also [11, Sec. 4.7.2]. ■

Concerning the regularity of the solution u of (3.1) we have:

Proposition 3.3. *Let $s \in \mathbb{N}_0$. Then there exist constants C, $\lambda > 0$ such that the following is true. For every $f \in H^s(I)$ and $g \in \mathbb{C}$ the solution u of (3.1) can be written as $u = u_{H^{s+2}} + u_{\mathcal{A}}$ where $u_{H^{s+2}} \in H^{s+2}(I)$ and $u_{\mathcal{A}}$ is analytic. Additionally,*

$$\begin{aligned} k^{s+2}\|u_{H^{s+2}}\|_{L^2(I)} + \|u_{H^{s+2}}\|_{H^{s+2}(I)} &\leq C\|f\|_{H^s(I)}, \\ \|u_{\mathcal{A}}\|_{\mathcal{H}} &\leq C\left[\|f\|_{L^2(I)} + |g|\right] \\ \|u_{\mathcal{A}}^{(n+2)}\|_{L^2(I)} &\leq C\lambda^n k^{-1}\max\{k,n\}^{n+2}\left[\|f\|_{L^2(I)} + |g|\right] \qquad \forall n \in \mathbb{N}_0. \end{aligned}$$

Proof. Follows by arguing as in the proof of [18, Thm. 4.5] and the appropriate modifications for the Dirichlet boundary conditions at $x = 0$ ([18, Thm. 4.5] considers (2.1) with Robin boundary conditions). ☐

3.2 Regularity in higher dimensions

3.2.1 Stability

The bilinear form B is not coercive in $H^1(\Omega)$. Thus the Lax-Milgram-Lemma is not applicable to show existence and uniqueness. Nevertheless, the bilinear form B satisfies a Gårding inequality, i.e., it has the form "coercive + compact perturbation", which makes the Fredholm theory applicable, and solvability follows from uniqueness. This was shown in [14, Prop. 8.1.3], that is, for every $f \in (H^1(\Omega))'$ and every $g \in H^{-1/2}(\partial\Omega)$, the variational problem (2.1) is uniquely solvable with the stability bound

$$\|u\|_{\mathcal{H}} \leq C(k)\left[\|f\|_{(H^1(\Omega))'} + \|g\|_{H^{-1/2}(\partial\Omega)}\right]$$

for a constant $C(k) > 0$, whose dependence on k is unspecified. For *convex* domains Ω, [14, Prop. 8.1.4] (for $d = 2$) and [8] (for $d = 3$) established the k-explicit stability bound

$$k^{-1}\|u\|_{H^2(\Omega)} + \|u\|_{\mathcal{H}} \quad \leq \quad C\left[\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right]$$

for a $C > 0$ that is independent of k. This motivates us to introduce the stability constant $C_{sol}(k)$ as follows: For the bounded Lipschitz domain Ω (not necessarily convex) and $k > 0$ we let $C_{sol}(k) > 0$ be the least constant such that for all $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ the solution u of (2.1) satisfies

$$\|u\|_{\mathcal{H}} \leq C_{sol}(k)\left[\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right]. \tag{3.6}$$

We mention that we are particularly interested in the case that $C_{sol}(k)$ is polynomially bounded, i.e.,

$$C_{sol}(k) \leq \widetilde{C}_{sol} k^{\theta} \tag{3.7}$$

for some $\widetilde{C}_{sol}$, $\theta \geq 0$ independent of k. As mentioned above, for convex Ω we have $\theta = 0$ whereas for general Lipschitz domain Ω we have $\theta = 5/2$ by [10, Thm. 2.4].

The stability bound (3.6) merely requires $f \in L^2(\Omega)$. As in the 1D situation, it is possible to obtain a better k-dependence by exploiting additional regularity of the data f. The following result shows this for the multi-dimensional case:

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$ be a bounded Lipschitz domain. Let $C_{sol}(k)$ be given by (3.6). Let $g = 0$. Then there exists $C > 0$ independent of f and k such that*

$$\|u\|_{\mathcal{H}} \leq Ck^{-1}(1 + C_{sol}(k))\|f\|_{H^1(\Omega)}.$$

Proof. Assume first $f \in H^2(\Omega)$. Define the function $u_0 := -k^{-2}f$. Then $\|u_0\|_{\mathcal{H}} \leq Ck^{-1}\|f\|_{L^2(\Omega)} + k^{-2}\|f\|_{H^1(\Omega)}$. Then the function $\delta := u - u_0$ satisfies

$$\begin{aligned} -\Delta\delta - k^2\delta &= f - (-\Delta u_0 - k^2 u_0) = +k^{-2}\Delta f, \qquad \text{in } \Omega \\ \partial_n\delta - \mathbf{i}k\delta &= 0 - (\partial_n u_0 - \mathbf{i}ku_0) \end{aligned}$$

By stability and generous trace estimates we have

$$\begin{aligned}\|\delta\|_{\mathcal{H}} &\leq C_{sol}(k)\left[k^{-2}\|\Delta f\|_{L^2(\Omega)} + k^{-2}\|\partial_n f\|_{L^2(\partial\Omega)} + k^{-1}\|f\|_{L^2(\partial\Omega)}\right]\\ &\leq CC_{sol}(k)\left[k^{-2}\|f\|_{H^2(\Omega)} + k^{-1}\|f\|_{H^1(\Omega)}\right]\end{aligned}$$

and conclude from the triangle inequality $\|u\|_{\mathcal{H}} \leq \|u_0\|_{\mathcal{H}} + \|\delta\|_{\mathcal{H}}$

$$\|u\|_{\mathcal{H}} \leq C\left[k^{-2}\|f\|_{\mathcal{H}} + C_{sol}(k)(k^{-2}\|f\|_{H^2(\Omega)} + k^{-1}\|f\|_{H^1(\Omega)})\right]. \tag{3.8}$$

In order to lower the regularity requirement for f from H^2 to H^1, we employ an interpolation argument. Recognizing that $H^1(\Omega)$ is the interpolation space $H^1(\Omega) = (L^2(\Omega), H^2(\Omega))_{1/2,2}$, we can write, for every $t > 0$, the function $f \in H^1(\Omega)$ as

$$f = (f - f_{H^2}) + f_{H^2},$$

where $f_{H^2} \in H^2(\Omega)$ and the following estimates are true (see [7] for details):

$$\begin{aligned}\|f - f_{H^2}\|_{L^2(\Omega)} + t\|f_{H^2}\|_{H^2(\Omega)} &\leq Ct^{1/2}\|f\|_{H^1(\Omega)},\\ \|f_{H^2}\|_{H^1(\Omega)} &\leq C\|f\|_{H^1(\Omega)}.\end{aligned}$$

Selecting $t = k^{-2}$, we arrive at

$$\|f - f_{H^2}\|_{L^2(\Omega)} \leq k^{-1}\|f\|_{H^1(\Omega)}, \qquad \|f_{H^2}\|_{H^1(\Omega)} + k^{-1}\|f_{H^2}\|_{H^2(\Omega)} \leq C\|f\|_{H^1(\Omega)},$$

We write $u = u_1 + u_2$, where u_1 and u_2 solve

$$\left\{\begin{array}{rcll} -\Delta u_1 - k^2 u_1 &=& f - f_{H^2} & \text{in } \Omega\\ \partial_n u_1 - \mathbf{i}ku_1 &=& 0 & \text{on } \partial\Omega\end{array}\right. \qquad \left\{\begin{array}{rcll} -\Delta u_2 - k^2 u_2 &=& f_{H^2} & \text{in } \Omega\\ \partial_n u_2 - \mathbf{i}ku_2 &=& 0 & \text{on } \partial\Omega\end{array}\right.$$

We conclude from (3.6) for u_1 and from (3.8) for u_2 that

$$\begin{aligned}\|u\|_{\mathcal{H}} &\leq \|u_1\|_{\mathcal{H}} + \|u_2\|_{\mathcal{H}}\\ &\leq C_{sol}(k)\|f - f_{H^2}\|_{L^2(\Omega)} + C\left[k^{-2}\|f_{H^2}\|_{\mathcal{H}} + C_{sol}(k)(k^{-2}\|f_{H^2}\|_{H^2(\Omega)} + k^{-1}\|f_{H^2}\|_{H^1(\Omega)})\right]\\ &\leq C(1 + C_{sol}(k))k^{-1}\|f\|_{H^1(\Omega)}.\end{aligned}$$

□

3.2.2 Regularity by decomposition

A key step in the arguments of [17] and likewise in [10,16,18] is decompose the solution of (2.1) into a part with finite regularity but k-independent bounds and an analytic part with k-explicit control over all derivatives. We cite from [18] the following version:

Proposition 3.5 ([18, Thm. 4.5])**.** *Let $\Omega \in \mathbb{R}^d$, $d \in \{2,3\}$ be a bounded Lipschitz domain. Assume additionally that Ω has an analytic boundary. Let $C_{sol}(k)$ be given by (3.6). Fix $s \in \mathbb{N}_0$. Then there exist constants C, $\lambda > 0$ independent of $k \geq k_0 > 0$ such that for every $f \in H^s(\Omega)$ and $g \in H^{s+1/2}(\partial\Omega)$ the solution $u = S(f,g)$ of the Helmholtz problem (2.1) can be written as $u = u_{H^{s+2}} + u_{\mathcal{A}}$, where, for all $n \in \mathbb{N}_0$,*

$$\|u_{\mathcal{A}}\|_{\mathcal{H},\Omega} \leq CC_{sol}(k)\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}\right) \tag{3.9}$$

$$\|\nabla^{n+2}u_{\mathcal{A}}\|_{L^2(\Omega)} \leq C\lambda^n k^{-1}C_{sol}(k)\max\{n,k\}^{n+2}\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}\right) \tag{3.10}$$

$$\|u_{H^{s+2}}\|_{H^{s+2}(\Omega)} + k^{s+2}\|u_{H^{s+2}}\|_{L^2(\Omega)} \leq C\left(\|f\|_{H^s(\Omega)} + \|g\|_{H^{s+1/2}(\partial\Omega)}\right). \tag{3.11}$$

As we have seen in the 1D case, it is possible to improve the estimates by one power of k for the special case of homogeneous boundary conditions and some additional regularity of the right-hand side f. This extends to the multi-dimensional case:

Theorem 3.6. *Let $\Omega \in \mathbb{R}^d$, $d \in \{2,3\}$ be a bounded Lipschitz domain. Assume additionally that Ω has an analytic boundary. Fix $s \in \mathbb{N}$. Then there exist constants C, $\lambda > 0$ independent of $k \geq k_0 > 0$ such that for every $f \in H^s(\Omega)$ the solution u of (2.1) with $g = 0$ can be written as $u = u_{H^{s+2}} + u_{\mathcal{A}}$, where, for all $n \in \mathbb{N}_0$,*

$$\|u_{\mathcal{A}}\|_{\mathcal{H},\Omega} \leq Ck^{-1}(1 + C_{sol}(k))\|f\|_{H^1(\Omega)} \tag{3.12}$$

$$\|\nabla^{n+2} u_{\mathcal{A}}\|_{L^2(\Omega)} \leq C\lambda^n k^{-2}(1 + C_{sol}(k)) \max\{n, k\}^{n+2}\|f\|_{H^1(\Omega)} \tag{3.13}$$

$$\|u_{H^{s+2}}\|_{H^{s+2}(\Omega)} + k^{s+2}\|u_{H^{s+2}}\|_{L^2(\Omega)} \leq C\|f\|_{H^s(\Omega)}. \tag{3.14}$$

Proof. We decompose the data f: Using the operators L_Ω and H_Ω of [17, (4.1b)], we can write

$$f = L_\Omega f + H_\Omega f =: f_L + f_H,$$

where, by [17, Lemma 4.2, Lemma 4.3], we have for some C, $\eta > 0$ independent of k the bounds

$$\begin{aligned} \|f_H\|_{H^{s_1}(\Omega)} &\leq Ck^{s_1 - s_2}\|f\|_{H^{s_2}(\Omega)}, && 0 \leq s_1 \leq s_2 \leq s, \\ \|\nabla^p f_L\|_{L^2(\Omega)} &\leq C(\eta k)^p \|f\|_{L^2(\Omega)} && \forall p \in \mathbb{N}_0, \\ \|\nabla^p f_L\|_{L^2(\Omega)} &\leq C(\eta k)^{p-s}\|f\|_{H^s(\Omega)} && \forall p \geq s. \end{aligned}$$

We denote by u_L and u_H the solutions to (2.1) with right-hand sides f_L and f_H, respectively. For u_H, we have $f_H \in H^s(\Omega)$ together with $\|f_H\|_{L^2(\Omega)} \leq Ck^{-1}\|f\|_{H^1(\Omega)}$. By Proposition 3.5, we may write $u_H = u_{H^{s+2}} + \widetilde{u}_{\mathcal{A}}$ with

$$\begin{aligned} k^{s+2}\|u_{H^{s+2}}\|_{L^2(\Omega)} + \|u_{H^{s+2}}\|_{H^{s+2}(\Omega)} &\leq C\|f_H\|_{H^s(\Omega)} \leq C\|f\|_{H^s(\Omega)} \\ \|\widetilde{u}_{\mathcal{A}}\|_{\mathcal{H}} &\leq C_{sol}(k)\|f_H\|_{L^2(\Omega)} \\ \|\nabla^n \widetilde{u}_{\mathcal{A}}\|_{L^2(\Omega)} &\leq C\lambda^p k^{-1} C_{sol}(k) \max\{k, n\}^p \|f_H\|_{L^2(\Omega)} \qquad \forall n \in \mathbb{N}_0 \end{aligned}$$

recalling that $\|f_H\|_{L^2(\Omega)} \leq Ck^{-1}\|f\|_{H^1(\Omega)}$, we see that $u_{H^{s+2}}$ and $\widetilde{u}_{\mathcal{A}}$ have the desired properties. We now turn to u_L. Since f_L and $\partial\Omega$ are analytic, the solution u_L is analytic. For bounds on the derivatives of u_L, we first note that Lemma 3.4 yields

$$\|u_L\|_{\mathcal{H}} \leq Ck^{-1}(1 + C_{sol}(k))\|f_L\|_{H^1(\Omega)}. \tag{3.15}$$

For higher order derivatives, we proceed as in the proof of [17, Lemma 4.13]: Upon setting $\varepsilon := 1/k$, we observe that u_L satisfies

$$-\varepsilon^2 \Delta u_L - u_L = \varepsilon^2 f_L \quad \text{in } \Omega, \qquad \varepsilon^2 \partial_n u_L - \mathbf{i}\varepsilon u_L = 0 \quad \text{on } \partial\Omega$$

with f_L satisfying the estimates above. Hence, the equation satisfied by u_L has the same structure as in the proof of [17, Lemma 4.13] making [15, Prop. 5.4.5, Rem. 5.4.6] applicable. The result is then

$$\|\nabla^{n+2} u_L\|_{L^2(\Omega)} \leq CK^{n+2} \max\{n, k\}^{n+2} \left[k^{-2}\|f_L\|_{L^2(\Omega)} + k^{-1}\|u_L\|_{\mathcal{H}}\right]$$

for a $K > 0$ independent of k and n. Inserting now the estimate (3.15) for u_L yields

$$\|\nabla^{n+2} u_L\|_{L^2(\Omega)} \leq CK^{n+2} \max\{n, k\}^{n+2} k^{-2} \left\{\|f_L\|_{L^2(\Omega)} + (1 + C_{sol}(k))\|f\|_{H^1(\Omega)}\right\}.$$

Using $\|f_L\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$ and setting $u_{\mathcal{A}} := \widetilde{u}_{\mathcal{A}} + u_L$ finishes the proof. □

4 Convergence analysis

4.1 *hp*-convergence analysis

We perform our convergence analysis for space V_N that have approximation properties typical of the $H^1(\Omega)$-conforming spaces of piecewise polynomials of degree p. Specifically, we will stipulate the following assumption, which formalizes the approximation properties of the *hp*-FEM spaces defined in [16, Sec. 5] and formulated explicitly in [17, Prop. 5.3]; essentially, those spaces are spaces of piecewise (mapped) polynomials of degree p on a mesh $\mathcal{T}$ whose elements have diameter h.

Assumption 4.1. *The space V_N depends on two discretization parameters h and p and has the following two properties:*

1. *Let $\gamma > 0$ be given. Then there exist constants C, $\sigma > 0$ independent of k and the discretization parameters h, p such that if u is analytic on Ω with*

$$\|\nabla^n u\|_{L^2(\Omega)} \leq C_u \gamma^n \max\{n,k\}^n \qquad \forall n \in \mathbb{N}_0$$

then

$$\inf_{v \in V_N} \|u - v\|_{\mathcal{H}} \leq C C_u \left(1 + \frac{kh}{p}\right) \left(\left(\frac{h}{h+\sigma}\right)^p + k \left(\frac{kh}{\sigma p}\right)^p \right)$$

2. *Let $s \in \mathbb{N}_0$ be given. Then there exists a constant C independent of k and the discretization parameters h, p such that if $u \in H^{s+1}(\Omega)$, then*

$$\inf_{v \in V_N} \|u - v\|_{\mathcal{H}} \leq C \left(1 + \frac{kh}{p}\right) \left(\frac{h}{p}\right)^s \|u\|_{H^{s+1}(\Omega)} \qquad p \geq s+1.$$

A further assumption on the parameters h and p is that

$$\frac{kh}{p} \leq \overline{C}$$

for a constant $\overline{C}$. This implies that the factors $(1 + kh/p)$ in the above estimates can be dropped.

It will be convenient to introduce the following shorthand:

$$\varepsilon(h,p,k) := \left(\frac{h}{h+\sigma}\right)^p + k \left(\frac{kh}{\sigma p}\right)^p \tag{4.1}$$

These approximation properties together with the regularity assertions of Proposition 3.5 and Theorem 3.6 allow us to estimate the quantities $\eta_N^{L^2}$ and $\eta_N^{H^1}$:

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ be a bounded Lipschitz domain with analytic boundary. Assume the approximation space V_N to have the properties stated in Assumption 4.1. Then for constants C, $\sigma > 0$ independent of h, k, and p:*

$$\eta_N^{L^2} \leq C \left[\frac{h}{p} + C_{sol}(k) k^{-1} \left\{ \left(\frac{h}{h+\sigma}\right)^p + k \left(\frac{kh}{\sigma p}\right)^p \right\} \right], \tag{4.2}$$

$$\eta_N^{H_0^1} \leq \eta_N^{H^1} \leq C \left[\left(\frac{h}{p}\right)^{\min\{2,p\}} + (1 + C_{sol}(k)) k^{-2} \left\{ \left(\frac{h}{h+\sigma}\right)^p + k \left(\frac{kh}{\sigma p}\right)^p \right\} \right]. \tag{4.3}$$

Proof. The estimate (4.2) has already been shown in [17, Prop. 5.3]; it follows from the approximation properties of V_N in combination with the regularity assertion Proposition 3.5. The first bound in (4.3) follows directly from the definition. For (4.3), let $f \in H^1(\Omega)$ be arbitrary. Then $u = S^* f = \overline{S(\overline{f}, 0)}$ can, according to Theorem 3.6 with $s = 1$, be written as

$$u = u_{H^3} + u_{\mathcal{A}},$$

where the contributions u_{H^3} and $u_{\mathcal{A}}$ have the regularity properties stated there. Therefore, we get from Assumption 4.1

$$\begin{aligned}
\inf_{v \in V_N} \|u_{H^3} - v\|_{\mathcal{H}} &\leq C \left(\frac{h}{p}\right)^{\min\{2,p\}} \|f\|_{H^1(\Omega)} \\
\inf_{v \in V_N} \|u_{\mathcal{A}} - v\|_{\mathcal{H}} &\leq C k^{-2} (1 + C_{sol}(k)) \|f\|_{H^1(\Omega)} \left[\left(\frac{h}{h+\sigma}\right)^p + k \left(\frac{hk}{\sigma p}\right)^p \right].
\end{aligned}$$

The result now follows. □

Remark 4.3. As mentioned above, for our model problem (2.1), the constant $C_{sol}(k)$ satisfies the polynomial bound (3.7) with $\theta = 5/2$. Hence, the crucial condition (2.6) is satisfied if, for a sufficiently small c_1 and a sufficiently large c_2, the following two conditions are satisfied:

$$\frac{kh}{p} \leq c_1 \qquad \text{and} \qquad p \geq c_2 \log k; \tag{4.4}$$

the constants c_1, c_2 depend only on C_c, σ, $\widetilde{C}_{sol}$, and θ. ■

Next, we formulate a best approximation result for data $(f, g) \in H^s(\Omega) \times H^{s+1/2}(\partial\Omega)$, which is proved very similarly to Theorem 4.2.

Theorem 4.4. *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$ be a bounded Lipschitz domain with analytic boundary. Let the approximation space V_N have the properties of Assumption 4.1. For the solution u of (2.1) we have:*

(i) If $s \in \mathbb{N}_0$ and $f \in H^s(\Omega)$ and $g \in H^{s+1/2}(\partial\Omega)$, then

$$\begin{aligned}
\inf_{v \in V_N} \|u - v\|_{\mathcal{H}} \leq\ & C \left(\frac{h}{p}\right)^{\min\{s+1,p\}} \left[\|f\|_{H^s(\Omega)} + \|g\|_{H^{s+1/2}(\partial\Omega)}\right] \\
& + C_{sol}(k) k^{-1} \left\{ \left(\frac{h}{h+\sigma}\right)^p + k \left(\frac{kh}{\sigma p}\right)^p \right\} \left[\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}\right].
\end{aligned}$$

(ii) If $s \in \mathbb{N}$ and $f \in H^s(\Omega)$ and $g = 0$, then

$$\begin{aligned}
\inf_{v \in V_N} \|u - v\|_{\mathcal{H}} \leq\ & C \left(\frac{h}{p}\right)^{\min\{s+1,p\}} \|f\|_{H^s(\Omega)} \\
& + (1 + C_{sol}(k)) k^{-2} \left\{ \left(\frac{h}{h+\sigma}\right)^p + k \left(\frac{kh}{\sigma p}\right)^p \right\} \|f\|_{H^1(\Omega)}.
\end{aligned}$$

Proof. (i) follows from Proposition 3.5 and the approximation properties of V_N. The estimate in (ii) is shown similarly, but we are able to exploit the improved k-dependence of Theorem 3.6. □

We are now in position to formulate some *a priori* error estimates. For simplicity of notation, we employ the abbreviation $\varepsilon(h, p, k)$ of (4.1):

Corollary 4.5. *Assume the hypotheses of Theorem 4.4. Assume in addition that h and p are such that condition (2.6) is satisfied.*

(i) If $s \in \mathbb{N}_0$, $f \in H^s(\Omega)$ and $g \in H^{s+1/2}(\partial\Omega)$, then with $C_{f,g} := \|f\|_{H^s(\Omega)} + \|g\|_{H^{s+1/2}(\partial\Omega)}$

$$\begin{aligned}
\|u - u_N\|_{\mathcal{H}} &\leq CC_{f,g} \left\{ \left(\frac{h}{p}\right)^{\min\{s+1,p\}} + k^{-1} C_{sol}(k) \varepsilon(h,p,k) \right\} \\
\|u - u_N\|_{L^2(\Omega)} &\leq C \left\{ \left(\frac{h}{p}\right) + k^{-1} C_{sol}(k) \varepsilon(h,p,k) \right\} \|u - u_N\|_{\mathcal{H}} \\
\|u - u_N\|_{H^{-1}(\Omega)} &\leq C \left\{ \left(\frac{h}{p}\right)^{\min\{2,p\}} + k^{-2}(1 + C_{sol}(k)) \varepsilon(h,p,k) \right\} \|u - u_N\|_{\mathcal{H}}
\end{aligned}$$

Furthermore, if the function z in (2.10) is in $H^{s'}(\Omega)$ ($s' \in \mathbb{N}$) then

$$|L(u) - L(u_N)| \leq C \left\{ \left(\frac{h}{p}\right)^{\min\{s'+1,p\}} + k^{-2}(1 + C_{sol}(k)) \varepsilon(h,p,k) \right\} \|u - u_N\|_{\mathcal{H}}$$

(ii) If $s \in \mathbb{N}$ and $f \in H^s(\Omega)$ and $g = 0$, then,

$$\begin{aligned}
\|u - u_N\|_{\mathcal{H}} &\leq CC_f \left\{ \left(\frac{h}{p}\right)^{\min\{s+1,p\}} + k^{-2}(1 + C_{sol}(k)) \varepsilon(h,p,k) \right\} \\
\|u - u_N\|_{L^2(\Omega)} &\leq C \left\{ \left(\frac{h}{p}\right) + k^{-2}(1 + C_{sol}(k)) \varepsilon(h,p,k) \right\} \|u - u_N\|_{\mathcal{H}} \\
\|u - u_N\|_{H^{-1}(\Omega)} &\leq C \left\{ \left(\frac{h}{p}\right)^{\min\{2,p\}} + k^{-2}(1 + C_{sol}(k)) \varepsilon(h,p,k) \right\} \|u - u_N\|_{\mathcal{H}}
\end{aligned}$$

Furthermore, if the function z in (2.10) is in $H^{s'}(\Omega)$ ($s' \in \mathbb{N}$) then

$$|L(u) - L(u_N)| \leq C \left\{ \left(\frac{h}{p}\right)^{\min\{s'+1,p\}} + k^{-2}(1 + C_{sol}(k)) \varepsilon(h,p,k) \right\} \|u - u_N\|_{\mathcal{H}}$$

Proof. Follows by combining the above results. □

4.2 h-convergence analysis: examples

The above considerations are formulated in a general hp-setting. We will now present some numerical examples in an h-FEM setting since this setting shows more clearly the k-dependence. For example, we can simplify

$$\varepsilon(h,p,k) = \left(\frac{h}{h+\sigma}\right)^p + k\left(\frac{kh}{\sigma p}\right)^p \leq C_p k^{p+1} h^p.$$

The next corollary follows from Corollary 4.5 by fixing p. Additionally, we assume explicitly the condition (3.7) in order to make the k-dependence more visible:

Corollary 4.6 (h-FEM)**.** *Assume the hypotheses of Corollary 4.5. Fix $p \in \mathbb{N}$. Then* *(i) If $s \in \mathbb{N}_0$, $f \in H^s(\Omega)$, and $g \in H^{s+1/2}(\partial\Omega)$, then with $C_{f,g} = \|f\|_{H^s(\Omega)} + \|g\|_{H^{s+1/2}(\partial\Omega)}$*

$$\begin{aligned}
\|u - u_N\|_{\mathcal{H}} &\leq CC_{f,g}\left[h^{\min\{s+1,p\}} + k^\theta (kh)^p\right], \\
\|u - u_N\|_{L^2(\Omega)} &\leq CC_{f,g}\left[h^{\min\{s+1,p\}} + k^\theta (kh)^p\right] h\left[1 + k^{\theta+1}(kh)^{p-1}\right], \\
\|u - u_N\|_{H^{-1}(\Omega)} &\leq CC_{f,g}\left[h^{\min\{s+1,p\}} + k^\theta (kh)^p\right] \begin{cases} h^2(1 + k^{\theta+1}(kh)^{p-2}) & \text{if } p \geq 2 \\ hk^\theta & \text{if } p = 1 \end{cases}
\end{aligned}$$

If the function z in (2.10) is in $H^{s'}(\Omega)$ ($s' \in \mathbb{N}$), then

$$|L(u) - L(u_N)| \leq CC_{f,g}\left[h^{\min\{s+1,p\}} + k^\theta (kh)^p\right]\left[h^{\min\{s'+1,p\}} + k^{\theta-1}(kh)^p\right].$$

(ii) If $s \in \mathbb{N}$ and $f \in H^s(\Omega)$ and $g = 0$, then with $C_f = \|f\|_{H^s(\Omega)}$

$$\begin{aligned}
\|u - u_N\|_{\mathcal{H}} &\leq CC_f\left[h^{\min\{s+1,p\}} + k^{\theta-1}(kh)^p\right] \\
\|u - u_N\|_{L^2(\Omega)} &\leq CC_f\left[h^{\min\{s+1,p\}} + k^{\theta-1}(kh)^p\right] h\left[1 + k^{\theta+1}(kh)^{p-1}\right] \\
\|u - u_N\|_{H^{-1}(\Omega)} &\leq CC_{f,g}\left[h^{\min\{s+1,p\}} + k^{\theta-1}(kh)^p\right] \begin{cases} h^2(1 + k^{\theta+1}(kh)^{p-2}) & \text{if } p \geq 2 \\ hk^\theta & \text{if } p = 1 \end{cases}
\end{aligned}$$

If the function z in (2.10) is in $H^{s'}(\Omega)$ ($s' \in \mathbb{N}$), then

$$|L(u) - L(u_N)| \leq CC_{f,g}\left[h^{\min\{s+1,p\}} + k^{\theta-1}(kh)^p\right]\left[h^{\min\{s'+1,p\}} + k^{\theta-1}(kh)^p\right].$$

Remark 4.7. A different way of phrasing the $L^2(\Omega)$-convergence result is as follows: If Ω has an analytic boundary and we assume that the exact solution $u \in H^{m+1}(\Omega)$ of (2.1) satisfies

$$|u|_{H^j(\Omega)} \sim k^j, \qquad j = 0, \ldots, m+1,$$

and the solvability condition (2.6) is satisfied, then

$$\|u - u_N\|_{L^2(\Omega)} \leq C_m \left(\frac{hk}{p}\right)^{m+1} \left\{1 + \left[1 + \frac{k}{\sigma}\left(\frac{hk}{\sigma p}\right)^{p-1}\right]\left(C_{sol}(k) + 1\right)\right\}. \tag{4.5}$$

This follows by combining the estimate for $\eta_N^{L^2}$ with the *a priori* bound $\inf_{v \in V_N} \|u - v\|_{\mathcal{H}} \leq Ch^m \|u\|_{H^{m+1}(\Omega)} \leq Ck(kh)^m$. The estimate (4.5) illustrates again that the special nature of the case $p = 1$. ∎

4.2.1 1D examples

We will show several numerical examples for the simple 1D model problem (3.1). Without proof we mention that the convergence results of Corollary 4.6 are valid in this case as well (in spite of the fact that Dirichlet boundary conditions are imposed on parts of the boundary) with the special choice $\theta = 0$; in particular, the different k-dependence for the cases $g \neq 0$ and $g = 0$ applies.

Example 4.8. We consider the smooth 1D example of (2.1) with $f \equiv 1$ and $g = 0$. With $\theta = 0$, Corollary 4.6 gives

$$\|u - u_N\|_{\mathcal{H}} \leq C\left(h^p + k^{-1}(kh)^p\right) \leq Ck^{p-1}h^p$$

In Fig. 1, we plot the relative error in the $H^1(I)$-seminorm. A closed form solution for u is available, (4.10), and shows $\|u'\|_{L^2(I)} \sim k^{-1}$. Hence, the relative error in the $H^1(I)$-seminorm (even in the

$\|\cdot\|_{\mathcal{H}}$-norm) is expected to behave like $O((kh)^p)$, i.e., order p in the number N_λ of degrees of freedom per wavelength. This is visible in Fig. 1, where the relative error in the $H^1(I)$-seminorm is plotted versus N_λ. Fig 2 shows the relative error in the $L^2(I)$-norm versus N_λ. From the close form solution (4.10) we infer $\|u\|_{L^2(I)} \sim k^{-2}$. Hence, Corollary 4.6 yields

$$\frac{\|u - u_N\|_{L^2(I)}}{\|u\|_{L^2(I)}} \leq C \frac{1}{k^{-2}} \{h^p + k^{-1}(kh)^p\} h \{1 + k(kh)^{p-1}\} \leq C(kh)^{p+1}(1 + k(kh)^{p-1})$$

This formula shows that, in its k-dependence, the asymptotic behavior of the case $p = 1$ is different from the case $p > 1$; this is again visible in Fig. 2, where we plot the relative error in $L^2(I)$ versus N_λ, the number of degrees of freedom per wavelength.

We finally turn the convergence behavior for a linear functional given by $z \equiv 1$ in (2.10). Since z is again given by (4.10), we have $|L(u)| \sim k^{-2}$ so that we expect by Cor. 4.6

$$\frac{|L(u) - L(u_N)|}{|L(u)|} \leq C \frac{1}{k^{-2}} k^{p-1} h^p k^{p-1} h^p \leq C(kh)^{2p},$$

which is again visible in Fig. 3. ∎

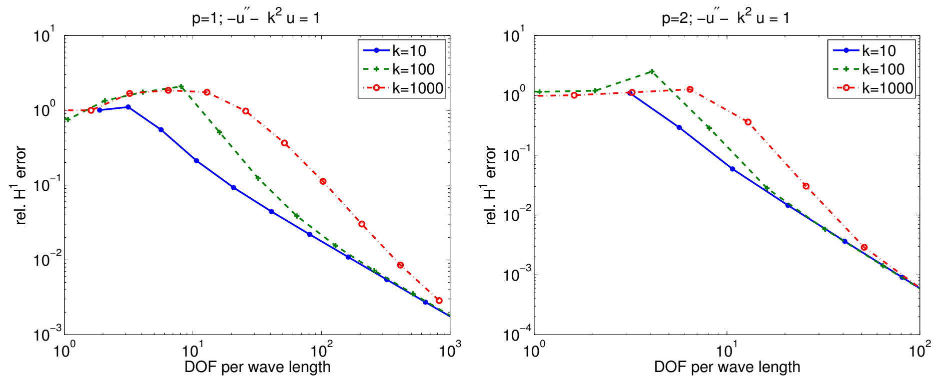


Figure 1: h-FEM for smooth right-hand side, relative error in H^1, $p = 1, 2$ (see Example 4.8).

Example 4.9. We consider our 1D model problem (3.1) with $f(x) = x^\alpha$ and $g = 0$. We take the specific choices $\alpha = -1/2$ and $\alpha = 1/2$. For $\alpha = -1/2$, the right-hand side f in $H^s(I)$ for every $s < 0$

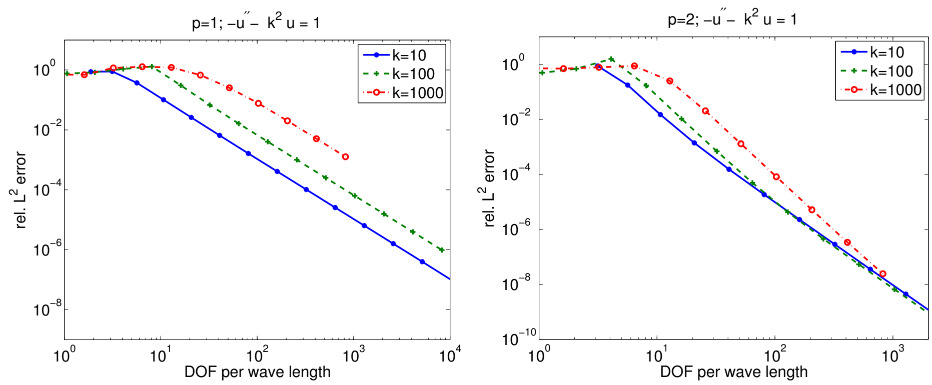


Figure 2: h-FEM for smooth right-hand side, relative error in L^2, $p = 1, 2$ (see Example 4.8).

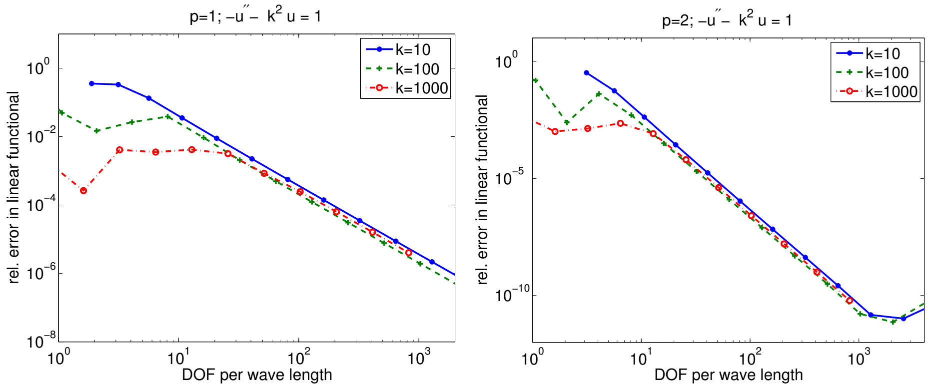


Figure 3: h-FEM for smooth right-hand side, error in smooth linear functional, $p = 1, 2$ (see Example 4.8).

and for $\alpha = 1/2$, the right-hand side f is in $H^s(I)$ for every $s < 1$. Hence, we expect Cor. 4.6 to be applicable with $s = 0$ and $s = 1$, respectively. We first consider the case $\alpha = -1/2$ (corresponding to $s = 0$). Then, only case (i) is applicable in Cor. 4.6, yielding

$$\|u - u_N\|_{\mathcal{H}} \;\leq\; C\left(h^{\min\{s+1,p\}} + k^p h^p\right) \leq C\left(h + (kh)^p\right) = Ch\left(1 + k^p h^{p-1}\right),$$

where we inserted $s = 0$. We note the pronounced difference between the cases $p = 1$ and $p = 2$, which is also visible in Fig. 4. For $\alpha = 1/2$, a similar situation arises. Applying case (ii) of Cor. 4.6 with $s = 1$, we expect the behavior

$$\|u - u_N\|_{\mathcal{H}} \;\leq\; C\left(h^{\min\{s+1,p\}} + k^{p-1} h^p\right) \leq C\left(h^2 + k^{p-1} h^p\right) = Ch^2\left(1 + k^{p-1} h^{p-2}\right),$$

which is again visible in Fig. 5. Figs. 4, 5 also present the k-scaled $L^2(I)$-norm error.

The absolute error in the linear functional of (2.10) with $z = 1$ is presented in Fig. 6. The top row corresponds to $\alpha = -1/2$ in conjunction with $p = 1$ and $p = 2$ whereas the bottom row depicts the case $\alpha = 1/2$ with $p = 2$ and $p = 3$. The first two cases ($p = 1$ for $\alpha = -1/2$ and $p = 2$ for $\alpha = 1/2$) are covered by Cor. 4.6. An improved estimate $O(h^{\min\{s+3/2,p\}+p})$ can be shown by exploiting the fact that the solution has a singularity at one point only. We refer to Lemma B.5 for details. ■

4.2.2 2D examples

In the following 3 two-dimensional examples, the solution is always smooth but the geometry differs: whereas the first case is a convex geometry, the second one is non-convex and the third one is not simply connected. All examples are computed with the hp-FEM software package NETGEN/NGSOLVE by J. Schöberl [19,20]. The curved geometry is resolved using high order approximations as provided by NETGEN/NGSOLVE.

Example 4.10. The domain consists of the square $[-1, 1]^2$ with two semicircular caps attached, i.e., $\overline{\Omega} = [-1, 1]^2 \cup \{(x, y) | (x \pm 1)^2 + y^2 \leq 1\}$ (see Fig. 7, (a)).
For the problem

$$\begin{aligned} -\Delta u - |\mathbf{k}|^2 u = 0 \quad & \text{in} \quad \Omega \\ \partial_n u - \mathbf{i}|\mathbf{k}| u = g \quad & \text{on} \quad \partial\Omega \end{aligned}$$

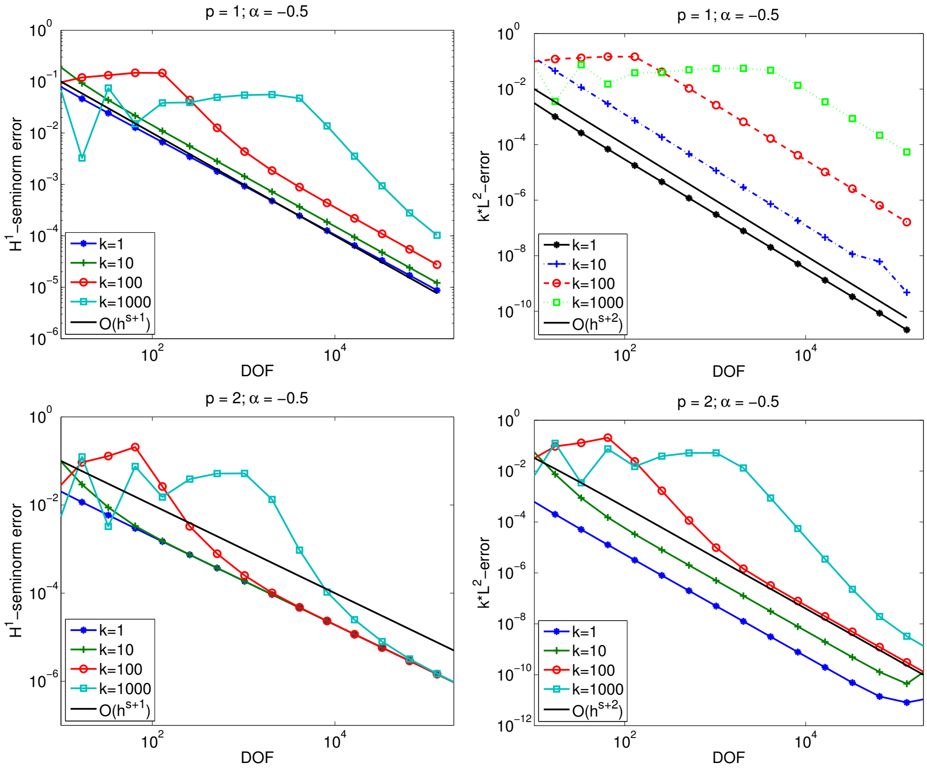


Figure 4: h-FEM for $\alpha = -1/2$: $p = 1$ (top row) and $p = 2$ (bottom row) (see Example 4.9).

The inhomogeneity g is chosen in such a way that the exact solution has the form $u(\mathbf{x}) = e^{i\mathbf{x}\cdot\mathbf{k}} = e^{\mathbf{i}(k_1x+k_2y)}$ where $k_1 = -k_2 = k/\sqrt{2}$ and $k \in \{1, 10, 100\}$. For fixed $p = 1$,2 we computed the L^2-error as well as the relative H^1-error.

In Fig. 8 we present the convergence of the h-FEM both in L^2 and H^1-seminorm versus the number of degrees of freedom per wavelength. We observe the same marked difference between the cases $p = 1$ and $p = 2$ that we have seen already in 1D and which is explained by Corollary 4.6. The convergence when evaluating a linear functional of the form (2.10) given by $z \equiv 1$ is presented in Fig. 9. ■

Example 4.11. The model problem remains the same as in Example 4.10. Only the geometry is modified to a non-convex domain, see Fig. 7, (b). The domain Ω is a subset of $[-2, 0.5] \times [-0.5, 2]$, thus diam$\Omega \lesssim 3.5$. The h-FEM (error in L^2 and H^1-seminorm as well as the error in a smooth linear functional) with $k = \{4, 20, 80\}$ and $p = 1$, 2 is presented in Figs. 10, 11. ■

Example 4.12. We consider the same problem as in Example 4.10 on a non-simply connected domain, see Fig. 7, (c). The h-FEM has been computed for $k = \{8, 20, 80\}$ and $p = 1$, 2 is shown in Figs. 12, 13. ■

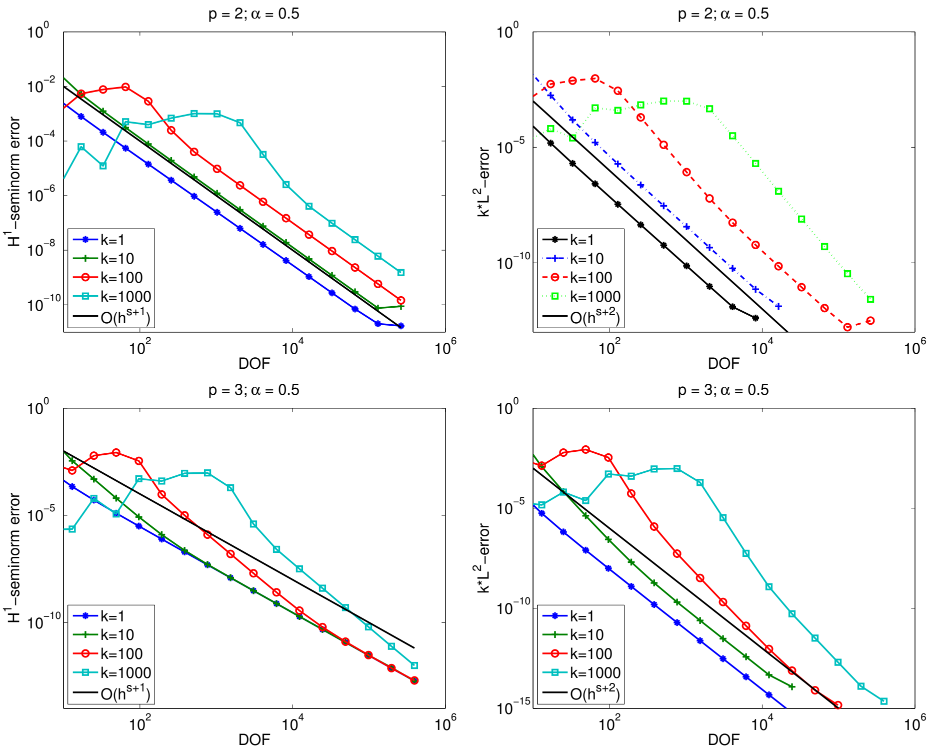


Figure 5: the case $\alpha = +1/2$: $p = 2$ (top row) and $p = 3$ (bottom row) (see Example 4.9).

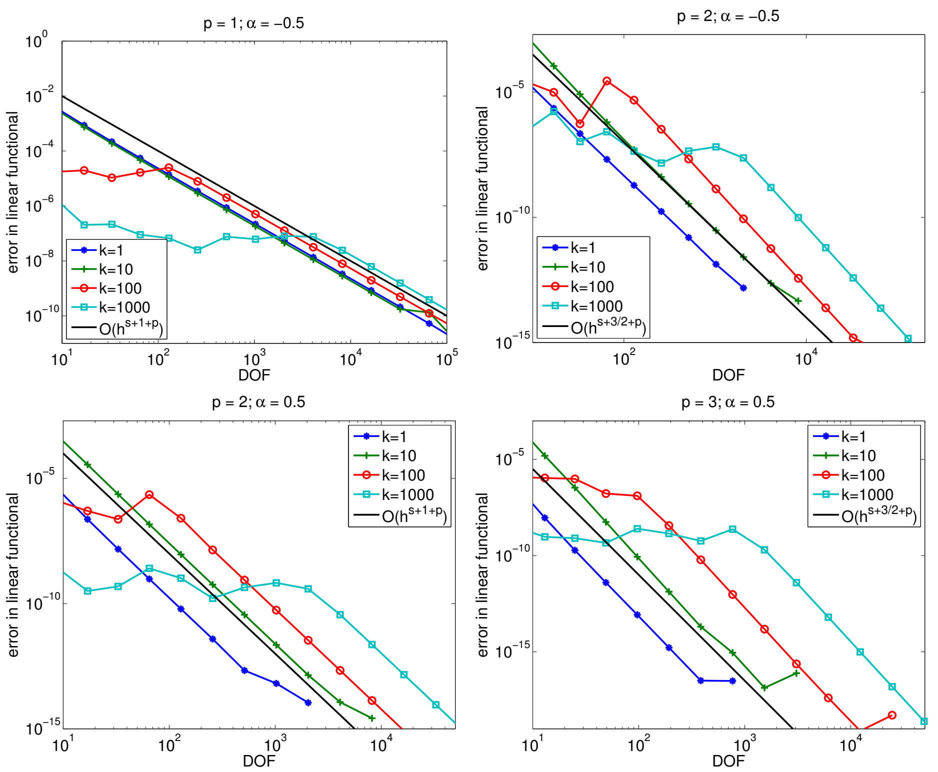
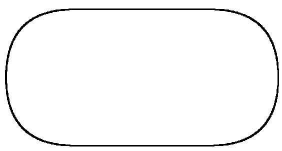
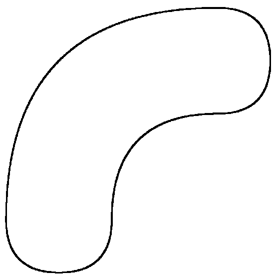


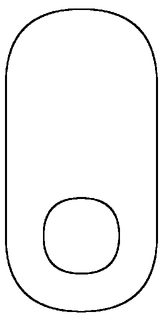
Figure 6: error in smooth linear functions: top row: the case $\alpha = -1/2$ with $p = 1$ (left) and $p = 2$ (right). bottom row: the case $\alpha = 1/2$ with $p = 2$ (left) and $p = 3$ (right). (see Example 4.9).



(a) Example 4.10



(b) Example 4.11



(c) Example 4.12

Figure 7: domain geometries

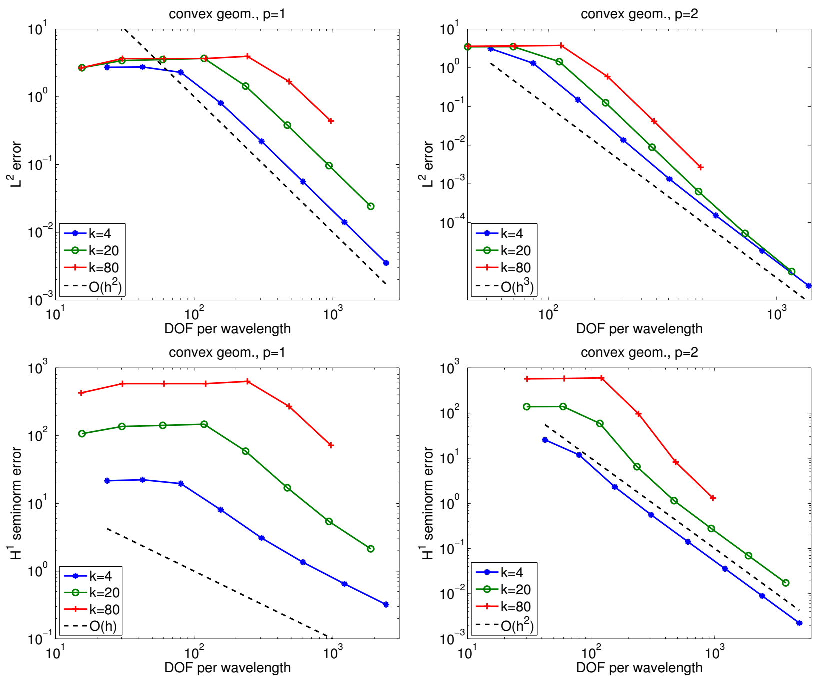


Figure 8: Convergence plot for Example 4.10 with convex domain. Top: L^2-error for h-FEM with $p = 1$ and $p = 2$. Bottom: H^1-seminorm error for h-FEM with $p = 1$ and $p = 2$.

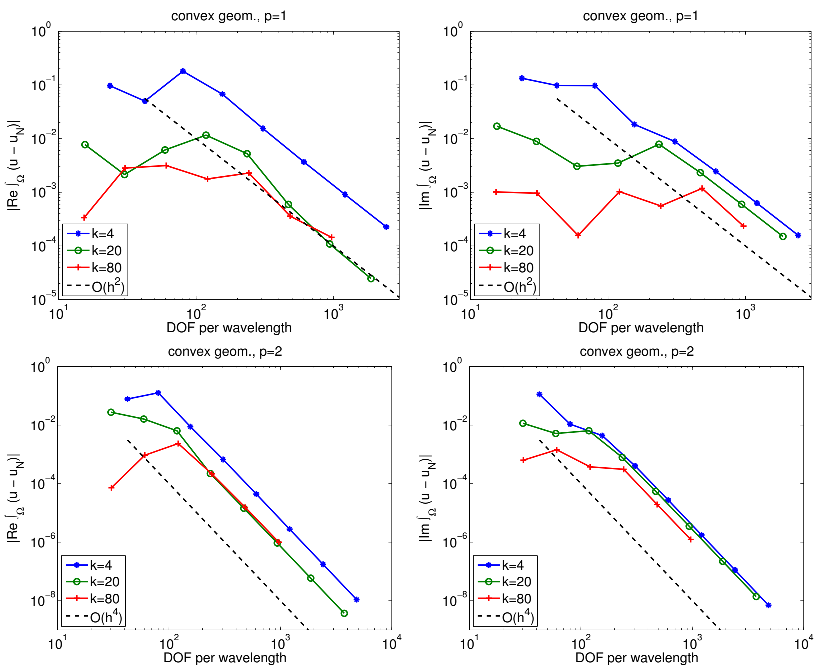


Figure 9: Error in a linear functional for Example 4.10 with convex domain. Top: $|\operatorname{Re}\int_\Omega(u-u_N)|$ and $|\operatorname{Im}\int_\Omega(u-u_N)|$ for $p = 1$. Bottom: $|\operatorname{Re}\int_\Omega(u-u_N)|$ and $|\operatorname{Im}\int_\Omega(u-u_N)|$ for $p = 2$.

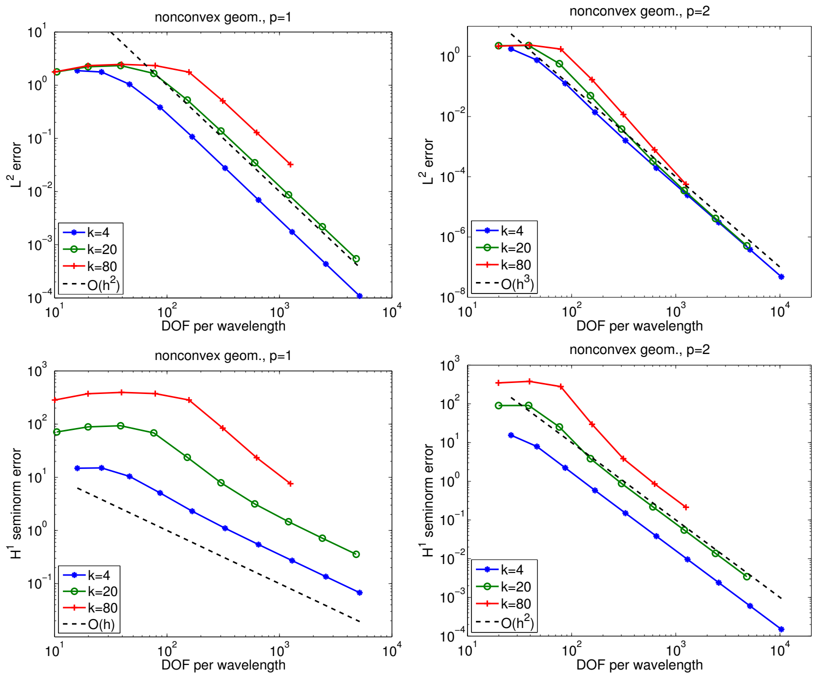


Figure 10: Convergence plot for Example 4.11 with nonconvex domain. Top: L^2-error for h-FEM with $p = 1$ and $p = 2$. Bottom: H^1-seminorm error for h-FEM with $p = 1$ and $p = 2$.

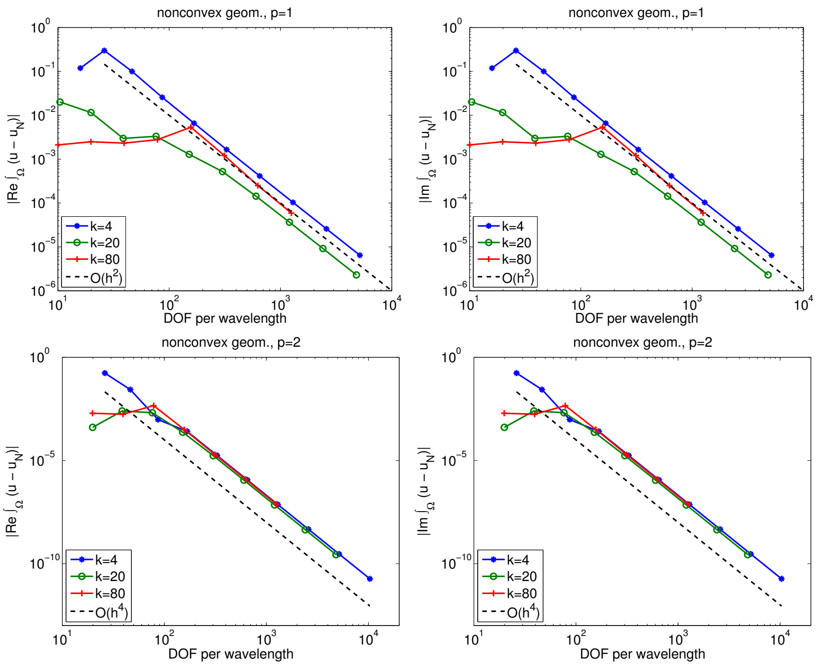


Figure 11: Error in a linear functional for Example 4.11 with nonconvex domain. Top: $|\operatorname{Re}\int_\Omega(u - u_N)|$ and $|\operatorname{Im}\int_\Omega(u - u_N)|$ for $p = 1$. Bottom: $|\operatorname{Re}\int_\Omega(u - u_N)|$ and $|\operatorname{Im}\int_\Omega(u - u_N)|$ for $p = 2$.

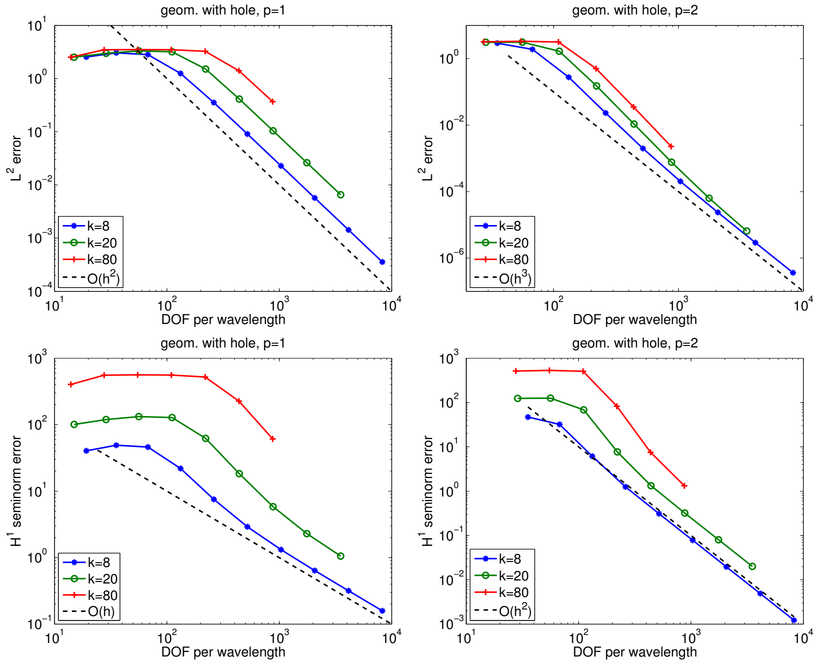


Figure 12: Convergence plot for Example 4.12 with domain with hole Top: L^2-error for h-FEM with $p = 1$ and $p = 2$. Bottom: H^1-seminorm error for h-FEM with $p = 1$ and $p = 2$.

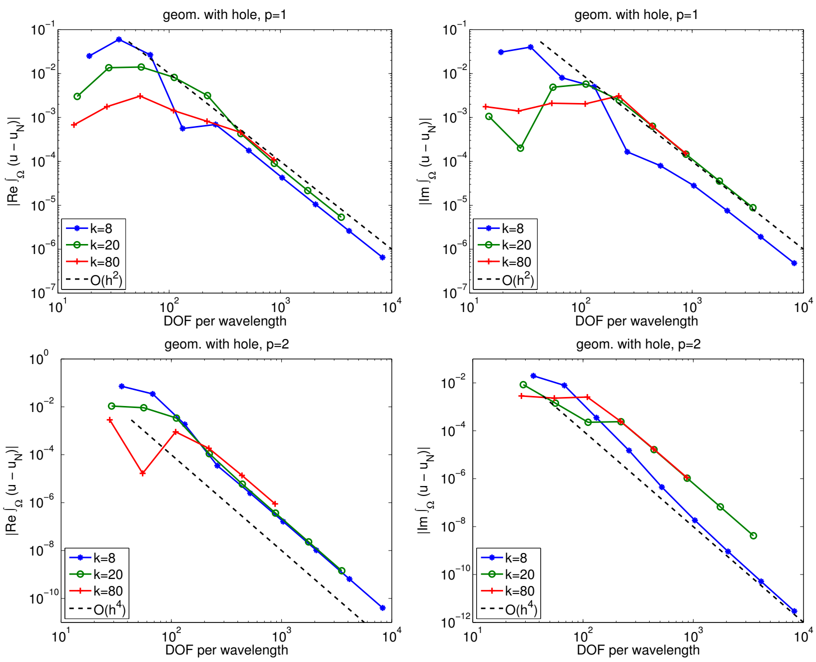


Figure 13: Error in a linear functional for Example 4.12 for computational domain with a hole. Top: $|\operatorname{Re}\int_\Omega (u - u_N)|$ and $|\operatorname{Im}\int_\Omega (u - u_N)|$ for $p = 1$. Bottom: $|\operatorname{Re}\int_\Omega (u - u_N)|$ and $|\operatorname{Im}\int_\Omega (u - u_N)|$ for $p = 2$.

The optimally blended FE-SE scheme

It is possible to analyze the dispersive properties of a numerical scheme on regular, translation-invariant meshes. Such an analysis is performed, for example, in [9,11–13] in an h-FEM context and in [2–4] in a p-FEM and in a spectral element context; [1] analyzes various DG-methods for their dispersion errors. A key idea of the recent work [4] is to exploit the different dispersive behaviors of two different schemes in order to design a new scheme with less pollution error. This idea is worked out in [4] by blending the Galerkin method (i.e., all bilinear forms are evaluated exactly) with the spectral element method (here: all bilinear forms are evaluated with the Gauß-Lobatto rule), giving the two terms the relative weight $1 : p$. That is, the bilinear form of the so-called "optimally blended scheme" is given by

$$B_{FE-SE}(u,v) = \frac{1}{p+1} B_{FEM}(u,v) + \left(1 - \frac{1}{p+1}\right) B_{SEM}(u,v),$$

where p is the polynomial degree employed and $B_{FEM}(\cdot,\cdot)$ and $B_{SEM}(\cdot,\cdot)$ are bilinear forms of the Galerkin method and the spectral element method respectively. On regular (infinite) grids, one can associate a discrete wavenumber $\tilde{k}$ with the numerical scheme and correspondingly measure the dispersion error by relating the discrete wavenumber $\tilde{k}$ to the exact wavenumber k. It is shown in [2–4] that

$$\begin{aligned} |k-\tilde{k}| &= kO((kh)^{2p}) && \text{for the Galerkin method,} \\ |k-\tilde{k}| &= kO((kh)^{2p}) && \text{for the spectral element method,} \\ |k-\tilde{k}| &= kO((kh)^{2p+2}) && \text{for the optimally blended spectral element method.} \end{aligned}$$

We will show in Theorem 4.13 below that such estimates for the dispersion error translate into actually error bounds. We will do this specifically for the lowest order case $p = 1$ and errors in the L^2-norm for the model problem (3.1). To that end, we will assume that the discretization has the form: Find $u_h^\alpha \in V_N$ such that

$$B_h(u_h^\alpha, v) = l(v) := \int_I f\varphi \qquad \forall \varphi \in V_N, \tag{4.6}$$

where $V_N \subset \{v \in H^1(I) \,|\, v(0) = 0\}$ consists of the classical piecewise linear functions on a regular mesh of mesh size h. After selecting the classical basis of hat functions, we assume that the stiffness matrix is tridiagonal. This setting includes the classical Galerkin FEM, the lowest order spectral element method as well as lowest order optimally blended scheme. The salient feature of the analysis is that such a discretization admits solution operators that can be expressed in terms of a discrete wavenumber $\tilde{k}$ (see the outline of the proof of Theorem 4.13 below). We will assume the following property of $\tilde{k}$:

$$|\tilde{k} - k| = kO(hk)^{2\alpha}, \quad \alpha \geq 1 \tag{4.7}$$

This assumption covers the classical Galerkin method with $\alpha = 1$ (this case has been analyzed previously in [11, Sec. 4.6.4]) and the optimally blended scheme with $\alpha = 2$. It is worth pointing out that the lowest order case is particularly striking in that the difference between the Galerkin method and the optimally blended method is most pronounced in this case.

We can state the following result:

Theorem 4.13. *Let u be the exact solution of the 1D-Helmholtz problem (3.1) with $g = 0$. Let u_h^α be the piecewise linear function solving (4.6). Then, for sufficiently smooth f and $kh = O(1)$ the error $u - u_h^\alpha$ can be estimated by*

$$\|u - u_h^\alpha\|_{L^2} \lesssim h^2\big(1 + k(hk)^{2(\alpha-1)}\big)C_f,$$

where C_f depends only on the data f.

Remark 4.14. We note the marked difference between the cases $\alpha = 1$ (covering the lowest order Galerkin and spectral element method) and $\alpha \geq 2$ (which includes the optimally blended scheme). If $\alpha = 1$, then $\|u - u_h\|_{L^2} \lesssim kh^2 C_f$. Assuming that the solution behaves like $\|u\|_{L^2} \sim k^{-2}$ (as can be ascertained for smooth f using the solution formula) this leads to a relative error

$$\frac{\|u - u_h\|_{L^2}}{\|u\|_{L^2}} \lesssim k(hk)^2. \tag{4.8}$$

Thus, the FEM converges at the optimal rate as measured in relative error versus N_λ, but the constant is $O(k)$. The case $\alpha = 2$ representing the optimally blended scheme gives

$$\|u - u_h\|_{L^2} \lesssim h^2(1 + k(hk)^2)C_f.$$

Thus in this case we arrive at

$$\frac{\|u - u_h\|_{L^2}}{\|u\|_{L^2}} \lesssim (hk)^2, \tag{4.9}$$

where the constant in front of N_λ^{-2} is bounded uniformly in k. ∎

The observation of Remark 4.14 is illustrated in the following numerical example.

Example 4.15. We consider (3.1) with $f = 1$ and $g = 0$. The exact solution u is given by

$$u(x) = \frac{1}{k^2}(e^{\mathbf{i}kx} - \mathbf{i}e^{\mathbf{i}k}\sin(kx) - 1). \tag{4.10}$$

The top leftmost plot in Fig. 14 shows the performance of the optimally blended scheme. The relative error in L^2 is plotted versus N_λ. We note the good agreement with the *a priori* estimate (4.9). The performance of the classical Galerkin FEM is shown in Fig. 2, which clearly shows the k-dependence predicted in (4.8). The remaining plots in Fig. 14 show the performance of the optimally blended scheme on non-uniform meshes. The mesh points of a regular mesh were randomly perturbed by n percent, where $n \in \{10, 20, 30, 40, 50\}$. Although the favorable properties of the optimally blended scheme are not proved under these circumstances, the numerical results indicate a certain robustness of the method under meshpoint disturbation. ∎

Example 4.16. We consider (3.1) with $f = x^\alpha$ and $g = 0$. For the case $\alpha = -1/2$ we present the relative error in L^2 versus the number N_λ of degrees of freedom per wavelength. We compare, for $p = 1$ and $p = 2$ the Galerkin method with the optimally blended scheme. Fig. 15. We remark in passing that the L^2-norm of the exact solution is observed numerically to scale like $O(k^{-3/2})$. Although this examples is not covered by Theorem 4.13, the optimally blended scheme is, in particular for the lowest order case, superior to the Galerkin method. ∎

Sketch of the proof of Theorem 4.13: As mentioned before the solution can be written in the form

$$u(x) = \int_\Omega G(x, y)f(y)dy$$

where the Green's function is defined in (3.2). Likewise, as described, for example, in [11], the nodal values $u_h^\alpha(x_i)$ of the discrete solution u_h^α can be expressed by means of a discrete Green's function

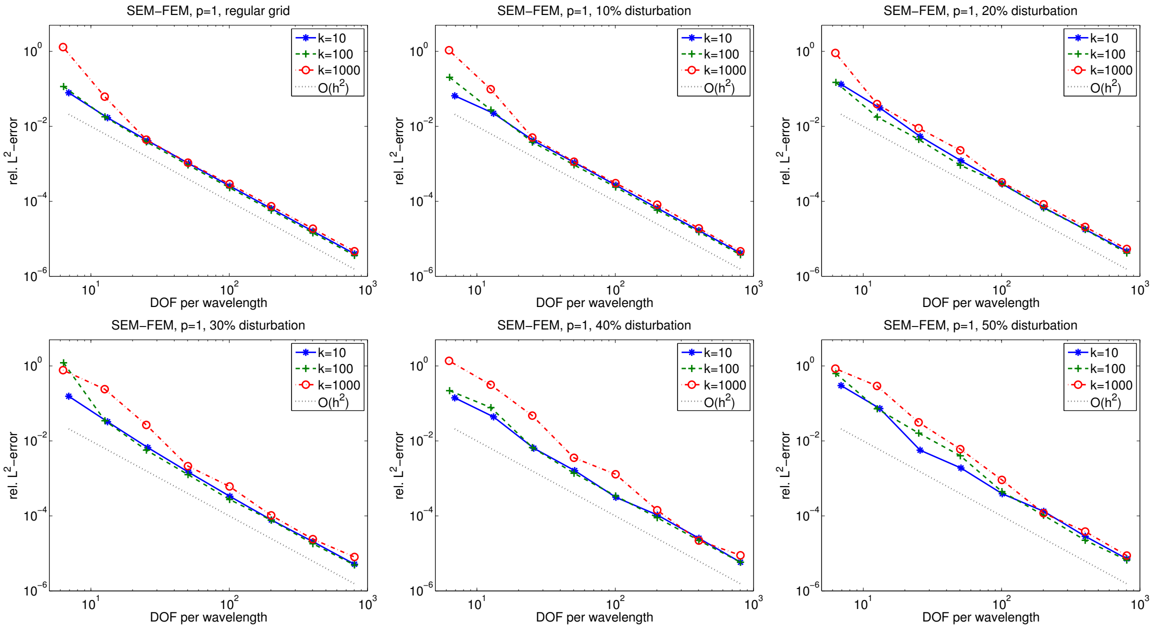


Figure 14: h-Method for the 1D blended spectral-finite element scheme on disturbed meshes

G_h^α in terms of the discrete wavenumber $\tilde{k}$:

$$
\begin{aligned}
u_h^\alpha(x_i) &= h \sum_{j=1}^{N} G_h^\alpha(x_i, x_j) r_h(x_j), \\
G_h^\alpha(x,y) &= \frac{1}{h \sin(h\tilde{k})} \begin{cases} \sin(\tilde{k}x)\big(A \sin(\tilde{k}y) + \cos(\tilde{k}y)\big) & 0 \le x \le y \le 1 \\ \sin(\tilde{k}y)\big(A \sin(\tilde{k}x) + \cos(\tilde{k}x)\big) & 0 \le y \le x \le 1 \end{cases} \\
A &:= A(k, \tilde{k}) = \frac{(hk)^2 \sin(\tilde{k}) \cos(\tilde{k}) + \mathbf{i}\sqrt{12}\sqrt{12 - (hk)^2}}{12 - (hk)^2 \cos^2(\tilde{k})} \\
r_h(x_j) &= h \int_\Omega f(y) \varphi_j(y) dy.
\end{aligned}
$$

Here, the nodes $x_i = ih$, $i = 0, \ldots, N$, represent the mesh and the functions φ_i, $i = 0, \ldots, N$ are the classical hat functions associated with the nodes x_i.

In particular we can see via Taylor expansion at $hk = 0$ that $A = \mathbf{i} + O(hk)^2$.

Let u_I be the linear interpolation of the solution u. From the triangle inequality $\|u - u_h^\alpha\|_{L^2(\Omega)} \le \|u - u_I\|_{L^2(\Omega)} + \|u_I - u_h^\alpha\|_{L^2(\Omega)}$ and the standard polynomial interpolation estimate

$$\|u - u_I\|_{L^2(\Omega)} \le Ch^2\big(\|f\|_\infty + \|f'\|_\infty\big) \lesssim h^2 C_f$$

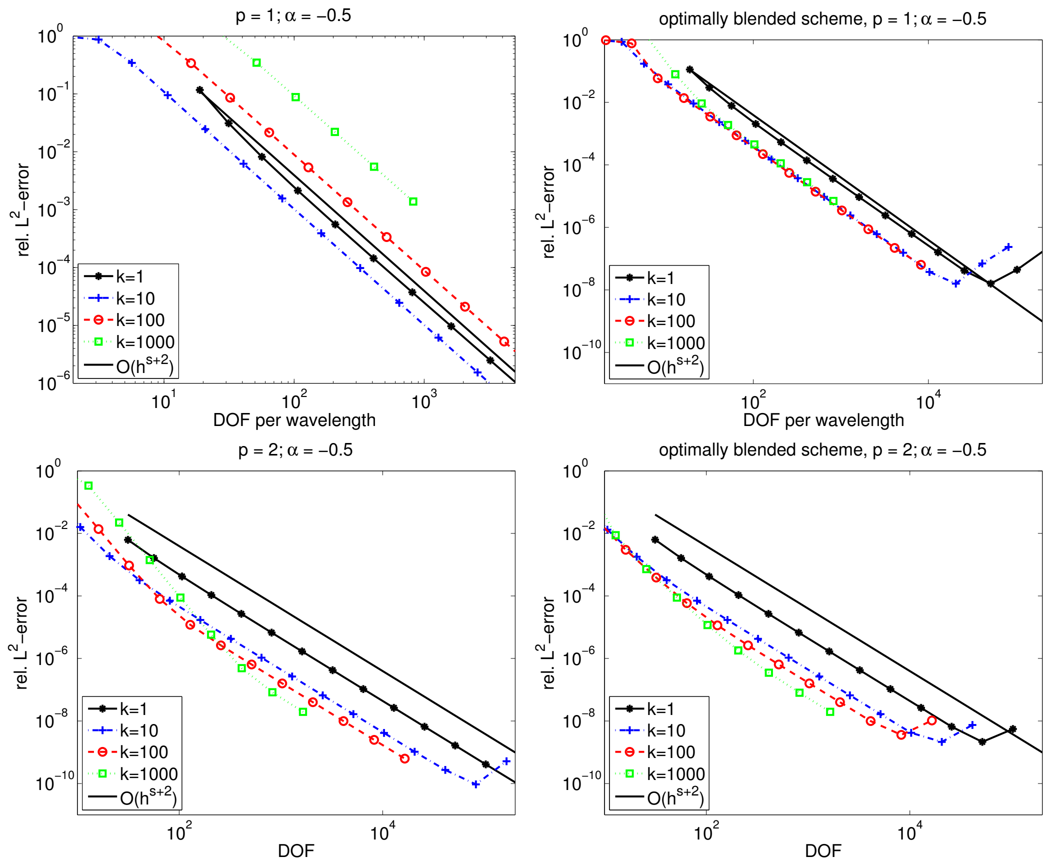


Figure 15: Galerkin FEM and optimally blended scheme for non-smooth right hand side. top: $p = 1$ (left: Galerkin, right: optimally blended), bottom: $p = 2$ (left: Galerkin, right: optimally blended).

we see that we have to estimate

$$\begin{aligned}
\|u_I - u_h^\alpha\|^2_{L^2(\Omega)} &= \int_\Omega |(u_I - u_h^\alpha)(x)|^2 dx \le h \sum_i |u(x_i) - u_h^\alpha(x_i)|^2 \qquad (4.11)\\
&\lesssim h \sum_i \Big| \int_\Omega G(x_i,y) f(y) dy - h \sum_j G_h^\alpha(x_i,x_j) r_h(x_j) \Big|^2 \\
&\lesssim h \sum_i \Big| \underbrace{\int_0^{x_i} G(x_i,y) f(y) dy - \Big(h \sum_{j=0}^{i-1} G_h^\alpha(x_i,x_j) r_h(x_j) + \frac{h}{2} G_h^\alpha(x_i,x_i) r_h(x_i) \Big) +}_{=:A_i} \\
&\quad \underbrace{+ \int_{x_i}^1 G(x_i,y) f(y) dy - \Big(h \sum_{j=i+1}^{N} G_h^\alpha(x_i,x_j) r_h(x_j) + \frac{h}{2} G_h^\alpha(x_i,x_i) r_h(x_i) \Big)}_{=:Bi} \Big|^2 .
\end{aligned}$$

We are thus left with bounding the sums $\sum_i A_i$ and $\sum_i B_i$. This is achieved using integration by parts and summation by parts, thus, exploiting regularity of the right-hand side f. Here, we give merely some key steps and refer to Appendix A for more details.

We will need the following observation:

Lemma 4.17. *With the abbreviation $x_{j+1/2} = (j+1/2)h$, we have*

$$\begin{aligned}
r_h(0) &= 0 \\
r_h(x_N) &= \frac{h^2}{2} f(x_N) - \frac{h^3}{6} f'(x_N) + O(h^4 \|f''\|_\infty) \\
r_h(j) &= h^2 f(x_j) + O(h^4\|f''\|_\infty), \qquad \textit{for } 0 < i < N \\
r_h(x_1) - r_h(0) &= r_h(x_1) \\
r_h(x_N) - r_h(x_{N-1}) &= -\frac{h^2}{2} f(x_{N-1/2}) + f'(x_{N-1/2}) \frac{7h^3}{12} + O(h^4\|f''\|_\infty) \\
r_h(x_{j+1}) - r_h(x_j) &= h^3 f'(x_{j+1/2}) + O(h^5 \|f'''\|_\infty) \textit{ for } 0 < j < N-2
\end{aligned}$$

We abbreviate a term that arises in the definition of G_h^α:

$$\alpha_i := A \sin(\tilde{k} x_i) + \cos(\tilde{k} x_i) = e^{\mathbf{i}\tilde{k}x_i} + (A - \mathbf{i}) \sin(\tilde{k} x_i) = e^{\mathbf{i}\tilde{k}x_i} + O((kh)^2). \qquad (4.12)$$

Via integration by parts as well as the summation by parts rule

$$\sum_{j=0}^N x_j y_j = y_N \sum_{j=0}^N x_j - \sum_{j=0}^{N-1} \sum_{l=0}^{j} x_l \big(y_{j+1} - y_j \big) \qquad (4.13)$$

and using the properties of the right-hand side we get (after a lengthy calculation) for the first term, A_i:

$$\begin{aligned}
A_i &= \frac{1}{k^2} e^{\mathbf{i}kx_i} f(0) - \frac{\tilde{\alpha}_i h^2}{4 \sin^2(\frac{h\tilde{k}}{2})} \big(f(0) + O(h^2 \|f''\|_\infty) \big) \\
&\quad - \Big[\frac{1}{k^2} e^{\mathbf{i}kx_i} \cos(kx_i) f(x_i) - \frac{\tilde{\alpha}_i h^2}{4\sin^2(\frac{h\tilde{k}}{2})} \cos(\tilde{k} x_i) \big(f(x_i) + O(h^2\|f''\|_\infty) \big) \Big] \\
&\quad + \frac{1}{k^2} e^{\mathbf{i}kx_i} \int_0^{x_i} \cos(ky) f'(y) dy - \frac{\tilde{\alpha}_i h^3}{2 \sin(h\tilde{k}) \sin(\frac{h\tilde{k}}{2})} \sum_{j=0}^{i-1} \cos(\tilde{k} x_{j+1/2}) \big(f'(x_{j+1/2}) + O(h^2 \|f'''\|_\infty) \big) \\
&=: ⓐ + ⓑ + ⓒ
\end{aligned}$$

Inserting the definition of asymptotics o $\tilde{\alpha}_i$ given in (4.12) and writing the discrete wavenumber $\tilde{k}$ in the form $\tilde{k} = k(1+\varepsilon)$ with small $\varepsilon = (hk)^{2\alpha}$, we get

$$\begin{aligned}
ⓐ &= \frac{1}{k^2} e^{\mathbf{i}kx_i} f(0) - \frac{h^2}{4\sin(\frac{h\tilde{k}}{2})^2}\big(e^{\mathbf{i}\tilde{k}x_i} + O(hk)^2\big)\big(f(0) + O(h^2\|f''\|_\infty)\big) \\
&= \frac{1}{k^2} e^{\mathbf{i}kx_i}\Big(1 - \frac{(hk)^2}{4\sin(\frac{h\tilde{k}}{2})^2} e^{\mathbf{i}(\tilde{k}-k)x_i}\Big) f(0) + O(h^2 f(0)) + O(h^2 k^{-2}\|f''\|_\infty) \\
&= O\big((h^2 + k^{-1}\varepsilon)\|f\|_\infty\big) + O(h^2 k^{-2}\|f''\|_\infty).
\end{aligned}$$

For ⓑ we get by inserting the asymptotics of $\tilde{\alpha}_i$ given in (4.12) and using $\tilde{k} - k = k\varepsilon$ for small ε:

$$\begin{aligned}
ⓑ &= -\frac{1}{k^2} e^{\mathbf{i}kx_i}\cos(kx_i) f(x_i) + \frac{h^2}{4\sin\big(\frac{h\tilde{k}}{2}\big)^2}\big(e^{\mathbf{i}\tilde{k}x_i} + O(hk)^2\big)\cos(\tilde{k}x_i)\Big[f(x_i) + O(h^2\|f''\|_\infty)\Big] \\
&= -\frac{1}{k^2} e^{\mathbf{i}kx_i}\Big(\cos(kx_i) - \frac{(hk)^2}{4\sin\big(\frac{h\tilde{k}}{2}\big)^2} e^{\mathbf{i}(\tilde{k}-k)x_i}\cos(\tilde{k}x_i)\Big) f(x_i) + O(h^2\|f\|_\infty) + O(h^2 k^{-2}\|f''\|_\infty) \\
&= O\big(h^2 + k^{-1}\varepsilon)\|f\|_\infty\big) + O(h^2 k^{-2}\|f''\|_\infty).
\end{aligned}$$

The treatment of the term ⓒ is more involved as is includes, as a first step, the discretization of the integral by the midpoint rule:

$$\frac{1}{k^2} e^{\mathbf{i}kx_i}\int_0^{x_i}\cos(ky) f'(y)\,dy = h\sum_{j=0}^{i-1}\cos(kx_{j+1/2}) f'(x_{j+1/2}) + h^2 O(k^2\|f'\|_\infty + k\|f''\|_\infty + \|f'''\|_\infty)$$

Inserting this result in the definition of ⓒ, we get after some manipulations

$$ⓒ \lesssim (h^2 + k^{-1}\varepsilon)\|f'\|_\infty + h^2(k^{-1}\|f''\|_\infty + k^{-2}\|f'''\|_\infty).$$

Thus this leads to

$$|A_i| = ⓐ + ⓑ + ⓒ \lesssim \big(h^2 + k^{-1}\varepsilon\big)\big(\|f\|_\infty + \|f'\|_\infty\big) + h^2\big(k^{-1}\|f''\|_\infty + k^{-2}\|f'''\|_\infty\big).$$

For the second term we proceed similarly: Summation by parts and use of the properties stated in Lemma 4.17 yields

$$\begin{aligned}
B_i &= \Big\{\frac{1}{\mathbf{i}k^2}\sin(kx_i) e^{\mathbf{i}k} f(1) + \frac{h^2(1+e^{\mathbf{i}\tilde{k}h})}{2\sin(h\tilde{k})(1-e^{\mathbf{i}\tilde{k}h})}\sin(\tilde{k}x_i) e^{\mathbf{i}\tilde{k}} f(1)\Big\} \\
&\quad + \Big\{\frac{h^2}{4\sin^2(\frac{\tilde{k}h}{2})}\sin(\tilde{k}x_i)\cos(\tilde{k}) f(1)(A - \mathbf{i}) \\
&\quad - \frac{1}{\mathbf{i}k^2}\sin(kx_i) e^{\mathbf{i}kx_i} f(x_i) - \frac{h^2(1+e^{\mathbf{i}\tilde{k}h})}{2\sin(h\tilde{k})(1-e^{\mathbf{i}\tilde{k}h})}\sin(\tilde{k}x_i) e^{\mathbf{i}\tilde{k}x_i} f(x_i)\Big\} \\
&\quad \Big\{-\frac{h^2}{4\sin^2(\frac{\tilde{k}h}{2})}\sin(\tilde{k}x_i)\cos(\tilde{k}x_i) f(x_i)(A - \mathbf{i}) \\
&\quad - \frac{1}{\mathbf{i}k^2}\sin(kx_i)\int_{x_i}^1 e^{\mathbf{i}ky} f'(y)dy - \frac{h^2 e^{\frac{\mathbf{i}\tilde{k}h}{2}}}{\sin(h\tilde{k})(1-e^{\mathbf{i}\tilde{k}h})}\sin(\tilde{k}x_i) h\sum_{j=i}^{N-1} e^{\mathbf{i}\tilde{k}x_{j+1/2}}\big(f'(x_{j+1/2}) + O(h^2\|f'''\|_\infty)\big) \\
&\quad + \frac{h^2}{2\sin(\frac{h\tilde{k}}{2})} h\sum_{j=i}^{N-1}\cos(\tilde{k}x_{j+1/2})\big(f'(x_{j+1/2}) + O(h^2\|f'''\|_\infty)\big)(A - \mathbf{i})\Big\} \\
&=: ⓓ + ⓔ + ⓕ
\end{aligned}$$

By similar arguments as in the case of the term A_i, one can show

$$\begin{aligned}
\text{ⓓ} &\lesssim |f(1)|(h^2 + k^{-1}\varepsilon) \\
\text{ⓔ} &\lesssim (h^2 + k^{-1}\varepsilon)\|f\|_\infty \\
\text{ⓕ} &\lesssim h^2(\|f\|_\infty + \|f'\|_\infty + h^2\|f'''\|_\infty) + (h^2 + k^{-1}\varepsilon)\|f'\|_\infty + h^2(k^{-1}\|f''\|_\infty + k^{-2}\|f'''\|_\infty)
\end{aligned}$$

This leads us again to

$$|B_i| = \text{ⓓ} + \text{ⓔ} + \text{ⓕ} \lesssim \big(h^2 + k^{-1}\varepsilon\big)\big(\|f\|_\infty + \|f'\|_\infty\big) + h^2\big(k^{-1}\|f''\|_\infty + k^{-2}\|f'''\|_\infty\big)$$

and we end up with

$$\|e_{poll}\|_{L^2}^2 \lesssim \sqrt{h\sum_i |A_i + B_i|^2} \lesssim \big(h^2 + k^{-1}\varepsilon\big)\big(\|f\|_\infty + \|f'\|_\infty\big) + h^2\big(k^{-1}\|f''\|_\infty + k^{-2}\|f'''\|_\infty\big)$$

Since we assumed $\varepsilon = (hk)^{2\alpha}, \alpha \geq 1$, we conclude for smooth f

$$\|u - u_h\|_{L^2} \lesssim \big(h^2 + k^{-1}(hk)^{2\alpha}\big)C_f \lesssim h^2\big(1 + k(hk)^{2(\alpha-1)}\big)C_f.$$

For more details on the estimates in $|A_i|$ and $|B_i|$ see the Appendix. □

A Proof of Theorem 4.13: detailed version

As discussed in Theorem 4.13 we need to estimate the pollution error $e_{poll} = u_I - u_h^\alpha$ in the L^2-norm. This will be done in (A.20).

In particular from (4.11) we have to estimate

$$\|u_I - u_h^\alpha\|_{L^2}^2 \lesssim h \sum_i |A_i + B_i|^2$$

where

$$A_i := \int_0^{x_i} G(x_i, y) f(y) dy - \Big(h \sum_{j=0}^{i-1} G_h^\alpha(x_i, x_j) r_h(x_j) + \frac{h}{2} G_h^\alpha(x_i, x_i) r_h(x_i) \Big)$$

and

$$B_i := \int_{x_i}^1 G(x_i, y) f(y) dy - \Big(h \sum_{j=i+1}^{N} G_h^\alpha(x_i, x_j) r_h(x_j) + \frac{h}{2} G_h^\alpha(x_i, x_i) r_h(x_i) \Big)$$

Our strategy now is to estimate both terms A_i and B_i by the same expression (independent of i).

A standing assumption will be that kh (and thus $\tilde{k}h$) is small. We will also use the abbreviation

$$\begin{aligned} \tilde{\alpha}_i &:= A(k, \tilde{k}) \sin(\tilde{k} x_i) + \cos(\tilde{k} x_i) \qquad \text{(A.1)} \\ A(k, \tilde{k}) &:= \frac{(hk)^2 \sin(\tilde{k}) \cos(\tilde{k}) + \mathbf{i}\sqrt{12}\sqrt{12 - (hk)^2}}{12 - (hk)^2 \cos^2(\tilde{k})} \end{aligned}$$

This readily implies for $kh \to 0$ (which also implies $\tilde{k}h \to 0$) the relation

$$\tilde{\alpha}_i = e^{\mathbf{i}\tilde{k}x_i} + (A(k, \tilde{k}) - \mathbf{i}) \sin(\tilde{k} x_i) = e^{\mathbf{i}\tilde{k}x_i} + O(kh)^2 = e^{\mathbf{i}\tilde{k}x_i}(1 + O((kh)^2)) \qquad \text{(A.2)}$$

The basic mechanism to obtain additional powers of k^{-1} on the continuous level is an integration by parts. On the discrete level, this role is taken by the summation by parts formula:

$$\sum_{j=0}^{N} x_j y_j = y_N \sum_{j=0}^{N} x_j - \sum_{j=0}^{N-1} \sum_{l=0}^{j} x_l \big(y_{j+1} - y_j \big) \qquad \text{(A.3)}$$

We will also require the identity

$$\sum_{l=0}^{j} \sin \tilde{k} x_l = \frac{\sin \frac{j\tilde{k}h}{2} \sin(\frac{j+1}{2}\tilde{k}h)}{\sin \frac{\tilde{k}h}{2}} = \frac{\cos \frac{\tilde{k}h}{2} - \cos(j + \frac{1}{2})\tilde{k}h}{2 \sin \frac{\tilde{k}h}{2}} \qquad \text{(A.4)}$$

We start by studying the discrete right-hand side defined by

$$r_h(x_j) = h \int_\Omega f(y) \varphi_j(y) dy$$

which has the following properties:

Lemma A.1. *(i)* $r_h(0) = 0$

(ii) $r_h(x_N) = \frac{h^2}{2} f(x_N) - \frac{h^3}{6} f'(x_N) + O(h^4 \|f''\|_\infty)$

(iii) $r_h(x_j) = h^2 f(x_j) + O(h^4 \|f''\|_\infty)$, *for* $0 < i < N$

(iv) $r_h(x_1) - r_h(0) = r_h(x_1)$

(v) $r_h(x_N) - r_h(x_{N-1}) = -\frac{h^2}{2} f(x_{N-1/2}) + f'(x_{N-1/2})\frac{7h^3}{12} + O(h^4\|f''\|_\infty)$

(vi) $r_h(x_{j+1}) - r_h(x_j) = h^3 f'(x_{j+1/2}) + \frac{h^5}{8} f'''(x_{j+1/2}) + O(h^6\|f^{(4)}\|_\infty)$ *for* $0 < j < N-2$

(vii) $r_h(x_{j+1}) - r_h(x_j) = h^3 f'(x_{j+1/2}) + O(h^4\|f''\|_\infty)$

(viii) $r_h(x_{j+1}) - r_h(x_j) = h^3 f'(x_{j+1/2}) + O(h^5\|f'''\|_\infty)$

Proof. In more detail, we have due to the Dirichlet boundary conditions on the left boundary that $r_h(0) = 0$. Further we can calculate

$$\begin{aligned}
r_h(N) &= h\int_0^1 f(y)\varphi_N(y)dy = h\int_{x_{N-1}}^{x_N} f(y)\frac{y-x_{N-1}}{h}dy \\
&= \int_{x_{N-1}}^{x_N} \big(f(x_N) + f'(x_N)(y-x_N) + f''(x_N)\frac{(y-x_N)^2}{2} + O((y-x_N)^3)\big)(y-x_{N-1})dy \\
&= f(x_N)\int_{x_{N-1}}^{x_N}(y-x_{N-1})dy + f'(x_N)\int_{x_{N-1}}^{x_N}(y-x_N)(y-x_{N-1})dy \\
&\quad + f''(x_N)\int_{x_{N-1}}^{x_N}\frac{(y-x_N)^2}{2}(y-x_{N-1}) + \int_{x_{N-1}}^{x_N} O((y-x_N)^3)(y-x_{N-1})dy \\
&= f(x_N)\frac{h^2}{2} - \frac{h^3}{6}f'(x_N) + O(h^4)
\end{aligned}$$

and

$$\begin{aligned}
r_h(j) &= h\int_0^1 f(y)\varphi_i(y)dy = h\int_{x_{j-1}}^{x_j} f(y)\frac{y-x_{j-1}}{h}dy + h\int_{x_j}^{x_{j+1}} f(y)\frac{x_{i+1}-y}{h}dy \\
&= \int_{x_{j-1}}^{x_j}\big(f(x_j) + f'(x_j)(y-x_j) + f''(x_j)\frac{(y-x_j)^2}{2} + O((y-x_j)^3)\big)(y-x_{j-1})dy \\
&\quad + \int_{x_j}^{x_{j+1}}\big(f(x_j) + f'(x_j)(y-x_j) + f''(x_j)\frac{(y-x_j)^2}{2} + O((y-x_j)^3)\big)(x_{j+1}-y)dy \\
&= f(x_j)\Big(\int_{x_{j-1}}^{x_j}(y-x_{j-1})dy + \int_{x_j}^{x_{j+1}}(x_{j+1}-y)dy\Big) \\
&\quad + f'(x_j)\Big(\int_{x_{j-1}}^{x_j}(y-x_j)(y-x_{j-1})dy + \int_{x_j}^{x_{j+1}}(y-x_j)(x_{j+1}-y)dy\Big) \\
&\quad + f''(x_j)/2\Big(\int_{x_{j-1}}^{x_j}(y-x_j)^2(y-x_{j-1})dy + \int_{x_j}^{x_{j+1}}(y-x_j)^2(x_{j+1}-y)dy\Big) + \ldots \\
&= f(x_j)h^2 + O(h^4),
\end{aligned}$$

for $0 < i < N$. Property (iv) follows obviously from (i). Next,

$$\begin{aligned}
&r_h(x_N) - r_h(x_{N-1}) = \\
&= h\int_0^1 f(y)\varphi_N(y)dy - h\int_0^1 f(y)\varphi_{N-1}(y)dy \\
&= -h\int_{x_{N-2}}^{x_{N-1}} f(y)\varphi_{N-1}(y)dy + h\int_{x_{N-1}}^{x_N} f(y)\big(\varphi_N(y) - \varphi_{N-1}(y)\big)dy \\
&= -\int_{x_{N-2}}^{x_{N-1}}\big(f(x_{N-1/2}) + f'(x_{N-1/2})(y-x_{N-1/2}) + O((y-x_{N-1/2})^2)\big)(y-x_{N-2})dy
\end{aligned}$$

$$
\begin{aligned}
&+\int_{x_{N-1}}^{x_N} \big(f(x_{N-1/2})+f'(x_{N-1/2})(y-x_{N-1/2})+O((y-x_{N-1/2})^2)\big)(2y-(x_N+x_{N-1}))dy\\
=\;& f(x_{N-1/2})\Big(-\int_{x_{N-2}}^{x_{N-1}}(y-x_{N-2})dy+\int_{x_{N-1}}^{x_N}(2y-(x_N+x_{N-1}))dy\Big)\\
&+f'(x_{N-1/2})\Big(-\int_{x_{N-2}}^{x_{N-1}}(y-x_{N-1/2})(y-x_{N-2})dy+\int_{x_{N-1}}^{x_N}(y-x_{N-1/2})(2y-(x_N+x_{N-1}))dy\Big)\\
&+\Big(\int_{x_{N-2}}^{x_{N-1}}O((y-x_{N-1/2})^2)(y-x_{N-2})dy+\int_{x_{N-1}}^{x_N}O((y-x_{N-1/2})^2)(2y-(x_N+x_{N-1}))dy\Big)\\
=\;& -f(x_{N-1/2})\frac{h^2}{2}+f'(x_{N-1/2})\frac{7h^3}{12}+O(h^4).
\end{aligned}
$$

Finally

$$
\begin{aligned}
r_h(x_{j+1})-r_h(x_j) &= h\int_0^1 f(y)\varphi_{j+1}(y)dy-h\int_0^1 f(y)\varphi_j(y)dy = h\int_0^1 f(y)\left(\varphi_{j+1}(y)-\varphi_j(y)\right)dy\\
&= h\int_0^1 \Big(f(x_{j+1/2})+f'(x_{j+1/2})(y-x_{j+1/2})+f''(x_{j+1/2})\,\frac{(y-x_{j+1/2})^2}{2}\\
&\qquad +f'''(x_{j+1/2})\,\frac{(y-x_{j+1/2})^3}{6}+O((y-x_{j+1/2})^4)\Big)\big(\varphi_{j+1}(y)-\varphi_j(y)\big)dy\\
&= hf(x_{j+1/2})\int_0^1 \varphi_{j+1}(y)-\varphi_j(y)dy\\
&\quad +hf'(x_{j+1/2})\int_0^1 (y-x_{j+1/2})(\varphi_{j+1}(y)-\varphi_j(y))dy\\
&\quad +hf''(x_{j+1/2})\int_0^1 \frac{(y-x_{j+1/2})^2}{2}(\varphi_{j+1}(y)-\varphi_j(y))dy\\
&\quad +hf'''(x_{j+1/2})\int_0^1 \frac{(y-x_{j+1/2})^3}{6}(\varphi_{j+1}(y)-\varphi_j(y))dy\\
&\quad +h\int_0^1 O((y-x_{j+1/2})^4)(\varphi_{j+1}(y)-\varphi_j(y))dy\\
&= h^3f'(x_{j+1/2})+\frac{h^5}{8}f'''(x_{j+1/2})+h\int_0^1 O((y-x_{j+1/2})^4)(\varphi_{j+1}(y)-\varphi_j(y))dy\\
&= h^3f'(x_{j+1/2})+\frac{h^5}{8}f'''(x_{j+1/2})+O(h^6)
\end{aligned}
$$

for $0<j<N-2$. □

A.1 The term A_i

In order to estimate the term A_i, we will need the following result:

Lemma A.2.

$$
\begin{aligned}
&\sum_{j=0}^{i-1}\sin(\tilde{k}x_j)r_h(x_j)+\frac{1}{2}\sin(\tilde{k}x_i)r_h(x_i)=\\
&=\frac{h^2\cos(\frac{h\tilde{k}}{2})}{2\sin(\frac{h\tilde{k}}{2})}\big[f(0)+\cos(\tilde{k}x_i)f(x_i)+O(h^2\|f''\|_\infty)\big]+\frac{h^2}{2\sin(\frac{h\tilde{k}}{2})}\sum_{j=0}^{i-1}\cos(\tilde{k}x_{j+1/2})\big(hf'(x_{j+1/2})+O(h^3\|f'''\|_\infty)\big)
\end{aligned}
$$

Proof. The essential ingredient of the proof is a summation by parts given by (A.3). Together with (A.4) we get

$$
\begin{aligned}
&\sum_{j=0}^{i-1} \sin(\tilde{k}x_j) r_h(x_j) + \frac{1}{2}\sin(\tilde{k}x_i) r_h(x_i) = \\
&= \left(\sum_{j=0}^{i-1} \sin(\tilde{k}x_j)\right) r_h(x_{i-1}) - \sum_{j=0}^{i-2}\left(\sum_{l=0}^{j} \sin(\tilde{k}x_l)\right)\big(r_h(x_{j+1}) - r_h(x_j)\big) + \frac{1}{2}\sin(\tilde{k}x_i) r_h(x_i) \\
&= \frac{1}{2\sin(\frac{h\tilde{k}}{2})}\big(\cos(\frac{h\tilde{k}}{2}) - \cos(\tilde{k}x_i - \frac{h\tilde{k}}{2})\big) r_h(x_{i-1}) \\
&\quad - \frac{1}{2\sin(\frac{h\tilde{k}}{2})}\sum_{j=0}^{i-2}\big(\cos(\frac{h\tilde{k}}{2}) - \cos(\tilde{k}x_j + \frac{h\tilde{k}}{2})\big)\big(r_h(x_{j+1}) - r_h(x_j)\big) + \frac{1}{2}\sin(\tilde{k}x_i) r_h(x_i) \\
&= \frac{1}{2\sin(\frac{h\tilde{k}}{2})}\Big[\big(\cos(\frac{h\tilde{k}}{2}) - \cos(\tilde{k}x_i - \frac{h\tilde{k}}{2})\big) r_h(x_{i-1}) - \cos(\frac{h\tilde{k}}{2})\sum_{j=0}^{i-2}\big(r_h(x_{j+1}) - r_h(x_j)\big) \\
&\quad + \sum_{j=0}^{i-2}\cos(\tilde{k}x_j + \frac{h\tilde{k}}{2})\big(r_h(x_{j+1}) - r_h(x_j)\big) + \sin(\frac{h\tilde{k}}{2})\sin(\tilde{k}x_i) r_h(x_i)\Big] \\
&= \frac{1}{2\sin(\frac{h\tilde{k}}{2})}\Big[\big(\cos(\frac{h\tilde{k}}{2}) - \cos(\tilde{k}x_i - \frac{h\tilde{k}}{2})\big) r_h(x_{i-1}) - \cos(\frac{h\tilde{k}}{2})\big(r_h(x_{i-1}) - r_h(0)\big) \\
&\quad + \sum_{j=1}^{i-1}\cos(\tilde{k}x_j + \frac{h\tilde{k}}{2})\big(r_h(x_{j+1}) - r_h(x_j)\big) - \cos(\tilde{k}x_i - \frac{h\tilde{k}}{2})\big(r_h(x_i) - r_h(x_{i-1})\big) \\
&\quad + \cos(\frac{h\tilde{k}}{2})\big(r_h(x_1) - r_h(0) + \sin(\frac{h\tilde{k}}{2})\sin(\tilde{k}x_i) r_h(x_i)\Big] \\
&= \frac{\cos(\frac{h\tilde{k}}{2})}{2\sin(\frac{h\tilde{k}}{2})} r_h(x_1) + \frac{\cos(\frac{h\tilde{k}}{2})}{2\sin(\frac{h\tilde{k}}{2})}\cos(\tilde{k}x_i) r_h(x_i) + \frac{1}{2\sin(\frac{h\tilde{k}}{2})}\sum_{j=1}^{i-1}\cos(\tilde{k}x_j + \frac{h\tilde{k}}{2})\big(r_h(x_{j+1}) - r_h(x_j)\big)
\end{aligned}
$$

Then, after applying the properties of the discrete right-hand side given in Lemma A.1, we get

$$
\begin{aligned}
&\sum_{j=0}^{i-1} \sin(\tilde{k}x_j) r_h(x_j) + \frac{1}{2}\sin(\tilde{k}x_i) r_h(x_i) = \\
&= \frac{h^2\cos(\frac{h\tilde{k}}{2})}{2\sin(\frac{h\tilde{k}}{2})}\big(f(x_1) + O(h^2\|f''\|_\infty)\big) + \frac{h^2\cos(\frac{h\tilde{k}}{2})}{2\sin(\frac{h\tilde{k}}{2})}\cos(\tilde{k}x_i)\big(f(x_i) + O(h^2\|f''\|_\infty)\big) \\
&\quad + \frac{1}{2\sin(\frac{h\tilde{k}}{2})}\sum_{j=1}^{i-1}\cos(\tilde{k}x_j + \frac{h\tilde{k}}{2})\big(h^3 f'(x_{j+1/2}) + O(h^5\|f'''\|_\infty)\big) \\
&= \frac{h^2\cos(\frac{h\tilde{k}}{2})}{2\sin(\frac{h\tilde{k}}{2})}\big(f(x_1) + O(h^2\|f''\|_\infty) - hf'(x_{1/2}) + O(h^3\|f'''\|_\infty)\big) + \frac{h^2\cos(\frac{h\tilde{k}}{2})}{2\sin(\frac{h\tilde{k}}{2})}\cos(\tilde{k}x_i)\big(f(x_i) + O(h^2\|f''\|_\infty)\big) \\
&\quad + \frac{h^2}{2\sin(\frac{h\tilde{k}}{2})}\sum_{j=0}^{i-1}\cos(\tilde{k}x_j + \frac{h\tilde{k}}{2})\big(hf'(x_{j+1/2}) + O(h^3\|f'''\|_\infty)\big) \\
&= \frac{h^2\cos(\frac{h\tilde{k}}{2})}{2\sin(\frac{h\tilde{k}}{2})}\big(f(0) + hf'(0) + O(h^2\|f''\|_\infty) - h(f'(0) + \frac{h}{2}f''(0) + O(h^2\|f'''\|_\infty)\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{h^2\cos(\frac{h\tilde{k}}{2})}{2\sin(\frac{h\tilde{k}}{2})}\cos(\tilde{k}x_i)\big(f(x_i)+O(h^2\|f''\|_\infty)\big)\\
&+\frac{h^2}{2\sin(\frac{h\tilde{k}}{2})}\sum_{j=0}^{i-1}\cos(\tilde{k}x_j+\frac{h\tilde{k}}{2})\big(hf'(x_{j+1/2})+O(h^3\|f'''\|_\infty)\big)\\
=\;&\frac{h^2\cos(\frac{h\tilde{k}}{2})}{2\sin(\frac{h\tilde{k}}{2})}\big(f(0)+O(h^2\|f''\|_\infty)\big)+\frac{h^2\cos(\frac{h\tilde{k}}{2})}{2\sin(\frac{h\tilde{k}}{2})}\cos(\tilde{k}x_i)\big(f(x_i)+O(h^2\|f''\|_\infty)\big)\\
&+\frac{h^2}{2\sin(\frac{h\tilde{k}}{2})}\sum_{j=0}^{i-1}\cos(\tilde{k}x_j+\frac{h\tilde{k}}{2})\big(hf'(x_{j+1/2})+O(h^3\|f'''\|_\infty)\big)
\end{aligned}
$$

□

Using Lemma A.2, we obtain with the definition of $\tilde{\alpha}_i$ in (A.1):

$$
\begin{aligned}
A_i :=\;& \int_0^{x_i} G(x_i,y)f(y)dy-\Big(h\sum_{j=0}^{i-1}G_h^\alpha(x_i,x_j)r_h(x_j)+\frac{1}{2}hG_h^\alpha(x_i,x_i)r_h(x_i)\Big)\\
=\;&\frac{1}{k}e^{\mathbf{i}kx_i}\int_0^{x_i}\sin(ky)f(y)dy\\
&-\Big(h\frac{1}{h\sin(h\tilde{k})}\tilde{\alpha}_i\sum_{j=0}^{i-1}\sin(\tilde{k}x_j)r_h(x_j)+\frac{h}{2h\sin(h\tilde{k})}\tilde{\alpha}_i\sin(\tilde{k}x_i)r_h(x_i)\Big)\\
=\;&\frac{1}{k}e^{\mathbf{i}kx_i}\Big[-\frac{1}{k}\cos(ky)f(y)\Big|_0^{x_i}+\frac{1}{k}\int_0^{x_i}\cos(ky)f'(y)dy\Big]\\
&-\frac{\tilde{\alpha}_i}{\sin(h\tilde{k})}\Big[\sum_{j=0}^{i-1}\sin(\tilde{k}x_j)r_h(x_j)+\frac{1}{2}\sin(\tilde{k}x_i)r_h(x_i)\Big]\\
=\;&\frac{1}{k^2}e^{\mathbf{i}kx_i}\Big[f(0)-\cos(kx_i)f(x_i)+\int_0^{x_i}\cos(ky)f'(y)dy\Big]\\
&-\frac{\tilde{\alpha}_i}{\sin(h\tilde{k})}\Big[\frac{h^2\cos(\frac{h\tilde{k}}{2})}{2\sin(\frac{h\tilde{k}}{2})}\{f(0)+\cos(\tilde{k}x_i)f(x_i)+O(h^2\|f''\|_\infty)\}+\\
&\frac{h^2}{2\sin(\frac{h\tilde{k}}{2})}\sum_{j=0}^{i-1}\cos(\tilde{k}x_{j+1/2})\{hf'(x_{j+1/2})+O(h^3\|f'''\|_\infty)\}\Big]\\
=\;&\left\{\frac{1}{k^2}e^{\mathbf{i}kx_i}f(0)-\frac{h^2}{4\sin^2(\frac{h\tilde{k}}{2})}\tilde{\alpha}_i\big(f(0)+O(h^2\|f''\|_\infty)\big)\right\}\qquad ⓐ\\
&\left\{-\Big[\frac{1}{k^2}e^{\mathbf{i}kx_i}\cos(kx_i)f(x_i)-\frac{h^2}{4\sin^2(\frac{h\tilde{k}}{2})}\tilde{\alpha}_i\cos(\tilde{k}x_i)\big(f(x_i)+O(h^2\|f''\|_\infty)\big)\Big]\right\}\qquad ⓑ\\
&\left\{+\frac{1}{k^2}e^{\mathbf{i}kx_i}\int_0^{x_i}\cos(ky)f'(y)dy-\frac{h^2}{2\sin(h\tilde{k})\sin(\frac{h\tilde{k}}{2})}\tilde{\alpha}_i h\sum_{j=0}^{i-1}\cos(\tilde{k}x_j+\frac{h\tilde{k}}{2})\big(f'(x_{j+1/2})+O(h^2\|f'''\|_\infty)\big)\right\}ⓒ
\end{aligned}
$$

Next, we aim to estimate each of the terms ⓐ, ⓑ, and ⓒ. Therefore we will need the following estimates:

Lemma A.3. *As $kh \to 0$ (and thus $\varepsilon \to 0$) we have (uniformly in x_i, x_j, x, $y \in [0,1]$)*

$$\frac{1}{k^2}\left|e^{\mathbf{i}kx_i} - \frac{(kh)^2}{4\sin^2(\frac{h\tilde{k}}{2})}\tilde{\alpha}_i\right| = O(h^2) + O(k^{-1}\varepsilon) \tag{A.5}$$

$$\frac{1}{k^2}\left|e^{\mathbf{i}kx_i}\cos(kx_j) - \frac{(kh)^2}{4\sin^2(\frac{h\tilde{k}}{2})}\tilde{\alpha}_i\cos(\tilde{k}x_j)\right| = O(h^2) + O(k^{-1}\varepsilon) \tag{A.6}$$

$$e^{\mathbf{i}\tilde{k}x} - e^{\mathbf{i}kx} = O(k\varepsilon) \tag{A.7}$$

$$e^{\mathbf{i}\tilde{k}x}\cos(\tilde{k}y) - e^{\mathbf{i}kx}\cos(ky) = O(k\varepsilon) \tag{A.8}$$

$$\frac{(hk)^2}{4\sin^2(\frac{h\tilde{k}}{2})}e^{\mathbf{i}(\tilde{k}-k)x_i} = (1 + O((kh)^2) + O(\varepsilon))(1 + O(k\varepsilon)). \tag{A.9}$$

Proof. We start with the proof of (A.7):

$$e^{\mathbf{i}\tilde{k}x} - e^{\mathbf{i}kx} = e^{\mathbf{i}kx}\left(e^{\mathbf{i}(\tilde{k}-k)x} - 1\right) = e^{\mathbf{i}kx}\left(e^{\mathbf{i}k\varepsilon x} - 1\right) = O(k\varepsilon), \qquad \forall \varepsilon \to 0. \tag{A.10}$$

(If $k\varepsilon$ is small, then the statement is shown by Taylor expansion; if $k\varepsilon$ is *not* small, then $e^{\mathbf{i}k\varepsilon x} - 1$ is $O(1)$ since k, ε, x are real).

Next, we observe that (A.10) implies

$$\cos(kx) - \cos(\tilde{k}x) = \frac{1}{2}\left(e^{\mathbf{i}kx} - e^{\mathbf{i}\tilde{k}x} + e^{-\mathbf{i}kx} - e^{\mathbf{i}\tilde{k}x}\right) = O(k\varepsilon). \tag{A.11}$$

The bounds (A.8) is shown similarly using (A.7), (A.11):

$$\begin{aligned}
e^{\mathbf{i}\tilde{k}x}\cos(ky) - e^{\mathbf{i}\tilde{k}x}\cos(\tilde{k}y) &= e^{\mathbf{i}\tilde{k}x}\left[\cos(ky) - e^{\mathbf{i}(\tilde{k}-k)x}\left(\cos(\tilde{k}y) - \cos(ky) + \cos(ky)\right)\right] \\
&= e^{\mathbf{i}\tilde{k}x}\left[\cos(ky) - (1 + O(k\varepsilon))(\cos(ky) + O(k\varepsilon))\right] \\
&= O(k\varepsilon).
\end{aligned}$$

(Here, we ignore the term $O(k^2\varepsilon^2)$ that would formally arise since if $k\varepsilon = O(1)$, then the left-hand side is also $O(1)$).

Next, we show (A.9):

$$\begin{aligned}
\frac{(hk)^2}{4\sin^2(\frac{h\tilde{k}}{2})}e^{\mathbf{i}(\tilde{k}-k)x_i} &= \left(\frac{k}{\tilde{k}}\right)^2 \frac{(h\tilde{k})^2}{4\sin^2(\frac{h\tilde{k}}{2})}e^{\mathbf{i}(\tilde{k}-k)x_i} = \left(\frac{1}{1+\varepsilon}\right)^2 \frac{(h\tilde{k})^2}{4\sin^2(\frac{h\tilde{k}}{2})}e^{\mathbf{i}\varepsilon kx_i} \\
&= (1 + O(\varepsilon))\left(1 + O(h\tilde{k})^2\right)(1 + O(k\varepsilon)) \\
&= \left(1 + O((kh)^2) + O(\varepsilon)\right)(1 + O(k\varepsilon)).
\end{aligned}$$

Turning to the proof of (A.5), we get in view of (A.1) and (A.9) that (as $kh \to 0$)

$$\begin{aligned}
\left|\frac{1}{k^2}e^{\mathbf{i}kx_i} - \frac{(kh)^2}{4\sin^2(\frac{h\tilde{k}}{2})}\tilde{\alpha}_i\right| &= \frac{1}{k^2}\left|1 - \frac{(hk)^2}{4\sin^2(\frac{h\tilde{k}}{2})}e^{\mathbf{i}(\tilde{k}-k)x_i}(1 + O(kh)^2)\right| \\
&= \frac{1}{k^2}\left|1 - \frac{(hk)^2}{4\sin^2(\frac{h\tilde{k}}{2})}e^{\mathbf{i}(\tilde{k}-k)x_i}\right| + O(h^2) \\
&= k^{-2}\left\{O((kh)^2) + O(\varepsilon) + (1 + O((kh)^2) + O(\varepsilon))O(k\varepsilon)\right\} + k^2 O(\varepsilon^2)\Big\} + O(h^2) \\
&= k^{-2}\left\{O((kh)^2) + O(k\varepsilon)\right\} + O(h^2) = O(h^2) + O(k^{-1}\varepsilon)
\end{aligned}$$

We finally show (A.6):

$$\frac{1}{k^2}\left|e^{\mathbf{i}kx_i}\cos(kx_j)-\frac{(kh)^2}{4\sin^2(\frac{h\tilde{k}}{2})}\tilde{\alpha}_i\cos(\tilde{k}x_j)\right| = \frac{1}{k^2}\left|e^{\mathbf{i}kx_i}\cos(kx_j)-\frac{(kh)^2}{4\sin^2(\frac{h\tilde{k}}{2})}(e^{\mathbf{i}\tilde{k}x_j}+O(kh)^2)\cos(\tilde{k}x_j)\right|$$
$$= \frac{1}{k^2}\left|e^{\mathbf{i}kx_i}\cos(kx_j)-\frac{(kh)^2}{4\sin^2(\frac{h\tilde{k}}{2})}e^{\mathbf{i}\tilde{k}x_j}\cos(\tilde{k}x_j)\right| + O(h^2)$$
$$= O(h^2)+O(k^{-1}\varepsilon).$$

□

With these estimates, we can analyze ⓐ, ⓑ, ⓒ. From (A.5) we get

$$\text{ⓐ} := \frac{1}{k^2}e^{\mathbf{i}kx_i}f(0)-\frac{h^2}{4\sin(\frac{h\tilde{k}}{2})^2}\tilde{\alpha}_i\big(f(0)+O(h^2\|f''\|_\infty)\big)$$
$$= \frac{1}{k^2}\left(e^{\mathbf{i}kx_i}-\frac{(kh)^2}{4\sin(\frac{h\tilde{k}}{2})^2}\tilde{\alpha}_i\right)f(0)+\frac{1}{k^2}\frac{(kh)^2}{4\sin^2(h\tilde{k}/2)}O(h^2\|f''\|_\infty)$$
$$\lesssim (h^2+k^{-1}\varepsilon)\|f\|_\infty + h^2k^{-2}\|f''\|_\infty.$$

Similarly we get with (A.6):

$$\text{ⓑ} := -\frac{1}{k^2}e^{\mathbf{i}kx_i}\cos(kx_i)f(x_i)+\frac{h^2}{4\sin\left(\frac{h\tilde{k}}{2}\right)^2}\tilde{\alpha}_i\cos(\tilde{k}x_i)\Big[f(x_i)+O(h^2\|f''\|_\infty)\Big]$$
$$= \left[-\frac{1}{k^2}e^{\mathbf{i}kx_i}\cos(kx_i)+\frac{h^2}{4\sin\left(\frac{h\tilde{k}}{2}\right)^2}\tilde{\alpha}_i\cos(\tilde{k}x_i)\right]f(x_i)+O(k^{-2}h^2\|f''\|_\infty)$$
$$\lesssim (h^2+k^{-1}\varepsilon)\|f\|_\infty + k^{-2}h^2\|f''\|_\infty.$$

For the third term, ⓒ, we start with the observation

$$\frac{1}{2\sin(h\tilde{k})\sin(\frac{h\tilde{k}}{2})} = \frac{1}{4\sin^2(\frac{h\tilde{k}}{2})}\frac{1}{\cos(\frac{h\tilde{k}}{2})} = \frac{1}{4\sin^2(\frac{h\tilde{k}}{2})}(1+O(kh)^2), \qquad kh\to 0. \tag{A.12}$$

For the term ⓒ, we discretize the integral by the midpoint rule to get

$$\frac{1}{k^2}e^{\mathbf{i}kx_i}\int_0^{x_i}\cos(ky)f'(y)\,dy = \frac{1}{k^2}e^{\mathbf{i}kx_i}h\sum_{j=0}^{i-1}\cos(kx_{j+1/2})f'(x_{j+1/2})+k^{-2}h^2O(k^2\|f'\|_\infty+k\|f''\|_\infty+\|f'''\|_\infty) \tag{A.13}$$

Using (A.12) and (A.13), (A.6) we get for the third term ⓒ:

$$\text{ⓒ} := \frac{1}{k^2}e^{\mathbf{i}kx_i}\int_0^{x_i}\cos(ky)f'(y)dy-\Big(\frac{1}{2\sin(h\tilde{k})\sin(\frac{h\tilde{k}}{2})}\tilde{\alpha}_i\sum_{j=0}^{i-1}\cos(\tilde{k}x_{j+1/2})\big(h^3f'(x_{j+1/2})+O(\frac{h^5}{8}\|f'''\|_\infty)\Big)$$
$$= \frac{1}{k^2}e^{\mathbf{i}kx_i}h\sum_{j=0}^{i-1}\cos(kx_{j+1/2})f'(x_{j+1/2}) \quad +h^2O(\|f'\|_\infty+k^{-1}\|f''\|_\infty+k^{-2}\|f'''\|_\infty)$$
$$-\frac{h^2}{2\sin(h\tilde{k})\sin(\frac{h\tilde{k}}{2})}\tilde{\alpha}_ih\sum_{j=0}^{i-1}\cos(\tilde{k}x_{j+1/2})\big(f'(x_{j+1/2})+O(\frac{h^2}{8}\|f'''\|_\infty))\big)$$
$$= \frac{1}{k^2}e^{\mathbf{i}kx_i}h\sum_{j=0}^{i-1}\cos(kx_{j+1/2})f'(x_{j+1/2})-\frac{h^2}{2\sin(h\tilde{k})\sin(\frac{h\tilde{k}}{2})}\tilde{\alpha}_ih\sum_{j=0}^{i-1}\cos(\tilde{k}x_{j+1/2})f'(x_{j+1/2})$$

$$
\begin{aligned}
&\quad + h^2 O(\|f'\|_\infty + k^{-1}\|f''\|_\infty + k^{-2}\|f'''\|_\infty) \\
&= h\sum_{j=0}^{i-1}\Big[\frac{1}{k^2}e^{\mathbf{i}kx_i}\cos(kx_{j+1/2}) - \frac{h^2}{4\sin^2(\frac{h\tilde k}{2})}(1+O(kh)^2)\tilde\alpha_i\cos(\tilde k x_{j+1/2})\Big]f'(x_{j+1/2}) \\
&\quad + h^2 O(\|f'\|_\infty + k^{-1}\|f''\|_\infty + k^{-2}\|f'''\|_\infty) \\
&\lesssim (h^2 + k^{-1}\varepsilon)\|f'\|_\infty + h^2k^{-1}\|f''\|_\infty + h^2k^{-2}\|f'''\|_\infty
\end{aligned}
$$

This leads us to

$$
A_i = ⓐ + ⓑ + ⓒ \lesssim \big(h^2 + k^{-1}\varepsilon\big)\big(\|f\|_\infty + \|f'\|_\infty\big) + k^{-1}h^2\|f''\|_\infty + k^{-2}h^2\|f'''\|_\infty\big) \tag{A.14}
$$

A.2 The term B_i

For the second term B_i we will need

Lemma A.4. *For $kh \to 0$ (and thus $\tilde k h \to 0$) we have*

$$
\begin{aligned}
\sum_{j=i+1}^{N} e^{\mathbf{i}\tilde k x_j} r_h(x_j) + \frac12 e^{\mathbf{i}\tilde k x_i} r_h(x_i) = &-\frac{h^2(1+e^{\mathbf{i}\tilde k h})}{2(1-e^{\mathbf{i}\tilde k h})}e^{\mathbf{i}\tilde k}\big(f(1) + O(h^2\|f''\|_\infty)\big) \\
&+ \frac{h^2(1+e^{\mathbf{i}\tilde k h})}{2(1-e^{\mathbf{i}\tilde k h})}e^{\mathbf{i}\tilde k x_i}\big(f(x_i) + O(h^2\|f''\|_\infty)\big) \\
&+ \frac{h^2 e^{\frac{i\tilde k h}{2}}}{(1-e^{\mathbf{i}\tilde k h})}h\sum_{j=i}^{N-1} e^{\mathbf{i}\tilde k x_{j+1/2}}\big(f'(x_{j+1/2}) + O(h^2\|f'''\|_\infty)\big)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{j=i+1}^{N}\sin(\tilde k x_j) r_h(x_j) + \frac12\sin(\tilde k x_i) r_h(x_i) = &-\frac{h^2\cos(\frac{h\tilde k}{2})}{2\sin(\frac{h\tilde k}{2})}\cos(\tilde k x_N) f(1) + O(h^3\|f'\|_\infty + \tilde k^{-1}h^3\|f''\|_\infty) \\
&+ \frac{h^2\cos(\frac{h\tilde k}{2})}{2\sin(\frac{h\tilde k}{2})}\cos(\tilde k x_i)\big(f(x_i) + O(h^2\|f''\|_\infty)\big) \\
&+ \frac{h^2}{2\sin(\frac{h\tilde k}{2})}h\sum_{j=i}^{N-1}\cos(\tilde k x_{j+1/2})\big(f'(x_{j+1/2}) + O(h^2\|f'''\|_\infty)\big)
\end{aligned}
$$

Proof. These two identities are again shown via summation by parts:

$$
\begin{aligned}
&\sum_{j=i+1}^{N} e^{\mathbf{i}\tilde k x_j} r_h(x_j) + \frac12 e^{\mathbf{i}\tilde k x_i} r_h(x_i) = \\
&= \left(\sum_{j=i+1}^{N} e^{\mathbf{i}\tilde k x_j}\right) r_h(x_N) - \sum_{j=i+1}^{N-1}\left(\sum_{l=i+1}^{j} e^{\mathbf{i}\tilde k x_l}\right)\big(r_h(x_{j+1}) - r_h(x_j)\big) + \frac12 e^{\mathbf{i}\tilde k x_i} r_h(x_i) \\
&= \frac{e^{\mathbf{i}\tilde k x_{i+1}} - e^{\mathbf{i}\tilde k x_{N+1}}}{1-e^{\mathbf{i}\tilde k h}} r_h(x_N) - \sum_{j=i+1}^{N-1}\frac{e^{\mathbf{i}\tilde k x_{i+1}} - e^{\mathbf{i}\tilde k x_{j+1}}}{1-e^{\mathbf{i}\tilde k h}}\big(r_h(x_{j+1}) - r_h(x_j)\big) + \frac12 e^{\mathbf{i}\tilde k x_i} r_h(x_i) \\
&= \frac{1}{(1-e^{\mathbf{i}\tilde k h})}\Big[(e^{\mathbf{i}\tilde k x_{i+1}} - e^{\mathbf{i}\tilde k x_{N+1}}) r_h(x_N) - e^{\mathbf{i}\tilde k x_{i+1}}\sum_{j=i+1}^{N-1}\big(r_h(x_{j+1}) - r_h(x_j)\big) \\
&+ \sum_{j=i+1}^{N-1} e^{\mathbf{i}\tilde k x_{j+1}}\big(r_h(x_{j+1}) - r_h(x_j)\big) + \frac12(1-e^{\mathbf{i}\tilde k h})e^{\mathbf{i}\tilde k x_i} r_h(x_i)\Big]
\end{aligned}
$$

$$= \frac{1}{(1-e^{\mathbf{i}\tilde{k}h})}\Big[(e^{\mathbf{i}\tilde{k}x_{i+1}} - e^{\mathbf{i}\tilde{k}x_{N+1}})r_h(x_N) - e^{\mathbf{i}\tilde{k}x_{i+1}}\big(r_h(x_N) - r_h(x_{i+1})\big)$$

$$+ \sum_{j=i}^{N-1} e^{\mathbf{i}\tilde{k}x_{j+1}}\big(r_h(x_{j+1}) - r_h(x_j)\big) - e^{\mathbf{i}\tilde{k}x_{i+1}}\big(r_h(x_{i+1}) - r_h(x_i)\big) + \frac{1}{2}(1-e^{\mathbf{i}\tilde{k}h})e^{\mathbf{i}\tilde{k}x_i}r_h(x_i)\Big]$$

$$= \frac{1}{(1-e^{\mathbf{i}\tilde{k}h})}\Big[-e^{\mathbf{i}\tilde{k}x_{N+1}}r_h(x_N) + e^{\mathbf{i}\tilde{k}x_{i+1}}r_h(x_i) + \frac{1}{2}(1-e^{\mathbf{i}\tilde{k}h})e^{\mathbf{i}\tilde{k}x_i}r_h(x_i)$$

$$+ \sum_{j=i}^{N-2} e^{\mathbf{i}\tilde{k}x_{j+1}}\big(r_h(x_{j+1}) - r_h(x_j)\big) + e^{\mathbf{i}\tilde{k}x_N}\big(r_h(x_N) - r_h(x_{N-1})\big)\Big]$$

$$= \frac{1}{(1-e^{\mathbf{i}\tilde{k}h})}\Big[e^{\mathbf{i}\tilde{k}}\Big(\big(r_h(x_N) - r_h(x_{N-1})\big) - e^{\mathbf{i}\tilde{k}h}r_h(x_N)\Big) + \frac{1}{2}(1+e^{\mathbf{i}\tilde{k}h})e^{\mathbf{i}\tilde{k}x_i}r_h(x_i) + \sum_{j=i}^{N-2} e^{\mathbf{i}\tilde{k}x_{j+1}}\big(r_h(x_{j+1}) - r_h(x_j)\big)\Big]$$

Applying the properties of r_h shown in Lemma A.1 produces

$$\sum_{j=i+1}^{N} e^{\mathbf{i}\tilde{k}x_j}r_h(x_j) + \frac{1}{2}e^{\mathbf{i}\tilde{k}x_i}r_h(x_i) =$$

$$= \frac{1}{(1-e^{\mathbf{i}\tilde{k}h})}\Big[e^{\mathbf{i}\tilde{k}}\Big(\big(-f(x_{N-1/2})\frac{h^2}{2} + f'(x_{N-1/2})\frac{7h^3}{12} + O(h^4\|f''\|_\infty)\big) - e^{\mathbf{i}\tilde{k}h}\big(\frac{h^2}{2}f(x_N) - \frac{h^3}{6}f'(x_N) + O(h^4\|f''\|_\infty)\big)\Big)$$

$$+ \frac{1}{2}(1+e^{\mathbf{i}\tilde{k}h})e^{\mathbf{i}\tilde{k}x_i}\big(f(x_i)h^2 + O(h^4\|f''\|_\infty)\big) + \sum_{j=i}^{N-1} e^{\mathbf{i}\tilde{k}x_{j+1}}\big(h^3 f'(x_{j+1/2}) + O(h^5\|f'''\|_\infty)\big)$$

$$- e^{\mathbf{i}\tilde{k}}\big(h^3 f'(x_{N-1/2}) + O(h^5\|f'''\|_\infty)\big)\Big]$$

$$= \frac{e^{\mathbf{i}\tilde{k}}}{(1-e^{\mathbf{i}\tilde{k}h})}\Big[-\frac{h^2}{2}\Big(f(x_N) - \frac{h}{2}f'(x_N)\Big) + \frac{7h^3}{12}\Big(f'(x_N)\Big) + O(h^4\|f''\|_\infty)$$

$$- e^{\mathbf{i}\tilde{k}h}\big(\frac{h^2}{2}f(x_N) - \frac{h^3}{6}f'(x_N)\big) - h^3\Big(f'(x_N)\Big) + O(h^4\|f'''\|_\infty)\Big]$$

$$+ \frac{1+e^{\mathbf{i}\tilde{k}h}}{2(1-e^{\mathbf{i}\tilde{k}h})}e^{\mathbf{i}\tilde{k}x_i}\big(f(x_i)h^2 + O(h^4\|f''\|_\infty)\big) + \frac{1}{(1-e^{\mathbf{i}\tilde{k}h})}\sum_{j=i}^{N-1} e^{\mathbf{i}\tilde{k}x_{j+1}}\big(h^3 f'(x_{j+1/2}) + O(h^5\|f'''\|_\infty)\big)$$

$$= \frac{e^{\mathbf{i}\tilde{k}}}{(1-e^{\mathbf{i}\tilde{k}h})}\Big[-\frac{h^2}{2}(1+e^{\mathbf{i}\tilde{k}h})f(x_N) - \frac{h^3}{6}(1-e^{\mathbf{i}\tilde{k}h})f'(x_N) + O(h^4\|f''\|_\infty)\Big]$$

$$+ \frac{1+e^{\mathbf{i}\tilde{k}h}}{2(1-e^{\mathbf{i}\tilde{k}h})}e^{\mathbf{i}\tilde{k}x_i}\big(f(x_i)h^2 + O(h^4\|f''\|_\infty)\big) + \frac{1}{(1-e^{\mathbf{i}\tilde{k}h})}\sum_{j=i}^{N-1} e^{\mathbf{i}\tilde{k}x_{j+1}}\big(h^3 f'(x_{j+1/2}) + O(h^5\|f'''\|_\infty)\big)$$

$$= -\frac{h^2(1+e^{\mathbf{i}\tilde{k}h})}{2(1-e^{\mathbf{i}\tilde{k}h})}e^{\mathbf{i}\tilde{k}}\,\big(f(1) + O(h^2\|f''\|_\infty)\big) - \frac{h^3}{6}f'(1)e^{\mathbf{i}\tilde{k}} + \frac{h^2(1+e^{\mathbf{i}\tilde{k}h})}{2(1-e^{\mathbf{i}\tilde{k}h})}e^{\mathbf{i}\tilde{k}x_i}\big(f(x_i) + O(h^2\|f''\|_\infty)\big)$$

$$+ \frac{e^{\frac{i\tilde{k}h}{2}}}{(1-e^{\mathbf{i}\tilde{k}h})}\sum_{j=i}^{N-1} e^{\mathbf{i}\tilde{k}x_{j+1/2}}\big(h^3 f'(x_{j+1/2}) + O(h^5\|f'''\|_\infty)\big)$$

$$= -\frac{h^2(1+e^{\mathbf{i}\tilde{k}h})}{2(1-e^{\mathbf{i}\tilde{k}h})}e^{\mathbf{i}\tilde{k}}\big(f(1) + O(h^2\|f''\|_\infty)\big) + \frac{h^2(1+e^{\mathbf{i}\tilde{k}h})}{2(1-e^{\mathbf{i}\tilde{k}h})}e^{\mathbf{i}\tilde{k}x_i}\big(f(x_i) + O(h^2\|f''\|_\infty)\big)$$

$$+ \frac{h^2 e^{\frac{i\tilde{k}h}{2}}}{(1-e^{\mathbf{i}\tilde{k}h})}h\sum_{j=i}^{N-1} e^{\mathbf{i}\tilde{k}x_{j+1/2}}\big(f'(x_{j+1/2}) + O(h^2\|f''\|_\infty)\big)$$

For the second identity of the lemma, we calculate with the summation by parts formula (A.3) and the trigonometric identity (A.4):

$$\sum_{j=i+1}^{N} \sin(\tilde{k}x_j) r_h(x_j) + \frac{1}{2}\sin(\tilde{k}x_i) r_h(x_i) =$$

$$= \left(\sum_{j=i+1}^{N} \sin(\tilde{k}x_j)\right) r_h(x_N) - \sum_{j=i+1}^{N-1}\left(\sum_{l=i+1}^{j} \sin(\tilde{k}x_l)\right)\big(r_h(x_{j+1}) - r_h(x_j)\big) + \frac{1}{2}\sin(\tilde{k}x_i) r_h(x_i)$$

$$= \frac{1}{2\sin(\frac{h\tilde{k}}{2})}\big(\cos(\tilde{k}x_{i+1/2}) - \cos(\tilde{k}x_{N+1/2})\big) r_h(x_N)$$

$$- \frac{1}{2\sin(\frac{h\tilde{k}}{2})}\sum_{j=i+1}^{N-1}\big(\cos(\tilde{k}x_{i+1/2}) - \cos(\tilde{k}x_{j+1/2})\big)\big(r_h(x_{j+1}) - r_h(x_j)\big) + \frac{1}{2}\sin(\tilde{k}x_i) r_h(x_i)$$

$$= \frac{1}{2\sin(\frac{h\tilde{k}}{2})}\Big[\big(\cos(\tilde{k}x_{i+1/2}) - \cos(\tilde{k}x_{N+1/2})\big) r_h(x_N) - \cos(\tilde{k}x_{i+1/2})\sum_{j=i+1}^{N-1}\big(r_h(x_{j+1}) - r_h(x_j)\big)$$

$$+ \sum_{j=i+1}^{N-1}\cos(\tilde{k}x_{j+1/2})\big(r_h(x_{j+1}) - r_h(x_j)\big) + \sin(\frac{h\tilde{k}}{2})\sin(\tilde{k}x_i) r_h(x_i)\Big]$$

$$= \frac{1}{2\sin(\frac{h\tilde{k}}{2})}\Big[\big(\cos(\tilde{k}x_{i+1/2}) - \cos(\tilde{k}x_{N+1/2})\big) r_h(x_N) - \cos(\tilde{k}x_{i+1/2})\big(r_h(x_N) - r_h(x_{i+1})\big)$$

$$+ \sum_{j=i}^{N-2}\cos(\tilde{k}x_{j+1/2})\big(r_h(x_{j+1}) - r_h(x_j)\big) - \cos(\tilde{k}x_{i+1/2})\big(r_h(x_{i+1}) - r_h(x_i)\big)$$

$$+ \cos(\tilde{k}x_{N-1/2})\big(r_h(x_N) - r_h(x_{N-1})\big) + \sin(\frac{h\tilde{k}}{2})\sin(\tilde{k}x_i) r_h(x_i)\Big]$$

$$= \frac{1}{2\sin(\frac{h\tilde{k}}{2})}\Big[\big(\cos(\tilde{k}x_{N-1/2}) - \cos(\tilde{k}x_{N+1/2})\big) r_h(x_N) - \cos(\tilde{k}x_{N-1/2}) r_h(x_{N-1})\big)$$

$$+ \sum_{j=i}^{N-2}\cos(\tilde{k}x_{j+1/2})\big(r_h(x_{j+1}) - r_h(x_j)\big) + \Big(\cos(\tilde{k}x_{i+1/2}) + \sin(\frac{h\tilde{k}}{2})\sin(\tilde{k}x_i)\Big) r_h(x_i)\Big]$$

$$= \sin(\tilde{k}x_N) r_h(x_N) - \frac{1}{2\sin(\frac{h\tilde{k}}{2})}\cos(\tilde{k}x_{N-1/2}) r_h(x_{N-1}) + \frac{\cos(\frac{h\tilde{k}}{2})}{2\sin(\frac{h\tilde{k}}{2})}\cos(\tilde{k}x_i) r_h(x_i)$$

$$+ \frac{1}{2\sin(\frac{h\tilde{k}}{2})}\sum_{j=i}^{N-2}\cos(\tilde{k}x_{j+1/2})\big(r_h(x_{j+1}) - r_h(x_j)\big)$$

$$= \sin(\tilde{k}x_N)\big(\frac{h^2}{2}f(x_N) - \frac{h^3}{6}f'(x_N) + O(h^4\|f''\|_\infty)\big) - \frac{1}{2\sin(\frac{h\tilde{k}}{2})}\cos(\tilde{k}x_{N-1/2})\big(h^2 f(x_{N-1}) + O(h^4\|f''\|_\infty)\big)$$

$$+ \frac{\cos(\frac{h\tilde{k}}{2})}{2\sin(\frac{h\tilde{k}}{2})}\cos(\tilde{k}x_i)\big(h^2 f(x_i) + O(h^4\|f''\|_\infty)\big) + \frac{1}{2\sin(\frac{h\tilde{k}}{2})}\sum_{j=i}^{N-1}\cos(\tilde{k}x_{j+1/2})\big(h^3 f'(x_{j+1/2}) + O(h^5\|f'''\|_\infty)\big)$$

$$- \frac{1}{2\sin(\frac{h\tilde{k}}{2})}\cos(\tilde{k}x_{N-1/2})\big(h^3 f'(x_{N-1/2}) + O(h^5\|f'''\|_\infty)\big)$$

$$= \sin(\tilde{k}x_N)\big(\frac{h^2}{2}f(x_N) - \frac{h^3}{6}f'(x_N) + O(h^4\|f''\|_\infty)\big)$$

$$
\begin{aligned}
&-\frac{1}{2\sin(\frac{h\tilde{k}}{2})}\cos(\tilde{k}x_{N-1/2})\big(h^2(f(x_N)-hf'(x_N)+O(h^2\|f''\|_\infty))+O(h^4\|f''\|_\infty)\big)\\
&+\frac{\cos(\frac{h\tilde{k}}{2})}{2\sin(\frac{h\tilde{k}}{2})}\cos(\tilde{k}x_i)\big(h^2f(x_i)+O(h^4\|f''\|_\infty)\big)+\frac{1}{2\sin(\frac{h\tilde{k}}{2})}\sum_{j=i}^{N-1}\cos(\tilde{k}x_{j+1/2})\big(h^3f'(x_{j+1/2})+O(h^5\|f'''\|_\infty)\big)\\
&-\frac{1}{2\sin(\frac{h\tilde{k}}{2})}\cos(\tilde{k}x_{N-1/2})\big(h^3(f'(x_N)-\frac{h}{2}f''(x_N)+O(h^2\|f'''\|_\infty)))+O(h^5\|f'''\|_\infty)\big)\\
&=-\frac{h^2\cos(\frac{h\tilde{k}}{2})}{2\sin(\frac{h\tilde{k}}{2})}\cos(\tilde{k}x_N)f(x_N)-\frac{h^3}{6}f'(1)\sin(\tilde{k})+\sin(\tilde{k})O(\tilde{k}^{-1}h^3\|f''\|_\infty)+\frac{h^2\cos(\frac{h\tilde{k}}{2})}{2\sin(\frac{h\tilde{k}}{2})}\cos(\tilde{k}x_i)\big(f(x_i)+O(h^2\|f''\|_\infty)\big)\\
&+\frac{1}{2\sin(\frac{h\tilde{k}}{2})}\sum_{j=i}^{N-1}\cos(\tilde{k}x_{j+1/2})\big(h^3f'(x_{j+1/2})+O(h^5\|f'''\|_\infty)\big)\\
&=-\frac{h^2\cos(\frac{h\tilde{k}}{2})}{2\sin(\frac{h\tilde{k}}{2})}\cos(\tilde{k}x_N)f(x_N)+O(h^3\|f'\|_\infty+\tilde{k}^{-1}h^3\|f''\|_\infty)+\frac{h^2\cos(\frac{h\tilde{k}}{2})}{2\sin(\frac{h\tilde{k}}{2})}\cos(\tilde{k}x_i)\big(f(x_i)+O(h^2\|f''\|_\infty)\big)\\
&+\frac{h^2}{2\sin(\frac{h\tilde{k}}{2})}h\sum_{j=i}^{N-1}\cos(\tilde{k}x_{j+1/2})\big(f'(x_{j+1/2})+O(h^2\|f'''\|_\infty)\big)
\end{aligned}
$$

□

Thus, using the structure of $\tilde{\alpha}_j$ given in (A.1) and Lemma A.4

$$
\begin{aligned}
B_i &:= \int_{x_i}^1 G(x_i,y)f(y)dy-\Big(h\sum_{j=i+1}^{N}G_h^\alpha(x_i,x_j)r_h(x_j)+\frac{h}{2}G_h^\alpha(x_i,x_i)r_h(x_i)\Big)\\
&=\frac{1}{k}\sin(kx_i)\int_{x_i}^1 e^{\mathbf{i}ky}f(y)dy-\Big(\frac{h}{h\sin(h\tilde{k})}\sin(\tilde{k}x_i)\sum_{j=i+1}^{N}\tilde{\alpha}_jr_h(x_j)+\frac{h}{2h\sin(h\tilde{k})}\sin(\tilde{k}x_i)\tilde{\alpha}_ir_h(x_i)\Big)\\
&=\frac{1}{k}\sin(kx_i)\Big[\frac{1}{\mathbf{i}k}e^{\mathbf{i}ky}f(y)\Big|_{x_i}^1-\frac{1}{\mathbf{i}k}\int_{x_i}^1 e^{\mathbf{i}ky}f'(y)dy\Big]\\
&\quad-\frac{1}{\sin(h\tilde{k})}\sin(\tilde{k}x_i)\Big[\sum_{j=i+1}^{N}\big(e^{\mathbf{i}\tilde{k}x_j}+\sin(\tilde{k}x_j)(A(k,\tilde{k})-\mathbf{i})\big)r_h(x_j)+\frac{1}{2}\big(e^{\mathbf{i}\tilde{k}x_i}+\sin(\tilde{k}x_i)(A(k,\tilde{k})-\mathbf{i})\big)r_h(x_i)\Big]\\
&=\frac{1}{\mathbf{i}k^2}\sin(kx_i)\Big[e^{\mathbf{i}k}f(1)-e^{\mathbf{i}kx_i}f(x_i)-\int_{x_i}^1 e^{\mathbf{i}ky}f'(y)dy\Big]\\
&\quad-\frac{1}{\sin(h\tilde{k})}\sin(\tilde{k}x_i)\Big[\sum_{j=i+1}^{N}e^{\mathbf{i}\tilde{k}x_j}r_h(x_j)+\frac{1}{2}e^{\mathbf{i}\tilde{k}x_i}r_h(x_i)+\Big(\sum_{j=i+1}^{N}\sin(\tilde{k}x_j)r_h(x_j)+\frac{1}{2}\sin(\tilde{k}x_i)r_h(x_i)\Big)(A(k,\tilde{k})-\mathbf{i})\Big]\\
&=\frac{1}{\mathbf{i}k^2}\sin(kx_i)\Big[e^{\mathbf{i}k}f(1)-e^{\mathbf{i}kx_i}f(x_i)-\int_{x_i}^1 e^{\mathbf{i}ky}f'(y)dy\Big]\\
&\quad-\frac{1}{\sin(h\tilde{k})}\sin(\tilde{k}x_i)\Big(-\frac{h^2(1+e^{\mathbf{i}\tilde{k}h})}{2(1-e^{\mathbf{i}\tilde{k}h})}\big(e^{\mathbf{i}\tilde{k}}f(1)-e^{\mathbf{i}\tilde{k}x_i}f(x_i)+O(h^2\|f''\|_\infty)\big)\\
&\quad+\frac{h^2e^{\frac{\mathbf{i}\tilde{k}h}{2}}}{(1-e^{\mathbf{i}\tilde{k}h})}h\sum_{j=i}^{N-1}e^{\mathbf{i}\tilde{k}x_{j+1/2}}\big(f'(x_{j+1/2})+O(h^2\|f''\|_\infty)\big)\Big)\\
&\quad-\frac{1}{\sin(h\tilde{k})}\sin(\tilde{k}x_i)(A(k,\tilde{k})-\mathbf{i})\Big\{-\frac{h^2\cos(\frac{h\tilde{k}}{2})}{2\sin(\frac{h\tilde{k}}{2})}\big(\cos(\tilde{k})f(1)-\cos(\tilde{k}x_i)f(x_i)\big)+O(h^3\|f'\|_\infty+\tilde{k}^{-1}h^3\|f''\|_\infty)
\end{aligned}
$$

$$+\frac{h^2}{2\sin(\frac{h\tilde{k}}{2})}h\sum_{j=i}^{N-1}\cos(\tilde{k}x_{j+1/2})\big(f'(x_{j+1/2})+O(h^2\|f'''\|_\infty)\big)\Big\}$$

$$=\Big\{\frac{1}{\mathbf{i}k^2}\sin(kx_i)e^{\mathbf{i}k}f(1)+\frac{h^2(1+e^{\mathbf{i}\tilde{k}h})}{2\sin(h\tilde{k})(1-e^{\mathbf{i}\tilde{k}h})}\sin(\tilde{k}x_i)e^{\mathbf{i}\tilde{k}}f(1)\Big\} \quad ⓓ$$

$$+\Big\{\frac{h^2}{4\sin^2(\frac{\tilde{k}h}{2})}\sin(\tilde{k}x_i)\cos(\tilde{k})f(1)(A(k,\tilde{k})-\mathbf{i})$$

$$-\frac{1}{\mathbf{i}k^2}\sin(kx_i)e^{\mathbf{i}kx_i}f(x_i)-\frac{h^2(1+e^{\mathbf{i}\tilde{k}h})}{2\sin(h\tilde{k})(1-e^{\mathbf{i}\tilde{k}h})}\sin(\tilde{k}x_i)e^{\mathbf{i}\tilde{k}x_i}f(x_i)\Big\} \quad ⓔ$$

$$\Big\{-\frac{h^2}{4\sin^2(\frac{\tilde{k}h}{2})}\sin(\tilde{k}x_i)\cos(\tilde{k}x_i)f(x_i)(A(k,\tilde{k})-\mathbf{i})$$

$$-\frac{1}{\mathbf{i}k^2}\sin(kx_i)\int_{x_i}^1 e^{\mathbf{i}ky}f'(y)dy-\frac{h^2e^{\frac{\mathbf{i}\tilde{k}h}{2}}}{\sin(h\tilde{k})(1-e^{\mathbf{i}\tilde{k}h})}\sin(\tilde{k}x_i)h\sum_{j=i}^{N-1}e^{\mathbf{i}\tilde{k}x_{j+1/2}}\big(f'(x_{j+1/2})+O(h^2\|f'''\|_\infty)\big)$$

$$+\frac{h^2}{2\sin(\frac{h\tilde{k}}{2})}h\sum_{j=i}^{N-1}\cos(\tilde{k}x_{j+1/2})\big(f'(x_{j+1/2})+O(h^2\|f'''\|_\infty)\big)(A(k,\tilde{k})-\mathbf{i})\Big\} \quad ⓕ$$

In order to simplify these three terms, we need a lemma:

Lemma A.5. *For $kh\to 0$ (and thus $\tilde{k}h\to 0$)*

$$A(k,\tilde{k})-\mathbf{i}=O((kh)^2) \tag{A.15}$$

$$\frac{1}{\mathbf{i}k^2}\sin(kx_i)e^{\mathbf{i}k}+\frac{h^2(1+e^{\mathbf{i}\tilde{k}h})}{2\sin(h\tilde{k})(1-e^{\mathbf{i}\tilde{k}h})}\sin(\tilde{k}x_i)e^{\mathbf{i}\tilde{k}}=O(k\varepsilon)+O(h^2) \tag{A.16}$$

$$\frac{1}{\mathbf{i}k^2}\sin(kx_i)e^{\mathbf{i}ky}+\frac{h^2e^{\mathbf{i}\tilde{k}h/2}}{\sin(h\tilde{k})(1-e^{\mathbf{i}h\tilde{k}})}\sin(\tilde{k}x_i)e^{\mathbf{i}\tilde{k}y}=O(h^2)+O(k^{-1}\varepsilon). \tag{A.17}$$

Proof. (A.15) follows from the definition of $A(k,\tilde{k})$ and a straight forward Taylor expansion.

For (A.16), we first note that the estimate is trivial if $k\varepsilon=O(1)$ (and kh is small). We may therefore assume that additionally $k\varepsilon$ is small. With $\tilde{k}=k(1+\varepsilon)$ we then have

$$\frac{1}{\mathbf{i}k^2}\sin(kx_i)e^{\mathbf{i}k}+\frac{h^2(1+e^{\mathbf{i}\tilde{k}h})}{2\sin(h\tilde{k})(1-e^{\mathbf{i}\tilde{k}h})}\sin(\tilde{k}x_i)e^{\mathbf{i}\tilde{k}}=$$
$$\frac{1}{\mathbf{i}k^2}e^{\mathbf{i}k}\left(\sin(kx_i)+\mathbf{i}\frac{(kh)^2(1+e^{\mathbf{i}kh(1+\varepsilon)})}{2\sin(kh(1+\varepsilon))(1-e^{\mathbf{i}kh(1+\varepsilon)})}\sin(k(1+\varepsilon)x_i)e^{\mathbf{i}k\varepsilon}\right)$$

We set $\delta=kh$ and perform a Taylor expansions (assuming δ and ε to be small) to get

$$\begin{aligned}\frac{(kh)^2(1+e^{\mathbf{i}kh(1+\varepsilon)})}{2\sin(kh(1+\varepsilon))(1-e^{\mathbf{i}kh(1+\varepsilon)})}&=\frac{\delta^2(2+\mathbf{i}\delta(1+\varepsilon)+O(\delta^2))}{2(\delta(1+\varepsilon)+O(\delta^3))(1-(1+\mathbf{i}\delta(1+\varepsilon)-\frac{1}{2}(\delta(1+\varepsilon))^2+O(\delta^3))}\\&=\frac{1+\frac{\mathbf{i}\delta(1+\varepsilon)}{2}+O(\delta^2)}{(1+\varepsilon+O(\delta^2))(-\mathbf{i}(1+\varepsilon)+\frac{1}{2}\delta(1+\varepsilon)^2+O(\delta^2))}\\&=\frac{1}{-\mathbf{i}(1+\varepsilon)}\frac{1+\frac{\mathbf{i}\delta(1+\varepsilon)}{2}+O(\delta^2)}{(1+\varepsilon+O(\delta^2))(1+\mathbf{i}\frac{1}{2}\delta(1+\varepsilon)+O(\delta^2))}\\&=\frac{1}{-\mathbf{i}(1+\varepsilon)^2}(1+O(\delta^2))=\frac{1}{-\mathbf{i}}(1+O(\delta^2)+O(\varepsilon))\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\frac{1}{\mathbf{i}k^2}e^{\mathbf{i}k}\left(\sin(kx_i)+\mathbf{i}\frac{(kh)^2(1+e^{\mathbf{i}kh(1+\varepsilon)})}{2\sin(kh(1+\varepsilon))(1-e^{\mathbf{i}kh(1+\varepsilon)})}\sin(k(1+\varepsilon)x_i)e^{\mathbf{i}k\varepsilon)}\right)\\
&=\frac{1}{\mathbf{i}k^2}e^{\mathbf{i}k}\left(\sin(kx_i)-(1+O(\delta^2)+O(\varepsilon))\sin(kx_i(1+\varepsilon))e^{\mathbf{i}k\varepsilon}\right)\\
&=\frac{1}{\mathbf{i}k^2}e^{\mathbf{i}k}\left(\sin(kx_i)-(1+O(\delta^2)+O(\varepsilon))(\sin(kx_i)+O(k\varepsilon))(1+O(k\varepsilon))\right)\\
&k^{-2}\left(O(\delta^2)+O(k\varepsilon)\right)
\end{aligned}$$

Recalling that $\delta = kh$ finishes the proof of (A.16).

We now show (A.17). Taylor expansion gives (for small δ and ε)

$$\frac{\delta^2 e^{\mathbf{i}\delta(1+\varepsilon)/2}}{\sin(\delta(1+\varepsilon))(1-e^{\mathbf{i}\delta(1+\varepsilon)})}=-1+O(\delta^2)+O(\varepsilon)$$

Hence, we get with the notation $\delta = kh$

$$\begin{aligned}
&\frac{1}{\mathbf{i}k^2}\sin(kx_i)e^{\mathbf{i}ky}+\frac{h^2e^{\mathbf{i}\tilde{k}h/2}}{\sin(h\tilde{k})(1-e^{\mathbf{i}h\tilde{k}})}\sin(\tilde{k}x_i)e^{\mathbf{i}\tilde{k}y}\\
&=\frac{1}{\mathbf{i}k^2}e^{\mathbf{i}ky}\left(\sin(kx_i)+(-1+O(\delta^2)+O(\varepsilon))\sin(k(1+\varepsilon)x_i)e^{\mathbf{i}k\varepsilon y}\right)\\
&=\frac{1}{\mathbf{i}k^2}e^{\mathbf{i}ky}\left(\sin(kx_i)+(-1+O(\delta^2)+O(\varepsilon))(\sin(kx_i)+O(k\varepsilon))(1+O(k\varepsilon))\right)\\
&=k^{-2}\left(O(\delta^2)+O(k\varepsilon)\right),
\end{aligned}$$

which concludes the proof of (A.17). □

With Lemma A.5 in hand, we can bounds the terms ⓓ, ⓔ, and ⓕ.

From (A.16), we get

$$|ⓓ|\leq C|f(1)|\left(h^2+k^{-1}\varepsilon\right).$$

Combining (A.15) and (A.16) yields

$$|ⓔ|\leq C|f(x_i)|\left(h^2+k^{-1}\varepsilon\right)+Ch^2|f(1)|.$$

The term ⓕ consists of three terms

$$ⓕ = ⓕ_1 + ⓕ_2 + ⓕ_3.$$

The terms $ⓕ_1$ and $ⓕ_2$ can be estimated using (A.15) by

$$|ⓕ_1|+|ⓕ_3|\leq Ch^2|f(x_i)|+Ch^2\left(\|f'\|_\infty+h^2\|f'''\|_\infty\right).$$

The term $ⓕ_2$ requires more care. Discretizing the integral in the term $ⓕ_2$ with the midpoint rule we get

$$-\frac{1}{\mathbf{i}k^2}\sin(kx_i)\int_{x_i}^1 e^{\mathbf{i}ky}f'(y)\,dy=-\frac{1}{\mathbf{i}k^2}\sin(kx_i)h\sum_{j=i}^{N-1}e^{\mathbf{i}kx_{j+1/2}}f'(x_{j+1/2})+h^2O(k^2\|f'\|_\infty+k\|f''\|_\infty+\|f'''\|_\infty). \tag{A.18}$$

With the aid of (A.17) and (A.18) we get for ⓒ:

$$ⓕ := -\frac{1}{\mathbf{i}k^2}\sin(kx_i)\int_{x_i}^1 e^{\mathbf{i}ky}f'(y)dy-\frac{h^2e^{\frac{i\tilde{k}h}{2}}}{\sin(h\tilde{k})(1-e^{\mathbf{i}\tilde{k}h})}\sin(\tilde{k}x_i)h\sum_{j=i}^{N-1}e^{\mathbf{i}\tilde{k}x_{j+1/2}}\left(f'(x_{j+1/2})+O(h^2\|f'''\|_\infty)\right)$$

$$
\begin{aligned}
&= -\frac{1}{\mathbf{i}k^2}\sin(kx_i)h\sum_{j=i}^{N} e^{\mathbf{i}kx_{j+1/2}} f'(x_{j+1/2}) \quad + k^{-2}h^2 O(k^2\|f'\|_\infty + k\|f''\|_\infty + \|f'''\|_\infty) \\
&\quad -\frac{h^2 e^{\frac{\mathbf{i}\tilde{k}h}{2}}}{\sin(h\tilde{k})(1-e^{\mathbf{i}\tilde{k}h})}\sin(\tilde{k}x_i)h\sum_{j=i}^{N-1} e^{\mathbf{i}\tilde{k}x_{j+1/2}}\big(f'(x_{j+1/2}) + O(h^2\|f'''\|_\infty)\big) \\
&= -h\sum_{j=i}^{N-1}\Big[\frac{1}{\mathbf{i}k^2}\sin(kx_i)e^{\mathbf{i}kx_{j+1/2}} + \frac{h^2 e^{\frac{\mathbf{i}\tilde{k}h}{2}}}{\sin(h\tilde{k})(1-e^{\mathbf{i}\tilde{k}h})}\sin(\tilde{k}x_i)e^{\mathbf{i}\tilde{k}x_{j+1/2}}\Big] f'(x_{j+1/2}) \\
&\qquad + k^{-2}h^2 O(k^2\|f'\|_\infty + k\|f''\|_\infty + \|f'''\|_\infty) \\
&= (O(h^2) + O(k^{-1}\varepsilon))\|f'\|_\infty + k^{-2}h^2 O(k^2\|f'\|_\infty + k\|f''\|_\infty + \|f'''\|_\infty).
\end{aligned}
$$

This leads us to

$$
B_i = \text{ⓓ} + \text{ⓔ} + \text{ⓕ} \lesssim \big(h^2 + k^{-1}\varepsilon\big)\big(\|f\|_\infty + \|f'\|_\infty\big) + h^2(k^{-1}\|f''\|_\infty + k^{-2}\|f'''\|_\infty). \tag{A.19}
$$

A.3 The final estimate

Combining (A.14) and (A.19) we arrive at

$$
\|e_{poll}\|_{L^2} \lesssim \sqrt{h\sum_i |A_i + B_i|^2} \lesssim \big(h^2 + k^{-1}\varepsilon\big)\big(\|f\|_\infty + \|f'\|_\infty\big) + h^2k^{-1}\|f''\|_\infty + h^2k^{-2}\|f'''\|_\infty \tag{A.20}
$$

B Duality arguments for specific right-hand sides

For simplicity of exposition, we do not work out the k-dependence explicitly. The argument follow standard lines as illustrated, for example, in [21, Sec. 1.5]. We start with the nodal error:

Lemma B.1. *Consider (3.1) and let V_N be the conforming subspace of piecewise polynomials of degree p. Let $u \in H^{p+1}(I)$ and $u_N \in V_N$ be the Galerkin approximation. Then for all mesh points x_i:*

$$|u(x_i) - u_N(x_i)| \le C_{p,k} h^{2p} \|u^{(p+1)}\|_{L^\infty},$$

where the constant $C_{p,k}$ is independent of h (by may depend on p and k).

Proof. Let G be the Green's function and $IG(x,\cdot) \in V_N$ be a piecewise polynomial approximation to it. Then by the piecewise smoothness of $G(x_i,\cdot)$

$$\|G(x_i,\cdot) - IG(x_i,\cdot)\|_{\mathcal{H}} \le Ch^p.$$

$$\begin{aligned} |u(x_i) - u_N(x_i)| &= B(u - u_N, G(x_i,\cdot)) = |B(u - u_N, G(x_i,\cdot) - IG(x_i,\cdot))| \\ &\le \|u - u_N\|_{\mathcal{H}} \|G(x_i,\cdot) - IG(x_i,\cdot)\|_{\mathcal{H}} \le C_{k,p} h^{2p}. \end{aligned}$$

□

For quasi-uniform meshes, we also have L^∞-bounds for the derivative:

Lemma B.2. *Assume the hypotheses of Lemma B.1. Assume additionally that the mesh is quasi-uniform. Then*

$$\|(u - u_N)'\|_{L^\infty(I)} \le Ch^p \|u^{(p+1)}\|_{L^\infty(I)}.$$

Proof. The proof follows [21, Thm. 1.5.1]. Fix an interval K of the mesh. Let $L_0, \ldots, L_{p-1}$ be the Legendre polynomials scaled to the interval $K = (x_i, x_{i+1})$. Write, on the interval K, the error $e'(x) = u'(x) - u_N'(x)$ as

$$e'(x) = \sum_{j=0}^{p-1} c_j L_j(x) + O(h^p),$$

where the $O(h^p)$-term involves only $\|u^{(p)}\|_{L^\infty(K)}$. We next show that the coefficients $c_j = O(h^p)$. To that end, we denote by χ_K the characteristic function of K and let

$$\mathcal{E}g := \int_0^x \chi_K(t) g(t)\, dt.$$

We note that if g is a polynomial of degree $p-1$, then $\mathcal{E}g \in V_N$. Hence, the Galerkin orthogonality gives us

$$\int_K e' g\, dt = k^2 \int_I e \mathcal{E}g\, dt$$

In particular, therefore,

$$\left|\int_K e' g\, dt\right| \le Ck^2 \|e\|_{L^2(I)} \|g\|_{L^2(K)}$$

Using the orthogonality of the Legendre polynomials, we get with $h_K \sim h$:

$$c_j \sim h_K^{-1} \int_K (e' + O(h^p)) L_j\, dt \lesssim h^{-1} \left(O(h^{p+1}) + k^2 \|e\|_{L^2(I)} \|L_j\|_{L^2(K)}\right) = O(h^p) + k^2 h^{p+1} h^{-1/2} = O(h^p).$$

□

We now turn to the evaluation of linear functionals:

Theorem B.3. *Let V_N consist of piecewise polynomials on a quasi-uniform mesh with mesh size h. Let u and u_N be the exact solution of (3.1) and its Galerkin approximation. Assume that $u \in W^{t,1}(I)$ with $t \geq 1$. Let ψ be smooth and ψ_N be its Galerkin approximation. Then*

$$|B(u - u_N, \psi)| \leq C_{p,k} h^{\min\{t-1,p\}+p}.$$

Proof. By Galerkin orthogonality, we have

$$|B(u-u_N,\psi)| = |B(u-u_N,\psi-\psi_N)| = |B(u-Iu,\psi-\psi_N)| \leq C_k\|u-Iu\|_{W^{1,1}(I)}\|\psi-\psi_N\|_{W^{1,\infty}(I)},$$

where Iu is an arbitrary interpolant of u. Inserting now the approximation properties for the interpolant Iu and using Lemma B.2 concludes the proof. □

Theorem B.3 may be applied as follows:

Lemma B.4. *Let u be of the form*

$$u(x) = x^{\alpha+2} + \tilde{u},$$

where $\tilde{u}$ is smooth. Then $u \in W^{t,1}(I)$ for any $0 < t < \alpha + 3$.

Proof. The proof follows standard lines as worked out, for example, in [5]. We just consider the case $\tilde{u} \equiv 0$. Let $\chi \in C^\infty(\mathbb{R})$ with $\operatorname{supp}\chi \subset [-1,1]$ and $\chi \equiv 1$ on $[-1/2, 1/2]$. For $\delta > 0$ define χ_δ by $\chi_\delta(x) := \chi(x/\delta)$. Consider the decomposition

$$u = \chi_\delta x^{\alpha+2} + (1-\chi_\delta)x^{\alpha+2} =: u_1 + u_2,$$

where the parameter δ will be chosen later. Then for any integer $k > \alpha + 3$

$$\|(u_2)^{(k)}\|_{L^1(I)} \leq \int_{\delta/2}^{\delta} x^{(\alpha+2-k)}\,dx + \sum_{j=0}^{k}\delta^{-(k-j)}\int_{\delta/2}^{\delta} x^{\alpha+2-j}\,dx + \int_{\delta}^{1} x^{\alpha+2-k}\,dx \sim \delta^{\alpha+3-k}$$

Hence, for $k > \alpha + 3$

$$\|u_1\|_{W^{k,1}(I)} \leq C\delta^{\alpha+3-k}$$

Next, for u_2 we have

$$\|u_1\|_{L^1(I)} \leq \int_0^{\delta} x^{\alpha+2}\,dx \sim \delta^{\alpha+3}$$

This implies for the K-functional:

$$K(u,t) \leq \|u_2\|_{L^1(I)} + t\|u_1\|_{W^{k,1}(I)} \leq C\delta^{\alpha+3}(1 + t\delta^{-k}).$$

Upon selecting $\delta = t^{1/k}$, we get

$$K(u,t) \leq Ct^{(\alpha+3)/k}.$$

We calculate

$$\int_{t=0}^{1} t^{-\sigma}K(u,t)\frac{dt}{t} \leq C\int_{t=0}^{1} t^{-1-\sigma+(\alpha+3)k)}\,dt < \infty$$

if

$$\sigma < (\alpha+3)/k$$

Given that $W^{t,1}(I) = (W^{t_1,1}(I), W^{t_2,1}(I))_{\theta,1}$ with $t = \theta t_1 + (1-\theta)t_2$ for integer t_1, t_2, we conclude that

$$u \in W^{\sigma,1}(I), \qquad \sigma < (\alpha+3)/k \cdot k = (\alpha+3).$$

□

Corollary B.5. *Let V_N consist of piecewise polynomials of degree p on a quasi-uniform mesh of mesh size h. Let u be solution of (3.1) with $f(x) = x^\alpha$. Then, for smooth z, the error in the linear functional given by (2.10) is expected to be*

$$|L(u) - L(u_N)| \leq Ch^{p+\min\{\alpha+2,p\}}$$

Proof. This follows by combining Theorem B.3 with Lemma B.4. Strictly speaking, this procedure only yields convergence $O(h^{p+\min\{\alpha+3,p\}-\varepsilon}$ for all $\varepsilon > 0$ but the proof shows that ε can be removed. □

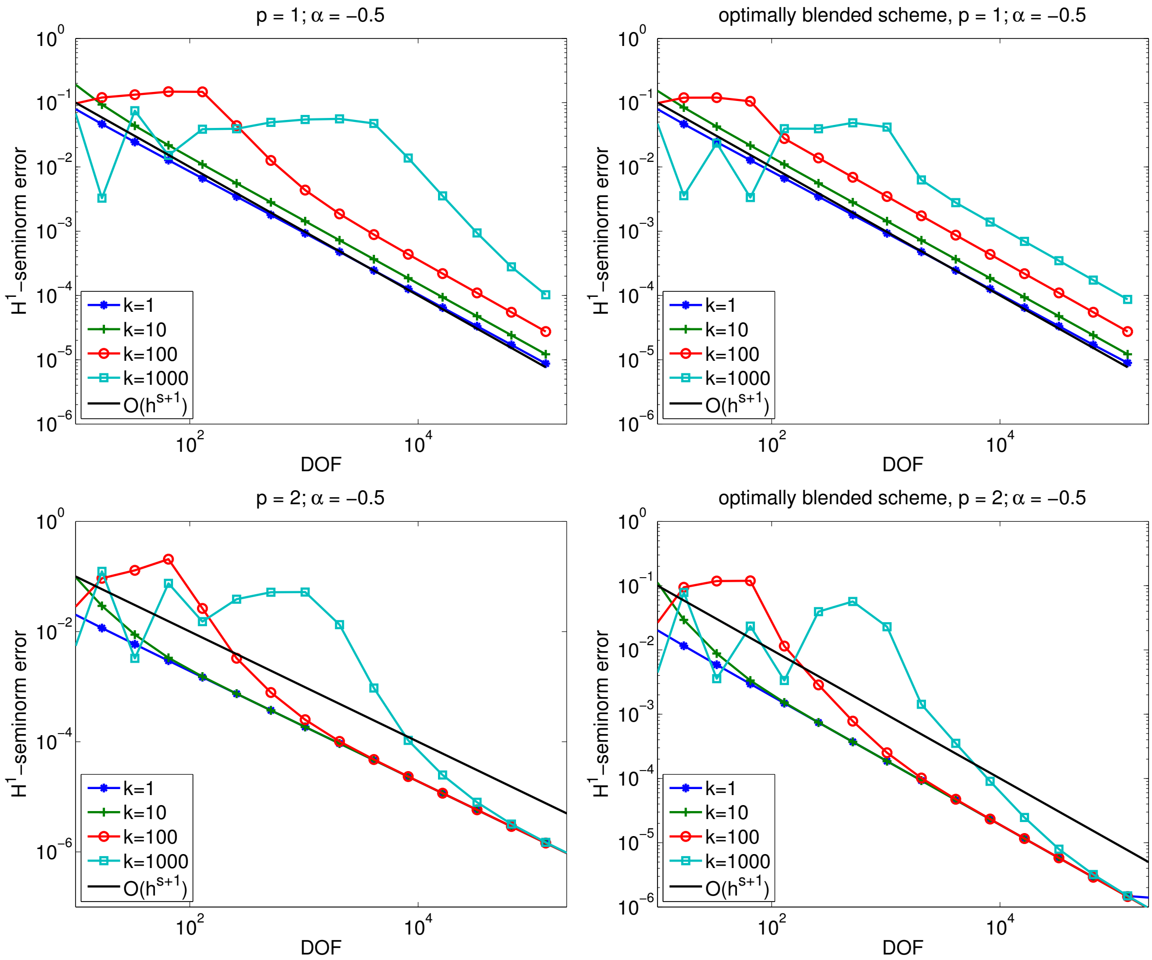


Figure 16: Galerkin FEM and optimally blended scheme for non-smooth right hand side. top: $p = 1$ (left: Galerkin, right: optimally blended), bottom: $p = 2$ (left: Galerkin, right: optimally blended).

C Further numerical examples

We collect additional numerical examples in this section.

Example C.1. We continue Example 4.16 for the case $\alpha = -1/2$ and compare the H^1-performance of the Galerkin method with the optimally blended scheme. We observe in Fig. 16 that both method have the same asymptotic behavior but that the optimally blended scheme is much more efficient at suppressing pollution, i.e., it reduces the preasymptotic regime.

Fig. 17 present a comparison of the Galerkin method with the optimally blended scheme for the evaluation of the average (i.e., the linear functional is given by (2.10) with $z = 1$). ∎

Example C.2. We continue Example 4.16 for the case $\alpha = +1/2$. In Fig. 18 and compare the relative L^2-performance of the Galerkin method with the optimally blended scheme for the cases $\alpha = 1/2$ (which corresponds to $s = 1$) and $p = 2$ as well as $p = 3$. We mention that the numerics suggest an $O(k^{-2})$-behavior for the L^2-norm of the exact solution.

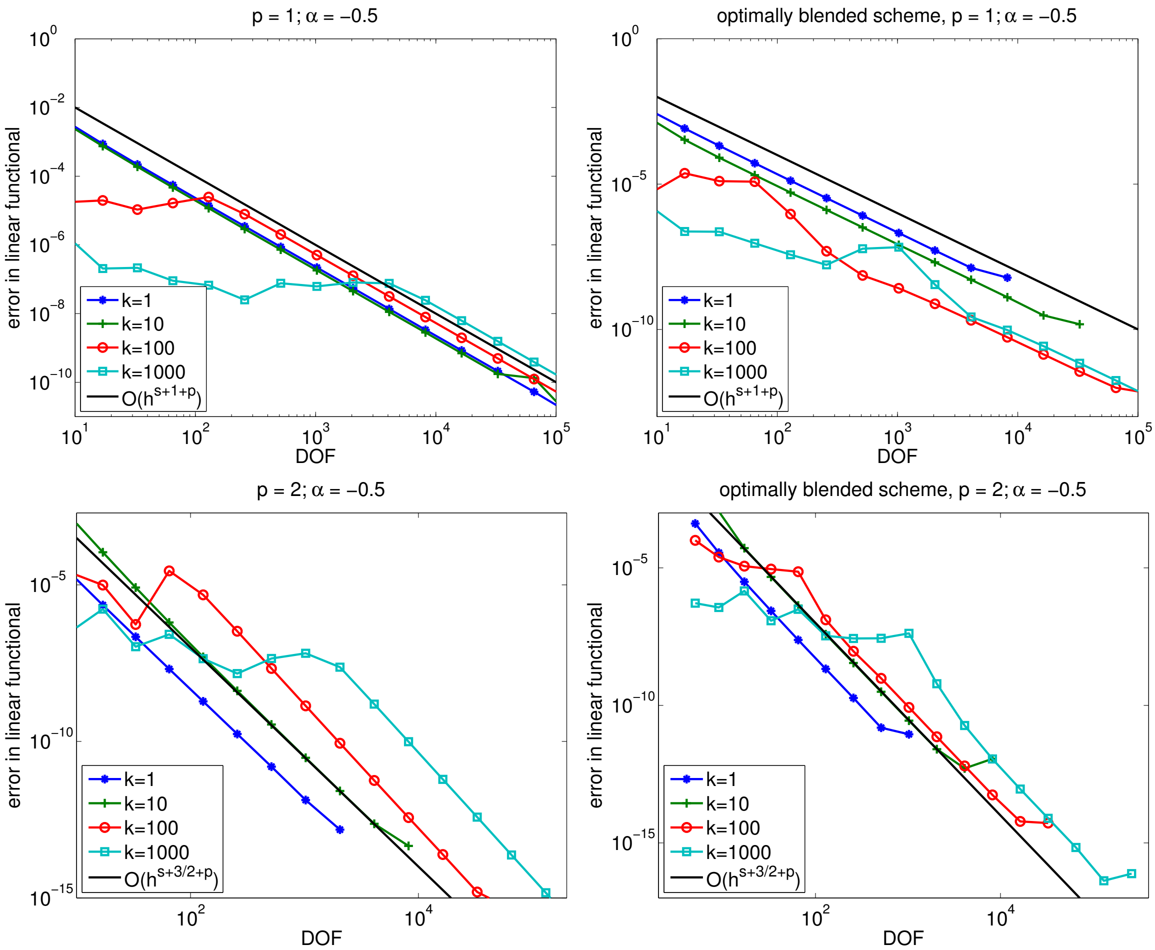


Figure 17: Galerkin FEM and optimally blended scheme for non-smooth right hand side. top: $p = 1$ (left: Galerkin, right: optimally blended), bottom: $p = 2$ (left: Galerkin, right: optimally blended).

Fig. 19 presents the comparison of the Galerkin method with the optimally blended scheme for the cases $p = 2$ and $p = 3$ as measured in the H^1-seminorm and Fig. 20 the comparison for the evaluation of a linear functional given by (2.10) with $z = 1$. ∎

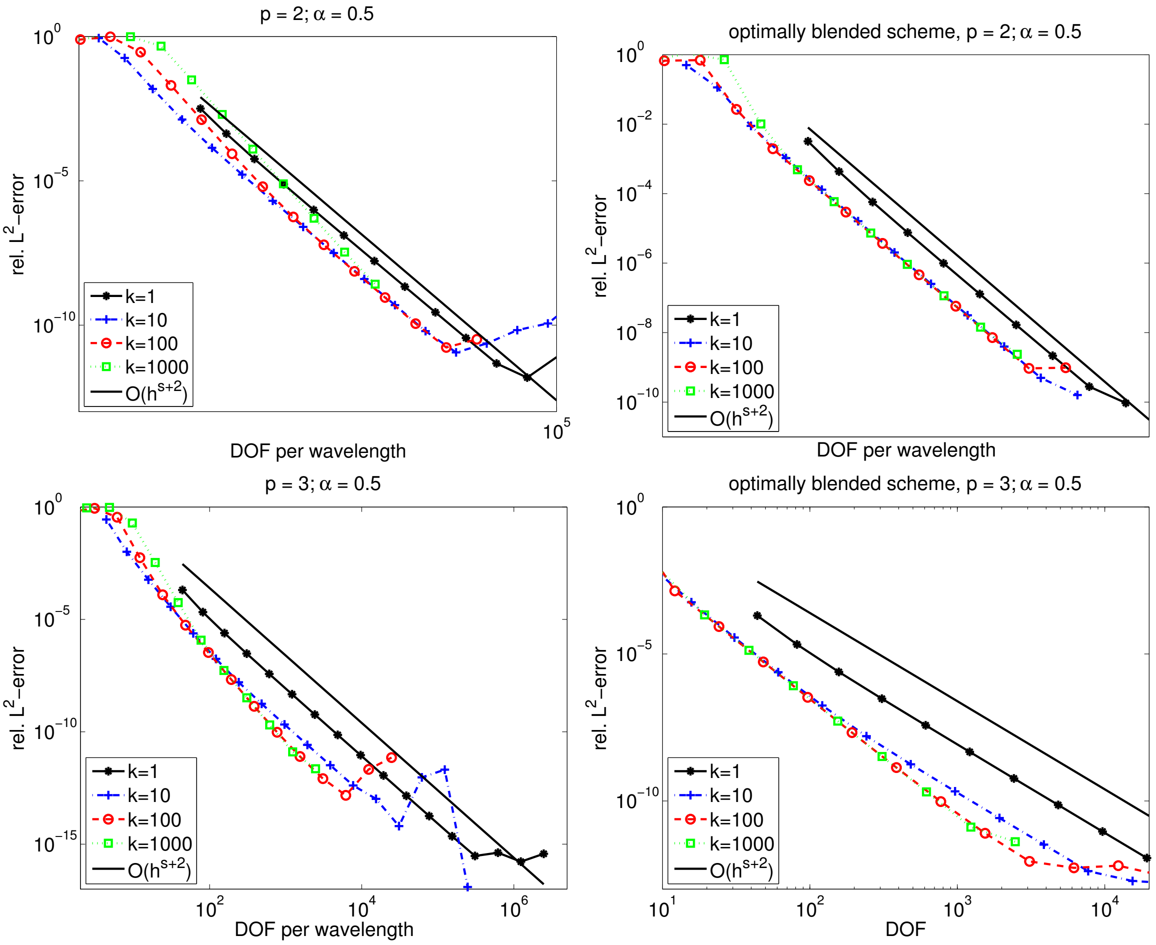


Figure 18: Galerkin FEM and optimally blended scheme for non-smooth right hand side. top: $p = 2$ (left: Galerkin, right: optimally blended), bottom: $p = 3$ (left: Galerkin, right: optimally blended) for $\alpha = 1/2$.

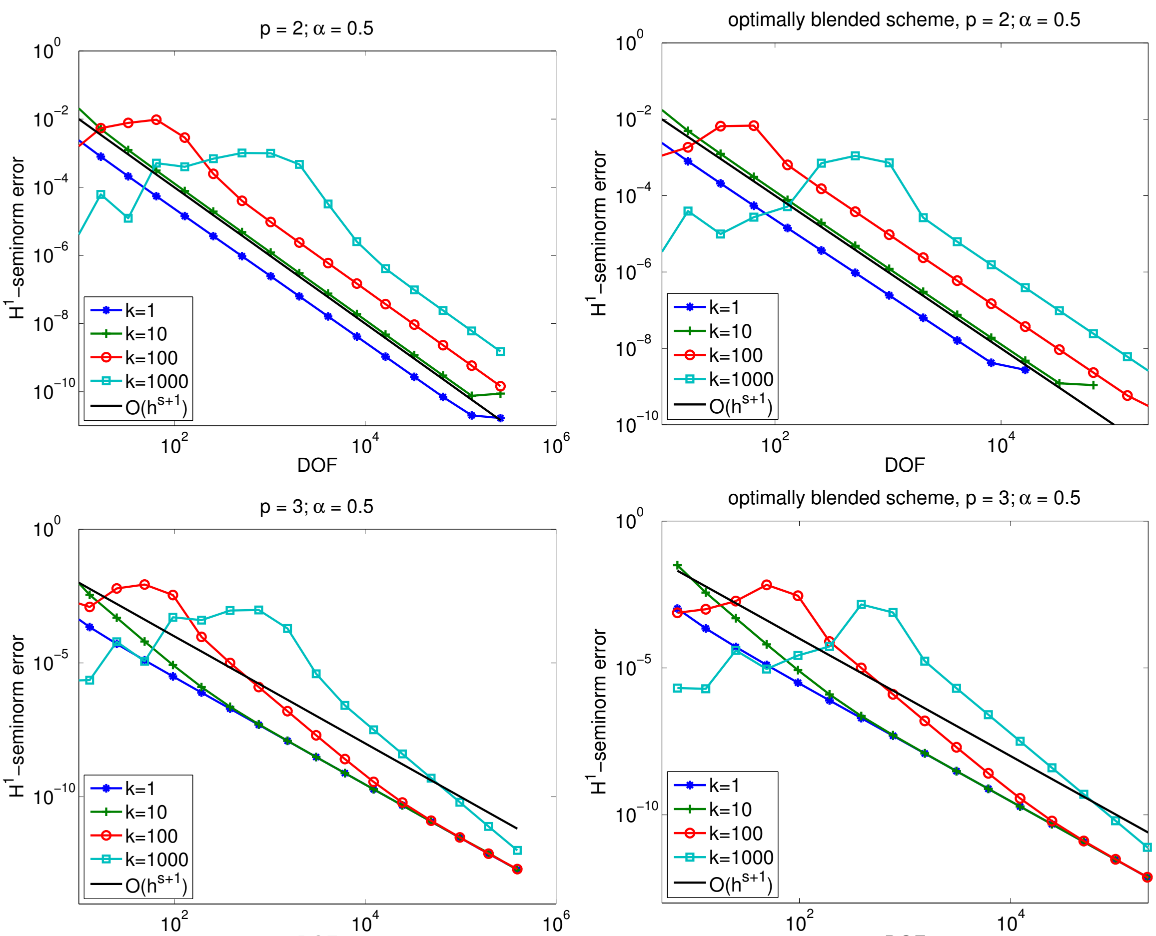


Figure 19: Galerkin FEM and optimally blended scheme for non-smooth right hand side. top: $p = 2$ (left: Galerkin, right: optimally blended), bottom: $p = 3$ (left: Galerkin, right: optimally blended) for $\alpha = 1/2$.

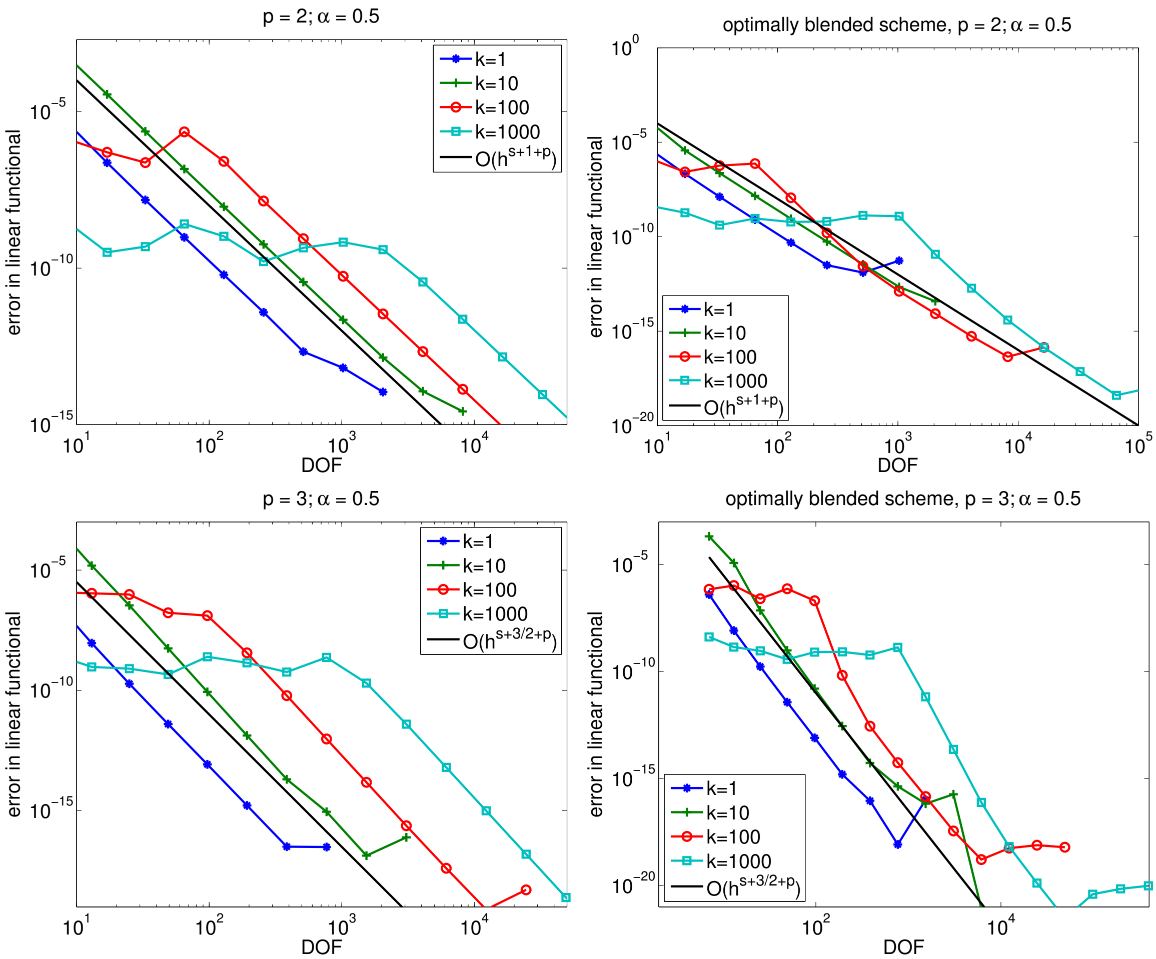


Figure 20: Galerkin FEM and optimally blended scheme for non-smooth right hand side. top: $p = 2$ (left: Galerkin, right: optimally blended), bottom: $p = 3$ (left: Galerkin, right: optimally blended) for $\alpha = 1/2$.

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