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J.M. Melenk and T.P. Wihler

Institute for Analysis and Scientific Computing Vienna University of Technology — TU Wien
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Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße 8–10
1040 Wien, Austria

E-Mail: `admin@asc.tuwien.ac.at`
WWW: `http://www.asc.tuwien.ac.at`
FAX: +43-1-58801-10196

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A POSTERIORI ERROR ANALYSIS OF hp-FEM FOR SINGULARLY PERTURBED PROBLEMS

JENS M. MELENK AND THOMAS P. WIHLER

Abstract. We consider the approximation of singularly perturbed linear second-order boundary value problems by hp-finite element methods. In particular, we include the case where the associated differential operator may not be coercive. Within this setting we derive an *a posteriori* error estimate for a natural residual norm. The error bound is robust with respect to the perturbation parameter and fully explicit with respect to both the local mesh size h and the polynomial degree p.

1. Introduction

A posteriori error estimation and adaptivity for low-order methods has seen a significant development in the last decades as witnessed by several monographs [1, 3, 28] on *a posteriori* error estimation, and on convergence and optimality of adaptive algorithms; see, e.g., [7, 26, 13]. The situation is less developed for high-order finite element methods (hp-FEM), where both the local mesh size can be reduced and the local approximation order can be increased to improve the accuracy.

In an hp-context, several adaptive strategies and algorithms have been proposed (see [23] for an overview and comparison). The first work on hp-adaptive strategies for finite element approximations of elliptic problems was presented in [25]. In addition, methods based on smoothness estimation techniques were proposed in [11, 15, 16, 19], or in the recent approach [12, 29, 30] involving Sobolev embeddings, which will also be exploited in the present article. Moreover, a prediction technique was developed in [22]. Further hp-adaptive approaches in the literature include, for example, the use of *a priori* knowledge, mesh optimization strategies, the Texas-3-step algorithm, or the application of reference solution strategies; see, e.g., [2, 8, 9, 14, 24]. Research focusing on the convergence of hp-adaptive FEM has been developed only recently in [5, 6].

In spite of the practical success of these hp-adaptive algorithms, *a posteriori* error estimation in hp-FEM is still a topic of active research, and several, structurally different *a posteriori* error estimators for hp-FEM for standard elliptic problems are available in the literature. We mention in particular the one of residual type, featuring a reliability-efficiency gap in the approximation order [10, 22], and the p-robust estimators of [4] which assumes the elliptic problem to be in divergence form.

Here, we present an *a posteriori* error estimator for hp-FEM that is suitable for singularly perturbed problems; it is of residual type and results from merging the techniques of [27] for singular perturbations with p-explicit estimators from [22]. More precisely, on an interval $\Omega = (a, b) \subset \mathbb{R}$, $a < b$, we consider the singularly perturbed boundary value problem

$$-\varepsilon u''(x) + d(x)u(x) = f(x), \qquad x \in \Omega, \tag{1}$$

$$u(a) = u(b) = 0. \tag{2}$$

Here, $\varepsilon > 0$ is a possibly small constant, $d \in L^\infty(\Omega)$ is a given function, and $f \in L^2(\Omega)$ is the right-hand side. We use standard notation: For an open set $D \subseteq \Omega$, we let $L^2(D)$ be the standard

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Lebesgue space of all square-integrable functions on D with norm $\|\cdot\|_{L^2(D)}$, and $L^\infty(D)$ is the space of all essentially bounded functions on D with norm $L^\infty(D)$.

We propose the following variational formulation of (1)–(2): Find $u \in H_0^1(\Omega)$, the standard L^2-based Sobolev space of first order with vanishing trace, such that

$$a(u,v) := \varepsilon \int_\Omega u'(x)v'(x)\,\mathsf{d}x + \int_\Omega d(x)u(x)v(x)\,\mathsf{d}x = \int_\Omega f(x)v(x)\,\mathsf{d}x \qquad \forall v \in H_0^1(\Omega). \tag{3}$$

Throughout this paper, we make the general assumption that the solution of (3) exists and is unique. Evidently, this the case if $d \geq 0$.

The article is organized as follows: In the following Section 2 we provide the *hp*-framework and *hp*-FEM for the discretization of (1)–(2). Furthermore, Section 3 contains some *hp*-quasi-interpolation results, and the *hp*-*a posteriori* error analysis. In addition, we present some numerical tests in Section 4. Finally, we summarize our work in Section 5.

2. *hp*-FEM Discretization

In order to discretize the boundary value problem (1)–(2) by means of an *hp*-finite element method, let us introduce a partition $\mathcal{T} = \{K_j\}_{j=1}^N$ of $N \geq 1$ (open) elements $K_j = (x_{j-1}, x_j)$, $j = 1, 2, \ldots, N$ on $\Omega = (a, b)$, with

$$a = x_0 < x_1 < x_2 < \ldots < x_{N-1} < x_N = b.$$

The length of an element K_j is denoted by $h_j = x_j - x_{j-1}$, $j = 1, 2, \ldots, N$. For each element $K_j \in \mathcal{T}$, it will be convenient to introduce the patch $\widetilde{K}_j = \bigcup\{K_i \in \mathcal{T} \,|\, \overline{K_i} \cap \overline{K_j} \neq \emptyset\}$ as the union of K_j and of the elements adjacent to it. In addition, to each element K_j we associate a polynomial degree $p_j \geq 1$, $j = 1, 2, \ldots, N$. These numbers are stored in a polynomial degree vector $\boldsymbol{p} = (p_1, p_2, \ldots, p_N)$. Then, we define an *hp*-finite element space by

$$V_{\mathrm{hp}}(\mathcal{T}, \boldsymbol{p}) = \left\{ v \in H_0^1(\Omega) : v|_{K_j} \in \mathbb{P}_{p_j}(K_j),\ j = 1, 2, \ldots, N \right\},$$

where, for $p \geq 1$, we denote by $\mathbb{P}_p$ the space of all polynomials of degree at most p. We say that the pair $(\mathcal{T}, \boldsymbol{p})$ of a partition $\mathcal{T}$ and of a degree vector $\boldsymbol{p}$ is μ-shape regular, for some constant $\mu > 0$ independent of j, if

$$\mu^{-1}h_{j+1} \leq h_j \leq \mu h_{j+1}, \qquad \mu^{-1}p_{j+1} \leq p_j \leq \mu p_{j+1}, \qquad j = 1, \ldots, N-1, \tag{4}$$

i.e., if both the element sizes and polynomial degrees of neighboring elements are comparable.

We can now discretize the variational formulation (3) by finding a numerical approximation $u_{\mathrm{hp}} \in V_{\mathrm{hp}}(\mathcal{T}, \boldsymbol{p})$ such that

$$a(u_{\mathrm{hp}}, v) = \int_\Omega f v \,\mathsf{d}x \qquad \forall v \in V_{\mathrm{hp}}(\mathcal{T}, \boldsymbol{p}). \tag{5}$$

As in the continuous case, we generally suppose that, for a given *hp*-space $V_{\mathrm{hp}}(\mathcal{T}, \boldsymbol{p})$, a unique numerical solution $u_{\mathrm{hp}} \in V_{\mathrm{hp}}(\mathcal{T}, \boldsymbol{p})$ of (5) exists.

Furthermore, let us introduce the following norm on $H_0^1(\Omega)$:

$$|||v|||^2 := \sum_{j=1}^N |||v|||_{K_j}^2 := \sum_{j=1}^N \left(\varepsilon \|v'\|_{L^2(K_j)}^2 + \left\| \sqrt{|d|}v \right\|_{L^2(K_j)}^2 \right). \tag{6}$$

We note that, if $d \geq 0$ on Ω, then the norm $|||\cdot|||$ equals the natural energy norm corresponding to the bilinear form $a(\cdot,\cdot)$ from (3). More precisely, in that case we have that $a(v,v) = |||v|||^2$ for any $v \in H_0^1(\Omega)$.

3. Robust A Posteriori Error Analysis

The goal of this section is to derive an *a posteriori* error analysis for the *hp*-FEM (5) with respect to the residual

$$\mathsf{R}_{\mathrm{hp}}[e_{\mathrm{hp}}] := \sup_{\substack{v \in H_0^1(\Omega) \\ v \not\equiv 0}} \frac{|a(u - u_{\mathrm{hp}}, v)|}{|||v|||},$$

where $u \in H_0^1(\Omega)$ and $u_{\mathrm{hp}} \in V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})$ are the exact and numerical solutions of (3) and (5), respectively, and $e_{\mathrm{hp}} = u - u_{\mathrm{hp}}$ signifies the error. Again, let us notice that, if $d \geq 0$, then the residual $\mathsf{R}_{\mathrm{hp}}[e_{\mathrm{hp}}]$ equals the norm $\|\!|e_{\mathrm{hp}}|\!\|$ of the error.

In order to state our main result, let us denote by Π_{K_j}, for $j = 1,2,\ldots,N$, the elementwise L^2-projection onto $\mathbb{P}_{p_j}(K_j)$. Moreover, let

$$[\![u_{\mathrm{hp}}']\!](x_j) = u_{\mathrm{hp}}'(x_j^+) - u_{\mathrm{hp}}'(x_j^-) = \lim_{x\searrow x_j} u'(x) - \lim_{x\nearrow x_j} u'(x), \qquad 1 \leq j \leq N-1,$$

signify the jump of u_{hp}' at the mesh point x_j, and define $[\![u_{\mathrm{hp}}']\!](x_0) = [\![u_{\mathrm{hp}}']\!](x_N) = 0$.

3.1. **Main Result.** We shall prove the following *a posteriori* error bound:

Theorem 3.1. *For the error $e_{\mathrm{hp}} = u - u_{\mathrm{hp}}$ between the exact solution $u \in H_0^1(\Omega)$ of (3) and its numerical approximation $u_{\mathrm{hp}} \in V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})$ from (5), there holds the following a posteriori error estimate:*

$$\mathsf{R}_{\mathrm{hp}}[e_{\mathrm{hp}}]^2 \leq C\sum_{j=1}^{N} \eta_{K_j}^2. \tag{7}$$

Here, for $j = 1,2,\ldots,N$,

$$\begin{aligned}\eta_{K_j}^2 := \alpha_j \left(\left\|\Pi_{K_j} f + \varepsilon u_{\mathrm{hp}}'' - d u_{\mathrm{hp}}\right\|_{L^2(K_j)}^2 + \left\|f - \Pi_{K_j} f\right\|_{L^2(K_j)}^2 \right) \\ + \frac{1}{2}\varepsilon^2\gamma_{j-1}\left|[\![u_{\mathrm{hp}}']\!](x_{j-1})\right|^2 + \frac{1}{2}\varepsilon^2\gamma_j\left|[\![u_{\mathrm{hp}}']\!](x_j)\right|^2\end{aligned} \tag{8}$$

are local error indicators, where we let

$$\alpha_j = \begin{cases} \min\left\{\varepsilon^{-1}h_j^2 p_j^{-2}, \|{}^1\!/\!_d\|_{L^\infty(\widetilde{K}_j)}\right\}, & \text{if } {}^1\!/\!_d \in L^\infty(\widetilde{K}_j), \\ \varepsilon^{-1}h_j^2 p_j^{-2}, & \text{otherwise}, \end{cases} \tag{9}$$

(with obvious modifications if $j = 0$ or $j = N$), and

$$\beta_j = \alpha_j h_j^{-1} + 2\sqrt{\varepsilon^{-1}\alpha_j}. \tag{10}$$

Moreover,

$$\gamma_j = \frac{\beta_j \beta_{j+1}}{\beta_j + \beta_{j+1}}, \tag{11}$$

for $1 \leq j \leq N-1$, and $\gamma_0 = \gamma_N = 0$. The constant $C > 0$ is independent of u, u_{hp}, f, ε, $\mathcal{T}$, and of $\boldsymbol{p}$.

3.2. *hp*-**Quasi-Interpolation.** The proof of Theorem 3.1 will require the construction of an *hp*-version quasi-interpolant that is both L^2- and H^1-stable. This is the subject of this section.

Proposition 3.2. *Let the pair $(\mathcal{T},\boldsymbol{p})$ be μ-shape regular (see (4)) and $v \in H_0^1(\Omega)$. Then, there exists a quasi-interpolant $\pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v \in V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})$ of v such that, for any $j = 1,2,\ldots,N$, there holds*

$$\begin{aligned}&\left\|v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v\right\|_{L^2(K_j)} \leq C_I\left\|v\right\|_{L^2(\widetilde{K}_j)}, \qquad \left\|v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v\right\|_{L^2(K_j)} \leq C_I\frac{h_j}{p_j}\left\|v'\right\|_{L^2(\widetilde{K}_j)}, \\ &\left\|(v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v)'\right\|_{L^2(K_j)} \leq C_I\left\|v'\right\|_{L^2(\widetilde{K}_j)}.\end{aligned} \tag{12}$$

Furthermore, we have the nodal estimates

$$\begin{aligned}|(v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v)(x_i)|^2 \leq C_I\Big[&\frac{1}{h_i + h_{i+1}}\|v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v\|_{L^2(K_i\cup K_{i+1})}^2 \\ &+ \|v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v\|_{L^2(K_i\cup K_{i+1})}\|(v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v)'\|_{L^2(K_i\cup K_{i+1})}^2\Big].\end{aligned}$$

Here, $C_I > 0$ is a constant that depends solely on μ; in particular, it is independent of v, $\mathcal{T}$, and of $\boldsymbol{p}$.

Proof. Let us, without loss of generality, assume that $\Omega = (0,1)$. The result can be shown with the techniques developed for the higher-dimensional case in [17, 18]. In the present, one-dimensional case, a simpler argument can be brought to bear. Let $x_{-1} = -h_1$ and $x_{N+1} = 1 + h_N$ and φ_i, $i = 0, \dots, N+1$ be the standard piecewise linear hat functions associated with the nodes x_i, $i = -1, \dots, N+1$. The extra nodes x_{-1} and x_{N+1} define in a natural way the elements K_0 and K_{N+1}. The (open) patches ω_i, $i = 0, \dots, N$, are given by the supports of the functions φ_i, i.e., $\omega_i = (\operatorname{supp}\varphi_i)^\circ = K_i \cup K_{i+1} \cup \{x_i\}$.

Polynomial approximation (see, e.g., [20, Proposition A.2]) gives the existence of a quasi-interpolation operator $J_p : L^2(-1,1) \to \mathbb{P}_p(-1,1)$ that is uniformly (in $p \geq 0$) stable, i.e., $\|J_p v\|_{L^2(-1,1)} \leq C\|v\|_{L^2(-1,1)}$ for all $v \in L^2(-1,1)$ and has the following properties for $v \in H^1(-1,1)$:

$$(p+1)\|v - J_p v\|_{L^2(-1,1)} + \|(v - J_p v)'\|_{L^2(-1,1)} \leq C\|v'\|_{L^2(-1,1)}.$$

Furthermore, if v is antisymmetric with respect to the midpoint $x = 0$, then $J_p v$ can be assumed to be antisymmetric as well, i.e., $(J_p v)(0) = 0$ (this follows from studying the antisymmetric part of the original function $J_p v$).

The approximation $\pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})} v$ is now constructed with the aid of a "partition of unity argument" as described in [21, Theorem 2.1]. For ω_0 and ω_N, extend v anti-symmetrically, i.e., $v(x) := -v(-x)$ for $x \in K_0$ and $v(x) := -v(1-x)$ for $x \in K_{N+1}$. Then v is defined on each patch ω_i, $i = 0, \dots, N$. For each patch ω_i, let $p_i' := \min\{p_i, p_{i+1}\}$ (with the understanding $p_0 = p_1$ and $p_{N+1} = p_N$). The above operator J_p then induces for each patch ω_i by scaling an operator $J^i : L^2(\omega_i) \to \mathcal{P}_{p_i'-1}(\omega_i)$ with the following properties:

$$\frac{p_i'+1}{h_i}\|v - J^i v\|_{L^2(\omega_i)} + \|(v - J^i v)'\|_{L^2(\omega_i)} \leq C\|v'\|_{L^2(\omega_i)};$$

here, we have exploited the μ-shape regularity of the mesh. We note that $(J^0 v)(0) = 0$ and $(J^N v)(1) = 0$. Also, the operators J^i are uniformly (in the polynomial degree) stable in $L^2(\omega_i)$. The approximation $\pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})} v$ is now taken to be $\pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})} v := \sum_{i=0}^N \varphi_i J^i v$. The desired approximation properties follow now from [21, Theorem 2.1].

Finally, the nodal estimate results from the observation that at the nodes, there holds the identity $\pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})} v(x_i) = (J^i v)(x_i)$, and from a multiplicative trace inequality (see Appendix, Lemma A.1). $\square$

The above proposition implies the following bounds.

Corollary 3.3. *For $v \in H_0^1(\Omega)$, the quasi-interpolant from Proposition 3.2 satisfies*

$$\left\|v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})} v\right\|_{L^2(K_j)}^2 \leq C_I^2 \alpha_j \,\|v\|_{\widetilde{K}_j}^2, \qquad j = 1, 2, \dots, N,$$

and

$$\left|(v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})} v)(x_j)\right|^2 \leq C_I^2 \gamma_j \left(\|v\|_{\widetilde{K}_j}^2 + \|v\|_{\widetilde{K}_{j+1}}^2\right), \qquad j = 1, 2, \dots, N-1,$$

where α_j and γ_j are defined in (9) *and* (11)*, respectively, and C_I is the constant from* (12).

Proof. We proceed along the lines of [27]. Using the bounds from Proposition 3.2, we have for each element $K_j \in \mathcal{T}$ that

$$\left\|v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})} v\right\|_{L^2(K_j)}^2 \leq C_I^2 \frac{h_j^2}{\varepsilon p_j^2} \varepsilon \left\|v'\right\|_{L^2(\widetilde{K}_j)}^2.$$

Furthermore, if $1/d \in L^\infty(\widetilde{K}_j)$, then

$$\left\|v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})} v\right\|_{L^2(K_j)}^2 \leq C_I^2 \left\|v\right\|_{L^2(\widetilde{K}_j)}^2 \leq C_I^2 \left\|1/d\right\|_{L^\infty(\widetilde{K}_j)} \left\|\sqrt{|d|}v\right\|_{L^2(\widetilde{K}_j)}^2.$$

Combining these two estimates, yields the first bound.

In order to prove the second estimate, we apply, for $1 \leq j \leq N-1$, a multiplicative trace inequality (see Appendix, Lemma A.1):

$$\begin{aligned}
&\left|(v-\pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v)(x_j)\right|^2 \\
&\quad\leq h_j^{-1}\left\|v-\pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v\right\|_{L^2(K_j)}^2+2\left\|v-\pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v\right\|_{L^2(K_j)}\left\|(v-\pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v)'\right\|_{L^2(K_j)}.
\end{aligned}$$

Then, invoking the above bounds as well as the estimates from Proposition 3.2, we get

$$\begin{aligned}
\left|(v-\pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v)(x_j)\right|^2 &\leq C_I^2\left(\alpha_j h_j^{-1}\|v\|_{\widetilde{K}_j}^2+2\sqrt{\alpha_j}\,\|v\|_{\widetilde{K}_j}\|v'\|_{L^2(\widetilde{K}_j)}\right)\\
&\leq C_I^2\left(\alpha_j h_j^{-1}\|v\|_{\widetilde{K}_j}^2+2\sqrt{\varepsilon^{-1}\alpha_j}\,\|v\|_{\widetilde{K}_j}^2\right)\\
&\leq C_I^2\beta_j\|v\|_{\widetilde{K}_j}^2,
\end{aligned}$$

with β_j from (10). Since x_j is also a boundary point of K_{j+1}, we similarly obtain that

$$\left|(v-\pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v)(x_j)\right|^2\leq C_I^2\beta_{j+1}\|v\|_{\widetilde{K}_{j+1}}^2.$$

Therefore,

$$\begin{aligned}
\left|(v-\pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v)(x_j)\right|^2 &= \frac{\beta_{j+1}}{\beta_j+\beta_{j+1}}\left|(v-\pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v)(x_j)\right|^2+\frac{\beta_j}{\beta_j+\beta_{j+1}}\left|(v-\pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v)(x_j)\right|^2\\
&\leq C_I^2\gamma_j\left(\|v\|_{\widetilde{K}_j}^2+\|v\|_{\widetilde{K}_{j+1}}^2\right),
\end{aligned}$$

with γ_j from (11). Thus, we have shown the second estimate. □

3.3. **Proof of Theorem 3.1.** We are now in a position to prove the *hp*-*a posteriori* error bound (7).

From the definitions of the exact solution u from (3) and the numerical solution u_{hp} defined in (5), it follows that, for any $v \in H_0^1(\Omega)$ and any $v_{\mathrm{hp}} \in V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})$,

$$\begin{aligned}
a(u,v)-a(u_{\mathrm{hp}},v) &= a(u,v-v_{\mathrm{hp}})-a(u_{\mathrm{hp}},v-v_{\mathrm{hp}})\\
&=\int_\Omega f(v-v_{\mathrm{hp}})\,\mathsf{d}x-\varepsilon\int_\Omega u_{\mathrm{hp}}'(v-v_{\mathrm{hp}})'\,\mathsf{d}x-\int_\Omega du_{\mathrm{hp}}(v-v_{\mathrm{hp}})\,\mathsf{d}x.
\end{aligned}$$

Integrating by parts elementwise in the second integral leads to

$$\begin{aligned}
\int_\Omega u_{\mathrm{hp}}'(v-v_{\mathrm{hp}})'\,\mathsf{d}x &= \sum_{j=1}^N\int_{K_j}u_{\mathrm{hp}}'(v-v_{\mathrm{hp}})'\,\mathsf{d}x\\
&=-\sum_{j=1}^N\int_{K_j}u_{\mathrm{hp}}''(v-v_{\mathrm{hp}})\,\mathsf{d}x+\sum_{j=1}^N\left(u_{\mathrm{hp}}'(x_j^-)(v-v_{\mathrm{hp}})(x_j)-u_{\mathrm{hp}}'(x_{j-1}^+)(v-v_{\mathrm{hp}})(x_{j-1})\right)\\
&=-\sum_{j=1}^N\int_{K_j}u_{\mathrm{hp}}''(v-v_{\mathrm{hp}})\,\mathsf{d}x-\sum_{j=1}^{N-1}[\![u_{\mathrm{hp}}']\!](x_j)(v-v_{\mathrm{hp}})(x_j),
\end{aligned}$$

and thus, choosing $v_{\mathrm{hp}}=\pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v$ to be the *hp*-interpolant from Section 3.2, we arrive at

$$\begin{aligned}
a(u,v)-a(u_{\mathrm{hp}},v) &= \sum_{j=1}^N\left(\Pi_{K_j}f+\varepsilon u_{\mathrm{hp}}''-du_{\mathrm{hp}}\right)(v-\pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v)\,\mathsf{d}x\\
&\quad+\sum_{j=1}^N\left(f-\Pi_{K_j}f\right)(v-\pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v)\,\mathsf{d}x+\varepsilon\sum_{j=1}^{N-1}[\![u_{\mathrm{hp}}']\!](x_j)(v-\pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v)(x_j).
\end{aligned}$$

Hence, applying the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
|a(u,v)-a(u_{\mathrm{hp}},v)| \le &\sum_{j=1}^{N} \left\|\Pi_{K_j} f+\varepsilon u_{\mathrm{hp}}''-du_{\mathrm{hp}}\right\|_{L^2(K_j)} \left\|v-\pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v\right\|_{L^2(K_j)}\\
&+\sum_{j=1}^{N} \left\|f-\Pi_{K_j} f\right\|_{L^2(K_j)} \left\|v-\pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v\right\|_{L^2(K_j)}\\
&+\sum_{j=1}^{N-1} \varepsilon \left|[\![u_{\mathrm{hp}}']\!](x_j)\right| \left|(v-\pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v)(x_j)\right|.
\end{aligned}
$$

The bounds from Corollary 3.3 lead to

$$
\begin{aligned}
|a(u,v)-a(u_{\mathrm{hp}},v)| \le &\, C_I\sum_{j=1}^{N} \sqrt{\alpha_j}\left\|\Pi_{K_j} f+\varepsilon u_{\mathrm{hp}}''-du_{\mathrm{hp}}\right\|_{L^2(K_j)} |\!|\!| v |\!|\!|_{\widetilde{K}_j}\\
&+C_I\sum_{j=1}^{N} \sqrt{\alpha_j}\left\|f-\Pi_{K_j} f\right\|_{L^2(K_j)} |\!|\!| v |\!|\!|_{\widetilde{K}_j}\\
&+C_I\sum_{j=1}^{N-1} \left(|\!|\!| v |\!|\!|^2_{\widetilde{K}_j}+|\!|\!| v |\!|\!|^2_{\widetilde{K}_{j+1}}\right)^{1/2} \varepsilon\sqrt{\gamma_j}\left|[\![u_{\mathrm{hp}}']\!](x_j)\right|.
\end{aligned}
$$

The Cauchy-Schwarz inequality yields

$$
\begin{aligned}
&|a(u,v)-a(u_{\mathrm{hp}},v)|\\
&\le C_I\left(\sum_{j=1}^{N}\alpha_j\left\|\Pi_{K_j} f+\varepsilon u_{\mathrm{hp}}''-du_{\mathrm{hp}}\right\|^2_{L^2(K_j)}+\alpha_j\left\|f-\Pi_{K_j} f\right\|^2_{L^2(K_j)}\right)^{1/2}\left(2\sum_{j=1}^{N}|\!|\!| v |\!|\!|^2_{\widetilde{K}_j}\right)^{1/2}\\
&\quad+C_I\left(\sum_{j=1}^{N-1}\varepsilon^2\gamma_j\left|[\![u_{\mathrm{hp}}']\!](x_j)\right|^2\right)^{1/2}\left(\sum_{j=1}^{N-1}\left(|\!|\!| v |\!|\!|^2_{\widetilde{K}_j}+|\!|\!| v |\!|\!|^2_{\widetilde{K}_{j+1}}\right)\right)^{1/2}
\end{aligned}
$$

Observing that

$$
\sum_{j=1}^{N}|\!|\!| v |\!|\!|^2_{\widetilde{K}_j}\le 3\,|\!|\!| v |\!|\!|^2, \qquad \sum_{j=1}^{N-1}\left(|\!|\!| v |\!|\!|^2_{\widetilde{K}_j}+|\!|\!| v |\!|\!|^2_{\widetilde{K}_{j+1}}\right)\le 6\,|\!|\!| v |\!|\!|^2,
$$

we finally see that

$$
|a(u,v)-a(u_{\mathrm{hp}},v)| \;\le \sqrt{12}C_I\left(\sum_{j=1}^{N}\eta^2_{K_j}\right)^{1/2}|\!|\!| v |\!|\!|,
$$

with η_{K_j} from (8). Dividing both sides of this inequality by $|\!|\!| v |\!|\!|$ and taking the supremum for all $v\in H_0^1(\Omega)$ shows Theorem 3.1.

Remark 3.4. In the case $d\ge 0$, following along the lines of [27], it is possible to prove ε-robust local lower bounds for the error in terms of the error indicators η_{K_j} and the data oscillation terms. This approach, however, results in efficiency bounds that will be slightly suboptimal with respect to the local polynomial degrees due to the need of applying p-dependent norm equivalences (involving cut-off functions); cf. [22].

4. Numerical Experiments

The purpose of this section is to illustrate the *a posteriori* error estimates from Theorem 3.1 by means of some numerical experiments. We will emphasize on the robustness of the error indicators with respect to ε as $\varepsilon\to 0$, and on the capability of *hp*-FEM to deliver exponential rates of convergence.

We shall apply an hp-adaptive algorithm which is based on the following ingredients:

- The *a posteriori* error estimate from Theorem 3.1.
- *Dörfler marking:* In order to mark elements for refinement, we fix a parameter $\theta \in (0,1)$ (in the experiments below we choose $\theta = 0.5$) such that
$$\theta \sum_{j=1}^{N} \eta_{K_j}^2 \leq \sum_{j'=1}^{M} \eta_{K_{j'}}^2, \tag{D}$$
where the indices j' are chosen such that the error indicators $\eta_{K_{j'}}$ from (8) are sorted in descending order, and M is minimal.
- *hp-refinement criterion:* The decision of whether a marked element is refined with respect to h (element bisection) or p (increasing the local polynomial order by 1) is based on the elementwise smoothness indicator
$$\mathcal{F}_j := \begin{cases} \dfrac{\sup_{x\in K_j}\left|\frac{\mathsf{d}^{p_j-1}}{\mathsf{d}x^{p_j-1}}u_{\mathrm{hp}}(x)\right|}{h_j^{-1/2}\left\|\frac{\mathsf{d}^{p_j-1}u_{\mathrm{hp}}}{\mathsf{d}x^{p_j-1}}\right\|_{K^2(K_j)} + \frac{1}{\sqrt{2}}h_j^{1/2}\left\|\frac{\mathsf{d}^{p_j}u_{\mathrm{hp}}}{\mathsf{d}x^{p_j}}\right\|_{K^2(K_j)}} & \text{if } \frac{\mathsf{d}^{p_j-1}}{\mathsf{d}x^{p-1}}u_{\mathrm{hp}}|_{K_j} \not\equiv 0 \\ 1 & \text{if } \frac{\mathsf{d}^{p_j-1}}{\mathsf{d}x^{p_j-1}}u_{\mathrm{hp}}|_{K_j} \equiv 0 \end{cases} \tag{F}$$
introduced in [12, Eq. (3)]. Here, the basic idea is to monitor the constant in the Sobolev embedding $H^1 \hookrightarrow L^\infty$, and thereby to decide whether or not the local solution is smooth. More precisely, if $u_{\mathrm{hp}}^{(p_j-1)}$ is nearly constant on K_j then $\mathcal{F}_j \approx 1$, and $\mathcal{F}_j$ is getting smaller if $u_{\mathrm{hp}}^{(p_j-1)}$ is less smooth. In this way, u_{hp} is classified smooth on K_j if $\mathcal{F}_j \geq \tau$ and otherwise nonsmooth, for a prescribed parameter τ (in our experiments we choose $\tau = 0.6$). For ease of evaluation, note that, by taking the derivative of order $p_j - 1$, the smoothness indicator $\mathcal{F}_j$ is applied to linear functions only; in this case, it can be shown that $\frac{1}{2} \approx \frac{\sqrt{3}}{\sqrt{6}+1} \leq \mathcal{F}_j \leq 1$; cf. [12, Section 2.2].

Combing the above ideas leads to the following hp-adaptive refinement algorithm:

Algorithm 4.1. *Choose prescribed parameters $\theta \in (0,1)$ and $\tau \in \left(\frac{\sqrt{3}}{\sqrt{6}+1}, 1\right)$ for the Dörfler marking as well as for the hp-decision process as described before, respectively. Furthermore, consider a (coarse) initial mesh $\mathcal{T}^0$, and an associated polynomial degree vector $\boldsymbol{p}^0$. Set $n = 0$. Then, perform the following iteration (until a given maximum iteration number is reached, or until the estimated error is sufficiently small):*

(1) Compute the numerical solution $u_{\mathrm{hp}}^n \in V_{\mathrm{hp}}(\mathcal{T}^n, \boldsymbol{p}^n)$ from (5)*, and evaluate the error indicators $\{\eta_{K_j}\}_{K_j\in\mathcal{T}^n}$ defined in* (8)*.*

(2) Mark the elements in $\mathcal{T}^n$ based on the Dörfler marking (D)*.*

(3) For each marked element K_j evaluate the smoothness indicator $\mathcal{F}_j$ from (F)*; if $\mathcal{F}_j \geq \tau$ then increase the polynomial degree p_j by 1, i.e., $p_j \leftarrow p_j + 1$, otherwise bisect K_j into two new elements (taking p_j for both elements).*

In the ensuing experiments, we will choose a uniform initial mesh consisting of 10 elements, and set the polynomial degree to be 1 on each of them.

4.1. **Example 1:** We begin by looking at the singularly perturbed reaction-diffusion problem
$$-\varepsilon u'' + u = 1 \quad \text{on } \Omega = (-1,1), \qquad u(-1) = u(1) = 0.$$
This problem is coercive and has exactly one (analytic) solution. For small $\varepsilon \ll 1$ the exact solution exhibits a boundary layer at $x = 0$ and $x = 1$ which needs to be resolved properly by the hp-adaptive FEM. In Figure 1 the hp-mesh after 24 adaptive refinement steps is displayed for $\varepsilon = 10^{-4}$. We observe that the boundary layer is resolved by some mild h-refinement and by increasing p in the same area. Moreover, the mesh remains unrefined in the center of the domain where the exact solution is nearly constant 1. In addition, in Figure 2 we show the errors measured with respect to the norm $|||\cdot|||$ from (6) as well as the estimated errors. The exponential

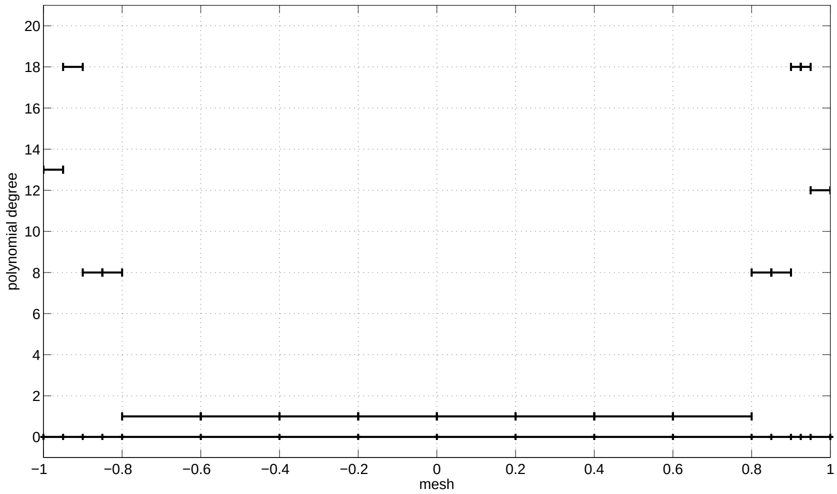

FIGURE 1. Example 1 for $\varepsilon = 10^{-4}$: Adaptively generated *hp*-mesh after 24 refinement steps (17 elements, maximal polynomial degree 18).

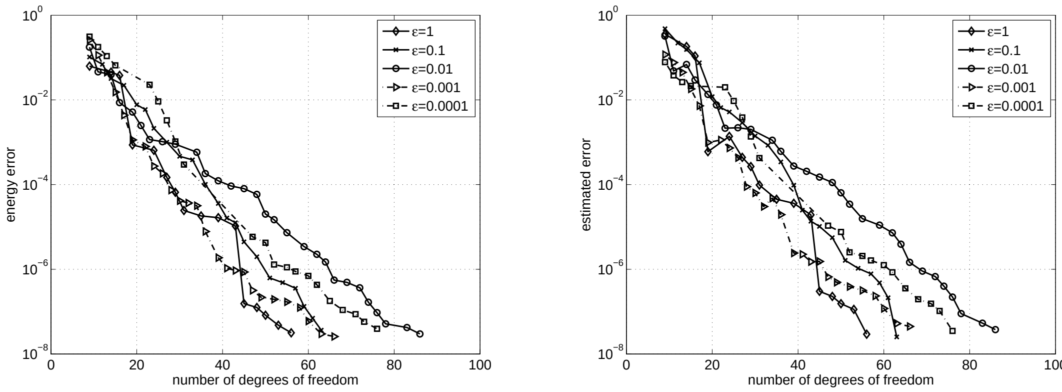

FIGURE 2. Example 1: Energy error (left) and estimated error (right) for different choices of ε.

decay of both quantities for different choices of ε becomes clearly visible in the semi-logarithmic plot. Finally, the efficiency indices, i.e., the ratio between the estimated and true errors, are depicted in Figure 3; they oscillate between 1 and 4, and do not deteriorate as $\varepsilon \to 0$, thereby clearly testifying to the robustness of the *a posteriori* error estimate from Theorem 3.1.

4.2. **Example 2:** In this experiment, we consider Airy's equation

$$-\varepsilon u'' + xu = 1 \quad \text{on } \Omega = (-1, 1), \qquad u(-1) = u(1) = 0.$$

The particularity of this example is that, for $0 < \varepsilon \ll 1$, the corresponding differential operator is coercive for $x \geq 1$, however, it becomes hyperbolic near $x = -1$; this becomes evident in Figure 4 (left), where the numerical solution is shown for $\varepsilon = 10^{-4}$. The oscillating regime for $x < 0$ requires a proper resolution by the *hp*-FEM as shown in the *hp*-mesh in Figure 5. The decay of the estimated error is plotted in Figure 4 (right) for various choices of ε. In particular, for small ε, we see that, after a number of initial refinements resolving the oscillations, the algorithm provides exponentially converging results.

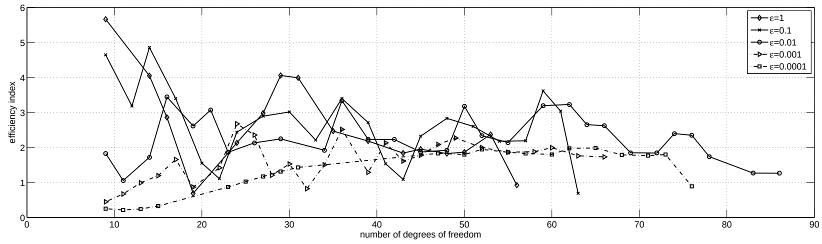

FIGURE 3. Example 1: Efficiency indices for different choices of ε.

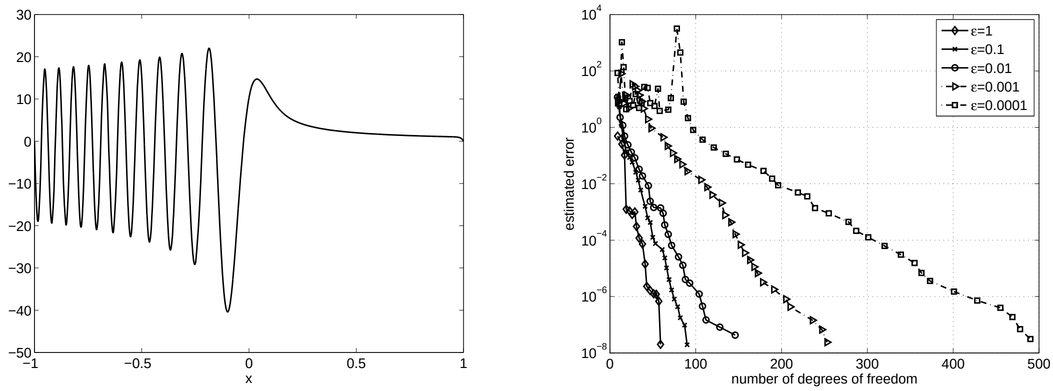

FIGURE 4. Example 2 for $\varepsilon = 10^{-4}$. Left: Numerical solution, which is highly oscillatory for $x < 0$. Right: Estimated errors for different choices of ε.

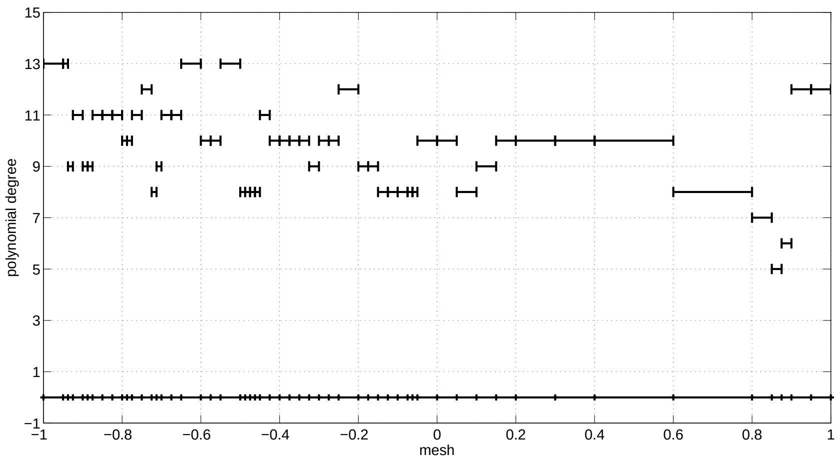

FIGURE 5. Example 2 for $\varepsilon = 10^{-4}$: Adaptively generated *hp*-mesh after 75 refinement steps (55 elements, maximal polynomial degree 13).

5. CONCLUSIONS

In this paper we have studied the numerical approximation of linear second-order boundary value problems (with possibly non-constant reaction coefficient) by the *hp*-FEM. In particular, we have derived an *a posteriori* error estimate for a natural residual-type norm that is robust with respect to the (possibly) small perturbation parameter and explicit with respect to the local mesh size and polynomial degree. Numerical experiments for both coercive as well as partly coercive differential equations underline the robustness of the error bound. In addition, an appropriate combination of the error estimate with a smoothness testing procedure reveals that the method is able to achieve exponential rates of convergence.

Appendix A. A Multiplicative Trace Inequality

Lemma A.1. *Let $h > 0$ and $w \in H^1(0, h)$. Then, the multiplicative trace inequality*

$$\max\{|w(0)|, |w(h)|\}^2 \leq h^{-1}\|w\|^2_{L^2(0,h)} + 2\|w\|_{L^2(0,h)}\|w'\|_{L^2(0,h)}$$

holds true.

Proof. By density of $C^\infty([0, h])$ in $H^1(0, h)$, we may suppose that w is smooth. There holds

$$w(0)^2 = \int_0^h \frac{\mathsf{d}}{\mathsf{d}x}\left[\left(h^{-1}x - 1\right)w(x)^2\right]\,\mathsf{d}x = h^{-1}\int_0^h w(x)^2\,\mathsf{d}x + 2\int_0^h \left(h^{-1}x - 1\right)w(x)w'(x)\,\mathsf{d}x.$$

Then, applying the Cauchy-Schwarz inequality and noticing that $\left|1 - h^{-1}x\right| < 1$ for $x \in (0, h)$, results in

$$|w(0)|^2 \leq h^{-1}\|w\|^2_{L^2(0,h)} + 2\|w\|_{L^2(0,h)}\|w'\|_{L^2(0,h)}.$$

By symmetry, the same bound can be obtained for $|w(h)|^2$. This completes the proof. □

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Institut für Analysis und Scientific Computing, TU Wien, Wiedner Hauptstrasse 8–10, A-1040 Wien, Austria

E-mail address: `melenk@tuwien.ac.at`

Mathematics Institute, University of Bern, CH-3012 Bern, Switzerland

E-mail address: `wihler@math.unibe.ch`