ASC Report No. 49/2012

Novel Inverse Estimates for Non-Local Operators

M. Feischl, T. Führer, M. Karkulik,
J.M. Melenk and D. Praetorius

Institute for Analysis and Scientific Computing
Vienna University of Technology — TU Wien
www.asc.tuwien.ac.at ISBN 978-3-902627-05-6

Most recent ASC Reports

48/2012 *M. Feischl, T. Führer, M. Karkulik, and D. Praetorius*
Quasi-Optimal Adaptive BEM

47/2012 *M. Feischl, T. Führer, M. Karkulik, J.M. Melenk, and D. Praetorius*
FEM-BEM Couplings without Stabilization

46/2012 *P. Dörsek, J.M. Melenk*
Symmetry-free, p-robust equilibrated error indication for the hp-version of the FEM in almost incompressible linear elasticity

45/2012 *P. Dörsek, J.M. Melenk*
A Numerical Study of Averaging Error Indicators in p-FEM

44/2012 *A. Jüngel, I. Stelzer*
Existence analysis of Maxwell-Stefan systems for multicomponent mixtures

43/2012 *M. Feischl, T. Führer, and D. Praetorius*
Adaptive FEM with optimal convergence rates for a certain class of non-symmetric and possibly non-linear problems

42/2012 *L. Banǎs, M. Page, D. Praetorius*
A general integrator for the Landau-Lifshitz-Gilbert equation

41/2012 *I. Rachunková, A. Spielauer, S. Staněk, E. Weinmüller*
Positive solutions of nonlinear Dirichlet BVPs in ODEs with time and space singularities

40/2012 *S. Simonov, H. Woracek*
Spectral multiplicity of selfadjoint Schrödinger operators on star-graphs with standard interface conditions

39/2012 *A. Sokolova, H. Woracek*
Congruences of Convex Algebras

Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße 8–10
1040 Wien, Austria

E-Mail: admin@asc.tuwien.ac.at
WWW: http://www.asc.tuwien.ac.at
FAX: +43-1-58801-10196

ISBN 978-3-902627-05-6

ASC
TU WIEN

NOVEL INVERSE ESTIMATES FOR NON-LOCAL OPERATORS

MICHAEL FEISCHL, THOMAS FÜHRER, MICHAEL KARKULIK,
JENS MARKUS MELENK, AND DIRK PRAETORIUS

1. Inverse estimates

Inverse estimates are a means to bound expressions in stronger norms than in the generic situation by exploiting additional structure. Two examples of such structures are the following:

(a) In FEM, strong norms of piecewise polynomials can be bounded in terms of weak norms. The key point is the ability to use norm equivalence on finite dimensional spaces on a reference configuration. Scaling arguments provide the correct powers of the local mesh size.

(b) In regularity theory for elliptic PDE, "interior regularity" of solutions can be seen as an inverse estimate. By using the underlying equation directly, strong norms of solutions can be controlled by weaker norms at the expense of slightly enlarging the domain.

In the following, $\mathcal{T}$ denotes a (even locally refined) mesh on a subset $\Gamma \subseteq \partial\Omega$ of the boundary $\partial\Omega$ of a polyhedral domain $\Omega \subset \mathbb{R}^d$. The local mesh size is denoted by $h \in \mathcal{P}^0(\mathcal{T})$ where $\mathcal{P}^p(\mathcal{T})$ is the space of piecewise polynomials of degree at most p. An example for (a) is given, e.g., in [6, Theorem 3.6]:

Theorem 1.1 (Inverse estimate for piecewise constants)**.** *There exists a constant $C > 0$, which depends only on an upper bound for the shape-regularity constant of $\mathcal{T}$ and the polynomial degree $p \geq 0$, such that*

$$\|h^{1/2}\Psi\|_{L_2(\Gamma)} \leq C\|\Psi\|_{H^{-1/2}(\Gamma)} \quad \textit{for all } \Psi \in \mathcal{P}^p(\mathcal{T}).$$

As an example for (b) serves [9, Lemma 5.7.1]:

Theorem 1.2 (Interior regularity/Caccioppoli inequality)**.** *There is $C > 0$ such that the following holds: If B_r, B_{r+h} are balls with radii $r, r+h > 0$ around a joint midpoint, and if $u \in H^1(B_{r+h})$ satisfies $\Delta u = 0 \in L_2(B_{r+h})$ for some $r, h > 0$, then $u \in H^2(B_r)$ with*

$$\|D^2 u\|_{L_2(B_r)} \leq Ch^{-1}\|\nabla u\|_{L_2(B_{r+h})}.$$

The research of the authors is supported through the FWF research project *Adaptive Boundary Element Method*, see http://www.asc.tuwien.ac.at/abem/, funded by the Austrian Science Fund (FWF) under grant P21732, as well as through the *Innovative Projects Initiative* of Vienna University of Technology. This support is thankfully acknowledged.

In the following, we present a possibility how to obtain inverse estimates involving boundary layer potentials by combining the two inverse estimates above. As a prototype for such a layer potential serves the simple-layer potential of the 3D-Laplacian, which is given by

$$V\psi(x) := \frac{1}{4\pi}\int_\Gamma \frac{\psi(y)}{|x-y|}ds_y. \tag{1}$$

Hereafter, K and K' denote the double layer potential and its adjoint, and W denotes the hypersingular integral operator. Our main result, which is taken from [1, 8], is the following.

Theorem 1.3 (Inverse estimates for boundary integral operators)**.** *There is $C>0$ which depends only on Γ and an upper bound for the shape-regularity constant of $\mathcal{T}$, such that*

$$\|h^{1/2}\nabla_\Gamma V\psi\|_{L_2(\Gamma)} \le C\left[\|\psi\|_{H^{-1/2}(\Gamma)} + \|h^{1/2}\psi\|_{L_2(\Gamma)}\right], \tag{2}$$

$$\|h^{1/2}K'\psi\|_{L_2(\Gamma)} \le C\left[\|\psi\|_{H^{-1/2}(\Gamma)} + \|h^{1/2}\psi\|_{L_2(\Gamma)}\right], \tag{3}$$

$$\|h^{1/2}\nabla_\Gamma Kv\|_{L_2(\Gamma)} \le C\left[\|v\|_{H^{1/2}(\Gamma)} + \|h^{1/2}\nabla_\Gamma v\|_{L_2(\Gamma)}\right], \tag{4}$$

$$\|h^{1/2}Wv\|_{L_2(\Gamma)} \le C\left[\|v\|_{H^{1/2}(\Gamma)} + \|h^{1/2}\nabla_\Gamma v\|_{L_2(\Gamma)}\right], \tag{5}$$

holds for all $\psi \in L_2(\Gamma)$ and $v \in H^1(\Gamma)$. Here, ∇_Γ denotes the surface gradient for $d \ge 3$, and the arclength derivative for $d = 2$.

Some observations regarding this result are the following.

- We stress that a difficulty in proving Theorem 1.3 lies in the consideration of *locally refined* meshes. If we consider a *globally uniform* mesh $\mathcal{T}$, we can use stability $V : L_2(\Gamma) \to H^1(\Gamma)$ to estimate, e.g.,
$$\begin{aligned}\|h^{1/2}\nabla_\Gamma V\psi\|_{L_2(\Gamma)} &= h^{1/2}\|\nabla_\Gamma V\psi\|_{L_2(\Gamma)} \lesssim h^{1/2}\|\psi\|_{L_2(\Gamma)}\\ &= \|h^{1/2}\psi\|_{L_2(\Gamma)}.\end{aligned}$$
- As already mentioned, inverse estimates typically require a space with some structure. Theorem 1.3 holds for $\psi \in L_2(\Gamma)$, which might not be regarded as a space with a rich structure. However, if $\psi = \Psi \in \mathcal{P}^p(\mathcal{T})$, we can use Theorem 1.1 to estimate, e.g.,
$$\|h^{1/2}\nabla_\Gamma V\Psi\|_{L_2(\Gamma)} \lesssim \|\Psi\|_{H^{-1/2}(\Gamma)}. \tag{6}$$
By stability of V and its inverse, the last estimate is equivalent to
$$\|h^{1/2}\nabla_\Gamma V\Psi\|_{L_2(\Gamma)} \lesssim \|V\Psi\|_{H^{1/2}(\Gamma)}, \tag{7}$$
which is indeed an inverse estimate.

An analogous result to Theorem 1.3 was proven independently in [5] for lowest-order discretizations and $C^{1,1}$ surfaces. In the next section, we present the ideas for proving the bound for the simple layer potential V in Theorem 1.3, and in the last section we comment on several applications.

2. Inverse Estimate for V

The main difficulty in proving (2) is the fact that V is a non-local operator. We restrict our considerations to local configurations, i.e. elements, via

$$\|h^{1/2}\nabla_\Gamma V\psi\|_{L_2(\Gamma)}^2 = \sum_{T\in\mathcal{T}} \|h^{1/2}\nabla_\Gamma V\psi\|_{L_2(T)}^2.$$

Even if we would restrict to $\psi = \Psi \in \mathcal{P}^0(\mathcal{T})$, it would be impossible to bound the contributions of $V\Psi$ on T, as the local dimension of this space is dominated by the whole mesh $\mathcal{T}$ – actually, V is non-local. On T, we therefore split the potential $u = V\psi$ in a part u_T^{near} with a bounded and small dimension, and a rest u_T^{far} via

$$V\psi = V(\psi_T) + V(\psi_{\Gamma\setminus T}) = u_T^{\mathrm{near}} + u_T^{\mathrm{far}}.$$

Here, $\psi_\omega := \psi\chi_\omega$, where χ_ω is the characteristic function of the set ω. We stress that the actual splitting that is used in the proofs of [1, 8] extends a little to the neighborhood of T, but we stick to this simplification for ease of presentation. We call u_T^{near} the *nearfield* and u_T^{far} the *farfield* and write

$$\|h^{1/2}\nabla_\Gamma V\psi\|_{L_2(\Gamma)}^2 \lesssim \sum_{T\in\mathcal{T}} \|h^{1/2}\nabla_\Gamma u_T^{\mathrm{near}}\|_{L_2(T)}^2 + \sum_{T\in\mathcal{T}} \|h^{1/2}\nabla_\Gamma u_T^{\mathrm{far}}\|_{L_2(T)}^2. \tag{8}$$

Due to the locality of the nearfield and stability $V : L_2(\Gamma) \to H^1(\Gamma)$,

$$\|\nabla_\Gamma u_T^{\mathrm{near}}\|_{L_2(T)}^2 \le \|\nabla_\Gamma u_T^{\mathrm{near}}\|_{L_2(\Gamma)}^2 \lesssim \|\psi_T\|_{L_2(\Gamma)}^2 = \|\psi\|_{L_2(T)}^2, \tag{9}$$

and a multiplication with the local mesh width h and a sum over all elements bounds the nearfield terms in (8). It remains to bound the farfield terms in (8), which is done by exploiting Theorem 1.2. We may do so, because

1. V is a potential in $\mathbb{R}^d$, i.e., $\Delta V(\psi_{\Gamma\setminus T}) = 0$ in Ω and in $\mathbb{R}^d \setminus \Omega$, and it is smooth in both parts,
2. and the farfield term is smooth in a d-dimensional neighborhood of T, as the density $\psi_{\Gamma\setminus T}$ vanishes on T.

Put differently, u_T^{far} is a potential that is induced by a density on $\Gamma \setminus T$. A standard trace inequality applied to $\nabla_\Gamma u_T^{\mathrm{far}}$ involves the second derivative of u_T^{far} in a volume U_T around T,

$$\|\nabla_\Gamma u_T^{\mathrm{far}}\|_{L_2(T)}^2 \lesssim h_T^{-1}\|\nabla u_T^{\mathrm{far}}\|_{L_2(U_T)}^2 + \|\nabla u_T^{\mathrm{far}}\|_{L_2(U_T)}\|D^2 u_T^{\mathrm{far}}\|_{L_2(U_T)}.$$

Due to reasons 1. and 2., we can bound the second derivative by Theorem 1.2 and obtain

$$\|\nabla_\Gamma u_T^{\mathrm{far}}\|_{L_2(T)}^2 \lesssim h_T^{-1}\|\nabla u_T^{\mathrm{far}}\|_{L_2(\widetilde{U}_T)}^2$$

with a slightly enlarged volume $\widetilde{U}_T$. A sum over T bounds the farfield terms in (8), which is then splitted into the difference $u_T^{\mathrm{far}} = V\psi - u_T^{\mathrm{near}}$ of the whole potential and the nearfield,

$$\sum_{T\in\mathcal{T}} \|h^{1/2}\nabla_\Gamma u_T^{\mathrm{far}}\|_{L_2(T)}^2 \lesssim \sum_{T\in\mathcal{T}} \|\nabla V\psi\|_{L_2(\widetilde{U}_T)}^2 + \sum_{T\in\mathcal{T}} \|\nabla u_T^{\mathrm{near}}\|_{L_2(\widetilde{U}_T)}^2$$

We ensure that only a bounded number of $\widetilde{U}_T$ overlap, such that the terms on the right hand side can be estimated by stability of the potential $V\psi$ and the locality of the nearfield analogous to (9).

3. Applications

3.1. **Convergence of adaptive BEM.** For given data f, the (unknown) solution ϕ of the weakly-singular integral equation

$$V\phi = f \tag{10}$$

can be approximated adaptively by a Galerkin method. To that end, we employ the following adaptive algorithm.

Algorithm 3.1. Input: coarse mesh $\mathcal{T}_0$, approximation order $p \in \mathbb{N}_0$, parameter $\theta \in (0,1)$, counter $\ell := 0$.

(i) compute Galerkin solution $\Phi_\ell \in \mathcal{P}^p(\mathcal{T}_\ell)$ of (10).

(ii) for every $T \in \mathcal{T}_\ell$, compute error indicator

$$\rho_\ell(T) := \|h_\ell^{1/2}\nabla_\Gamma(V\Phi_\ell - f)\|_{L_2(T)}.$$

(iii) choose a set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ of minimal cardinality such that

$$\theta \sum_{T\in\mathcal{T}} \rho_\ell(T)^2 \leq \sum_{T\in\mathcal{M}_\ell} \rho_\ell(T)^2.$$

(iv) refine at least the elements $\mathcal{M}_\ell$ in $\mathcal{T}_\ell$ and obtain $\mathcal{T}_{\ell+1}$.

(v) increase counter $\ell := \ell + 1$ and goto (i).

Output: sequence of solutions $(\Phi_\ell)_{\ell\in\mathbb{N}_0}$, sequence of estimators, $(\rho_\ell)_{\ell\in\mathbb{N}_0}$.

We can employ the inverse estimate for V to show that the adaptive Algorithm 3.1 converges, cf. [7]:

Theorem 3.2. *The sequence of Galerkin solutions $(\Phi_\ell)_{\ell\in\mathbb{N}_0}$ computed by Algorithm 3.1 converges to ϕ, i.e.,*

$$\|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} \to 0.$$

Two important observations regarding algorithm 3.1 are the following:

- In [4], it is shown that the estimator ρ_ℓ, employed in Algorithm 3.1, is reliable, i.e.,

$$\|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} \lesssim \rho_\ell := \Big(\sum_{T\in\mathcal{T}} \rho_\ell(T)^2\Big)^{1/2}. \tag{11}$$

- Arguments from [3] show that Algorithm 3.1 converges *a priori*, i.e.,

$$\|\Phi_{\ell+1} - \Phi_\ell\|_{H^{-1/2}(\Gamma)} \to 0. \tag{12}$$

Using the contraction of the mesh size on refined elements, it is possible to show that Algorithm 3.1 yields the so-called *estimator reduction*

$$\rho_{\ell+1} \leq \kappa\rho_\ell + C_{\rm red}\|h_{\ell+1}^{1/2}\nabla_\Gamma V(\Phi_{\ell+1} - \Phi_\ell)\|_{L_2(\Gamma)} \tag{13}$$

with some $\kappa \in (0,1)$ and $C_{\rm red} > 0$. This means that ρ_ℓ is a contraction up to a perturbation term, which consists of the difference of two successive Galerkin solutions, measured in a stronger norm compared to (12). We use the inverse estimate (6) to bound the perturbation term by the weaker $H^{-1/2}$ norm and use the a priori convergence (12),

$$\|h_{\ell+1}^{1/2}\nabla_\Gamma V(\Phi_{\ell+1}-\Phi_\ell)\|_{L_2(\Gamma)} \lesssim \|\Phi_{\ell+1}-\Phi_\ell\|_{H^{-1/2}(\Gamma)} \to 0. \tag{14}$$

Hence, we conclude from (13) and (14) that ρ_ℓ is a contraction up to a zero sequene. From basic calculus, we infer that $\rho_\ell \to 0$. Due to reliability (11), we conclude the statement of Theorem 3.2.

3.2. **Convergence of adaptive FEM-BEM coupling.** We adaptively solve a Laplace transmission problem with given Dirichlet and Neumann jumps u_0 and ϕ_0. We use the symmetric FEM-BEM coupling formulation and seek $u \in H^1(\Omega)$ and $\phi \in H^{-1/2}(\Gamma)$ s.t.

$$\begin{aligned}\langle \nabla u\,,\,\nabla v\rangle_\Omega + \langle Wu + (K'-1/2)\phi\,,\,v\rangle_\Gamma &= \langle f\,,\,v\rangle_\Omega + \langle \phi_0 + Wu_0\,,\,v\rangle_\Gamma\\ \langle \psi\,,\,V\phi - (K-1/2)u\rangle_\Gamma &= -\langle \psi\,,\,(K-1/2)u_0\rangle_\Gamma,\end{aligned}$$

hold true for all $v \in H^1(\Omega)$ and $\psi \in H^{-1/2}(\Gamma)$. We stress that the following arguments also apply to other couplings, e.g. the Johnson-Nédélec or Bielak-McCamy coupling. For a posteriori error estimation, we use a combination η_ℓ of the residual FEM error estimator and the weighted residual BEM estimator ρ_ℓ from Subsection 3.1, cf. [1]. Again, an adaptive algorithm of the form of Algorithm 3.1 exhibits the estimator reduction for η_ℓ, where strong norms involving all 4 boundary integral operators appear in the perturbation terms. This strong norms can be estimated by the inverse estimates from Theorem 1.3, such that we obtain $\eta_\ell \to 0$, cf. Subsection 3.1, which results in convergence of the adaptive coupling due to the reliability of η_ℓ, cf. [1].

3.3. **Efficiency of weighted residual estimates in BEM.** The inverse estimate for V can be used to show the efficiency of the residual error estimate ρ_ℓ from Subsection 3.1 for $d = 2$. We recall that ϕ is the exact solution of the weakly singular integral equation (10), and $\Phi_\ell \in \mathcal{P}^0(\mathcal{T}_\ell)$ denotes a lowest-order Galerkin solution. The estimate for V in Theorem 1.3 states that

$$\rho_\ell = \|h_\ell^{1/2}\nabla_\Gamma V(\phi-\Phi_\ell)\|_{L_2(\Gamma)} \lesssim \|\phi-\Phi_\ell\|_{H^{-1/2}(\Gamma)} + \|h_\ell^{1/2}(\phi-\Phi_\ell)\|_{L_2(\Gamma)}.$$

By an explicite use of the singular behaviour of ϕ on polygonal boundaries, we can bound the last term in the preceding estimate and obtain the following, cf. [2].

Theorem 3.3 (Efficiency of ρ_ℓ in 2D)**.** *If $f \in H^s(\Gamma)$ for $s > 2$, then*

$$\rho_\ell \lesssim \|\phi-\Phi_\ell\|_{H^{-1/2}(\Gamma)} + \mathrm{hot}_\ell,$$

where, for all $\varepsilon > 0$,

$$\mathrm{hot}_\ell^2 = \sum_{T\in\mathcal{T}} \mathrm{hot}_\ell(T)^2 \quad \text{and} \quad \mathrm{hot}_\ell(T) \le C_{\mathrm{hot}} h_\ell(T)^{\min(s,5/2)-1/2-\varepsilon}.$$

The constant C_{hot} depends only on Γ, an upper bound of the shape regularity constant of $\mathcal{T}_\ell$, s, and ε.

If we again restrict to globally uniform meshes, we see that the preceding Theorem yields, for some $\varepsilon > 0$,

$$\rho_\ell \lesssim \|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} + \mathcal{O}(h_\ell^{3/2+\varepsilon}).$$

As the optimal rate of convergence for lowest order BEM is $\mathcal{O}(h^{3/2})$, we obtain efficiency of ρ_ℓ up to terms of higher order.

References

[1] M. Aurada, M. Feischl, T. Führer, M. Karkulik, J. M. Melenk, and D. Praetorius, *Inverse estimates for elliptic boundary integral operators and their application to the adaptive coupling of FEM and BEM*, ASC Report 07/2012, Institute for Analysis and Scientific Computing, Vienna University of Technology, Wien, 2012.

[2] M. Aurada, M. Feischl, T. Führer, M. Karkulik, and D. Praetorius, *Efficiency and optimality of some weighted-residual error estimator for adaptive 2D boundary element method*, ASC Report 15/2012, Institute for Analysis and Scientific Computing, Vienna University of Technology, Wien, 2012.

[3] I. Babuska and M. Vogelius, *Feedback and adaptive finite element solution of one-dimensional boundary value problems*, Numer. Math., 44 (1984), 75–102.

[4] C. Carstensen, M. Maischak, and E. Stephan, *A posteriori error estimate and h-adaptive algorithm on surfaces for Symm's integral equation*, Numer. Math., 90 (2001), 197–213.

[5] T. Gantumur, *Adaptive boundary element methods with convergence rates*, to appear in Numer. Math., cf. arXiv:1107.0524.

[6] I. Graham, W. Hackbusch, and S. Sauter, *Finite elements on degenerate meshes: Inverse-type inequalities and applications*, IMA J. Numer. Anal. 25 (2005), 379–407.

[7] M. Feischl, M. Karkulik, J. Melenk, and D. Praetorius, *Quasi-optimal convergence rate for an adaptive boundary element method*, submitted to SIAM J. Numer. Anal.

[8] M. Karkulik, *Zur Konvergenz und Quasioptimalität adaptiver Randelementmethoden*, PhD thesis (in German), Institute for Analysis and Scientific Computing, Vienna University of Technology, Wien, 2012.

[9] J. C. B. Morrey, *Multiple integrals in the calculus of variations*, Classics in Mathematics. Springer, Berlin, 2008. Reprint of the 1966 edition.

Vienna University of Technology, Institute for Analysis and Scientific Computing, Wiedner Hauptstrasse 8-10, 1040 Vienna

Pontificia Universidad Católica de Chile, Facultad De Matemáticas, Avenida Vicuña Mackenna 4860, Santiago, Chile

E-mail address: michael.karkulik@tuwien.ac.at