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DISSERTATION

Stability analysis and a dissipative FEM for an Euler-Bernoulli beam with tip body and passivity-based boundary control

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Abstract

The Euler-Bernoulli beam equation is used to model many mechanical systems from industry and engineering. The need to control the dynamics of these systems has made stabilization, stability analysis and simulation of such systems an important research area. In this thesis, a model for the time evolution of a cantilever with tip body is considered. It is assumed that the cantilever can be modeled by the Euler-Bernoulli beam equation. This system belongs to the class of passive infinite dimensional systems and hence a passivity based feedback controller may be applied at the free end to include damping into the system. The feedback controller is considered to be dynamic and hence a hybrid PDE-ODE system is obtained. The main questions studied in this thesis are the well-posedness of such control systems and the long-term behavior of their solutions, in particular the asymptotic stability.

In order to perform the stability analysis, the system is posed as an evolution problem and treated within semigroup framework. Identifying an appropriate Lyapunov functional for the system proves to be fundamental in the present approach. The stability proof proceeds in two steps. First, it is demonstrated that the system operator generates a strongly continuous semigroup of uniformly bounded operators. Next, by demonstrating the precompactness of system trajectories, the asymptotic stability follows from La Salle's invariance principle.

The Euler-Bernoulli beam system with linear and nonlinear dynamic control is treated separately. From the literature it is known that the system with linear dynamic feedback control is asymptotically stable. However, by means of spectral analysis it is proved that this system is not exponentially stable. Alternatively, in case when the control law includes nonlinearities, the proof for the precompactness property of the system trajectories is far from obvious and a novel approach is developed. For this purpose, a toy-model is introduced first: an Euler-Bernoulli beam with a tip body and attached to a spring and a damper, both nonlinear. For this system it is shown that the trajectories of classical solutions are precompact and that, for almost all moments of inertia of the tip body, the trajectories tend to zero as time goes to infinity. However, for countably many values of the moment of inertia, the trajectories tend to a time-periodic solution. For given initial conditions it is possible to characterize this asymptotic limit explicitly, including its phase. The developed method for showing the precompactness of trajectories is further extended from the toy-model to the case with the nonlinear dynamic boundary control where the asymptotic stability of the system is demonstrated for all classical solutions.

Another research topic considered in this thesis is a numerical method for the Euler-Bernoulli beam system with dynamic boundary control or nonlinear spring and damper attached at the end. The goal is to derive a dissipative numerical method which conserves the dissipativity property of the Lyapunov functional. The discretization of the system is performed in two steps: first a semi-discrete numerical method is obtained utilizing the finite element method for the discretization in space, and in the second step a fully discrete numerical scheme is obtained using the Crank-Nicolson scheme for discretization in time. It is demonstrated that this numerical method leads to energy dissipation, analogous to the continuous case and that the method is well-defined and stable. In the linear case the convergence of the method is shown and a-priori error estimates are obtained. In order to illustrate the effectiveness and above mentioned properties of the developed numerical method, simulation results are presented. For a finite element space, the piecewise cubic Hermitian shape functions are chosen in the simulations, and the advantages of this choice are discussed.

Kurzfassung

Der Euler-Bernoulli-Balken wurde oft verwendet, um in der Industrie und in den Ingenieurwissenschaften oft auftretende mechanische Systeme zu modellieren. Mit der Herausforderung die Regelung dieser Systeme zu verbessern und weiterzuentwickeln, sind die Stabilisierung, Stabilitätsanalyse und Simulation dieser Systeme auch zu einem wichtigen Forschungsbereich geworden. Diese Dissertation befasst sich mit einem Modell für das dynamische Verhalten eines Kragbalkens mit einem Starrkörper am Balkenende. Dabei kann die Biegung des Balkens mit der Euler-Bernoulli Gleichung beschrieben werden. Dieses System gehört zur Klasse passiver unendlich-dimensionaler Systeme. Damit das System dissipativ wird, wurde eine passivitätsbasierte Rückkopplung am freien Ende des Balkens durchgeführt. Die Rückkopplung wurde als ein dynamischer Regler entworfen, und folglich erhält man ein hybrides PDGL-GDGL System. In der vorliegenden Doktorarbeit wurde nachgeprüft, ob dieses rückwärtsgeregelte System ein korrekt gestelltes Problem ist. Ebenfalls werden das Langzeitverhalten und die asymptotische Stabilität untersucht.

Für die mathematische Behandlung, sowie für die Stabilitätsanalyse, wurde das System als eine Evolutionsgleichung formuliert und in diesem Rahmen die Halbgruppentheorie betrachtet. Ein grundlegender Schritt der Analyse ist die Identifikation einer geeigneten Lyapunov-Funktion des Systems. Der Beweis zur asymptotischen Stabilität besteht aus zwei Schritten. Erst wird gezeigt, dass der Systemoperator der infinitesimale Generator einer stark stetigen Halbgruppe von gleichmäßig beschränkten Operatoren ist. Falls die Präkompaktheit der Lösungstrajektorien des Systems nachgewiesen werden kann, folgt darauf die asymptotische Stabilität direkt aus dem La Salle'schen Invarianz-Prinzip.

Das Euler-Bernoulli-Balken-System mit dem linearen und nichtlinearen dynamischen Regler wurde getrennt behandelt. Aus der Literatur ist bekannt, dass das System mit linearer dynamischen Rückkopplung asymptotisch stabil ist. Dennoch wird mit Hilfe der Spektralanalyse gezeigt, dass das System nicht exponentiell stabil ist. Wenn die Regelung auch Nichtlinearitäten enthält, ist der Nachweis für die Präkompaktheit der Lösungstrajektorien schwierig und es wurde ein neuer alternativer Ansatz entwickelt. Hierzu wird zuerst ein einfacheres Modell betrachtet: Ein Euler-Bernoulli-Balken mit einem Starrkörper am Ende sowie ein am Balkenende befestigtes nichtlineares Feder - Dämpfer System. Für dieses System wurde gezeigt, dass die Trajektorien der klassischen Lösungen präkompakt sind, und dass für fast alle Trägheitsmomente des Starrkörpers, die Lösung im Langzeitverhalten gegen Null konvergiert. Demgegenüber wurde für abzählbar viele Werte des Trägheitsmoments gezeigt, dass die Trajektorie sich einer zeitperiodischen Lösung nähert. Angenommen, dass die Anfangsbedingung bekannt ist, ist es möglich diesen Grenzwert, sowie seine Phase explizit festzustellen. Die entwickelte Methode für die Präkompaktheit der Lösungstrajektorien wurde weiterhin auf das System mit nichtlinearen dynamischen Reglern ausgeweitet. Ebenso wurde die asymptotische Stabilität des Systems für alle klassische Lösungen gezeigt.

Neben der mathematischen Analyse wurde eine weitere Fragestellung in dieser Dissertation behandelt. Dabei handelt es sich um eine numerische Methode für das Euler-Bernoulli-Balken-System mit einem dynamischen Regler oder mit einem nichtlinearen Feder-Dämpfer-System am Balkenende. Das Ziel ist es ein dissipatives numerisches Verfahren abzuleiten, welches die dissipative Eigenschaft der Lyapunov Funktion erhält. Die Diskretisierung des Systems wird in zwei Schritten durchgeführt. Zuerst wurde zur Ortsdiskretisierung die Methode der Finiten Elemente angewendet, woraus in weiterer Folge eine halb-diskrete numerische Methode entwickelt wurde. Im zweiten Schritt wurde das Crank-Nicolson Schema für die Zeitdiskretisierung ausgeführt. Diese numerischen Methoden führen zur Energiedissipation, welche dem Beispiel aus dem kontinuierlichen Fall entspricht. Im linearen Fall wurde die Konvergenz des Verfahrens nachgewiesen und eine a-priori-Fehlerabschätzung bewiesen. In mehreren Simulationsbeispielen wird die Effizienz des entwickelten numerischen Verfahrens illustriert.

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Contents

Chapter 1

Introduction

This thesis is concerned with analytical and numerical aspects of mechanical systems with control mechanisms. In particular, the Euler-Bernoulli beam (EBB) with one end clamped and a tip body attached to the free end shall be considered. As a stabilization and for damping of the system, several variants of boundary control at the free end shall be analyzed.

The EBB equation with a tip body is a well-established model with a wide range of applications: satellites with flexible appendages [3, 5], flexible robot arms [46], oscillations of telecommunication antennas, flexible wings of micro air vehicles [10], tall buildings due to external forces [42], and even vibrations of railway structures [64]. These are some of the many examples arising in engineering and industry, which demonstrates that the stabilization and tracking control of EBB indeed is an important reseach area. The interest of engineers and mathematicians in this problem has been greatly stimulated in the 1980s, when The National Aeronautics and Space Administration (NASA) started a Spacecraft Control Laboratory Experiment (SCOLE), see e.g. [45, 4, 5], with the goal to control the dynamics of large flexible spacecraft. The structures comprised within the SCOLE project include an offset-feed antenna, attached to the space shuttle by a flexible mast, modeled by an EBB with a tip body.

Since the demand on high precision performance for these systems continuously grows, it is of great interest to extend the existing stability results to the case of dynamic linear and nonlinear boundary control. These problems will be the focus of this thesis. First, the existing analysis for dynamic linear boundary control of EBB with tip body is completed. A new strategy needs to be developed to extend the stability results to the nonlinear case. Hence, the long-term behavior of a toy model is analyzed first: an EBB with a spring and a damper (both nonlinear) attached to its end. Another aim of this thesis is to design a numerical method for the dissipative systems under consideration. The method is derived in such a way that the discretized systems preserve dissipativity. For the discretization in space, finite element method is used, and Crank-Nicolson method for the discretization in time. The numerical method is validated by various simulation examples, and in the linear case its convergence is demonstrated.

1.1 Piezoelectric cantilever with tip body

The system under consideration was derived in [40] to model the bending motion of a piezoelectric cantilever with tip body at the free end. The mass of the tip body is denoted by M, and its momentum of inertia by J. The system consists of a piezoelectric cantilever of length L, clamped at the left end $x = 0$, and a tip body fixed at the tip $x = L$. In its reference state, the mid-axis of the beam lies on the x-axis, as illustrated in Figure 1.1. The cantilever is composed of thin piezoelectric layers, each of length L, and width

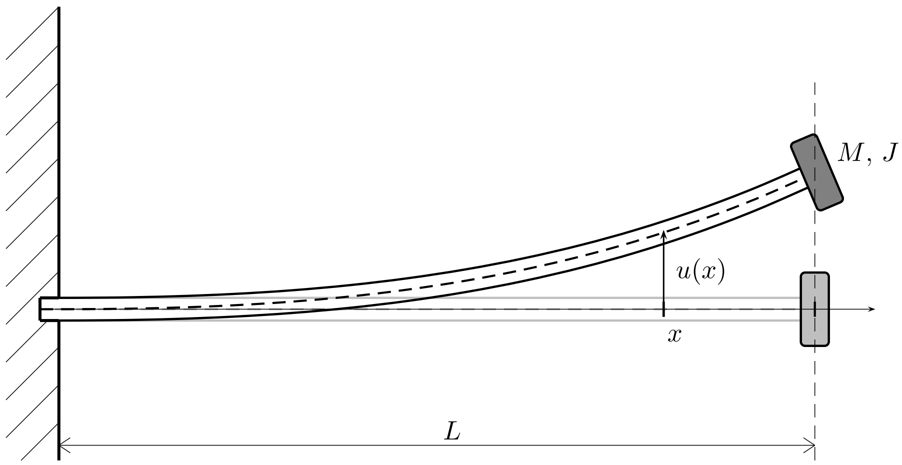


Figure 1.1: The beam is depicted in both its reference state and when deflected. It is clamped at $x = 0$, and there is a rigid body fixed at the other end $x = L$. Deflection of the beam at x is denoted by $u(x)$

B, see Figure 1.2. Some of the layers are covered by thin, appropriately shaped metallic electrodes, and are used as actuators, or as sensors. The third type of layers, called substrate layers, are not covered with electrodes, and their purpose is to provide isolation between the electrodes. Furthermore, all of the layers come in couples, and are placed symmetrically with respect to the mid-axis, as depicted in Figure 1.2. The authors in [40] use shape of the electrode layers as an additional degree of freedom in the controller design. The sensor layers were given rectangular and triangular shaped electrodes, so that the difference of the charges measured on the sensor layer couple at $x = 0$ is proportional to the tip deflection $u(t, L)$ and the tip angle of the beam, respectively $u_x(t, L)$. Also the actuator layers were assumed to be covered with both rectangular and triangular shaped electrodes, with the following motivation: A voltage supplied actuator layer couple with rectangular (or triangular) shaped electrodes acts in the same way on the structure as a bending moment (or force) at the tip of the beam.

Such piezo-actuation of the elastic cantilever is used for motion planning of the ho-

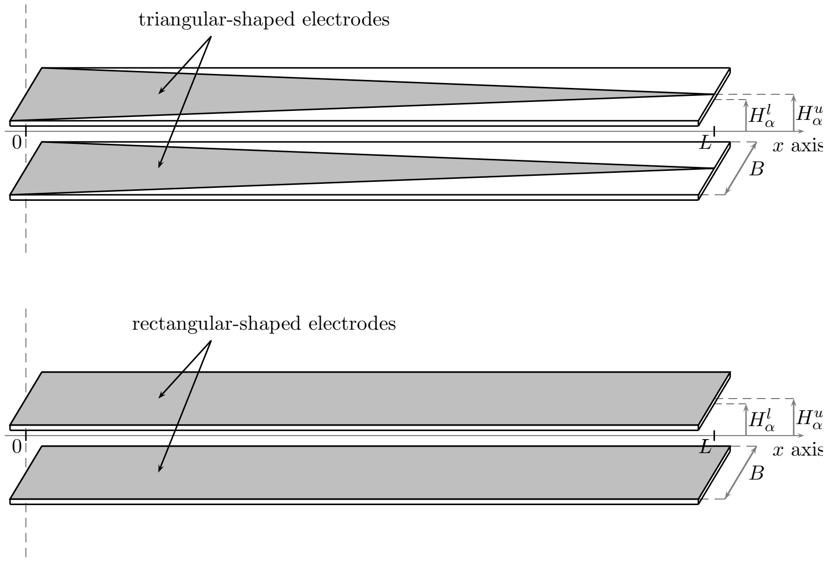


Figure 1.2: Rectangular-/triangular- shaped piezoelectric layer couples, that are utilized for actuation and sensing

mogeneous beam system. More precisely, a feed-forward tracking control is derived which causes the tip position and the tip angle of the beam to closely follow prescribed trajectories. The feed-forward control inputs Θ_1^{d} and Θ_2^{d} represent the voltage applied at $x = 0$ on rectangular- and triangular-shaped electrodes respectively, up to a constant that depends on system parameters. The desired prescribed trajectory is denoted by $u^{\mathrm{d}}(t, x)$. A very common approach for solving trajectory planning problem is used, the so-called method of differential flatness. For more details, the reader is referred to [40].

In the following, linear system (1.1)–(1.5) represents the evolution of the trajectory error system: function $u(t, x)$ denotes the deviation of the actual beam deflection from the desired reference trajectory $u^{\mathrm{d}}(t, x)$. Similarly, $\Theta_{1,2}(t)$ denote the difference between the applied voltages to the electrodes of the piezoelectric layers and $\Theta_{1,2}^{\mathrm{d}}$ specified by the feed-forward controller. Note that due to the linearity, the beam trajectory and the error of the beam trajectory solve the same equations of motion.

$$\begin{aligned}
\mu u_{tt} + \Lambda u_{xxxx} &= 0, \quad 0 < x < L,\, t > 0, && (1.1)\\
u(t, 0) &= 0, \quad t > 0, && (1.2)\\
u_x(t, 0) &= 0, \quad t > 0, && (1.3)\\
J u_{xtt}(t, L) + \Lambda u_{xx}(t, L) + \Theta_1(t) &= 0, \quad t > 0, && (1.4)\\
M u_{tt}(t, L) - \Lambda u_{xxx}(t, L) + \Theta_2(t) &= 0, \quad t > 0. && (1.5)
\end{aligned}$$

Here, mass density and the flexural rigidity of the beam are positive constants, denoted by $\mu > 0$, and $\Lambda > 0$, respectively. They are calculated from the geometrical properties (length L, width B, hight $H_\alpha^l - H_\alpha^u$), and material specifications of the piezoelectric layers. In the above system, the equations of motion for the elastic beam and an attached body consist of a partial differential equation (1.1) which describes the deviation of the beam, coupled to the ordinary differential equations (1.4) and (1.5) which govern the motion of the tip body. Therefore, in literature, the system (1.1)–(1.5) is often called *hybrid* [45]. Equation (1.4) states that the beam bending moment at $x = L$ (i.e. $\Lambda u_{xx}(t, L)$) plus the bending moment of the tip body (i.e. $J u_{xtt}(t, L)$) is balanced by the control input $-\Theta_1$. Similarly, equation (1.5) states that the total force at the free end, which is equal to shear force at the tip (i.e. $-\Lambda u_{xxx}(t, L)$) plus the tip mass force $M u_{tt}$, cancels with the control input Θ_2.

The control inputs Θ_1 and Θ_2 need to devise a stable feedback control for that beam, such that the beam evolves very close by to a desired trajectory, in the sense that the error system (1.1)–(1.5) approaches the zero state $u \equiv 0$ (as $t \to \infty$). However, when designing the control inputs, only $u(t, L)$, $u_x(t, L)$ and their time derivatives can be employed, in order to make the system technically realizable with the aforementioned piezoelectric sensors. Furthermore, the control laws should be such that the resulting closed-loop system is a well-posed problem, i.e. it has a unique solution.

1.2 Dynamic feedback boundary control

Various boundary control laws for EBB systems have been devised and mathematically analyzed in the literature – with the stabilization of the system being a key objective (cf. [45]). Soon afterwords, also exponentially stable controllers were developed which require, however, higher order boundary controls for an EBB with tip body [58]. On the other hand, if only a tip mass is applied, lower order controls are sufficient for exponential stabilization [15]. In spite of this progress, and due to its widespread technological applications, considerable research on EBB-control problems is still underway: In the more recent papers [31, 29] exponential stability of related control systems was established by verifying the Riesz basis property. For the exponential stability of a more general class of boundary control systems (including the Timoshenko beam) in the port-Hamiltonian approach, refer to [69].

As a supplement to the feed-forward control, feedback control which have the goal to drive the error system to the zero state is introduced. The objective of this section is to review linear feedback control laws for (1.1)–(1.5) introduced in [40], and extend it further to nonlinear feedback control. The controllers are taken to be dynamic, rather than static, since the dynamic controller has the advantage of better disturbance rejection in comparison to the static controller (see [51] and [43]).

For the controller design, it is essential to observe the total energy of the system:

$$E_{\text{beam}} := \frac{\Lambda}{2}\int_0^L |u_{xx}(x)|^2\,dx + \frac{\mu}{2}\int_0^L |u_t(x)|^2\,dx + \frac{M}{2}u_t(L)^2 + \frac{J}{2}(u_{tx}(L))^2, \quad (1.6)$$

where the first term represents its potential, and the remaining ones its kinetic energy. Assuming sufficient regularity of u, the time derivative of energy of the system can be written as:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}E_{\mathrm{beam}} &= \Lambda\int_0^L u_{xx}u_{xxt}\,dx + \mu\int_0^L u_t u_{tt}\,dx + Mu_t(L)u_{tt}(L) + Ju_{tx}(L)u_{ttx}(L)\\ &= -\Theta_1 u_{tx}(L) - \Theta_2 u_t(L),\end{aligned} \tag{1.7}$$

whereby partial integration and identities from (1.1)–(1.5) have been employed. This identity serves as a motivation for the design of the control inputs Θ_1 and Θ_2, which needs to ensure that energy of the system decays in time. Furthermore, (1.7) implies that the system (1.1)–(1.5) is passive [47].

An effective strategy the for control design is to couple the Euler-Bernoulli beam system with a passive system in the feedback path [40, 67, 47]. The motivation for such control design is the fact that, in the finite dimensional case, the feedback interconnection of a passive systems yields a stable closed-loop system (for the concept of passivity based controller design see [38] and [39]). This principle of passivity-based controller design has recently been generalized to the infinite dimensional case, to systems frequently considered in the literature (such as wave equation, Euler-Bernoulli and Timoshenko beam [47]). The passivity-based linear and nonlinear feedback controllers are further discussed in the subsections 1.2.1 and 1.2.2.

1.2.1 Linear controller

The approach used in [40], takes a *strictly positive real* (SPR) controller[1] as the passive controller in the feedback loop. Consequently, the proposed linear controller has a dynamic design, thus coupling the governing PDEs of the beam with a system of ODEs:

$$\begin{aligned}(\zeta_1)_t(t) &= A_1\zeta_1(t) + b_1 u_{xt}(t,L),\\ (\zeta_2)_t(t) &= A_2\zeta_2(t) + b_2 u_t(t,L),\\ \Theta_1(t) &= k_1 u_x(t,L) + c_1\cdot\zeta_1(t) + d_1 u_{xt}(t,L),\\ \Theta_2(t) &= k_2 u(t,L) + c_2\cdot\zeta_2(t) + d_2 u_t(t,L),\end{aligned} \tag{1.8}$$

with the auxiliary variables $\zeta_1, \zeta_2 \in C([0,\infty);\mathbb{R}^n)$ and $\Theta_1, \Theta_2 \in C[0,\infty)$. Moreover, $A_1, A_2 \in \mathbb{R}^{n\times n}$ are Hurwitz[2] matrices, $b_j, c_j \in \mathbb{R}^n$, $k_j, d_j \in \mathbb{R}$ for $j = 1,2$, and coefficients k_1 and k_2 are assumed to be positive. It is also assumed the transfer functions $\mathcal{G}_j(s) := (sI - A_j)^{-1}b_j\cdot c_j + d_j$ for $j = 1,2$ satisfy

$$Re(\mathcal{G}_j(i\omega)) \geq d_j > \delta_j > 0 \quad \forall\omega \geq 0,$$

[1] A SPR controller is defined as a controller with SPR transfer function.

[2] A square matrix is called a Hurwitz matrix if all its eigenvalues have negative real parts.

for some constants δ_1 and δ_2. This assumption yields that the transfer functions are SPR (for its definition refer to [36], [47]), and hence the feedback control system (1.8) is passive. It follows from the Kalman-Yakubovich-Popov Lemma (see [36], [47]) that there exist symmetric positive definite matrices P_j, positive scalars ε_j, and vectors $q_j \in \mathbb{R}^n$ such that

$$\begin{aligned} P_jA_j + A_j^\top P_j &= -q_jq_j^\top - \varepsilon_j P_j, \\ P_jb_j &= c_j - q_j\sqrt{2(d_j - \delta_j)}, \end{aligned} \tag{1.9}$$

for $j = 1, 2$. In [40], it was shown, using (1.9), that (1.8) introduces damping into the system. In order to see this, an energy functional for the controller is defined:

$$E^{\mathrm{L}}_{\mathrm{control}} \;:=\; \frac{1}{2}\zeta_1^\top P_1\zeta_1 + \frac{k_1}{2}u_x(L)^2 + \frac{1}{2}\zeta_2^\top P_2\zeta_2 + \frac{k_2}{2}u(L)^2.$$

The time derivative of the energy functional read as follows:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}E^{\mathrm{L}}_{\mathrm{control}} &= \zeta_1^\top P_1(\zeta_1)_t + k_1u_x(L)u_{xt}(L) + \zeta_2^\top P_2(\zeta_2)_t + k_2u(L)u_t(L) \\ &= \zeta_1^\top P_1[A_1\zeta_1 + b_1u_{xt}(L)] + \zeta_2^\top P_2[A_2\zeta_2 + b_2u_t(L)] \\ &\quad + u_{xt}(L)[\Theta_1 - c_1\cdot\zeta_1 - d_1u_{xt}(L)] + u_{xt}(L)[\Theta_2 - c_2\cdot\zeta_2 - d_2u_t(L)] \\ &= \Theta_1u_{xt}(L) - \frac{\varepsilon_1}{2}\zeta_1^\top P_1\zeta_1 - \delta_1u_{xt}(L)^2 - \frac{1}{2}\left(\zeta_1\cdot q_1 + \tilde{\delta}_1u_{xt}(L)\right)^2 \\ &\quad + \Theta_2u_t(L) - \frac{\varepsilon_2}{2}\zeta_2^\top P_2\zeta_2 - \delta_2u_t(L)^2 - \frac{1}{2}\left(\zeta_2\cdot q_2 + \tilde{\delta}_2u_t(L)\right)^2, \end{aligned}$$

where equations (1.8) and (1.9) were used. Hence, defining

$$E^L_{\mathrm{total}} := E_{\mathrm{beam}} + E^{\mathrm{L}}_{\mathrm{control}}, \tag{1.10}$$

gives

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}E^{\mathrm{L}}_{\mathrm{total}} &= -\frac{\varepsilon_1}{2}\zeta_1^\top P_1\zeta_1 - \delta_1u_{xt}(L)^2 - \frac{1}{2}\left(\zeta_1\cdot q_1 + \tilde{\delta}_1u_{xt}(L)\right)^2 \\ &\quad -\frac{\varepsilon_2}{2}\zeta_2^\top P_2\zeta_2 - \delta_2u_t(L)^2 - \frac{1}{2}\left(\zeta_2\cdot q_2 + \tilde{\delta}_2u_t(L)\right)^2 \le 0. \end{aligned} \tag{1.11}$$

Since the expression in (1.11) is always non-positive, it follows that due to (1.8) the energy of the system indeed decays, and it implies that the functional $E^{\mathrm{L}}_{\mathrm{total}}$ is a good candidate for the Lyapunov functional of the system (1.1)–(1.5) and (1.8).

Equations (1.1)–(1.5) and (1.8) constitute a coupled PDE–ODE system for the beam deflection $u(x, t)$, the position of its tip $u(t, L)$, and its slope $u_x(t, L)$, as well as the two control variables $\zeta_1(t)$, $\zeta_2(t)$. The main mathematical difficulty of this system stems from the high order boundary conditions (involving both x- and t- derivatives) which makes the analytical and numerical treatment far from obvious. Well-posedness of this system and asymptotic stability of the zero state were established in [40] using semigroup theory on an equivalent first order system (in time), Lyapunov functional as in (1.10), and LaSalle's invariance principle.

In Chapter 2, a more general case of inhomogeneous EBB is considered. In Section 2.1 the stability of the system shall be analyzed further and it shall be shown that this unique steady state is *not* exponentially stable, thus extending the results of Rao [58] to dynamic control of inhomogeneous Euler-Bernoulli beams.

1.2.2 Nonlinear controller

Although considerable attention has been paid to the stability analysis of flexible beams, most results deal with the situation in which the control is linear, and in general the respective stability analysis uses results from linear functional analysis. Extending the boundary control to a class of nonlinear dynamic controllers increases greatly the stabilization possibilities of flexible beam systems. Also it enables one to choose among different options in order to find one with best disturbance rejection, depending on the practical problem at hand. This is necessary due to the fact that in real-life applications, the sensors and actuators do not perform as precisely as in theory, and therefore the system input and the system output contain some disturbances. However, the analysis of the nonlinear boundary control is not straightforward in most cases, since the linear techniques do not apply in this situation any more. In particular, up to the knowledge of the author the only models with nonlinear boundary control considered in the literature do not have a tip body (see [13, 18, 19]). Thus the model introduced here is a first step toward closing this gap, with the goal of investigating possible approaches for demonstrating asymptotic stability.

In this subsection, a SPR nonlinear control law is proposed to asymptotically stabilize the EBB system (1.1)–(1.5):

$$\begin{aligned} (\zeta_1)_t(t) &= a_1(\zeta_1(t)) + b_1(\zeta_1(t))u_{xt}(t,L), \\ (\zeta_2)_t(t) &= a_2(\zeta_2(t)) + b_2(\zeta_2(t))u_t(t,L), \\ \Theta_1(t) &= k_1(u_x(t,L)) + c_1(\zeta_1(t)) + d_1(\zeta_1(t))u_{xt}(t,L), \\ \Theta_2(t) &= k_2(u(t,L)) + c_2(\zeta_2(t)) + d_2(\zeta_2(t))u_t(t,L), \end{aligned} \tag{1.12}$$

where $a_j, b_j \in C^2(\mathbb{R}^n;\mathbb{R}^n)$, $c_j, d_j \in C^1(\mathbb{R}^n;\mathbb{R})$, $k_j \in C^2(\mathbb{R},\mathbb{R}^+)$, $j = 1,2$ and the following condition is satisfied:

$$k_j(x)x \geq 0, \qquad j = 1,2. \tag{1.13}$$

In particular, the Kalman-Yakubovich-Popov Lemma implies that there exist functions $V_j \in C^2(\mathbb{R}^n,\mathbb{R})$, such that:

$$V_j(\zeta_j) \;\geq\; 0, \quad \forall \zeta_j \in \mathbb{R}^n,$$

$$\begin{aligned} V_j(0) &= 0, \\ \lim_{\|\zeta_j\|\to\infty} V_j(\zeta_j) &= \infty, \end{aligned} \tag{1.14}$$

and that the coefficient functions satisfy:

$$\begin{aligned} \nabla V_j(\zeta_j)\cdot a_j(\zeta_j) &< 0, \qquad \zeta_j \neq 0, \\ \nabla V_j(\zeta_j)\cdot b_j(\zeta_j) &= c_j(\zeta_j), \\ d_j(\zeta_j) &> 0, \end{aligned} \tag{1.15}$$

for all $\zeta_j \in \mathbb{R}^n$, $j = 1, 2$. The demonstration of the decay of the energy of the system will serve as the justification for a control law given by (1.12). With this purpose in mind, an energy functional for the controller is given by:

$$E^{\mathrm{NL}}_{\mathrm{control}} := V_1(\zeta_1) + \int_0^{u_x(L)} k_1(\sigma)\,\mathrm{d}\sigma + V_2(\zeta_2) + \int_0^{u(L)} k_2(\sigma)\,\mathrm{d}\sigma,$$

which, due to (1.13) and (1.14), is always non-negative. Then it follows:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} E^{\mathrm{NL}}_{\mathrm{control}} &= \nabla V_1(\zeta_1)(\zeta_1)_t + k_1(u_x(L))u_{xt}(L) + \nabla V_2(\zeta_2)(\zeta_2)_t + k_2(u(L))u_t(L) \\ &= \nabla V_1(\zeta_1)[a_1(\zeta_1(t)) + b_1(\zeta_1(t))u_{xt}(t,L)] + k_1(u_x(L))u_{xt}(L) \\ &\quad +\nabla V_2(\zeta_2)[a_2(\zeta_2(t)) + b_2(\zeta_2(t))u_t(t,L)] + k_2(u(L))u_t(L) \\ &\leq \Theta_1 u_{xt}(L) - d_1(\zeta_1)u_{xt}(L)^2 + \Theta_2 u_t(L) - d_2(\zeta_2)u_t(L)^2. \end{aligned}$$

where (1.12) and (1.15) were used. Therefore, the functional

$$E^{\mathrm{NL}}_{\mathrm{total}} := E_{\mathrm{beam}} + E^{\mathrm{NL}}_{\mathrm{control}},$$

is a good candidate for the Lyapunov functional of the system (1.1)–(1.5) and (1.12), since

$$\frac{\mathrm{d}}{\mathrm{dt}} E^{\mathrm{NL}}_{\mathrm{total}} < -d_1(\zeta_1)u_{xt}(L)^2 - d_2(\zeta_2)u_t(L)^2 \leq 0. \tag{1.16}$$

It is a common strategy to formulate the Euler-Bernoulli beam with high order nonlinear boundary conditions as a nonlinear evolution equation in an appropriate (infinite-dimensional) Hilbert space. In general, showing that every mild solution tends to zero as time goes to infinity consists of two steps, namely showing the precompactness of the trajectories and proving that the only possible limit is the zero solution. In the linear case verifying the precompactness is straightforward by showing that the resolvent of the system operator is compact [47]. For the nonlinear case, the inspection of the precompactness property is more complex. The most commonly used criteria for the precompactness of trajectories can be found in [23, 55, 54, 70], and further generalizations in [20, 66]. There the authors split the system operator into the sum of two operators $A+\mathcal{N}$ (where A is its linear, and $\mathcal{N}$ its nonlinear part) and infer precompactness under the following conditions. In [23] A is required to be m-dissipative and $\mathcal{N}$ applied to a trajectory is L^1 in time. In [54] the requirement on $\mathcal{N}$ is loosened by assuming uniform local integrability of $\mathcal{N}$ applied to a trajectory, however the linear semigroup e^{tA} needs additionally to be compact in order to still ensure precompactness. Finally, in [70] operator $\mathcal{N}$ needs to map into a compact set, and A needs to generate an exponentially stable linear C_0-semigroup. These strategies have successfully been applied in the literature to the Euler-Bernoulli beam without tip payload and with nonlinear boundary control: in [18] the precompactness of the trajectories follows directly from the m-dissipativity of the system operator, and in [13] from the L^1-integrability of the nonlinearity.

In contrast to the mentioned literature, the nonlinear boundary control considered in this thesis does not fall into any of these sets of assumptions. In this thesis, A shall be m-dissipative, but not compact and it does not generate an exponentially stable semigroup. On the other hand, the operator $\mathcal{N}$ does not necessarily satisfy the strong assumptions either, for it is compact, but L^1-integrability can not be guaranteed. Thus the properties of the system operator considered here are too weak in order to apply the mentioned standard results. However, in this thesis the precompactness of the trajectories is demonstrated in a novel way, thus extending the available methods.

1.3 Coupling to nonlinear spring-damper system

In order to tackle the challenges arising from stability analysis of the EBB with nonlinear boundary terms (as introduced in Section 1.2.2) first a toy model is analyzed. An Euler-Bernoulli beam is considered, which is clamped at one end, and at the tip of the beam there is a payload of mass $M > 0$, which has the moment of inertia $J > 0$ (see Figure 1.3). Moreover, the beam has mass density $\mu > 0$ and length L. The beam is parametrized with $x \in [0, L]$, and is described by its deviation $u(t, x)$ from the horizontal (as depicted in Figure 1.3). The constant flexural rigidity is $\Lambda > 0$, and the tension is assumed to be zero. It is assumed that only two forces act upon the beam. First, the tip is assumed to be attached to a non-linear spring, producing the restoring force $-s(u(t, L))$. Second, there is a nonlinear damping force, given by $-d(u_t(t, L))$. Furthermore, it is assumed that $s \in C^2(\mathbb{R})$, $d \in C^1(\mathbb{R})$, and

$$\int_0^z s(w)\,\mathrm{d}w \geq 0, \quad \forall z \in \mathbb{R}, \tag{1.17}$$

$$d'(z) \geq 0, \ d(0) = 0, \quad \forall z \in \mathbb{R}. \tag{1.18}$$

Additionally, the following is assumed:

$$|d(z)| \geq Dz^2, \quad \forall z \in \mathcal{U}, \tag{1.19}$$

for some positive constant $D > 0$, on a small neighborhood $\mathcal{U} := [-\delta, \delta]$ around zero. Note that (1.17) implies $k_1(0) = 0$. For the derivation of the model, the approach in [26] and [40]

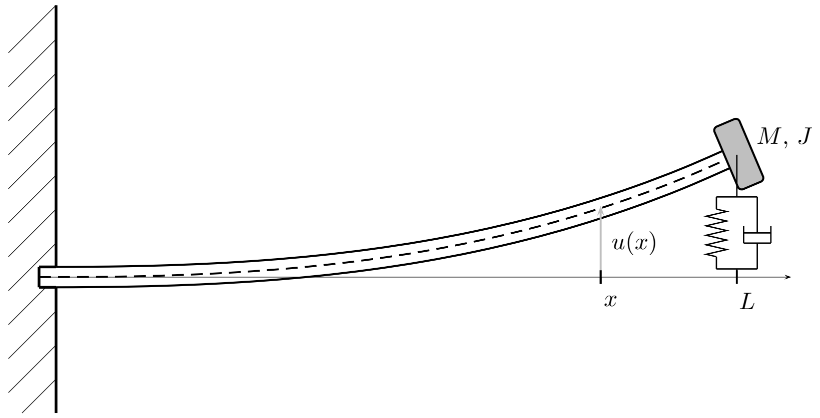


Figure 1.3: At the end $x = L$, beam is attached to a nonlinear damper and a spring

is followed, whereby it is assumed that the beam satisfies the Euler-Bernoulli assumption. The equations of motion can be derived according to Hamilton's principle, i.e. they are the Euler-Lagrange equations corresponding to the action functional. In the present model the kinetic energy E_k and the potential (strain) energy E_p are

$$E_k = \frac{\mu}{2}\int_0^L u_t(x)^2\,\mathrm{d}x + \frac{M}{2}u_t(L)^2 + \frac{J}{2}u_{tx}(L)^2, \qquad E_p = \frac{\Lambda}{2}\int_0^L u_{xx}(x)^2\,\mathrm{d}x.$$

Additionally, the virtual work δW of the external forces reads:

$$\delta W = -s(u(L))\delta u(L) - d(u_t(L))\delta u(L).$$

Taking into account the boundary conditions $u(0) = u_x(0) = 0$ of the clamped end, the Hamilton's principle implies that u solves the following system:

$$\mu u_{tt}(t,x) + \Lambda u_{xxxx}(t,x) = 0, \quad 0 < x < L,\ t > 0, \tag{1.20a}$$

$$u(t,0) = u_x(t,0) = 0, \quad t > 0, \tag{1.20b}$$

$$-\Lambda u_{xxx}(t,L) + Mu_{tt}(t,L) + s(u(t,L)) + d(u_t(t,L)) = 0, \quad t > 0, \tag{1.20c}$$

$$\Lambda u_{xx}(t,L) + Ju_{ttx}(t,L) = 0, \quad t > 0. \tag{1.20d}$$

Due to the damping, it is expected that the total energy of the beam will decrease in time. The total energy of the system is given by

$$E_{\text{total}} = E_k + E_p + E_s, \tag{1.21}$$

where $E_s := \int_0^{u(L)} s(w)\,\mathrm{d}w$ represents the potential energy stored in the nonlinear spring. Now (1.17) ensures that this integral always stays non-negative. The time derivative of the total energy is computed using the Euler-Lagrange equations (1.20):

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E_{\text{total}} &= \Lambda \int_0^L u_{xx}u_{txx}\,\mathrm{d}x + \mu \int_0^L u_t u_{tt}\,\mathrm{d}x + Mu_t(L)u_{tt}(L) + Ju_{tx}(L)u_{ttx}(L) \\
&\quad + s(u(L))u_t(L) \\
&= \Lambda \int_0^L u_{xxxx}u_t\,\mathrm{d}x + \Lambda u_{xx}u_{tx}\big|_0^L - \Lambda u_{xxx}u_t\big|_0^L + \mu \int_0^L u_t u_{tt}\,\mathrm{d}x \\
&\quad + Mu_t(L)u_{tt}(L) + Ju_{tx}(L)u_{ttx}(L) + s(u(L))u_t(L) \\
&= \Lambda u_{xx}(L)u_{tx}(L) - \Lambda u_{xxx}(L)u_t(L) + Mu_t(L)u_{tt}(L) + Ju_{tx}(L)u_{ttx}(L) \\
&\quad + s(u(L))u_t(L) \\
&= -\Lambda u_{xxx}(L)u_t(L) + Mu_t(L)u_{tt}(L) + s(u(L))u_t(L) \\
&= -d(u_t(L))u_t(L) \leq 0. \qquad (1.22)
\end{aligned}$$

The decay of the total energy of the system makes it a good candidate for a Lyapunov function, and it will be used to show the stability of the system in Chapter 3. Furthermore, it will be shown that the trajectories of the classical solutions are precompact and that for almost all moments of inertia $J > 0$ the trajectories tend to zero as time goes to infinity. Interestingly it is found that, for countably many values of the parameter J, the trajectories tend to a time-periodic solution. For given initial conditions it is possible to characterize this asymptotic limit explicitly, including its phase. Let it be stated here, that precompactness of the trajectories does not follow from any standard criteria found it the literature. Instead, the novel method, introduced in this thesis, is used as for the EBB with nonlinear dynamic boundary control described in Subsection 1.2.2.

A possible application of the method developed here is the nonlinear extension of the linear theory in [7], describing a model for a flexible micro-gripper used for DNA manipulation (the DNA-bundle model consists of a damper, spring and a load). Studying the stability of the system, when nonlinear phenomena for the controller and DNA-bundle are included, is a goal for future research set in [7]. The analysis and the results on the asymptotic behavior obtained in Chapter 3 of this thesis can be considered as a step in this direction.

1.4 Numerical method for EBB with tip body

In general, the solution of the EBB coupled to a control system or some mechanical system at the boundary can not be obtained explicitly, and hence it is important to develop an efficient numerical method for these systems. Such a method proves to be necessary, since the available simulation tools are often not apt for simulating complex dynamical boundary control problems. The EBB systems described in Subsections 1.2.1, 1.2.2, and 1.3 are dissipative systems, as seen in (1.11), (1.16), and (1.22), respectively. The goal of the second part of this thesis is to design the numerical method in such a way that the discretized systems are dissipative as well. In the rest of this subsection, several numerical strategies for the EBB from the literature are briefly reviewed and compared against the numerical methods introduced in this thesis.

1.4.1 Linear boundary conditions

In [68] the authors propose a conditionally stable, central difference method for both space and time discretization of the EBB equation. Their system models a beam, which has a tip mass with moment of inertia on the free end. At the fixed end a boundary control is applied in form of a control torque. Due to higher order boundary conditions, fictitious nodes are needed at both boundaries. In [22] the authors consider a damped, cantilevered EBB, with one end clamped into a moving base (as a boundary control) and a tip mass with moment of inertia placed at the other. For their numerical treatment they considered a finite number of modes, thus obtaining an ODE system. Also [40, 41] are based on a finite dimensional modal approximation of (1.1)–(1.8). In [43] the EBB with one free end (without tip mass, but with boundary torque control) was solved in the frequency domain: After Laplace transformation in time, the resulting ODEs could be solved explicitly. However, this approach has a disadvantage that in addition a numerical method for the inverse Laplace transformation is necessary. The more elaborate approaches are based on FEMs: In [16] the authors present a semi-discrete (using cubic splines) and fully discrete Galerkin scheme (based on the Crank-Nicolson method) for the strongly damped, extensible beam equation with both ends hinged. In [4] the authors consider a EBB with tip mass at the free end, yielding a conservative hyperbolic system. They analyze a cubic B-spline based Galerkin method (including convergence analysis of the spatial semi-discretization) and put special emphasis on the subsequent parameter identification problem. Their extended model in [5] involves a viscoelastic damping (in the equation), hence leading to an abstract parabolic system. All these FEMs are for models *without boundary control*. In this thesis, the coupled hyperbolic system (1.1)-(1.8) will be considered, where the damping only appears due to the boundary control. Hence, the focus of this thesis is on the correct large-time behavior (i.e. dissipativity) in the numerical scheme. To this end a Crank-Nicolson scheme in time is used, which was also the appropriate approach for the decay of discretized parabolic equations [2]. Let it be noted that the modeling and discretization of boundary control systems as port-Hamiltonian systems also has this flavor of preserving the structure: For a general methodology on this spatial semi-discretization (leading to mixed finite elements) and its application to the telegrapher's equations, the reader is referred to [27].

1.4.2 Nonlinear boundary conditions

Concerning the numerical simulations of the Euler-Bernoulli beam with nonlinearities, the contributions in the literature are much fewer. Thereby a common approach is to use the Galerkin method: In [6] two space-time spectral element methods are employed to solve a simply supported, nonlinear, modified EBB subjected to forced lateral vibrations but with no mass attached: There, Hermitian polynomials, both in space and time, lead to strict stability limitations. But a mixed discontinuous Galerkin formulation with Hermitian cubic polynomials in space and Lagrangian spectral polynomials in time yields an unconditionally stable scheme. As the result of the discretization, nonlinear systems of equations are obtained, which are solved using the Picard method. In [72] the authors use spectral Tchebyshev technique for the spatial discretization of Euler-Bernoulli and Timoshenko beams without tip mass. The spatially discretized equations of motion are obtained applying Galerkin's method with Tchebychev polynomials as spatial basis functions. The authors do not propose a method for full discretization in time, hence the obtained equations, which form a system of ODEs, are solved by commercial ODE solvers, in order to demonstrate numerical efficiency and accuracy of the semi-discrete method.

In this thesis, the numerical method for the EBB with linear boundary control is adapted in order to numerically handle nonlinear boundary conditions: FEM approximation in space and Crank-Nicolson in time is utilized. This approach will prove to be unconditionally stable in both linear and nonlinear case. Moreover, it is structure preserving in the sense that the finite difference of the energy functional of the fully-discrete solution is always non-positive and it corresponds to the (also non-positive) time derivative of the energy functional of the solution to the continuous problem. Furthermore, the dissipativity property and stability of the method are independent of the choice of the finite dimensional approximation space.

1.5 Organization and the summary of the thesis

This thesis is organized as follows: In Chapter 2 linear dynamic boundary control for an inhomogeneous EBB will be considered. Section 2.1 is dedicated to discussion of the stability of the closed-loop system. Firstly, the analysis of [40] is completed, proving that despite asymptotic stability, this system is *not* exponentially stable. Toward this analysis the asymptotic behavior of the eigenvalues and eigenfunctions of the coupled system is inspected. Obtained results are an extension of Rao's analysis [58] to dynamic controllers and inhomogeneous beams. Further, the Riesz basis property and spectrum-determined growth condition has been demonstrated. To the knowledge of the author, there exist no such results in the literature for the non-homogeneous beam with tip body and dynamic controller. In Section 2.2 the weak formulation of the closed-loop system is discussed. The techniques of Lions [44] are used to demonstrate the existence and uniqueness for the weak solution to the initial-boundary value problem (1.1)-(1.8). In Section 2.3 an unconditionally stable FEM (along with a Crank-Nicolson scheme in time) is developed, which dissipates an appropriate energy functional independently of the chosen FEM basis. Error estimates (second order in space and time) of the numerical scheme are derived. Chapter 3 considers a problem of a cantilevered Euler-Bernoulli beam attached to a nonlinear spring and a damper, introduced in Section 1.3. In Section 3.1 the system is written as an evolution problem and its well-posedness is analyzed. In Section 3.2 the precompactness of the trajectories is proved for all classical solutions, and the long-term behavior and stability of the system are discussed in Section 3.3. Thereby, possible ω-limit sets are characterized, proving that any regular solution tends either to zero or to a periodic solution, depending on the prescribed value of the moment of inertia J. Section 3.4 is concerned with the weak formulation of the system, and in Section 3.5 a dissipative numerical method is developed. In Chapter 4 an EBB system coupled to nonlinear feedback boundary control is analyzed. Section 4.1 discusses well-posedness and the stability of the system. In Section 4.2 a weak formulation of the problem is introduced, and in Section 4.3 a dissipative numerical method is developed. For all three cases (i.e., coupling the beam to a dynamic linear and nonlinear control, and a nonlinear spring-damper system), it has been shown that the appropriate numerical method, which conserves dissipation of the system, is combining a FEM discretization in space, and the Crank-Nicolson discretization method in time, as presented in Sections 2.3, 4.3, and 3 respectively. Finally, in Chapter 5, the simulation results for the numerical methods are presented, and their implementation in MATLAB is discussed. For easier understanding of the thesis, some results and lengthy proofs are deferred to Appendix A. For completeness, the Appendix B states the most important results from the literature used in this thesis.

Chapter 2

Linear dynamic boundary control

In this chapter, the system (1.1)–(1.5) will be generalized to the case where the mass density $\mu \in C^4[0, L]$ and flexural rigidity of the beam $\Lambda \in C^4[0, L]$ are inhomogeneous:

$$
\begin{aligned}
\mu(x)u_{tt} + (\Lambda(x)u_{xx})_{xx} &= 0, \quad 0 < x < L,\, t > 0, && (2.1)\\
u(t,0) &= 0, \quad t > 0, && (2.2)\\
u_x(t,0) &= 0, \quad t > 0, && (2.3)\\
Ju_{xtt}(t,L) + (\Lambda u_{xx})(t,L) + \Theta_1(t) &= 0, \quad t > 0, && (2.4)\\
Mu_{tt}(t,L) - (\Lambda u_{xx})_x(t,L) + \Theta_2(t) &= 0, \quad t > 0, && (2.5)
\end{aligned}
$$

where, it is assumed $\mu(x), \Lambda(x) > 0$, for all $x \in [0, L]$. For the feedback boundary control the dynamic linear SPR controller is considered, as designed in [40], and described in Subsection 1.2.1:

$$
\begin{aligned}
(\zeta_1)_t(t) &= A_1\zeta_1(t) + b_1 u_{xt}(t,L),\\
(\zeta_2)_t(t) &= A_2\zeta_2(t) + b_2 u_t(t,L),\\
\Theta_1(t) &= k_1 u_x(t,L) + c_1 \cdot \zeta_1(t) + d_1 u_{xt}(t,L),\\
\Theta_2(t) &= k_2 u(t,L) + c_2 \cdot \zeta_2(t) + d_2 u_t(t,L).
\end{aligned} \qquad (2.6)
$$

This chapter is organized as follows. In Section 2.1 the system (2.1)–(2.5), (2.6) is formulated as an evolution problem and studied in semigroup framework. In order to examine if the system is exponentially stable, the spectrum of the system operator is analyzed and it is demonstrated that the generalized eigenvalues of the operator form an Riesz basis in the corresponding state space. Next, in Section 2.2 the weak formulation of the system is defined, and the existence and uniqueness of the weak solution are demonstrated. This formulation is used in Section 2.3 to develop a dissipative numerical method for the system. The results of this chapter were published in [48], with exception of the subsections 2.1.4 and 2.1.5.

2.1 Stability of the closed-loop system

Well-posedness of the closed-loop system (2.1)–(2.6) and asymptotic stability of the zero state were established in [40] for constant μ and Λ using semigroup theory, a carefully designed Lyapunov functional, and LaSalle's invariance principle. In order to perform the stability analysis of the system, the authors formulate the problem as an evolution problem first.

2.1.1 Semigroup formulation

The theory of semigroups is vital for investigating the properties of solutions to partial differential operators. In particular, semigroups generated by the system operator of an abstract Cauchy problem, can be used to completely characterize the well-posedness and the stability of its solution. Hence, the following formulation provides an efficient tool for the discussion on asymptotic and exponential stability. Let $\tilde{H}_0^k(0,L)$ for $k \geq 2$ be defined by:

$$\tilde{H}_0^k(0,L) := \{u \in H^k(0,L)|\ u(0) = u_x(0) = 0\}.$$

The analytical setting for (2.1)–(2.6) in the framework of semigroup theory is revised from [40]. The Hilbert space is defined by:

$$\mathcal{H} := \{z = (u, v, \zeta_1, \zeta_2, \xi, \psi)^\top : u \in \tilde{H}_0^2(0,L), v \in L^2(0,L), \zeta_1, \zeta_2 \in \mathbb{R}^n, \xi, \psi \in \mathbb{R}\},$$

with the inner product

$$\begin{aligned} \langle z, \breve{z} \rangle \ &:= \ \frac{1}{2}\int_0^L \Lambda\, u_{xx}\breve{u}_{xx}\, dx + \frac{1}{2}\int_0^L \mu\, v\breve{v}\, dx + \frac{1}{2J}\xi\breve{\xi} + \frac{1}{2M}\psi\breve{\psi} \\ &+ \ \frac{1}{2}k_1 u_x(L)\breve{u}_x(L) + \frac{1}{2}k_2 u(L)\breve{u}(L) + \frac{1}{2}\zeta_1^\top P_1\breve{\zeta}_1 + \frac{1}{2}\zeta_2^\top P_2\breve{\zeta}_2, \end{aligned}$$

where $\|z\|_{\mathcal{H}}$ denotes the corresponding norm. Let $\mathcal{A}\colon D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ be a linear operator with the domain

$$D(\mathcal{A}) = \{z \in \mathcal{H}\colon u \in \tilde{H}_0^4(0,L), v \in \tilde{H}_0^2(0,L), \xi = Jv_x(L), \psi = Mv(L)\}, \tag{2.7}$$

defined by

$$\mathcal{A}\begin{bmatrix} u \\ v \\ \zeta_1 \\ \zeta_2 \\ \xi \\ \psi \end{bmatrix} = \begin{bmatrix} v \\ -\frac{1}{\mu}(\Lambda u_{xx})_{xx} \\ A_1\zeta_1 + b_1\frac{\xi}{J} \\ A_2\zeta_2 + b_2\frac{\psi}{M} \\ -\Lambda(L)u_{xx}(L) - k_1 u_x(L) - c_1\cdot\zeta_1 - d_1\frac{\xi}{J} \\ (\Lambda u_{xx})_x(L) - k_2 u(L) - c_2\cdot\zeta_2 - d_2\frac{\psi}{M} \end{bmatrix}.$$

Now (2.1)-(2.5), and (2.6) can be written formally as a first order evolution equation:

$$\begin{aligned} z_t &= \mathcal{A}z, \\ z(0) &= z_0 \in \mathcal{H}. \end{aligned} \tag{2.8}$$

Notice that in order to incorporate the higher order boundary conditions (2.4), (2.5) and the boundary terms on the r.h.s. of (2.6), it shows to be essential to introduce $u_t(t, L)$, $u_{xt}(t, L)$ as separate variables, see (2.7). More precisely, $\psi = Mv(L)$ is the vertical momentum, and $J = Jv_x(L)$ the angular momentum of the tip mass, where $v = u_t$ is the velocity of the beam's deflection.

Theorem 2.1. *Operator $\mathcal{A}$ is densely defined (i.e. $D(\mathcal{A})$ is dense in $\mathcal{H}$), and it generates a C_0-semigroup of contractions, denoted by $\{T(t)\}_{t\geq 0}$.*

Proof. The proof that $D(\mathcal{A})$ is a dense subset, and the operator $\mathcal{A}$ is a dissipative, is identical as in [40]. On the other hand, since in [40] the functions μ and Λ are constant, the inverse $\mathcal{A}^{-1}$ can be explicitly determined in order to show that $\mathcal{A}^{-1}$ is compact. However, in the case when the beam is inhomogeneous, the inverse of $\mathcal{A}$ is not explicitly known. Still compactness of $\mathcal{A}^{-1}$ can be shown as in the proof of Lemma 2.23. Now according to Lümer-Phillips Theorem, the statement of the theorem follows. □

Before discussion on well-posedness and stability of (2.8), a definition of a classical solution is given.

Definition 2.2. A function $z\colon [0, \infty) \to \mathcal{H}$ is said to be a classical solution of (2.8) if $z \in C([0, \infty); D(\mathcal{A})) \cap C^1((0, \infty); \mathcal{H})$, and z satisfies the initial conditions and (2.8) on $(0, \infty)$.

The existence and uniqueness result for the classical solution follows immediately from Theorem B.1 in Appendix B:

Theorem 2.3. *For all $z_0 \in D(\mathcal{A})$, there exists a classical solution to* (2.8)*, and it is given by $z(t) = T(t)z_0$.*

Furthermore, a more general solution will be considered, when z_0 is not necessarily in $D(\mathcal{A})$. Then (2.8) is not guaranteed to have a classical solution at all. For this purpose, a notion of mild solution to (2.8) is introduced, which is also called the generalized solution.

Definition 2.4. Let $\mathcal{A}$ be the infinitesimal generator of a C_0-semigroup $T(t)$ on a Banach space X. For $z_0 \in X$, mild solution of (2.8) is defined by $z(t) = T(t)z_0$.

Next result follows directly from Theorem 2.1:

Theorem 2.5. *For any $z_0 \in \mathcal{H}$, (2.8) has a unique mild solution $z \in C([0, \infty); \mathcal{H})$.*

Notice that the contractivity of the semigroup also implies that $\|.\|_{\mathcal{H}}$ is a candidate for the Lyapunov functional for (2.8). More precisely, let the functional $V\colon \mathcal{H} \to \mathbb{R}$ be defined by:

$$\begin{aligned} V(z) &:= \|z\|_{\mathcal{H}} = \frac{1}{2}\int_0^L \Lambda\, u_{xx}^2\, dx + \frac{1}{2}\int_0^L \mu\, v^2\, dx + \frac{\xi^2}{2J} + \frac{\psi^2}{2M} \\ &+ \frac{1}{2}k_1 u_x(L)^2 + \frac{1}{2}k_2 u(L)^2 + \frac{1}{2}\zeta_1^\top P_1 \zeta_1 + \frac{1}{2}\zeta_2^\top P_2 \zeta_2. \end{aligned} \tag{2.9}$$

Analogously as in (1.11), for all classical solutions z it follows that:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}V(z) &= -\frac{\varepsilon_1}{2}\zeta_1^\top P_1\zeta_1 - \delta_1\left(\frac{\xi}{J}\right)^2 - \frac{1}{2}\left(\zeta_1\cdot q_1 + \tilde{\delta}_1\frac{\xi}{J}\right)^2 \\ &-\frac{\varepsilon_2}{2}\zeta_2^\top P_2\zeta_2 - \delta_2\left(\frac{\psi}{M}\right)^2 - \frac{1}{2}\left(\zeta_2\cdot q_2 + \tilde{\delta}_2\frac{\psi}{M}\right)^2 \le 0, \end{aligned} \tag{2.10}$$

hence time evolution of the functional V along the classical solutions is non-increasing. For the mild solutions, due to the lack of regularity, the time derivative is generalized:

Definition 2.6. The generalized time derivative of V along the mild solution $z(t)$ of (2.8) to the initial value $z_0 \in \mathcal{H}$ is defined as:

$$\dot{V}(z_0) := \limsup_{t\searrow 0} \frac{V(z(t)) - V(z_0)}{t},$$

which may take the value $-\infty$.

Definition 2.7. Functional $V\colon \mathcal{H} \to \mathbb{R}$ is a called a Lyapunov functional of the evolution problem (2.8) if the following holds:

i) $V(z) > 0, \quad \forall z \in \mathcal{H}\setminus\{0\}$,

ii) $V(0) = 0$,

iii) $\dot{V}(z_0) \le 0, \quad \forall z_0 \in \mathcal{H}$.

Since $\{T(t)\}_{t\ge 0}$ is a linear semigroup of contractions, the decay of V along the trajectories can easily be extended to mild solutions (see [40]), and hence V is the Lyapunov functional for (2.8). Moreover, the largest invariant subset of

$$\mathcal{M} := \{z \in \mathcal{H}\colon \dot{V}(z) = 0\}$$

contains only zero solution (for the proof when the beam is homogeneous see [40], in the inhomogeneous case see the proof of Theorem 4.17). Now, applying La Salle's invariance principle (stated in Appendix B, Theorem B.2) the central stability result obtained in [40] follows:

Theorem 2.8. *Let $z(t)$ be the mild solution to (2.8), for some $z_0 \in \mathcal{H}$. Then $z(t) \overset{t\to\infty}{\longrightarrow} 0$ in $\mathcal{H}$.*

Therefore, the system (2.1)-(2.5) and (2.6) is asymptotically stable. However, there remains an open question if the system is exponentially stable as well. This question is tackled in the remainder of this section.

2.1.2 Spectral analysis for the operator $\mathcal{A}$

Spectral analysis has often been used in the past century to determine dynamic behavior of vibrating systems. In particular, [29], [31], and [15], are some of the examples in the literature in which stability analysis of a cantilever beam with tip mass (or tip body) and boundary control has been performed solely by means of spectral analysis. In general, stability problems of infinite dimensional systems are much more complicated than those of the finite dimensional systems. Asymptotic stability, exponential stability, as well as the property that all eigenvalues of $\mathcal{A}$ are located on the open left-half complex plane are equivalent in finite dimensions. For infinite dimensional linear systems, however, these equivalences do not hold in general. Two different stability types will be studied here, for which definitions are given in a semigroup framework:

Definition 2.9. A C_0-semigroup $T(t)$ is said to be *asymptotically stable* if for every $z \in \mathcal{H}$,

$$\lim_{t\to\infty} T(t)z = 0.$$

A C_0-semigroup $T(t)$ is said to be *exponentially stable* if there exist constants $M \geq 1$, and $\omega > 0$ such that

$$\|T(t)\| \leq Me^{-\omega t}.$$

As can be seen in Theorem 2.5, asymptotic stability for (2.8) has already been demonstrated in [40]. Furthermore, from the proof of Theorem 2.1 it is known that $\mathcal{A}^{-1}$ is compact. The asymptotic stability and compact resolvent property of the operator $\mathcal{A}$, offer more information about the spectrum of $\mathcal{A}$:

Theorem 2.10. *For all $\lambda \in \sigma(\mathcal{A})$, $Re(\lambda) < 0$.*

Proof. Statement follows directly from Theorem B.3. □

However, contrary to the finite dimensional case, exponential stability for the infinite dimensional systems can not be deduced solely from the fact that the spectra of the system lies in the open left-half complex plane. Additional necessary conditions are needed, and these are considered in the next subsection.

2.1.3 Non-exponential stability

The focus of this subsection will be the study of the exponential stability of system (2.8), which has remained an open question. For this purpose, a commonly used criteria due to Huang [33] is stated.

Definition 2.11. Let $\mathcal{B}$ be a linear operator. The spectral bound of $\mathcal{B}$ is defined by:

$$r(\mathcal{B}) = \sup\{\mathrm{Re}(\lambda)\colon \lambda \in \sigma(\mathcal{B})\},$$

where $r(\mathcal{B})$ may take value ∞.

Theorem 2.12. *Let $S(t)$ be a uniformly bounded C_0-semigroup on a Hilbert space with infinitesimal generator $\mathcal{B}$. Then $S(t)$ is exponentially stable if and only if*

$$r(\mathcal{B}) < 0 \tag{2.11}$$

and

$$\sup_{\lambda\in\mathbb{R}} \|R(i\lambda, \mathcal{B})\| < \infty \tag{2.12}$$

holds.

This method of examining exponential stability of a semigroup, as presented in Theorem 2.12, is also called *frequency domain criteria*. Some of the first articles dealing with the construction and analysis of linear boundary control for an Euler-Bernoulli beam *without* tip body [11, 12, 51] show exponential stability of the system using frequency domain criteria. However, in this thesis this criteria will be utilized to demonstrate the lack of exponential stability. This result does not come as a surprise, since it is already known from the literature that the linear boundary feedback controller composed of lower order derivatives does not exponentially stabilize an Euler-Bernoulli beam with tip body. First such result was shown in [45], for a specifically chosen controller parameters, and a more general result, for arbitrarily chosen parameters, is presented in [58]. The following theorem, which is the main result of this section, can be seen as an extension of work in [58] to inhomogeneous beam and dynamic control.

Theorem 2.13. *The operator $\mathcal{A}$ has eigenvalue pairs λ_n and $\overline{\lambda}_n$, $n \in \mathbb{N}$, with the following asymptotic behavior when $n \to \infty$:*

$$\lambda_n = i\left[\left(\frac{(2n-1)\pi}{2h}\right)^2 + \frac{4hM^{-1}\mu(L)^{\frac{3}{4}}\Lambda(L)^{\frac{1}{4}} - I}{2h^2}\right] + \mathcal{O}(n^{-1}), \tag{2.13}$$

where

$$h := \int_0^L \left(\frac{\mu(w)}{\Lambda(w)}\right)^{\frac{1}{4}} \, \mathrm{d}w, \tag{2.14}$$

and I is a real constant given by (2.39)*. Therefore,*

$$\sup\{\mathrm{Re}(\lambda)\colon \lambda \in \sigma(\mathcal{A})\} = 0,$$

and hence the evolution problem (2.8) is not *exponentially stable.*

Proof. It is already known that the operator $\mathcal{A}$ has a compact resolvent. Thus, its spectrum $\sigma(\mathcal{A})$ consists entirely of isolated eigenvalues, at most countably many, and each eigenvalue has a finite algebraic multiplicity [35]. Since $\mathcal{A}$ also generates an asymptotically stable C_0-semigroup of contractions, it follows (see Theorem B.3 in Appendix B):

$$\mathrm{Re}\lambda < 0, \quad \forall \lambda \in \sigma(\mathcal{A}).$$

The matrices A_1 and A_2 are Hurwitz matrices and therefore only have eigenvalues with negative real parts. The set $\sigma(\mathcal{A}) \cap (\sigma(A_1) \cup \sigma(A_2)) \subset \mathbb{C}$ is therefore empty or finite. Hence, it suffices to consider only such eigenvalues λ of the operator $\mathcal{A}$ that are not eigenvalues of A_1 or A_2. Now $z = (u, v, \zeta_1, \zeta_2, \xi, \psi)^\top \in D(\mathcal{A})$ is a corresponding eigenvector if and only if:

$$\begin{aligned} v &= \lambda u, \\ \zeta_1 &= \lambda u_x(L) \left(\lambda I - A_1\right)^{-1} b_1, \\ \zeta_2 &= \lambda u(L) \left(\lambda I - A_2\right)^{-1} b_2, \end{aligned}$$

and u satisfies the following boundary value problem:

$$\begin{aligned} (\Lambda u_{xx})_{xx} + \mu\lambda^2 u &= 0, && (2.15) \\ u(0) &= 0, && (2.16) \\ u_x(0) &= 0, && (2.17) \\ \Lambda(L)u_{xx}(L) + (k_1 + \lambda \left(\lambda I - A_1\right)^{-1} b_1 \cdot c_1 + \lambda d_1 + \lambda^2 J)u_x(L) &= 0, && (2.18) \\ -\left(\Lambda u_{xx}\right)_x(L) + (k_2 + \lambda \left(\lambda I - A_2\right)^{-1} b_2 \cdot c_2 + \lambda d_2 + \lambda^2 M)u(L) &= 0. && (2.19) \end{aligned}$$

In order to solve (2.15)–(2.19), spatial transformations as introduced in [30] are performed, which convert (2.15) into a more convenient form. For this reason, (2.15) is firstly rewritten as:

$$u_{xxxx} + \frac{2\Lambda_x}{\Lambda} u_{xxx} + \frac{\Lambda_{xx}}{\Lambda} u_{xx} + \frac{\mu}{\Lambda}\lambda^2 u = 0. \tag{2.20}$$

In order to transform the coefficient function appearing with u in (2.20) into a constant, a space transformation is introduced. Let $u(x) = \breve{u}(y)$, where

$$y = y(x) := \frac{1}{h}\int_0^x \left(\frac{\mu(w)}{\Lambda(w)}\right)^{\frac{1}{4}} \mathrm{d}w, \tag{2.21}$$

with h defined as in (2.14). From (2.16)–(2.20) it follows that $\breve{u}$ satisfies:

$$\begin{aligned} \breve{u}_{yyyy} + \alpha_3 \breve{u}_{yyy} + \alpha_2 \breve{u}_{yy} + \alpha_1 \breve{u}_y + h^4\lambda^2 \breve{u} &= 0, \\ \breve{u}(0) &= 0, \\ \breve{u}_y(0) &= 0, \\ \breve{u}_{yy}(1) + \breve{u}_y(1)\left(\beta_0 + \kappa_1(\lambda)\right) &= 0, \\ -\breve{u}_{yyy}(1) + \beta_1 \breve{u}_{yy}(1) + \beta_2 \breve{u}_y(1) + \kappa_2(\lambda)\breve{u}(1) &= 0, \end{aligned} \tag{2.22}$$

with

$$\alpha_3(y) = h\left(\frac{\mu(x)}{\Lambda(x)}\right)^{-\frac{1}{4}}\left(\frac{3}{2}\frac{\mu_x(x)}{\mu(x)}+\frac{1}{2}\frac{\Lambda_x(x)}{\Lambda(x)}\right), \tag{2.23}$$

$$\begin{aligned}\alpha_2(y) &= \frac{1}{h^2}\left\{-\frac{9}{16}\left(\frac{\mu(x)}{\Lambda(x)}\right)^{-\frac{3}{2}}\left[\left(\frac{\mu(x)}{\Lambda(x)}\right)_x\right]^2+\left(\frac{\mu(x)}{\Lambda(x)}\right)^{-\frac{1}{2}}\left(\frac{\mu(x)}{\Lambda(x)}\right)_{xx}\right.\\ &\quad\left.+\frac{3}{2}\frac{\Lambda_x(x)}{\Lambda(x)}\left(\frac{\mu(x)}{\Lambda(x)}\right)^{-\frac{1}{2}}\left(\frac{\mu(x)}{\Lambda(x)}\right)_x+\frac{\Lambda_{xx}(x)}{\Lambda(x)}\left(\frac{\mu(x)}{\Lambda(x)}\right)^{\frac{1}{2}}\right\}, \end{aligned}\tag{2.24}$$

and α_1 being a smooth function of h, $\frac{\mathrm{d}^k\Lambda}{\mathrm{d}x^k}$, and $\frac{\mathrm{d}^k\mu}{\mathrm{d}x^k}$ for $k=0,1,2,3$. The coefficients β_0,β_1,β_2 are constants, depending on h, $\frac{\mathrm{d}^k\Lambda}{\mathrm{d}x^k}(L)$, and $\frac{\mathrm{d}^k\mu}{\mathrm{d}x^k}(L)$ for $k=0,1,2$. Furthermore, the following notation has been introduced:

$$\begin{aligned}\kappa_1(\lambda) &:= \frac{h}{\Lambda(L)}\left(\frac{\mu(L)}{\Lambda(L)}\right)^{-\frac{1}{4}}\left(k_1+\lambda\left((\lambda I-A_1)^{-1}b_1\right)\cdot c_1+\lambda d_1+\lambda^2 J\right),\\ \kappa_2(\lambda) &:= \frac{h^3}{\Lambda(L)}\left(\frac{\mu(L)}{\Lambda(L)}\right)^{-\frac{3}{4}}\left(k_2+\lambda\left((\lambda I-A_2)^{-1}b_2\right)\cdot c_2+\lambda d_2+\lambda^2 M\right).\end{aligned}$$

In order to solve (2.22), the strategy as in Chapter 2, Section 4 of [52] is used. Hence, to eliminate the third derivative term $\alpha_3\breve{u}_{yyy}$, a new invertible space transformation is introduced:

$$\breve{u}(y) = e^{-\frac{1}{4}\int_0^y \alpha_3(z)\,\mathrm{d}z}\tilde{u}(y). \tag{2.25}$$

Boundary value problem (2.22) can be written as:

$$\tilde{u}_{yyyy}+\tilde{\alpha}_2\tilde{u}_{yy}+\tilde{\alpha}_1\tilde{u}_y+\tilde{\alpha}_0\tilde{u}+h^4\lambda^2\tilde{u} = 0, \tag{2.26}$$

$$\tilde{u}(0) = 0, \tag{2.27}$$

$$\tilde{u}_y(0) = 0, \tag{2.28}$$

$$\tilde{u}_{yy}(1)+\tilde{u}_y(1)\left(\beta_3+\kappa_1(\lambda)\right)+\tilde{u}(1)\left(\beta_4-\frac{1}{4}\alpha_3(1)\kappa_1(\lambda)\right) = 0, \tag{2.29}$$

$$-\tilde{u}_{yyy}(1)+\beta_5\tilde{u}_{yy}(1)+\beta_6\tilde{u}_y(1)+\left(\beta_7+\kappa_2(\lambda)\right)\tilde{u}(1) = 0, \tag{2.30}$$

where

$$\tilde{\alpha}_2(y) = \alpha_2(y)-\frac{3}{8}\alpha_3(y)^2-\frac{3}{2}(\alpha_3)_y(y), \tag{2.31}$$

and $\tilde{\alpha}_1$, $\tilde{\alpha}_0$ are smooth functions of h, $\frac{\mathrm{d}^k\Lambda}{\mathrm{d}x^k}$, and $\frac{\mathrm{d}^k\mu}{\mathrm{d}x^k}$ for $k=0,\dots,4$. The constant coefficients $\beta_3,\dots,\beta_7$ depend on h, $\frac{\mathrm{d}^k\Lambda}{\mathrm{d}x^k}(L)$, and $\frac{\mathrm{d}^k\mu}{\mathrm{d}x^k}(L)$ for $k=0,\dots,3$. Due to the invertibility of the above transformations, the obtained problem (2.26)–(2.30) is equivalent to the original problem (2.15)–(2.19).

Since the eigenvalues of $\mathcal{A}$ come in complex conjugated pairs, and have negative real parts, it suffices to consider only those λ in the upper-left quarter-plane, i.e. such that $\arg\lambda \in (\frac{\pi}{2}, \pi]$. Note that $\tau \in \mathbb{C}$ is uniquely determined with $\mathrm{Re}(\tau) \geq 0$, and $\lambda = i\frac{\tau^2}{h^2}$. It can be seen that $\arg\tau \in (0, \frac{\pi}{4}]$. Now, the solution to (2.26) can be approximated by the solution to the differential equation with the dominant terms only, i.e. $\tilde{u}_{xxxx} + \lambda^2 h^4 \tilde{u} = 0$. More precisely, it holds (by adaptation of *Satz 1*, pp. 42 of [52]; and the last result of Lemma 2.14 is stated in the proof of *Satz 1*):

Lemma 2.14. *For $\tau \in (0, \frac{\pi}{4}]$, and $|\tau|$ large enough, there exist linearly independent solutions $\{\gamma_j\}_{j=1}^4$, to (2.26), such that:*

$$\begin{aligned} \gamma_j(y) &= e^{\omega_j \tau y}\left(1 + f_j(y)\right), \\ \frac{\mathrm{d}^k}{\mathrm{d}y^k}\gamma_j(y) &= (\omega_j\tau)^k e^{\omega_j\tau y}\left(1 + f_j(y) + \mathcal{O}(|\tau|^{-2})\right), \quad k \in \{1,2,3\}, \end{aligned} \tag{2.32}$$

where $\omega_1 = 1$, $\omega_2 = i$, $\omega_3 = -1$, $\omega_4 = -i$, and

$$f_j(y) = -\frac{\int_0^y \tilde{\alpha}_2(w)\,\mathrm{d}w}{4\,\omega_j\tau} + \mathcal{O}(|\tau|^{-2}), \ \ as\ |\tau| \to \infty,\ j = 1, \dots, 4.$$

Furthermore, the functions $\frac{\mathrm{d}^k}{\mathrm{d}y^k}\gamma_j$ are analytically dependent on τ, for $|\tau|$ large enough, $j = 1, \dots, 4$ and $k = 0, \dots, 3$.

Now, due to Lemma 2.14, the solution to (2.26)–(2.30) can be written as:

$$\tilde{u}(y) = C_1\gamma_1(y) + C_2\gamma_2(y) + C_3\gamma_3(y) + C_4\gamma_4(y),$$

where the constants $\{C_j\}_{j=1}^4$ are determined by the boundary conditions (2.27) – (2.30), and therefore satisfy the following linear system:

$$\begin{aligned} 0 &= C_1\gamma_1(0) + C_2\gamma_2(0) + C_3\gamma_3(0) + C_4\gamma_4(0), \\ 0 &= C_1(\gamma_1)_y(0) + C_2(\gamma_2)_y(0) + C_3(\gamma_3)_y(0) + C_4(\gamma_4)_y(0), \\ 0 &= \sum_{i=1}^4 C_i m_{3\,i}, \\ 0 &= \sum_{i=1}^4 C_i m_{4\,i}, \end{aligned} \tag{2.33}$$

where

$$m_{3\,i} := (\gamma_i)_{yy}(1) + (\beta_3 + \kappa_1(\lambda))(\gamma_i)_y(1) + (\beta_4 - \frac{1}{4}\alpha_3(1)\kappa_1(\lambda))\gamma_i(1),$$

$$m_{4i} := -(\gamma_i)_{yyy}(1) + \beta_5(\gamma_i)_{yy}(1) + \beta_6(\gamma_i)_y(1) + (\beta_7 + \kappa_2(\lambda))\gamma_i(1).$$

From (2.32) easily follows:

$$\begin{aligned}
&\gamma_j(0) = 1 + f_j(0), \quad (\gamma_j)_y(0) = \omega_j\tau(1 + f_j(0) + \mathcal{O}(|\tau|^{-2})), \quad j = 1, \dots, 4,\\
&m_{31} = e^{\tau}\left((l_1\tau^5 + l_2\tau^4)(1 + f_1(1)) + \mathcal{O}(|\tau|^3)\right),\\
&m_{41} = e^{\tau}\left((l_3\tau^4 - \tau^3)(1 + f_1(1)) + \mathcal{O}(|\tau|^3)\right),\\
&m_{32} = e^{i\tau}\left((il_1\tau^5 + l_2\tau^4)(1 + f_2(1)) + \mathcal{O}(|\tau|^3)\right),\\
&m_{42} = e^{i\tau}\left((l_3\tau^4 + i\tau^3)(1 + f_2(1)) + \mathcal{O}(|\tau|^2)\right),\\
&m_{33} = e^{-\tau}\left((-l_1\tau^5 + l_2\tau^4)(1 + f_3(1)) + \mathcal{O}(|\tau|^3)\right),\\
&m_{43} = e^{-\tau}\left((l_3\tau^4 + \tau^3)(1 + f_3(1)) + \mathcal{O}(|\tau|^2)\right)),\\
&m_{34} = e^{-i\tau}\left((-il_1\tau^5 + l_2\tau^4)(1 + f_4(1)) + \mathcal{O}(|\tau|^3)\right),\\
&m_{44} = e^{-i\tau}\left((l_3\tau^4 - i\tau^3)(1 + f_4(1)) + \mathcal{O}(|\tau|^2)\right),
\end{aligned} \tag{2.34}$$

with

$$l_1 := -\frac{J}{h^3\Lambda(L)}\left(\frac{\mu(L)}{\Lambda(L)}\right)^{-\frac{1}{4}}, \; l_2 := \frac{J\alpha_3(1)}{4h^3\Lambda(L)}\left(\frac{\mu(L)}{\Lambda(L)}\right)^{-\frac{1}{4}}, \; l_3 := -\frac{M}{h\Lambda(L)}\left(\frac{\mu(L)}{\Lambda(L)}\right)^{-\frac{3}{4}}.$$

For $\tilde{u}$ to be nontrivial, the determinant of the system (2.33) has to vanish:

$$\begin{vmatrix}
\gamma_1(0) & \gamma_2(0) & \gamma_3(0) & \gamma_4(0)\\
(\gamma_1)_y(0) & (\gamma_2)_y(0) & (\gamma_3)_y(0) & (\gamma_4)_y(0)\\
m_{31} & m_{32} & m_{33} & m_{34}\\
m_{41} & m_{42} & m_{43} & m_{44}
\end{vmatrix} = 0 \tag{2.35}$$

Next (2.35) shall be written in an asymptotic form when $\mathrm{Re}(\tau)$ is large:

$$B_1(m_{31}m_{44} - m_{41}m_{34}) + B_2(m_{31}m_{42} - m_{41}m_{32}) + \mathcal{O}(|\tau|^{10}) = 0, \tag{2.36}$$

where

$$\begin{aligned}
B_1 :=& -(1+i)\left[1 + f_2(1) + f_3(1)\right] + \mathcal{O}(|\tau|^{-2}),\\
B_2 :=& (1-i)\left[1 + f_3(1) + f_4(1)\right] + \mathcal{O}(|\tau|^{-2}).
\end{aligned} \tag{2.37}$$

Noting only the terms with leading powers of τ in (2.36), and after division by $e^{\tau}\tau^{10}$, it is obtained

$$\cos\tau - \tau^{-1}\left((\frac{I}{4} + \frac{1}{l_3})(\cos\tau + \sin\tau)\right) + \mathcal{O}(|\tau|^{-2}) \;=\; 0, \tag{2.38}$$

where

$$I := \int_0^1 \tilde{\alpha}_2(w)\;\mathrm{d}w. \tag{2.39}$$

Let $k = n - \frac{1}{2}$ for $n \in \mathbb{N}$ be sufficiently large and the equation (2.38) for τ in a neighborhood of $k\pi$ be considered. Rouché's Theorem (see [37], e.g.) is applied to the equation (2.38), written as

$$\cos\tau + f(\tau) = 0, \tag{2.40}$$

where $f(\tau) = \mathcal{O}(|\tau|^{-1})$. Consider $\cos\tau$ on a simple closed contour $K \subset \{(n-1)\pi \leq \mathrm{Re}(\tau) \leq n\pi\}$ "around" $\tau = k\pi$ such that $|\cos\tau| \geq 1$ on K. For n large enough, the holomorphic function f satisfies $|f(z)| < 1 \leq |\cos\tau|$ on K. Since $\tau = k\pi$ is the only zero of $\cos\tau$ inside K, Rouché's Theorem implies that (2.40) has also exactly one solution inside K:

$$\tau_n = k\pi + h_n. \tag{2.41}$$

Then, $\cos\tau_n = (-1)^n \sin h_n$. Furthermore, (2.40) implies $h_n = \mathcal{O}(n^{-1})$. To make the asymptotic behavior of h_n more precise, note that

$$\begin{aligned} \sin\tau_n &= -(-1)^n \cos h_n = -(-1)^n + \mathcal{O}(n^{-2}), \\ \cos\tau_n &= (-1)^n\, h_n + \mathcal{O}(n^{-3}). \end{aligned}$$

Using this in (2.38), it follows

$$h_n + \tau^{-1}(\frac{1}{l_3} + \frac{I}{4}) + \mathcal{O}(n^{-2}) = 0.$$

Finally, this yields

$$h_n = \frac{4hM^{-1}\mu(L)^{\frac{3}{4}}\Lambda(L)^{\frac{1}{4}} - I}{4k\pi} + \mathcal{O}(n^{-2}),$$

and (2.41) implies

$$\lambda_n = i\left(\frac{\tau_n}{h}\right)^2 = i\left[\left(\frac{k\pi}{h}\right)^2 + \frac{4hM^{-1}\mu(L)^{\frac{3}{4}}\Lambda(L)^{\frac{1}{4}} - I}{2h^2}\right] + \mathcal{O}(n^{-1}). \tag{2.42}$$

Hence, condition (2.11) fails and $T(t)$ is *not* exponentially stable. □

In Figure 2.1 the eigenvalue pairs corresponding to the first simulation example from Section 5.1 (depicted in Figures 5.1 and 5.2) are shown. They were obtained by application of Newton's method to the equation (2.35).

Remark 2.15. Let us compare this result to a similar system studied in [50] and Section 5.3 of [47], which also consists of an EBB coupled to a passivity based dynamic boundary control, but without the tip mass. Then, that system is exponentially stable.

Remark 2.16. Note that the dominant term of the system eigenvalues (2.13) for large n depends only on geometrical and physical properties of the beam and the tip body. Therefore, the asymptotic behavior of the eigenvalues is independent of the choice of the dynamic linear controller.

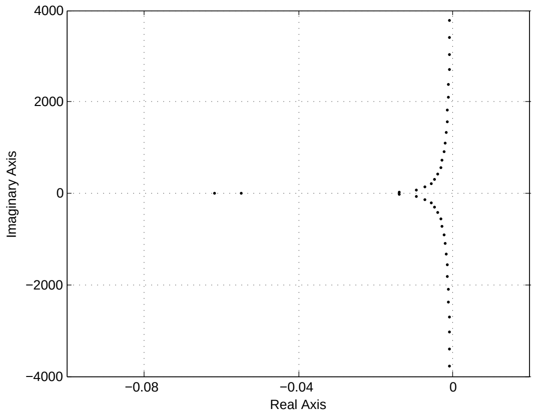


Figure 2.1: The eigenvalues λ_n of the system approach the imaginary axis as $n \to \infty$.

2.1.4 Riesz Basis Property

The Riesz basis property is an elegant way to obtain stability results and it is ever more employed in the literature [14, 17, 29, 31]. In order to closely inspect this property, a definition for Riesz basis is revised.

Definition 2.17. A sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ is called a Riesz basis for $\mathcal{H}$ if there exists an orthonormal basis $\{\Phi_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ and a linear bounded invertible operator T such that

$$T(\varphi_n) = \Phi_n, \qquad \forall n \in \mathbb{N}. \tag{2.43}$$

Definition 2.18. Let $\mathcal{B} : D(\mathcal{B}) \subset \mathcal{H} \to \mathcal{H}$ be a closed linear operator. Then $z \in \mathcal{H}$ is said to be a *generalized eigenfunction* corresponding to an eigenvalue $\lambda \in \sigma(\mathcal{B})$ with finite algebraic multiplicity, if

$$(\lambda I - \mathcal{B})^n z = 0,$$

for some $n \in \mathbb{N}$. Furthermore, it is said that $\mathcal{B}$ satisfies *Riesz basis property* if the generalized eigenfunctions of $\mathcal{B}$ form a Riesz basis for $\mathcal{H}$.

When the operator of the evolution equation satisfies the Riesz basis property, it permits one to deduce many important features of the system. Examples are the optimal decay rate, as well as spectrum-determined growth condition, that has both theoretical and practical significance, which are stated next.

Definition 2.19. Let the linear operator $\mathcal{B}$ be the infinitesimal generator of a C_0-semigroup $S(t)$. The growth rate of $S(t)$ is defined as:

$$\omega_0(\mathcal{B}) := \inf_{t>0} \frac{\ln \|S(t)\|}{t}.$$

From the definition it follows that there exists a constant $M > 0$ such that $\|S(t)\| \leq Me^{(\omega_0(\mathcal{B})+\varepsilon)t}$, for any $\varepsilon > 0$. Furthermore, if $\omega_0(\mathcal{B}) = r(\mathcal{B})$, it is said that $S(t)$ satisfies the *spectrum-determined growth (SDG) condition*.

It follows easily that $\omega_0(\mathcal{B}) \geq r(\mathcal{B})$, but the equality does not hold in general. Consequently, if the SDG condition holds, the exponential stability of $S(t)$ is equivalent to the condition that $r(\mathcal{B}) < 0$. Thus the SDG condition gives a practical criterion when the exponential stability of $S(t)$ is completely determined by the spectrum of $\mathcal{B}$. Such method for studying the exponential stability is also called *spectral analysis method*. The most frequent approach in the literature for showing that the SDG condition holds, is verifying that the system satisfies the Riesz basis property. A system that satisfies the Riesz basis property, is usually referred to as Riesz spectral system (see [71]).

Note that the condition (2.43) from the Definition 2.17 is equivalent to the generalized eigenfunctions of the system $\{\varphi_n\}_{n=1}^{\infty}$ being approximately normalized, i.e. there exist $o_1, o_2 > 0$ such that for all sufficiently large n the following holds:

$$o_1 \leq \|\varphi_n\|_{\mathcal{H}} \leq o_2. \tag{2.44}$$

However, the Riesz basis property is often not straightforward to verify for infinite dimensional systems, not even for flexible beam systems which have already been greatly studied in the literature. The main difficulty for such a verification is usually the non self-adjointness of the system operator. However, recently a new approach has been introduced in [32] for studying the Riesz basis property of a system. An advantage of the aforementioned method is that only the asymptotic behavior of the eigenfunctions needs to be considered. This turns out to be a very helpful result, since in the case of a beam with variable coefficients, it is not possible to obtain an explicit expression for the solution of the characteristic equation nor the system eigenfunctions. The method is presented in the following lemma, which is a corollary of the Bari Theorem (stated in the [32], Theorem 1, pp. 243).

Lemma 2.20. *Let $\mathcal{B}$ be a densely defined operator in a Hilbert space $\mathcal{H}$ with compact resolvent. Let $\{w_n\}_{n=1}^{\infty}$ be a Riesz basis for $\mathcal{H}$. If there exist an $N \geq 0$, and a sequence of generalized eigenvectors $\{z_n\}_{n=N+1}^{\infty}$ of $\mathcal{B}$ such that*

$$\sum_{n=N+1}^{\infty} \|w_n - z_n\|^2 < \infty, \tag{2.45}$$

then:

i) There exist $M > N$ and generalized eigenvectors $\{z_{n0}\}_{n=1}^{M}$ of $\mathcal{B}$ such that $\{z_{n0}\}_{n=1}^{M} \cup \{z_n\}_{n=M+1}^{\infty}$ forms a Riesz basis for $\mathcal{H}$.

ii) Consequently, let $\{z_{n0}\}_{n=1}^{M} \cup \{z_n\}_{n=M+1}^{\infty}$ correspond to eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ of $\mathcal{B}$. Then $\sigma(\mathcal{B}) = \{\lambda_n\}_{n=1}^{\infty}$, where λ_n is counted according to its algebraic multiplicity.

iii) If there exists $M_0 > 0$ such that $\lambda_n \neq \lambda_m$ for all $m, n > M_0$, then there is an $N_0 > M_0$ such that all λ_n, for $n > N_0$, are algebraically simple.

The aim of the rest of this subsection is to apply Lemma 2.20 to the operator $\mathcal{A}$, in order to demonstrate that $\mathcal{A}$ has the Riesz basis property. First, the asymptotic behavior of the eigenfunctions z_n corresponding to eigenvalue λ_n of the operator $\mathcal{A}$ when $n \to \infty$ is studied. Since the system matrix (2.35) has rank 3 for every n large enough, it follows that there exists only one linearly independent solution to (2.26)–(2.30) for $\tau = \tau_n$. Therefore, all eigenvalues λ_n, for n sufficiently large, are geometrically simple. Furthermore, the function $\tilde{u}_n$ has the form (see [52] and Proof of Theorem 2.13):

$$\tilde{u}_n(y) = \begin{vmatrix} \gamma_1(0) & \gamma_2(0) & \gamma_3(0) & \gamma_4(0) \\ (\gamma_1)_y(0) & (\gamma_2)_y(0) & (\gamma_3)_y(0) & (\gamma_4)_y(0) \\ m_{31} & m_{32} & m_{33} & m_{34} \\ \gamma_1(y) & \gamma_2(y) & \gamma_3(y) & \gamma_4(y), \end{vmatrix},$$

up to a multiplicative constant. Using the Laplace expansion, scaling the expression with $-e^{-\tau}\tau^{-8}\frac{h^2}{2l_1}$, and considering only the terms with leading powers of τ, it can be seen that, for n large,

$$\tilde{u}_n(y) = \lambda_n^{-1}\left[e^{-(n-\frac{1}{2})\pi y} - \cos\left((n-\frac{1}{2})\pi y\right) + \sin\left((n-\frac{1}{2})\pi y\right) + (-1)^n e^{(n-\frac{1}{2})\pi(y-1)} + \mathcal{O}(n^{-1})\right], \tag{2.46}$$

$y \in [0,1]$. Therefore, the following result holds:

Theorem 2.21. *The function u_n corresponding to the eigenvalue λ_n (solving (2.15)–(2.19)) has the following asymptotic property as $n \to n$:*

$$\begin{aligned} u_n(x) &= \lambda_n^{-1} e^{\frac{1}{4}\int_0^y \alpha_3(z)\,dz}\left[e^{-(n-\frac{1}{2})\pi y} - \cos\left((n-\frac{1}{2})\pi y\right) + \sin\left((n-\frac{1}{2})\pi y\right)\right. \\ &\quad \left. +(-1)^n e^{(n-\frac{1}{2})\pi(y-1)} + \mathcal{O}(n^{-1})\right], \end{aligned} \tag{2.47}$$

where $0 \le x \le L$, with $y = y(x)$ and α_3 as in (2.21) and (2.23). Hence the eigenfunction corresponding to λ_n has the form

$$z_n = \begin{bmatrix} u_n \\ \lambda_n u_n \\ \lambda_n (u_n)_x(L)\,(\lambda_n I - A_1)^{-1} b_1 \\ \lambda_n u_n(L)\,(\lambda_n I - A_2)^{-1} b_2 \\ J\lambda_n (u_n)_x(L) \\ M\lambda_n u_n(L) \end{bmatrix}. \tag{2.48}$$

Additionally, $\overline{z_n}$ are the eigenfunctions corresponding to conjugated eigenvalues $\overline{\lambda}_n$, $n \in \mathbb{N}$.

Remark 2.22. It is interesting to note that the asymptotic behavior of $\tilde{u}_n(y)$ in (2.46) is the same as of the first coordinate of the eigenfunctions for a homogeneous beam, with no tip mass attached, and only control torque applied at the boundary, as considered in [32].

The next step in showing that $\mathcal{A}$ has the Riesz basis property will be to choose an appropriate reference Riesz basis $\{w_n\}_{n\in\mathbb{N}}$. To this end, the system (2.1)-(2.5) coupled to a simplified control law which does not include damping into the system:

$$\begin{aligned}(\zeta_1)_t(t) &= b_1 u_{xt}(t,L),\\ (\zeta_2)_t(t) &= b_2 u_t(t,L),\\ \tilde{\Theta}_1(t) &= k_1 u_x(t,L) + P_1 b_1 \cdot \zeta_1(t),\\ \tilde{\Theta}_2(t) &= k_2 u(t,L) + P_2 b_2 \cdot \zeta_2(t),\end{aligned} \tag{2.49}$$

is considered. The system is written as an evolution problem

$$z_t = \mathcal{A}_c z, \tag{2.50}$$

where the associated operator is given by

$$\mathcal{A}_c \begin{bmatrix} u \\ v \\ \zeta_1 \\ \zeta_2 \\ \xi \\ \psi \end{bmatrix} = \begin{bmatrix} v \\ -\frac{1}{\mu}(\Lambda u_{xx})_{xx} \\ b_1 \frac{\xi}{J} \\ b_2 \frac{\psi}{M} \\ -\Lambda(L) u_{xx}(L) - k_1 u_x(L) - P_1 b_1 \cdot \zeta_1 \\ (\Lambda u_{xx})_x(L) - k_2 u(L) - P_2 b_2 \cdot \zeta_2 \end{bmatrix},$$

and $D(\mathcal{A}_c) = D(\mathcal{A})$. This system is conservative, since for any $z_0 \in D(\mathcal{A}_c)$ it is easily demonstrated that $\frac{d}{dt}\|z(t)\|^2_{\mathcal{H}} = 0$. Moreover, the following holds:

Lemma 2.23. *Generalized eigenfunctions of $\mathcal{A}_c$ form an orthogonal basis for $\mathcal{H}$. Furthermore, the eigenvalues $\{\nu_n, \overline{\nu}_n\}$ of $\mathcal{A}_c$ have the following asymptotic behavior when $n \to \infty$:*

$$\nu_n = i\left[\left(\frac{(2n-1)\pi}{2h}\right)^2 + \frac{4hM^{-1}\mu(L)^{\frac{3}{4}}\Lambda(L)^{\frac{1}{4}} - I}{2h^2}\right] + \mathcal{O}(n^{-1}), \tag{2.51}$$

where h and I are the same real constants as in (2.14), (2.39). *The eigenfunction corresponding to ν_n has the form:*

$$w_n = \begin{bmatrix} u_n^c \\ \nu_n u_n^c \\ (u_n^c)_x(L)\, b_1 \\ u_n^c(L)\, b_2 \\ J\nu_n (u_n^c)_x(L) \\ M\nu_n u_n^c(L) \end{bmatrix}, \tag{2.52}$$

with

$$u_n^c(x) = \nu_n^{-1} e^{\frac{1}{4}\int_0^y \alpha_3(z)\,dz} \left[e^{-(n-\frac{1}{2})\pi y} - \cos\left((n-\frac{1}{2})\pi y\right) + \sin\left((n-\frac{1}{2})\pi y\right)\right.$$

$$+(-1)^n e^{(n-\frac{1}{2})\pi(y-1)} + \mathcal{O}(n^{-1})\Big]. \tag{2.53}$$

Hence, for the dissipative system (2.8) *and for the conservative system* (2.50) *(i.e. with* $A_{1,2} = 0, d_{1,2} = 0$*), the asymptotic behavior of the eigenvalues and the eigenfunctions is the same.*

Proof. For any $z, \breve{z} \in D(\mathcal{A}_c)$ one can obtain

$$\begin{aligned}
\langle \mathcal{A}_c z, \breve{z} \rangle &= \frac{1}{2}\int_0^L \Lambda(v_{xx}\breve{u}_{xx} - \breve{v}_{xx}u_{xx})\,\mathrm{d}x \\
&+ \frac{1}{2}k_1(-u_x(L)\breve{v}_x(L) + \breve{u}_x(L)v_x(L)) + \frac{1}{2}k_2(-u(L)\breve{v}(L) + \breve{u}(L)v(L)) \\
&+ \frac{1}{2}\left(-\zeta_1^\top P_1 b_1\,\breve{v}_x(L) + \breve{\zeta}_1^\top P_1 b_1\,v_x(L)\right) + \frac{1}{2}\left(-\zeta_2^\top P_2 b_2\,\breve{v}(L) + \breve{\zeta}_2^\top P_2 b_2\,v(L)\right) \\
&= -\langle z, \mathcal{A}_c \breve{z} \rangle,
\end{aligned}$$

hence $\mathcal{A}_c$ is skew-symmetric. Next it is demonstrated that $\mathcal{A}_c$ has a compact resolvent. This result is much more tedious than in the case of an inhomogeneous beam with a non-dynamic controller (cf. [29], [28], [15]), where the inverse of the operator can be obtained in the closed form. To proceed, let $\breve{z} = [f\ g\ \Upsilon_1\ \Upsilon_2\ \Xi\ \Psi]^\top \in \mathcal{H}$ be given, and $z \in D(\mathcal{A}_c)$ is to be determined such that $(\lambda I - \mathcal{A}_c)z = \breve{z}$, for some $\lambda \in \mathbb{C}$. This problem is equivalent to

$$\begin{aligned}
v &= \lambda u - f, \\
\xi &= J v_x(L), \\
\psi &= M v(L), \\
\zeta_1 &= \lambda^{-1}(\Upsilon_1 + b_1 v_x(L)), \\
\zeta_2 &= \lambda^{-1}(\Upsilon_2 + b_2 v(L)),
\end{aligned} \tag{2.54}$$

where

$$(\Lambda u_{xx})_{xx} + \mu\lambda^2 u = \mu(g + \lambda f), \tag{2.55}$$
$$u(0) = 0, \tag{2.56}$$
$$u_x(0) = 0, \tag{2.57}$$
$$\Lambda(L)u_{xx}(L) + (k_1 + b_1^\top P_1 b_1 + \lambda^2 J)u_x(L) = B_1, \tag{2.58}$$
$$-(\Lambda u_{xx})_x(L) + (k_2 + b_2^\top P_2 b_2 + \lambda^2 M)u(L) = B_2, \tag{2.59}$$

with $B_{1,2}$ introduced as

$$B_1 := \Xi + \lambda J f_x(L) - \frac{1}{\lambda}b_1^\top P_1 \Upsilon_1 + \frac{1}{\lambda}b_1^\top P_1 b_1\, f_x(L),$$

and

$$B_2 := \Psi + \lambda M f(L) - \frac{1}{\lambda}b_2^\top P_2 \Upsilon_2 + \frac{1}{\lambda}b_2^\top P_2 b_2\, f(L).$$

Notice that $|B_i| \leq C\|\check{z}\|_{\mathcal{H}}$, $i = 1, 2$, for some $C = C(\lambda) > 0$. Now, it is argued that there exists a particular solution $u_p \in \tilde{H}_0^4(0, L)$ to (2.55). Due to the Lax-Milgram Lemma, there exists a unique weak solution $u_p \in H_0^2(0, L)$ to (2.55), and it holds

$$(\Lambda(u_p)_{xx})_{xx} = -\mu\lambda u_p^2 + \mu(g + \lambda f) \in L^2(0, L).$$

Hence

$$\Lambda(u_p)_{xxx} = (\Lambda(u_p)_{xx})_x - \Lambda_x(u_p)_{xx} \in L^2(0, L).$$

Since $\Lambda(x) \geq \Lambda_0 > 0$ for some positive constant Λ_0, it follows that $\Lambda \in L^\infty(0, L)$. Furthermore, $\Lambda \in W^{2,\infty}(0, L)$ implies

$$(u_p)_{xxx} = \frac{1}{\Lambda} \cdot \Lambda(u_p)_{xxx} \in L^2(0, L).$$

Therefore,

$$\Lambda(u_p)_{xxxx} = -2\Lambda_x(u_p)_{xxx} - \Lambda_{xx}(u_p)_{xx} - \mu\lambda u_p^2 + \mu(g + \lambda f) \in L^2(0, L).$$

This finally implies $(u_p)_{xxxx} = \frac{1}{\Lambda}\big(\Lambda(u_p)_{xxxx}\big) \in L^2(0, L)$. The solution u can now be written in the form $u = u_h + u_p$, where u_h is a solution of the homogeneous equation, as in (2.15). It follows that u_h satisfies the boundary conditions (2.56),(2.57), and

$$\Lambda(L)u_{xx}(L) + (k_1 + b_1^\top P_1 b_1 + \lambda^2 J)u_x(L) = \tilde{B}_1, \tag{2.60}$$

$$-\left(\Lambda u_{xx}\right)_x(L) + (k_2 + b_2^\top P_2 b_2 + \lambda^2 M)u(L) = \tilde{B}_2, \tag{2.61}$$

where

$$\tilde{B}_1 = B_1 - \Lambda(L)(u_p)_{xx}(L) - (k_1 + b_1^\top P_1 b_1 + \lambda^2 J)(u_p)_x(L),$$

and

$$\tilde{B}_2 = B_2 + \left(\Lambda(u_p)_{xx}\right)_x(L) - (k_2 + b_2^\top P_2 b_2 + \lambda^2 M)u_p(L).$$

As before, it can be easily seen that $|\tilde{B}_i| \leq C(\lambda)\|\check{z}\|_{\mathcal{H}}$, $i = 1, 2$. Therefore, the boundary value problem for u_h can be compared to (2.15)-(2.19), and the same solution strategy applies. The space transformations as in (2.21) and (2.25) can be performed to obtain an equivalent problem: $\tilde{u}_h$ is sought such that it satisfies the following boundary value problem

$$(\tilde{u}_h)_{yyyy} + \tilde{\alpha}_2(\tilde{u}_h)_{yy} + \tilde{\alpha}_1(\tilde{u}_h)_y + \tilde{\alpha}_0(\tilde{u}_h) + h^4\lambda^2(\tilde{u}_h) = 0, \tag{2.62}$$

$$\tilde{u}_h(0) = 0, \tag{2.63}$$

$$(\tilde{u}_h)_y(0) = 0, \tag{2.64}$$

$$(\tilde{u}_h)_{yy}(1) + (\tilde{u}_h)_y(1)\left(\beta_3 + \tilde{\kappa}_1(\lambda)\right) + \tilde{u}_h(1)\left(\beta_4 - \frac{1}{4}\alpha_3(1)\tilde{\kappa}_1(\lambda)\right) = \tilde{B}_1, \tag{2.65}$$

$$-(\tilde{u}_h)_{yyy}(1) + \beta_5(\tilde{u}_h)_{yy}(1) + \beta_6(\tilde{u}_h)_y(1) + (\beta_7 + \tilde{\kappa}_2(\lambda))\,(\tilde{u}_h)(1) \;=\; \tilde{B}_2, \qquad (2.66)$$

with

$$\begin{aligned}\tilde{\kappa}_1(\lambda) &:= \frac{h}{\Lambda(L)}\left(\frac{\mu(L)}{\Lambda(L)}\right)^{-\frac{1}{4}}\left(k_1 + b_1^\top P_1 b_1 + \lambda^2 J\right),\\ \tilde{\kappa}_2(\lambda) &:= \frac{h^3}{\Lambda(L)}\left(\frac{\mu(L)}{\Lambda(L)}\right)^{-\frac{3}{4}}\left(k_2 + b_2^\top P_2 b_2 + \lambda^2 M\right).\end{aligned}$$

Here, the functions $\tilde{\alpha}_i$, $i = 0, 1, 2$ and constants $\beta_3, \ldots, \beta_7$ are the same as in the proof of Theorem 2.13. Hence, $\tilde{u}_h$ can be written as $\tilde{u}_h = \sum_{i=1}^4 \tilde{C}_i\gamma_i$, where γ_i are the linearly independent solutions to the homogeneous equation (2.62), given in Lemma 2.14. The coefficients $\tilde{C}_i$ are determined by the boundary conditions, and they satisfy the following linear system

$$\begin{bmatrix} \gamma_1(0) & \gamma_2(0) & \gamma_3(0) & \gamma_4(0) \\ (\gamma_1)_y(0) & (\gamma_2)_y(0) & (\gamma_3)_y(0) & (\gamma_4)_y(0) \\ \tilde{m}_{31} & \tilde{m}_{32} & \tilde{m}_{33} & \tilde{m}_{34} \\ \tilde{m}_{41} & \tilde{m}_{42} & \tilde{m}_{43} & \tilde{m}_{44} \end{bmatrix} \begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_2 \\ \tilde{C}_3 \\ \tilde{C}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, \qquad (2.67)$$

with $m_{3\,i}$, $m_{4\,i}$ given by:

$$\tilde{m}_{3\,i} := (\gamma_i)_{yy}(1) + (\beta_3 + \tilde{\kappa}_1(\lambda))(\gamma_i)_y(1) + (\beta_4 - \frac{1}{4}\alpha_3(1)\tilde{\kappa}_1(\lambda))\gamma_i(1),$$

$$\tilde{m}_{4\,i} := -(\gamma_i)_{yyy}(1) + \beta_5(\gamma_i)_{yy}(1) + \beta_6(\gamma_i)_y(1) + (\beta_7 + \tilde{\kappa}_2(\lambda))\gamma_i(1).$$

Two cases will be distinguished:

(i)–the determinant in (2.67) *is zero*: This is true for a given λ if and only if λ is an eigenvalue of $\mathcal{A}_c$. Since $\mathcal{A}_c$ is skew-symmetric, its eigenvalues are purely imaginary. Furthermore, they come in complex-conjugated pairs. Proceeding as in the proof of Theorem 2.13, it follows easily that the eigenvalues $\{\nu_n\}_{n\in\mathbb{N}}$ have the asymptotic behavior as in (2.51). Let w_n denote the eigenfunction corresponding to ν_n, and let u_n^c denote its first coordinate. As in the Theorem 2.13, it shows that the asymptotic behavior of u_n^c is given by (2.53). The eigenfunction corresponding to the conjugate eigenvalue $\overline{\nu}_n$ is $\overline{w_n}$.

(ii)–the determinant in (2.67) *is not zero*: Then a unique solution $\tilde{u}_h$ exists, and it holds that $|\tilde{C}_i| \le C(\lambda)\|\check{z}\|_{\mathcal{H}}$, $i = 1, \ldots, 4$. Due to Rouché's Theorem, it is known that for all λ in some neighborhood $\mathcal{U}_n$ around ν_n, the determinant in (2.67) is not zero. For a fixed $\lambda \in \mathcal{U}_n$, there exists a solution u to (2.55)-(2.59), and $\|u\|_{H^2(0,L)} \le C(\lambda)\|\check{z}\|_{\mathcal{H}}$ holds. From here follows that $\|z\|_{\mathcal{H}} \le C(\lambda)\|\check{z}\|_{\mathcal{H}}$, and hence $\lambda \in \rho(\mathcal{A}_c)$. Moreover, it is easily shown that $\|u\|_{H^4(0,L)}, \|v\|_{H^2(0,L)} \le C(\lambda)\|\check{z}\|_{\mathcal{H}}$ as well. Thus, due to compact embeddings $H^4(0,L) \hookrightarrow\hookrightarrow H^2(0,L) \hookrightarrow\hookrightarrow L^2(0,L)$, it follows that $R(\lambda, \mathcal{A}_c)$ is compact. Hence, according to the corollary of Theorem VII.3.1 in [73], $\mathcal{A}_c$ is skew-adjoint. Furthermore, the

spectrum $\sigma(\mathcal{A}_c)$ consists of countably many isolated eigenvalues (cf. Theorem III.6.26 in [35]), and hence eigenfunctions of $\mathcal{A}_c$ form an orthogonal basis for $\mathcal{H}$ (Theorem V.2.10 in [35]). □

At this point, everything is prepared for stating the main result of this subsection.

Theorem 2.24. *There exists a sequence of generalized eigenfunctions of the operator $\mathcal{A}$ which forms a Riesz basis for the state space $\mathcal{H}$. Furthermore, for the semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A}$, the spectrum determined growth condition holds: $\omega_0(\mathcal{A}) = r(\mathcal{A})$.*

Proof. It suffices to show that eigenfunctions w_n and z_n satisfy (2.44) and (2.45). From (2.47), it follows directly that:

$$\begin{aligned} u_n &= \mathcal{O}(n^{-2}), \\ u_n(L) &= \mathcal{O}(n^{-3}). \end{aligned} \tag{2.68}$$

Further, $(u_n)_x(L)$ and $(u_n)_{xx}$ need to be considered. It easily follows that:

$$\frac{d^k}{dy^k}(\tilde{u}_n)(y) = -\tau_n^{-8} e^{-\tau_n} \frac{1}{2l_1} \begin{vmatrix} \gamma_1(0) & \gamma_2(0) & \gamma_3(0) & \gamma_4(0) \\ (\gamma_1)_y(0) & (\gamma_2)_y(0) & (\gamma_3)_y(0) & (\gamma_4)_y(0) \\ m_{31} & m_{32} & m_{33} & m_{34} \\ \frac{d^k}{dy^k}\gamma_1(y) & \frac{d^k}{dy^k}\gamma_2(y) & \frac{d^k}{dy^k}\gamma_3(y) & \frac{d^k}{dy^k}\gamma_4(y) \end{vmatrix}, \quad k \in \mathbb{N}. \tag{2.69}$$

Inspection of the dominant terms for large n gives

$$\begin{aligned} (\tilde{u}_n)_y(y) &= -ih^2\tau_n^{-1}\Big[-e^{-(n-\frac{1}{2})\pi y} + \cos\Big((n-\frac{1}{2})\pi y\Big) + \sin\Big((n-\frac{1}{2})\pi y\Big) \\ &\quad + (-1)^n e^{(n-\frac{1}{2})\pi(y-1)} + \mathcal{O}(n^{-1})\Big], \end{aligned} \tag{2.70}$$

and

$$\begin{aligned} (\tilde{u}_n)_{yy}(y) &= -ih^2\Big[e^{-(n-\frac{1}{2})\pi y} + \cos\Big((n-\frac{1}{2})\pi y\Big) - \sin\Big((n-\frac{1}{2})\pi y\Big) \\ &\quad + (-1)^n e^{(n-\frac{1}{2})\pi(y-1)} + \mathcal{O}(n^{-1})\Big], \end{aligned} \tag{2.71}$$

$y \in [0,1]$. Immediately, due to (2.70), $(\tilde{u}_n)_y(y) = \mathcal{O}(n^{-1})$, $\forall y \in [0,1]$ and $(\tilde{u}_n)_y(1) = \mathcal{O}(n^{-2})$. Furthermore,

$$\begin{aligned} (u_n)_x(x) &= \frac{1}{h}\Big(\frac{\mu}{\Lambda}(x)\Big)^{\frac{1}{4}} e^{-\frac{1}{4}\int_0^y \alpha_3}\Big[(\tilde{u}_n)_y(y) - \frac{1}{4}\alpha_3(y)\tilde{u}_n(y)\Big] \\ &= -ih\Big(\frac{\mu}{\Lambda}(x)\Big)^{\frac{1}{4}} e^{-\frac{1}{4}\int_0^y \alpha_3}\Big\{\tau_n^{-1}\Big[-e^{-(n-\frac{1}{2})\pi y} + \cos\Big((n-\frac{1}{2})\pi y\Big) \\ &\quad + \sin\Big((n-\frac{1}{2})\pi y\Big) + (-1)^n e^{(n-\frac{1}{2})\pi(y-1)}\Big] + \mathcal{O}(n^{-2})\Big\} \end{aligned} \tag{2.72}$$

and hence

$$J\lambda_n u_x(L) = \mathcal{O}(1). \tag{2.73}$$

From (2.21) and (2.25) follows

$$\begin{aligned}
(u_n)_{xx}(x) &= \frac{1}{h^2}\left(\frac{\mu}{\Lambda}(x)\right)^{\frac{1}{2}} e^{-\frac{1}{4}\int_0^y \alpha_3} \\
&\quad \cdot \left[(\tilde{u}_n)_{yy}(y) - \frac{1}{2}\alpha_3(y)(\tilde{u}_n)_y(y) + \tilde{u}_n(y)\left(-\frac{1}{4}(\alpha_3)_y(y) + \frac{1}{16}\alpha_3^2(y)\right)\right] \\
&\quad + \frac{1}{4h}\left(\frac{\mu}{\Lambda}(x)\right)_x \left(\frac{\mu}{\Lambda}(x)\right)^{-\frac{3}{4}} e^{-\frac{1}{4}\int_0^y \alpha_3}\left((\tilde{u}_n)_y(y) - \frac{1}{4}\alpha_3(y)\tilde{u}_n(y)\right) \\
&= -i\left(\frac{\mu}{\Lambda}(x)\right)^{\frac{1}{2}} e^{-\frac{1}{4}\int_0^y \alpha_3}\left[e^{-(n-\frac{1}{2})\pi y} + \cos\left((n-\frac{1}{2})\pi y\right)\right. \\
&\quad \left. - \sin\left((n-\frac{1}{2})\pi y\right) + (-1)^n e^{(n-\frac{1}{2})\pi(y-1)}\right] + \mathcal{O}(n^{-1}).
\end{aligned} \tag{2.74}$$

Similarly, for u_n^c given in (2.53), it can be obtained that $(u_n^c)_x(x)$ and $(u_n^c)_{xx}(x)$ have the same asymptotic expression for large n as in (2.68), (2.72), (2.73) and (2.74), respectively. Hence, the sequences $\{z_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ are approximately normalized, i.e. satisfy (2.44). Therefore $\{w_n\}_{n=1}^{\infty}$ is a Riesz basis for $\mathcal{H}$. These results, together with (2.51), imply that for n large enough

$$\|z_n - w_n\|_{\mathcal{H}}^2 = \mathcal{O}(n^{-2})$$

holds. Therefore, the condition in (2.45) is satisfied. According to Lemma 2.24 it can be concluded that the operator $\mathcal{A}$ has the Riesz basis property and that the eigenvalues of $\mathcal{A}$ with sufficiently large modulus are algebraically simple. Consequently, the spectrum-determined growth condition holds. Furthermore, algebraic simplicity also implies that the asymptotic behavior for the generalized eigenfunctions of $\mathcal{A}$ is fully provided by (2.47) and (2.48). □

Remark 2.25. In the analysis above, approximately normalized eigenfunctions of the system (2.50) (as in (2.44)) and not normalized eigenfunctions itself have been taken. The reason for this is that the condition (2.45) is easier to verify in the case of approximatively normalized eigenvalues, since in case of normalized eigenvalues, the asymptotic behavior of norms $\|w_n\|_{\mathcal{H}}$ and $\|z_n\|_{\mathcal{H}}$ when $n \to \infty$ would need to be considered as well.

2.1.5 Frequency domain criteria

Frequency domain criteria, as presented in Theorem 2.12, is a common technique in the literature for proving exponential stability of a beam system. According to another variation of Huang's theorem, it even suffices to show that the imaginary axis belongs to the resolvent set and that resolvent norm is bounded along imaginary axis:

Theorem 2.26. *Let $\mathcal{A}$ be a linear operator on a Hilbert space $\mathcal{H}$. Assume that $\mathcal{A}$ generates a bounded C_0-semigroup $T(t)$ on $\mathcal{H}$. Then $T(t)$ is exponentially stable if and only if the following holds:*

i) imaginary axis belongs to the resolvent set of $\mathcal{A}$

ii) the following resolvent estimate holds:

$$\sup_{\omega\in\mathbb{R}} \|(i\omega I - \mathcal{A})^{-1}\| < \infty \tag{2.75}$$

However, in this subsection it will be shown that the condition (2.75) does not hold for the system (2.8), which offers another evidence for the lack of exponential stability. For simplicity it is assumed that μ and Λ are constant on $[0, L]$. Let $\breve{z} = [f\, g\, \Upsilon_1\, \Upsilon_2\, \Xi\, \Psi]^\top \in \mathcal{H}$ and $\lambda \in \mathbb{R}$ be given. We consider the resolvent equation: find $z \in D(\mathcal{A})$ such that

$$(i\lambda I - \mathcal{A})\, z = \breve{z}.$$

In vector form, the equation reads:

$$\begin{bmatrix} i\lambda u - v \\ i\lambda v + \frac{1}{\mu}\Lambda u_{xxxx} \\ i\lambda\zeta_1 - A_1\zeta_1 - b_1 v_x(L) \\ i\lambda\zeta_2 - A_2\zeta_2 - b_2 v(L) \\ i\lambda\xi + \Lambda u_{xx}(L) + k_1 u_x(L) + c_1\cdot\zeta_1 + d_1 v_x(L) \\ i\lambda\psi - \Lambda u_{xxx}(L) + k_2 u(L) + c_2\cdot\zeta_2 + d_2 v(L) \end{bmatrix} = \begin{bmatrix} f \\ g \\ \Upsilon_1 \\ \Upsilon_2 \\ \Xi \\ \Psi \end{bmatrix}. \tag{2.76}$$

From (2.76) follows that z is uniquely determined by u:

$$\begin{aligned} v &= i\lambda u - f, \\ \zeta_1 &= (i\lambda I - A_1)^{-1}(b_1 v_x(L) + \Upsilon_1), \\ \zeta_2 &= (i\lambda I - A_2)^{-1}(b_2 v(L) + \Upsilon_2), \\ \xi &= J v_x(L), \\ \psi &= M v(L). \end{aligned}$$

Furthermore, from (2.76) it follows that u satisfies forth order boundary problem:

$$\Lambda u_{xxxx} - \mu\lambda^2 u = \mu(i\lambda f + g), \tag{2.77}$$
$$u(0) = 0, \tag{2.78}$$
$$u_x(0) = 0, \tag{2.79}$$
$$\Lambda u_{xx}(L) + (k_1 - J\lambda^2 + i\lambda\mathcal{G}_1(i\lambda))u_x(L) = h_1, \tag{2.80}$$
$$-\Lambda u_{xxx}(L) + (k_2 - M\lambda^2 + i\lambda\mathcal{G}_2(i\lambda))u(L) = h_2, \tag{2.81}$$

where

$$h_1 = \Xi + (i\lambda J + \mathcal{G}_1(i\lambda))f_x(L) - c_1\cdot(i\lambda I - A_1)^{-1}\Upsilon_1,$$

and

$$h_2 = \Psi + (i\lambda M + \mathcal{G}_2(i\lambda))f(L) - c_2 \cdot (i\lambda I - A_2)^{-1}\Upsilon_2.$$

It is enough to consider $\lambda > 0$, to show that (2.12) is not true. Let $\tau = \sqrt{\lambda}$ and $\alpha = \sqrt[4]{\frac{\mu}{\Lambda}}$. SUch τ is uniquely determined. Equation (2.77) can be rewritten as:

$$u_{xxxx} - \alpha^4\tau^4 u = \alpha^4(i\tau^2 f + g).$$

Taking into account the domain boundary conditions (2.78) and (2.79), implies that the general solution for (2.77) is of the form

$$u(x) = \tilde{u}(x) + u_p(x)$$

where

$$\tilde{u}(x) = A\left(\cosh\alpha\tau x - \cos\alpha\tau x\right) + B\left(\sinh\alpha\tau x - \sin\alpha\tau x\right)$$

and u_p is a particular solution

$$u_p(x) = \frac{\alpha}{2\tau^3}\int_0^x (\sinh\alpha\tau(x-\sigma) - \sin\alpha\tau(x-\sigma))\left(i\tau^2 f(\sigma) + g(\sigma)\right)\,\mathrm{d}\sigma.$$

Taking the remaining boundary conditions (2.80) and (2.81), a linear system in A and B is obtained:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \frac{r_1}{\alpha^2\tau^2} \\ \frac{r_2}{\alpha^2\tau^2} \end{bmatrix}, \tag{2.82}$$

where

$$m_{11} = \Lambda(\cosh\alpha\tau L + \cos\alpha\tau L) + \frac{1}{\alpha\tau}(\sinh\alpha\tau L + \sin\alpha\tau L)(k_1 - J\tau^4 + i\tau^2\mathcal{G}_1(i\tau^2)),$$
$$m_{12} = \Lambda(\sinh\alpha\tau L + \sin\alpha\tau L) + \frac{1}{\alpha\tau}(\cosh\alpha\tau L - \cos\alpha\tau L)(k_1 - J\tau^4 + i\tau^2\mathcal{G}_1(i\tau^2)),$$
$$m_{21} = -\Lambda\alpha\tau(\sinh\alpha\tau L - \sin\alpha\tau L) + \left(\frac{1}{\alpha\tau}\right)^2(\cosh\alpha\tau L - \cos\alpha\tau L)(k_2 - M\tau^4 + i\tau^2\mathcal{G}_2(i\tau^2)),$$
$$m_{22} = -\Lambda\alpha\tau(\cosh\alpha\tau L + \cos\alpha\tau L) + \left(\frac{1}{\alpha\tau}\right)^2(\sinh\alpha\tau L - \sin\alpha\tau L)(k_2 - M\tau^4 + i\tau^2\mathcal{G}_2(i\tau^2)),$$

and

$$\begin{aligned} r_1 &= \Xi + f_x(L)\left(i\tau^2 J + \mathcal{G}_1(i\tau^2)\right) - c_1^\top(i\tau^2 I - A_1)^{-1}\Upsilon_1 \\ &\quad -\frac{\Lambda\alpha^3}{2\tau}\int_0^L(\sinh\alpha\tau(L-\sigma) + \sin\alpha\tau(L-\sigma))(i\tau^2 f(\sigma) + g(\sigma))\,\mathrm{d}\sigma \\ &\quad -(k_1 - J\tau^4 + i\tau^2\mathcal{G}_1(i\tau^2))\frac{\alpha^2}{2\tau^2}\int_0^L(\cosh\alpha\tau(L-\sigma) - \cos\alpha\tau(L-\sigma))(i\tau^2 f(\sigma) + g(\sigma))\,\mathrm{d}\sigma, \end{aligned}$$

$$r_2 = \Psi + f(L)\left(i\tau^2 M + \mathcal{G}_2(i\tau^2)\right) - c_2^\top(i\tau^2 I - A_2)^{-1}\Upsilon_2$$

$$+\frac{\Lambda\alpha^4}{2}\int_0^L (\cosh\alpha\tau(L-\sigma)+\cos\alpha\tau(L-\sigma))(i\tau^2 f(\sigma)+g(\sigma))\,\mathrm{d}\sigma$$
$$-(k_2-M\tau^4+i\tau^2\mathcal{G}_2(i\tau^2))\frac{\alpha}{2\tau^3}\int_0^L (\sinh\alpha\tau(L-\sigma)-\sin\alpha\eta(L-\sigma))(i\tau^2 f(\sigma)+g(\sigma))\,\mathrm{d}\sigma.$$

Now, introducing following notations

$$\begin{aligned}
I_1 &= \int_0^L e^{-\alpha\tau\sigma}(if_{xx}(\sigma)+g(\sigma))\,\mathrm{d}\sigma,\\
I_2 &= e^{-\alpha\tau L}\int_0^L e^{\alpha\tau\sigma}(if_{xx}(\sigma)+g(\sigma))\,\mathrm{d}\sigma,\\
I_3 &= \int_0^L \sin\alpha\tau(L-\sigma)(-if_{xx}(\sigma)+g(\sigma))\,\mathrm{d}\sigma,\\
I_4 &= \int_0^L \cos\alpha\tau(L-\sigma)(-if_{xx}(\sigma)+g(\sigma))\,\mathrm{d}\sigma,
\end{aligned}$$

it holds that

$$\begin{aligned}
r_1 &= \Xi + f_x(L)\frac{ik_1\alpha^2}{\tau^2} - c_1^\top(i\tau^2 I - A_1)^{-1}\Upsilon_1\\
&\quad -\frac{\Lambda\alpha^3}{4\tau}I_1 e^{\alpha\tau L} + \frac{\Lambda\alpha^3}{4\tau}I_2 - \frac{\Lambda\alpha^3}{2\tau}I_3 - (k_1 - J\tau^4 + i\tau^2\mathcal{G}_1(i\tau^2))\frac{\alpha^2}{4\tau^2}I_1 e^{\alpha\tau L}\\
&\quad -(k_1 - J\tau^4 + i\tau^2\mathcal{G}_1(i\tau^2))\frac{\alpha^2}{4\tau^2}I_2 + (k_1 - J\tau^4 + i\tau^2\mathcal{G}_1(i\tau^2))\frac{\alpha^2}{2\tau^2}I_4,
\end{aligned}$$

and

$$\begin{aligned}
r_2 &= \Psi + f(L)\frac{ik_2\alpha^2}{\tau^2} - c_2^\top(i\tau^2 I - A_2)^{-1}\Upsilon_2\\
&\quad +\frac{\Lambda\alpha^4}{4}I_1 e^{\alpha\tau L} + \frac{\Lambda\alpha^4}{4}I_2 + \frac{\Lambda\alpha^4}{2}I_4 - (k_2 - M\tau^4 + i\tau^2\mathcal{G}_2(i\tau^2))\frac{\alpha}{4\tau^3}I_1 e^{\alpha\tau L}\\
&\quad +(k_2 - M\tau^4 + i\tau^2\mathcal{G}_2(i\tau^2))\frac{\alpha}{4\tau^3}I_2 + (k_2 - M\tau^4 + i\tau^2\mathcal{G}_2(i\tau^2))\frac{\alpha}{2\tau^3}I_3.
\end{aligned}$$

Note that

$$I_1 = I_2 = \mathcal{O}(\tau^{-\frac{1}{2}}(\|f_{xx}\|_2 + \|g\|_2)).$$

Let the determinant of the linear system given in (2.82) be denoted by D. Then the following is obtained:

$$\begin{aligned}
D &= m_{11}m_{22} - m_{12}m_{21}\\
&= e^{\tau L}\big\{JM\alpha^{-3}\tau^5\cos\alpha\tau L + \Lambda J\tau^4(\cos\alpha\tau L + \sin\alpha\tau L) - M\Lambda\alpha^{-2}\tau^2(\cos\alpha\tau L - \sin\alpha\tau L)\\
&\quad -i(Md_1 + Jd_2)\alpha^{-3}\tau^3\cos\alpha\tau L - i\Lambda d_1\tau^2(\cos\alpha\tau L + \sin\alpha\tau L)\big\} + \mathcal{O}(\tau e^{\alpha\tau L}).
\end{aligned}$$

Furthermore, it holds:

$$
\begin{aligned}
A &= \frac{1}{D\alpha^2\tau^2}\left(r_1 m_{22} - r_2 m_{12}\right), \\
B &= \frac{1}{D\alpha^2\tau^2}\left(-r_1 m_{21} + r_2 m_{11}\right).
\end{aligned}
$$

The second derivative of the solution is of the form:

$$
\begin{aligned}
u_{xx}(x) &= \frac{\alpha^3}{4\tau} e^{\alpha\tau x} I_1 + \mathcal{O}(\tau^{-1}(\|f_{xx}\|_2 + \|g\|_2)) \\
&+ A\alpha^2\tau^2(\cosh\alpha\tau x + \cos\alpha\tau x) + B\alpha^2\tau^2(\sinh\alpha\tau x + \sin\alpha\tau x) \\
&= C_1 e^{\alpha\tau x} + C_2 e^{-\alpha\tau x} + C_3\cos\alpha\tau x + C_4 \sin\alpha\tau x + \mathcal{O}(\tau^{-1}(\|f_{xx}\|_2 + \|g\|_2)),
\end{aligned}
$$

where

$$
\begin{aligned}
C_1 &= \frac{\alpha^2}{2}\left((A+B)\tau^2 + \frac{\alpha}{2\tau} I_1\right), \\
C_2 &= (A-B)\frac{\alpha^2\tau^2}{2}, \\
C_3 &= A\alpha^2\tau^2, \\
C_4 &= B\alpha^2\tau^2.
\end{aligned}
$$

Considering only the dominant terms of τ, the following is obtained:

$$
\begin{aligned}
C_1 D &= JM\frac{1}{4}\left(-2I_1 + I_2\sin\alpha\tau L + I_3(\sin\alpha\tau L + \cos\alpha\tau L)\right. \\
&\quad \left. + I_4(-\sin\alpha\tau L + \cos\alpha\tau L)\right)\tau^4 + \mathcal{O}(\tau^3\|\breve{z}\|_{\mathcal{H}}), \\
C_2 D &= JM\frac{1}{4}\left(I_1\sin\alpha\tau L - I_2 - I_3 + I_4\right)\tau^4 e^{\tau L} + \mathcal{O}(\tau^3 e^{\alpha\tau L}\|\breve{z}\|_{\mathcal{H}}), \\
C_3 D &= JM\frac{1}{4}\left(I_1(\sin\alpha\tau L - \cos\alpha\tau L) - I_2 - I_3 + I_4\right)\tau^4 e^{\alpha\tau L} + \mathcal{O}(\tau^3 e^{\alpha\tau L}\|\breve{z}\|_{\mathcal{H}}), \\
C_4 D &= JM\frac{1}{4}\left(-I_1(\sin\alpha\tau L + \cos\alpha\tau L) + I_2 + I_3 - I_4\right)\tau^4 e^{\alpha\tau L} + \mathcal{O}(\tau^3 e^{\alpha\tau L}\|\breve{z}\|_{\mathcal{H}}).
\end{aligned}
$$

Moreover, since

$$
\begin{aligned}
|I_1| &= \mathcal{O}(\tau^{-\frac{1}{2}}\|\breve{z}\|_{\mathcal{H}}), \\
|I_2| &= \mathcal{O}(\tau^{-\frac{1}{2}}\|\breve{z}\|_{\mathcal{H}}), \\
|I_3| &= \mathcal{O}(\|\breve{z}\|_{\mathcal{H}}), \\
|I_4| &= \mathcal{O}(\|\breve{z}\|_{\mathcal{H}}),
\end{aligned}
$$

it follows that

$$
\|u_{xx}\|_2 = \frac{1}{D}\sqrt{\frac{L}{2}\left(|C_3 D|^2 + |C_4 D|^2\right) + \mathcal{O}(\tau^7 e^{2\alpha\tau L}\|\breve{z}\|_{\mathcal{H}}^2)} + \mathcal{O}(\tau^{-1}\|\breve{z}\|_{\mathcal{H}})
$$

$$= \frac{JM\tau^4}{4D}\sqrt{L|I_4 - I_3|^2 e^{2\alpha\tau L} + \mathcal{O}(\tau^{-\frac{1}{2}} e^{2\alpha\tau L}\|\breve{z}\|_{\mathcal{H}})} + \mathcal{O}(\tau^{-1}\|\breve{z}\|_{\mathcal{H}}^2).$$

For every τ large enough, a function $\breve{z}_\tau = (f_\tau, 0, 0, 0, 0, 0)$ can be chosen with $f_\tau \in H_0^2(0, L)$ such that

$$\sqrt{L|I_4 - I_3|^2 + \mathcal{O}(\tau^{-\frac{1}{2}}\|\breve{z}\|_{\mathcal{H}}^2)} \geq K\|\breve{z}_\tau\|_{\mathcal{H}}, \tag{2.83}$$

where constant K does not depend on τ.

For this purpose, let f_τ be defined with

$$f_\tau(x) = -\frac{1}{\alpha^2\tau^2}\sin(\alpha\tau(L-x) - \frac{\pi}{4}) - \frac{x}{\alpha\tau}\cos(\alpha\tau L - \frac{\pi}{4}) + \frac{1}{\alpha^2\tau^2}\sin(\alpha\tau L - \frac{\pi}{4}).$$

Then

$$(f_\tau)_{xx}(x) = \sin(\alpha\tau(L-x) - \frac{\pi}{4}),$$

and

$$\|(f_\tau)_{xx}\|_2^2 = \frac{L}{2} + \mathcal{O}(\tau^{-1}). \tag{2.84}$$

Hence

$$\begin{aligned}\|\breve{z}_\tau\|_{\mathcal{H}}^2 &= \frac{\Lambda}{2}\|(f_\tau)_{xx}\|_2^2 + \frac{k_1}{2}(f_\tau)_x(L)^2 + \frac{k_2}{2}f_\tau(L)^2 \\ &= \frac{\Lambda L}{4} + \mathcal{O}(\tau^{-1}),\end{aligned} \tag{2.85}$$

which implies that for all τ large enough, $\|\breve{z}_\tau\|_{\mathcal{H}}$ is bounded by come constant independent of τ. There holds:

$$\begin{aligned}I_4 - I_3 &= i\int_0^L (\sin\alpha\tau(L-\sigma) - \cos\alpha\tau(L-\sigma))(f_\tau)_{xx}(\sigma)d\sigma \\ &= i\sqrt{2}\int_0^L \sin\left(\alpha\tau(L-\sigma) - \frac{\pi}{4}\right)^2 d\sigma \\ &= i\sqrt{2}\|(f_\tau)_{xx}\|_2^2.\end{aligned}$$

Therefore (2.83) follows easily from (2.84) and (2.85) for τ large enough. Moreover

$$\|(u_\tau)_{xx}\|_2 \geq K\frac{JM\tau^4 e^{\tau L}}{4D}\|\breve{z}_\tau\|_{\mathcal{H}} + \mathcal{O}(\tau^{-1}\|\breve{z}_\tau\|_{\mathcal{H}}),$$

for all τ large enough. Hence, a sufficiently large τ can always be found so that

$$D \leq Se^{\tau L}\tau^3,$$

where constant $S > 0$ does not depend on τ. For such τ, there holds:

$$\|(u_\tau)_{xx}\|_2 \geq \tau\frac{KJM}{4S}\|\breve{z}_\tau\|_{\mathcal{H}} + \mathcal{O}(\tau^{-1}\|\breve{z}_\tau\|_{\mathcal{H}}).$$

This implies that there exists some constant $\tilde{M} > 0$ independent of τ such that

$$\|\left(i\tau^2 - \mathcal{A}\right)^{-1}\| \geq \frac{\|z_\tau\|_{\mathcal{H}}}{\|\breve{z}_\tau\|_{\mathcal{H}}} \geq \tilde{M}\tau.$$

Therefore (2.75) does not hold, and (2.8) can not be exponentially stable.

2.2 Weak formulation

In this section the system of equations (2.1)–(2.5) and (2.6) is written in the weak form, and the existence and uniqueness of the weak solution is demonstrated. This reformulation will be used in Section 2.3 to develop a dissipative finite element method for the observed system.

2.2.1 Definition of a weak solution

In order to derive the weak formulation, the following initial conditions are assumed:

$$u(0) = u_0 \in \tilde{H}_0^2(0,L), \tag{2.86a}$$
$$u_t(0) = v_0 \in L^2(0,L), \tag{2.86b}$$
$$\zeta_1(0) = \zeta_{1,0} \in \mathbb{R}^n, \tag{2.86c}$$
$$\zeta_2(0) = \zeta_{2,0} \in \mathbb{R}^n. \tag{2.86d}$$

Moreover, let $v_0(L)$ and $(v_0)_x(L)$ be given in addition to the function v_0, and *not* as its trace. Multiplying (2.1) by $w \in \tilde{H}_0^2(0,L)$, integrating over $[0,L]$, and taking into account the given boundary conditions (2.2)-(2.5) yields:

$$\begin{aligned}
&\int_0^L \mu\, u_{tt} w\, dx + \int_0^L \Lambda\, u_{xx} w_{xx}\, dx + M u_{tt}(t,L) w(L) + J u_{ttx}(t,L) w_x(L) \\
&+ k_1 u_x(t,L) w_x(L) + k_2 u(t,L) w(L) + d_1 u_{tx}(t,L) w_x(L) + d_2 u_t(t,L) w(L) \\
&+ c_1 \cdot \zeta_1(t)\; w_x(L) + c_2 \cdot \zeta_2(t)\; w(L) = 0, \qquad \forall w \in \tilde{H}_0^2(0,L),\ t > 0.
\end{aligned} \tag{2.87}$$

This identity will motivate the weak formulation. The first step in the definition of the weak formulation is the appropriate space setting. Let the Hilbert space H with its inner product be defined by:

$$\begin{aligned}
H &:= \mathbb{R} \times \mathbb{R} \times L^2(0,L), \\
(\hat{\varphi}, \hat{\nu})_H &:= J\,({}^1\hat{\varphi})\,({}^1\hat{\nu}) + M\,({}^2\hat{\varphi})\,({}^2\hat{\nu}) + (\mu\,{}^3\hat{\varphi},\, {}^3\hat{\nu})_{L^2},
\end{aligned} \tag{2.88}$$

for $\hat{\varphi} = ({}^1\hat{\varphi},\, {}^2\hat{\varphi},\, {}^3\hat{\varphi})$, $\hat{\nu} \in H$. Next, the Hilbert space V with its inner product is introduced as follows:

$$\begin{aligned}
V &:= \{\hat{w} = (w_x(L), w(L), w) \colon w \in \tilde{H}_0^2(0,L)\}, \\
(\hat{w}_1, \hat{w}_2)_V &:= ((w_1)_{xx}, (w_2)_{xx})_{L^2}.
\end{aligned} \tag{2.89}$$

It can easily be shown that V is densely embedded in H. Therefore taking H as a pivot space, a Gelfand triple $V \subset H \subset V'$ is obtained. Furthermore, let the bilinear forms $a : V \times V \to \mathbb{R}$, $b : H \times H \to \mathbb{R}$ and $e_1, e_2 : \mathbb{R}^n \times V \to \mathbb{R}$ be given by

$$a(\hat{w}_1, \hat{w}_2) = (\Lambda\, \hat{w}_1, \hat{w}_2)_V + k_1 (w_1)_x(L)(w_2)_x(L) + k_2 w_1(L) w_2(L),$$

$$\begin{aligned} b(\hat{\varphi},\hat{\nu}) &= d_1\,{}^1\hat{\varphi}\,{}^1\hat{\nu} + d_2\,{}^2\hat{\varphi}\,{}^2\hat{\nu},\\ e_1(\zeta_1,\hat{w}) &= (c_1\cdot\zeta_1)\,w_x(L),\\ e_2(\zeta_2,\hat{w}) &= (c_2\cdot\zeta_2)\,w(L). \end{aligned}$$

Definition 2.27. Let $T > 0$ be fixed. Functions $\hat{u} = (u_x(L), u(L), u)$ and ζ_1, ζ_2 are said to be the *weak solution* to (2.1)–(2.6) on $[0, T]$ if

$$\hat{u} \in L^2(0,T;V) \cap H^1(0,T;H) \cap H^2(0,T;V'),$$

$$\zeta_1, \zeta_2 \in H^1(0,T;\mathbb{R}^n),$$

and they satisfy:

$$_{V'}\langle \hat{u}_{tt}, \hat{w}\rangle_V + a(\hat{u},\hat{w}) + b(\hat{u}_t,\hat{w}) + e_1(\zeta_1,\hat{w}) + e_2(\zeta_2,\hat{w}) = 0, \tag{2.90}$$

for a.e. $t \in (0,T), \forall \hat{w} \in V$. Here the bilinear form $_{V'}\langle .,.\rangle_V$ is the duality pairing between V and V', which is a natural extension of the inner product in H. Equation (2.90) is coupled to the ODEs

$$\begin{aligned} (\zeta_1)_t(t) &= A_1\zeta_1(t) + b_1\,{}^1\hat{u}_t(t),\\ (\zeta_2)_t(t) &= A_2\zeta_2(t) + b_2\,{}^2\hat{u}_t(t), \end{aligned} \tag{2.91}$$

with initial conditions

$$\hat{u}(0) = \hat{u}_0 = ((u_0)_x(L), u_0(L), u_0) \in V, \tag{2.92a}$$

$$\hat{u}_t(0) = \hat{v}_0 = ((v_0)_x(L), v_0(L), v_0) \in H, \tag{2.92b}$$

$$\zeta_1(0) = \zeta_{1,0} \in \mathbb{R}^n, \tag{2.92c}$$

$$\zeta_2(0) = \zeta_{2,0} \in \mathbb{R}^n. \tag{2.92d}$$

In (2.92a) the first two components of the right hand side are the boundary traces of $u_0 \in \tilde{H}_0^2(0,L)$, but in (2.92b) they are additionally given values. Note that in the case when $\hat{u} \in H^2(0,T;V)$, formulation (2.90) is equivalent to identity (2.87). This weak formulation is an extension of [4] to the case where the beam with the tip-body is additionally coupled to the first order ODE controller system. Here, terms $u_t(L)$ and $u_{tx}(L)$ also need to be considered. And these additional first order boundary terms (in t), included in $b(.,.)$, require a slight generalization of the standard theory (as presented in Chapter 8 of [44], e.g.).

2.2.2 Existence and uniqueness results

In order to give a meaning to the initial conditions (2.92a), (2.92b) the following lemma shall be used (special case of Theorem 3.1 in [44]).

Lemma 2.28. *Let X and Y be two Hilbert spaces, such that X is dense and continuously embedded in Y. Assume that*

$$\begin{aligned} u &\in L^2(0,T;X), \\ u_t &\in L^2(0,T;Y). \end{aligned}$$

Then

$$u \in C([0,T];[X,Y]_{\frac{1}{2}}),$$

after, possibly, a modification on a set of measure zero. Here, the definition of intermediate spaces *as given in [44], Section 2.1, was assumed.*

Additionally, the following 'Duality Theorem' (see [44], Chapter 6.2, pp. 29) will be needed in the proof of Theorem 2.30.

Lemma 2.29. *Let X and Y be two Hilbert spaces, such that X is dense and continuously embedded in Y. For all $\theta \in (0,1)$,*

$$[X,Y]'_\theta = [Y',X']_{1-\theta}$$

holds.

Theorem 2.30.

(a) The weak formulation (2.90) – (2.92) has a unique solution $(\hat{u},\zeta_1,\zeta_2)$.

(b) The weak solution has the additional regularity

$$\hat{u} \in L^\infty(0,T;V), \quad \hat{u}_t \in L^\infty(0,T;H), \tag{2.93}$$

$$\zeta_1,\zeta_2 \in C([0,T];\mathbb{R}^n), \tag{2.94}$$

$$\hat{u} \in C([0,T];[V,H]_{\frac{1}{2}}). \tag{2.95}$$

$$\hat{u}_t \in C([0,T];[V,H]'_{\frac{1}{2}}). \tag{2.96}$$

The following proof is an adaption of the proof of Theorem 8.1 in [44], for the system studied here. It is included for the sake of completeness.

Proof. (a)–existence: Let $\{\hat{w}_k\}_{k=1}^\infty$ be a sequence of functions that is an orthonormal basis for H, and an orthogonal basis for V. Existence and construction for such basis is given by Theorem A.1 in Appendix A. Finite dimensional spaces are introduced as follows:

$$\hat{W}_m := \operatorname{span}\{\hat{w}_1,\dots,\hat{w}_m\}, \quad \forall m \in \mathbb{N}.$$

Furthermore, let sequences $\hat{u}_{m0},\hat{v}_{m0} \in \hat{W}_m$ be given so that

$$\begin{aligned} \hat{u}_{m0} &\to \hat{u}_0 \quad \text{in } V, \\ \hat{v}_{m0} &\to \hat{v}_0 \quad \text{in } H. \end{aligned} \tag{2.97}$$

For a fixed $m \in \mathbb{N}$, the Galerkin approximation

$$\hat{u}_m(t) = ((u_m)_x(L), u_m(L), u_m) = \sum_{k=1}^{m} d_m^k(t)\hat{w}_k$$

is considered, with $d_m^k(t) \in \mathbb{R}$, which solves the formulation (2.87) on $\hat{W}_m$:

$$((\hat{u}_m)_{tt}, \hat{w})_H + a(\hat{u}_m, \hat{w}) + b((\hat{u}_m)_t, \hat{w}) + e_1(\zeta_{1,m}, \hat{w}) + e_2(\zeta_{2,m}, \hat{w}) = 0, \quad \forall \hat{w} \in \hat{W}_m \quad (2.98)$$

and $\zeta_{1,m}, \zeta_{2,m}$ solve the ODE system

$$\begin{aligned} (\zeta_{1,m})_t(t) &= A_1\zeta_{1,m}(t) + b_1\,{}^1(\hat{u}_m)_t(t), \\ (\zeta_{2,m})_t(t) &= A_2\zeta_{2,m}(t) + b_2\,{}^2(\hat{u}_m)_t(t), \end{aligned} \quad (2.99)$$

with the initial conditions

$$\begin{aligned} \hat{u}_m(0) &= \hat{u}_{m0}, \\ (\hat{u}_m)_t(0) &= \hat{v}_{m0}, \\ \zeta_{1,m}(0) &= \zeta_{1,0}, \\ \zeta_{2,m}(0) &= \zeta_{2,0}. \end{aligned}$$

Thus a linear system of second order differential equations is obtained. After rewriting it as a system of first order differential equations, standard existence theory for linear differential equations implies that there exists a unique solution satisfying $\hat{u}_m \in C^2([0,T];V)$ and $\zeta_{1,m}, \zeta_{2,m} \in C^1([0,T];\mathbb{R}^n)$. Next, an energy functional is defined analogous to (2.9), for the *trajectory* $(\hat{u}, \zeta_1, \zeta_2)$:

$$\begin{aligned} \hat{E}(t; \hat{u}, \zeta_1, \zeta_2) &:= \frac{1}{2}\|\sqrt{\Lambda}\,\hat{u}(t)\|_V^2 + \frac{k_1}{2}({}^1\hat{u}(t))^2 + \frac{k_2}{2}({}^2\hat{u}(t))^2 + \frac{1}{2}\|\hat{u}_t(t)\|_H^2 \\ &\quad + \frac{1}{2}\zeta_1^\top(t)P_1\zeta_1(t) + \frac{1}{2}\zeta_2^\top(t)P_2\zeta_2(t) \\ &= \|(u, u_t, \zeta_1, \zeta_2, Ju_{tx}(J), Mu_t(L))\|_{\mathcal{H}}. \end{aligned} \quad (2.100)$$

Taking $\hat{w} = (\hat{u}_m)_t$ in (2.98) and using the smoothness of $\hat{u}_m, \zeta_{1,m}, \zeta_{2,m}$, a straightforward calculation yields

$$\begin{aligned} \frac{d}{dt}\hat{E}(t; \hat{u}_m, \zeta_{1,m}, \zeta_{2,m}) &= -\delta_1({}^1(\hat{u}_m)_t)^2 - \frac{1}{2}\left(\zeta_{1,m}\cdot q_1 + \tilde{\delta}_1({}^1(\hat{u}_m)_t)\right)^2 \\ &\quad -\delta_2({}^2(\hat{u}_m)_t)^2 - \frac{1}{2}\left(\zeta_{2,m}\cdot q_2 + \tilde{\delta}_2({}^2(\hat{u}_m)_t)\right)^2 \\ &\quad -\frac{\varepsilon_1}{2}(\zeta_{1,m})^\top P_1\zeta_{1,m} - \frac{\varepsilon_2}{2}(\zeta_{2,m})^\top P_2\zeta_{2,m} \\ &=: \hat{F}(t; \hat{u}_m, \zeta_{1,m}, \zeta_{2,m}) \le 0, \end{aligned} \quad (2.101)$$

which is analogous to (2.10) for the continuous solution. Hence

$$\hat{E}(t;\hat{u}_m,\zeta_{1,m},\zeta_{2,m}) \le \hat{E}(0;\hat{u}_{m0},\zeta_{1,0},\zeta_{2,0}), \quad t \ge 0,$$

which implies

$$\begin{aligned} &\{\hat{u}_m\}_{m\in\mathbb{N}} \quad \text{is bounded in } C([0,T];V),\\ &\{(\hat{u}_m)_t\}_{m\in\mathbb{N}} \quad \text{is bounded in } C([0,T];H),\\ &\{\zeta_{1,m}\}_{m\in\mathbb{N}}, \quad \{\zeta_{2,m}\}_{m\in\mathbb{N}} \ \text{are bounded in } C([0,T];\mathbb{R}^n). \end{aligned} \tag{2.102}$$

Due to these boundedness results, it holds $\forall \hat{w} \in V$:

$$|a(\hat{u}_m(t),\hat{w}) + b((\hat{u}_m)_t(t),\hat{w}) + e_1(\zeta_{1,m}(t),\hat{w}) + e_2(\zeta_{2,m}(t),\hat{w})| \ \le \ D_1\|\hat{w}\|_V,$$

a.e. on $(0,T)$, with some constant $D_1 > 0$ which does not depend on m. Now, let $m \in \mathbb{N}$ be fixed. Furthermore, let $\hat{w} \in V$, and $\hat{w} = \hat{\varphi}_1 + \hat{\varphi}_2$, such that $\hat{\varphi}_1 \in W_m$ and $\hat{\varphi}_2$ orthogonal to $\hat{W}_m$ in H. Equation (2.98) yields:

$$\begin{aligned} ((\hat{u}_m)_{tt},\hat{w})_H &= ((\hat{u}_m)_{tt},\hat{\varphi}_1)_H\\ &= -a(\hat{u}_m,\hat{\varphi}_1) - b((\hat{u}_m)_t,\hat{\varphi}_1) - e_1(\zeta_{1,m},\hat{\varphi}_1) - e_2(\zeta_{2,m},\hat{\varphi}_1)\\ &\le D_1\|\hat{\varphi}_1\|_V \le D_1\|\hat{w}\|_V. \end{aligned}$$

This implies that also $(\hat{u}_m)_{tt}$ is bounded in $C([0,T];V')$. Furthermore, from (2.99) it trivially follows that $\{(\zeta_{1,m})_t\}_{m\in\mathbb{N}}$ and $(\{\zeta_{2,m})_t\}_{m\in\mathbb{N}}$ are also bounded in $C([0,T];\mathbb{R}^n)$.

According to the Eberlein-Šmuljan Theorem, there exist subsequences $\{\hat{u}_{m_l}\}_{l\in\mathbb{N}}$, $\{\zeta_{1,m_l}\}_{l\in\mathbb{N}}$, $\{\zeta_{2,m_l}\}_{l\in\mathbb{N}}$, and $\hat{u} \in L^2(0,T;V)$, with $\hat{u}_t \in L^2(0,T;H)$, $\hat{u}_{tt} \in L^2(0,T;V')$, and $\zeta_1,\zeta_2 \in H^1(0,T;\mathbb{R}^n)$ such that

$$\begin{aligned} \{\hat{u}_{m_l}\} &\rightharpoonup \hat{u} \text{ in } L^2(0,T;V),\\ \{(\hat{u}_{m_l})_t\} &\rightharpoonup \hat{u}_t \text{ in } L^2(0,T;H),\\ \{(\hat{u}_{m_l})_{tt}\} &\rightharpoonup \hat{u}_{tt} \text{ in } L^2(0,T;V'),\\ \{\zeta_{1,m_l}\} &\to \zeta_1 \text{ in } L^2(0,T;\mathbb{R}^n),\\ \{\zeta_{2,m_l}\} &\to \zeta_2 \text{ in } L^2(0,T;\mathbb{R}^n),\\ \{(\zeta_{1,m_l})_t\} &\to (\zeta_1)_t \text{ in } L^2(0,T;\mathbb{R}^n), \end{aligned} \tag{2.103}$$

$$\{(\zeta_{2,m_l})_t\} \to (\zeta_2)_t \text{ in } L^2(0,T;\mathbb{R}^n),$$

Furthermore, (2.103) yields

$$\begin{aligned} \{{}^i(\hat{u}_{m_l})_t(t)\} &\to {}^i\hat{u}_t(t), \\ \{\zeta_{i,m_l}(t)\} &\to \zeta_i(t), \\ \{(\zeta_{i,m_l})_t(t)\} &\to (\zeta_i)_t(t), \end{aligned} \tag{2.104}$$

for $i = 1, 2$, and for almost every $t \in [0, T]$. Taking $m = m_l$ in (2.99), and passing to the limit $l \to \infty$, it follows that (2.91) holds. Let now $m_0 \in \mathbb{N}$. For all functions $\hat{\varphi} \in L^2(0, T; \hat{W}_{m_0})$ of the form

$$\hat{\varphi}(t, x) = \sum_{j=1}^{m_0} \alpha_j(t) w_j(x), \tag{2.105}$$

where $\alpha_j \in L^2(0, T; \mathbb{R})$, and for all $m_l \geq m_0$, equation (2.98) yields

$$\int_0^T ((\hat{u}_{m_l})_{tt}, \hat{\varphi})_H + a(\hat{u}_{m_l}, \hat{\varphi}) + b((\hat{u}_{m_l})_t, \hat{\varphi}) + e_1(\zeta_{1,m_l}, \hat{\varphi}) + e_2(\zeta_{2,m_l}, \hat{\varphi}) \,\mathrm{d}t = 0. \tag{2.106}$$

Therefore, passing to the limit in (2.106), convergence results (2.103) give:

$$\int_0^T {}_{V'}\langle \hat{u}_{tt}, \hat{\varphi}\rangle_V + a(\hat{u}, \hat{\varphi}) + b((\hat{u})_t, \hat{\varphi}) + e_1(\zeta_1, \hat{\varphi}) + e_2(\zeta_2, \hat{\varphi}) \,\mathrm{d}t = 0. \tag{2.107}$$

However, functions of the form (2.105) are dense in $L^2(0, T; V)$, and hence (2.107) holds for all $\hat{\varphi} \in L^2(0, T; V)$. This implies that (2.90) is satisfied almost everywhere on $[0, T]$. Therefore $\hat{u}$ and $\zeta_{1,2}$ solve the weak formulation.

(b)–additional regularity: From $\zeta_1, \zeta_2 \in H^1(0, T; \mathbb{R}^n)$ follows the continuity of the controller functions, i.e. (2.94). It is easily seen from the construction of the weak solution and (2.102) that $\hat{u}$ satisfies (2.93). Result (2.95) follows immediately due to Lemma 2.28, after, possibly, a modification on a set of measure zero. Moreover, regularity (2.96) follows from Lemma 2.28 and Lemma 2.29.

(a)–initial conditions, uniqueness: It remains to show that $\hat{u}$, ζ_1, and ζ_2 satisfy the initial conditions. For this purpose, equation (2.90) is integrated by parts (in time), with $\hat{w} \in C^2([0, T]; V)$ such that $\hat{w}(T) = 0$ and $\hat{w}_t(T) = 0$:

$$\begin{aligned} &\int_0^T \left[(\hat{u}, \hat{w}_{tt})_H + a(\hat{u}, \hat{w}) + b(\hat{u}_t, \hat{w}) + e_1(\zeta_1, \hat{w}) + e_2(\zeta_2, \hat{w})\right] d\tau \\ &\quad = -(\hat{u}(0), \hat{w}_t(0))_H + {}_{V'}\langle \hat{u}_t(0), \hat{w}(0)\rangle_V. \end{aligned} \tag{2.108}$$

Similarly, for a fixed m it follows from (2.98):

$$\begin{aligned}\int_0^T & [(\hat{u}_m, \hat{w}_{tt})_H + a(\hat{u}_m, \hat{w}) + b((\hat{u}_m)_t, \hat{w}) + e_1(\zeta_{1m}, \hat{w}) + e_2(\zeta_{2m}, \hat{w})]\, d\tau \\ & = -(\hat{u}_{m0}, \hat{w}_t(0))_H + (\hat{v}_{m0}, \hat{w}(0))_H.\end{aligned} \tag{2.109}$$

Due to (2.97) and (2.103), passing to the limit in (2.109) along the convergent subsequence $\{\hat{u}_{m_l}\}$ gives

$$\begin{aligned}\int_0^T & [(\hat{u}, \hat{w}_{tt})_H + a(\hat{u}, \hat{w}) + b(\hat{u}_t, \hat{w}) + e_1(\zeta_1, \hat{w}) + e_2(\zeta_2, \hat{w})]\, d\tau \\ & = -(\hat{u}_0, \hat{w}_t(0))_H + (\hat{v}_0, \hat{w}(0))_H.\end{aligned} \tag{2.110}$$

Comparing (2.108) with (2.110), implies $\hat{u}(0) = \hat{u}_0$ and $\hat{u}_t(0) = \hat{v}_0$. Analogously, $\zeta_1(0) = \zeta_{1,0}$ and $\zeta_2(0) = \zeta_{2,0}$ is obtained.

In order to show uniqueness, let $(\hat{u}, \zeta_1, \zeta_2)$ be a solution to (2.90) and (2.91) with zero initial conditions. Let $s \in (0, T)$ be fixed, and set

$$\hat{U}(t) := \begin{cases} \int_t^s \hat{u}(\tau)\, d\tau, & t < s, \\ 0, & t \geq s, \end{cases}$$

and

$$Z_i(t) := \int_0^t \zeta_i(\tau)\, d\tau,$$

for $i = 1, 2$. Integrating (2.91) over $(0, t)$ yields with (1.9)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(Z_i^\top P_i Z_i)(t) &= -\frac{1}{2}\varepsilon_i Z_i^\top(t) P_i Z_i(t) - \frac{1}{2}(q_i \cdot Z_i(t) + \tilde{\delta}_i({}^i\hat{u}(t)))^2 \\ & \quad + (d_i - \delta_i)({}^i\hat{u}(t))^2 + Z_i(t) \cdot c_i({}^i\hat{u}(t)),\end{aligned} \tag{2.111}$$

for $0 \leq t \leq T$, $i = 1, 2$. Equation (2.90) is then integrated over $[0, T]$ with $\hat{w} = \hat{U}$. Performing partial integration in time, yields:

$$\begin{aligned}\int_0^s & (\hat{u}_t(\tau), \hat{u}(\tau))_H - a(\hat{U}_t(\tau), \hat{U}(\tau)) + b(\hat{u}(\tau), \hat{u}(\tau))\, d\tau \\ & + \sum_{i=1}^2 \int_0^s Z_i(\tau) \cdot c_i({}^i\hat{u}(\tau))\, d\tau = 0.\end{aligned} \tag{2.112}$$

From (2.111) and (2.112) follows

$$\int_0^s \frac{d}{dt}\left(\frac{1}{2}\|\hat{u}(\tau)\|_H^2 - \frac{1}{2}a(\hat{U}(\tau), \hat{U}(\tau)) + \frac{1}{2}\sum_{i=1}^2 Z_i^\top(\tau) P_i Z_i(\tau)\right) d\tau$$

$$= -\sum_{i=1}^{2}\int_0^s \left(\delta_i({}^i\hat{u}(\tau))^2 + \frac{\varepsilon_i}{2}Z_i^\top(\tau)P_iZ_i(\tau) + \frac{1}{2}(q_i\cdot Z_i(\tau) + \tilde{\delta}_i({}^i\hat{u})(\tau))^2\right)\,d\tau.$$

Therefore,

$$\frac{1}{2}\|\hat{u}(s)\|_H^2 + \frac{1}{2}a(\hat{U}(0),\hat{U}(0)) + \sum_{i=1}^{2}\frac{1}{2}Z_i^\top(s)P_iZ_i(s) \quad\leq\quad 0.$$

The matrices $P_j, j = 1,2$ are positive definite, and the bilinear form $a(.,.)$ is coercive. Hence $\hat{u}(s) = 0$, $\hat{U}(0) = 0$, and $Z_i(s) = 0$. Since $s \in (0,T)$ was arbitrary, $\hat{u} \equiv 0$, $\zeta_i \equiv 0$, $i = 1,2$ follows. □

2.2.3 Higher regularity results

In this subsection, it will be demonstrated that even stronger continuity holds for the weak solution $\hat{u}$ solving (2.90) – (2.92).

Theorem 2.31. *After, possibly, a modification on a set of measure zero, the weak solution $\hat{u}$ of (2.90)-(2.92) satisfies*

$$\hat{u} \in C([0,T];V), \tag{2.113}$$

$$\hat{u}_t \in C([0,T];H), \tag{2.114}$$

$$\zeta_j \in C^1([0,T];\mathbb{R}^n). \tag{2.115}$$

Before the proof of the continuity in time of the weak solution, a definition and a lemma are stated.

Definition 2.32. Let Y be a Banach space. Then

$$C_w([0,T];Y) \;:=\; \{w \in L^\infty(0,T;Y)\colon\ t \mapsto \langle f, w(t)\rangle \text{ is continuous on } [0,T],\ \forall f \in Y'\}.$$

denotes the space of *weakly continuous functions* with values in Y.

The following Lemma was stated and proved in [44] (Chapter 8.4, pp. 275).

Lemma 2.33. *Let X, Y be Banach spaces, $X \subset Y$ with continuous injection, X reflexive. Then*

$$L^\infty(0,T;X) \cap C_w(0,T;Y) = C_w(0,T;X).$$

Proof of Theorem 2.31 . Note that it suffices to show that (2.113) and (2.114) holds. Regularity (2.115) then follows easily from (2.91). This proof is an adaption of standard strategies to the situation at hand (cf. Section 8.4 in [44] and Section 2.4 in [65]). Using Lemma 2.33 with $X = V$, $Y = H$, it follows from (2.93) and (2.95) that $\hat{u} \in C_w([0,T];V)$. Similarly, (2.93) and (2.96) imply $\hat{u}_t \in C_w([0,T];H)$.

Next, the scalar cut-off function $O_I \in C^\infty(\mathbb{R})$ is taken, such that it equals one on some interval $I \subset\subset [0,T]$, and zero on $\mathbb{R} \setminus [0,T]$. Then the functions $O_I\hat{u} : \mathbb{R} \to V$ and $O_I\zeta_1, O_I\zeta_2 : \mathbb{R} \to \mathbb{R}^n$ are compactly supported. Let $\eta^\varepsilon : \mathbb{R} \to \mathbb{R}$ be a standard mollifier in time. For example, η^ε may be given by

$$\eta^\varepsilon(t) := \frac{1}{\varepsilon}\eta\left(\frac{t}{\varepsilon}\right),$$

where

$$\eta(t) := \begin{cases} e^{\frac{1}{1-|t|^2}}, & |t| < 1, \\ 0, & |t| \geq 1. \end{cases}$$

Following definitions are introduced:

$$\begin{aligned} \hat{u}^\varepsilon &:= \eta^\varepsilon * O_I\hat{u} \in C_c^\infty(\mathbb{R}, V), \\ \zeta_1^\varepsilon &:= \eta^\varepsilon * O_I\zeta_1 \in C_c^\infty(\mathbb{R}, \mathbb{R}^n), \\ \zeta_2^\varepsilon &:= \eta^\varepsilon * O_I\zeta_2 \in C_c^\infty(\mathbb{R}, \mathbb{R}^n). \end{aligned}$$

Now ζ_1^ε and ζ_2^ε converge uniformly on I to ζ_1 and ζ_2, respectively. Moreover, $\hat{u}^\varepsilon$ converges to $\hat{u}$ in V, and $\hat{u}_t^\varepsilon$ to $\hat{u}_t$ in H a.e. on I. Then, $\hat{E}(t;\hat{u}^\varepsilon,\zeta_1^\varepsilon,\zeta_2^\varepsilon)$ converges to $\hat{E}(t;\hat{u},\zeta_1,\zeta_2)$ a.e. on I as well. Since $\hat{u}^\varepsilon, \zeta_1^\varepsilon, \zeta_2^\varepsilon$ are smooth, a straightforward calculation on I yields

$$\frac{d}{dt}\hat{E}(t;\hat{u}^\varepsilon,\zeta_1^\varepsilon,\zeta_2^\varepsilon) = \hat{F}(t;\hat{u}^\varepsilon,\zeta_1^\varepsilon,\zeta_2^\varepsilon), \tag{2.116}$$

with $\hat{F}$ defined in (2.101). Passing to the limit in (2.116) as $\varepsilon \to 0$

$$\frac{d}{dt}\hat{E}(t;\hat{u},\zeta_1,\zeta_2) = \hat{F}(t;\hat{u},\zeta_1,\zeta_2) \tag{2.117}$$

holds in the sense of distributions on I. Since I was arbitrary, (2.117) holds on all compact subintervals of $(0,T)$. Now $t \mapsto \hat{E}(t;\hat{u},\zeta_1,\zeta_2)$ is an integral of an L^1-function. Note that the input functions of $\hat{F}$ satisfy ${}^1\hat{u}_t, {}^2\hat{u}_t \in L^2(0,T))$, so $\hat{F}$ is absolutely continuous.

For a fixed t, let $\lim_{n\to\infty} t_n = t$ and let the sequence χ_n be defined by

$$\begin{aligned} \chi_n := {} & \frac{1}{2}\|\sqrt{\Lambda}(\hat{u}(t) - \hat{u}(t_n))\|_V^2 + \frac{1}{2}\|\hat{u}_t(t) - \hat{u}_t(t_n)\|_H^2 \\ & + \frac{k_1}{2}({}^1\hat{u}(t) - {}^1\hat{u}(t_n))^2 + \frac{k_2}{2}({}^2\hat{u}(t) - {}^2\hat{u}(t_n))^2 \\ & + \frac{1}{2}(\zeta_1(t) - \zeta_1(t_n))^\top P_1(\zeta_1(t) - \zeta_1(t_n)) \\ & + \frac{1}{2}(\zeta_2(t) - \zeta_2(t_n))^\top P_2(\zeta_2(t) - \zeta_2(t_n)). \end{aligned}$$

Then

$$\chi_n = \hat{E}(t;\hat{u},\zeta_1,\zeta_2) + \hat{E}(t_n;\hat{u},\zeta_1,\zeta_2) - (\Lambda\hat{u}(t),\hat{u}(t_n))_V - (\hat{u}_t(t),\hat{u}_t(t_n))_H$$

$$-k_1\, {}^1\hat{u}(t){}^1\hat{u}(t_n) - k_2\, {}^2\hat{u}(t){}^2\hat{u}(t_n) - \zeta_1(t)^\top P_1 \zeta_1(t_n) - \zeta_2(t)^\top P_2 \zeta_2(t_n).$$

Due to the t-continuity of the energy function, weak continuity of $\hat{u}, \hat{u}_t$, and continuity of ζ_1, ζ_2, it follows

$$\lim_{n\to\infty} \chi_n = 0.$$

Finally, this implies that

$$\begin{aligned} \lim_{n\to\infty} \|\hat{u}_t(t) - \hat{u}_t(t_n)\|_H^2 &= 0, \\ \lim_{n\to\infty} \|\hat{u}(t) - \hat{u}(t_n)\|_V^2 &= 0, \end{aligned}$$

which proves the theorem. □

2.3 Dissipative FEM method

The goal of this section is to develop a stable and convergent numerical method which faithfully describes the behavior of the system (2.1)–(2.6). From (2.10) it is known that the norm of the solution $z(t)$ of the evolution formulation (2.8) decreases in time:

$$\begin{aligned} \frac{d}{dt}\|z\|_{\mathcal{H}}^2 &= -\delta_1 \left(\frac{\xi}{J}\right)^2 - \frac{1}{2}\left(\zeta_1 \cdot q_1 + \tilde{\delta}_1 \frac{\xi}{J}\right)^2 \\ &\quad -\delta_2 \left(\frac{\psi}{M}\right)^2 - \frac{1}{2}\left(\zeta_2 \cdot q_2 + \tilde{\delta}_2 \frac{\psi}{M}\right)^2 \\ &\quad -\frac{\varepsilon_1}{2}\zeta_1^\top P_1 \zeta_1 - \frac{\varepsilon_2}{2}\zeta_2^\top P_2 \zeta_2 \le 0, \end{aligned} \tag{2.118}$$

where $\tilde{\delta}_j = \sqrt{2(d_j - \delta_j)}$, $j = 1, 2$. Note that the r.h.s. of (2.118) only involves boundary terms of the beam and the control variables. Hence, $\frac{d}{dt}\|z\|_{\mathcal{H}}^2 = 0$ does *not* imply $z = 0$ (which can easily be verified from (2.8)).

Therefore, it is important that the corresponding numerical method also preserves this structural property of dissipativity. The importance of this feature is twofold: For long-time computations, the numerical scheme must of course be convergent in the classical sense (i.e. on finite time intervals) but also yield the correct large-time limit. Moreover, dissipativity of the scheme implies immediately unconditional stability.

In this section first a time-continuous and then a time-discrete FEM shall be developed, such that they dissipate the norm in time. The main results on the convergence of the numerical schemes are stated in Theorems 2.35 and 2.38.

The different options to proceed shall be briefly discussed. Evolution formulation (2.8) is an inconvenient starting point for deriving a weak formulation due to the high boundary traces of u at $x = L$: The natural regularity of a weak solution would be $u \in C([0,\infty); \tilde{H}_0^2(0,L))$, $v = u_t \in C([0,\infty); L^2(0,L))$. Hence, the terms $u_{xx}(t,L)$, $u_{xxx}(t,L)$ in (2.8) could only be incorporated by resorting to the boundary conditions (2.4), (2.5). Therefore, in this approach it is rather started from the original second order system (2.1)–(2.6).

2.3.1 Semi-discrete scheme

In this subsection, first a FEM method for discretization in space is presented, followed by the dissipativity argumentation. Finally, a choice of an appropriate finite element space for the proposed method is discussed and a-priori error estimates are obtained.

2.3.1.1 Space discretization

Let $W_h \subset \tilde{H}_0^2(0,L)$ be an arbitrarily chosen finite dimensional space. It follows that its elements are globally $C^1[0,L]$, due to Sobolev embedding. Furthermore, let $w_j, j = 1,\dots,N$ be some fixed basis for W_h. As already seen in the proof of Theorem 2.30, the Galerkin approximation of (2.90) reads: Find $u_h \in C^2([0,\infty), W_h)$, i.e. $\hat{u}_h = ((u_h)_x(L), u_h(L), u_h) \in C^2([0,\infty), V)$, and $\tilde{\zeta}_{1,2} \in C^1([0,\infty), \mathbb{R}^n)$ with

$$\begin{aligned}
&\int_0^L \mu\,(u_h)_{tt} w_j\,dx + \int_0^L \Lambda\,(u_h)_{xx}(w_j)_{xx}\,dx + M(u_h)_{tt}(L)w_j(L) + J(u_h)_{xtt}(L)(w_j)_x(L)\\
&+\big[k_1(u_h)_x(L) + d_1(u_h)_{xt}(L) + c_1\cdot\tilde{\zeta}_1(t)\big](w_j)_x(L)\\
&+\big[k_2 u_h(L) + d_2(u_h)_t(L) + c_2\cdot\tilde{\zeta}_2(t)\big]w_j(L) = 0, \qquad j = 1,\dots,N,\ t>0,
\end{aligned} \tag{2.119}$$

coupled to the analogue of (2.91):

$$\begin{aligned}
(\tilde{\zeta}_1)_t &= A_1\tilde{\zeta}_1 + b_1(u_h)_{xt}(L),\\
(\tilde{\zeta}_2)_t &= A_2\tilde{\zeta}_2 + b_2(u_h)_t(L),
\end{aligned} \tag{2.120}$$

and the initial conditions

$$\begin{aligned}
u_h(0,.) &= u_{h,0} \in W_h,\\
(u_h)_t(0,.) &= v_{h,0} \in W_h,\\
\tilde{\zeta}_1(0) &= \zeta_{1,0} \in \mathbb{R}^n,\\
\tilde{\zeta}_2(0) &= \zeta_{2,0} \in \mathbb{R}^n.
\end{aligned} \tag{2.121}$$

Equation (2.119) is a second order ODE-system in time. Its solution can be expanded in the chosen basis, i.e.

$$u_h(t,x) = \sum_{i=1}^N U_i(t) w_i(x),$$

and its coefficients denoted by the vector

$$\mathbb{U} = \begin{bmatrix} U_1 & U_2 & \dots & U_N \end{bmatrix}^\top.$$

It will be said that $\mathbb{U}$ is the *vector representation* of the function u_h. This notation yields that (2.119) is equivalent to the vector equation:

$$\mathbb{A}\mathbb{U}_{tt} + \mathbb{B}\mathbb{U}_t + \mathbb{K}\mathbb{U} + (\mathbb{W}_1 \otimes c_1)\zeta_1 + (\mathbb{W}_2 \otimes c_2)\zeta_2 = 0, \tag{2.122}$$

where its coefficient matrices are defined by

$$\begin{aligned}
\mathbb{A}_{i,j} &:= \int_0^L \mu\, w_i w_j \, dx + M w_i(L) w_j(L) + J (w_i)_x(L)(w_j)_x(L), \\
\mathbb{B}_{i,j} &:= d_1 (w_i)_x(L)(w_j)_x(L) + d_2 w_i(L) w_j(L), \\
\mathbb{K}_{i,j} &:= \int_0^L \Lambda\, (w_i)_{xx}(w_j)_{xx} \, dx + k_1 (w_i)_x(L)(w_j)_x(L) + k_2 w_i(L) w_j(L),
\end{aligned}$$

for $i, j = 1, \dots, N$, and the coefficient vectors as

$$\begin{aligned}
\mathbb{W}_1 &:= [(w_1)_x(L)\, (w_2)_x(L)\, \dots (w_N)_x(L)]^\top, \\
\mathbb{W}_2 &:= [w_1(L)\, w_2(L)\, \dots w_N(L)]^\top.
\end{aligned}$$

The matrix $\mathbb{K}$ is symmetric positive definite, since there holds $k_{1,2} > 0$. Since $\mathbb{A}$ is symmetric positive definite, one sees very easily that the initial value problem (2.119), (2.120), and (2.121) is uniquely solvable.

2.3.1.2 Dissipativity of the method

Next, the dissipativity of the semi-discrete scheme is demonstrated. As an analogue of the norm $\|z(t)\|_{\mathcal{H}}$ defined in Subsection 2.1.1, the following time dependent functional for a trajectory $u \in C^2([0,\infty); \tilde{H}_0^2(0,L))$ and $\zeta_{1,2} \in C^1([0,\infty); \mathbb{R}^n)$ is defined first:

$$\begin{aligned}
E(t; u, \zeta_1, \zeta_2) &:= \frac{1}{2} \int_0^L \left(\Lambda u_{xx}(t,x)^2 + \mu u_t(t,x)^2 \right) dx + \frac{M}{2} u_t(t,L)^2 + \frac{J}{2} u_{xt}(t,L)^2 \\
&\quad + \frac{k_1}{2} u_x(t,L)^2 + \frac{k_2}{2} u(t,L)^2 + \frac{1}{2} \zeta_1^\top(t) P_1 \zeta_1(t) + \frac{1}{2} \zeta_2^\top(t) P_2 \zeta_2(t).
\end{aligned}$$

For a classical solution of (2.8) in $D(\mathcal{A})$, it holds $E(t; u, \zeta_1, \zeta_2) = \|z(t)\|_{\mathcal{H}}^2$.

Theorem 2.34. *Let $u_h \in C^2([0,\infty); \tilde{H}_0^2(0,L))$ and $\tilde{\zeta}_{1,2} \in C^1([0,\infty); \mathbb{R}^n)$ solve* (2.119), (2.120). *Then it holds for $t > 0$:*

$$\begin{aligned}
\frac{d}{dt} E(t; u_h, \tilde{\zeta}_1, \tilde{\zeta}_2) &= -\frac{\varepsilon_1}{2} \tilde{\zeta}_1^\top P_1 \tilde{\zeta}_1 - \frac{1}{2} \left(\tilde{\zeta}_1 \cdot q_1 + \tilde{\delta}_1 (u_h)_{xt}(L) \right)^2 - \delta_1 (u_h)_{xt}(L)^2 \\
&\quad - \frac{\varepsilon_2}{2} \tilde{\zeta}_2^\top P_2 \tilde{\zeta}_2 - \frac{1}{2} \left(\tilde{\zeta}_2 \cdot q_2 + \tilde{\delta}_2 (u_h)_t(L) \right)^2 - \delta_2 (u_h)_t(L)^2 \le 0.
\end{aligned}$$

Proof. Equation (2.119) with the test function $w_h = (u_h)_t$ is used in the following computation:

$$\begin{aligned}
\frac{d}{dt}E(t;u_h,\tilde{\zeta}_1,\tilde{\zeta}_2) &= \int_0^L \Lambda\,(u_h)_{xx}(u_h)_{xxt}\,dx + \int_0^L \mu\,(u_h)_t(u_h)_{tt}\,dx \\
&\quad + M(u_h)_t(L)(u_h)_{tt}(L) + J(u_h)_{tx}(L)(u_h)_{ttx}(L) \\
&\quad + k_1(u_h)_x(L)(u_h)_{xt}(L) + k_2(u_h)(L)(u_h)_t(L) \\
&\quad + \tilde{\zeta}_1^\top P_1(\tilde{\zeta}_1)_t + \tilde{\zeta}_2^\top P_2(\tilde{\zeta}_2)_t \\
&= -d_1(u_h)_{xt}(L)^2 - d_2(u_h)_t(L)^2 \\
&\quad - c_1\cdot\tilde{\zeta}_1(u_h)_{xt}(L) - c_2\cdot\tilde{\zeta}_2(u_h)_t(L) + \tilde{\zeta}_1^\top P_1(\tilde{\zeta}_1)_t + \tilde{\zeta}_2^\top P_2(\tilde{\zeta}_2)_t,
\end{aligned}$$

and the result follows with (2.120) and (1.9). □

Note that it has been shown in the proof of Theorem 2.30, that the energy functional for the weak solution $(\hat{u},\zeta_1,\zeta_2)$ of (2.90) - (2.92) has an analogous dissipative property (2.117).

2.3.1.3 Piecewise cubic Hermite polynomials

In this subsection, the choice of an appropriate discrete space for the FEM is discussed. For notational simplicity, a uniform distribution of nodes on $[0,L]$ is assumed:

$$x_m = mh, \quad m \in \{0,1,\dots,P\},$$

where $h = \frac{L}{P}$. A standard choice for the discrete space W_h is a space of piecewise cubic polynomials with both displacement and slope continuity across element boundaries, also called Hermitian cubic polynomials (see e.g. [60], [6]). They are not only employed for the Euler-Bernoulli beam, but often for Timoshenko beams (cf. [25]) as well. This space is often denoted by $H_3(\pi)$, where $\pi = (x_m)_{m=0}^P$ stands for the discretization of the domain (notation as in [57]). In particular, for a fixed $s \in H_3(\pi)$, it holds that $p_m := s|_{[x_{m-1},x_m]} \in P_3([x_{m-1},x_m])$, $m = 1,\dots,P$. Due to the continuity of s and its derivative across the nodes, the following needs to hold:

$$\begin{aligned}
p_m(x_m) &= s(x_m) = p_{m+1}(x_m), \\
p'_m(x_m) &= s'(x_m) = p'_{m+1}(x_m),
\end{aligned}$$

$m = 1,\dots P$. Therefore, s is uniquely determined by $\{s(x_m), s'(x_m),\ m = 1,\dots,P\}$. Hence, the nodal values of a function and of its derivative are the associated degrees of freedom.

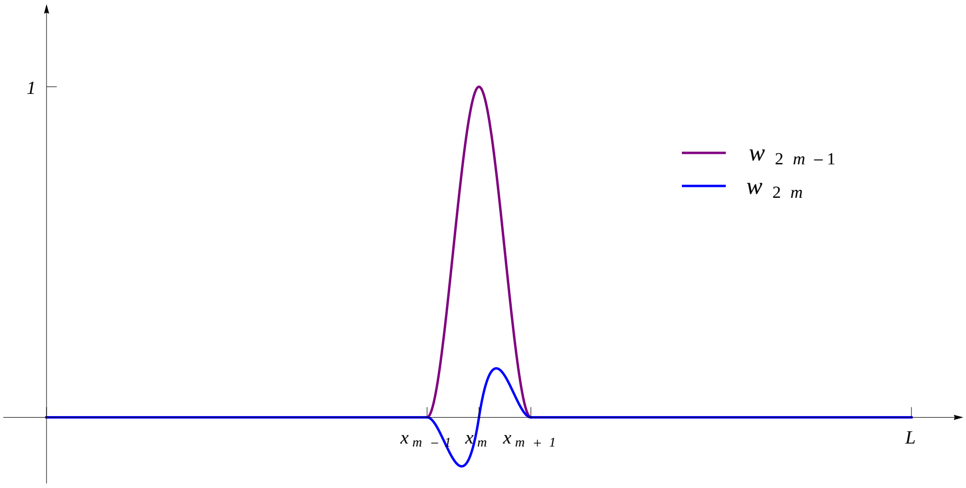


Figure 2.2: Basis functions w_{2m-1}, w_{2m} associated to discretization node x_m

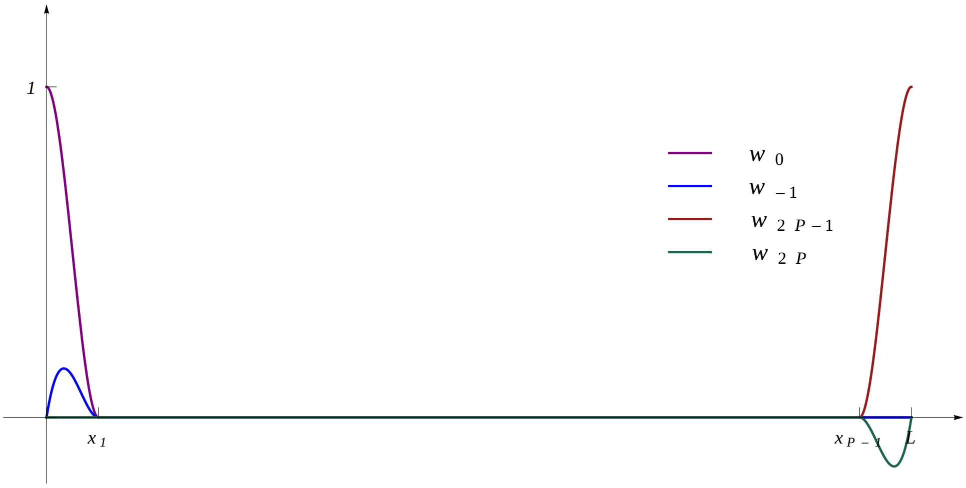


Figure 2.3: Basis functions associated to $x_0 = 0$ and $x_P = L$

To define a corresponding basis for $H_3(\pi)$, two piecewise cubic functions are associated with each node $x_m,\ m \geq 1$:

$$
\begin{aligned}
w_{2m-1}(x_k) &= \begin{cases} 1, & m = k \\ 0, & m \neq k \end{cases} \qquad & w'_{2m-1}(x_k) = 0, \\
w'_{2m}(x_k) &= \begin{cases} 1, & m = k \\ 0, & m \neq k \end{cases} \qquad & w_{2m}(x_k) = 0,
\end{aligned}
\tag{2.123}
$$

for all $k = 0, \ldots, P$. Note that such functions exist and are unique (see Figure 2.2). Because of the property (2.123), they are known as the cardinal basis. Moreover, they can be written in their explicit form:

$$
\begin{aligned}
w_{2m-1}(x) &= \begin{cases} \left(\frac{|x-x_m|}{h}-1\right)^2\left(2\frac{|x-x_m|}{h}+1\right), & x\in[x_{m-1},x_{m+1}] \\ 0, & \text{otherwise} \end{cases} \\
w_{2m}(x_k) &= \begin{cases} \left(\frac{|x-x_m|}{h}-1\right)^2(x-x_m), & x\in[x_{m-1},x_{m+1}] \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
\tag{2.124}
$$

for $1 \le m \le P-1$. For $m = 0$, and $m = P$ the same expression holds, only the intervals on which the functions are nontrivial are restricted to $[x_0, x_1]$, and $[x_{P-1}, x_P]$, respectively (see Figure 2.3). Thus, $\forall s \in H_3(\pi)$

$$s(x) = \sum_{m=1}^{P} (s(x_m)w_{2m-1}(x) + s'(x_m)w_{2m}(x)).$$

Due to the boundary conditions at $x = 0$ in $W_h \subset \tilde{H}_0^2(0,L)$, the functions w_{-1} and w_0 associated to the node $x_0 = 0$ can be excluded from the basis set. Thus, $N = 2P$. For the coupling to the control variables in (2.120), the boundary values of u_h at $x = L$ shall be employed. An advantage of this choice of discrete space and its basis is that it yields the simple relations $u_h(t,L) = U_{N-1}(t)$, $(u_h)_x(t,L) = U_N(t)$. Moreover, the compact support of the basis functions $\{w_j\}_{j=1}^N$, leads to a sparse structure of the matrices $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{K}$: $\mathbb{A}$ and $\mathbb{K}$ are tridiagonal, $\mathbb{B}$ is diagonal with only two non-zero elements $\mathbb{B}_{N-1,N-1} = d_2$, $\mathbb{B}_{N,N} = d_1$. And the vector $\mathbb{C} := (\mathbb{W}_1 \otimes c_1)\zeta_1 + (\mathbb{W}_2 \otimes c_2)\zeta_2$ has all zero entries except for $\mathbb{C}_{N-1} = c_2 \cdot \tilde{\zeta}_2$, $\mathbb{C}_N = c_1 \cdot \tilde{\zeta}_1$.

2.3.1.4 A-priori error estimates

In this subsection, the a-priori error estimates for the Galerkin solution to (2.119) and (2.120) shell be derived, where the discrete space W_h is the space of Hermite cubic polynomials as introduced in Subsection 2.3.1.3. Thereby, the common method for obtaining error estimates (cf. [16]) will be adjusted to the problem at hand. Since using cubic polynomials for the space approximation, accuracy of order two in space (in $H^2(0,L)$) is obtained. The Hermite interpolation of the weak solution u to W_h is denoted by $\tilde{u}$:

$$\tilde{u}(t,x) = \sum_{m=1}^{P} u(t,x_m)w_{2m-1}(x) + \sum_{m=1}^{P} u_x(t,x_m)w_{2m}(x).$$

Assuming that

$$
\begin{aligned}
&u \in C([0,T]; \tilde{H}_0^4(0,L)), \\
&u_t \in L^2(0,T; \tilde{H}_0^4(0,L)),
\end{aligned}
\tag{2.125}
$$

$$u_{tt} \in L^2(0,T;\tilde{H}_0^2(0,L)),$$

it can be seen (e.g. in [8], [16]) that a.e. in t:

$$\begin{aligned}
\|u-\tilde{u}\|_{H^2(0,L)} &\leq Ch^2\|u\|_{H^4(0,L)}, \\
\|u_t-\tilde{u}_t\|_{H^2(0,L)} &\leq Ch^2\|u_t\|_{H^4(0,L)}, \\
\|u_{tt}-\tilde{u}_{tt}\|_{L^2(0,L)} &\leq Ch^2\|u_{tt}\|_{H^2(0,L)}.
\end{aligned} \tag{2.126}$$

The error of the semi-discrete solution $(u_h, \tilde{\zeta}_1, \tilde{\zeta}_2)$ is defined as $\varepsilon_h := u_h - \tilde{u} \in W_h$ and $\zeta_i^e := \tilde{\zeta}_i - \zeta_i,\ i = 1,2$. Utilizing equations (2.119)–(2.120), it follows:

$$\begin{aligned}
&\int_0^L \mu\,(\varepsilon_h)_{tt} w \ \mathrm{d}x + \int_0^L \Lambda\,(\varepsilon_h)_{xx} w_{xx} \ \mathrm{d}x + M(\varepsilon_h)_{tt}(L)w(L) + J(\varepsilon_h)_{xtt}(L)w_x(L) \\
&\quad + \big(k_1(\varepsilon_h)_x(L) + d_1(\varepsilon_h)_{xt}(t,L) + c_1 \cdot \zeta_1^e(t)\big)\, w_x(L) \\
&\quad + \big(k_2\varepsilon_h(t,L) + d_2(\varepsilon_h)_t(t,L) + c_2 \cdot \zeta_2^e(t)\big)\, w(L) \\
&= \int_0^L \mu\,(u_{tt}-\tilde{u}_{tt}) w \ \mathrm{d}x + \int_0^L \Lambda\,(u_{xx}-\tilde{u}_{xx}) w_{xx} \ \mathrm{d}x, \qquad \forall w \in W_h,\ t>0,
\end{aligned}$$

coupled to:

$$\begin{aligned}
(\zeta_1^e)_t(t) &= A_1\zeta_1^e(t) + b_1(\varepsilon_h)_{xt}(t,L), \\
(\zeta_2^e)_t(t) &= A_2\zeta_2^e(t) + b_2(\varepsilon_h)_t(t,L).
\end{aligned}$$

Using $w = (\varepsilon_h)_t$ and proceeding as in the proof of Theorem 2.34 yields:

$$\frac{1}{2}\frac{d}{dt}E(t;\varepsilon_h,\zeta_1^e,\zeta_2^e) \leq \int_0^L \mu\,(u_{tt}-\tilde{u}_{tt})(\varepsilon_h)_t \ \mathrm{d}x + \int_0^L \Lambda\,(u_{xx}-\tilde{u}_{xx})(\varepsilon_h)_{txx} \ \mathrm{d}x, \tag{2.127}$$

for a.e. $t \in [0,T]$. Integrating (2.127) in time, and performing partial integration, it follows:

$$\begin{aligned}
E(t;\varepsilon_h,\zeta_1^e,\zeta_2^e) &\leq E(0;\varepsilon_h(0),\zeta_1^e(0),\zeta_2^e(0)) \\
&+ 2\int_0^t\int_0^L \mu\,(u_{tt}(s,x)-\tilde{u}_{tt}(s,x))(\varepsilon_h)_t(s,x) \ \mathrm{d}x \ \mathrm{d}s \\
&+ 2\int_0^L \Lambda\,(u_{xx}(t,x)-\tilde{u}_{xx}(t,x))(\varepsilon_h)_{xx}(t,x) \ \mathrm{d}x \\
&+ 2\int_0^L \Lambda\,(u_{xx}(0,x)-\tilde{u}_{xx}(0,x))(\varepsilon_h)_{xx}(0,x) \ \mathrm{d}x
\end{aligned} \tag{2.128}$$

$$- \quad 2\int_0^t \int_0^L \Lambda\,(u_{txx}(s,x) - \tilde{u}_{txx}(s,x))(\varepsilon_h)_{xx}(s,x)\ \mathrm{d}x\ \mathrm{d}s.$$

Applying Cauchy-Schwarz to (2.128) yields:

$$\begin{aligned} E(t;\varepsilon_h,\zeta_1^e,\zeta_2^e) \ &\leq\ E(0;\varepsilon_h(0),\zeta_1^e(0),\zeta_2^e(0)) \\ &+\ \overline{\mu}\|u_{tt} - \tilde{u}_{tt}\|^2_{L^2(0,T;L^2(0,L))} + \overline{\mu}\int_0^t \|(\varepsilon_h)_t(s,.)\|^2_{L^2(0,L)}\ \mathrm{d}s \\ &+\ 8\overline{\Lambda}\|u_{xx}(t,.) - \tilde{u}_{xx}(t,.)\|^2_{L^2(0,L)} + \frac{\overline{\Lambda}}{8}\|(\varepsilon_h)_{xx}(t,.)\|^2_{L^2(0,L)} \qquad (2.129) \\ &+\ 8\overline{\Lambda}\|u_{xx}(0,.) - \tilde{u}_{xx}(0,.)\|^2_{L^2(0,L)} + \frac{\overline{\Lambda}}{8}\|(\varepsilon_h)_{xx}(0,.)\|^2_{L^2(0,L)} \\ &+\ \overline{\Lambda}\|u_t - \tilde{u}_t\|^2_{L^2(0,T;H^2(0,L))} + \Lambda\int_0^t \|(\varepsilon_h)_{xx}(s,.)\|^2_{L^2(0,L)}\ \mathrm{d}s, \end{aligned}$$

where $\overline{\Lambda} := \max_{x\in[0,L]} \Lambda(x)$ and $\overline{\mu} := \max_{x\in[0,L]} \mu(x)$. Utilizing equation (2.126), it is obtained that:

$$\begin{aligned} \frac{3}{4}E(t;\varepsilon_h,\zeta_1^e,\zeta_2^e) \ &\leq\ \frac{5}{4}E(0;\varepsilon_h(0),\zeta_1^e(0),\zeta_2^e(0)) + 2\int_0^t E(s;\varepsilon_h,\zeta_1^e,\zeta_2^e)\ \mathrm{d}s \\ &+Ch^4\left(\|u\|^2_{C([0,T];H^4(0,L))} + \|u_t\|^2_{L^2(0,T;H^4(0,L))} + \|u_{tt}\|^2_{L^2(0,T;H^2(0,L))}\right). \end{aligned} \qquad (2.130)$$

Gronwall's inequality applied to (2.130) gives:

$$\begin{aligned} E(t;\varepsilon_h,\zeta_1^e,\zeta_2^e) \ &\leq\ C\Big(E(0;\varepsilon_h(0),\zeta_{1e}(0),\zeta_{2e}(0)) \\ &+ h^4\left(\|u\|^2_{C([0,T];H^4(0,L))} + \|u_t\|^2_{L^2(0,T;H^4(0,L))} + \|u_{tt}\|^2_{L^2(0,T;H^2(0,L))}\right)\Big). \end{aligned} \qquad (2.131)$$

Finally, the following result holds:

Theorem 2.35. *Assume (2.125), and take W_h to be the space of the piecewise cubic Hermite polynomials. The following error estimate holds for $u_h \in C^2([0,T];W_h)$ and $\zeta_{1,2} \in C^1([0,T];\mathbb{R}^n)$ solving* (2.119), (2.120)*:*

$$\begin{aligned} E(t;u_h - u,\tilde{\zeta}_1 - \zeta_1,\tilde{\zeta}_2 - \zeta_2)^{\frac{1}{2}} \leq C\Big(E(0;\varepsilon_h(0),\zeta_{1e}(0),\zeta_{2e}(0))^{\frac{1}{2}} \\ + h^2\left(\|u_{tt}\|_{L^2(0,T;H^2(0,L))} + \|u_t\|_{L^2(0,T;H^4(0,L))} + \|u\|_{C([0,T];H^4(0,L))}\right)\Big), \end{aligned} \qquad (2.132)$$

$0 \leq t \leq T$. Furthermore, if $\tilde{\zeta}_1(0)$ and $\tilde{\zeta}_2(0)$ are chosen as in (2.121)*, $u_{h,0}$ and $v_{h,0}$ are Hermite interpolations of u_0 and v_0 respectively, then:*

$$\begin{aligned} E(t; u_h - u, \tilde{\zeta}_1 - \zeta_1, \tilde{\zeta}_2 - \zeta_2)^{\frac{1}{2}} &\leq Ch^2 \left(\|u_{tt}\|_{L^2(0,T;H^2(0,L))} \right. \\ &\left. + \|u_t\|_{L^2(0,T;H^4(0,L))} + \|u\|_{C([0,T];H^4(0,L))} \right), \end{aligned}$$

Proof. The result follows from (2.126), (2.131), and the triangle inequality. □

2.3.2 Fully-discrete scheme

The goal of this subsection is to perform discretization of the system (2.119)-(2.120), i.e. (2.122) in time, in such a way that the dissipation of the system energy is preserved. For this purpose, the system is first written as a first order system and then the Crank-Nicolson scheme is used, which is shown to be crucial for the dissipativity of the scheme. Lastly, the a-priori error estimated are obtained.

In order to write the system as an first order ODE, $v_h := (u_h)_t$ is introduced, and furthermore let $\mathbb{V} := \mathbb{U}_t = [\, V_1 \ V_2 \ \ldots \ V_N \,]^\top$ be its representation in the basis $\{w_j\}$. In what follows, the solution of the semi-discretized system (2.119), (2.120) is denoted in a vector form: $z_h = [\, u_h \ v_h \ \tilde{\zeta}_1 \ \tilde{\zeta}_2 \,]^\top$. In contrast to Subsection 2.1.1, the boundary traces $v_h(L)$, $(v_h)_x(L)$ need not to be included since in the finite dimensional case both u_h and v_h are in $\tilde{H}_0^2(0, L)$. In analogy to Subsection 2.1.1, the natural norm of $z_h = z_h(t)$ is defined as

$$\begin{aligned} \|z_h\|^2 &:= \frac{1}{2}\int_0^L \Lambda\, (u_h)_{xx}^2 \, dx + \frac{1}{2}\int_0^L \mu\, v_h^2 \, dx + \frac{M}{2} v_h^2(L) + \frac{J}{2}(v_h)_x^2(L) \\ &\quad + \frac{k_1}{2}(u_h)_x^2(L) + \frac{k_2}{2} u_h^2(L) + \frac{1}{2}\tilde{\zeta}_1^\top P_1 \tilde{\zeta}_1 + \frac{1}{2}\tilde{\zeta}_2^\top P_2 \tilde{\zeta}_2. \end{aligned}$$

2.3.2.1 Crank-Nicolson scheme

The time interval $[0, T]$ is discretized into S equidistant subintervals, for a fixed $S \in \mathbb{N}$. Let $\Delta t := T/S$ denote the time step and

$$t_k = k\Delta t, \quad \forall k \in \{0, 1, \ldots, S\}, \tag{2.133}$$

represent the nodes of the discretization. For the approximation of the solution z_h at time $t = t_k$, the notation $z^k = [u^k \ v^k \ \zeta_1^k \ \zeta_2^k]^\top$ shall be used. Let $\mathbb{U}^k, \mathbb{V}^k$ be the vector representations (in $\{w_j\}_{j=1}^N$) of u^k and v^k, respectively.

Furthermore, let the vector $\mathbb{C}^k$ be defined by:

$$\mathbb{C}^k := (\mathbb{W}_1 \otimes c_1)\zeta_1^k + (\mathbb{W}_2 \otimes c_2)\zeta_2^k.$$

The Crank-Nicolson scheme for (2.122), (2.120) then reads:

$$\frac{\mathbb{U}^{k+1}-\mathbb{U}^k}{\Delta t} = \frac{1}{2}(\mathbb{V}^{k+1}+\mathbb{V}^k), \tag{2.134}$$

$$\frac{\mathbb{A}\mathbb{V}^{k+1}-\mathbb{A}\mathbb{V}^k}{\Delta t} = -\frac{1}{2}(\mathbb{K}\mathbb{U}^{k+1}+\mathbb{K}\mathbb{U}^k)-\frac{1}{2}(\mathbb{B}\mathbb{V}^{k+1}+\mathbb{B}\mathbb{V}^k) -\frac{1}{2}(\mathbb{C}^{k+1}+\mathbb{C}^k), \tag{2.135}$$

$$\frac{\zeta_1^{k+1}-\zeta_1^k}{\Delta t} = A_1\frac{\zeta_1^{k+1}+\zeta_1^k}{2}+b_1\frac{v_x^{k+1}(L)+v_x^k(L)}{2}, \tag{2.136}$$

$$\frac{\zeta_2^{k+1}-\zeta_2^k}{\Delta t} = A_2\frac{\zeta_2^{k+1}+\zeta_2^k}{2}+b_2\frac{v^{k+1}(L)+v^k(L)}{2}. \tag{2.137}$$

Notice that if the chosen basis $\{w_j\}$ is the cardinal basis for the space of piecewise cubic Hermite polynomials as given in Subsection 2.3.1.3, the last term of (2.136), (2.137) reads $\left(V_N^{k+1}+V_N^k\right)/2$ and $\left(V_{N-1}^{k+1}+V_{N-1}^k\right)/2$, respectively.

2.3.2.2 Dissipativity of the method

In the following, it is shown that the fully discrete scheme (2.134)-(2.137) dissipates the norm. The somewhat lengthy proof is deferred to the Appendix A.

Theorem 2.36. *For $k\in\mathbb{N}_0$ it holds for the norm from* (2.133)*:*

$$\begin{aligned}\|z^{k+1}\|^2 &= \|z^k\|^2-\Delta t\left\{\delta_1\left(\frac{u_x^{k+1}(L)-u_x^k(L)}{\Delta t}\right)^2\right.\\ &+\frac{1}{2}\left(q_1\cdot\frac{\zeta_1^{k+1}+\zeta_1^k}{2}+\tilde{\delta}_1\frac{u_x^{k+1}(L)-u_x^k(L)}{\Delta t}\right)^2\\ &+\delta_2\left(\frac{u^{k+1}(L)-u^k(L)}{\Delta t}\right)^2+\frac{1}{2}\left(q_2\cdot\frac{\zeta_2^{k+1}+\zeta_2^k}{2}+\tilde{\delta}_2\frac{u^{k+1}(L)-u^k(L)}{\Delta t}\right)^2\\ &\left.+\frac{\varepsilon_1}{2}\frac{(\zeta_1^{k+1}+\zeta_1^k)^\top}{2}P_1\frac{\zeta_1^{k+1}+\zeta_1^k}{2}+\frac{\varepsilon_2}{2}\frac{(\zeta_2^{k+1}+\zeta_2^k)^\top}{2}P_2\frac{\zeta_2^{k+1}+\zeta_2^k}{2}\right\}.\end{aligned}$$

This decay of the norm is consistent (as $\Delta t\to 0$) with the decay (2.118) for the continuous case, and with the decay of the semi-discrete solution stated in Theorem 2.34. For the uncontrolled beam (i.e. $\Theta_1=\Theta_2=0$), Theorem 2.36 shows that $\|z^k\|$ is constant in k. This justifies the choice of the Crank-Nicolson time discretization.

Remark 2.37. Note that the scheme (2.134)–(2.137) and the norm dissipation property from Theorem 2.36 were written independently of the basis $\{w_j\}$. Hence, this decay property applies to any choice of the subspace $W_h\subset\tilde{H}_0^2(0,L)$. Same remark applies to Theorem 2.34.

2.3.2.3 A-priori error estimates

In this subsection, a-priori error estimates are given for the scheme (2.134)–(2.137). Thereby, additional regularity of the weak solutions u, ζ_1 and ζ_2 shall be assumed. Suppose that $u \in H^4(0,T;\tilde{H}_0^2(0,L))$ and $\zeta_1, \zeta_2 \in H^3(0,T;\mathbb{R}^n)$. Let $\breve{u} \in W_h$ be defined as the projection of the weak solution u, such that

$$a(\breve{u}(t), w_h) = a(u(t), w_h), \qquad \forall w_h \in W_h,$$

$\forall t \in [0,T]$. One easily verifies that it holds $\breve{u} \in H^4(0,T;\tilde{H}_0^2(0,L))$, since the projection $u \mapsto \breve{u}$ is bounded in $\tilde{H}_0^2(0,L)$. Furthermore, let $u^e := u - \breve{u}$ denote the error of the projection. Assuming $u \in H^2(0,T;\tilde{H}_0^4(0,L))$, the error estimates for $\breve{u}$ are obtained (cf. [61]):

$$\begin{aligned} \|u^e\|_{H^2(0,L)} &\leq Ch^2 \|u\|_{H^4(0,L)}, \\ \|u_t^e\|_{H^2(0,L)} &\leq Ch^2 \|u_t\|_{H^4(0,L)}, \\ \|u_{tt}^e\|_{H^2(0,L)} &\leq Ch^2 \|u_{tt}\|_{H^4(0,L)}. \end{aligned} \tag{2.138}$$

Let $z(t_k) = [u(t_k)\ u_t(t_k)\ \zeta_1(t_k)\ \zeta_2(t_k)]^\top$ denotes the weak solution (2.90) at time $t = t_k$, and $z^k = [u^k\ v^k\ \zeta_1^k\ \zeta_2^k]^\top$ the k-th iteration of the fully discrete scheme (2.134)–(2.137), approximating $z(t_k)$. Then the approximation error is defined by

$$\begin{aligned} \varepsilon^k &:= u^k - \breve{u}(t_k), \\ \Phi^k &:= v^k - \breve{u}_t(t_k), \\ \zeta_{e,i}^k &:= \zeta_i^k - \zeta_i(t_k), \quad i = 1, 2, \end{aligned}$$

and $z_e^k := [\varepsilon^k\ \Phi^k\ \zeta_{e,1}^k\ \zeta_{e,2}^k]^\top$, for every $k \in \{0, 1, \ldots, S\}$.

The second order error estimate (both in space and time) of the fully discrete scheme is obtained. However, due to the length of the proof, it is deferred to Appendix A.

Theorem 2.38. *Assume $u \in H^2(0,T;\tilde{H}_0^4(0,L)) \cap H^4(0,T;\tilde{H}_0^2(0,L))$ and $\zeta_1, \zeta_2 \in H^3([0,T];\mathbb{R}^n)$. Furthermore, let $k \in \{1, \ldots, S\}$. Then the following estimate holds:*

$$\begin{aligned} \|z^k - z(t_k)\| \ &\leq\ C\big[\|z_e^0\| + h^2\|u\|_{H^2(0,T;H^4(0,L))} + (\Delta t)^2 \big(\|u_{tt}\|_{L^2(0,T;H^4(0,L))} \\ &+\ \|u_{tt}\|_{H^2(0,T;H^2(0,L))} + \|(\zeta_1)_{tt}\|_{H^1(0,T;\mathbb{R}^n)} + \|(\zeta_2)_{tt}\|_{H^1(0,T;\mathbb{R}^n)}\big)\big]. \end{aligned}$$

Chapter 3

Euler-Bernoulli beam attached to a non-linear spring and a damper

In this chapter, the asymptotic behavior and numerical method for the system introduced in Section 1.3 is considered. First the equations of motion of the system are revised. The system consists of an EBB clamped at $x = 0$, and attached to a nonlinear spring and a nonlinear damper at $x = L$, as depicted in Figure 1.3. It is assumed that the force of the spring acting at the tip of the beam is given by $-s(u(t, L))$, and the force of the damper by $-d(u_t(t, L))$. Furthermore, functions $s, d \in C^2(\mathbb{R})$ are assumed to satisfy the following assumptions:

$$\int_0^z s(w)\,\mathrm{d}w \geq 0, \quad \forall z \in \mathbb{R}, \tag{3.1}$$

$$d'(z) \geq 0, \quad \forall z \in \mathbb{R}. \tag{3.2}$$

$$d(0) = 0, \tag{3.3}$$

and

$$|d(z)| \geq Dz^2, \qquad \text{for } |z| < \delta, \tag{3.4}$$

for some constants $D, \delta > 0$. The equations of motion of the system, which have been derived in Section 1.3, read as follows:

$$\mu u_{tt}(t, x) + \Lambda u_{xxxx}(t, x) = 0, \quad 0 < x < L,\ t > 0, \tag{3.5a}$$

$$u(t, 0) = u_x(t, 0) = 0, \quad t > 0, \tag{3.5b}$$

$$-\Lambda u_{xxx}(t, L) + m u_{tt}(t, L) + s(u(t, L)) + d(u_t(t, L)) = 0, \quad t > 0, \tag{3.5c}$$

$$\Lambda u_{xx}(t, L) + J u_{ttx}(t, L) = 0, \quad t > 0. \tag{3.5d}$$

From (3.5d) it can be seen that there is no external moment of inertia acting on the top of the beam.

This chapter is organized as follows. In Section 3.1 the existence and the uniqueness of the real-valued mild solution u of (3.5) will be demonstrated, followed by the study of the precompactness of the solution trajectory in Section 3.2, and its long-time behavior in Section 3.3. Sections 3.1–3.3 are joint work with Dipl. Ing. Dominik Stürzer, and the obtained results are presented in [49]. In addition to these results, in Section 3.4 a weak formulation of system (3.5) is introduced, followed by a dissipative numerical method presented in Section 3.5.

3.1 Existence and uniqueness of the mild solution

Since the nonlinear spring and damping force are defined on $\mathbb{R}$, only real-valued solutions of (3.5) are considered. Hence, if not explicitly stated, all functions occurring in this chapter are considered to be real-valued.

For this reason, subspaces of $L^2(0,L)$ and $H^k(0,L)$ which contain only real-valued functions are introduced:

$$
\begin{aligned}
L^2_{\mathbb{R}}(0,L) &:= \{f \in L^2(0,L) \,|\, f : [0,L] \to \mathbb{R}\},\\
H^k_{\mathbb{R}}(0,L) &:= \{f \in H^k(0,L) \,|\, f : [0,L] \to \mathbb{R}\}.
\end{aligned}
$$

However, all linear operators appearing in this chapter are assumed to be defined on a dense subset of a complex Hilbert space.

The aim of this section is to show that, for given sufficiently regular initial conditions $u(0,x) = u_0(x)$ and $u_t(0,x) = v_0(x)$, the system (3.5) has a unique (mild) solution u. Therefore, the problem is written as an evolution problem in the standard state space setting for Euler-Bernoulli beam with tip payload (as introduced in [40] and in Section 2.1). However, since the functions s and d are defined on $\mathbb{R}$ only, the following real Hilbert space is introduced:

$$
\mathcal{H} := \{y = [u, v, \xi, \psi]^\top : u \in \tilde{H}^2_{0,\mathbb{R}}(0,L), v \in L^2_{\mathbb{R}}(0,L), \xi, \psi \in \mathbb{R}\},
$$

where $\tilde{H}^n_{0,\mathbb{R}}(0,L) := \{f \in H^n_{\mathbb{R}}(0,L) : f(0) = f_x(0) = 0\}$ for $n \geq 2$. The inner product of the space $\mathcal{H}$ is given with

$$
\langle y_1, y_2\rangle_{\mathcal{H}} := \frac{\Lambda}{2}\int_0^L (u_1)_{xx}(u_2)_{xx}\,\mathrm{d}x + \frac{\mu}{2}\int_0^L v_1 v_2\,\mathrm{d}x + \frac{1}{2J}\xi_1\xi_2 + \frac{1}{2M}\psi_1\psi_2, \quad \forall y_1, y_2 \in \mathcal{H}.
$$

Let the linear operator A on $\mathcal{H}$ be given by:

$$
A(y) := \begin{bmatrix} v \\ -\frac{\Lambda}{\mu} u_{xxxx} \\ -\Lambda u_{xx}(L) \\ \Lambda u_{xxx}(L), \end{bmatrix} \tag{3.6}
$$

and defined on the dense domain

$$
D(A) := \{y \in \mathcal{H} : u \in \tilde{H}^4_{0,\mathbb{R}}(0,L), v \in \tilde{H}^2_{0,\mathbb{R}}(0,L), \xi = Jv_x(L), \psi = Mv(L)\}.
$$

The proof of the following Lemma is deferred to the Appendix A.

Lemma 3.1. *The linear operator A generates a C_0-semigroup $\left(e^{tA}\right)_{t\geq 0}$ of unitary operators in $\mathcal{H}$.*

Furthermore, a bounded nonlinear operator $\mathcal{N}$ on $\mathcal{H}$ is defined by:

$$\mathcal{N}(y) := \begin{bmatrix} 0 \\ 0 \\ 0 \\ -s(u(L)) - d(\frac{\psi}{m}) \end{bmatrix}.$$

With this notation the system (3.5) can be written formally as the following evolution equation in $\mathcal{H}$:

$$y_t = \mathcal{A}y, \tag{3.7a}$$

$$y(0) = y_0, \tag{3.7b}$$

for some initial condition $y_0 \in \mathcal{H}$, where the nonlinear operator $\mathcal{A} := A + \mathcal{N}$ is defined on the domain $D(\mathcal{A}) = D(A)$.

Definition 3.2. A solution $y(t)$ is said to be a classical solution of (3.7) on $[0, T]$ if

$$y \in C^1((0, T], \mathcal{H}) \cap C([0, T], D(A)),$$

initial condition (3.7b) is satisfied, and (3.7a) holds for all $t \in (0, T]$. Furthermore, a continuous function $y \in C([0, T], \mathcal{H})$ which satisfies the Duhamel formula

$$y(t) = \mathrm{e}^{tA}y_0 + \int_0^t \mathrm{e}^{(t-\tau)A}\mathcal{N}y(\tau)\,\mathrm{d}\tau, \quad t \in (0, T), \tag{3.8}$$

is called a mild solution of (3.7) on $[0, T]$, see [56].

A properly defined Lyapunov function on $\mathcal{H}$ shall prove to be essential for well-posedness and stability analysis of the system. In the Section 1.3 a candidate for a Lyapunov was obtained by (1.21). Therefore, for mild solutions of (3.7) the following functional is defined:

$$V(y) := \frac{\Lambda}{2}\int_0^L (u_{xx})^2\,\mathrm{d}x + \frac{\mu}{2}\int_0^L v^2\,\mathrm{d}x + \frac{1}{2M}\psi^2 + \frac{1}{2J}\xi^2 + \int_0^{u(L)} s(w)\,\mathrm{d}w. \tag{3.9}$$

Its derivative along the classical solutions of (3.7) satisfies:

$$\frac{\mathrm{d}}{\mathrm{d}t}V(y(t)) = -d\big(\frac{\psi}{M}(t)\big)\frac{\psi}{M}(t) \leq 0, \tag{3.10}$$

where the non-positivity is ensured by (3.2) and (3.3). It can trivially be seen that the functional V satisfies the following properties.

Lemma 3.3. *The function $V : \mathcal{H} \to \mathbb{R}$ is continuous with respect to the norm $\| \cdot \|_{\mathcal{H}}$. Moreover, for any $Y \subset \mathcal{H}$ there holds:*

$$\sup \{V(y) \colon y \in Y\} < \infty \quad \Leftrightarrow \quad \sup \{\|y\|_{\mathcal{H}} \colon y \in Y\} < \infty$$

Next result states the existence and uniqueness of the local mild solution.

Proposition 3.4. *For every $y_0 \in \mathcal{H}$ there exists a unique mild solution $y : [0, T_{\max}(y_0)) \to \mathcal{H}$, where $T_{\max}(y_0)$ is the maximal time interval for which the solution exists. If $T_{\max}(y_0) < \infty$ then a blow-up occurs, i.e.*

$$\lim_{t \nearrow T_{\max}} \|y(t)\|_{\mathcal{H}} = \infty.$$

Proof. Due to the assumptions made on d and s it follows that $\mathcal{N}$ is continuously differentiable on $\mathcal{H}$, and thus locally Lipschitz continuous. Furthermore, A generates a C_0-semigroup. Hence, according to Theorem B.5 stated in Appendix B, a unique mild solution exists on $[0, T_{\max})$, for some maximal $0 < T_{\max}(y_0) \leq \infty$. Moreover, if $T_{\max}(y_0) < \infty$ then $\lim_{t \nearrow T_{\max}} \|y(t)\|_{\mathcal{H}} = \infty$. □

Moreover, if the solution is classical, it is also global.

Lemma 3.5. *If $y_0 \in D(\mathcal{A})$ then the corresponding mild solution $y(t)$ is a classical solution. Furthermore $y(t)$ is a global solution, i.e. $T_{\max}(y_0) = \infty$.*

Proof. Since $\mathcal{N}$ is continuously differentiable, Theorem B.6 stated in Appendix B implies that $y(t)$ is a classical solution. Therefore (3.10) holds and implies:

$$V(y(t)) \leq V(y_0), \quad \forall t \in [0, T_{\max}).$$

Thus, according to Lemma 3.3 the norm $\|y(t)\|_{\mathcal{H}}$ stays uniformly bounded. Consequently, no blow-up occurs and $T_{\max} = \infty$. □

The following result is a consequence of Proposition B.7 stated in the Appendix B:

Proposition 3.6. *Let $y : [0, T] \to \mathcal{H}$ be a mild solution of (3.7) for some $y_0 \in \mathcal{H}$, and $T < \infty$. Also, let $\{y_{n0}\}_{n \in \mathbb{N}} \subset D(\mathcal{A})$ be such that $y_{n0} \to y_0$ in $\mathcal{H}$. Denote by $y_n(t)$ the classical solution of (3.7) to the initial value y_{n0}. Then $y_n \to y$ in $C([0, T]; \mathcal{H})$.*

Finally, this leads to the main result of the section, which states that the mild solution is global for any initial conditions in $\mathcal{H}$.

Theorem 3.7. *For every $y_0 \in \mathcal{H}$ the initial value problem (3.7) has a unique global mild solution, which is classical if $y_0 \in D(\mathcal{A})$. Moreover, the function $t \mapsto V(y(t))$ is non-increasing, and $\|y(t)\|_{\mathcal{H}}$ is uniformly bounded on $\mathbb{R}_0^+$.*

Proof. For an approximating sequence $\{y_n\}_{n\in\mathbb{N}}$ as in Proposition 3.6, it holds that

$$V(y(t)) = \lim_{n\to\infty} V(y_n(t)), \qquad \forall t \in [0, T_{\max}(y_0)),$$

since V is continuous. Due to (3.10), for the classical solution it follows that $t \mapsto V(y_n(t))$ is non-increasing for each fixed $n \in \mathbb{N}$, i.e.

$$V(y_n(t_1)) \geq V(y_n(t_2)) \qquad 0 \leq t_1 \leq t_2.$$

Passing on to the limit $n \to \infty$ in this inequality shows that $t \mapsto V(y(t))$ is non-increasing on $[0, T_{max})$. In particular this implies $\sup_{t\in[0,T_{max})} \|y(t)\|_{\mathcal{H}} < \infty$. Hence no blow-up can occur at $t = T_{max}$, and thus $T_{\max}(y_0) = \infty$. □

Corollary 3.8. *The function $V : \mathcal{H} \to \mathbb{R}$ is a Lyapunov function for the initial value problem* (3.7)*.*

Proof. According to Theorem 3.7, it follows that $t \mapsto V(y(t))$ is non-increasing for all $y_0 \in \mathcal{H}$. This implies $\dot{V}(y_0) \leq 0$, which proves the statement. □

Furthermore, Theorem 9.3.2 in [9] implies the following result.

Theorem 3.9. *Let the family of operators $\{S(t)\}_{t\leq 0}$, be defined by*

$$S(t)y_0 := y(t),$$

for every $y_0 \in \mathcal{H}$, and for all $t \geq 0$, where $y(t)$ is the mild solution of (3.7) *corresponding to the initial condition y_0. Then $\{S(t)\}_{t\geq 0}$ is a strongly continuous semigroup of nonlinear operators in $\mathcal{H}$.*

In the next two sections, the asymptotic stability of the nonlinear semigroup S will be considered, whereby first the precompactness property of the trajectories will be demonstrated. Secondly, it will be shown that the semigroup is asymptotically stable, except for countably many values of the parameter J. For these exceptional values of J, it is demonstrated that there exist non-trivial solutions which are periodic in time, and therefore do not decay. Explicit formulas for such solutions are also obtained, see (3.68) below.

3.2 Precompactness of the trajectories

In this section the precompactness of the trajectories of (3.7) is investigated. Thereby, for given $y_0 \in \mathcal{H}$ the corresponding trajectory is denoted by $\gamma(y_0)$ and defined by:

$$\gamma(y_0) := \bigcup_{t\geq 0} S(t)y_0.$$

First, the precompactness property is demonstrated for solutions that are twice differentiable in time. This result is then extended to all classical solutions. Therefore the following lemma is necessary:

Lemma 3.10. *Let $y_0 \in D(\mathcal{A}^2)$ and let y be the corresponding solution of* (3.7)*. Then $y \in C^2([0,\infty),\mathcal{H})$ and $y_t(t) \in D(\mathcal{A})$ for all $t > 0$.*

Proof. Note that if $y \in C^2([0,\infty),\mathcal{H})$, then it follows that $\tilde{y} := y_t$ is a solution of the following evolution equation:

$$\tilde{y}_t = A\tilde{y} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -s'(u(L))\frac{\psi}{M} - d'(\frac{\psi}{M})\frac{\tilde{\psi}}{M} \end{bmatrix}. \tag{3.11}$$

According to Lemma 3.5, it holds that $y \in C^1([0,\infty),\mathcal{H})$. Further, $y_0 \in D(\mathcal{A}^2)$ implies $\mathcal{A}y(0) = y_t(0) \in D(\mathcal{A})$. Motivated by (3.11), let the following functions be defined for a fixed $y(t)$:

$$F(t) := -s'(u(t,L))\frac{\psi(t)}{M} \in C^1([0,\infty)),$$
$$G(t,z) := -d'\Big(\frac{\psi(t)}{M}\Big)\frac{\chi}{M} \equiv g(t)\chi,$$

where $z = [U, V, \zeta, \chi]^\top \in \mathcal{H}$. Since $y(t)$ is a classical solution, both $F(t)$ and $g(t)$ are continuously differentiable. Consequently, the operator $\tilde{\mathcal{N}} : [0,\infty) \times \mathcal{H} \to \mathcal{H}$, defined by $\tilde{\mathcal{N}}(t,z) := [0,0,0,F(t) + G(t,z)]^\top$, is also continuously differentiable (in time). Furthermore, $\tilde{\mathcal{N}}$ is Lipschitz continuous in $\mathcal{H}$, uniformly in $t \in [0,T]$ for every $T > 0$. The following linear, non-autonomous, initial value problem is considered:

$$z_t = Az + \tilde{\mathcal{N}}(t,z), \tag{3.12a}$$
$$z(0) = z_0 \in \mathcal{H}. \tag{3.12b}$$

Applying Theorem 6.1.2 in [56] yields that there exists a unique global mild solution $z(t)$ of (3.12) for every $z_0 \in \mathcal{H}$. Furthermore, if $z_0 \in D(A)$, then $z(t)$ is a classical solution.

Next, it is shown that for the given classical solution $y(t)$ the function $y_t(t)$ is a mild solution of (3.12) for $z_0 = \mathcal{A}y_0$. Clearly, $y(t)$ satisfies the Duhamel formula (3.8), and differentiation with respect to t yields

$$y_t(t) = \mathrm{e}^{tA}Ay_0 + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathrm{e}^{(t-s)A}\mathcal{N}y(s)\,\mathrm{d}s. \tag{3.13}$$

According to the proof of Corollary 4.2.5 in [56] there holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathrm{e}^{(t-s)A}\mathcal{N}y(s)\,\mathrm{d}s = \mathrm{e}^{tA}\mathcal{N}y_0 + \int_0^t \mathrm{e}^{(t-s)A}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{N}y(s)\,\mathrm{d}s.$$

Inserting the above equation in (3.13) proves that $y_t(t)$ fulfills the Duhamel formula for (3.12), and as a consequence $y_t(t)$ is the unique mild solution of (3.12) to the initial condition $z_0 = \mathcal{A}y_0$. However from the beginning of this proof, it follows that the mild solution $z(t) = y_t(t)$ is a classical solution of (3.12) if $\mathcal{A}y_0 \in D(\mathcal{A})$, i.e. $y_0 \in D(\mathcal{A}^2)$. Therefore $y_t \in C^1([0,\infty),\mathcal{H})$ and $y \in C^2([0,\infty),\mathcal{H})$. □

Remark 3.11. The above result is straightforward in the situation where the evolution equation is linear and autonomous, i.e. $\mathcal{N} = 0$ in this case, and it is standard in the literature. This argumentation depends on commutative property of the time derivative and the linear operators, and can not in general be applied in the nonlinear case. According to Section II.5.a in [24] the density of $D(A^2)$ in $\mathcal{H}$ also immediately follows. Since in this case $D(\mathcal{A}^2)$ it is not a linear subset of $\mathcal{H}$ (see (3.19)), its density needs to be checked separately:

Lemma 3.12. *For any $y \in D(\mathcal{A})$, there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ in $D(\mathcal{A}^2)$ such that $\lim_{n\to\infty} y_n = y$ and $\lim_{n\to\infty} \mathcal{A}y_n = \mathcal{A}y$ in $\mathcal{H}$.*

Proof. First the set $D(\mathcal{A}^2)$ is characterized. It holds that $y \in D(\mathcal{A}^2)$ if and only if $y \in D(\mathcal{A})$ and $\mathcal{A}y \in D(\mathcal{A})$, or equivalently

$$v \in \tilde{H}^4_{0,\mathbb{R}}(0, L), \tag{3.14}$$

$$u \in \tilde{H}^6_{0,\mathbb{R}}(0, L), \quad u_{xxxx}(0) = u_{xxxxx}(0) = 0, \tag{3.15}$$

$$\xi = Jv_x(L), \tag{3.16}$$

$$\psi = Mv(L), \tag{3.17}$$

$$u_{xx}(L) = \frac{J}{\mu} u_{xxxxx}(L), \tag{3.18}$$

$$\Lambda u_{xxx}(L) - s(u(L)) - d\big(\frac{\psi}{M}\big) = -\frac{M\Lambda}{\mu} u_{xxxx}(L). \tag{3.19}$$

It suffices to show that for an arbitrary $y \in D(\mathcal{A})$, a sequence $\{y_n\}_{n\in\mathbb{N}} \subset D(\mathcal{A}^2)$ can be constructed such that $y_n = [u_n\, v_n\, \xi_n\, \psi_n]^\top$ converges to y in the space $H^4(0, L) \times H^2(0, L) \times \mathbb{R}^2$. Since the space

$$\tilde{C}^\infty_{0,\mathbb{R}}(0, L) := \{f \in C^\infty([0, L], \mathbb{R}) \colon f^{(k)}(0) = 0, \forall k \in \{0\} \cup \mathbb{N}\}$$

is dense in $\tilde{H}^2_{0,\mathbb{R}}(0, L)$ (see Theorem 3.17 in [1]), there exists a sequence $\{v_n\}_{n\in\mathbb{N}} \subset \tilde{C}^\infty_{0,\mathbb{R}}(0, L)$ such that $\lim_{n\to\infty} v_n = v$ in $H^2(0, L)$. Further, function v_n satisfies (3.14), for all $n \in \mathbb{N}$. Defining $\xi_n := J(v_n)_x(L)$ and $\psi_n := Mv_n(L)$ ensures that y_n satisfies (3.16) and (3.17). Moreover, the Sobolev embedding $H^2(0, L) \hookrightarrow C^1(0, L)$ implies that $\lim_{n\to\infty} \xi_n = \xi$ and $\lim_{n\to\infty} \psi_n = \psi$. As a final step, a sequence $\{u_n\}_{n\in\mathbb{N}} \subset C^\infty([0, L], \mathbb{R})$ is constructed such that u_n satisfies (3.15), (3.18), and (3.19) for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} u_n = u$ in $H^4(0, L)$. For this purpose, first the polynomial

$$h_n(x) := h_{2,n}x^2 + h_{3,n}x^3 + h_{6,n}x^6 + h_{7,n}x^7 + h_{8,n}x^8 + h_{9,n}x^9 + h_{10,n}x^{10} + h_{11,n}x^{11}$$

is introduced, for all $n \in \mathbb{N}$, where $h_{2,n}, \ldots, h_{11,n} \in \mathbb{R}$ are to be determined. In the following, the notation $k^{\underline{l}} := k \cdot (k-1) \cdot \cdots \cdot (k-l+1)$ is used, for $k, l \in \mathbb{N}$, $k \geq l$. Notice that

$$h_n(0) = (h_n)_x(0) = (h_n)_{xxxx}(0) = (h_n)_{xxxxx}(0) = 0, \tag{3.20}$$

holds. Let $h_{2,n} = \frac{u_{xx}(0)}{2}$ and $h_{3,n} = \frac{u_{xxx}(0)}{6}$, which is equivalent to

$$(h_n)_{xx}(0) = u_{xx}(0), \; (h_n)_{xxx}(0) = u_{xxx}(0). \tag{3.21}$$

Assume further that

$$h_n^{(k)}(L) = u^{(k)}(L), \qquad k \in \{0, 1, 2, 3\},$$

or equivalently:

$$h_{n,6} + h_{n,7}L + h_{n,8}L^2 + h_{n,9}L^3 + h_{n,10}L^4 + h_{n,11}L^5 = r_1, \tag{3.22a}$$
$$6h_{n,6} + 7h_{n,7}L + 8h_{n,8}L^2 + 9h_{n,9}L^3 + 10h_{n,10}L^4 + 11h_{n,11}L^5 = r_2, \tag{3.22b}$$
$$6^2 h_{n,6} + 7^2 h_{n,7}L + 8^2 h_{n,8}L^2 + 9^2 h_{n,9}L^3 + 10^2 h_{n,10}L^4 + 11^2 h_{n,11}L^5 = r_3 \tag{3.22c}$$
$$6^3 h_{n,6} + 7^3 h_{n,7}L + 8^3 h_{n,8}L^2 + 9^3 h_{n,9}L^3 + 10^3 h_{n,10}L^4 + 11^3 h_{n,11}L^5 = r_4, \tag{3.22d}$$

where

$$r_1 = \frac{u(L)}{L^6} - \frac{u_{xx}(0)}{2L^4} - \frac{u_{xxx}(0)}{6L^3}, \quad r_2 = \frac{u_x(L)}{L^5} - \frac{u_{xx}(0)}{L^4} - \frac{u_{xxx}(0)}{2L^3},$$
$$r_3 = \frac{u_{xx}(L)}{L^4} - \frac{u_{xx}(0)}{L^4} - \frac{u_{xxx}(0)}{L^3}, \quad r_4 = \frac{u_{xxx}(L)}{L^3} - \frac{u_{xxx}(0)}{L^3}.$$

Additional conditions are imposed on h_n:

$$\frac{M\Lambda}{\mu}(h_n)_{xxxx}(L) = -\Lambda u_{xxx}(L) + s(u(L)) + d(\frac{\psi_n}{M}), \tag{3.23}$$
$$\frac{J}{\mu}(h_n)_{xxxxx}(L) = u_{xx}(L). \tag{3.24}$$

Equations (3.23) and (3.24) are equivalent to:

$$6^4 h_{n,6} + 7^4 h_{n,7}L + 8^4 h_{n,8}L^2 + 9^4 h_{n,9}L^3 + 10^4 h_{n,10}L^4 + 11^4 h_{n,11}L^5 = r_5, \tag{3.25a}$$
$$6^5 h_{n,6} + 7^5 h_{n,7}L + 8^5 h_{n,8}L^2 + 9^5 h_{n,9}L^3 + 10^5 h_{n,10}L^4 + 11^2 h_{n,11}L^5 = r_6, \tag{3.25b}$$

with

$$r_5 = \mu\frac{-\Lambda u_{xxx}(L) + s(u(L)) + d(\frac{\psi_n}{M})}{\Lambda M L^2}, \quad r_6 = \frac{\mu u_{xx}(L)}{JL}.$$

The linear system (3.22) and (3.25) has a strictly positive determinant. Hence, its solution h_n exists and is unique. Consequently, (3.20), (3.21), and (3.22) imply that $u - h_n \in H^4_{0,\mathbb{R}}(0, L)$, for all $n \in \mathbb{N}$. Since $C^\infty_{0,\mathbb{R}}(0, L)$ is dense in $H^4_{0,\mathbb{R}}(0, L)$, there exists a sequence $\{\tilde{u}_n\}_{n\in\mathbb{N}} \subset C^\infty_{0,\mathbb{R}}(0, L)$ such that $\|\tilde{u}_n - (u - h_n)\|_{H^4} < \frac{1}{n}$, $\forall n \in \mathbb{N}$. Here the following definitions have been used:

$$C^\infty_{0,\mathbb{R}}(0, L) := \{f \in C^\infty_0(0, L) \,|\, f : [0, L] \to \mathbb{R}\},$$
$$H^k_{0,\mathbb{R}}(0, L) := \{f \in H^k_0(0, L) \,|\, f : [0, L] \to \mathbb{R}\}, \quad \forall k \in \mathbb{N}.$$

Now defining $u_n = \tilde{u}_n + h_n$, gives $\lim_{n\to\infty} u_n = u$ in $H^4(0, L)$. Obviously u_n satisfies (3.15) for all $n \in \mathbb{N}$. Also due to (3.23) and (3.24), u_n satisfies (3.18) and (3.19), as well. The statement follows. □

The next result is the main result of this section, and it states that all classical solutions have precompact trajectories.

Theorem 3.13. *The trajectory $\gamma(y_0)$ is precompact for $y_0 \in D(\mathcal{A})$.*

Proof. For a fixed $y_0 \in D(\mathcal{A})$ it shall be demonstrated that the corresponding trajectory $y(t)$ is precompact in $\mathcal{H}$. As seen in Lemma 3.5, the solution y is classical. Due to the compact embeddings $H^4(0,L) \hookrightarrow\hookrightarrow H^2(0,L) \hookrightarrow\hookrightarrow L^2(0,L)$ it is sufficient to show that

$$\sup_{t>0} \|\mathcal{A}y(t)\|_{\mathcal{H}} < \infty.$$

Moreover, since $y_t = \mathcal{A}y$, it is equivalent to show that y_t is uniformly bounded in $\mathcal{H}$.

Part 1: In the first part of this proof, it is assumed that $y_0 \in D(\mathcal{A}^2)$. According to Lemma 3.10, the time derivative y_t of the corresponding solution is a classical solution of the system (3.5) differentiated in time once:

$$\mu u_{ttt} + \Lambda u_{xxxxt} = 0, \tag{3.26a}$$
$$u_t(t,0) = 0, \tag{3.26b}$$
$$u_{tx}(t,0) = 0, \tag{3.26c}$$
$$M u_{ttt}(L) - \Lambda u_{txxx}(L) + s'(u(L))u_t(L) + d'(u_t(L))u_{tt}(L) = 0, \tag{3.26d}$$
$$J u_{tttx}(L) + \Lambda u_{txx}(L) = 0. \tag{3.26e}$$

Next, the time derivative of $V(y_t)$ is calculated:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V(y_t) &= \mu \int_0^L u_{ttt}u_{tt}\,\mathrm{d}x + \Lambda \int_0^L u_{ttxx}u_{txx}\,\mathrm{d}x + Ju_{tttx}(L)u_{ttx}(L) + Mu_{ttt}(L)u_{tt}(L) \\
&\quad + s(u_t(L))u_{tt}(L) \\
&= u_{tt}(L)\big(Mu_{ttt}(L) - \Lambda u_{txxx}(L) + s(u_t(L))\big) \\
&\quad + u_{ttx}(L)\big(\Lambda u_{txx}(L) + Ju_{tttx}(L)\big) \\
&= u_{tt}(L)\big(s(u_t(L)) - s'(u(L))u_t(L) - d'(u_t(L))u_{tt}(L)\big),
\end{aligned} \tag{3.27}$$

where the partial integration in x was performed twice and the equations (3.26b)-(3.26e) were used. Due to (3.2), it holds

$$-d'(u_t(L))u_{tt}(L)^2 \leq 0, \qquad \forall t \geq 0.$$

Integrating (3.27) in time gives

$$V(y_t(t)) \leq V(y_t(0)) + \int_0^t u_{tt}(\tau,L)\left[s(u_t(\tau,L)) - s'(u(\tau,L))u_t(\tau,L)\right]\mathrm{d}\tau. \tag{3.28}$$

The first integral on the right hand side, which is

$$\int_0^t u_{tt}(\tau,L)s(u_t(\tau,L))\,\mathrm{d}\tau = \int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^{u_t(\tau,L)} s(w)\,\mathrm{d}w\,\mathrm{d}\tau$$

$$= \int_0^{u_t(t,L)} s(w)\,\mathrm{d}w - \int_0^{u_t(0,L)} s(w)\,\mathrm{d}w, \tag{3.29}$$

is uniformly bounded, since $u_t(t, L) = \frac{\psi(t)}{M}$ is uniformly bounded, see Theorem 3.7. For the second term on the right hand side in (3.28) it holds:

$$\int_0^t u_{tt}(\tau, L)s'(u(\tau, L))u_t(\tau, L)\,\mathrm{d}\tau = \int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{(u_t(\tau, L))^2}{2}\right) s'(u(\tau, L))\,\mathrm{d}\tau$$
$$= \frac{u_t(t, L)^2}{2} s'(u(t, L)) - \frac{u_t(0, L)^2}{2} s'(u(0, L)) - \int_0^t \frac{u_t(\tau, L)^3}{2} s''(u(\tau, L))\,\mathrm{d}\tau. \tag{3.30}$$

Due to the Sobolev embedding $H^2(0, L) \hookrightarrow C(0, L)$, the estimate $|u(t, L)| \le C\|u\|_{H^2} \le C\|y\|_{\mathcal{H}}$ holds. Therefore $s''(u(t, L))$ is also uniformly bounded for $t \in [0, \infty)$. Together with the previously shown uniform boundedness of $u_t(t, L)$, it follows that the first two terms in (3.30) are uniformly bounded, and the remaining integral satisfies the following inequality

$$\Big|\int_0^t \frac{u_t(\tau, L)^3}{2} s''(u(\tau, L))\,\mathrm{d}\tau\Big| \le C \int_0^t |u_t(\tau, L)|^3\,\mathrm{d}\tau.$$

Due to (3.4), and considering $u_t(t, L)$ is uniformly bounded, there exists a positive constant $C > 0$ such that $|d(u_t(t, L))| \ge C u_t(t, L)^2$ for all $t \ge 0$. This yields

$$\int_0^\infty |u_t(t, L)|^3\,\mathrm{d}t \le C \int_0^\infty d(u_t(t, L))u_t(t, L)\,\mathrm{d}t,$$

and since $\frac{\mathrm{d}}{\mathrm{d}t}(V(y(t))) = -d(u_t(t, L))u_t(t, L)$ is integrable on $[0, \infty)$, it follows $u_t(., L) \in L^3(0, \infty)$. Therefore, all terms in (3.30) are uniformly bounded. Together with the uniform boundedness of (3.29), inequality (3.28) yields that $V(y_t(t)) \in L^\infty(0, \infty)$, and therefore $t \mapsto \|y_t(t)\|_{\mathcal{H}}$ is uniformly bounded, see Lemma 3.3. Therefore, $\gamma(y_0)$ is precompact. Moreover, notice that:

$$\sup_{t\ge0} \|y_t(t)\|_{\mathcal{H}} \le C(\|y_0\|_{\mathcal{H}}, \|y_t(0)\|_{\mathcal{H}}), \tag{3.31}$$

where the constant C depends continuously on $\|y_0\|_{\mathcal{H}}$ and $\|y_t(0)\|_{\mathcal{H}}$.

Part 2: In the second part of the proof, a more general case $y_0 \in D(\mathcal{A})$ is considered. According to Lemma 3.12, there exists a sequence $\{y_{n0}\}_{n\in\mathbb{N}} \subset D(\mathcal{A}^2)$ such that

$$\begin{aligned} \lim_{n\to\infty} y_{n0} &= y_0, \\ \lim_{n\to\infty} \mathcal{A}y_{n0} &= \mathcal{A}y_0. \end{aligned} \tag{3.32}$$

Taking $y_n(t) := S(t)y_{n0}$, there holds $y_n \in C^1([0, \infty); \mathcal{H}) \cap C^2((0, \infty); \mathcal{H})$ for all $n \in \mathbb{N}$ and hence $(y_n)_t(0) = \mathcal{A}y_{n0}$. This implies

$$\lim_{n\to\infty} (y_n)_t(0) = \mathcal{A}y_0 \;\text{ in }\; \mathcal{H}. \tag{3.33}$$

Therefore (3.32) and (3.33) imply that the both sequences $\{y_{n0}\}_{n\in\mathbb{N}}$ and $\{(y_n)_t(0)\}_{n\in\mathbb{N}}$ are bounded in $\mathcal{H}$. Together with (3.31), this yields that there holds:

$$\sup_{t\geq 0, n\in\mathbb{N}} \|(y_n)_t(t)\|_{\mathcal{H}} \leq C,$$

i.e. $(y_n)_t$ is bounded in $L^\infty([0,\infty);\mathcal{H})$. Now the Banach-Alaoglu Theorem (see Theorem I.3.15 in [59]) shows that there exists $\tilde{y} \in L^\infty([0,\infty);\mathcal{H})$ and a subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ such that

$$(y_{n_k})_t \stackrel{*}{\rightharpoonup} \tilde{y} \ \text{ in } \ L^\infty((0,\infty);\mathcal{H}).$$

Now let $z \in \mathcal{H}$ and $t \geq 0$ be arbitrary. Then

$$\lim_{k\to\infty} \int_0^t \langle (y_{n_k})_t(\tau), z\rangle_{\mathcal{H}} \, \mathrm{d}\tau = \int_0^t \langle \tilde{y}(\tau), z\rangle_{\mathcal{H}} \, \mathrm{d}\tau,$$

which is equivalent to

$$\lim_{k\to\infty} \langle y_{n_k}(t) - y_{n_k}(0), z\rangle_{\mathcal{H}} = \Big\langle \int_0^t \tilde{y}(\tau) \, \mathrm{d}\tau, z \Big\rangle_{\mathcal{H}}.$$

According to Proposition 3.6, $\lim_{n\to\infty} y_n(\tau) = y(\tau)$ in $\mathcal{H}$, $\forall \tau \in [0,\infty)$, and hence

$$\langle y(t) - y(0), z\rangle_{\mathcal{H}} = \Big\langle \int_0^t \tilde{y}(\tau) \, \mathrm{d}\tau, z \Big\rangle_{\mathcal{H}}.$$

Due to z being arbitrary, it follows that

$$y(t) - y(0) = \int_0^t \tilde{y}(\tau) \, \mathrm{d}\tau. \tag{3.34}$$

Now, since $y \in C^1([0,\infty);\mathcal{H})$, the time derivative of (3.34) can be calculated, and obtained that $y_t \equiv \tilde{y}$. This implies $y_t \in L^\infty((0,\infty);\mathcal{H})$, i.e. $\|y_t(.)\|_{\mathcal{H}}$ is uniformly bounded, which proves the precompactness of $\gamma(y_0)$. □

Remark 3.14. The question of precompactness for trajectories of the mild solutions which are not classical, remains open. In contrast, in the case of uniformly bounded linear semigroup, i.e. $\mathcal{N} = 0$, the proof of trajectory precompactness of mild solutions is much simpler, and it follows from precompactness property of the classical solution. Namely, for classical solutions $y(t)$ it holds that $Ay(t) = A\mathrm{e}^{tA}y_0 = \mathrm{e}^{tA}Ay_0$, so $Ay(t)$ is uniformly bounded. Since A^{-1} is compact, this proves the precompactness for classical solutions. Considering that $\{\mathrm{e}^{tA}\}_{t\geq 0}$ is bounded, any mild solution can uniformly be approximated by classical solutions, which implies the trajectory precompactness also for mild solutions.

However, in the case when s is linear, and d is non-linear, the precompactness property of the mild solution can still be verified. If the linear term s and the linear part of d are incorporated into A, this operator still generates a contraction semigroup, is invertible and has a compact resolvent. For the remaining nonlinear term it can be shown that $\mathcal{N}(y(.)) \in L^1([0,\infty);\mathcal{H})$, using (3.10) and (3.35). Then the prerequisites of Theorem B.8 stated in Appendix B are fulfilled, and the precompactness of the trajectories for all mild solutions follows.

3.3 ω-limit set and asymptotic stability

In the study of the asymptotic behavior of systems, the analysis of the ω-limit sets is vital. Therefore, their properties are examined at the beginning of this section.

Definition 3.15. Given the semigroup S, the *ω-limit set* for $y_0 \in \mathcal{H}$ is denoted by $\omega(y_0)$, and defined by:

$$\omega(y_0) := \{y \in \mathcal{H} \colon \exists \{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+,\ \lim_{n\to\infty} t_n = \infty \wedge \lim_{n\to\infty} S(t_n)y_0 = y\}$$

In general, it can hold $\omega(y_0) = \emptyset$. However, in the case that $\omega(y_0)$ is non-empty, Proposition 9.1.7 in [9] states:

Lemma 3.16. *For $y_0 \in \mathcal{H}$, the set $\omega(y_0)$ is S-invariant, i.e. $S(t)\omega(y_0) \subset \omega(y_0)$ for all $t \geq 0$.*

For some fixed $y_0 \in \mathcal{H}$, the function $t \mapsto V(S(t)y_0)$ is monotonically non-increasing, as seen in Theorem 3.7, and bounded below by 0. Therefore, $V(S(t)y_0)$ converges for $t \to \infty$ and the following definition is introduced:

$$\nu(y_0) := \lim_{t\to\infty} V(S(t)y_0) \geq 0. \tag{3.35}$$

Lemma 3.17. *Assuming $\omega(y_0) \neq \emptyset$, there holds*

$$V(y) = \nu(y_0), \qquad \forall y \in \omega(y_0).$$

Hence $\dot{V}(y) = 0$ for all $y \in \omega(y_0)$.

Proof. Let $y \in \omega(y_0)$. There exists a sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$, with $t_n \to \infty$, such that $\lim_{n\to\infty} S(t_n)y_0 = y$. Since V is continuous,

$$\lim_{n\to\infty} V(S(t_n)y_0) = V(y).$$

Due to (3.35), $V(y) = \nu(y_0)$, and the result follows. □

Hence the possible ω-limit sets may be identified by investigating trajectories along which the Lyapunov function V is constant. For this purpose, let the set $\Omega \subset \mathcal{H}$ be defined as the largest S-invariant subset of $\{y \in \mathcal{H} : \dot{V}(y) = 0\}$. There holds

$$\omega(y_0) \subset \Omega, \quad \forall y_0 \in \mathcal{H}, \tag{3.36}$$

therefore it is important to characterize the set Ω. The following proposition follows directly from Theorem A.3, stated and proven in Appendix A:

Proposition 3.18. *For every $y_0 \in \mathcal{H}$ the following holds, for all $t > 0$:*

$$\int_0^t S(\tau) y_0 \,\mathrm{d}\tau \in D(\mathcal{A}), \tag{3.37}$$

and

$$S(t)y_0 - y_0 = A \int_0^t S(\tau) y \,\mathrm{d}\tau + \int_0^t \mathcal{N} S(\tau) y_0 \,\mathrm{d}\tau. \tag{3.38}$$

Now the following result can be obtained.

Proposition 3.19. *For all $y = [u, v, \xi, \psi]^\top \in \Omega$, there holds $\psi = u(L) = 0$.*

Proof. For a fixed $y_0 \in \Omega$, let $y(t) = S(t)y_0$. Since Ω is S-invariant, it follows that $V(y(t)) = \nu(y_0)$ for all $t \geq 0$. First it is shown that

$$\psi(t) = 0, \qquad \forall t \geq 0. \tag{3.39}$$

In the case when $y_0 \in \Omega \cap D(\mathcal{A})$, (3.39) follows easily since (3.10) implies

$$\dot{V}(y(t)) = 0 \quad \Leftrightarrow \quad \psi(t) = 0.$$

Next the case when $y_0 \in \Omega \setminus D(\mathcal{A})$ will be investigated. Then there exists a sequence $\{y_{n0}\}_{n\in\mathbb{N}} \subset D(\mathcal{A})$ such that $\lim_{n\to\infty} y_{n0} = y_0$ in $\mathcal{H}$. Theorem 3.7 implies $y_n(t) \to y(t)$ in $C([0,T];\mathcal{H})$ for any $T > 0$, where $y_n(t) = S(t)y_{n0}$. In particular

$$\psi_n(t) \to \psi(t) \quad \text{in} \quad C([0,T];\mathbb{R}). \tag{3.40}$$

Together with (3.10) this implies

$$\left\{ \frac{\mathrm{d}}{\mathrm{d}t} V(y_n(t)) \right\}_{n\in\mathbb{N}}$$

is a Cauchy sequence in $C([0,T];\mathbb{R})$. Since V is locally Lipschitz continuous in $\mathcal{H}$ it follows that $\{V(y_n(t))\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];\mathbb{R})$. Thereby, $\{V(y_n(t))\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^1([0,T];\mathbb{R})$. Therefore, there exists a unique $w(t) \in C^1([0,T];\mathbb{R})$ such that

$$V(y_n(t)) \to w(t) \quad \text{in} \quad C^1([0,T];\mathbb{R}). \tag{3.41}$$

On the other hand, there holds $V(y_n(t)) \to V(y(t)) = \nu(y_0)$, for every $t \geq 0$, and hence $w(t) \equiv \nu(y_0)$. Together with (3.41) this implies $\frac{\mathrm{d}V}{\mathrm{d}t}(y_n(t)) = -d(\frac{\psi_n}{m})\frac{\psi_n}{m}$ converges uniformly to 0 on $[0,T]$. With (3.40) this now yields (3.39).

Next, $u(t,L) = 0$, $\forall t \geq 0$ is demonstrated. From (3.37) it follows that

$$M\left(\int_0^t v(\tau)\,\mathrm{d}\tau\right)\bigg|_{x=L} = \int_0^t \psi(\tau)\,\mathrm{d}\tau = 0.$$

Together with first equation in (3.38), it implies

$$0 = \left(\int_0^t v(\tau)\,\mathrm{d}\tau \right) \Big|_{x=L} = u(t,L) - u(0,L).$$

Therefore $u(t,L)$ is constant along $y(t)$, which gives

$$\int_0^t u(s,L)\,\mathrm{d}s = u_0(L)t, \quad t \geq 0. \tag{3.42}$$

Since $\{v(t) \colon t \geq 0\}$ is bounded in $L^2(0,L)$, the second component of (3.38) implies

$$\sup_{t \geq 0} \Big\| \Big(\int_0^t u(s)\,\mathrm{d}s \Big)_{xxxx} \Big\|_{L^2(0,L)} < \infty. \tag{3.43}$$

The Gagliardo–Nirenberg inequality (cf. [53]) is applied next, which guarantees the existence of $C > 0$ such that for all $t \geq 0$ there holds

$$\Big\| \int_0^t u(s)\,\mathrm{d}s \Big\|_{L^\infty(0,L)} \leq C \Big\| \Big(\int_0^t u(s)\,\mathrm{d}s \Big)_{xxxx} \Big\|_{L^2(0,L)}^{\frac{1}{8}} \Big\| \int_0^t u(s)\,\mathrm{d}s \Big\|_{L^2(0,L)}^{\frac{7}{8}}. \tag{3.44}$$

The first factor on the right hand side is uniformly bounded due to (3.43). For the second factor, according to Theorem 3.7 it follows that $t \mapsto \|u(t)\|_{L^2(0,L)}$ is uniformly bounded, and therefore $t \mapsto \| \int_0^t u(s)\,\mathrm{d}s\|_{L^2(0,L)}$ increases in time at most linearly. Altogether this implies in (3.44) that $t \mapsto \int_0^t u(s,L)\,\mathrm{d}s$ increases in time at most as $t^{\frac{7}{8}}$. However, this contradicts (3.42) unless $u_0(L) = 0$. Hence $u(t,L) = 0$ for all $t \geq 0$.

□

This result allows to represent any solution $S(t)y_0$ which lies in Ω (i.e. $y_0 \in \Omega$) as a solution to a simpler, linear system, which thus characterizes Ω. By inserting the result of Proposition 3.19 in the equation (3.38), it is obtained that any mild solution y of (3.7) with $y(t) \in \Omega$, $\forall t \geq 0$, satisfies the boundary condition

$$u(t,L) = 0, \ \forall t \geq 0, \tag{3.45}$$

and the following system:

$$u(t) - u(0) = \int_0^t v(\tau)\,\mathrm{d}\tau, \tag{3.46a}$$

$$v(t) - v(0) = -\frac{\Lambda}{\mu} \left(\int_0^t u(\tau)\,\mathrm{d}\tau \right)_{xxxx}, \tag{3.46b}$$

$$\xi(t) - \xi(0) = -\Lambda \left(\int_0^t u(\tau)\,\mathrm{d}\tau \right)_{xx} \Big|_{x=L}, \tag{3.46c}$$

$$0 = \left(\int_0^t u(\tau)\,\mathrm{d}\tau \right)_{xxx} \Big|_{x=L}. \tag{3.46d}$$

It shall be demonstrated that this system is overdetermined. The system (3.46) can be interpreted as a mild formulation of a linear evolution equation in a Hilbert space $\tilde{\mathcal{H}}$:

$$\begin{aligned} w_t &= \mathcal{B}w, \\ w(0) &= w_0, \end{aligned} \tag{3.47}$$

with $w = [u, v, \xi]^\top \in \tilde{\mathcal{H}}$ and where $w_0 = [u_0, v_0, \xi_0]^\top$. Thereby $\tilde{\mathcal{H}}$ is the Hilbert space

$$\tilde{\mathcal{H}} := \{w = [u, v, \xi]^\top : u \in \tilde{H}^2_{0,\mathbb{R}}(0, L), v \in L^2_{\mathbb{R}}(0, L), \xi \in \mathbb{R}\},$$

and $\mathcal{B}$ is the following linear operator in $\tilde{\mathcal{H}}$:

$$\mathcal{B} \begin{bmatrix} u \\ v \\ \xi \end{bmatrix} = \begin{bmatrix} v \\ -\frac{\Lambda}{\mu} u_{xxxx} \\ -\Lambda u_{xx}(L) \end{bmatrix}, \tag{3.48}$$

with the domain

$$D(\mathcal{B}) := \{y \in \mathcal{H} \colon u \in \tilde{H}^4_{0,\mathbb{R}}(0, L), v \in \tilde{H}^2_{0,\mathbb{R}}(0, L), \xi = Jv_x(L), u_{xxx}(L) = 0\}.$$

The space $\tilde{\mathcal{H}}$ is equipped with the inner product

$$\langle\langle w_1, w_2 \rangle\rangle := \frac{\Lambda}{2} \int_0^L (u_1)_{xx} (u_2)_{xx} \, \mathrm{d}x + \frac{\mu}{2} \int_0^L v_1 v_2 \, \mathrm{d}x + \frac{1}{2J} \xi_1 \xi_2.$$

As shown in Proposition A.4 in Appendix A, $\mathcal{B}$ is skew-adjoint and generates a C_0-semigroup of unitary operators. The eigenvalues $\{\lambda_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ are purely imaginary, and come in complex conjugated pairs, i.e. $\lambda_{-n} = \overline{\lambda_n}$, for $n \in \mathbb{N}$. Zero is not an eigenvalue since $\mathcal{B}$ is invertible, see [40]. The corresponding eigenfunctions $\{\Phi_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ form an orthonormal basis for the space $\tilde{\mathcal{X}}$ (extension of $\tilde{\mathcal{H}}$ to complex functions). They are given by

$$\Phi_n = \begin{bmatrix} u_n \\ \lambda_n u_n \\ \lambda_n J (u_n)_x(L) \end{bmatrix}, \tag{3.49}$$

where u_n is the unique real valued solution of

$$\mu \lambda_n^2 u_n + \Lambda (u_n)_{xxxx} = 0, \tag{3.50a}$$

$$(u_n)_{xxx}(L) = 0, \tag{3.50b}$$

$$J \lambda_n^2 (u_n)_x(L) + \Lambda (u_n)_{xx}(L) = 0, \tag{3.50c}$$

normalized such that $\|\Phi_n\|_{\tilde{\mathcal{X}}} = 1$. Note that $\lambda_n^2 < 0$. From (3.49) it follows that $\Phi_{-n} = \overline{\Phi_n}$, and therefore $u_{-n} = u_n$, $\forall n \in \mathbb{N}$. The complete spectral analysis of $\mathcal{B}$ is performed in the Proposition A.4 in the Appendix A.

In the following lemma, the solutions to (3.50) will be closely examined.

Lemma 3.20. *There exists a non-trivial solution u_n of the system (3.50) that satisfies $u_n(L) = 0$ if and only if*

$$J = \mu \left(\frac{L}{\ell\pi}\right)^3 \frac{(-1)^\ell + \cosh \ell\pi}{\sinh \ell\pi}, \qquad \text{for some } \ell \in \mathbb{N}. \tag{3.51}$$

In this case, u_n is unique up to normalization, and $\lambda_n^2 = -\frac{\Lambda}{\mu}\left(\frac{\ell\pi}{L}\right)^4$. The uniquely determined index is denoted by $n = n^*(\ell) > 0$.

Proof. A solution $\varphi \in \tilde{H}_0^4(0, L)$ to (3.50a) for some $\lambda \in i\mathbb{R}$ is of the form

$$u_n(x) = C_1[\cosh px - \cos px] + C_2[\sinh px - \sin px], \tag{3.52}$$

with $p = \left(\frac{-\mu\lambda^2}{\Lambda}\right)^{\frac{1}{4}} > 0$. The boundary conditions (3.50b) and (3.50c) are now equivalent to the following equations for C_1 and C_2:

$$C_1 (\sinh pL - \sin pL) + C_2 (\cosh pL + \cos pL) = 0, \tag{3.53}$$

and

$$\begin{aligned} &C_1 \left[J\mu^2 (\sinh pL + \sin pL) + p\Lambda (\cosh pL + \cos pL)\right] \\ &+C_2 \left[J\mu^2 (\cosh pL - \cos pL) + p\Lambda (\sinh pL + \sin pL)\right] = 0. \end{aligned} \tag{3.54}$$

Furthermore, notice that the condition $\varphi(L) = 0$ is equivalent to

$$C_1 (\cosh pL - \cos pL) + C_2 (\sinh pL - \sin pL) = 0. \tag{3.55}$$

First, it is assumed that $\varphi(L) = 0$. In order for φ to be non-zero the determinant of the linear system formed by (3.53) and (3.55) needs to vanish, i.e.

$$\begin{aligned} (\sinh pL - \sin pL)^2 - (\cosh pL - \cos pL)(\cosh pL + \cos pL) = \\ -2 \sinh pL \sin pL = 0. \end{aligned}$$

Since $pL > 0$, this is true if and only if $p = \frac{\ell\pi}{L}$, for some $\ell \in \mathbb{N}$. Hence $\lambda^2 = -\frac{\Lambda}{\mu}\left(\frac{\ell\pi}{L}\right)^4$. Now (3.53) gives that $C_2 = -C_1 \frac{\sinh \ell\pi}{\cosh \ell\pi + (-1)^\ell}$. Multiplying (3.54) by $\frac{(-1)^\ell \cosh \ell\pi + 1}{2C_1}$, it follows

$$-J\frac{\Lambda}{\mu}\left(\frac{\ell\pi}{L}\right)^4 \sinh \ell\pi + \frac{\ell\pi\Lambda}{L}[\cosh \ell\pi + (-1)^\ell] = 0,$$

and equivalently

$$J = \mu \left(\frac{L}{\ell\pi}\right)^3 \frac{\cosh \ell\pi + (-1)^\ell}{\sinh \ell\pi}.$$

Reversely, let (3.51) for some $\ell \in \mathbb{N}$, and let φ be defined by:

$$\varphi(x) := \left(\cosh \frac{\ell\pi x}{L} - \cos \frac{\ell\pi x}{L}\right) - \frac{\sinh \ell\pi}{\cosh \ell\pi + (-1)^\ell}\left(\sinh \frac{\ell\pi x}{L} - \sin \frac{\ell\pi x}{L}\right). \tag{3.56}$$

It needs to be verified that φ satisfies both (3.50) and $\varphi(L) = 0$. It follows immediately:

$$\begin{aligned}\varphi(L) &= \cosh \ell\pi - (-1)^\ell - \frac{(\sinh \ell\pi)^2}{\cosh \ell\pi + (-1)^\ell}\\ &= \frac{(\cosh \ell\pi)^2 - 1 - (\sinh \ell\pi)^2}{\cosh \ell\pi + (-1)^\ell} = 0,\end{aligned}$$

and

$$\varphi_{xxx}(L) = \left(\frac{\ell\pi}{L}\right)^3 \left[\sinh \ell\pi - \frac{\sinh \ell\pi}{\cosh \ell\pi + (-1)^\ell}(\cosh \ell\pi + (-1)^\ell)\right] = 0.$$

Moreover, there holds:

$$\begin{aligned}\varphi_x(L) &= \frac{\ell\pi}{L}\left[\sinh \ell\pi - \frac{\sinh \ell\pi}{\cosh \ell\pi + (-1)^\ell}(\cosh \ell\pi - (-1)^\ell)\right]\\ &= \frac{\ell\pi}{L}\frac{\sinh \ell\pi(\cosh \ell\pi + (-1)^\ell - \cosh \ell\pi + (-1)^\ell)}{\cosh \ell\pi + (-1)^\ell}\\ &= \frac{\ell\pi}{L}\frac{2(-1)^\ell \sinh \ell\pi}{\cosh \ell\pi + (-1)^\ell},\end{aligned}$$

and

$$\begin{aligned}\varphi_{xx}(L) &= \left(\frac{\ell\pi}{L}\right)^2 \left[\cosh \ell\pi + (-1)^\ell - \frac{(\sinh \ell\pi)^2}{\cosh \ell\pi + (-1)^\ell}\right]\\ &= \left(\frac{\ell\pi}{L}\right)^2 \frac{(\cosh \ell\pi)^2 + 2(-1)^\ell \cosh \ell\pi + 1 - (\sinh \ell\pi)^2}{\cosh \ell\pi + (-1)^\ell}\\ &= 2(-1)^\ell \left(\frac{\ell\pi}{L}\right)^2.\end{aligned}$$

With $\lambda^2 = -\frac{\Lambda}{\mu}\left(\frac{\ell\pi}{L}\right)^4$ this yields

$$\begin{aligned}\Lambda\varphi_{xx}(L) + J\lambda^2\varphi_x(L) &= \Lambda\varphi_{xx}(L) - J\frac{\Lambda}{\mu}\left(\frac{\ell\pi}{L}\right)^4 \varphi_x(L)\\ &= 2\left(\frac{\ell\pi}{L}\right)^2 \left(\Lambda(-1)^\ell - \Lambda\frac{(-1)^\ell + \cosh \ell\pi}{\sinh \ell\pi}\frac{\sinh \ell\pi(-1)^\ell}{\cosh \ell\pi + (-1)^\ell}\right)\\ &= 0.\end{aligned}$$

Therefore φ is the sought $u_{n^*(\ell)}$, which concludes the proof. □

In accordance to Lemma 3.20, the set

$$\mathscr{J} := \left\{\mu\left(\frac{L}{\ell\pi}\right)^3 \frac{(-1)^\ell + \cosh \ell\pi}{\sinh \ell\pi} : \ell \in \mathbb{N}\right\} \tag{3.57}$$

is introduced, and its ℓ-th entry is denoted by J_ℓ.

In the following theorem, the characterization of the set Ω is given in dependence on the parameter J.

Theorem 3.21. *Concerning the set Ω, it will be distinguished between two situations:*

i) Assume that parameter $J \notin \mathscr{J}$. Then $w = [u, v, \xi]^\top \equiv 0$ is the only solution to (3.47) with $u(L) = 0$, and therefore $\Omega = \{0\}$.

ii) Otherwise if $J \in \mathscr{J}$, then Ω is

$$\operatorname{span}_{\mathbb{R}}\{[u_{n^*}, 0, 0, 0]^\top, [0, u_{n^*}, J(u_{n^*})_x(L), 0]^\top\}.$$

Thereby u_{n^*} is the non-trivial solution from Lemma 3.20.

Proof. This proof closely follows the argumentation in [18]. According to Corollary A.5 in the Appendix A, the solution of the linear evolution equation (3.47) can be written as

$$w(t) = e^{t\mathcal{B}} w_0 = \sum_{n\in\mathbb{Z}\setminus\{0\}} \langle\!\langle w_0, \overline{\Phi_n} \rangle\!\rangle_{\tilde{\mathcal{X}}} e^{\lambda_n t} \Phi_n, \tag{3.58}$$

where $\{\lambda_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ are the (imaginary) eigenvalues of $\mathcal{B}$, and the Φ_n are the corresponding eigenfunctions, see Lemma A.4. Thereby $\langle\!\langle ., . \rangle\!\rangle_{\tilde{\mathcal{X}}}$ is the inner product on $\tilde{\mathcal{X}}$ defined in Appendix A. Let $c_n := \langle\!\langle w_0, \Phi_n \rangle\!\rangle_{\tilde{\mathcal{X}}}$ for all $n \in \mathbb{Z}$. Due to the orthonormality of the eigenfunctions $\{\Phi_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ and the fact that $\{\lambda_n\}_{n\in\mathbb{Z}\setminus\{0\}} \subset \mathrm{i}\mathbb{R}$ it holds for all $N \in \mathbb{N}$:

$$\Big\| \sum_{|n|\geq N} c_n e^{\lambda_n t} \Phi_n \Big\|^2_{\tilde{\mathcal{X}}} = \sum_{|n|\geq N} |c_n|^2. \tag{3.59}$$

Due to Parseval's identity it follows that $\sum_{n\in\mathbb{Z}\setminus\{0\}} |\langle\!\langle w_0, \Phi_n \rangle\!\rangle_{\tilde{\mathcal{X}}}|^2 = \|w_0\|^2_{\tilde{\mathcal{X}}}$. As a consequence, the right hand side in (3.59) tends to zero as $N \to \infty$. So, for every $\varepsilon > 0$ there exists some $N > 0$ such that

$$\sup_{t\geq 0} \Big\| \sum_{|n|\geq N} c_n e^{\lambda_n t} \Phi_n \Big\|_{\tilde{\mathcal{X}}} < \varepsilon. \tag{3.60}$$

The first component of the series (3.58) converges in $H^2(0, L)$ and therefore also in $C([0, L])$. Thus it holds

$$u(t, L) = \sum_{n\in\mathbb{Z}\setminus\{0\}} c_n \mathrm{e}^{\lambda_n t} u_n(L), \quad \forall t \geq 0. \tag{3.61}$$

Using this representation formula, those $u(t)$ that satisfy $u(t, L) = 0$ for all times are investigated. It immediately follows for every $N \in \mathbb{N}$:

$$\Big| \sum_{|n|\geq N} c_n \mathrm{e}^{\lambda_n t} u_n(L) \Big| \leq C \Big\| \sum_{|n|\geq N} c_n \mathrm{e}^{\lambda_n t} u_n \Big\|_{H^2(0,L)}$$

$$\leq C \Big\| \sum_{|n| \geq N} c_n \mathrm{e}^{\lambda_n t} \Phi_n \Big\|_{\tilde{\mathcal{X}}}.$$

According to (3.60) this implies that, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ large enough such that

$$\sup_{t \geq 0} \Big| \sum_{\substack{n=-N \\ n \neq 0}}^{N} c_n \mathrm{e}^{\lambda_n t} u_n(L) \Big| < \varepsilon, \tag{3.62}$$

provided that $u(t, L) = 0$ for all $t \geq 0$.

Let some $k \in \mathbb{Z} \setminus \{0\}$ and $\varepsilon > 0$ be fixed, and let $N \in \mathbb{N}$ be so large that $|k| < N$ and (3.62) is satisfied. Next, the finite sum in (3.62) is multiplied by $\mathrm{e}^{-\lambda_k t}$ and integrated over $[0, T]$:

$$\frac{1}{T} \int_0^T \sum_{\substack{n=-N \\ n \neq 0}}^{N} c_n \mathrm{e}^{\lambda_n t} u_n(L) \mathrm{e}^{-\lambda_k t} \, \mathrm{d}t = \sum_{\substack{n=-N \\ n \neq 0}}^{N} c_n u_n(L) \frac{1}{T} \int_0^T \mathrm{e}^{(\lambda_n - \lambda_k)t} \, \mathrm{d}t.$$

Due to (3.62), this expression has modulus less than ε. Now let $T \to \infty$. Since all eigenvalues λ_n of $\mathcal{B}$ are distinct (see Proposition A.4), all terms in the integral vanish except for the term where $n = k$, and it follows

$$|c_k \, u_k(L)| < \varepsilon.$$

Since ε was arbitrary, the following can be concluded:

$$c_k \, u_k(L) = 0, \quad \forall k \in \mathbb{Z} \setminus \{0\}. \tag{3.63}$$

There needs to be distinguish between two situations: Either $J \notin \mathscr{J}$ or $J \in \mathscr{J}$. In the first case, due to Lemma 3.20, $u_n(L) \neq 0$ for all $n \in \mathbb{Z}$. Then (3.63) implies that $c_k = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$, and consequently $w(t) \equiv 0$ for all $t > 0$. Therefore $\Omega = \{0\}$. In the case $J = J_\ell \in \mathscr{J}$, according to Lemma 3.20 it holds that $u_k(L) = 0$ if and only if $k \neq \pm n^*(\ell)$.Therefore, (3.63) yields:

$$c_k = 0, \quad \forall k \in \mathbb{Z} \setminus \{\pm n^*(\ell)\}, \tag{3.64}$$

$$c_{n^*} \in \mathbb{C} \quad \text{arbitrary}, \tag{3.65}$$

and $c_{-n^*} = \overline{c_{n^*}}$. Therefore, Ω is given by

$$\mathrm{Re}\Big(\mathrm{span}_{\mathbb{C}}\Big\{ \begin{bmatrix} \Phi_{-n^*} \\ 0 \end{bmatrix}, \begin{bmatrix} \Phi_{n^*} \\ 0 \end{bmatrix} \Big\}\Big) = \mathrm{span}_{\mathbb{R}} \left\{ \begin{bmatrix} u_{n^*} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ u_{n^*} \\ J(u_{n^*})_x(L) \\ 0 \end{bmatrix} \right\}. \tag{3.66}$$

□

Remark 3.22. An alternative approach is to consider the system (3.46a)-(3.46c) together with (3.45), and consider the condition (3.46d) afterward. The system (3.47) can be formulated on the space

$$\tilde{\mathcal{H}}_1 := \{w \in \tilde{\mathcal{H}} : u(L) = 0\}$$

instead of $\tilde{\mathcal{H}}$, where the system operator $\mathcal{B}$ has a different domain:

$$D_1(\mathcal{B}) := \{w \in \tilde{\mathcal{H}}_1 : u \in \tilde{H}^4_{0,\mathbb{R}}(0,L), v \in \tilde{H}^2_{0,\mathbb{R}}(0,L), \xi = Jv_x(L), v(L) = 0\}.$$

Analogously to the Proposition A.4 one finds that the operator $(\mathcal{B}, D_1(\mathcal{B}))$ is again skew-adjoint, generates a C_0-semigroup of unitary operators, and its eigenfunctions form an orthogonal basis. The first component u_n of the eigenfunction has again the same form (3.52). Applying the remaining condition (3.46d), the same characterization of Ω is obtained.

In the case where $J \in \mathscr{J}$, it has been seen (in Theorem 3.21) that $\Omega = \mathrm{span}_{\mathbb{R}}\{[\Phi_{\pm n^*}, 0]^\top\}$. From the definition of the $\Phi_{\pm n^*}$, it follows that they are precisely the two common eigenfunctions of $(\mathcal{B}, D(\mathcal{B}))$ and $(\mathcal{B}, D_1(\mathcal{B}))$. Therefore, in order to determine the ω-limit set, the two approaches using either $(\mathcal{B}, D(\mathcal{B}))$ or $(\mathcal{B}, D_1(\mathcal{B}))$ are equivalent. They only differ in the order in which the boundary conditions $u_{xxx}(L) = 0$ and $u(L) = 0$ are applied.

At this point, the prerequisites to show the main result are obtained.

Theorem 3.23. *Assume $J \notin \mathscr{J}$. For every $y_0 \in D(\mathcal{A})$,*

$$\lim_{t\to\infty} y(t) = 0,$$

i.e. the system (3.7) is asymptotically stable.

Proof. Due to Lemma 3.13, the trajectory $\gamma(y_0)$ is precompact, therefore $\omega(y_0)$ is non-empty. Furthermore, according to (3.36) it follows $\omega(y_0) \subset \Omega$. Hence $\omega(y_0) = \{0\}$, due to Theorem 3.21. Therefore, there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} y(t_n) = 0$. This implies $\lim_{n\to\infty} V(y(t_n)) = 0$, and since V is non-increasing, it implies $\lim_{t\to\infty} V(y(t)) = 0$. Since $\|y\|_{\mathcal{H}} \le V(y)$, for all $y \in \mathcal{H}$, the statement follows. □

Theorem 3.24. *Let $J = J_\ell \in \mathscr{J}$ for some $\ell \in \mathbb{N}$. Given an initial condition $y_0 \in \mathcal{H}$, the corresponding solution $y(t)$ of (3.7) approaches the solution to the initial condition $\Pi^* y_0$ as $t \to \infty$. Thereby Π^* is the orthogonal projection from $\mathcal{H}$ onto Ω, and is given by*

$$\Pi^* y = \begin{bmatrix} \Lambda\langle u_{xx}, (u_{n^*})_{xx}\rangle_{L^2} u_{n^*} \\ |\lambda_{n^*}|^2\big(\mu\langle v, u_{n^*}\rangle_{L^2} + \xi(u_{n^*})_x(L)\big) u_{n^*} \\ J|\lambda_{n^*}|^2\big(\mu\langle v, u_{n^*}\rangle_{L^2} + \xi(u_{n^*})_x(L)\big)(u_{n^*})_x(L) \\ 0 \end{bmatrix}, \tag{3.67}$$

where $\langle .,. \rangle_{L^2}$ denotes the standard inner product on $L^2(0,L)$.

Proof. Let $n^*(\ell)$ be as in Lemma 3.20. According to (3.66) the ω-limit set is a subset of the real part of the (complex) span of the two vectors $\Psi_{\pm n^*} = [\Phi_{\pm n^*}, 0]^\top$, where $\Phi_{-n^*} = \overline{\Phi_{n^*}}$. Since $\Phi_{\pm n^*} \in D(\mathcal{B})$, there holds that $(u_{n^*})_{xxx}(L) = 0$, and so the $\Psi_{\pm n^*}$ are eigenvectors of A to the eigenvalues $\pm\lambda_{n^*}$.The orthogonal projection may be defined first in $\mathcal{X}$ (see the Appendix A):

$$\Pi^* := \langle ., \Psi_{-n^*}\rangle_{\mathcal{X}} \Psi_{-n^*} + \langle ., \Psi_{n^*}\rangle_{\mathcal{X}} \Psi_{n^*}.$$

According to Theorem A.2 the eigenvectors of A form an orthogonal basis in $\mathcal{X}$, so Π^* commutes with A, and $\mathcal{X} = \ker \Pi^* \oplus \operatorname{ran} \Pi^*$ is an orthogonal, A-invariant decomposition of $\mathcal{X}$. In the following the restriction of Π^* to $\mathcal{H}$ shall be considered, and the same notation is kept. The explicit representation of Π^* is given by (3.67).

In the next step, it is shown that Π^* commutes with the nonlinearity $\mathcal{N}$. Since the first component u_{n^*} of Ψ_{n^*} satisfies $u_{n^*}(L) = 0$, it is clear that $\mathcal{N}\Psi_{\pm n^*} = 0$ and thus $\mathcal{N}\Pi^* = 0$. Let now $y \in \mathcal{X}$, then

$$\mathcal{N}y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -s(u(L)) - d(\frac{\psi}{M}) \end{bmatrix}.$$

Due to $u_n(L) = 0$ for the first component of $\Psi_{\pm n^*}$ it follows immediately $\langle \Psi_{\pm n^*}, \mathcal{N}y\rangle = 0$, i.e. $\Pi^*\mathcal{N}y = 0$.

As a consequence, the decomposition $\mathcal{H} = \ker \Pi^* \oplus \operatorname{ran} \Pi^*$ is invariant under the nonlinear semigroup S generated by $\mathcal{A}$. The trajectories of $S|_{\ker \Pi^*}$ lying in $D(\mathcal{A})$ are precompact. Theorem 3.21 implies that any ω-limit set of $S|_{\ker \Pi^*} \subset S$ has to be a subset of $\operatorname{ran} \Pi^*$. But on the other hand any trajectory and limit of $S|_{\ker \Pi^*}$ has to lie within $\ker \Pi^*$, which is orthogonal to $\operatorname{ran} \Pi^*$. Thus the only possible ω-limit set for $S|_{\ker \Pi^*}$ is $\{0\} = \operatorname{ran} \Pi^* \cap \ker \Pi^*$. And therefore $S(t)y_0$ approaches $S(t)\Pi^* y_0$ as $t \to \infty$. □

Remark 3.25. The asymptotic limit described in Theorem 3.24 can explicitly be computed. If $J = J_\ell$ for some $\ell \in \mathbb{N}$, it follows from (3.61), (3.64) and Lemma 3.20 that all real non-decaying solutions u of (3.5) are given by

$$u_p(t, x) = T(t) u_{n^*}(x), \tag{3.68}$$

where

$$T(t) = a \cos \sqrt{\frac{\Lambda}{\mu}} \left(\frac{\ell\pi}{L}\right)^2 t + b \sin \sqrt{\frac{\Lambda}{\mu}} \left(\frac{\ell\pi}{L}\right)^2 t, \quad a, b \in \mathbb{R},$$

and u_{n^*} is defined with (3.56).

In particular, it follows from Theorem 3.24 that for a given initial condition y_0, solution u of (3.5) approaches the solution u_p given in (3.68), with the coefficients a and b determined by:

$$a := \Lambda \langle (u_0)_{xx}, (u_{n^*})_{xx}\rangle_{L^2},$$

and

$$b := -\sqrt{\frac{\Lambda}{\mu}} \left(\frac{l\pi}{L}\right)^2 \big(\mu \langle v_0, u_{n^*}\rangle_{L^2} + \xi_0 (u_{n^*})_x(L)\big).$$

3.4 Weak formulation

In this section a weak formulation of the system (3.5) is defined. The Section is organized as follows. In Subsection 3.4.1 the weak solution is defined, and its existence is discussed in Subsection 3.4.2. In order to incorporate the nonlinearities appearing in the boundary conditions, the strategy used in Subsection 2.2.2 for the linear weak formulation is adapted.

3.4.1 Motivation and definition of the weak solution

For the definition of the weak solution, a motivation formulation is considered first. Assuming that $u : [0, \infty) \to \mathbb{R}$ is a classical solution, partially integrating (3.5a), and using (3.5b)–(3.5d), one obtains:

$$\mu \int_0^L u_{tt} w \, \mathrm{d}x + \Lambda \int_0^L u_{xx} w_{xx} \, \mathrm{d}x + J u_{ttx}(t, L) w_x(L) + \Big[M u_{tt}(t, L) + s(u(t, L)) + d(u_t(t, L)) \Big] w(L) = 0,$$

for all $w \in \tilde{H}_0^2(0, L)$ and $t > 0$.

Let $T > 0$ be fixed, and Hilbert spaces H and V as introduced by (2.88) and (2.89) in Section 2.2. The following nonlinear forms $a_{ds} : V \times V \to \mathbb{R}$ and $b_{ds} : H \times H \to \mathbb{R}$ are defined:

$$\begin{aligned} a_{ds}(\hat{w}_1, \hat{w}_2) &:= \Lambda((w_1)_{xx}, (w_2)_{xx})_{L^2} + s\big(w_1(L)\big) w_2(L), \\ b_{ds}(\hat{\varphi}, \hat{\nu}) &:= d({}^2\hat{\varphi})\, {}^2\hat{\nu}. \end{aligned}$$

Definition 3.26. A function $\hat{u} \in L^2(0, T; V) \cap H^1(0, T; H) \cap H^2(0, T; V')$ is said to be a *weak solution* to (3.5) on the time interval $[0, T]$ if it satisfies:

$$a_{ds}(\hat{u}, \hat{w}) + b_{ds}(\hat{u}_t, \hat{w}) + {}_{V'}\langle \hat{u}_{tt}, \hat{w} \rangle_V = 0, \quad \forall \hat{w} \in V \tag{3.69}$$

for a.e. $t \in (0, T)$, with initial conditions

$$\hat{u}(0) = \hat{u}_0 = ((u_0)_x(L), u_0(L), u_0) \in V, \tag{3.70a}$$

$$\hat{u}_t(0) = \hat{v}_0 = ((v_0)_x(L), v_0(L), v_0) \in H. \tag{3.70b}$$

The existence of the weak solution is discussed in the next subsection.

3.4.2 Existence and regularity results

The strategy for the existence of the weak solution closely follows the approach in the linear case, Section 2.2. Thereby, a sequence of Galerkin approximations is constructed and the limit of Galerkin weakly-convergent subsequence is identified as the corresponding weak solution.

Theorem 3.27. *There exists a weak solution $\hat{u}$ to the weak formulation (3.69) with initial conditions* (3.70)*. Moreover, $\hat{u}$ has the additional regularity $\hat{u} \in C([0,T];V)$, and $\hat{u}_t \in C([0,T];H)$.*

Proof. Let $\{\hat{w}_k\}_{k=1}^{\infty}$ be an orthonormal basis for H, and an orthogonal basis for V. For a fixed $m \in \mathbb{N}$, let $\widehat{W}_m = \operatorname{span}\{\hat{w}_1, \dots, \hat{w}_m\}$, and $\hat{u}_m \in C^2([0,T];\widehat{W}_m)$ be the Galerkin approximation which solves:

$$((\hat{u}_m)_{tt}, \hat{w}_k)_H + a_{ds}(\hat{u}_m, \hat{w}_k) + b_{ds}((\hat{u}_m)_t, \hat{w}_k) = 0, \tag{3.71}$$

for all $k \in \{1, \dots, m\}$, with the initial conditions

$$\begin{aligned} \hat{u}_m(0) &= \hat{u}_{m0}, \\ (\hat{u}_m)_t(0) &= \hat{v}_{m0}. \end{aligned} \tag{3.72}$$

It is assumed that the sequences $\hat{u}_{m0}, \hat{v}_{m0} \in \widehat{W}_m$ are such that

$$\begin{aligned} \hat{u}_{m0} &\to \hat{u}_0 \text{ in } V, \\ \hat{v}_{m0} &\to \hat{v}_0 \text{ in } H. \end{aligned} \tag{3.73}$$

In order to prove global solvability of (3.71)–(3.72), the system is written as a nonlinear system of first order differential equations. Introducing a new variable $\hat{v}_m := (\hat{u}_m)_t$, yields the following system:

$$\begin{aligned} (\hat{u}_m)_t &= \hat{v}_m \\ (\hat{v}_m)_t &= -\sum_{j=1}^{m} \left[a_{ds}(\hat{u}_m, \hat{w}_j) + b_{ds}(\hat{v}_m, \hat{w}_j)\right]\hat{w}_j \end{aligned} \tag{3.74}$$

Let $\widehat{E} : \mathbb{R} \times V \times H \to \mathbb{R}$ be the analogue of the Lyapunov functional as defined by (3.9):

$$\widehat{E}(t;\hat{u},\hat{v}) := \frac{\Lambda}{2}\|\hat{u}(t)\|_V^2 + \frac{1}{2}\|\hat{v}(t)\|_H^2 + \int_0^{2\hat{u}(t)} s(\psi)\,\mathrm{d}\psi \tag{3.75}$$

Assuming that there exists a solution $\hat{u}_m \in C^2([0,\tau];\widehat{W}_m)$ to (3.71) on some interval $[0,\tau]$, a straightforward calculation yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{E}(t;\hat{u}_m,\hat{v}_m) = -d(v_m(L))v_m(L) \tag{3.76}$$

$\forall t \in (0,\tau)$. Dissipation of the functional $\widehat{E}$ corresponds to the decay in (3.10) for the classical solution. This implies uniform boundedness of the solution on $[0,\tau]$:

$$\widehat{E}(t;\hat{u}_m,\hat{v}_m) \le \widehat{E}(0;\hat{u}_{m0},\hat{v}_{m0}), \quad t \ge 0. \tag{3.77}$$

Next, let $f_m : \widehat{W}_m \times \widehat{W}_m \to \widehat{W}_m \times \widehat{W}_m$ be defined with

$$f_m\left(\begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix}\right) := \begin{bmatrix} \hat{v} \\ -\sum_{j=1}^m \left[a_{ds}(\hat{u}, \hat{w}_j) + b_{ds}(\hat{v}, \hat{w}_j)\right]\hat{w}_j \end{bmatrix}.$$

Denoting $\hat{z}_m := [\hat{u}_m \, \hat{v}_m]^\top$, equation (3.74) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{z}_m(t) = f_m\left(\hat{z}_m(t)\right), \tag{3.78}$$

with

$$\hat{z}_m(0) = \hat{z}_{m0} := [\hat{u}_{m0} \, \hat{v}_{m0}]^\top. \tag{3.79}$$

Due to the regularity of the coefficient functions, it follows that f_m is continuously differentiable, and hence locally Lipschitz. Let $T_{\max}$ be defined by:

$$T_{\max} = \min\{T, \frac{1}{2L(2\|\hat{z}_{m0}\|)}\},$$

whereby $L(2\|\hat{z}_{m0}\|)$ denotes the Lipschitz constant for f_m on the ball with center at zero and radius $2\|\hat{z}_{m0}\|$. Additionally, let the mapping

$$F_m : C([0, T_{\max}]; \widehat{W}_m \times \widehat{W}_m) \to C([0, T_{\max}]; \widehat{W}_m \times \widehat{W}_m)$$

be defined by:

$$[F_m(\hat{z})](t) := \hat{z}_{m0} + \int_0^t f_m(\hat{z}(\tau)) \, \mathrm{d}\tau$$

It follows that solving the system (3.78)–(3.79) on $[0, T_{\max}]$ is equivalent to solving a fixed point problem for F_m. It can be shown that F_m maps from $B(0, 2\|\hat{z}_{m0}\|)$ to itself:

$$\begin{aligned} \|F_m(\hat{z})(t)\| &\leq \|\hat{z}_{m0}\| + \int_0^t \|f_m(\hat{z}(\tau))\| \, \mathrm{d}\tau \\ &\leq \|\hat{z}_{m0}\| + \int_0^t L(2\|\hat{z}_{m0}\|) \, \|\hat{z}(\tau)\| \, \mathrm{d}\tau \\ &\leq \|\hat{z}_{m0}\| + tL(2\|\hat{z}_{m0}\|)2\|\hat{z}_{m0}\| \\ &\leq 2\|\hat{z}_{m0}\|, \end{aligned}$$

for all $t \in [0, T_{\max}]$. Furthermore, F_m is a contraction on $B(0, 2\|\hat{z}_{m0}\|)$, since it holds:

$$\begin{aligned} \|F_m(\hat{z}_1)(t) - F_m(\hat{z}_2)(t)\| &\leq \int_0^t \|f_m(\hat{z}_1(\tau)) - f_m(\hat{z}_2(\tau))\| \, \mathrm{d}\tau \\ &\leq \int_0^t L(2\|\hat{z}_{m0}\|) \, \|\hat{z}_1(\tau) - \hat{z}_2(\tau)\| \, \mathrm{d}\tau \\ &\leq tL(2\|\hat{z}_{m0}\|)\|\hat{z}_1 - \hat{z}_2\|_{C([0,T_{\max}];\widehat{W}_m^2)} \end{aligned}$$

$$\leq \frac{1}{2}\|\hat{z}_1 - \hat{z}_2\|_{C([0,T_{\max}];\widehat{W}_m^2)},$$

$\forall \hat{z}_1, \hat{z}_2 \in B(0, 2\|\hat{z}_{m0}\|)$. Now, the Banach's fixed point theorem implies that F_m has a unique fixed point $\hat{z}$ in $B(0, 2\|\hat{z}_{m0}\|)$. Applying the above procedure, any solution $\hat{z}$ on the time interval $[0, \tau]$ can be extended to $[0, \tau + \delta(\hat{z}(\tau))]$, where $\delta(\hat{z}(\tau)) = \frac{1}{2L(\|2\hat{z}(\tau)\|)} \geq \frac{1}{2L(2C(\|\hat{z}_0\|))}$. Therefore, the solution can be extended to the global unique solution on the whole interval $[0, T]$. Furthermore, due to (3.73) there exists a constant $C > 0$ such that

$$\widehat{E}(0; \hat{u}_{m0}, \hat{v}_{m0}) \leq C\,\widehat{E}(0; \hat{u}_0, \hat{v}_0), \quad \text{for all } m \in \mathbb{N}, \tag{3.80}$$

and for fixed $\hat{u}_0$, $\hat{v}_0$, and fixed sequences $\{\hat{u}_{m0}\}$, $\{\hat{v}_{m0}\}$. Therefore (3.77) and (3.80) yield

$$\widehat{E}(t; \hat{u}_m, (\hat{u}_m)_t) \leq C\,\widehat{E}(0; \hat{u}_0, \hat{v}_0), \tag{3.81}$$

which implies

$$\begin{aligned} \{\hat{u}_m\}_{m\in\mathbb{N}} &\quad \text{is bounded in } C([0,T]; V), \\ \{(\hat{u}_m)_t\}_{m\in\mathbb{N}} &\quad \text{is bounded in } C([0,T]; H). \end{aligned}$$

Analogous to the proof of Theorem 2.30, this yields that $(\hat{u}_m)_{tt}$ is bounded in $L^2(0, T; V')$. For this purpose, let $\hat{w} \in V$, and $\hat{w} = \hat{\varphi}_1 + \hat{\varphi}_2$, such that $\hat{\varphi}_1 \in W_m$ and $\hat{\varphi}_2$ orthogonal to $\hat{W}_m$ in H. From (3.71) it follows:

$$\begin{aligned} ((\hat{u}_m)_{tt}, \hat{w})_H &= ((\hat{u}_m)_{tt}, \hat{\varphi}_1)_H \\ &= -a_{ds}(\hat{u}_m, \hat{\varphi}_1) - b_{ds}((\hat{u}_m)_t, \hat{\varphi}_1) \\ &\leq D_1\|\hat{\varphi}_1\|_V \leq D_1\|\hat{w}\|_V, \end{aligned}$$

for some $D_1 > 0$. Hence $(\hat{u}_m)_{tt}$ is bounded in $C([0,T]; V')$. Therefore, as stated in the Eberlein-Šmuljan Theorem, there exists a subsequence $\{\hat{u}_{m_l}\}_{l\in\mathbb{N}}$, and $\hat{u} \in L^2(0,T;V)$, such that $\hat{u}_t \in L^2(0,T;H)$, $\hat{u}_{tt} \in L^2(0,T;V')$ and the following holds:

$$\begin{aligned} \{\hat{u}_{m_l}\} &\rightharpoonup \hat{u} \text{ in } L^2(0,T;V), \\ \{(\hat{u}_{m_l})_t\} &\rightharpoonup \hat{u}_t \text{ in } L^2(0,T;H), \\ \{(\hat{u}_{m_l})_{tt}\} &\rightharpoonup \hat{u}_{tt} \text{ in } L^2(0,T;V'). \end{aligned} \tag{3.82}$$

Furthermore (3.82) implies

$$\{{}^2\hat{u}_{m_l}\} \to {}^2\hat{u} \text{ in } L^2(0,T;\mathbb{R}),$$

$$\{^2(\hat{u}_{m_l})_t\} \to {}^2\hat{u}_t \text{ in } L^2(0,T;\mathbb{R}).$$

Now it is justified to pass on to the limit in (3.71) for $m = m_l$, when $l \to \infty$, since all the nonlinear terms are continuous, and their arguments converge strongly. This yields that $\hat{u}$ is a solution to (3.69) on $[0,T]$.

The argumentation to show that the weak solution $\hat{u}$ satisfies the initial conditions and additional regularity follows closely the lines of the proof of Theorem 2.30, and will therefore be omitted. □

3.5 Dissipative numerical method

The goal of this section is to derive a numerical method for (3.5), which conserves the dissipativity property for the Lyapunov functional of the system:

$$\frac{\mathrm{d}}{\mathrm{d}t}V(y(t)) = -d\Big(\frac{\psi}{M}\Big)\frac{\psi}{M} \leq 0. \tag{3.83}$$

The strategy is to divide this problem into two steps: In the Subsection 3.5.1 first discretization in space is performed, to obtain a dissipative semi-discrete method. Secondly, in the Subsection 3.5.2 a fully-discrete dissipative scheme is obtained by discretization in time.

3.5.1 Semi-discrete scheme: space discretization

Let $W_h \subset \tilde{H}_0^2(0,L)$ be a N-dimensional space, and let $\{w_j\}_{j=1}^N$ be its basis. The semi-discrete solution $u^h \in C^2([0,\infty), W_h)$ is defined as the solution of a FEM:

$$\begin{aligned} \mu \int_0^L u_{tt}^h w_j \, dx + \Lambda \int_0^L u_{xx}^h (w_j)_{xx} \, dx + J u_{xtt}^h(t,L)(w_j)_x(L) \\ + \big[M u_{tt}^h(t,L) + s(u^h(t,L)) + d(u_t^h(t,L))\big] w_j(L) = 0, \end{aligned} \tag{3.84}$$

for $j = 1, \ldots, N$, and $t > 0$, which solves the initial conditions

$$\begin{aligned} u^h(0) &= u_{0,h}, \\ u_t^h(0) &= v_{0,h}. \end{aligned}$$

An analogue of the Lyapunov functional given by (3.9) for semi-discrete solution u^h is defined by:

$$E(t;u^h) := \frac{\Lambda}{2}\int_0^L (u_{xx}^h)^2 \, \mathrm{d}x + \frac{\mu}{2}\int_0^L (u_t^h)^2 \, \mathrm{d}x + \frac{M}{2}(u_t^h(t,L))^2 + \frac{J}{2}(u_{tx}^h(t,L))^2 + \int_0^{u^h(L)} s(w) \, \mathrm{d}w.$$

Theorem 3.28. *Let $u^h \in C^2([0,\infty); \tilde{H}_0^2(0,L))$ solve* (3.84)*. Then it holds for $t > 0$:*

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t; u^h) \quad = \quad -d(u_t^h(t,L)) u_t^h(t,L) \leq 0,$$

hence u_h is uniformly bounded on $[0,\infty)$*.*

Proof. Taking $w = u_t^h$ as the test function in (3.84) proves the statement. □

Theorem 3.29. *The system* (3.84)*, has a unique, global solution.*

Proof. Equation (4.90) is written as a first order differential equation. For this purpose, let the vector function

$$\mathbb{U}(t) = \begin{bmatrix} U_1(t) & U_2(t) & \dots & U_N(t) \end{bmatrix}^\top$$

be the vector representation of u^h in the basis $\{w_i\}_{i=1}^N$, i.e.

$$u^h(t,x) = \sum_{i=1}^N U_i(t) w_i(x).$$

Then (4.90) can be written equivalently as a semi-linear vector equation:

$$\mathbb{A}\mathbb{U}_{tt} + \mathbb{B}(\mathbb{U}_t) + \mathbb{K}\mathbb{U} + \mathbb{C}(\mathbb{U}) = 0, \tag{3.85}$$

with coefficient matrices defined as

$$\begin{aligned} \mathbb{A}_{i,j} &:= \mu \int_0^L w_i w_j \, dx + M w_i(L) w_j(L) + J (w_i)_x(L) (w_j)_x(L), \\ \mathbb{K}_{i,j} &:= \Lambda \int_0^L (w_i)_{xx} (w_j)_{xx} \, dx, \end{aligned}$$

and nonlinear vectors functions have the entries:

$$\begin{aligned} \mathbb{B}(\mathbb{U}_t)_j &:= d(u_t^h(L)) w_j(L), \\ \mathbb{C}(\mathbb{U})_j &:= s(u^h(L)) w_j(L), \end{aligned}$$

where $i,j \in \{1,\dots,N\}$. Now, let $\mathbb{V} := \mathbb{U}_t$. Since $\mathbb{A}$ is symmetric positive definite, the equation (3.85) can be written as

$$\left(\begin{bmatrix} \mathbb{U} \\ \mathbb{V} \end{bmatrix} \right)_t = f\left(\begin{bmatrix} \mathbb{U} \\ \mathbb{V} \end{bmatrix} \right), \tag{3.86}$$

with initial conditions

$$\begin{bmatrix} \mathbb{U} \\ \mathbb{V} \end{bmatrix} (0) = Z_0 := \begin{bmatrix} \mathbb{U}_0 \\ \mathbb{V}_0 \end{bmatrix}, \tag{3.87}$$

where $\mathbb{U}_0$, $\mathbb{V}_0$ are vector representations of $u_{0,h}$, $v_{0,h}$ respectively, and where $f\colon \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ is given by:

$$f\left(\begin{bmatrix} \mathbb{U} \\ \mathbb{V} \end{bmatrix}\right) = \begin{bmatrix} \mathbb{V} \\ -\mathbb{A}^{-1}\left(\mathbb{B}(\mathbb{V}) + \mathbb{K}\mathbb{U} + \mathbb{C}(\mathbb{U})\right) \end{bmatrix}. \tag{3.88}$$

Due to the regularity of the coefficient functions, it follows that f is continuously differentiable. Assuming that $Z(t) = [\mathbb{U}(t)\ \mathbb{V}(t)]^\top$ is the solution of (3.86)– (3.87) on some time interval $[0,T]$, it follows from Theorem 3.28 that Z is uniformly bounded, i.e. there exists $C > 0$ such that:

$$\|Z(t)\| \leq C(\|Z_0\|),\ t \in [0,T],$$

where C does not depend on T. Next, let the mapping $F : C([0,T];\mathbb{R}^{2N}) \to C([0,T];\mathbb{R}^{2N})$ be defined by:

$$F(Z)(t) := Z_0 + \int_0^t f(Z(\tau))\ \mathrm{d}\tau. \tag{3.89}$$

Definition (3.89) implies that solving the system (3.86), and (3.87) on $[0,T]$ is equivalent to solving a fixed point problem for F. Let T be defined as

$$T := \frac{1}{2L(2\|Z_0\|)}.$$

Furthermore, F maps the set $B := \{Z \in C([0,T];\mathbb{R}^{2N})\ :\ \|Z\|_{C([0,T];\mathbb{R}^{2N})} < 2\|Z_0\|\}$ to itself, since it holds:

$$\begin{aligned} \|F(Z)(t)\| &\leq \|Z_0\| + \int_0^t \|f(Z(\tau))\|\ \mathrm{d}\tau \\ &\leq \|Z_0\| + \int_0^t L(2\|Z_0\|)\,\|Z(\tau)\|\ \mathrm{d}\tau \\ &\leq \|Z_0\| + TL(2\|Z_0\|)\,2\|Z_0\| = 2\|Z_0\|, \end{aligned}$$

for all $Z \in C([0,T];\mathbb{R}^{2N})$ such that $\|Z\|_{C([0,T];\mathbb{R}^{2N})} \leq 2\|Z_0\|$. Moreover, F is a contraction on B:

$$\begin{aligned} \|F(Z_1)(t) - F(Z_2)(t)\| &\leq \int_0^t \|f(Z_1(\tau)) - f(Z_2(\tau))\|\ \mathrm{d}\tau \\ &\leq \int_0^t L(2\|Z_0\|)\,\|Z_1(\tau) - Z_2(\tau)\|\ \mathrm{d}\tau \\ &\leq TL(2\|Z_0\|)\,\|Z_1 - Z_2\|_{C([0,T];\mathbb{R}^{2N})} \\ &= \frac{1}{2}\|Z_1 - Z_2\|_{C([0,T];\mathbb{R}^{2N})}. \end{aligned}$$

Now, the Banach's fixed point theorem implies that F has a unique fixed point Z in B, which also solves (3.86) and (3.87) on $[0,T]$. More generally, by applying the above procedure, any solution Z on the time interval $[0,\tau]$ can be extended to $[0,\tau+\delta(Z(\tau))]$, where $\delta(Z(\tau)) = \frac{1}{2L(\|Z(\tau)\|)} \geq \frac{1}{2L(C(\|Z_0\|))}$. This implies that the solution can be extended to the whole $[0,\infty)$.

□

3.5.2 Fully-discrete scheme: time discretization

In this subsection (3.84) shall be discretized in time. Let Δt denote the time step of the discretization and define $t_n := n\Delta t$, $n \in \mathbb{N}$. Furthermore, let u^n and v^n denote the approximation for u and u_t, at $t = t_n$, respectively. For the time discretization of (3.84) Crank-Nicolson scheme is utilized:

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{v^{n+1} + v^n}{2}, \tag{3.90}$$

$$\mu \int_0^L \frac{v^{n+1} - v^n}{\Delta t} w_h \, \mathrm{d}x + \Lambda \int_0^L \frac{u_{xx}^{n+1} + u_{xx}^n}{2} (w_h)_{xx} \, \mathrm{d}x + (w_h)_x(L) J \frac{v_x^{n+1}(L) - v_x^n(L)}{\Delta t}$$
$$+ w_h(L) \left(M \frac{v^{n+1}(L) - v^n(L)}{\Delta t} + H(u^{n+1}(L), u^n(L)) + d\Big(\frac{v^{n+1}(L) + v^n(L)}{2}\Big) \right) = 0, \tag{3.91}$$

for all $w_h \in W_h$, where

$$H(\breve{\psi}, \psi) := \begin{cases} \frac{\int_\psi^{\breve{\psi}} s(w) \, \mathrm{d}w}{\breve{\psi} - \psi}, & \breve{\psi} \neq \psi \\ s(\psi), & \breve{\psi} = \psi \end{cases} \tag{3.92}$$

Remark 3.30. Note that, for a fixed $\psi \in \mathbb{R}$, the mapping $\tilde{H}_\psi \colon \breve{\psi} \to H(\breve{\psi}, \psi)$ is continuous on $\mathbb{R}$. Namely, although the expression

$$\frac{\int_\psi^{\breve{\psi}} s(w) \, \mathrm{d}w}{\breve{\psi} - \psi}$$

is not defined when $\breve{\psi} = \psi$, due to continuity of s, it follows

$$\lim_{\breve{\psi} \to \psi} \frac{\int_\psi^{\breve{\psi}} s(w) \, \mathrm{d}w}{\breve{\psi} - \psi} = s(\psi).$$

Hence, $\tilde{H}_\psi$ can be extended to a continuous function on $\mathbb{R}$ with $\tilde{H}_\psi(\psi) = H(\psi, \psi) := s(\psi)$.

Theorem 3.31. *Let*

$$\tilde{V}\left(\begin{bmatrix} u \\ v \end{bmatrix} \right) := \frac{\Lambda}{2} \int_0^L u_{xx}^2 \, \mathrm{d}x + \frac{\mu}{2} \int_0^L v^2 \, \mathrm{d}x + \frac{M}{2} v(L)^2 + \frac{J}{2} v_x(L)^2 + \int_0^{u(L)} s(w) \, \mathrm{d}w,$$

be the analogue of the Lyapunov functional in (3.9)*. For $n \in \mathbb{N}$, let $z^n := [u^n \; v^n]^\top$ and $z^{n+1} := [u^{n+1} \; v^{n+1}]^\top$ satisfy* (3.90) *and* (3.91)*. Then the following holds:*

$$\frac{\tilde{V}(z^{n+1}) - \tilde{V}(z^n)}{\Delta t} = -\frac{v^{n+1}(L) + v^n(L)}{2} \, d\Big(\frac{v^{n+1}(L) + v^n(L)}{2}\Big) \leq 0,$$

Proof. There holds:

$$\begin{aligned}\tilde{V}(z^{n+1}) - \tilde{V}(z^n) &= \frac{\Lambda}{2}(\|u_{xx}^{n+1}\|^2 - \|u_{xx}^n\|^2) + \frac{\mu}{2}(\|v^{n+1}\|^2 - \|v^n\|^2) \\ &\quad + \frac{M}{2}\left((v^{n+1}(L))^2 - (v^n(L))^2\right) + \frac{J}{2}\left((v_x^{n+1}(L))^2 - (v_x^n(L))^2\right) \\ &\quad + \int_{u^n(L)}^{u^{n+1}(L)} s(w)\, \mathrm{d}w.\end{aligned}$$

Next, (3.90) is multiplied by $\mu(v^{n+1} - v^n)$, and integrated over $[0, L]$, to obtain

$$\frac{\mu}{2}(\|v^{n+1}\|^2 - \|v^n\|^2) = \mu \int_0^L \frac{u^{n+1} - u^n}{\Delta t}(v^{n+1} - v^n)\, \mathrm{d}x.$$

Taking $w_h = u^{n+1} - u^n$ in (3.91) gives:

$$\begin{aligned}&\frac{\Lambda}{2}(\|u_{xx}^{n+1}\|^2 - \|u_{xx}^n\|^2) = -\mu \int_0^L \frac{v^{n+1} - v^n}{\Delta t}(u^{n+1} - u^n)\, \mathrm{d}x \\ &\quad -(u_x^{n+1}(L) - u_x^n(L)) \left(J \frac{v_x^{n+1}(L) - v_x^n(L)}{\Delta t} \right) \\ &\quad -(u^{n+1}(L) - u^n(L)) \Big(M \frac{v^{n+1}(L) - v^n(L)}{\Delta t} + \frac{\int_{u^n(L)}^{u^{n+1}(L)} s(w)\, \mathrm{d}w}{u^{n+1}(L) - u^n(L)} + d\Big(\frac{v^{n+1} + v^n}{2}\Big)\Big).\end{aligned}$$

Finally, equation (3.90) yields:

$$\tilde{V}(z^{n+1}) - \tilde{V}(z^n) = -\Delta t \left(\frac{v^{n+1} + v^n}{2}\right) d\Big(\frac{v^{n+1} + v^n}{2}\Big),$$

which proves the statement. □

Theorem 3.32. *Let $n \in \mathbb{N}$ and $\Delta t > 0$. Moreover, let an arbitrary $z^n \in W_h \times W_h$ be given. Then there exists a solution z^{n+1} to the system* (3.90)–(3.91).

Proof. First, equations (3.90) and (3.91) are rewritten in their vector representation:

$$\frac{\mathbb{U}^{n+1} - \mathbb{U}^n}{\Delta t} = \frac{\mathbb{V}^{n+1} + \mathbb{V}^n}{2} \tag{3.93}$$

$$\mathbb{A}\frac{\mathbb{V}^{n+1} - \mathbb{V}^n}{\Delta t} = -\mathbb{K}\frac{\mathbb{U}^{n+1} + \mathbb{U}^n}{2} - \mathbb{B}\left(\frac{\mathbb{V}^{n+1} + \mathbb{V}^n}{2}\right) - \tilde{\mathbb{C}}(\mathbb{U}^{n+1}, \mathbb{U}^n), \tag{3.94}$$

with $\tilde{\mathbb{C}}$ defined with:

$$\tilde{\mathbb{C}}(\mathbb{U}^{n+1}, \mathbb{U}^n)_j = (w_j)(L)\, H(u^{n+1}(L), u^n(L)).$$

Furthermore, let $g : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ be defined as:

$$g\left(\begin{bmatrix} \Delta\mathbb{U} \\ \Delta\mathbb{V} \end{bmatrix}\right) := \begin{bmatrix} \Delta t \left(\mathbb{V}^n + \frac{\Delta\mathbb{V}}{2}\right) \\ -\Delta t\, \mathbb{A}^{-1}\left(\mathbb{K}(\mathbb{U}^n + \frac{\Delta\mathbb{U}}{2}) + \mathbb{B}(\mathbb{V}^n + \frac{\Delta\mathbb{V}}{2}) + \tilde{\mathbb{C}}(\mathbb{U}^n + \Delta\mathbb{U}, \mathbb{U}^n)\right) \end{bmatrix}$$

It can be seen that $[\mathbb{U}^{n+1}\ \mathbb{V}^{n+1}]^\top$ solves (3.93)–(3.94), if and only if

$$\begin{bmatrix} \Delta\mathbb{U} \\ \Delta\mathbb{V} \end{bmatrix} := \begin{bmatrix} \mathbb{U}^{n+1} \\ \mathbb{V}^{n+1} \end{bmatrix} - \begin{bmatrix} \mathbb{U}^{n} \\ \mathbb{V}^{n} \end{bmatrix}$$

is a fixed point of g. Next, let the subset $S \subset \mathbb{R}^{2N}$ be defined with:

$$S := \{\Delta Z \in \mathbb{R}^{2N} : \Delta Z = \lambda g(\Delta Z),\ \lambda \in [0, 1]\}.$$

It can be shown that the set S is bounded. For this purpose, let $\Delta Z = [\Delta\mathbb{U}\ \Delta\mathbb{V}]^\top \in S$ be arbitrary. Moreover, let $u, v \in W_h$ be such that their vector representations are $\mathbb{U}^n + \Delta\mathbb{U}$ and $\mathbb{V}^n + \Delta\mathbb{V}$, respectively. Then the following holds:

$$\frac{u - u^n}{\Delta t} = \lambda \frac{v + v^n}{2},$$

$$\mu \int_0^L \frac{v - v^n}{\Delta t} w_h \, \mathrm{d}x + M \frac{v(L) - v^n(L)}{\Delta t} w_h(L) + J \frac{v_x(L) - v_x^n(L)}{\Delta t}(w_h)_x(L) =$$

$$-\lambda \left[\Lambda \int_0^L \frac{u_{xx} + u_{xx}^n}{2}(w_h)_{xx} \, \mathrm{d}x + \left(H(u(L), u^n(L)) + d\left(\frac{v(L) + v^n(L)}{2}\right)\right) w_h(L)\right],$$

$\forall w_h \in W_h$. Following the lines of the proof for Theorem 3.31, it follows that:

$$\lambda\left(\tilde{V}(z) - \tilde{V}(z^n)\right) = -\lambda^2 \frac{v(L) + v^n(L)}{2} d\left(\frac{v(L) + v^n(L)}{2}\right) \le 0. \tag{3.95}$$

If $\lambda = 0$, then it is trivial to see $z = z^n$. For $\lambda \in (0, 1]$, it follows $\tilde{V}(z) \le \tilde{V}(z^n)$. Thus S is bounded. Due to discussion in Remark 3.30, the function g is continuous, therefore trivially g is also compact. According to the Leray-Schauder fixed point theorem, g has a fixed point. This means that (3.93)–(3.94) is solvable, which proves the theorem. □

Remark 3.33. The nonlinear schemes, semi-discretization in space (3.84), and full-discretization in space and time (3.90)-(3.91), developed here are stable dissipative numerical schemes (Theorem (3.28) and (3.31)). Furthermore, they are solvable on $[0, \infty)$ and at each time step t_n respectively (Theorem 3.29 and 3.32). Finally, their dissipativity and solvability properties do not depend on the choice of the finite dimensional space for the Galerkin approximation. Moreover, the same scheme can be applied to a nonhomogeneous beam, i.e. when $\Lambda = \Lambda(x)$ and $\mu = \mu(x)$ are not constant on the interval $[0, L]$. The question of uniqueness of the fully discrete solution has not been considered here.

Remark 3.34. To the knowledge of the author, there are not many numerical methods for treating Euler–Bernoulli beam with nonlinearities at the boundary available in the literature. A similar approach has been introduced in [6], however the authors use FEM for discretization in both time and space. The idea to use Crank–Nicolson discretization in time as introduced in this work is novel. Furthermore, it allows for a straightforward proof of stability and dissipativity of the scheme.

Chapter 4

Nonlinear dynamic boundary control

In this chapter, the stability of an Euler-Bernoulli beam with tip body and nonlinear dynamic boundary control is analyzed. The results obtained in Section 4.1 are joint work with Dipl. Ing. Dominik Stürzer, and appear in [63]. First, the EBB equations are revised:

$$\begin{aligned}
\mu(x)u_{tt} + \Lambda u_{xxxx} &= 0, \quad 0 < x < L,\, t > 0, && (4.1)\\
u(t,0) &= 0, \quad t > 0, && (4.2)\\
u_x(t,0) &= 0, \quad t > 0, && (4.3)\\
Ju_{xtt}(t,L) + \Lambda u_{xx}(t,L) + \Theta_1(t) &= 0, \quad t > 0, && (4.4)\\
Mu_{tt}(t,L) - \Lambda u_{xxx}(t,L) + \Theta_2(t) &= 0, \quad t > 0. && (4.5)
\end{aligned}$$

The nonlinear control law, as introduced in Subsection 1.2.2, reads:

$$\begin{aligned}
(\zeta_1)_t(t) &= a_1(\zeta_1(t)) + b_1(\zeta_1(t))u_{xt}(t,L),\\
(\zeta_2)_t(t) &= a_2(\zeta_2(t)) + b_2(\zeta_2(t))u_t(t,L),\\
\Theta_1(t) &= k_1(u_x(t,L)) + c_1(\zeta_1(t)) + d_1(\zeta_1(t))u_{xt}(t,L),\\
\Theta_2(t) &= k_2(u(t,L)) + c_2(\zeta_2(t)) + d_2(\zeta_2(t))u_t(t,L).
\end{aligned} \qquad (4.6)$$

First the asymptotic stability of such closed loop system will be stated. Due to the lack of exponential stability in the linear closed-loop system, it is expected that the nonlinear controller does not lead to exponential stability of the system either. However, this question will not be discussed in this thesis. Instead, the weak formulation of the system and the development of a dissipative numerical method for the system are considered.

4.1 Stability of the closed-loop system

In this chapter, notation b_j, c_j, d_j will be used to denote nonlinear functions of the controller variable, unlike Chapter 2, where this notation was used for vectors and constants, respectively. However, the same notation is kept to emphasize the natural extension of the SPR linear dynamic controller to a nonlinear one.

4.1.1 Evolution formulation and dissipativity of the system

In this section, the following regularity of the coefficient functions of the controller law shall be required. It is assumed that $a_j, b_j \in C^2(\mathbb{R}^n;\mathbb{R}^n)$, $c_j, d_j \in C^2(\mathbb{R}^n;\mathbb{R})$ and $k_j \in C^2(\mathbb{R})$, $j = 1,2$. Furthermore, it is assumed that the control law given by (4.6) is strictly positive real. Hence, there exist some $V_j \in C^3(\mathbb{R}^n,\mathbb{R})$ such that

$$V_j(\zeta_j) > 0, \quad \forall \zeta_j \in \mathbb{R}^n \setminus \{0\} \tag{4.7a}$$
$$V_j(0) = 0, \tag{4.7b}$$
$$\lim_{|\zeta_j|\to\infty} V_j(\zeta_j) = \infty, \tag{4.7c}$$

for $j = 1,2$. Moreover, nonlinear coefficient functions satisfy:

$$\nabla V_j(\zeta_j)\cdot a_j(\zeta_j) \;<\; 0, \quad \text{if } \zeta_j \neq 0, \tag{4.8}$$
$$\nabla V_j(\zeta_j)\cdot b_j(\zeta_j) \;=\; c_j(\zeta_j), \tag{4.9}$$
$$d_j(\zeta_j) \;>\; 0, \tag{4.10}$$

for all $\zeta_j \in \mathbb{R}^n$, where $j = 1,2$. Furthermore, the following definition is introduced:

$$P_j := H(V_j)(0) > 0, \tag{4.11}$$

where $H(V_j)$ denotes the Hessian of V_j, $j = 1,2$. For the coefficient functions, the following assumptions are made. There exist *regular* matrices $A_j \in \mathbb{R}^{n\times n}$ such that for all $\zeta_j \in \mathbb{R}^n$:

$$a_j(\zeta_j) = A_j\zeta_j + \alpha_j(\zeta_j), \tag{4.12a}$$
$$|\alpha_j(\zeta_j)| = \mathcal{O}(|\zeta_j|^2) \quad \text{as } \zeta_j \to 0. \tag{4.12b}$$

Note that (4.8) implies

$$\zeta_j^T(P_jA_j)\zeta_j \le 0, \quad \forall \zeta_j \in \mathbb{R}^n, \tag{4.12c}$$
$$|\nabla V_j(\zeta_j)\cdot a_j(\zeta_j)| \ge C|\zeta_j|^2 \quad \text{as } \zeta_j \to 0. \tag{4.12d}$$

Furthermore, let $B_j := b_j(0) \in \mathbb{R}^n$. Then, for all $\zeta_j \in \mathbb{R}^n$ there holds:

$$b_j(\zeta_j) = B_j + \beta_j(\zeta_j), \tag{4.13a}$$
$$\beta_j(0) = 0. \tag{4.13b}$$

Equality (4.9) implies that $c_j(0) = 0$. Defining $C_j := \nabla c_j(0) \in \mathbb{R}^n$, there holds for all $\zeta_j \in \mathbb{R}^n$:

$$c_j(\zeta_j) = C_j\cdot\zeta_j + \gamma_j(\zeta_j), \tag{4.14a}$$
$$|\gamma_j(\zeta_j)| = \mathcal{O}(|\zeta_j|^2) \quad \text{as } \zeta_j \to 0. \tag{4.14b}$$

Note that (4.9) implies

$$P_j B_j = C_j. \tag{4.14c}$$

Let the constant $D_j > 0$ be defined by $D_j := d_j(0)$. Then for all $\zeta_j \in \mathbb{R}^n$ there holds

$$d_j(\zeta_j) = D_j + \delta_j(\zeta_j), \tag{4.15a}$$
$$\delta_j(0) = 0, \tag{4.15b}$$

and that the scalar functions k_j satisfy for all $s \in \mathbb{R}$:

$$k_j(s) = K_j s + \kappa_j(s), \tag{4.16a}$$
$$\int_0^s k_j(\sigma)\,\mathrm{d}\sigma \geq 0, \tag{4.16b}$$

for some $K_j > 0$, $j = 1, 2$. The nonlinear system (4.1)-(4.6) can be written as an evolution equation

$$\begin{aligned} z_t &= \mathcal{A} z, \\ z(0) &= z_0, \end{aligned} \tag{4.17}$$

with the nonlinear operator $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ given by

$$\mathcal{A}\begin{bmatrix} u \\ v \\ \zeta_1 \\ \zeta_2 \\ \xi \\ \psi \end{bmatrix} = \begin{bmatrix} v \\ -\frac{\Lambda}{\mu} u_{xxxx} \\ a_1(\zeta_1) + b_1(\zeta_1)\frac{\xi}{J} \\ a_2(\zeta_2) + b_2(\zeta_2)\frac{\psi}{M} \\ -\Lambda u_{xx}(L) - k_1(u_x(L)) - c_1(\zeta_1) - d_1(\zeta_1)\frac{\xi}{J} \\ \Lambda u_{xxx}(L) - k_2(u(L)) - c_2(\zeta_2) - d_2(\zeta_2)\frac{\psi}{M} \end{bmatrix}.$$

The domain $D(\mathcal{A})$ and the state space $\mathcal{H}$ are as defined in Subsection 2.1.1 for the dynamic linear boundary control:

$$\mathcal{H} := \{z = (u, v, \zeta_1, \zeta_2, \xi, \psi)^\top : u \in \tilde{H}_0^2(0, L), v \in L^2(0, L), \zeta_1, \zeta_2 \in \mathbb{R}^n, \xi, \psi \in \mathbb{R}\},$$

$$D(\mathcal{A}) = \{z \in \mathcal{H} \colon u \in \tilde{H}_0^4(0, L), v \in \tilde{H}_0^2(0, L), \zeta_1, \zeta_2 \in \mathbb{R}^n, \xi = J v_x(L), \psi = M v(L)\}.$$

Space $\mathcal{H}$ is a Hilbert space with respect to the inner product

$$\begin{aligned} \langle z, \breve{z} \rangle \;:=\; & \frac{1}{2}\int_0^L \Lambda\, u_{xx}\breve{u}_{xx}\, dx + \frac{1}{2}\int_0^L \mu\, v\breve{v}\, dx + \frac{1}{2J}\xi\breve{\xi} + \frac{1}{2M}\psi\breve{\psi} \\ + \; & \frac{K_1}{2}u_x(L)\breve{u}_x(L) + \frac{K_2}{2}u(L)\breve{u}(L) + \frac{1}{2}\zeta_1^\top P_1\breve{\zeta}_1 + \frac{1}{2}\zeta_2^\top P_2\breve{\zeta}_2, \end{aligned} \tag{4.18}$$

where P_j is given by (4.11), $j = 1, 2$.

As seen in Section 1.2.2, the functional $V : \mathcal{H} \to \mathbb{R}$ defined with

$$\begin{aligned} V(z) := \frac{1}{2}\int_0^L \left(\Lambda u_{xx}(x)^2 + \mu v(x)^2\right) dx + \frac{1}{2M}\psi^2 + \frac{1}{2J}\xi^2 \\ + \int_0^{u_x(L)} k_1(\sigma)\,\mathrm{d}\sigma + \int_0^{u(L)} k_2(\sigma)\,\mathrm{d}\sigma + V_1(\zeta_1(t)) + V_2(\zeta_2(t)), \end{aligned} \tag{4.19}$$

for $z = [u\; v\; \zeta_1\; \zeta_2\; \xi\; \psi]^\top \in \mathcal{H}$, is a good candidate for a Lyapunov functional of the system (4.17). It was shown in (1.16), that the derivative of function $t \to V(z(t))$ for classical solutions $z(t)$ reads:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}V(z(t)) &= \nabla V_1(\zeta_1)\cdot A_1(\zeta_1) - d_1(\zeta_1)\left(\frac{\xi}{J}\right)^2 + \nabla V_2(\zeta_2)\cdot A_2(\zeta_2) - d_2(\zeta_2)\left(\frac{\psi}{M}\right)^2 \\ &\leq -d_1(\zeta_1)\left(\frac{\xi}{J}\right)^2 - d_2(\zeta_2)\left(\frac{\psi}{M}\right)^2. \end{aligned} \tag{4.20}$$

With the notation above for the coefficient functions, the operator $\mathcal{A}$ can be decomposed into a linear and a nonlinear part. The linear part of $\mathcal{A}$ is denoted by A, and defined as the linearization of $\mathcal{A}$ around the origin:

$$A : \begin{bmatrix} u \\ v \\ \zeta_1 \\ \zeta_2 \\ \xi \\ \psi \end{bmatrix} \mapsto \begin{bmatrix} v \\ -\frac{\Lambda}{\mu}u_{xxxx} \\ A_1\zeta_1 + \frac{1}{J}B_1\xi \\ A_2\zeta_2 + \frac{1}{M}B_2\psi \\ -\Lambda u_{xx}(L) - [C_1\zeta_1 + \frac{1}{J}D_1\xi + K_1u_x(L)] \\ \Lambda u_{xxx}(L) - [C_2\zeta_2 + \frac{1}{M}D_2\psi + K_2u(L)] \end{bmatrix},$$

with the domain $D(A) = D(\mathcal{A})$. The nonlinear part is denoted by $\mathcal{N}$, and it is defined as the difference $\mathcal{N} := \mathcal{A} - A$:

$$\mathcal{N} : \begin{bmatrix} u \\ v \\ \zeta_1 \\ \zeta_2 \\ \xi \\ \psi \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \\ \alpha_1(\zeta_1) + \frac{1}{J}\beta_1(\zeta_1)\xi \\ \alpha_2(\zeta_2) + \frac{1}{M}\beta_2(\zeta_2)\psi \\ -\gamma_1(\zeta_1) - \frac{1}{J}\delta_1(\zeta_1)\xi - \kappa_1(u_x(L)) \\ -\gamma_2(\zeta_2) - \frac{1}{M}\delta_2(\zeta_2)\psi - \kappa_2(u(L)) \end{bmatrix}.$$

Under the above conditions, the linear part A generates a C_0-semigroup of contractions in $\mathcal{H}$. In order to see this, the discussion from [40] and Section 2.1 shall be closely followed.

Lemma 4.1. *The operator A is dissipative in $\mathcal{H}$ with respect to the inner product* (4.18).

Proof. This result has already been shown in Section 4.2 in [40]. In particular, a brief calculation for $z \in D(A)$ using (4.12c) yields:

$$\langle Az, z\rangle_{\mathcal{H}} = \zeta_1^T(P_1A_1)\zeta_1 + \zeta_2^T(P_2A_2)\zeta_2 - D_1|v_x(L)|^2 - D_2|v(L)|^2 \leq 0.$$

□

Lemma 4.2. *The inverse A^{-1} exists and is compact.*

A sketch of the proof can be found in Section 4.2 in [40], for a detailed proof, see the Appendix A. Now applying the Lumer-Phillips theorem, the following result is obtained.

Theorem 4.3. *The linear operator A with domain $D(A)$ generates a C_0-semigroup of contractions, denoted by $(\mathrm{e}^{tA})_{t\geq 0}$.*

Remark 4.4. Since A is the infinitesimal generator of a C_0-semigroup of contractions, A is dissipative and $\operatorname{ran}(\lambda - A) = \mathcal{H}$ for all $\lambda > 0$, in particular $\operatorname{ran}(I - A) = \mathcal{H}$. So A is *maximal dissipative* according to Theorem 2.2 in [21].

4.1.2 Existence and uniqueness of the mild solution

The following initial value problem in $\mathcal{H}$ shall be considered:

$$z_t(t) = \mathcal{A}z(t) = Az(t) + \mathcal{N}z(t), \tag{4.21a}$$

$$z(0) = z_0 \in \mathcal{H}. \tag{4.21b}$$

A function $z\colon [0,T) \to \mathcal{H}$ is said to be a mild solution if it satisfies the Duhamel's formula:

$$z(t) = \mathrm{e}^{tA}z_0 + \int_0^t \mathrm{e}^{(t-s)A}\mathcal{N}z(s)\,\mathrm{d}s, \quad 0 \leq t < T. \tag{4.22}$$

The result on existence of local solutions can be immediately obtained.

Proposition 4.5. *For every $z_0 \in \mathcal{H}$, there exists some maximal $0 < T_{\max}(z_0) \leq \infty$ such that (4.21) has a unique mild solution $z(t)$ on $[0, T_{\max}(z_0))$. If $z_0 \in D(\mathcal{A})$, the corresponding mild solution $z(t)$ is a classical solution. If $T_{\max}(z_0) < \infty$, then $\lim_{t\nearrow T_{\max}(z_0)} \|z(t)\|_{\mathcal{H}} = \infty$.*

Proof. By assumption, the functions $\alpha_j, \beta_j, \gamma_j, \delta_j$ and κ_j are continuously differentiable, so $\mathcal{N} : \mathcal{H} \to \mathcal{H}$ is also continuously differentiable, and thus locally Lipschitz continuous. Furthermore, A is the generator of a C_0-semigroup. Now Theorem B.5 in Appendix B yields the existence of a unique mild solution, and the blow-up at $T_{\max}(y_0)$. Since $\mathcal{N}$ is continuously differentiable, according to Theorem B.6 in Appendix B any mild solution for $z_0 \in D(\mathcal{A})$ is a classical solution. □

Next, it will be demonstrated that the functional V is a Lyapunov function for the system (4.21). Obviously $V(z) \geq 0$ for all $z \in \mathcal{H}$.

Lemma 4.6. *The function V is continuous in $\mathcal{H}$.*

Proof. The continuity of the terms in V is immediate, except for the terms with functions k_j. However, due to the continuous embedding $H^2 \hookrightarrow C^1$ the continuity of the remaining terms with k_j follows as well. □

Lemma 4.7. *Under the assumption* (4.7c)*, it holds for any sequence* $(z_k)_{k\in\mathbb{N}} \subset \mathcal{H}$*:*

$$\sup_{k\in\mathbb{N}} V(z_k) < \infty \quad \Leftrightarrow \quad \sup_{k\in\mathbb{N}} \|z_k\|_{\mathcal{H}} < \infty.$$

Proof. It suffices to notice that $\{V_j((\zeta_j)_k)\}_{k\in\mathbb{N}}$ is unbounded if and only if $\{\|(\zeta_j)_k\|_{\mathbb{R}^n}\}_{k\in\mathbb{N}}$ is unbounded. □

Again, the generalized time derivative $\dot{V}(z_0)$ of V along the mild solution $z(t)$ of (4.21) with initial condition $z_0 \in \mathcal{H}$ shall be considered. For classical solutions, equation (4.20) directly implies the following lemma.

Lemma 4.8. *For* $z_0 \in D(\mathcal{A})$ *there holds* $\dot{V}(z_0) \leq 0$*.*

However, it shall be demonstrated that $\dot{V}(z_0) \leq 0$ for all $z_0 \in \mathcal{H}$. This is achieved by uniform approximation of mild solutions by classical solutions, see Theorem 4.11 below.

Corollary 4.9. *For* $z_0 \in D(\mathcal{A})$ *the corresponding classical solution* $z(t)$ *of* (4.21) *is global, i.e. it exists for all* $t \in [0, \infty)$*.*

Proof. According to Lemma 4.8, V is non-increasing along $z(t)$. Thus according to Lemma 4.7, no blow-up occurs in $z(t)$, and therefore according to Proposition 4.5, it follows that $T_{\max}(z_0) = \infty$. □

Since the classical solutions are global and $D(\mathcal{A}) \subset \mathcal{H}$ is dense, classical solutions can be utilized to approximate mild (non-classical) solutions:

Proposition 4.10. *Let* $z_0 \in \mathcal{H}$ *and* $(z_{n,0})_{n\in\mathbb{N}} \subset D(\mathcal{A})$ *be such that* $z_{n,0} \to z_0$ *in* $\mathcal{H}$*. Denote by* $z_n(t)$ *the classical solution of* (4.21) *to the initial value* $z_{n,0}$ *and let* $z : [0, T] \to \mathcal{H}$ *be the mild solution corresponding to the initial value* z_0*. Then the sequence* $z_n(t)$ *converges to* $z(t)$ *in* $C([0, T]; \mathcal{H})$*.*

Proof. The result follows from the Proposition B.7 in Appendix B, since $\mathcal{N}$ is locally Lipschitz continuous. □

Theorem 4.11. *For any* $z_0 \in \mathcal{H}$ *the corresponding solution* $z(t)$ *of the initial value problem* (4.21) *is global in time. Furthermore,* $t \mapsto V(z(t))$ *is non-increasing on* $[0, \infty)$ *and* z *is uniformly bounded in* $\mathcal{H}$ *on* $[0, \infty)$*.*

Proof. Consider $z_0 \in \mathcal{H}$ and a sequence $(z_{n,0}) \subset D(\mathcal{A})$ with $z_{n,0} \to z_0$ in $\mathcal{H}$. Due to the convergence $z_n(t) \to z(t)$ for all $t \in [0, T_{max}(z_0))$ shown in Proposition 4.10 and the continuity of V, it holds $V(z_n(t)) \to V(z(t))$ for all $0 \leq t < T_{max}(z_0)$. Since V is non-increasing along every $z_n(t)$, this implies that $t \mapsto V(z(t))$ is non-increasing on $[0, T(z_0))$.

Thus, according to Lemma 4.7, no blow-up of $z(t)$ can occur at $t = T_{max}(z_0)$. Hence, according to Proposition 4.5 the solution is global in time. Uniform boundedness of z follows from the Lemma 4.7.

□

As a consequence of the results above, the following is shown.

Corollary 4.12. *The function V is a Lyapunov function for the initial value problem (4.21).*

Let a family of nonlinear operators $\{S(t)\}_{t\geq 0}$ on $\mathcal{H}$ be defined by $S(t)z_0 := z(t)$ for all $t \geq 0$ and for every $z_0 \in \mathcal{H}$, where $z(t)$ is the mild solution corresponding to the initial condition z_0. Then, it follows that the family $S \equiv (S(t))_{t\geq 0}$ is a strongly continuous semigroup of nonlinear (bounded, continuous) operators in $\mathcal{H}$, cf. Theorem 9.3.2 in [9].

Remark 4.13. Since (4.7c) is only needed to show that no blow-up of the solution occurs, it may be replaced by the weaker assumption

$$\lim_{|\zeta_j|\to\infty} V_j(\zeta_j) > V(z_0), \tag{4.23}$$

depending on the initial condition z_0 for the problem (4.21). In order to demonstrate this, note that according to Theorem 4.11 the function $t \mapsto V(z(t))$ is non-increasing (this is independent of (4.7c)), which ensures that no blow-up can occur in any component of $z(t)$ except for ζ_j. However, if $\zeta_j(t)$ would blow-up, (4.23) would imply that $\lim_{t\to\infty} V(z(t)) > V(z_0)$. So $V(z(t))$ could not be monotonically decreasing, which is a contradiction. Therefore, (4.23) suffices to show that no blow-up occurs and that the solution is global in time.

4.1.3 Characterization of the ω-limit Set

In this subsection the properties of ω-limit sets are investigated. It is possible that $\omega(z_0) = \emptyset$, but their existence shall be discussed later. As defined in the previous subsection, S is the strongly continuous (nonlinear) semigroup generated by $\mathcal{A}$ on $\mathcal{H}$.

As already seen from the Section 3.3 where a different nonlinear semigroup was considered, there holds:

Lemma 4.14. *Let $z_0 \in \mathcal{H}$ be fixed. The set $\omega(z_0)$ is S-invariant, i.e. $S(t)\omega(z_0) \subseteq \omega(z_0)$ for all $t \geq 0$. Moreover, the following limit exists:*

$$\nu(z_0) := \lim_{t\to\infty} V(S(t)z_0) \geq 0. \tag{4.24}$$

Furthermore, if $\omega(z_0) \neq \emptyset$ then there holds

$$\forall z \in \omega(z_0): \quad V(z) = \nu(z_0).$$

In particular, $\dot{V}(z) = 0$ for all $z \in \omega(z_0)$.

Proof. The first statement follows according to Proposition 9.1.7 in [9]. According to the results of Section 4.1.2, the function $t \mapsto V(S(t)z_0)$ is monotonically decreasing, and bounded from below by 0. Therefore, the limit in (4.24) exists. For every $z \in \omega(z_0)$ there exists a sequence $(t_n) \subset \mathbb{R}^+$ such that $S(t_n)z_0 \to z$. Since V is continuous, cf. Section 4.1.2, this implies that $V(z) = \lim_{n\to\infty} V(S(t_n)z_0)$. Due to (4.24) the right hand side equals $\nu(z_0)$, and the result follows.

□

Lemma 4.14 shall be used to identify the possible ω-limit sets by investigating trajectories along which the Lyapunov function V is constant.

Lemma 4.15. *Let $z_0 \in \mathcal{H}$ be such that $V(S(t)z_0) = \nu(z)$ for all $t \geq 0$, i.e. V is constant along $\gamma(z_0)$. Then $\gamma(z_0) \subset \{z \in \mathcal{H} : z = [u, v, 0, 0, 0, 0]^\top\}$.*

Proof. First, let $z_0 \in D(\mathcal{A})$. From Lemma 4.8 and the corresponding proof, it follows that

$$\dot{V}(S(t)z_0) = a_1(\zeta_1) \cdot \nabla V_1(\zeta_1) + a_2(\zeta_2) \cdot \nabla V_2(\zeta_2) - d_1(\zeta_1)|v_x(L)|^2 - d_2(\zeta_2)|v(L)|^2, \quad \forall t \geq 0, \tag{4.25}$$

where $[u, v, \zeta_1, \zeta_2, Jv_x(L), Mv(L)]^\top \equiv S(t)z_0$. Since it is required that (4.25) is equal to zero, according to (4.8) and (4.10) this holds if and only if $\xi = \psi = \zeta_1 = \zeta_2 = 0$. Let now $z_0 \in \mathcal{H} \setminus D(\mathcal{A})$. Then there is a sequence $(z_{n,0}) \subset D(\mathcal{A})$ such that $z_{n,0} \to z_0$ as $n \to \infty$. According to Proposition 4.10, the sequence $S(t)z_{n,0}$ converges to $S(t)z_0$ uniformly on $[0, T]$. Therefore, also for the components of $S(t)z_0$ there holds:

$$\zeta_{j,n}(t) \to \zeta_j(t), \quad \text{in } C([0,T];\mathbb{R}^n), \tag{4.26}$$

$$Mv_n(t, L) \to \psi(t), \quad \text{in } C([0,T];\mathbb{R}), \tag{4.27}$$

$$J(v_n)_x(t, L) \to \xi(t), \quad \text{in } C([0,T];\mathbb{R}). \tag{4.28}$$

Together with (4.25) this implies

$$\left(\dot{V}(S(t)z_{n,0})\right)_{n\in\mathbb{N}}$$

is a Cauchy sequence in $C([0,T];\mathbb{R})$. Since V is locally Lipschitz in $\mathcal{H}$, it also holds that $(V(S(t)z_{n,0}))_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];\mathbb{R})$. Hence $(V(S(t)z_{n,0}))_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^1([0,T];\mathbb{R})$. Now, there exists a unique $v(t) \in C^1([0,T];\mathbb{R})$ such that

$$V(S(t)z_{n,0}) \to v(t) \quad \text{in } C^1([0,T];\mathbb{R}). \tag{4.29}$$

On the other hand, it holds that $\lim_{n\to\infty} V(S(t)z_{n,0}) = V(S(t)z_0) = \nu(z_0)$ for every $t \geq 0$, and hence $v(t) \equiv \nu(z_0)$. Together with (4.29) this implies $\dot{V}(S(t)z_{n,0}) \to 0$ uniformly on $[0,T]$. By using (4.25) for every $z_{n,0}$ this now yields that in (4.26)–(4.28) the limits $\zeta_j(t) = \xi(t) = \psi(t) = 0$ are obtained. Therefore $S(t)z_0$ is of the form $S(t)z_0 = [u(t), v(t), 0, 0, 0, 0]^\top$. □

In order to show that the ω-limit set consists only of the zero solution, the following proposition will be used.

Proposition 4.16. *Let $z_0 \in \mathcal{H}$. Then for all $t > 0$ the following holds:*

$$\int_0^t S(\tau)z_0 \,\mathrm{d}\tau \in D(\mathcal{A}),$$

and

$$S(t)z_0 - z_0 = A\int_0^t S(\tau)z \,\mathrm{d}\tau + \int_0^t \mathcal{N}S(\tau)z_0 \,\mathrm{d}\tau.$$

This proposition follows directly from Theorem A.3 in Appendix A. Now the following can be demonstrated.

Theorem 4.17. *Let $\Omega \subset \mathcal{H}$ be the largest S-invariant subset of $\mathcal{M}$, where $\mathcal{M}$ is the set on which V is constant:*

$$\mathcal{M} := \{z \in \mathcal{H} \colon \dot{V}(z) = 0\}.$$

Then $\Omega = \{0\}$. In particular, for any $z_0 \in \mathcal{H}$ either $\omega(z_0) = \emptyset$ or $\omega(z_0) = \{0\}$.

Proof. Step 1 (linear system for $u(t)$, $v(t)$): Take a fixed $z_0 \in \Omega$, and let $z(t)$ be the corresponding mild solution. First let it be noted that, according to Proposition 4.16, there holds for all $t \geq 0$:

$$\begin{aligned} 0 &= \int_0^t \psi(s)\,\mathrm{d}s = M \int_0^t v(s,L)\,\mathrm{d}s = M\big(u(t,L) - u_0(L)\big),\\ 0 &= \int_0^t \xi(s)\,\mathrm{d}s = J\Big(\int_0^t v(s,x)\,\mathrm{d}s\Big)_x\Big|_{x=L} = J\big(u_x(t,L) - (u_0)_x(L)\big). \end{aligned}$$

Thus $u(t,L)$ and $u_x(t,L)$ are constant in time. Proposition 4.16 also implies that the (projected) mild solution $y_p(t) = [u(t), v(t)]^\top$ satisfies the following system:

$$u(t) - u_0 = \int_0^t v(s)\,\mathrm{d}s, \tag{4.30a}$$

$$v(t) - v_0 = -\frac{\Lambda}{\mu}\Big(\int_0^t u(s)\,\mathrm{d}s\Big)_{xxxx} \tag{4.30b}$$

$$0 = \Lambda\Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)_{xx}\Big|_{x=L} + K_1 \cdot \Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)_x\Big|_{x=L} + \int_0^t \kappa_1(u_x(s,L))\,\mathrm{d}s, \tag{4.30c}$$

$$0 = -\Lambda\Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)_{xxx}\Big|_{x=L} + K_2 \cdot \Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)\Big|_{x=L} + \int_0^t \kappa_2(u(s,L))\,\mathrm{d}s. \tag{4.30d}$$

Mild solutions satisfy $u \in C(\mathbb{R}^+; \tilde{H}_0^2(0,L))$. Hence, the integration and differentiation in the last term of (4.30c) can be interchanged. Since $u_x(t,L)$ is constant, there holds (for $(u_0)_x(L) \neq 0$):

$$\int_0^t \kappa_1(u_x(s,L))\,\mathrm{d}s = t\kappa_1((u_0)_x(L)) = \frac{\kappa_1((u_0)_x(L))}{(u_0)_x(L)}\Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)_x\Big|_{x=L}.$$

Next, the following constants are defined (since $\kappa_j(0) = 0$):

$$\begin{aligned} \tilde{K}_1 &:= K_1 + \frac{\kappa_1((u_0)_x(L))}{(u_0)_x(L)}, \quad \text{if } (u_0)_x(L) \neq 0, \quad \text{else } \tilde{K}_1 := K_1,\\ \tilde{K}_2 &:= K_2 + \frac{\kappa_2(u_0(L))}{u_0(L)}, \quad \text{if } u_0(L) \neq 0, \quad \text{else } \tilde{K}_2 := K_2. \end{aligned} \tag{4.31}$$

With this notation (4.30) can be rewritten as

$$u(t) - u_0 = \int_0^t v(s)\,\mathrm{d}s, \tag{4.32a}$$

$$v(t) - v_0 = -\frac{\Lambda}{\mu}\Big(\int_0^t u(s)\,\mathrm{d}s\Big)_{xxxx} \tag{4.32b}$$

$$0 = \Lambda\Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)_{xx}\Big|_{x=L} + \tilde{K}_1\Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)_x\Big|_{x=L}, \tag{4.32c}$$

$$0 = -\Lambda\Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)_{xxx}\Big|_{x=L} + \tilde{K}_2\int_0^t u(s,x)\,\mathrm{d}s\Big|_{x=L}, \tag{4.32d}$$

making this system linear. Thus, the projected vector $y_p(t) = [u(t), v(t)]^\top$ is the unique mild solution of

$$(y_p)_t = A_p y_p, \tag{4.33a}$$

$$y_p(0) = [u_0, v_0]^\top, \tag{4.33b}$$

with the operator

$$A_p : \begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} v \\ -\frac{\Lambda}{\mu}u_{xxxx} \end{bmatrix}.$$

The equations (4.32c) and (4.32d) are incorporated in the domain $D(A_p)$. For further details on the operator A_p in the space $\mathcal{H}_p$ see the Appendix A.

Step 2 (proof of $u(t,L) = u_x(t,L) = 0$): Now the solutions of the projected problem (4.33) with the additional property that $u(t,L)$ and $u_x(t,L)$ are constant in time are investigated. Since the semigroup e^{tA_p} is unitary in $\mathcal{H}_p$, it is known that $\|v(t)\|_{L^2} \le C = \frac{1}{\mu}\|y_p(0)\|_{\mathcal{H}_p}$ for all $t \ge 0$ (cf. (A.32)). Applying the norm to (4.32b) this yields

$$\sup_{t\ge 0}\Big\|\Big(\int_0^t u(s)\,\mathrm{d}s\Big)_{xxxx}\Big\|_{L^2(0,L)} < \infty. \tag{4.34}$$

Next, the following Gagliardo–Nirenberg inequalities are applied (cf. [53]), which guarantee the existence of a $C > 0$ such that there holds for all $t \ge 0$:

$$\begin{aligned} \Big\|\int_0^t u(s)\,\mathrm{d}s\Big\|_{L^\infty(0,L)} &\le C\Big\|\Big(\int_0^t u(s)\,\mathrm{d}s\Big)_{xxxx}\Big\|_{L^2(0,L)}^{\frac{1}{8}}\Big\|\int_0^t u(s)\,\mathrm{d}s\Big\|_{L^2(0,L)}^{\frac{7}{8}}, \\ \Big\|\int_0^t u_x(s)\,\mathrm{d}s\Big\|_{L^\infty(0,L)} &\le C\Big\|\Big(\int_0^t u(s)\,\mathrm{d}s\Big)_{xxxx}\Big\|_{L^2(0,L)}^{\frac{3}{8}}\Big\|\int_0^t u(s)\,\mathrm{d}s\Big\|_{L^2(0,L)}^{\frac{5}{8}}. \end{aligned} \tag{4.35}$$

The first factor on the right hand side in both inequalities is uniformly bounded (with respect to t) due to (4.34). The second factor $t \mapsto \|u(t)\|_{L^2(0,L)}$ is uniformly bounded according to Theorem 4.11, and therefore $t \mapsto \|\int_0^t u(s)\,\mathrm{d}s\|_{L^2(0,L)}$ grows at most linearly. Hence, (4.35) implies that $t \mapsto \|\int_0^t u(s,L)\,\mathrm{d}s\|_{L^\infty(0,L)}$ is of order at most $t^{\frac{7}{8}}$ in time and

$t \mapsto \| \int_0^t u_x(s,L)\,\mathrm{d}s\|_{L^\infty(0,L)}$ of order $t^{\frac{5}{8}}$ at most as $t \to \infty$. But this contradicts the fact that $u(t,L)$ and $u_x(t,L)$ are constant, unless $u_0(L) = (u_0)_x(L) = 0$. This shows that $u(t,L) = u_x(t,L) = 0$ for all $t \geq 0$.

Step 3 (Holmgren's Theorem): By repeating integration in time, C^4-solutions of (4.33a) shall now be constructed, to which the Holmgren Uniqueness Theorem can be applied [34, Section 3.5]. Let $y_1(t) \equiv [u_1(t), v_1(t)]^\top := \int_0^t y_p(s)\,\mathrm{d}s + A_p^{-1}[u_0, v_0]^\top$. Due to Theorem 1.2.4 in [56] and Lemma A.6 it follows that $y_1(t) \in D(A_p)$ for all $t \geq 0$. So y_1 is a classical solution of (4.33a) to the initial condition $y_1(0) = A_p^{-1}[u_0, v_0]^\top$. Furthermore, because of $u(t,L) = u_x(t,L) = 0$, again $u_1(t,L), (u_1)_x(t,L)$ are constant in time. Analogously, it can be shown that $u_1(t,L) = (u_1)_x(t,L) = 0$.

Next, solutions of higher regularity shall be constructed. The previously described step is repeated and the function y_n is recursively defined: $y_n(t) \equiv [u_n(t), v_n(t)]^\top := \int_0^t y_{n-1}(s)\,\mathrm{d}s + A_p^{-n}[u_0, v_0]^\top$. Function y_n solves (4.33a) classically with the initial condition $y_n(0) = A_p^{-n}[u_0, v_0]^\top$. Again there holds $u_n(t,L) = (u_n)_x(t,L) = 0$. Furthermore, by definition on the one hand it follows $A_p y_n(t) = y_{n-1}(t)$. On the other hand $A_p[u_n, v_n]^\top = [v_n, -\frac{\Lambda}{\mu}(u_n)_{xxxx}]^\top$, therefore it can be shown inductively that $y_n \in C(\mathbb{R}^+, \tilde{H}_0^{2n+2}(0,L) \times \tilde{H}_0^{2n}(0,L))$. Now, let the solution u_n for $n \geq 2$ be considered. It satisfies the following partial differential equation with boundary conditions:

$$(u_n)_{tt} = -\frac{\Lambda}{\mu}(u_n)_{xxxx}, \tag{4.36a}$$

$$[u_n(0,x), (u_n)_t(0,x)]^\top = A_p^{-n}[u_0, v_0]^\top, \tag{4.36b}$$

$$u_n(t,0) = (u_n)_x(t,0) = 0, \tag{4.36c}$$

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k} u_n(t,L) = 0, \quad k = 0,1,2,3. \tag{4.36d}$$

From equation (4.36a), $u_n \in C(\mathbb{R}^+; \tilde{H}_0^{2n+2}(0,L))$, and the fact that $(u_n)_t = v_n \in C(\mathbb{R}^+; \tilde{H}_0^{2n}(0,L))$, following properties for the mixed fourth order space-time derivatives of u_n are obtained:

$$\begin{aligned}
(u_n)_{xxxx} &\in C(\mathbb{R}^+, \tilde{H}_0^{2n-2}(0,L)),\\
(u_n)_{txxx} &\in C(\mathbb{R}^+, \tilde{H}_0^{2n-3}(0,L)),\\
(u_n)_{ttxx} &= -\frac{\Lambda}{\mu}\frac{\mathrm{d}^6}{\mathrm{d}x^5} u_n \in C(\mathbb{R}^+, \tilde{H}_0^{2n-4}(0,L)),\\
(u_n)_{tttx} &= -\frac{\Lambda}{\mu}\frac{\mathrm{d}^5}{\mathrm{d}x^5} v_n \in C(\mathbb{R}^+, \tilde{H}_0^{2n-5}(0,L)),\\
(u_n)_{tttt} &= \frac{\Lambda^2}{\mu^2}\frac{\mathrm{d}^8}{\mathrm{d}x^8} u_n \in C(\mathbb{R}^+, \tilde{H}_0^{2n-6}(0,L)).
\end{aligned}$$

So for $n \geq 4$, all mixed derivatives of u_n of order four lie in $C(\mathbb{R}^+; \tilde{H}_0^2(0,L)) \subset C(\mathbb{R}^+ \times [0,L])$. Thus $u_n(t,x)$ is a C^4-solution of (4.36).

Now the Holmgren Uniqueness Theorem [34, Section 3.5] can be applied on the strip $\mathbb{R}^+ \times (0, L)$. Due to (4.36d) all partial derivatives up to order 3 of u_4 vanish on the line $\mathbb{R}^+ \times \{L\}$. Therefore, Holmgren's Uniqueness Theorem implies that $u_4 = 0$ has to hold everywhere on $\mathbb{R}^+ \times (0, L)$. (See also the proof of Lemma 3 in [45] for a similar result, but without a detailed proof.) Therefore $A_p^{-4}[u_0, v_0]^\top = 0$ has to hold, and since A_p^{-1} is injective, this yields $[u_0, v_0]^\top = 0$. Since $y_p(t) = \mathrm{e}^{tA_p}[u_0, v_0]^\top$, it follows that $u(t) = v(t) = 0$ for all $t \geq 0$. □

As a consequence, convergence to zero for trajectories with $\omega(z_0) \neq \emptyset$ is obtained:

Corollary 4.18. *If $\omega(z_0) \neq \emptyset$ for some $z_0 \in \mathcal{H}$, then*

$$\lim_{t\to\infty} \|S(t)z_0\| = 0.$$

Proof. If $\omega(z_0) \neq \emptyset$ then there exists a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \to \infty$ such that $\lim_{n\to\infty} S(t_n)z_0 = 0$. Due to the continuity of the Lyapunov function V this implies that

$$\lim_{n\to\infty} V(S(t_n)z_0) = 0.$$

But since $t \mapsto V(S(t)z_0)$ is monotonically decreasing, this implies that

$$\lim_{t\to\infty} V(S(t)z_0) = 0.$$

Due to the continuity of V this implies that $\|S(t)z_0\| \to 0$ as $t \to \infty$. □

Remark 4.19. Note that to demonstrate that the ω-limit set is non-empty, it suffices to show that the solution trajectories are precompact.

Therefore, in order to demonstrate the asymptotic convergence of the system (4.17), the precompactness of the trajectories shall be discussed in the next two subsections.

4.1.4 Asymptotic stability for nonlinear k_j

According to Corollary 4.18, any trajectory with a non-empty ω-limit, is asymptotically stable. Thus, in order to complete the stability analysis for 4.17, it is shown in this subsection that any classical trajectory possesses a non-empty ω-limit. This is achieved by proving that every classical trajectory is precompact. To this end, the strategy introduced in Chapter 3 and [49] is closely followed. Specifically, the trajectory precompactness property is first demonstrated for all the solutions z with the higher regularity $z \in C^2([0, \infty), \mathcal{H})$. Thereby, the following result will be used:

Lemma 4.20. *Let z be the solution of* (4.21)*, such that $z_0 \in D(\mathcal{A}^2) := \{\tilde{z} \in D(\mathcal{A})\colon \mathcal{A}(\tilde{z}) \in D(\mathcal{A})\}$. Then $z \in C^2([0, \infty), \mathcal{H})$ and $z_t(t) \in D(\mathcal{A})$ for all $t > 0$.*

Proof. Under the assumption $z \in C^2([0,\infty),\mathcal{H})$, it follows that $\tilde{z} := z_t$ satisfies

$$\tilde{z}_t = A\tilde{z} + \begin{bmatrix} 0 \\ 0 \\ \alpha_1'(\zeta_1)\tilde{\zeta}_1 + \frac{1}{J}[\beta_1'(\zeta_1)\tilde{\zeta}_1\xi + \beta_1(\zeta_1)\tilde{\xi}] \\ \alpha_2'(\zeta_2)\tilde{\zeta}_2 + \frac{1}{M}[\beta_2'(\zeta_2)\tilde{\zeta}_2\psi + \beta_2(\zeta_2)\tilde{\psi}] \\ -\gamma_1'(\zeta_1)\tilde{\zeta}_1 - \frac{1}{J}[\delta_1'(\zeta_1)\tilde{\zeta}_1\xi + \delta_1(\zeta_1)\tilde{\xi}] - \kappa_1'(u_x(L))\tilde{u}_x(L) \\ -\gamma_2'(\zeta_2)\tilde{\zeta}_2 - \frac{1}{M}[\delta_2'(\zeta_2)\tilde{\zeta}_2\xi + \delta_2(\zeta_2)\tilde{\psi}] - \kappa_2'(u(L))\tilde{u}(L) \end{bmatrix}. \tag{4.37}$$

Since $z_0 \in D(\mathcal{A}^2) \subset D(\mathcal{A})$, Corollary 4.9 implies that $z \in C^1([0,\infty),\mathcal{H})$, but no higher regularity is guaranteed. Motivated by (4.37), the following functions for fixed $z(t) = [u\,v\,\zeta_1\,\zeta_2\,\xi\,\psi]^\top$ are defined:

$$\begin{aligned}
G_1(t,Z) &:= \alpha_1'(\zeta_1)Z_1 + \frac{1}{J}[\beta_1'(\zeta_1)Z_1\xi + \beta_1(z_1)\Xi], \\
G_2(t,Z) &:= \alpha_2'(\zeta_2)Z_2 + \frac{1}{M}[\beta_2'(\zeta_2)Z_2\psi + \beta_2(z_2)\Psi], \\
G_3(t,Z) &:= -\gamma_1'(\zeta_1)Z_1 - \frac{1}{J}[\delta_1'(\zeta_1)Z_1\xi + \delta_1(\zeta_1)\Xi] - \kappa_1'(u_x(L))U_x(L), \\
G_4(t,Z) &:= -\gamma_2'(\zeta_2)Z_2 - \frac{1}{M}[\delta_2'(\zeta_2)Z_2\xi + \delta_2(\zeta_2)\Psi] - \kappa_2'(u(L))U(L),
\end{aligned}$$

where $Z = [U, V, Z_1, Z_2, \Xi, \Psi]^\top \in \mathcal{H}$. Since $z(t)$ is a classical solution, it follows that the function $t \mapsto G_j(t,Z)$ is continuously differentiable for $j = 1,\dots,4$. The operator $\tilde{N}\colon [0,T) \to \mathbb{R}$ defined by:

$$\tilde{N}(t,Z) := \begin{bmatrix} 0 \\ 0 \\ G_1(t,Z) \\ G_2(t,Z) \\ G_3(t,Z) \\ G_4(t,Z) \end{bmatrix}$$

is consequently differentiable with respect to t for all $Z \in \mathcal{H}$, and linear with respect to $\mathcal{H}$. Now the linear, non-autonomous initial value problem:

$$\begin{aligned} Z_t &= AZ + \tilde{N}(t,Z), \\ Z(0) &= Z_0 \in \mathcal{H}, \end{aligned} \tag{4.38}$$

is considered. According to Theorem 6.1.2 in [56], (4.38) has a unique, global mild solution $Z(t)$ for every $Z_0 \in \mathcal{H}$. Moreover, if $Z_0 \in D(\mathcal{A})$, then according to Theorem B.5 in Appendix B the solution Z is classical. Function $z(t)$ is differentiable and satisfies the Duhamel formula (4.22), therefore it can be obtained:

$$z_t(t) = e^{At}z_0 + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t e^{(t-s)A}\mathcal{N}z(s)\,\mathrm{d}s. \tag{4.39}$$

Proceeding the same as in proof of Corollary 4.2.5 in [56], the following holds:

$$\frac{\mathrm{d}}{\mathrm{d}x} e^{(t-s)A} \mathcal{N} z(s)\,\mathrm{d}s = e^{tA} \mathcal{N} z_0 + \int_0^t e^{(t-s)A} \frac{\mathrm{d}}{\mathrm{d}s} \mathcal{N} z(s)\,\mathrm{d}s. \tag{4.40}$$

Using equation (4.40) in (4.39), it follows that z_t satisfies the Duhamel formula for (4.38). Moreover, z_t is the unique solution of (4.38) to the initial condition $Z_0 = \mathcal{A} z_0$. However, since $\mathcal{A} z_0 \in D(\mathcal{A})$ it follows that $Z(t) = z_t(t)$ is a classical solution. Hence $z_t \in C^1([0,\infty);\mathcal{H})$ and $z \in C^2([0,\infty);\mathcal{H})$.

□

Lemma 4.21. *The trajectory $\gamma(z_0)$ is precompact in $\mathcal{H}$, for $z_0 \in D(\mathcal{A}^2)$. Moreover, there exists a constant $C(\|z_0\|_{\mathcal{H}}, \|z_t(0)\|_{\mathcal{H}}) > 0$, such that*

$$\|z_t(t)\|_{\mathcal{H}} \le C, \qquad \forall t \ge 0, \tag{4.41}$$

where C depends continuously on $\|z_0\|_{\mathcal{H}}$ and $\|z_t(0)\|_{\mathcal{H}}$.

Proof. According to Lemma 4.20, $z(t) \in C^2([0,\infty);\mathcal{H})$. Differentiating (1.20) with respect to time implies that z_t is the classical solution to the following system:

$$\mu u_{ttt} + \Lambda u_{txxxx} = 0, \tag{4.42a}$$

$$u_t(t,0) = u_{tx}(t,0) = 0, \tag{4.42b}$$

$$J u_{tttx}(t,L) + \Lambda u_{txx}(t,L) + (\Theta_1)_t(t) = 0, \tag{4.42c}$$

$$M u_{ttt}(t,L) - \Lambda u_{txxx}(t,L) + (\Theta_2)_t(t) = 0, \tag{4.42d}$$

with

$$\begin{aligned} (\Theta_1)_t &= \Big[\nabla c_1(\zeta_1) + u_{tx}(L)\nabla d_1(\zeta_1)\Big](\zeta_1)_t + d_1(\zeta_1)u_{ttx}(L) + k_1'(u_x(L))u_{tx}(L), \\ (\Theta_2)_t &= \Big[\nabla c_2(\zeta_2) + u_t(L)\nabla d_2(z_2)\Big](\zeta_2)_t + d_2(\zeta_2)u_{tt}(L) + k_2'(u(L))u_t(L). \end{aligned} \tag{4.43}$$

Furthermore, there holds:

$$(\zeta_1)_{tt} = \Big[J_{a_1}(\zeta_1) + u_{tx}(L)J_{b_1}(\zeta_1)\Big](\zeta_1)_t + b_1(\zeta_1)u_{ttx}(L), \tag{4.44a}$$

$$(\zeta_2)_{tt} = \Big[J_{a_2}(\zeta_2) + u_t(L)J_{b_2}(\zeta_2)\Big](\zeta_2)_t + b_2(\zeta_2)u_{tt}(L), \tag{4.44b}$$

where J_{a_j}, J_{b_j} denote the Jacobian matrices of the functions a_j, b_j, respectively. From Lemma 4.8, it follows that $\zeta_j(.), u_t(\,.\,,L) = \frac{\psi}{M}, u_{tx}(\,.\,,L) = \frac{\xi}{J} \in L^2(\mathbb{R}^+)$, and therefore (4.6) implies $(\zeta_j)_t \in L^2(\mathbb{R}^+)$. In order to prove the precompactness of the trajectory, it suffices to show that

$$\sup_{t>0} \|\mathcal{A} z(t)\|_{\mathcal{H}} < \infty,$$

due to the compact embeddings $H^4(0,L) \hookrightarrow\hookrightarrow H^2(0,L) \hookrightarrow\hookrightarrow L^2(0,L)$. However, this is equivalent to showing that z_t is uniformly bounded in $\mathcal{H}$, since $z_t = \mathcal{A}z$. Since $(\zeta_j)_t$ is uniformly bounded in t as well, $j = 1, 2$. Therefore, it suffices to show that the functional

$$\tilde{V}(z_t) = \frac{\mu}{2}\int_0^L u_{tt}^2 \,\mathrm{d}x + \frac{\Lambda}{2}\int_0^L u_{txx}^2 \,\mathrm{d}x + \frac{J}{2}u_{ttx}(L)^2 + \frac{M}{2}u_{tt}(L)^2,$$

is uniformly bounded on $[0, \infty)$. There holds:

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\tilde{V}(z_t) &= \mu \textstyle\int_0^L u_{ttt}u_{tt}\,\mathrm{d}x + \Lambda \int_0^L u_{ttxx}u_{txx}\,\mathrm{d}x + Ju_{tttx}(L)u_{ttx}(L) + Mu_{ttt}(L)u_{tt}(L) \\
&= u_{tt}(L)\big(Mu_{ttt}(L) - \Lambda u_{txxx}(L)\big) + u_{ttx}(L)\big(Ju_{tttx}(L) + \Lambda u_{txx}(L)\big) \\
&= -u_{tt}(L)\big((\zeta_2)_t^\top [\nabla c_2(\zeta_2) + u_t(L)\nabla d_2(\zeta_2)] + k_2'(u(L))u_t(L)\big) \\
&\quad -u_{ttx}(L)\big((\zeta_1)_t^\top [\nabla c_1(\zeta_1) + u_{tx}(L)\nabla d_1(\zeta_1)] + k_1'(u_x(L))u_{tx}(L)\big) \\
&\quad -d_2(\zeta_2)(u_{tt}(L))^2 - d_1(\zeta_1)(u_{ttx}(L))^2,
\end{aligned} \tag{4.45}$$

where integration in x was performed twice, and the equations (4.42) and (4.43), were used. Integrating (4.45) on the time interval $[0, t]$, for some arbitrary $t \in \mathbb{R}^+$, it follows

$$\tilde{V}(z_t(t)) \;\le\; \tilde{V}(z_t(0)) + I_1(t) + I_2(t), \tag{4.46}$$

where

$$\begin{aligned}
I_1(t) &:= -\int_0^t u_{ttx}(L)\Big((\zeta_1)_t^\top [\nabla c_1(\zeta_1) + u_{tx}(L)\nabla d_1(\zeta_1)] + k_1'(u_x(L))u_{tx}(L)\Big)\,\mathrm{d}\tau, \\
I_2(t) &:= -\int_0^t u_{tt}(L)\Big((\zeta_2)_t^\top [\nabla c_2(\zeta_2) + u_t(L)\nabla d_2(z_2)] + k_2'(u(L))u_t(L)\Big)\,\mathrm{d}\tau.
\end{aligned}$$

Next, uniform boundedness is shown for each component of I_2:

$$\begin{aligned}
-\int_0^t u_{tt}(L)k_2'(u(L))u_t(L)\,\mathrm{d}\tau = &-\frac{1}{2}u_t(t,L)^2 c_2(\zeta_2(t)) + \frac{1}{2}u_t(0,L)^2 c_2(\zeta_2(0)) \\
&+ \frac{1}{2}\int_0^t u_t(L)^3 k_2''(u(L))\,\mathrm{d}\tau \le C, \quad \forall t \ge 0.
\end{aligned}$$

Further, it holds:

$$\begin{aligned}
-\int_0^t u_{tt}(L)(\zeta_2)_t^\top \nabla c_2(\zeta_2)\,\mathrm{d}\tau = &-u_t(t,L)(\zeta_2)_t(t)^\top \nabla c_2(\zeta_2(t)) + u_t(0,L)(\zeta_2)_t(0)^\top \nabla c_2(\zeta_2(0)) \\
&+ \int_0^t u_t(L)[(\zeta_2)_t^\top H_{c_2}(\zeta_2)(\zeta_2)_t + (\zeta_2)_{tt}^\top \nabla c_2(\zeta_2)]\,\mathrm{d}\tau,
\end{aligned}$$

Here, H_{c_2} denotes the Hessian of the function c_2. Since $c_2 \in C^2(\mathbb{R}^n;\mathbb{R})$, it follows that

$$\left|\int_0^t u_t(L)(\zeta_2)_t^\top H_{c_2}(\zeta_2)(\zeta_2)_t\,\mathrm{d}\tau\right| \le C\int_0^t \|(\zeta_2)_t\|^2\,\mathrm{d}\tau,$$

and (with (4.44)):

$$\begin{aligned}\int_0^t u_t(L)(\zeta_2)_{tt}^\top \nabla c_2(\zeta_2)\,\mathrm{d}\tau =& \int_0^t u_t(L)[J_{a_2}(\zeta_2)(\zeta_2)_t + u_t(L)J_{b_2}(\zeta_2)(\zeta_2)_t]^\top \nabla c_2(\zeta_2)\,\mathrm{d}\tau \\ &+ \int_0^t b_2(\zeta_2)^\top \nabla c_2(\zeta_2) u_{tt}(L)u_t(L)\,\mathrm{d}\tau \\ =& \int_0^t u_t(L)[J_{a_2}(\zeta_2)(\zeta_2)_t + u_t(L)J_{b_2}(\zeta_2)(\zeta_2)_t]^\top \nabla c_2(\zeta_2)\,\mathrm{d}\tau \\ &+ \frac{1}{2}b_2(\zeta_2(t))^\top \nabla c_2(\zeta_2(t))u_t(t,L)^2 - \frac{1}{2}b_2(\zeta_2(0))^\top \nabla c_2(\zeta_2(0))u_t(0,L)^2 \\ &- \frac{1}{2}\int_0^t u_t(L)^2(\zeta_2)_t^\top \left[J_{b_2}(\zeta_2)^\top \nabla c_2(\zeta_2) + H_{c_2}(\zeta_2)b_2(\zeta_2)\right]\mathrm{d}\tau \\ \le& C\int_0^t |u_t(L)|^2 + \|(\zeta_2)_t\|^2\,\mathrm{d}\tau + \frac{1}{2}b_2(\zeta_2(t))u_t(t,L)^2 \\ &- \frac{1}{2}b_2(\zeta_2(0))u_t(0,L)^2\end{aligned}$$

For the last component of I_2 there holds:

$$\begin{aligned}-\int_0^t u_{tt}(L)u_t(L)(\zeta_2)_t^\top \nabla d_2(\zeta_2)\,\mathrm{d}\tau =& -\frac{1}{2}u_t(t,L)^2(\zeta_2(t))_t^\top \nabla d_2(\zeta_2(t)) \\ &+ \frac{1}{2}u_t(0,L)^2(\zeta_2(0))_t^\top \nabla d_2(\zeta_2(0)) \\ &+ \frac{1}{2}\int_0^t u_t(L)^2[(\zeta_2)_{tt}^\top \nabla d_2(\zeta_2) + (\zeta_2)_t^\top H_{d_2}(\zeta_2)(\zeta_2)_t]\,\mathrm{d}\tau,\end{aligned}$$

where H_{d_2} denotes the Hessian of d_2. This term is also uniformly bounded for $t \ge 0$, since $d_2 \in C^2(\mathbb{R}^n;\mathbb{R})$,

$$\left|\int_0^t u_t(L)^2(\zeta_2)_t^\top H_{d_2}(\zeta_2)(\zeta_2)_t\,\mathrm{d}\tau\right| \le C\int_0^t \|(\zeta_2)_t\|^2\,\mathrm{d}\tau,$$

and (with (4.44)):

$$\begin{aligned}\int_0^t u_t(L)^2(\zeta_2)_{tt}^\top \nabla d_2(\zeta_2)\,\mathrm{d}\tau =& \int_0^t u_t(L)^2[J_{a_2}(\zeta_2)(\zeta_2)_t + u_t(L)J_{b_2}(\zeta_2)(\zeta_2)_t]^\top \nabla d_2(\zeta_2)\,\mathrm{d}\tau \\ &+ \int_0^t u_t(L)^2 u_{tt}(L)b_2(\zeta_2)^\top \nabla d_2(\zeta_2)\,\mathrm{d}\tau \\ \le& C\int_0^t |u_t(L)|^2\,\mathrm{d}\tau + \int_0^t u_t(L)^2 u_{tt}(L)b_2(\zeta_2)^\top \nabla d_2(\zeta_2)\,\mathrm{d}\tau,\end{aligned}$$

where the following holds:

$$
\begin{aligned}
3\int_0^t u_t(L)^2 u_{tt}(L) b_2(\zeta_2)^\top \nabla d_2(\zeta_2)\,\mathrm{d}\tau &= u_t(t,L)^3 b_2(\zeta_2(t))^\top \nabla d_2(\zeta_2(t)) \\
&\quad - u_t(0,L)^3 b_2(\zeta_2(0))^\top \nabla d_2(\zeta_2(0)) \\
&\quad - \int_0^t u_t(L)^3 (\zeta_2)_t^\top [J_{b_2}(\zeta_2)\nabla d_2(\zeta_2) + H_{d_2}(\zeta_2) b_2(\zeta_2)]\,\mathrm{d}\tau \\
&\leq C, \quad \forall t \geq 0.
\end{aligned}
$$

The uniform boundedness of I_1 follows analogously. Hence, $\tilde{V}(z_t(t))$ is uniformly bounded in time. Also it is immediately seen that all the positive constants C which appear in the inequalities, depend continuously on the initial conditions. This proves the statement of the lemma. □

In order to extend this result to all classical solutions, the following density argument shall be used.

Lemma 4.22. *For any $z \in D(\mathcal{A})$, there is a sequence $\{z_n\}_{n\in\mathbb{N}}$ in $D(\mathcal{A}^2)$ such that*

$$\lim_{n\to\infty} z_n = z$$

and

$$\lim_{n\to\infty} \mathcal{A} z_n = \mathcal{A} z.$$

Proof. Let an arbitrary $z \in D(\mathcal{A})$ be fixed. Notice that it suffices to show that there exists a sequence $z_n = [u_n\, v_n\, \zeta_{1n}\, \zeta_{2n}\, \xi_n \psi_n]^\top \in D(\mathcal{A}^2)$ such that $\lim_{n\to\infty} z_n = z$ in $H^4(0,L) \times H^2(0,L) \times \mathbb{R}^{2n+2}$. The set $D(\mathcal{A}^2) := \{z \in D(\mathcal{A}) : \mathcal{A}z \in D(\mathcal{A})\}$ is equivalent to:

$$u \in \tilde{H}_0^6(0,L) \wedge u_{xxxx}(0) = u_{xxxxxx}(0) = 0, \tag{4.47}$$

$$v \in \tilde{H}_0^4(0,L), \tag{4.48}$$

$$\xi = J v_x(L), \tag{4.49}$$

$$\psi = M v(L), \tag{4.50}$$

$$\Lambda u_{xx}(L) + [c_1(\zeta_1) + \frac{1}{J} d_1(\zeta_1)\xi + k_1(u_x(L))] = \frac{\Lambda J}{\mu} u_{xxxxx}(L), \tag{4.51}$$

$$-\Lambda u_{xxx}(L) + [c_2(\zeta_2) + \frac{1}{M} d_2(\zeta_2)\psi + k_2(u(L))] = \frac{\Lambda M}{\mu} u_{xxxx}(L). \tag{4.52}$$

Since $\tilde{C}_0^\infty(0,L) := \{f \in C^\infty[0,L] \colon f^{(k)}(0) = 0, \forall k \in \mathbb{N}_0\}$ is dense in $\tilde{H}_0^2(0,L)$ (see Theorem 3.17 in [1]), there exists a sequence $\{v_n\}_{n\in\mathbb{N}} \subset \tilde{C}_0^\infty(0,L)$ such that $\lim_{n\to\infty} v_n = v$ in $H^2(0,L)$. Also, v_n satisfies (4.48), for all $n \in \mathbb{N}$. Defining $\xi_n := J(v_n)_x(L)$ and $\psi_n := M v_n(L)$ ensures that z_n satisfies (4.49) and (4.50). Moreover, the Sobolev embedding $H^2(0,L) \hookrightarrow C^1[0,L]$ implies that $\lim_{n\to\infty} \xi_n = \xi$ and $\lim_{n\to\infty} \psi_n = \psi$ as well. Next, let $\zeta_{1n} := \zeta_1$ and $\zeta_{2n} := \zeta_2$ for all $n \in \mathbb{N}$.

Finally, the sequence $\{u_n\}_{n\in\mathbb{N}} \subset C^\infty[0,L]$ will be constructed such that u_n satisfies (4.47), (4.51), and (4.52) for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} u_n = u$ in $H^4(0,L)$. To this end, an auxiliary sequence of polynomial functions is introduced as follows:

$$h_n(x) := h_{2,n}x^2 + h_{3,n}x^3 + h_{6,n}x^6 + h_{7,n}x^7 + h_{8,n}x^8 + h_{9,n}x^9 + h_{10,n}x^{10} + h_{11,n}x^{11},$$

for all $n \in \mathbb{N}$, where $h_{2,n},\ldots,h_{11,n} \in \mathbb{R}$ are to be determined. It immediately follows that

$$h_n(0) = (h_n)_x(0) = (h_n)_{xxxx}(0) = (h_n)_{xxxxx}(0) = 0. \tag{4.53}$$

Let $h_{2,n} = \frac{u_{xx}(0)}{2}$ and $h_{3,n} = \frac{u_{xxx}(0)}{6}$, which is equivalent to

$$(h_n)_{xx}(0) = u_{xx}(0),\ (h_n)_{xxx}(0) = u_{xxx}(0). \tag{4.54}$$

Further conditions are imposed on h_n:

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}h_n(L) = \frac{\mathrm{d}^k}{\mathrm{d}x^k}u(L), \qquad k \in \{0,1,2,3\}.$$

This can equivalently be written in terms of coefficients:

$$h_{n,6} + h_{n,7}L + h_{n,8}L^2 + h_{n,9}L^3 + h_{n,10}L^4 + h_{n,11}L^5 = r_1, \tag{4.55a}$$
$$6h_{n,6} + 7h_{n,7}L + 8h_{n,8}L^2 + 9h_{n,9}L^3 + 10h_{n,10}L^4 + 11h_{n,11}L^5 = r_2, \tag{4.55b}$$
$$6^{\underline{2}}h_{n,6} + 7^{\underline{2}}h_{n,7}L + 8^{\underline{2}}h_{n,8}L^2 + 9^{\underline{2}}h_{n,9}L^3 + 10^{\underline{2}}h_{n,10}L^4 + 11^{\underline{2}}h_{n,11}L^5 = r_3 \tag{4.55c}$$
$$6^{\underline{3}}h_{n,6} + 7^{\underline{3}}h_{n,7}L + 8^{\underline{3}}h_{n,8}L^2 + 9^{\underline{3}}h_{n,9}L^3 + 10^{\underline{3}}h_{n,10}L^4 + 11^{\underline{3}}h_{n,11}L^5 = r_4, \tag{4.55d}$$

with

$$r_1 = \frac{u(L)}{L^6} - \frac{u_{xx}(0)}{2L^4} - \frac{u_{xxx}(0)}{6L^3}, \quad r_2 = \frac{u_x(L)}{L^5} - \frac{u_{xx}(0)}{L^4} - \frac{u_{xxx}(0)}{2L^3},$$
$$r_3 = \frac{u_{xx}(L)}{L^4} - \frac{u_{xx}(0)}{L^4} - \frac{u_{xxx}(0)}{L^3}, \quad r_4 = \frac{u_{xxx}(L)}{L^3} - \frac{u_{xxx}(0)}{L^3}.$$

It is further required that h_n satisfies:

$$\frac{\Lambda M}{\mu}(h_n)_{xxxx}(L) = -\Lambda u_{xxx}(L) + [c_2(\zeta_2) + \frac{1}{M}d_2(\zeta_2)\psi_n + k_2(u(L))] := r_5, \tag{4.56}$$
$$\frac{\Lambda J}{\mu}(h_n)_{xxxxx}(L) = \Lambda u_{xx}(L) + [c_1(\zeta_1) + \frac{1}{J}d_1(\zeta_1)\xi_n + k_1(u_x(L))] := r_6. \tag{4.57}$$

Equations (4.56) and (4.57) are equivalent to:

$$6^{\underline{4}}h_{n,6} + 7^{\underline{4}}h_{n,7}L + 8^{\underline{4}}h_{n,8}L^2 + 9^{\underline{4}}h_{n,9}L^3 + 10^{\underline{4}}h_{n,10}L^4 + 11^{\underline{4}}h_{n,11}L^5 = r_5\frac{\mu}{\Lambda M L^2}, \tag{4.58a}$$
$$6^{\underline{5}}h_{n,6} + 7^{\underline{5}}h_{n,7}L + 8^{\underline{5}}h_{n,8}L^2 + 9^{\underline{5}}h_{n,9}L^3 + 10^{\underline{5}}h_{n,10}L^4 + 11^{\underline{2}}h_{n,11}L^5 = r_6\frac{\mu}{\Lambda J L}. \tag{4.58b}$$

Such h_n exists and is unique, due to the fact that linear system (4.55) and (4.58) has strictly positive determinant. Consequently, (4.53), (4.54), and (4.55) imply that $u - h_n \in H_0^4(0, L)$, for all $n \in \mathbb{N}$. Since $C_0^\infty(0, L)$ is dense in $H_0^4(0, L)$, there exists a sequence $\{\tilde{u}_n\}_{n\in\mathbb{N}} \subset C_0^\infty(0, L)$ such that $\|\tilde{u}_n - (u - h_n)\|_{H^4} < \frac{1}{n}$, $\forall n \in \mathbb{N}$. Now defining $u_n := \tilde{u}_n + h_n$, gives $\lim_{n\to\infty} u_n = u$ in $H^4(0, L)$. Obviously u_n satisfies (4.47) for all $n \in \mathbb{N}$. Also, due to (4.56) and (4.57), u_n satisfies (4.51) and (4.52), as well. Hence, the statement follows. □

Theorem 4.23. *For all $z_0 \in D(\mathcal{A})$, the trajectory $\gamma(z_0)$ is precompact in $\mathcal{H}$.*

Proof. Let $z_0 \in D(\mathcal{A})$ be chosen arbitrarily, and let $\{z_{n0}\}_{n\in\mathbb{N}} \subset D(\mathcal{A}^2)$ be an approximating sequence as in Lemma 4.22. Then there holds:

$$\lim_{n\to+\infty} \mathcal{A}z_{n0} = \mathcal{A}z_0 \tag{4.59}$$

Applying Proposition 4.10, it follows that for an arbitrary $T > 0$ the approximating solutions $z_n(t)$ converge to $z(t)$ in $C([0, T]; \mathcal{H})$. Since $z_n(t) \in C^1([0, \infty); \mathcal{H})$ and solves (4.21) for all $n \in \mathbb{N}$, (4.59) yields

$$\lim_{n\to+\infty} (z_n)_t(0) = \mathcal{A}z_0 \quad \text{in} \quad \mathcal{H}. \tag{4.60}$$

Hence, (4.41) and (4.60) imply that there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$:

$$\sup_{t\geq 0} \|(z_n)_t(t)\|_{\mathcal{H}} \leq C(\|z_0\|_{\mathcal{H}}, \|\mathcal{A}z_0\|_{\mathcal{H}}),$$

where the constant C does not depend on n. From here it follows that $(z_n)_t$ is bounded in $L^\infty((0, +\infty); \mathcal{H})$. Hence, the Banach-Alaoglu Theorem (see Theorem I.3.15 in [59]) implies that there exists $w \in L^\infty((0, \infty); \mathcal{H})$ and a subsequence $\{z_{n_k}\}_{k\in\mathbb{N}}$ such that

$$(z_{n_k})_t \overset{*}{\rightharpoonup} w \quad \text{in} \quad L^\infty((0, \infty); \mathcal{H}).$$

For arbitrary $\tilde{z} \in \mathcal{H}$ and $t \geq 0$ there holds

$$\lim_{k\to\infty} \int_0^t \langle (z_{n_k})_t(\tau), \tilde{z}\rangle_{\mathcal{H}} \,\mathrm{d}\tau = \int_0^t \langle w(\tau), \tilde{z}\rangle_{\mathcal{H}} \,\mathrm{d}\tau,$$

which is equivalent to

$$\lim_{k\to\infty} \langle z_{n_k}(t) - z_{n_k}(0), \tilde{z}\rangle_{\mathcal{H}} = \langle \int_0^t w(\tau) \,\mathrm{d}\tau, \tilde{z}\rangle_{\mathcal{H}}.$$

Since $\lim_{n\to\infty} z_n(\tau) = z(\tau)$ in $\mathcal{H}$, $\forall \tau \in [0, +\infty)$, it follows that

$$\langle z(t) - z(0), \tilde{z}\rangle_{\mathcal{H}} = \langle \int_0^t w(\tau) \,\mathrm{d}\tau, \tilde{z}\rangle_{\mathcal{H}}.$$

Since $\tilde{z} \in \mathcal{H}$ is arbitrary, it is obtained

$$z(t) - z(0) = \int_0^t w(\tau) \,\mathrm{d}\tau, \quad \forall t \geq 0. \tag{4.61}$$

Due to continuous differentiability of z, the time derivative of (4.61) can be taken, which yields $z_t \equiv w$. This implies $z_t \in L^\infty((0, \infty); \mathcal{H})$, i.e. $\|z_t(.)\|_{\mathcal{H}}$ is uniformly bounded, which proves the theorem. □

4.1.5 Asymptotic stability for linear k_j

In the previous subsection, the precompactness property of the trajectories has been demonstrated for classical solutions. However, in the case where the k_j are linear, it is possible to show precompactness for the mild, non-classical solutions. This will yield that the ω-limit set is always non-empty and hence the asymptotic stability of the nonlinear semigroup S. This is the main objective of this subsection. The following lemma shall prove to be an essential step to achieve this.

Lemma 4.24. *Let $z_0 \in \mathcal{H}$, and $z(t)$ be the corresponding mild solution of* (4.21). *Let $\kappa_j \equiv 0$, $j = 1, 2$. Then $\mathcal{N}z(t) \in L^1([0,\infty); \mathcal{H})$.*

Proof. First, let $z_0 \in D(\mathcal{A})$, and hence $z(t)$ is a classical solution. It follows from Theorem 4.11 that $V(z(t))$ is non-increasing and integrating (4.25) with respect to time, yields:

$$V(z(T)) - V(z_0) = \int_0^T \left[- \frac{d_1(\zeta_1)|\xi|^2}{J^2} - \frac{d_2(\zeta_2)|\psi|^2}{M^2} + a_1(z_1) \cdot \nabla V_1(\zeta_1) + a_2(\zeta_2) \cdot \nabla V_2(\zeta_2) \right] \mathrm{d}t$$
$$=: I_T(z_0), \tag{4.62}$$

where all terms on the right hand side include elements of the vector $z(t)$, thus depend on t. Observing the limit when $T \to \infty$, it follows that $V(z(T))$ converges to $\nu(z_0)$, and hence the integral $I_\infty(z_0)$ is finite. Next, let the case when $z_0 \in \mathcal{H}$ be considered and let $z(t)$ be the corresponding mild solution of (4.21). Further, let $(z_{0,n})_{n\in\mathbb{N}} \subset D(\mathcal{A})$ be a sequence with $z_{0,n} \to z_0$. According to Proposition 4.10 and the corresponding classical solutions $z_n(t)$ converge to $z(t)$ in $C([0,T]; \mathcal{H})$ for all $T > 0$. Therefore $I_T(z_{0,n}) \to I_T(z_0)$, cf. (4.62). Due to continuity of V, also $V(z_n(T)) - V(z_{0,n}) \to V(z(T)) - V(z_0)$ as $n \to \infty$. Thus, (4.62) also holds for mild solutions for any $T > 0$. Since $V(z(T)) \to \nu(z_0) \in [0, V(z_0)]$ as $T \to \infty$, the integral $I_\infty(z_0)$ is finite. Hence, for any (mild) solution $z(t)$ the integral $I_\infty(z_0)$ is finite. Since all the terms in the integrand of (4.62) are non-positive, under the assumptions (4.12d) and (4.10) it can be concluded that

$$\begin{aligned} \psi(t), \xi(t) &\in L^2([0,\infty); \mathbb{R}), \\ \zeta_j(t) &\in L^2([0,\infty); \mathbb{R}^n). \end{aligned} \tag{4.63}$$

For (4.63) it was used that the uniform boundedness of $z(t)$ implies that $d_j(\zeta_j(t)) \geq \tilde{d}_j > 0$ for all $t \geq 0$. Under the assumptions made in Section 4.1 for the coefficient functions in the nonlinear operator $\mathcal{N}$, note that

$$\|\beta_j(\zeta_j)\| + |\delta_j(\zeta_j)| = \mathcal{O}(\|\zeta_j\|), \quad \text{as } \zeta_j \to 0.$$

Now, the properties (4.63) immediately imply $\mathcal{N}z(t) \in L^1([0,\infty); \mathcal{H})$. □

Note that (4.62) does not give any control on $u(t, L)$ and $u_x(t, L)$ (it the sense of (4.63)). Hence, the linearity assumption $\kappa_j = 0$, $J = 1, 2$ was crucial for the above proof.

Theorem 4.25. *For any* $z_0 \in \mathcal{H}$ *there holds* $\lim_{t\to\infty} S(t)z_0 = 0$, *i.e. the semigroup* S *is asymptotically stable.*

Proof. According to Remark 4.4 the linear part A of $\mathcal{A}$ is a maximal dissipative operator on $\mathcal{H}$. Clearly $A(0) = 0$, and according to Lemma 4.2, the inverse A^{-1} exists and is compact. Since A generates a C_0-semigroup of contractions, $(\lambda - A)^{-1}$ exists and is compact for all $\lambda > 0$. Due to these facts, Theorem B.8 in Appendix B can be applied with $f(t) := \mathcal{N}y(t)$. This demonstrates that the ω-limit set $\omega(z_0)$ is non-empty (in fact the trajectory $\gamma(z_0)$ is precompact). Thus, due to Corollary 4.18 and Theorem 4.17, it can be concluded that $\omega(z_0) = \{0\}$ and the mild solution $z(t)$ converges to 0. □

4.2 Weak formulation

In this section a weak formulation for the system consisting of the boundary controlled Euler-Bernoulli beam (4.1) – (4.5) coupled with a nonlinear boundary controller (4.6) is introduced and the existence of the weak solution is demonstrated. This will serve as a basis for the numerical method developed in Section 4.3.

4.2.1 Motivation and space setting

For the weak formulation, the initial conditions are given by:

$$u(0) = u_0 \in \tilde{H}_0^2(0, L), \tag{4.64a}$$
$$u_t(0) = v_0 \in L^2(0, L), \tag{4.64b}$$
$$\zeta_1(0) = \zeta_{1,0} \in \mathbb{R}^n, \tag{4.64c}$$
$$\zeta_2(0) = \zeta_{2,0} \in \mathbb{R}^n. \tag{4.64d}$$

Moreover, the values $v_0(L)$ and $(v_0)_x(L)$ need to be given additionally to the function v_0. The motivation for the weak solution is obtained analogous to Section 2.2.1: Multiplying (4.1) by $w \in \tilde{H}_0^2(0, L)$, integrating over $[0, L]$, and taking into account the given boundary conditions (4.2)-(4.5), yields:

$$\begin{aligned}
&\mu \int_0^L u_{tt} w \, \mathrm{d}x + \Lambda \int_0^L u_{xx} w_{xx} \, \mathrm{d}x \\
&+\Big(M u_{tt}(t, L) + k_2(u(t, L)) + c_2(\zeta_2(t)) + d_2(\zeta_2(t)) u_t(t, L)\Big) w(L) \\
&+\Big(J u_{ttx}(t, L) + k_1(u_x(t, L)) + c_1(\zeta_1(t)) + d_1(\zeta_1(t)) u_{tx}(t, L)\Big) w_x(L) = 0,
\end{aligned} \tag{4.65}$$

for all $w \in \tilde{H}_0^2(0, L),\ t > 0$.

Let the same space setting be introduced as in Subsection 2.2.1, i.e. two Hilbert spaces H and V are defined by (2.88) and (2.89). Also, the following nonlinear forms $a_{nl} : V \times V \to$

$\mathbb{R}$, $b_{nl} : \mathbb{R}^{2n} \times H \times H \to \mathbb{R}$ and $e_{1,nl}, e_{2,nl} : \mathbb{R}^n \times V \to \mathbb{R}$ are introduced by:

$$\begin{aligned}
a_{nl}(\hat{w_1}, \hat{w_2}) &= \Lambda((w_1)_{xx}, (w_2)_{xx})_{L^2} \\
&+ k_1\big((w_1)_x(L)\big)(w_2)_x(L) + k_2\big(w_1(L)\big)w_2(L), \\
b_{nl}(\zeta_1, \zeta_2, \hat{\varphi}, \hat{\nu}) &= d_1(\zeta)\, {}^1\hat{\varphi}\, {}^1\hat{\nu} + d_2(\zeta)\, {}^2\hat{\varphi}\, {}^2\hat{\nu}, \\
e_{1,nl}(\zeta_1, \hat{w}) &= c_1(\zeta_1)w_x(L), \\
e_{2,nl}(\zeta_2, \hat{w}) &= c_2(\zeta_2)w(L).
\end{aligned}$$

Definition 4.26. Let $T > 0$ be fixed. Functions $\hat{u} \in L^2(0,T;V) \cap H^1(0,T;H) \cap H^2(0,T;V')$, and $\zeta_1, \zeta_2 \in H^1(0,T;\mathbb{R}^n)$ are said to be a *weak solution* to (4.1)–(4.6) and (4.64) on the time interval $[0,T]$ if they satisfy:

$$a_{nl}(\hat{u}, \hat{w}) + b_{nl}(\hat{u}_t, \hat{w}) + {}_{V'}\langle \hat{u}_{tt}, \hat{w}\rangle_V + e_{1,nl}(\zeta_1, \hat{w}) + e_{2,nl}(\zeta_2, \hat{w}) = 0, \tag{4.66}$$

for a.e. $t \in (0,T), \forall \hat{w} \in V$, and

$$\begin{aligned}
(\zeta_1)_t(t) &= A_1(\zeta_1(t)) + b_1\,({}^1\hat{u}_t(t)), \\
(\zeta_2)_t(t) &= A_2(\zeta_2(t)) + b_2\,({}^2\hat{u}_t(t)),
\end{aligned} \tag{4.67}$$

with initial conditions

$$\hat{u}(0) = \hat{u}_0 = ((u_0)_x(L), u_0(L), u_0) \in V, \tag{4.68a}$$
$$\hat{u}_t(0) = \hat{v}_0 = ((v_0)_x(L), v_0(L), v_0) \in H, \tag{4.68b}$$
$$\zeta_1(0) = \zeta_{1,0} \in \mathbb{R}^n, \tag{4.68c}$$
$$\zeta_2(0) = \zeta_{2,0} \in \mathbb{R}^n. \tag{4.68d}$$

Notice that Lemma 2.28 gives interpretation to the initial conditions (4.68a) and (4.68b).

4.2.2 Existence and higher regularity of the weak solution

Theorem 4.27. *(a) There exists a solution $(\hat{u}, \zeta_1, \zeta_2)$ to the weak formulation (4.66) – (4.68).*

(b) The weak solution has the additional regularity

$$\hat{u} \in L^\infty(0,T;V), \quad \hat{u}_t \in L^\infty(0,T;H), \tag{4.69}$$

$$\zeta_1, \zeta_2 \in C([0,T];\mathbb{R}^n), \tag{4.70}$$

$$\hat{u} \in C([0,T];[V,H]_{\frac{1}{2}}), \tag{4.71}$$

$$\hat{u}_t \in C([0,T];[V,H]'_{\frac{1}{2}}). \tag{4.72}$$

This existence and regularity result for the weak solution proceeds similarly as in the case of linear boundary as stated in Theorem 2.30. However, since the forms in (4.66) depend non-linearly on the weak solution, the proof needs to be adapted.

Proof. (a)–existence: Let $\{\hat{w}_k\}_{k=1}^{\infty}$ be an orthonormal basis for H, and an orthogonal basis for V. Let $\widehat{W}_m = \operatorname{span}\{\hat{w}_1, \dots, \hat{w}_m\}$, for every $m \in \mathbb{N}$. For a fixed $m \in \mathbb{N}$, let $\hat{u}_m$, $\zeta_{1,m}$ and $\zeta_{2,m}$ be the Galerkin approximation that solves:

$$((\hat{u}_m)_{tt}, \hat{w}_k)_H + a_{nl}(\hat{u}_m, \hat{w}_k) + b_{nl}((\hat{u}_m)_t, \hat{w}_k) + e_{1,nl}(\zeta_{1,m}, \hat{w}_k) + e_{2,nl}(\zeta_{2,m}, \hat{w}_k) = 0, \quad (4.73)$$

for all $k \in \{1, \dots, m\}$ and

$$\begin{aligned} (\zeta_{1,m})_t(t) &= a_1(\zeta_{1,m}(t)) + b_1(\zeta_{1,m}(t))\,{}^1(\hat{u}_m)_t(t), \\ (\zeta_{2,m})_t(t) &= a_2(\zeta_{2,m}(t)) + b_2(\zeta_{2,m}(t))\,{}^2(\hat{u}_m)_t(t), \end{aligned} \quad (4.74)$$

with the initial conditions

$$\begin{aligned} \hat{u}_m(0) &= \hat{u}_{m0}, \\ (\hat{u}_m)_t(0) &= \hat{v}_{m0}, \\ \zeta_{1,m}(0) &= \zeta_{1,0}, \\ \zeta_{2,m}(0) &= \zeta_{2,0}, \end{aligned}$$

where the sequences $\hat{u}_{m0}, \hat{v}_{m0} \in \widehat{W}_m$ are such that

$$\begin{aligned} \hat{u}_{m0} &\to \hat{u}_0 \text{ in } V, \\ \hat{v}_{m0} &\to \hat{v}_0 \text{ in } H. \end{aligned} \quad (4.75)$$

In order to prove global solvability of (4.73)-(4.75), this problem is written as a nonlinear system of first order differential equations. Introducing a new variable $\hat{v}_m := (\hat{u}_m)_t$, yields:

$$\begin{aligned} (\hat{u}_m)_t &= \hat{v}_m, \\ (\hat{v}_m)_t &= -\sum_{j=1}^m \left[a_{nl}(\hat{u}_m, \hat{w}_j) + b_{nl}(\hat{v}_m, \hat{w}_j) + e_{1,nl}(\zeta_{1,m}, \hat{w}_j) + e_{2,nl}(\zeta_{2,m}, \hat{w}_j)\right]\hat{w}_j, \\ \zeta_{1,m} &= a_1(\zeta_{1,m}(t)) + b_1(\zeta_{1,m}(t))\,{}^1(\hat{u}_m)_t(t), \\ \zeta_{2,m} &= a_2(\zeta_{2,m}(t)) + b_2(\zeta_{2,m}(t))\,{}^2(\hat{u}_m)_t(t). \end{aligned} \quad (4.76)$$

Let $\widehat{E}_{nl} \colon V \times H \times \mathbb{R}^{2n} \to \mathbb{R}$ be the analogue to the Lyapunov functional as defined in (4.19):

$$\begin{aligned} \widehat{E}_{nl}(t; \hat{u}, \zeta_1, \zeta_2) := \frac{\Lambda}{2}\|\hat{u}(t)\|_V^2 + \frac{1}{2}\|\hat{u}_t(t)\|_H^2 + \int_0^{{}^1\hat{u}(t)} k_1(w)\,\mathrm{d}w + \int_0^{{}^2\hat{u}(t)} k_2(w)\,\mathrm{d}w \\ + V_1(\zeta_1(t)) + V_2(\zeta_2(t)). \end{aligned} \quad (4.77)$$

Assuming that there exists a solution $\hat{u}_m \in C^2([0,\tau];V)$ and $\zeta_{1,m}, \zeta_{2,m} \in C^1([0,\tau];\mathbb{R}^n)$ to (4.76) on some interval $[0,\tau]$, straightforward calculation yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{E}_{nl}(t;\hat{u}_m,\hat{v}_m,\zeta_{1,m},\zeta_{2,m}) \leq -d_1(\zeta_{1,m})u_{tx}(L)^2 - d_2(\zeta_{2,m})u_t(L)^2 \tag{4.78}$$

$\forall t \in (0,\tau)$. Dissipation of the functional $\widehat{E}_n$ corresponds to the decay in (4.20) for the continuous solution. This implies uniform boundedness of the solution on $[0,\tau]$:

$$\widehat{E}_{nl}(t;\hat{u}_m,\hat{v}_m,\zeta_{1,m},\zeta_{2,m}) \leq \widehat{E}_{nl}(0;\hat{u}_{m0},\hat{v}_{m0},\zeta_{1,0},\zeta_{2,0}), \quad t \geq 0. \tag{4.79}$$

Next, let $f_m : \widehat{W}_m \times \widehat{W}_m \times \mathbb{R}^{2n} \to \widehat{W}_m \times \widehat{W}_m \times \mathbb{R}^{2n}$ be defined by:

$$f_m\left(\begin{bmatrix}\hat{u}\\ \hat{v}\\ \zeta_1\\ \zeta_2\end{bmatrix}\right) := \begin{bmatrix}\hat{v}\\ -\sum_{j=1}^m\left[a_{nl}(\hat{u},\hat{w}_j)+b_{nl}(\hat{v},\hat{w}_j)+e_{1,nl}(\zeta_1,\hat{w}_j)+e_{2,nl}(\zeta_2,\hat{w}_j)\right]\hat{w}_j\\ a_1(\zeta_1)+b_1(\zeta_1)(v_h)_x(L)\\ a_2(\zeta_1)+b_2(\zeta_1)v_h(L)\end{bmatrix}.$$

Denoting $\hat{z}_m := [\hat{u}_m\,\hat{v}_m\,\zeta_{1m}\,\zeta_{2m}]^\top$, system (4.76) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{z}_m(t) = f_m\left(\hat{z}_m(t)\right), \tag{4.80}$$

with

$$\hat{z}_m(0) = \hat{z}_{0m} := [\hat{u}_{0m}\,\hat{v}_{0m}\,\zeta_{1,0}\,\zeta_{2,0}]^\top. \tag{4.81}$$

Due to the regularity of the coefficient functions, it easily follows that f_m is continuously differentiable, and hence locally Lipschitz. Let

$$T_{\max} = \min\{T, \frac{1}{2L(2\|\hat{z}_{0m}\|)}\}.$$

Additionally, if the mapping $F_m : C([0,T_{\max}];\widehat{W}_m \times \widehat{W}_m \times \mathbb{R}^{2n}) \to C([0,T_{\max}];\widehat{W}_m \times \widehat{W}_m \times \mathbb{R}^{2n})$ is defined by:

$$[F_m(\hat{z})](t) := \hat{z}_{0m} + \int_0^t f_m(\hat{z}(\tau))\ \mathrm{d}\tau$$

then solving the system (4.80), and (4.81) on $[0,T_{\max}]$ is equivalent to solving a fixed point problem for F_m. Applying the same procedure as in the proof of Theorem 3.27 in Subsection 3.4.2, yields that F_m is a contraction on $B(0,2\|\hat{z}_{0m}\|)$ and according to Banach's fixed point theorem, F_m has a unique fixed point $\hat{z}$. Applying the above procedure iteratively, any solution $\hat{z}$ on the time interval $[0,\tau]$ can be extended to $[0,\tau+\delta(\hat{z}(\tau))]$, where $\delta(\hat{z}(\tau)) = \frac{1}{2L(\|\hat{z}(\tau)\|)} \geq \frac{1}{2L(C(\|\hat{z}_0\|))}$. Therefore, the solution can be extended to the global unique solution on the whole $[0,T]$. Furthermore, due to (4.75) there exists a constant $C > 0$ such that

$$\widehat{E}_{nl}(0;\hat{u}_{m0},\hat{v}_{m0},\zeta_{1,0},\zeta_{2,0}) \leq C\widehat{E}_{nl}(0;\hat{u}_0,\hat{v}_0,\zeta_{1,0},\zeta_{2,0}). \tag{4.82}$$

Therefore (4.79) and (4.82) yield

$$\widehat{E}_{nl}(t;\hat{u}_m,\hat{v}_m,\zeta_{1,m},\zeta_{2,m}) \leq C\widehat{E}_{nl}(0;\hat{u}_0,\hat{v}_0,\zeta_{1,0},\zeta_{2,0}), \tag{4.83}$$

which implies

$$\begin{array}{rl} \{\hat{u}_m\}_{m\in\mathbb{N}} & \text{is bounded in } C([0,T];V), \\ \{(\hat{u}_m)_t\}_{m\in\mathbb{N}} & \text{is bounded in } C([0,T];H), \\ \{\zeta_{1,m}\}_{m\in\mathbb{N}},\{\zeta_{2,m}\}_{m\in\mathbb{N}} & \text{are bounded in } C([0,T];\mathbb{R}^n). \end{array} \tag{4.84}$$

As in the proof of Theorem 2.30, this implies that $(\hat{u}_m)_{tt}$ is bounded in $L^2(0,T;V')$ and $\{(\zeta_{1,m})_t\}_{m\in\mathbb{N}}$ and $(\{\zeta_{2,m})_t\}_{m\in\mathbb{N}}$ are bounded in $L^2(0,T;\mathbb{R}^n)$.

According to the Eberlein–Šmuljan Theorem, there exist subsequences $\{\hat{u}_{m_l}\}_{l\in\mathbb{N}}$, $\{\zeta_{1,m_l}\}_{l\in\mathbb{N}}$, $\{\zeta_{2,m_l}\}_{l\in\mathbb{N}}$, and $\hat{u}\in L^2(0,T;V)$, with $\hat{u}_t\in L^2(0,T;H)$, $\hat{u}_{tt}\in L^2(0,T;V')$, and $\zeta_1,\zeta_2\in H^1(0,T;\mathbb{R}^n)$ such that:

$$\begin{aligned} \{\hat{u}_{m_l}\} &\rightharpoonup \hat{u} \text{ in } L^2(0,T;V), \\ \{(\hat{u}_{m_l})_t\} &\rightharpoonup \hat{u}_t \text{ in } L^2(0,T;H), \\ \{(\hat{u}_{m_l})_{tt}\} &\rightharpoonup \hat{u}_{tt} \text{ in } L^2(0,T;V'), \\ \{\zeta_{1,m_l}\} &\to \zeta_1 \text{ in } L^2(0,T;\mathbb{R}^n), \\ \{\zeta_{2,m_l}\} &\to \zeta_2 \text{ in } L^2(0,T;\mathbb{R}^n), \\ \{(\zeta_{1,m_l})_t\} &\to (\zeta_1)_t \text{ in } L^2(0,T;\mathbb{R}^n), \\ \{(\zeta_{2,m_l})_t\} &\to (\zeta_2)_t \text{ in } L^2(0,T;\mathbb{R}^n). \end{aligned} \tag{4.85}$$

Furthermore $\{\hat{u}_{m_l}\}\rightharpoonup\hat{u}$ in $L^2(0,T;V)$ and $\{(\hat{u}_{m_l})_t\}\rightharpoonup\hat{u}_t$ in $L^2(0,T;H)$ imply

$$\begin{aligned} \{{}^1\hat{u}_{m_l}\} &\to {}^1\hat{u} \text{ in } L^2(0,T;\mathbb{R}), \\ \{{}^2\hat{u}_{m_l}\} &\to {}^2\hat{u} \text{ in } L^2(0,T;\mathbb{R}), \\ \{{}^1(\hat{u}_{m_l})_t\} &\to {}^1\hat{u}_t \text{ in } L^2(0,T;\mathbb{R}), \\ \{{}^2(\hat{u}_{m_l})_t\} &\to {}^2\hat{u}_t \text{ in } L^2(0,T;\mathbb{R}), \end{aligned}$$

Therefore, one may pass on to the limit in (4.73) and (4.74), since all the nonlinear terms are continuous, and their arguments converge strongly. This yields that $\hat{u}$ and ζ_1,ζ_2 solve (4.66) and (4.67).

(b)–additional regularity, (a)-initial conditions: Follows as in the proof of Theorem 2.30.

□

Furthermore, as in Subsection 2.2.3, stronger continuity for the weak solution can be shown:

Theorem 4.28. *After, possibly, a modification on a set of measure zero, a weak solution $\hat{u}$ of (4.66)-(4.68) satisfies*

$$\hat{u} \in C([0,T];V), \tag{4.86}$$

$$\hat{u}_t \in C([0,T];H), \tag{4.87}$$

$$\zeta_1, \zeta_2 \in C^1([0,T];\mathbb{R}^n). \tag{4.88}$$

Proof. As in the proof of Theorem 2.31, it follows that

$$t \mapsto \hat{E}(t; \hat{u}, \zeta_1, \zeta_2)$$

is absolutely continuous. Again, let $t \in [0, \infty)$ be fixed, and let $\lim_{n\to\infty} t_n = t$. Now the sequence χ_n is defined by

$$\chi_n := \frac{\Lambda}{2}\|\hat{u}(t) - \hat{u}(t_n)\|_V^2 + \frac{1}{2}\|\hat{u}_t(t) - \hat{u}_t(t_n)\|_H^2.$$

Then

$$\begin{aligned}
\chi_n = {} & \hat{E}(t; \hat{u}, \zeta_1, \zeta_2) + \hat{E}(t_n; \hat{u}, \zeta_1, \zeta_2) - \Lambda(\hat{u}(t), \hat{u}(t_n))_V - (\hat{u}_t(t), \hat{u}_t(t_n))_H \\
& - \int_0^{1\hat{u}(t)} k_1(\sigma)\, \mathrm{d}\sigma - \int_0^{1\hat{u}(t_n)} k_1(\sigma)\, \mathrm{d}\sigma - \int_0^{2\hat{u}(t)} k_2(\sigma)\, \mathrm{d}\sigma - \int_0^{2\hat{u}(t_n)} k_2(\sigma)\, \mathrm{d}\sigma \\
& - V_1(\zeta_1(t)) - V_1(\zeta_1(t_n)) - V_2(\zeta_2(t)) - V_2(\zeta_2(t_n)).
\end{aligned}$$

As the energy function is t-continuous, $\hat{u}, \hat{u}_t$ are weakly continuous, and ζ_1, ζ_2 continuous functions. It follows

$$\lim_{n\to\infty} \chi_n = 0.$$

Hence,

$$\begin{aligned}
\lim_{n\to\infty} \|\hat{u}_t(t) - \hat{u}_t(t_n)\|_H^2 &= 0, \\
\lim_{n\to\infty} \|\hat{u}(t) - \hat{u}(t_n)\|_V^2 &= 0.
\end{aligned}$$

Therefore (4.86) and (4.87) holds. (4.88) now follows from (4.67). □

4.3 Dissipative numerical method

In this section the goal is to develop a numerical method for (4.1)-(4.6) in such a way that the decay of the Lyapunov function V is preserved. As it was done in Section 2.3 for the linear controller, the first step towards this method is the discretization of the system in space to obtain the semi-discrete scheme, and then in time, obtaining the fully-discrete scheme. As a result, a system of nonlinear algebraic equations is obtained.

4.3.1 Discretization in space

Assuming that u is a classical solution, (4.65) gives

$$\begin{aligned}\int_0^L \mu\, u_{tt} w\, dx + \int_0^L \Lambda\, u_{xx} w_{xx}\, dx + M u_{tt}(t,L) w(L) + J u_{ttx}(t,L) w_x(L) \\ + \big[k_1(u_x(t,L)) + d_1(\zeta_1(t)) u_{tx}(t,L) + c_1(\zeta_1(t))\big] w_x(L) \\ + \big[k_2(u(t,L)) + d_2(\zeta_1(t)) u_t(t,L) + c_2(\zeta_2(t))\big] w(L) = 0,\end{aligned} \tag{4.89}$$

for all $w \in \tilde{H}_0^2(0,L),\ t > 0$. In the next subsection, the finite element method is applied to the formulation (4.89).

4.3.1.1 Finite element method

Let $W_h \subset \tilde{H}_0^2(0,L)$ be a N-dimensional space, with basis $\{w_j\}_{j=1}^N$. The finite element method for (4.89) yields: Find $u_h \in C^2([0,\infty), W_h)$, and $\tilde{\zeta}_{1,2} \in C^1([0,\infty), \mathbb{R}^n)$ with

$$\begin{aligned}\int_0^L \mu\, (u_h)_{tt} w\, dx + \int_0^L \Lambda\, (u_h)_{xx} w_{xx}\, dx + M (u_h)_{tt}(t,L) w(L) + J (u_h)_{ttx}(t,L) w_x(L) \\ + \big[k_1((u_h)_x(t,L)) + d_1(\zeta_1(t)) (u_h)_{tx}(t,L) + c_1(\zeta_1(t))\big] w_x(L) \\ + \big[k_2((u_h)(t,L)) + d_2(\zeta_1(t)) (u_h)_t(t,L) + c_2(\zeta_2(t))\big] w(L) = 0,\end{aligned} \tag{4.90}$$

for all $w \in W_h,\ t > 0$, coupled to the:

$$\begin{aligned}(\tilde{\zeta}_1)_t &= a_1(\tilde{\zeta}_1) + b_1(\tilde{\zeta}_1)(u_h)_{xt}(.,L), \\ (\tilde{\zeta}_2)_t &= a_2(\tilde{\zeta}_2) + b_2(\tilde{\zeta}_2)(u_h)_t(.,L),\end{aligned} \tag{4.91}$$

with the initial conditions

$$\begin{aligned} u_h(0) &= u_{0,h}, & (u_h)_t(0) &= v_{0,h}, \\ \tilde{\zeta}_1(0) &= \zeta_{1,0}, & \tilde{\zeta}_2(0) &= \zeta_{2,0}. \end{aligned}$$

4.3.1.2 Vector representation

Let $\mathbb{U}$ be the vector representation of the function u_h. Then (4.90) is equivalent to the following vector equation:

$$\mathbb{A}\mathbb{U}_{tt} + \tilde{\mathbb{B}}(\tilde{\zeta}_1, \tilde{\zeta}_2)\mathbb{U}_t + \tilde{\mathbb{K}}\mathbb{U} + \mathbb{G}(\mathbb{U}) + \tilde{\mathbb{C}}(\tilde{\zeta}_1, \tilde{\zeta}_2) = 0. \tag{4.92}$$

The corresponding matrices and matrix functions are given by:

$$\begin{aligned}
\mathbb{A}_{i,j} &= \mu \int_0^L w_i w_j \, \mathrm{d}x + M w_i(L) w_j(L) + J (w_i)_x(L)(w_j)_x(L), \\
\tilde{\mathbb{K}}_{i,j} &:= \Lambda \int_0^L (w_i)_{xx}(w_j)_{xx} \, \mathrm{d}x, \\
\tilde{\mathbb{B}}_{i,j}(\tilde{\zeta}_1, \tilde{\zeta}_2) &:= d_1(\tilde{\zeta}_1)(w_i)_x(L)(w_j)_x(L) + d_2(\tilde{\zeta}_2) w_i(L) w_j(L),
\end{aligned}$$

for $i, j = 1, \dots, N$, and the vectors functions are given by:

$$\begin{aligned}
\mathbb{G}(\mathbb{U})_j &:= k_1(u_x(L))(w_j)_x(L) + k_2(u(L)) w_j(L), \\
\tilde{\mathbb{C}}(\zeta_1, \zeta_2)_j &:= c_1(\tilde{\zeta}_1)\,(w_j)_x(L) + c_2(\tilde{\zeta}_2)\, w_j(L),
\end{aligned}$$

for $j = 1, \dots, N$.

4.3.1.3 Dissipativity of the semi-discrete scheme

In order to show that the scheme given by (4.90)–(4.91) is dissipative, first a time dependent energy functional E_n for a trajectory $u \in C^2([0,\infty); \tilde{H}_0^2(0,L))$ and $\zeta_{1,2} \in C^1([0,\infty); \mathbb{R}^n)$ is defined as an analogue of the Lyapunov functional $V : \mathcal{H} \to \mathbb{R}$:

$$\begin{aligned}
E_{nl}(t; u, \zeta_1, \zeta_2) := {} & \frac{1}{2} \int_0^L \left(\Lambda u_{xx}(t,x)^2 + \mu u_t(t,x)^2 \right) dx + \frac{M}{2} u_t(t,L)^2 + \frac{J}{2} u_{xt}(t,L)^2 \\
& + \int_0^{u_x(t,L)} k_1(\sigma)\,\mathrm{d}\sigma + \int_0^{u(t,L)} k_2(\sigma)\,\mathrm{d}\sigma + V_1(\zeta_1(t)) + V_2(\zeta_2(t)).
\end{aligned}$$

Theorem 4.29. *Let $u_h \in C^2([0,\infty); \tilde{H}_0^2(0,L))$ and $\tilde{\zeta}_{1,2} \in C^1([0,\infty); \mathbb{R}^n)$ solve* (4.90)–(4.91)*. Then it holds for $t > 0$:*

$$\begin{aligned}
\frac{d}{dt} E_{nl}(t; u_h, \tilde{\zeta}_1, \tilde{\zeta}_2) &= -d_1(\tilde{\zeta}_1)(u_h)_{xt}(L)^2 - d_2(\tilde{\zeta}_2)(u_h)_t(L)^2 \\
&\quad - \nabla V_1(\tilde{\zeta}_1) \cdot a_1(\tilde{\zeta}_1) - \nabla V_2(\tilde{\zeta}_2) \cdot a_2(\tilde{\zeta}_2) \le 0
\end{aligned}$$

Proof. Taking the function w_h in (4.90) to be $w_h = (u_h)_t$, and the statement follows. □

Theorem 4.29 yields the boundedness of the semi-discrete solution. Therefore, the global existence of the solution can be proved.

Theorem 4.30. *The system* (4.90)–(4.91) *has a unique, global solution.*

The proof for the Theorem 4.30 follows analogously as the proof of Theorem 4.27, and will therefore be omitted.

4.3.2 Discretization in time

In this subsection, (4.90) and (4.91) shall be discretized in time. The time interval $[0, T]$ is discretized into S equidistant subintervals, for a fixed $S \in \mathbb{N}$. The time steps of the discretization are given by $t_k = k\Delta t, \forall k \in \{0, 1, \ldots, S\}$, where $\Delta t := T/S$.

Let $z_h = [u_h \; v_h \; \tilde{\zeta}_1 \; \tilde{\zeta}_2]^\top$ denote the solution of the system (4.90)–(4.91), and $z^k = [u^k \; v^k \; \zeta_1^k \; \zeta_2^k]^\top$ the approximation of the solution at time $t = t_k$.

4.3.2.1 Crank-Nicolson scheme

The Crank-Nicolson discretization of (4.90) is defined as follows:

$$\frac{u^{k+1} - u^k}{\Delta t} = \frac{v^{k+1} + v^k}{2}, \tag{4.93}$$

$$\begin{aligned}
&\mu \int_0^L \frac{v^{k+1} - v^k}{\Delta t} w_h \, \mathrm{d}x + \Lambda \int_0^L \frac{u_{xx}^{k+1} + u_{xx}^k}{2} (w_h)_{xx} \, \mathrm{d}x \\
&\quad + w_h(L) \Bigg(M \frac{v^{k+1}(L) - v^k(L)}{\Delta t} + H_2(u^{k+1}(L), u^k(L)) \\
&\quad + \frac{d_2(\zeta_2^{k+1}) + d_2(\zeta_2^k)}{2} \frac{v^{k+1}(L) + v^k(L)}{2} + c_2(\frac{\zeta_2^{k+1} + \zeta_2^k}{2}) \Bigg) \\
&\quad + (w_h)_x(L) \Bigg(J \frac{v_x^{k+1}(L) - v_x^k(L)}{\Delta t} + H_1(u_x^{k+1}(L), u_x^k(L)) \\
&\quad + \frac{d_1(\zeta_1^{k+1}) + d_1(\zeta_1^k)}{2} \frac{v_x^{k+1}(L) + v_x^k(L)}{2} + c_1(\frac{\zeta_1^{k+1} + \zeta_1^k}{2}) \Bigg) = 0,
\end{aligned} \tag{4.94}$$

$\forall w_h \in W_h$, where

$$H_1(\xi, \breve{\xi}) := \begin{cases} \frac{\int_{\breve{\xi}}^{\xi} k_1(\sigma) \, \mathrm{d}\sigma}{\xi - \breve{\xi}}, & \xi \neq \breve{\xi} \\ k_1(\xi), & \xi = \breve{\xi} \end{cases}$$

and

$$H_2(\psi, \breve{\psi}) := \begin{cases} \frac{\int_{\breve{\psi}}^{\psi} k_2(\sigma) \, \mathrm{d}\sigma}{\psi - \breve{\psi}}, & \psi \neq \breve{\psi} \\ k_2(\psi), & \psi = \breve{\psi} \end{cases}$$

Next, discretization in time for (4.91) is defined as:

$$\begin{aligned}
\frac{\zeta_1^{k+1} - \zeta_1^k}{\Delta t} &= a_1\Big(\frac{\zeta_1^{k+1} + \zeta_1^k}{2}\Big) + b_1\Big(\frac{\zeta_1^{k+1} + \zeta_1^k}{2}\Big) \frac{v_x^{k+1}(L) + v_x^k(L)}{2}, \\
\frac{\zeta_2^{k+1} - \zeta_2^k}{\Delta t} &= a_2\Big(\frac{\zeta_2^{k+1} + \zeta_2^k}{2}\Big) + b_2\Big(\frac{\zeta_2^{k+1} + \zeta_2^k}{2}\Big) \frac{v^{n+1}(L) + v^n(L)}{2}.
\end{aligned} \tag{4.95}$$

Remark 4.31. Notice that the mappings $\breve{\xi} \to H_1(\breve{\xi}, \xi)$, and $\breve{\psi} \to H_2(\breve{\psi}, \psi)$ are continuous on $\mathbb{R}$ for all $\psi, \xi \in \mathbb{R}$.

4.3.2.2 Dissipativity of the solution

In order to obtain dissipativity of the solution to fully-discrete scheme (4.93), (4.94), and (4.95), the following assumption is introduced:

$$V_i(\zeta_i) = \frac{1}{2}\zeta_i^\top P_i \zeta_i, \quad i = 1,\ 2, \tag{4.96}$$

where P_i is a symmetric positive definite matrix.

Theorem 4.32. *Assume that* (4.96) *holds. Then for all* $k \in \mathbb{N}$*:*

$$\begin{aligned}\frac{V(z^{k+1}) - V(z^k)}{\Delta t} \le &- \left(\frac{v^{k+1}(L) + v^k(L)}{2}\right)^2 \frac{d_2(\zeta_2^{k+1}) + d_2(\zeta_2^k)}{2} \\ &- \left(\frac{v_x^{k+1}(L) + v_x^k(L)}{2}\right)^2 \frac{d_1(\zeta_1^{k+1}) + d_1(\zeta_1^k)}{2} \le 0,\end{aligned}$$

Proof. It follows that:

$$\begin{aligned} V(z^{k+1}) - V(z^k) &= \frac{\Lambda}{2}(\|u_{xx}^{k+1}\|^2 - \|u_{xx}^k\|^2) + \frac{\mu}{2}(\|v^{k+1}\|^2 - \|v^k\|^2) \\ &+ \frac{M}{2}\left((v^{k+1}(L))^2 - (v^k(L))^2\right) + \frac{J}{2}\left((v_x^{k+1}(L))^2 - (v_x^k(L))^2\right) \\ &+ \int_{u_x^k(L)}^{u_x^{k+1}(L)} k_1(\sigma)\,\mathrm{d}\sigma + \int_{u^k(L)}^{u^{k+1}(L)} k_2(\sigma)\,\mathrm{d}\sigma \\ &+ V_1(\zeta_1^{k+1}) - V_1(\zeta_1^k) + V_2(\zeta_2^{k+1}) - V_2(\zeta_2^k). \end{aligned}$$

Taking $w_h = \mu(v^{k+1} - v^k)$ in (4.93) yields:

$$\frac{\mu}{2}(\|v^{k+1}\|^2 - \|v^k\|^2) = \mu \int_0^L \frac{u^{k+1} - u^k}{\Delta t}(v^{k+1} - v^k)\,\mathrm{d}x.$$

Next, taking $w_h = u^{k+1} - u^k$ in (4.94) gives:

$$\begin{aligned}\frac{\Lambda}{2}(\|u_{xx}^{k+1}\|^2 - \|u_{xx}^k\|^2) = &-\mu \int_0^L \frac{v^{k+1} - v^k}{\Delta t}(u^{k+1} - u^k)\,\mathrm{d}x \\ &-(u^{k+1}(L) - u^k(L))\left(M\frac{v^{k+1}(L) - v^k(L)}{\Delta t} + c_2\Big(\frac{\zeta_2^{k+1} + \zeta_2^k}{2}\Big)\right. \\ &\left.+\frac{d_2(\zeta_2^{k+1}) + d_2(\zeta_2^k)}{2}\,\frac{v^{k+1}(L) + v^k(L)}{2} + \frac{\int_{u^k(L)}^{u^{k+1}(L)} k_2(\sigma)\,\mathrm{d}\sigma}{u^{k+1}(L) - u^k(L)}\right)\end{aligned}$$

$$-(u_x^{k+1}(L)-u_x^k(L))\left(J\frac{v_x^{k+1}(L)-v_x^k(L)}{\Delta t}+c_1\Big(\frac{\zeta_1^{k+1}+\zeta_1^k}{2}\Big)\right.$$
$$\left.+\frac{d_1(\zeta_1^{k+1})+d_1(\zeta_1^k)}{2}\ \frac{v_x^{k+1}(L)+v_x^k(L)}{2}+\frac{\int_{u_x^k(L)}^{u_x^{k+1}(L)}k_1(\sigma)\,\mathrm{d}\sigma}{u_x^{k+1}(L)-u_x^k(L)}\right).$$

This yields:

$$\begin{aligned}
&V(z^{k+1})-V(z^k)=\\
&\quad -(u^{k+1}(L)-u^k(L))\Big(c_2\Big(\frac{\zeta_2^{k+1}+\zeta_2^k}{2}\Big)+\frac{d_2(\zeta_2^{k+1})+d_2(\zeta_2^k)}{2}\ \frac{v^{k+1}(L)+v^k(L)}{2}\Big)\\
&\quad -(u_x^{k+1}(L)-u_x^k(L))\Big(c_1\Big(\frac{\zeta_1^{k+1}+\zeta_1^k}{2}\Big)+\frac{d_1(\zeta_1^{k+1})+d_1(\zeta_1^k)}{2}\ \frac{v_x^{k+1}(L)+v_x^k(L)}{2}\Big)\\
&\quad +V_1(\zeta_1^{k+1})-V_1(\zeta_1^k)+V_2(\zeta_2^{k+1})-V_2(\zeta_2^k)
\end{aligned}$$

Multiplying equations in (4.95) with $\nabla V_1(\frac{\zeta_1^{k+1}+\zeta_1^k}{2})$, and $\nabla V_2(\frac{\zeta_2^{k+1}+\zeta_2^k}{2})$ respectively, yields :

$$\begin{aligned}
-\frac{v_x^{k+1}(L)+v_x^k(L)}{2}\ \nabla V_1\Big(\frac{\zeta_1^{k+1}+\zeta_1^k}{2}\Big)\cdot b_1\Big(\frac{\zeta_1^{k+1}+\zeta_1^k}{2}\Big)+\nabla V_1\Big(\frac{\zeta_1^{k+1}+\zeta_1^k}{2}\Big)\cdot\frac{\zeta_1^{k+1}-\zeta_1^k}{\Delta t}\\
=\nabla V_1\Big(\frac{\zeta_1^{k+1}+\zeta_1^k}{2}\Big)\cdot a_1\Big(\frac{\zeta_1^{k+1}+\zeta_1^k}{2}\Big),\\
-\frac{v^{k+1}(L)+v^k(L)}{2}\ \nabla V_2\Big(\frac{\zeta_2^{k+1}+\zeta_2^k}{2}\Big)\cdot b_2\Big(\frac{\zeta_2^{k+1}+\zeta_2^k}{2}\Big)+\nabla V_2\Big(\frac{\zeta_2^{k+1}+\zeta_2^k}{2}\Big)\cdot\frac{\zeta_2^{k+1}-\zeta_2^k}{\Delta t}\\
=\nabla V_2\Big(\frac{\zeta_2^{k+1}+\zeta_2^k}{2}\Big)\cdot a_2\Big(\frac{\zeta_2^{k+1}+\zeta_2^k}{2}\Big),
\end{aligned}$$

which is, due to (4.8), equvivalent to:

$$\begin{aligned}
-\frac{v_x^{k+1}(L)+v_x^k(L)}{2}\ c_1\Big(\frac{\zeta_1^{k+1}+\zeta_1^k}{2}\Big)\le-\nabla V_1\Big(\frac{\zeta_1^{k+1}+\zeta_1^k}{2}\Big)\cdot\frac{\zeta_1^{k+1}-\zeta_1^k}{\Delta t}\\
-\frac{v^{k+1}(L)+v^k(L)}{2}\ c_2\Big(\frac{\zeta_2^{k+1}+\zeta_2^k}{2}\Big)\le-\nabla V_2\Big(\frac{\zeta_2^{k+1}+\zeta_2^k}{2}\Big)\cdot\frac{\zeta_2^{k+1}-\zeta_2^k}{\Delta t}
\end{aligned}\tag{4.97}$$

Therefore, from (4.97) follows:

$$\begin{aligned}
\frac{\|z^{k+1}\|_{\mathcal{H}}^2-\|z^k\|_{\mathcal{H}}^2}{\Delta t}\ \le\ &-\Big(\frac{v^{k+1}(L)+v^k(L)}{2}\Big)^2\frac{d_2(\zeta_2^{k+1})+d_2(\zeta_2^k)}{2}\\
&-\Big(\frac{v_x^{k+1}(L)+v_x^k(L)}{2}\Big)^2\frac{d_1(\zeta_1^{k+1})+d_1(\zeta_1^k)}{2}
\end{aligned}$$

$$+\frac{V_1(\zeta_1^{k+1})-V_1(\zeta_1^k)}{\Delta t}-\nabla V_1\Big(\frac{\zeta_1^{k+1}+\zeta_1^k}{2}\Big)\cdot\frac{\zeta_1^{k+1}-\zeta_1^k}{\Delta t}$$
$$+\frac{V_2(\zeta_2^{k+1})-V_2(\zeta_2^k)}{\Delta t}-\nabla V_2\Big(\frac{\zeta_2^{k+1}+\zeta_2^k}{2}\Big)\cdot\frac{\zeta_2^{k+1}-\zeta_2^k}{\Delta t}.$$

Finally, (4.96) implies

$$\begin{aligned}&\frac{V_i(\zeta_i^{k+1})-V_i(\zeta_i^k)}{\Delta t}-\nabla V_i\Big(\frac{\zeta_i^{k+1}+\zeta_i^k}{2}\Big)\cdot\frac{\zeta_i^{k+1}-\zeta_i^k}{\Delta t}\\&\quad=\frac{(\zeta_i^{k+1})^\top P_i\zeta_i^{k+1}-(\zeta_i^k)^\top P_i\zeta_i^k}{2\Delta t}-\Big(\frac{\zeta_i^{k+1}+\zeta_i^k}{2}\Big)^\top P_i\frac{\zeta_i^{k+1}-\zeta_i^k}{\Delta t}=0,\end{aligned}$$

for $i=1,2$, and the statement follows. □

4.3.2.3 Solvability of the fully-discrete method

In this subsection, it is investigated if the fully-discrete scheme (4.93)–(4.95) is solvable.

Theorem 4.33. *Assume condition* (4.96) *holds. Let $k\in\mathbb{N}$ be fixed and $z^k\in W_h\times W_h\times\mathbb{R}^{2n}$. Then there exists a solution to* (4.93)–(4.95)*.*

Proof. First, (4.93) and (4.94) are rewritten in their vector forms:

$$\frac{\mathbb{U}^{k+1}-\mathbb{U}^k}{\Delta t}=\frac{\mathbb{V}^{k+1}+\mathbb{V}^k}{2}\tag{4.98}$$

$$\begin{aligned}\mathbb{A}\frac{\mathbb{V}^{k+1}-\mathbb{V}^k}{\Delta t}&=-\frac{\tilde{\mathbb{B}}(\zeta_1^{k+1},\zeta_2^{k+1})+\tilde{\mathbb{B}}(\zeta_1^k,\zeta_2^k)}{2}\frac{\mathbb{V}^{k+1}+\mathbb{V}^k}{2}-\tilde{\mathbb{H}}(\mathbb{U}^{k+1},\mathbb{U}^k)\\&\quad-\tilde{\mathbb{K}}\frac{\mathbb{U}^{k+1}+\mathbb{U}^k}{2}-\tilde{\mathbb{C}}\Big(\frac{\zeta_1^{k+1}+\zeta_1^k}{2},\frac{\zeta_2^{k+1}+\zeta_2^k}{2}\Big),\end{aligned}\tag{4.99}$$

with vector function $\tilde{\mathbb{H}}$ defined by:

$$\tilde{\mathbb{H}}(\mathbb{U}^{k+1},\mathbb{U}^k)_j=H_1(u_x^{k+1}(L),u_x^k(L))(w_j)_x(L)+H_2(u^{k+1}(L),u^k(L))w_j(L).$$

Further, let $g:\mathbb{R}^{2N+2n}\to\mathbb{R}^{2N+2n}$ be defined as:

$$g\left(\begin{bmatrix}\Delta\mathbb{U}\\\Delta\mathbb{V}\\\Delta\zeta_1\\\Delta\zeta_2\end{bmatrix}\right)=\begin{bmatrix}\Delta t\left(\mathbb{V}^k+\frac{\Delta\mathbb{V}}{2}\right)\\-\Delta t\mathbb{A}^{-1}\mathbb{P}\\\Delta t\ a_1(\zeta_1^k+\frac{\Delta\zeta_1}{2})+b_1(\zeta_1^k+\frac{\Delta\zeta_1}{2})\Delta\mathbb{U}_x(L)\\\Delta t\ a_2(\zeta_2^k+\frac{\Delta\zeta_2}{2})+b_2(\zeta_2^k+\frac{\Delta\zeta_2}{2})\Delta\mathbb{U}(L)\end{bmatrix}$$

where the vector $\mathbb{P}$ is defined by:

$$\mathbb{P}=\frac{\mathbb{B}(\zeta_1^k+\Delta\zeta_1,\zeta_2^k+\Delta\zeta_1)+\mathbb{B}(\zeta_1^k,\zeta_2^k)}{2}(\mathbb{V}^k+\frac{\Delta\mathbb{V}}{2})+\tilde{\mathbb{K}}(\mathbb{U}^k+\frac{\Delta\mathbb{U}}{2})$$

$$+\tilde{\mathbb{H}}(\mathbb{U}^k+\Delta\mathbb{U},\mathbb{U}^k)+\mathbb{C}(\zeta_1^k+\frac{\Delta\zeta_1}{2},\zeta_2^k+\frac{\Delta\zeta_2}{2}).$$

Then it is easily seen that $[\Delta\mathbb{U}\,\Delta\mathbb{V}\,\Delta\zeta_1\,\Delta\zeta_2]^\top$ is a fixed point of g, if and only if

$$\begin{aligned}\mathbb{U}^{k+1} &:= \Delta\mathbb{U}+\mathbb{U}^k,\\ \mathbb{V}^{k+1} &:= \Delta\mathbb{V}+\mathbb{V}^k,\\ \zeta_1^{k+1} &:= \Delta\zeta_1+\zeta_1^k,\\ \zeta_2^{k+1} &:= \Delta\zeta_2+\zeta_2^k,\end{aligned}$$

solves (4.98), (4.99) and (4.95). Moreover, according to Remark 4.31, function g is continuous, and hence compact (since the domain and the range of g are both finite dimensional). Next, let the subset $S\subset\mathbb{R}^{2N+2n}$ be defined with:

$$S:=\{\Delta Z\in\mathbb{R}^{2N+2n}\colon \Delta Z=\lambda g(\Delta Z),\ \lambda\in[0,1]\}.$$

In the following, it is demonstrated that the set S is bounded. Namely, let $\Delta Z=[\Delta\mathbb{U}\,\Delta\mathbb{V}\,\Delta\zeta_1\,\Delta\zeta_2]^\top\in S$ be arbitrary. Then u and v in W_h are defined so that their vector representation is $\mathbb{U}^k+\Delta\mathbb{U}$ and $\mathbb{V}^k+\Delta\mathbb{V}$, respectively. Furthermore, let $\zeta_i=\zeta_i^k+\Delta\zeta_i$, for $i=1,2$. Then the following holds:

$$\int_0^L (u-u^k)\,w_h\ \mathrm{d}x=\lambda\Delta t\int_0^L\frac{v+v^k}{2}\,w_h\ \mathrm{d}x,$$

$$\begin{aligned}&\mu\int_0^L(v-v^k)w_h\ \mathrm{d}x+M(v(L)-v^k(L))w_h(L)+J(v_x(L)-v_x^k(L))(w_h)_x(L)=\\ &+\lambda\Delta t\Bigg[-\Lambda\int_0^L\frac{u_{xx}+u_{xx}^k}{2}(w_h)_{xx}\ \mathrm{d}x-H_2(u(L),u^k(L))w_h(L)-H_1(u_x(L),u_x^k(L))(w_h)_x(L)\\ &-w_h(L)\left(\frac{d_2(\zeta_2)+d_2(\zeta_2^k)}{2}\frac{v(L)+v^k(L)}{2}+c_2\Big(\frac{\zeta_2+\zeta_2^k}{2}\Big)\right)\\ &-(w_h)_x(L)\left(\frac{d_1(\zeta_1)+d_1(\zeta_1^k)}{2}\frac{v_x(L)+v_x^k(L)}{2}+c_1\Big(\frac{\zeta_1+\zeta_1^k}{2}\Big)\right)\Bigg]=0,\end{aligned}$$

for all $w_h\in W_h$ and

$$\begin{aligned}\zeta_1-\zeta_1^k &= \lambda\Delta t\left(a_1\Big(\frac{\zeta_1+\zeta_1^k}{2}\Big)+b_1\Big(\frac{\zeta_1+\zeta_1^k}{2}\Big)\frac{v_x(L)+v_x^k(L)}{2}\right),\\ \zeta_2-\zeta_2^k &= \lambda\Delta t\left(a_2\Big(\frac{\zeta_2+\zeta_2^k}{2}\Big)+b_2\Big(\frac{\zeta_2+\zeta_2^k}{2}\Big)\frac{v(L)+v^k(L)}{2}\right).\end{aligned}$$

Following the lines of the proof for Theorem 4.32, it follows that:

$$\lambda\left(\|z\|_{\mathcal{H}}^2-\|z^k\|_{\mathcal{H}}^2\right)\le 0.$$

If $\lambda=0$, then it is trivial to see $z=z^k$. For $\lambda\in(0,1]$, it follows $\|z\|\le\|z^k\|$. Thus S is bounded. According to Leray–Schauder fixed point theorem, g has a fixed point, and the statement of the theorem follows. □

Remark 4.34. Up to the knowledge of the author, the numerical method for EBB system with a nonlinear controller is novel. Further, it would be of interest to see if it can be extended to the case when condition (4.96) does not hold.

Chapter 5

Simulations

In this chapter, the numerical schemes developed in Subsections 2.3.2, 3.5.2, and 4.3.2, will be implemented, and simulation results presented. The dissipativity of the numerical methods and their stability will be verified. Numerical methods developed in Subsections 3.5.2, and 4.3.2 result in nonlinear algebraic equations which can be solved utilizing Picard or Newton–Raphson method in the implementations. In all simulation examples, the following values for the system coefficients are taken: $\mu = \Lambda = L = 1$, $M = J = 0.1$.

5.1 Linear boundary control

The numerical method from Subsection 2.3.2 for the Euler-Bernoulli beam with linear boundary control is implemented in this section. A part of the simulation results presented here, also appears in [48]. It is taken throughout this section, that

$$\begin{aligned} A_1 &= A_2 = -I \in \mathbb{R}^{n \times n}, \\ b_1 &= b_2 = c_1 = c_2 = [1\ 1\ \ldots\ 1]^\top \in \mathbb{R}^n, \end{aligned} \tag{5.1}$$

where I denotes the identity matrix, and $n \in \mathbb{N} \cup \{0\}$ is the dimension of the controller variables $\zeta_{1,2}$. Moreover, let $k_1 = k_2 = 0.01$ $d_1 = d_2 = 0.02$.

In the first simulation example, $n = 3$ is taken. The initial conditions are taken as follows:

$$\begin{aligned} u_0(x) &= -0.6 \left(\frac{x}{L}\right)^2 + 0.4 \left(\frac{x}{L}\right)^3, \\ v_0 &\equiv 0, \\ \zeta_{1,0} &= \zeta_{2,0} = [0\ 0\ 0]^\top. \end{aligned}$$

Furthermore, let the time step be $\Delta t = 0.01$, and the spatial discretization step $h = 0.01$. Figure 5.1 shows the damped oscillations of the beam $u(t, x)$ on $x \in [0, L]$ and its convergence to the steady state $u \equiv 0$ on the time interval $t \in [0, 50]$. Figure 5.2 illustrates the (slower then exponential) energy $\|z(t)\|_{\mathcal{H}}$ dissipation of the coupled control system, on $t \in [0, 50]$.

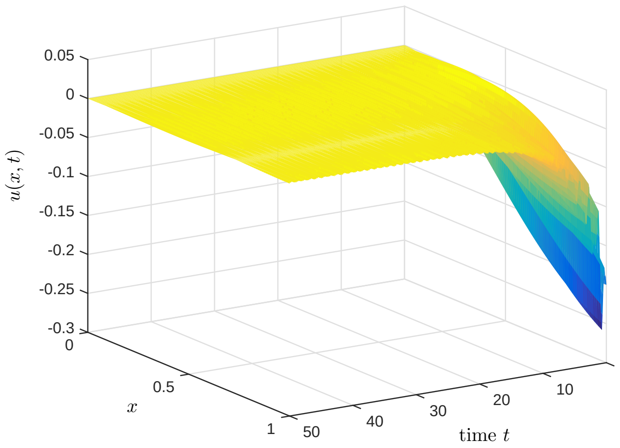


Figure 5.1: Deflection $u(t, x)$

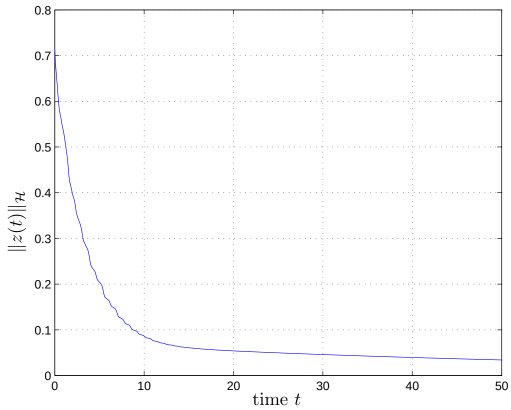


Figure 5.2: Norm dissipation: $\|u(t)\|_{\mathcal{H}}$

In order to verify the order of convergence (o.o.c.) proved in Subsection 2.3.2.3, simulations are performed for different time and space discretization steps. In Table 5.1, the l^2-error norms of z_e are listed. In the left table, the o.o.c. results for fixed $\Delta t = 0.01$ and varying space discretization step h on the time interval $[0, 1]$ are given. In the right table the o.o.c. results on the time interval $[0, 0.00041]$ for varying time steps Δt and $h = 1/50$ fixed, are presented. Note that the results from Table 5.1, confirm the order of convergence 2 of the numerical method in both space and time.

Table 5.1: Experimental convergence rates

Δt	h	$\|z_e\|_{l^2}$	o.o.c.
10^{-2}	$\frac{1}{4}$	$1.75 * 10^{-2}$	$--$
10^{-2}	$\frac{1}{8}$	$5.5 * 10^{-3}$	1.67
10^{-2}	$\frac{1}{16}$	$7.92 * 10^{-4}$	2.80
10^{-2}	$\frac{1}{32}$	$1.39 * 10^{-4}$	2.51
10^{-2}	$\frac{1}{64}$	$3.38 * 10^{-5}$	2.04
10^{-2}	$\frac{1}{128}$	$8.24 * 10^{-6}$	2.04

Δt	h	$\|z_e\|_{l^2}$	o.o.c.
$6.4 * 10^{-6}$	$\frac{1}{50}$	$2.58 * 10^{-6}$	$--$
$3.2 * 10^{-6}$	$\frac{1}{50}$	$6.87 * 10^{-7}$	1.91
$1.6 * 10^{-6}$	$\frac{1}{50}$	$1.73 * 10^{-7}$	1.99
$8 * 10^{-7}$	$\frac{1}{50}$	$4.27 * 10^{-8}$	2.02
$4 * 10^{-7}$	$\frac{1}{50}$	$1.02 * 10^{-8}$	2.07
$2 * 10^{-7}$	$\frac{1}{50}$	$2.03 * 10^{-9}$	2.32

In order to examine which effect does the dimension of the controller variable n has on the damping of the beam, three cases will be considered:

a) the static controller, or equivalently $n = 0$,

b) dynamic controller with dimension $n = 5$,

c) dynamic controller with dimension $n = 10$,

Again, let $\Delta t = 0.01$ and $h = 0.01$. Hereby, other system parameters stay as above, and the initial conditions are given as

$$\begin{aligned} u_0(x) &= -0.6\left(\frac{x}{L}\right)^2 + 0.4\left(\frac{x}{L}\right)^3, \\ v_0 &\equiv 0, \\ \zeta_{1,0} &= \zeta_{2,0} = [0.3\ 0.3\ 0.3]^\top. \end{aligned}$$

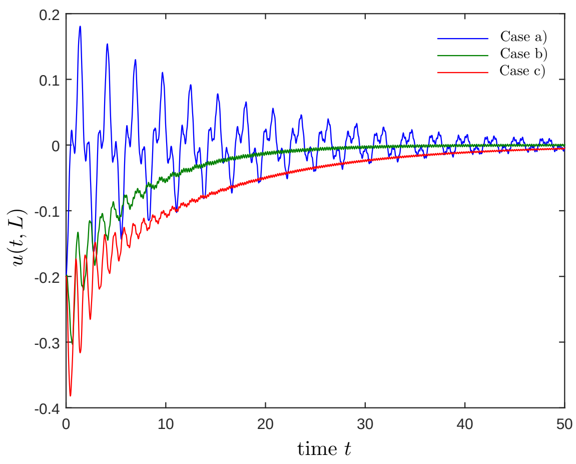


Figure 5.3: Tip position comparison

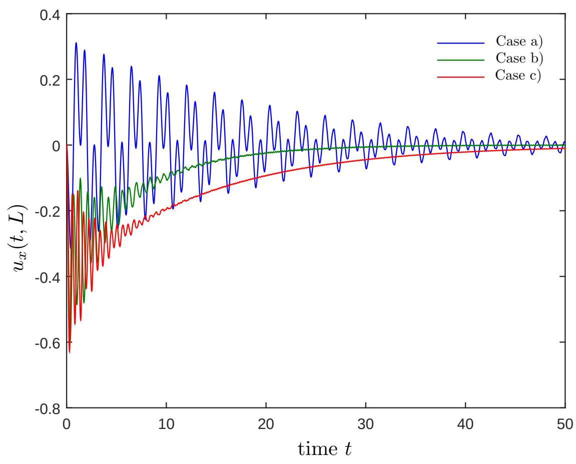


Figure 5.4: Tip angle comparison

In the Figures 5.3 and 5.4, the tip position $u(x, L)$ and tip angle $u_x(t, L)$ in all three cases on the time interval $[0, 50]$ are compared. They illustrate how coupling of the beam with a dynamic controller affect its deflection, as opposed to the static controller. Here it can be seen that dynamic controller suppresses the vibrations of the beams tip, but also can slow down its convergence to the steady-state if the dimension of the controller is taken too large.

5.2 Nonlinear damper and spring

In this subsection, the simulation results of the numerical method (3.90) and (3.91) for the Euler-Bernoulli beam with the nonlinear spring and damper, as introduced in Subsection 3.5.2, are presented. The time step $\Delta t = 0.01$ and the spatial discretization $h = 0.01$ are taken. Furthermore, Newton's method is utilized to solve the nonlinear system (3.90) and (3.91). Initial conditions are taken to be $u_0(x) = -0.6\left(\frac{x}{L}\right)^2 + 0.4\left(\frac{x}{L}\right)^3$, and $v_0(x) \equiv 0$. The simulations are performed for two different cases, first taking a polynomial, and then a trigonometric nonlinearity:

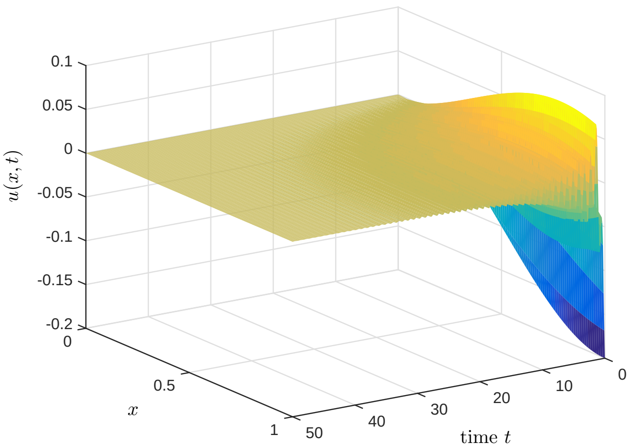


Figure 5.5: Case a): Deflection $u(t, L)$

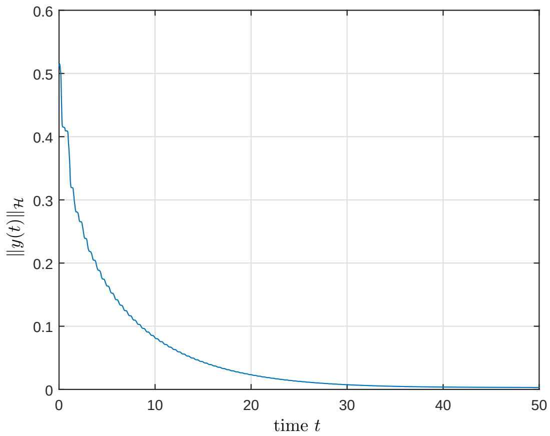


Figure 5.6: Case a): Lyapunov dissipation

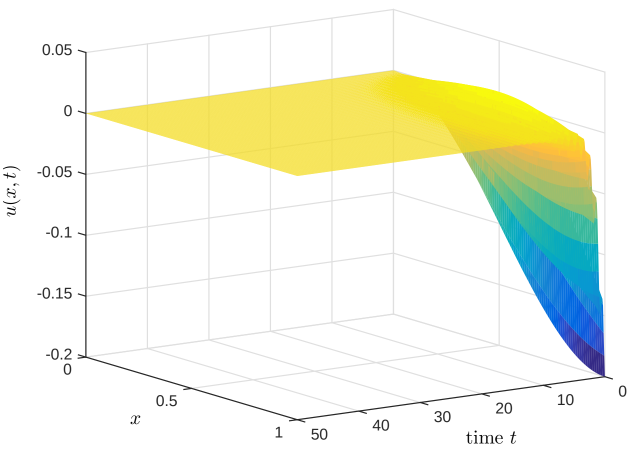


Figure 5.7: Case b): Deflection $u(t, L)$

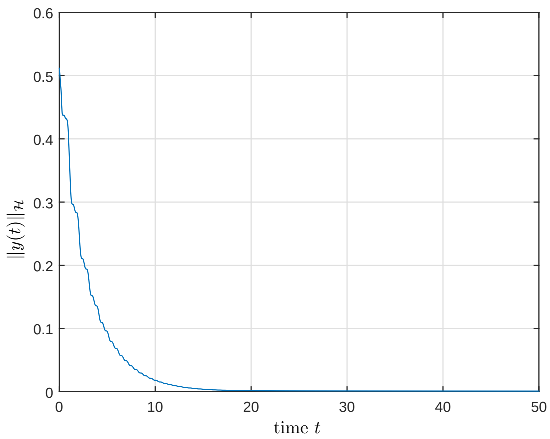


Figure 5.8: Case b): Lyapunov dissipation

a) $s(x) = 0.1x + x^3,\ d(x) = 0.5x + 5x^3$

b) $s(x) = 0.1x + \sin x,\ d(x) = 0.5x + 5 \tan x$

Figures 5.5 and 5.7 represent the deflection of the beam $u(t, x)$, and Figure 5.6 and Figure 5.8 represent the decay of the Lyapunov function $\|y(t)\|_{\mathcal{H}}$ on the time interval $[0, 50]$ for cases a) and b) respectively. Next, these results are compared to the simulation results in the case when the spring and damper are linear:

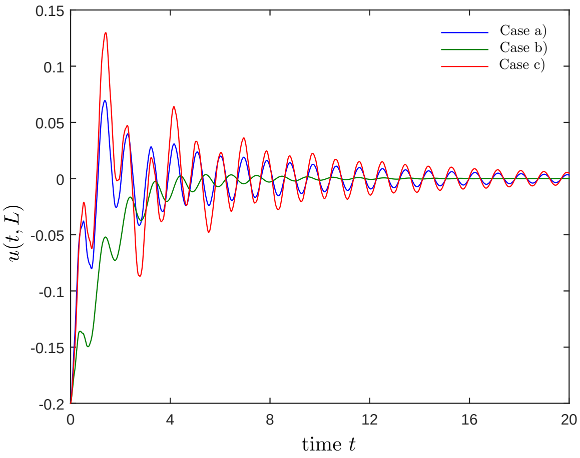


Figure 5.9: Tip position comparison

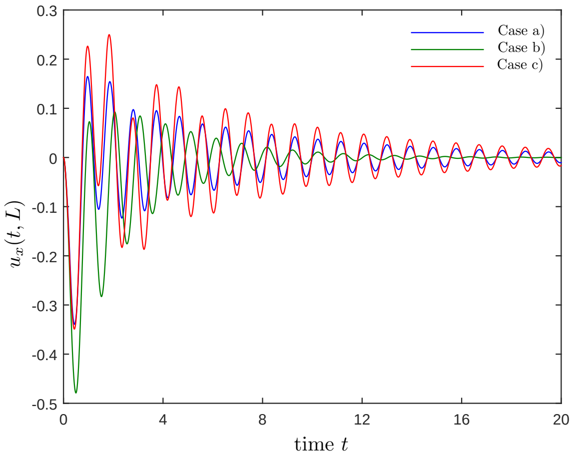


Figure 5.10: Tip angle comparison

c) $s(x) = 0.1x,\ d(x) = 0.5x.$

In the Figures 5.9 and 5.10, position $u(t, L)$ and the angle of the tip of the beam $u_x(t, L)$ are compared on the time interval $[0, 20]$ for the nonlinear cases a) and b), and the linear case c).

5.3 Nonlinear boundary control

Finally, in this section the simulation results for the numerical method (4.93) and (4.94) for the Euler-Bernoulli beam with nonlinear dynamic controller, introduced in Subsection 4.3.2, are presented. Again, Newton's method is utilized to solve the nonlinear system of equations (4.93) and (4.94), for $\Delta t = 0.01$ and $h = 0.01$. The parameter functions of the nonlinear control law are defined as follows:

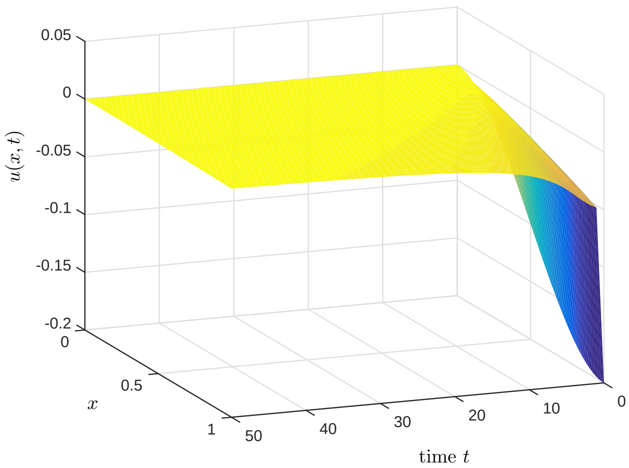
$$\begin{aligned} a_j(w) &= -[w_1^3, w_2^3, w_3^3]^\top, \\ V_j(w) &= \|w\|^2, \\ b_j(w) &= [w_1^2, w_2^2, w_3^2]^\top, \\ c_j(w) &= w_1^3 + w_2^3 + w_3^3, \end{aligned}$$

for $w = [w_1, w_2, w_3]^\top \in \mathbb{R}^3$, and $j = 1, 2$.

Two different choices for the functions $k_{1,2}$ and $d_{1,2}$ of the nonlinear controller will be considered:

a) $k_j(x) = 0.1x + x^3,\ d_j(w) = 0.5 + 5\|w\|^3,$

b) $k_j(x) = 0.1x + \sinh(x),\ d_j(w) = 0.5 + 5\sinh(\|w\|^2),$



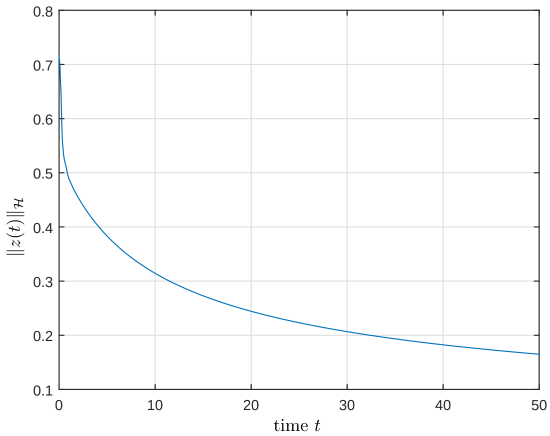
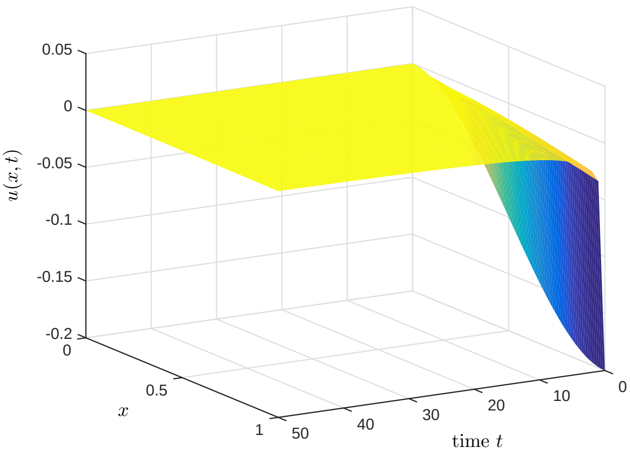


Figure 5.11: Case a): Deflection $u(t, L)$

Figure 5.12: Case a): Lyapunov dissipation



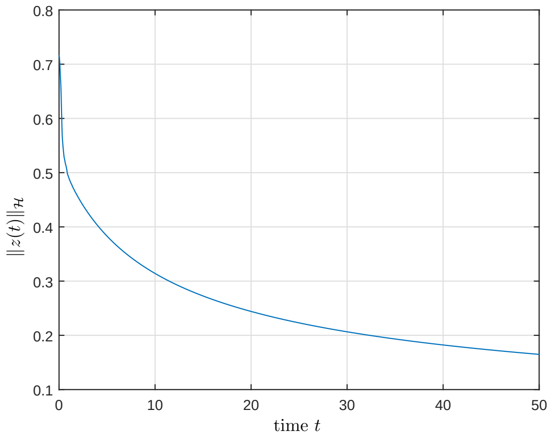


Figure 5.13: Case b): Deflection $u(t, L)$

Figure 5.14: Case b): Lyapunov dissipation

where $x \in \mathbb{R}$, and $w \in \mathbb{R}^3$, for $j = 1, 2$. Initial conditions are taken to be the same in both examples:

$$u_0(x) = -0.6\left(\frac{x}{L}\right)^2 + 0.4\left(\frac{x}{L}\right)^3, \quad v_0(x) \equiv 0, \quad \zeta_{1,2}(0) = [0.3\ 0.3\ 0.3]^\top.$$

In Figures 5.11 and 5.13 the beam deflection $u(t, x)$ on $[0, L]$ is represented, and it can be seen how the oscillations of the beam are damped out on the time interval $[0, 50]$, for cases

a) and b) respectively. The decay of the Lyapunov functional $\|z(t)\|_{\mathcal{H}}$ on time interval $[0, 50]$ for cases a) and b), is shown in Figures 5.12 and 5.14. The comparison of the tip

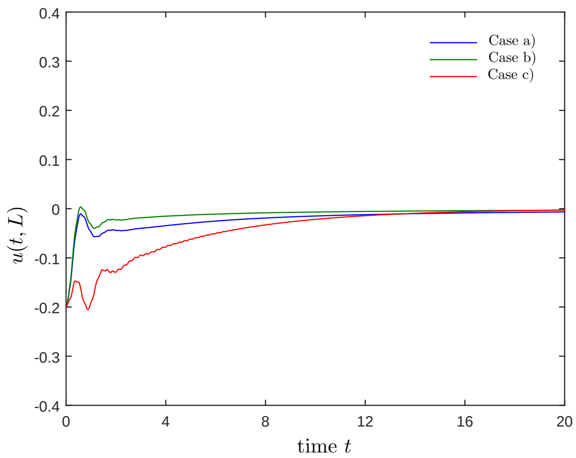


Figure 5.15: Tip position comparison

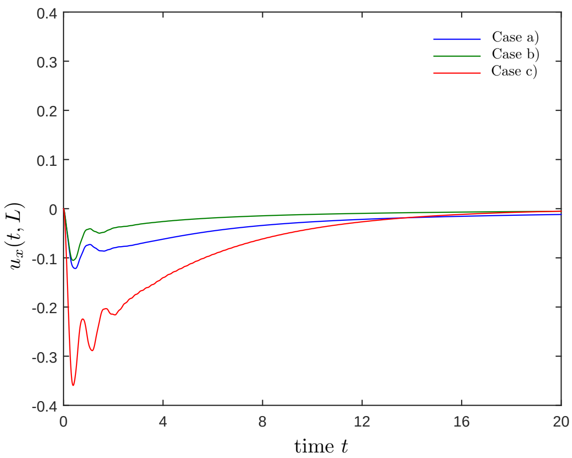


Figure 5.16: Tip angle comparison

position $u(t, L)$ and the tip angle $u_x(t, L)$ for these two examples and the case when the controller is linear, with (5.1) and for $j = 1, 2$:

c) $k_j(x) = 0.1x,\ d_j(x) = 0.5x,$

is illustrated in the Figures 5.15 and 5.16 for the time interval $[0, 20]$. It may be noticed that in this particular case, nonlinearity of the controller has resulted in faster decay of the beam and suppressed oscillations of the tip faster than for the linear control law.

5.4 Notes on the implementation

In this section, the implementation in MATLAB of the numerical methods developed in Subsection 2.3.2, 3.5.2, and 4.3.2 is presented.

5.4.1 Linear boundary control

The implementation of the numerical method for the EBB with linear boundary control given in Subsection 2.3.2 is described first. For this purpose, (2.134) – (2.137) are written in a compact vector form:

$$\mathbb{M}\mathbb{Z}^{n+1} = \mathbb{S}\mathbb{Z}^{n}, \tag{5.2}$$

where

$$\mathbb{Z}^{n} = [\mathbb{U}^{n}\ \mathbb{V}^{n}\ \zeta_1^{n}\ \zeta_2^{n}\ ]^{\top}, \tag{5.3}$$

and

$$\mathbb{M} = \begin{bmatrix} \frac{\mathbb{I}}{\Delta t} & -\frac{\mathbb{I}}{2} & 0 & 0 \\ \frac{\mathbb{K}}{2} & \frac{\mathbb{A}}{\Delta t} + \frac{\mathbb{B}}{2} & \frac{1}{2}\mathbb{W}_1 \otimes c_1 & \frac{1}{2}\mathbb{W}_2 \otimes c_2 \\ 0 & -\frac{1}{2} b_1 \otimes \mathbb{W}_1 & \frac{\mathbb{I}}{\Delta t} - \frac{\mathbb{A}_1}{2} & 0 \\ 0 & -\frac{1}{2} b_2 \otimes \mathbb{W}_2 & 0 & \frac{\mathbb{I}}{\Delta t} - \frac{\mathbb{A}_2}{2} \end{bmatrix}, \tag{5.4}$$

$$\mathbb{S} = \begin{bmatrix} \frac{\mathbb{I}}{\Delta t} & \frac{\mathbb{I}}{2} & 0 & 0 \\ -\frac{\mathbb{K}}{2} & \frac{\mathbb{A}}{\Delta t} - \frac{\mathbb{B}}{2} & -\frac{1}{2}\mathbb{W}_1 \otimes c_1 & -\frac{1}{2}\mathbb{W}_2 \otimes c_2 \\ 0 & \frac{1}{2} b_1 \otimes \mathbb{W}_1 & \frac{\mathbb{I}}{\Delta t} + \frac{\mathbb{A}_1}{2} & 0 \\ 0 & \frac{1}{2} b_2 \otimes \mathbb{W}_2 & 0 & \frac{\mathbb{I}}{\Delta t} + \frac{\mathbb{A}_2}{2} \end{bmatrix}. \tag{5.5}$$

The first step of the implementation is defining all the constants in the system. Here `mu` denotes μ, the mass density of the beam, and `lambda` its flexural rigidity Λ:

```
mu=1;
lambda=1;
M=0.1;
J=0.1;
```

Then, spatial step `h`, and a vector `x` which contains all the nodes of the spatial discretization are defined. Furthermore, `N` is the dimension of the space $H_3(\pi)$:

```
L=1;
P=50;
h=L/P;
x=linspace(0,L,P+1);
N=2*P;
```

Next, the variables of the controller law are defined. Here, variable `c_dim` denotes n, the dimension of the controller variables:

```
k_1=0.01;
k_2=0.01;
d_1=0.02;
d_2=0.02;

c_dim=10;
b_1=ones(c_dim,1);
b_2=ones(c_dim,1);
c_1=ones(c_dim,1);
c_2=ones(c_dim,1);
```

```
A_1=-eye(c_dim);
A_2=-eye(c_dim);
```

Time step Δt is denoted by `dt`, and `ts` is the number of time steps of the Crank-Nicolson scheme to be performed:

```
dt=10^(-7);
ts=512;
```

These definitions are enough to form the system matrix $\mathbb{M}$ from (5.4). In order to obtain a smaller condition number for the system matrix, the system is multiplied by Δt. Additionally, the following equalities are used for this step:

$$\begin{array}{lcl}
\int_{x_{m-1}}^{x_m} (w''_{2m-3}(x))^2 \ \mathrm{d}x &=& \frac{12}{h^3}, \\
\int_{x_{m-1}}^{x_m} (w''_{2m-2}(x))^2 \ \mathrm{d}x &=& \frac{4}{h}, \\
\int_{x_{m-1}}^{x_m} (w''_{2m-1}(x))^2 \ \mathrm{d}x &=& \frac{12}{h^3}, \\
\int_{x_{m-1}}^{x_m} (w''_{2m}(x))^2 \ \mathrm{d}x &=& \frac{4}{h},
\end{array}$$

$$\begin{array}{lcl}
\int_{x_{m-1}}^{x_m} w''_{2m-3}(x)\, w''_{2m-2}(x) \ \mathrm{d}x &=& \frac{6}{h^2}, \\
\int_{x_{m-1}}^{x_m} w''_{2m-3}(x)\, w''_{2m-1}(x) \ \mathrm{d}x &=& -\frac{12}{h^3}, \\
\int_{x_{m-1}}^{x_m} w''_{2m-3}(x)\, w''_{2m}(x) \ \mathrm{d}x &=& \frac{6}{h^2}, \\
\int_{x_{m-1}}^{x_m} w''_{2m-2}(x)\, w''_{2m-1}(x) \ \mathrm{d}x &=& -\frac{6}{h^2}, \\
\int_{x_{m-1}}^{x_m} w''_{2m-2}(x)\, w''_{2m}(x) \ \mathrm{d}x &=& \frac{2}{h}, \\
\int_{x_{m-1}}^{x_m} w''_{2m-1}(x)\, w''_{2m}(x) \ \mathrm{d}x &=& -\frac{6}{h^2},
\end{array}$$

$$\begin{array}{lcl}
\int_{x_{m-1}}^{x_m} (w_{2m-3}(x))^2 \ \mathrm{d}x &=& \frac{13h}{35}, \\
\int_{x_{m-1}}^{x_m} (w_{2m-2}(x))^2 \ \mathrm{d}x &=& \frac{h^3}{105}, \\
\int_{x_{m-1}}^{x_m} (w_{2m-1}(x))^2 \ \mathrm{d}x &=& \frac{13h}{35}, \\
\int_{x_{m-1}}^{x_m} (w_{2m}(x))^2 \ \mathrm{d}x &=& \frac{h^3}{105},
\end{array}$$

$$\begin{array}{lcl}
\int_{x_{m-1}}^{x_m} w_{2m-3}(x)\, w_{2m-2}(x) \ \mathrm{d}x &=& \frac{11h^2}{210}, \\
\int_{x_{m-1}}^{x_m} w_{2m-3}(x)\, w_{2m-1}(x) \ \mathrm{d}x &=& \frac{9h}{70}, \\
\int_{x_{m-1}}^{x_m} w_{2m-3}(x)\, w_{2m}(x) \ \mathrm{d}x &=& -\frac{13h^2}{420}, \\
\int_{x_{m-1}}^{x_m} w_{2m-2}(x)\, w_{2m-1}(x) \ \mathrm{d}x &=& \frac{13h^2}{420}, \\
\int_{x_{m-1}}^{x_m} w_{2m-2}(x)\, w_{2m}(x) \ \mathrm{d}x &=& -\frac{h^3}{140}, \\
\int_{x_{m-1}}^{x_m} w_{2m-1}(x)\, w_{2m}(x) \ \mathrm{d}x &=& -\frac{11h^2}{210},
\end{array}$$

for all $m = 1 \dots P$. Hence:

```
M=zeros(2*N+2*c_dim,2*N+2*c_dim);

M(1,1)=1; M(2,2)=1;
M(1,N+1)=-0.5*dt; M(2,N+2)=-0.5*dt;
```

```
for k = 2:P

    M(2*k,2*k)=1;  M(2*k-1,2*k-1)=1;
    M(2*k,N+2*k)=-0.5*dt;  M(2*k-1,N+2*k-1)=-0.5*dt;

    M(N+2*k-3,2*k-3)=M(N+2*k-3,2*k-3)+lambda*0.5*12*dt/(h^3);
    M(N+2*k-2,2*k-2)=M(N+2*k-2,2*k-2)+lambda*0.5*4*dt/(h);
    M(N+2*k-1,2*k-1)=M(N+2*k-1,2*k-1)+lambda*0.5*12*dt/(h^3);
    M(N+2*k,2*k)=M(N+2*k,2*k)+lambda*0.5*4*dt/(h);

    M(N+2*k-3,2*k-2)=M(N+2*k-3,2*k-2)+lambda*0.5*6*dt/(h^2);
    M(N+2*k-2,2*k-3)=M(N+2*k-2,2*k-3)+lambda*0.5*6*dt/(h^2);

    M(N+2*k-3,2*k-1)=M(N+2*k-3,2*k-1)-lambda*0.5*12*dt/(h^3);
    M(N+2*k-1,2*k-3)=M(N+2*k-1,2*k-3)-lambda*0.5*12*dt/(h^3);

    M(N+2*k-3,2*k)=M(N+2*k-3,2*k)+lambda*0.5*6*dt/(h^2);
    M(N+2*k,2*k-3)=M(N+2*k,2*k-3)+lambda*0.5*6*dt/(h^2);

    M(N+2*k-2,2*k-1)=M(N+2*k-2,2*k-1)-lambda*0.5*6*dt/(h^2);
    M(N+2*k-1,2*k-2)=M(N+2*k-1,2*k-2)-lambda*0.5*6*dt/(h^2);

    M(N+2*k-2,2*k)=M(N+2*k-2,2*k)+lambda*0.5*2*dt/(h);
    M(N+2*k,2*k-2)=M(N+2*k,2*k-2)+lambda*0.5*2*dt/(h);

    M(N+2*k-1,2*k)=M(N+2*k-1,2*k)-lambda*0.5*6*dt/(h^2);
    M(N+2*k,2*k-1)=M(N+2*k,2*k-1)-lambda*0.5*6*dt/(h^2);

    M(N+2*k-3,N+2*k-3)=M(N+2*k-3,N+2*k-3)+ mu*h*(13/35);
    M(N+2*k-2,N+2*k-2)=M(N+2*k-2,N+2*k-2)+ mu*h^3*(1/105);
    M(N+2*k-1,N+2*k-1)=M(N+2*k-1,N+2*k-1)+ mu*h*(13/35);
    M(N+2*k,N+2*k)=M(N+2*k,N+2*k)+ mu*h^3*(1/105);

    M(N+2*k-3,N+2*k-2)=M(N+2*k-3,N+2*k-2)+ mu*h^2*(11/210);
    M(N+2*k-2,N+2*k-3)=M(N+2*k-2,N+2*k-3)+ mu*h^2*(11/210);

    M(N+2*k-3,N+2*k-1)=M(N+2*k-3,N+2*k-1)+ mu*h*(9/70);
    M(N+2*k-1,N+2*k-3)=M(N+2*k-1,N+2*k-3)+ mu*h*(9/70);

    M(N+2*k-3,N+2*k)=M(N+2*k-3,N+2*k)+ mu*h^2*(-13/420);
    M(N+2*k,N+2*k-3)=M(N+2*k,N+2*k-3)+ mu*h^2*(-13/420);

    M(N+2*k-2,N+2*k-1)=M(N+2*k-2,N+2*k-1)+ mu*h^2*(13/420);
    M(N+2*k-1,N+2*k-2)=M(N+2*k-1,N+2*k-2)+ mu*h^2*(13/420);

    M(N+2*k-2,N+2*k)=M(N+2*k-2,N+2*k)+ mu*h^3*(-1/140);
    M(N+2*k,N+2*k-2)=M(N+2*k,N+2*k-2)+ mu*h^3*(-1/140);

    M(N+2*k-1,N+2*k)=M(N+2*k-1,N+2*k)+ mu*h^2*(-11/210);
    M(N+2*k,N+2*k-1)=M(N+2*k,N+2*k-1)+ mu*h^2*(-11/210);
```

```
end


M(N+1,1)=M(N+1,1)+lambda*0.5*12*dt/(h^3);
M(N+2,2)=M(N+2,2)+lambda*0.5*4*dt/(h);

M(N+1,2)=M(N+1,2)-lambda*0.5*6*dt/(h^2);
M(N+2,1)=M(N+2,1)-lambda*0.5*6*dt/(h^2);

M(2*N-1,N-1)=M(2*N-1,N-1)+0.5*k_2*dt;
M(2*N,N)=M(2*N,N)+0.5*k_1*dt;

M(N+1,N+1)=M(N+1,N+1)+ mu*h*(13/35);
M(N+2,N+2)=M(N+2,N+2)+ mu*h^3*(1/105);

M(N+1,N+2)=M(N+1,N+2)- mu*h^2*(11/210);
M(N+2,N+1)=M(N+2,N+1)- mu*h^2*(11/210);

M(2*N-1,2*N-1)=M(2*N-1,2*N-1)+ M + 0.5*d_2*dt;
M(2*N,2*N)=M(2*N,2*N)+ J + 0.5*d_1*dt;


for j= 1 : c_dim
     M(N+N,2*N+j)=0.5*c_1(j,1)*dt;
     M(N+N-1,2*N+c_dim+j)=0.5*c_2(j,1)*dt;
     M(2*N+j,2*N)= -0.5*b_1(j,1)*dt;
     M(2*N+c_dim+j,2*N-1)= -0.5*b_2(j,1)*dt;
end

for i = 1 : c_dim
    for j = 1 : c_dim

       M(2*N+i,2*N+j)=kroneckerDelta(i, j)  - 0.5*A_1(i,j)*dt;
       M(2*N+c_dim+i,2*N+c_dim+j)=kroneckerDelta(i, j) - 0.5*A_2(i,j)*dt;

    end
end
```

Construction of the right hand side $\mathbb{S}$ as in (5.5) follows:

```
S=zeros(2*N+2*c_dim,2*N+2*c_dim);

S(1,1)=1; S(2,2)=1;
S(1,N+1)=0.5*dt; S(2,N+2)=0.5*dt;

for k = 2:nn

    S(2*k,2*k)=1;  S(2*k-1,2*k-1)=1;
    S(2*k,N+2*k)=0.5*dt;  S(2*k-1,N+2*k-1)=0.5*dt;
```

```
    S(N+2*k-3,2*k-3)=S(N+2*k-3,2*k-3)-lambda*0.5*12*dt/(h^3);
    S(N+2*k-2,2*k-2)=S(N+2*k-2,2*k-2)-lambda*0.5*4*dt/(h);
    S(N+2*k-1,2*k-1)=S(N+2*k-1,2*k-1)-lambda*0.5*12*dt/(h^3);
    S(N+2*k,2*k)=S(N+2*k,2*k)-lambda*0.5*4*dt/(h);

    S(N+2*k-3,2*k-2)=S(N+2*k-3,2*k-2)-lambda*0.5*6*dt/(h^2);
    S(N+2*k-2,2*k-3)=S(N+2*k-2,2*k-3)-lambda*0.5*6*dt/(h^2);

    S(N+2*k-3,2*k-1)=S(N+2*k-3,2*k-1)+lambda*0.5*12*dt/(h^3);
    S(N+2*k-1,2*k-3)=S(N+2*k-1,2*k-3)+lambda*0.5*12*dt/(h^3);

    S(N+2*k-3,2*k)=S(N+2*k-3,2*k)-lambda*0.5*6*dt/(h^2);
    S(N+2*k,2*k-3)=S(N+2*k,2*k-3)-lambda*0.5*6*dt/(h^2);

    S(N+2*k-2,2*k-1)=S(N+2*k-2,2*k-1)+lambda*0.5*6*dt/(h^2);
    S(N+2*k-1,2*k-2)=S(N+2*k-1,2*k-2)+lambda*0.5*6*dt/(h^2);

    S(N+2*k-2,2*k)=S(N+2*k-2,2*k)-lambda*0.5*2*dt/(h);
    S(N+2*k,2*k-2)=S(N+2*k,2*k-2)-lambda*0.5*2*dt/(h);

    S(N+2*k-1,2*k)=S(N+2*k-1,2*k)+lambda*0.5*6*dt/(h^2);
    S(N+2*k,2*k-1)=S(N+2*k,2*k-1)+lambda*0.5*6*dt/(h^2);

    S(N+2*k-3,N+2*k-3)=S(N+2*k-3,N+2*k-3)+ mu*h*(13/35);
    S(N+2*k-2,N+2*k-2)=S(N+2*k-2,N+2*k-2)+ mu*h^3*(1/105);
    S(N+2*k-1,N+2*k-1)=S(N+2*k-1,N+2*k-1)+ mu*h*(13/35);
    S(N+2*k,N+2*k)=S(N+2*k,N+2*k)+ mu*h^3*(1/105);

    S(N+2*k-3,N+2*k-2)=S(N+2*k-3,N+2*k-2)+ mu*h^2*(11/210);
    S(N+2*k-2,N+2*k-3)=S(N+2*k-2,N+2*k-3)+ mu*h^2*(11/210);

    S(N+2*k-3,N+2*k-1)=S(N+2*k-3,N+2*k-1)+ mu*h*(9/70);
    S(N+2*k-1,N+2*k-3)=S(N+2*k-1,N+2*k-3)+ mu*h*(9/70);

    S(N+2*k-3,N+2*k)=S(N+2*k-3,N+2*k)+ mu*h^2*(-13/420);
    S(N+2*k,N+2*k-3)=S(N+2*k,N+2*k-3)+ mu*h^2*(-13/420);

    S(N+2*k-2,N+2*k-1)=S(N+2*k-2,N+2*k-1)+ mu*h^2*(13/420);
    S(N+2*k-1,N+2*k-2)=S(N+2*k-1,N+2*k-2)+ mu*h^2*(13/420);

    S(N+2*k-2,N+2*k)=S(N+2*k-2,N+2*k)+ mu*h^3*(-1/140);
    S(N+2*k,N+2*k-2)=S(N+2*k,N+2*k-2)+ mu*h^3*(-1/140);

    S(N+2*k-1,N+2*k)=S(N+2*k-1,N+2*k)+ mu*h^2*(-11/210);
    S(N+2*k,N+2*k-1)=S(N+2*k,N+2*k-1)+ mu*h^2*(-11/210);

end

S(N+1,1)=S(N+1,1)-lambda*0.5*12*dt/(h^3);
S(N+2,2)=S(N+2,2)-lambda*0.5*4*dt/(h);
```

```
S(N+1,2)=S(N+1,2)+lambda*0.5*6*dt/(h^2);
S(N+2,1)=S(N+2,1)+lambda*0.5*6*dt/(h^2);

S(2*N-1,N-1)=S(2*N-1,N-1)-0.5*k_1*dt;
S(2*N,N)=S(2*N,N)-0.5*k_2*dt;

S(N+1,N+1)=S(N+1,N+1)+ mu*h*(13/35);
S(N+2,N+2)=S(N+2,N+2)+ mu*h^3*(1/105);

S(N+1,N+2)=S(N+1,N+2)- mu*h^2*(11/210);
S(N+2,N+1)=S(N+2,N+1)- mu*h^2*(11/210);

S(2*N-1,2*N-1)=S(2*N-1,2*N-1)+ M - 0.5* d_2*dt;
S(2*N,2*N)=S(2*N,2*N)+ J - 0.5* d_1*dt;

for j= 1 : c_dim
     S(N+N,2*N+j)=-0.5*c_1(j,1)*dt;
     S(N+N-1,2*N+c_dim+j)=-0.5*c_2(j,1)*dt;
     S(2*N+j,2*N)= 0.5*b_1(j,1)*dt;
     S(2*N+c_dim+j,2*N-1)= 0.5*b_2(j,1)*dt;
end

for i = 1 : c_dim
    for j = 1 : c_dim

        S(2*N+i,2*N+j)=kroneckerDelta(i,j) + 0.5*dt*A_1(i,j);
        S(2*N+c_dim+i,2*N+c_dim+j)=kroneckerDelta(i,j) + 0.5*dt*A_2(i,j);

    end
end
```

Vector `Z` is introduced to store the solution of the scheme. Particularly, `Z(k+1,:)` contains $\mathbb{Z}^k$ defined with (5.3) approximating the solution at $t = t_k$. Also, initial conditions are introduced:

$$\begin{aligned} u_0 &= 0.2\left(-3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3\right), \\ v_0 &= 0, \\ \zeta_{1,0} &= 0, \\ \zeta_{2,0} &= 0. \end{aligned}$$

```
Z = zeros(ts+1, 2*N+2*c_dim);

for k=1:P
    Z(1,N+2*k-1)= 0;
    Z(1,N+2*k)= 0;
    Z(1,2*k)=0.2*(-6*(x(k+1)/L)/L + 6*(x(k+1)/L)^2/L);
    Z(1,2*k-1)=0.2*(-3*(x(k+1)/L)^2 + 2*(x(k+1)/L)^3);
```

```
end

    for k = 1 : c_dim
        Z(1,2*N + k)=0;
        Z(1,2*N + c_dim + k)=0;
    end
```

Finally, solving the linear system (5.2), $\mathbb{Z}^k$ is calculated for all time steps:

```
for k=1:ts

     Z(k+1,:)=linsolve(M,S*Z(k,:));

end
```

The simulation results of the above implementation are presented in Section 5.1. For the definition of the system matrix and the right hand side, a two parameter function `kroneckerDelta` is used, which returns 1 if the parameters are equal, 0 otherwise:

```
function f = kroneckerDelta (i,j)

if i == j
    f = 1;
else
    f=0;
end
```

5.4.2 EBB with a spring and a damper

For implementation of the numerical method for Euler-Bernoulli beam attached to a spring and damper described in Section 3.5.2, two cases a) and b) given in Section 5.2 are considered. A function `nonlin_spring_damper` is defined which as output gives the solution, tip position, tip angle, and Lyapunov function at all time steps. First the linear part of the system is formulated:

```
function [Z,pos,ang,l,nn,ts,dt] = nonl_spring_damper ()

mu=1; lambda=1; L=1;
M=0.1; J=0.1;

nn=100; h=L/nn; N=2*nn;
x=linspace(0,L,nn+1);

dt=10^(-2); ts=5000;

k_1=0.1; k_2=0.5;
```

```
k_11=1; k_22=5;

pos = zeros(ts+1,1);
ang = zeros(ts+1,1);
Z=zeros(nn+1,ts+1);
l=zeros(ts+1,1);

%matrices A and K

A=zeros(N,N);
K=zeros(N,N);

for k = 2:nn

    K(2*k-3,2*k-3)=K(2*k-3,2*k-3)+lambda*12/(h^3);

    K(2*k-2,2*k-2)=K(2*k-2,2*k-2)+lambda*4/(h);

    K(2*k-1,2*k-1)=K(2*k-1,2*k-1)+lambda*12/(h^3);

    K(2*k,2*k)=K(2*k,2*k)+lambda*4/(h);

    K(2*k-3,2*k-2)=K(2*k-3,2*k-2)+lambda*6/(h^2);
    K(2*k-2,2*k-3)=K(2*k-2,2*k-3)+lambda*6/(h^2);

    K(2*k-3,2*k-1)=K(2*k-3,2*k-1)-lambda*12/(h^3);
    K(2*k-1,2*k-3)=K(2*k-1,2*k-3)-lambda*12/(h^3);

    K(2*k-3,2*k)=K(2*k-3,2*k)+lambda*6/(h^2);
    K(2*k,2*k-3)=K(2*k,2*k-3)+lambda*6/(h^2);

    K(2*k-2,2*k-1)=K(2*k-2,2*k-1)-lambda*6/(h^2);
    K(2*k-1,2*k-2)=K(2*k-1,2*k-2)-lambda*6/(h^2);

    K(2*k-2,2*k)=K(2*k-2,2*k)+lambda*2/(h);
    K(2*k,2*k-2)=K(2*k,2*k-2)+lambda*2/(h);

    K(2*k-1,2*k)=K(2*k-1,2*k)-lambda*6/(h^2);
    K(2*k,2*k-1)=K(2*k,2*k-1)-lambda*6/(h^2);


    %mass matrix

    A(2*k-3,2*k-3)=A(2*k-3,2*k-3)+ mu*h*(13/35);

    A(2*k-2,2*k-2)=A(2*k-2,2*k-2)+ mu*h^3*(1/105);

    A(2*k-1,2*k-1)=A(2*k-1,2*k-1)+ mu*h*(13/35);

    A(2*k,2*k)=A(2*k,2*k)+ mu*h^3*(1/105);
```

```
    A(2*k-3,2*k-2)=A(2*k-3,2*k-2)+ mu*h^2*(11/210);
    A(2*k-2,2*k-3)=A(2*k-2,2*k-3)+ mu*h^2*(11/210);

    A(2*k-3,2*k-1)=A(2*k-3,2*k-1)+ mu*h*(9/70);
    A(2*k-1,2*k-3)=A(2*k-1,2*k-3)+ mu*h*(9/70);

    A(2*k-3,2*k)=A(2*k-3,2*k)+ mu*h^2*(-13/420);
    A(2*k,2*k-3)=A(2*k,2*k-3)+ mu*h^2*(-13/420);

    A(2*k-2,2*k-1)=A(2*k-2,2*k-1)+ mu*h^2*(13/420);
    A(2*k-1,2*k-2)=A(2*k-1,2*k-2)+ mu*h^2*(13/420);

    A(2*k-2,2*k)=A(2*k-2,2*k)+ mu*h^3*(-1/140);
    A(2*k,2*k-2)=A(2*k,2*k-2)+ mu*h^3*(-1/140);

    A(2*k-1,2*k)=A(2*k-1,2*k)+ mu*h^2*(-11/210);
    A(2*k,2*k-1)=A(2*k,2*k-1)+ mu*h^2*(-11/210);

end

%stiffness matrix

K(1,1)=K(1,1)+lambda*12/(h^3);

K(2,2)=K(2,2)+lambda*4/(h);

K(1,2)=K(1,2)-lambda*6/(h^2);
K(2,1)=K(2,1)-lambda*6/(h^2);

K(N-1,N-1) = K(N-1,N-1) + k_1;


%mass matrix

A(1,1)=A(1,1)+ mu*h*(13/35);

A(2,2)=A(2,2)+ mu*h^3*(1/105);

A(1,2)=A(1,2)- mu*h^2*(11/210);

A(2,1)=A(2,1)- mu*h^2*(11/210);

A(N-1,N-1)=A(N-1,N-1)+ M;

A(N,N)=A(N,N)+ J;


aux1=zeros(2*N,2*N);
     aux1(1:N,1:N) = eye(N);
     aux1(1:N,N+1:2*N) = -0.5*dt*eye(N);
     aux1(N+1:2*N,1:N) = dt*0.5*K;
```

```
aux1(N+1:2*N,N+1:2*N) = A;
aux1(2*N-1,2*N-1) = aux1(2*N-1,2*N-1)+k_2*0.5*dt;

aux2=zeros(2*N,2*N);
aux2(1:N,1:N) = eye(N);
aux2(1:N,N+1:2*N) = 0.5*dt*eye(N);
aux2(N+1:2*N,1:N) = -dt*0.5*K;
aux2(N+1:2*N,N+1:2*N) = A;
aux2(2*N-1,2*N-1) = aux2(2*N-1,2*N-1)-k_2*0.5*dt;
```

The initialization of the system follows, and the initial state is saved in corresponding vectors:

```
z0 = zeros(2*N,1);

for k=1:nn
    z0(N+2*k-1,1)= 0;
    z0(N+2*k,1)= 0;
    z0(2*k,1)=0.2*(-6*(x(k+1)/L)/L + 6*(x(k+1)/L)^2/L);
    z0(2*k-1,1)=0.2*(-3*(x(k+1)/L)^2 + 2*(x(k+1)/L)^3);

end

pos(1,1)=z0(2*nn-1,1);
ang(1,1)=z0(2*nn,1);
l(1,1) = lyapunov(nn,z0);

for k=1:nn
    Z(k+1,1)=z0(2*k-1,1);
end
```

Next, the system is solved over `ts` time steps, where the nonlinear system is solved using Newton-Rapson method. When $k_j(x) = 0.1x + x^3$, $d_j(w) = 0.5 + 5\|w\|^3$, the method proceeds as follows:

```
 w0 = z0;

for k=1:ts


     f=aux1*w0 - aux2*z0;

     f(2*N-1) = f(2*N-1)+dt*k_22*0.125*(w0(2*N-1,1)+z0(2*N-1,1))^3;

     f(2*N-1) = f(2*N-1)+dt*k_11*0.25*...
                      (w0(N-1,1)+z0(N-1,1))*(w0(N-1,1)^2+z0(N-1,1)^2);

     it = 0;
```

```
    while (norm(f,2) > 10^(-12) && it < 100)

       it = it+1;

       Df = aux1;

         Df(2*N-1,2*N-1) = Df(2*N-1,2*N-1)+ k_22*dt*0.5*0.25*3*...
                                    (w0(2*N-1,1)+z0(2*N-1,1))^2;

       Df(2*N-1,N-1) = Df(2*N-1,N-1)+ k_11*0.25*dt*...
                        (3*w0(N-1,1)^2+z0(N-1,1)^2+2*w0(N-1,1)*z0(N-1,1));

       w1 = linsolve(Df, Df*w0 - f);

       w0 = w1;

       f=aux1*w0 - aux2*z0;

       f(2*N-1) = f(2*N-1)+dt*k_22*0.125*(w0(2*N-1,1)+z0(2*N-1,1))^3;

       f(2*N-1) = f(2*N-1)+k_11*0.25*(w0(N-1,1)+...
                        z0(N-1,1))*(w0(N-1,1)^2+z0(N-1,1)^2);


    end

    z0=w0;

    for j=1 : nn
        Z(j+1,k+1)=z0(2*j-1,1);
    end

    l(k+1,1) = lyapunov(nn,z0);

    pos(k+1,1)=z0(2*nn-1,1);

    ang(k+1,1)=z0(2*nn,1);
end

end
```

In case $k_j(x) = 0.1x+\sinh(x)$, $d_j(w) = 0.5+5\sinh(\|w\|^2)$, the following implementation was used:

```
 w0 = z0;


for k=1:ts
```

```
f=aux1*w0 - aux2*z0;

f(2*N-1) = f(2*N-1)+dt*k_22*0.125*(w0(2*N-1,1)+z0(2*N-1,1))^3;

f(2*N-1) = f(2*N-1)+dt*k_11*0.25*...
                w0(N-1,1)+z0(N-1,1))*(w0(N-1,1)^2+z0(N-1,1)^2);

it = 0;


while (norm(f,2) > 10^(-12) && it < 100)

   it = it+1;

   Df = aux1;

   Df(2*N-1,2*N-1) = Df(2*N-1,2*N-1)+ k_22*dt*...
                        cos(0.5*w0(2*N-1,1)+0.5*z0(2*N-1,1))^(-2)*0.5;

   if w0(N-1,1) == z0(N-1,1)

        Df(2*N-1,N-1) = Df(2*N-1,N-1)+k_11*dt*0.5*cos(z0(N-1,1));
   else
        Df(2*N-1,N-1) = Df(2*N-1,N-1)+k_11*dt* ...
               (sin(w0(N-1,1))*(w0(N-1,1)-z0(N-1,1))+ ...
               cos(w0(N-1,1))-cos(z0(N-1,1)))/(w0(N-1,1)-z0(N-1,1))^2;
   end

   w1 = linsolve(Df, Df*w0 - f);

   w0 = w1;

   f=aux1*w0 - aux2*z0;

   f(2*N-1) = f(2*N-1)+k_22*dt*tan(0.5*w0(2*N-1,1)+0.5*z0(2*N-1,1));

   if w0(N-1,1) == z0(N-1,1)

        f(2*N-1) = f(2*N-1)+k_11*dt*sin(z0(N-1,1));
   else
        f(2*N-1) = f(2*N-1)-k_11*dt*...
               (cos(w0(N-1,1))-cos(z0(N-1,1)))/(w0(N-1,1)-z0(N-1,1));
   end
end

z0=w0;

for j=1 : nn

    Z(j+1,k+1)=z0(2*j-1,1);
```

```
    end

    l(k+1,1) = lyapunov(nn,z0);

    pos(k+1,1)=z0(2*nn-1,1);

    ang(k+1,1)=z0(2*nn,1);

end

end
```

The function `lyapunov` calculates the Lyapunov function of the system, and is defined as follows

```
function n = lyapunov(rnn,z1)

mu = 1; lambda = 1; L = 1;

M = 0.1; J = 0.1;

h=L/rnn; N=2*rnn;

k_1 = 0.1; k_11 = 1;

n = 0;

 %adding integral of u_xx^2

      for l=2: rnn
         n = n + 0.5*(z1(2*l-3,1)^2 *12 + ...
                    z1(2*l-2,1)^2 * 4*h^2 + ...
               z1(2*l-1,1)^2 *12 + ...
                    z1(2*l,1)^2 *4*h^2 + ...
               z1(2*l-3,1)*z1(2*l-2,1)*2*6*h + ...
                    z1(2*l-3,1)*z1(2*l-1,1)*2*(-12) + ...
               z1(2*l-3,1)*z1(2*l,1)*2*6*h + ...
                    z1(2*l-2,1)*z1(2*l-1,1)*2*(-6*h) + ...
               z1(2*l-2,1)*z1(2*l,1)*2*2*h^2 + ...
                    z1(2*l-1,1)*z1(2*l,1)*2*(-6*h))*lambda/(h^3);
      end

     %adding integral of u_xx^2 on [x0,x1]

      n = n + lambda*0.5*(12*z1(1,1)^2 ...
            - 2*6*h *z1(1,1)*z1(2,1) + 4*h^2*z1(2,1)^2)/(h^3);

     %adding integral of v^2

     for l=2: rnn
```

```
        n = n + 0.5*mu*(z1(N+2*l-3,1)^2*156 + ...
                z1(N+2*l-2,1)^2*4*h^2 + ...
            z1(N+2*l-1,1)^2*156 + ...
                z1(N+2*l,1)^2*4*h^2 + ...
            z1(N+2*l-3,1)*z1(N+2*l-2,1)*2*22*h + ...
                z1(N+2*l-3,1)*z1(N+2*l-1,1)*2*54 + ...
            z1(N+2*l-3,1)*z1(N+2*l,1)*2*(-13)*h + ...
                z1(N+2*l-2,1)*z1(N+2*l-1,1)*2*13*h + ...
            z1(N+2*l-2,1)*z1(N+2*l,1)*2*(-3)*h^2 + ...
                z1(N+2*l-1,1)*z1(N+2*l,1)*2*(-22)*h)*h/420;
    end

    %adding integral of v^2 on [x0,x1]
    n = n + 0.5*mu*(z1(N+1,1)^2 *156 + z1(N+2,1)^2*4*h^2 + ...
        2 * z1(N+1,1) * z1(N+2,1)*(-22)*h)*h/420;

    n = n + M*0.5*z1(2*N-1,1)^2+ J*0.5*z1(2*N,1)^2;
```

When $k_j(x) = 0.1x + x^3,\ d_j(w) = 0.5 + 5\|w\|^3$, the following is added:

```
     n = n + M*0.5*z1(2*N-1,1)^2+ J*0.5*z1(2*N,1)^2;

        n = n + k_1*0.5*z1(N-1,1)^2+ k_11*0.25*z1(N-1,1)^4;

        n = sqrt(n);
end
```

and in case $k_j(x) = 0.1x + \sinh(x),\ d_j(w) = 0.5 + 5\sinh(\|w\|^2)$

```
        n = n + M*0.5*z1(2*N-1,1)^2+ J*0.5*z1(2*N,1)^2;

        n  = n + k_1*0.5*z1(N-1,1)^2+ k_11*(cosh(z1(N-1,1))-1);

        n = sqrt(n);
end
```

5.4.3 Nonlinear boundary control

In this subsection, the implementation of the numerical method introduced in Subsection 4.3.2 for an EBB with nonlinear controller is presented. The output of the function `nonlinear_controller` is the solution of the numerical method `Z`, tip position `pos`, tip angle `ang`, and Lyapunov function `l` at all time steps. First, the constants to be used in the implementation are defined and the output vectors initialized:

```
function [Z,pos,ang,l,nn,ts,dt] = nonlinear_controller ()
```

```
mu = 1; lambda = 1; M = 0.1; J = 0.1;

L = 1; nn = 100; h = L/nn; N=2*nn;
x = linspace(0,L,nn+1);

c_dim = 3;

k_1 = 0.1; k_2 = 0.1; k_11 = 1; k_22 = 1;
d_1 = 0.5; d_2 = 0.5; d_11 = 5; d_22 = 5;

dt = 10^(-2); ts = 5000;

pos = zeros(ts+1,1);

ang = zeros(ts+1,1);

l = zeros(ts+1,1);

Z = zeros(nn+1,ts/skip+1);
```

Then, the linear part of the system matrices is defined:

```
%matrix A
A=zeros(2*N+2*c_dim,2*N+2*c_dim);

A(1,1)=1; A(2,2)=1;
A(1,N+1)=-0.5*dt; A(2,N+2)=-0.5*dt;

for k = 2:nn

    A(2*k,2*k)=1;  A(2*k-1,2*k-1)=1;
    A(2*k,N+2*k)=-0.5*dt;  A(2*k-1,N+2*k-1)=-0.5*dt;

    %%stiffness matrix

    A(N+2*k-3,2*k-3)=A(N+2*k-3,2*k-3)+lambda*0.5*12*dt/(h^3);

    A(N+2*k-2,2*k-2)=A(N+2*k-2,2*k-2)+lambda*0.5*4*dt/(h);

    A(N+2*k-1,2*k-1)=A(N+2*k-1,2*k-1)+lambda*0.5*12*dt/(h^3);

    A(N+2*k,2*k)=A(N+2*k,2*k)+lambda*0.5*4*dt/(h);

    A(N+2*k-3,2*k-2)=A(N+2*k-3,2*k-2)+lambda*0.5*6*dt/(h^2);
    A(N+2*k-2,2*k-3)=A(N+2*k-2,2*k-3)+lambda*0.5*6*dt/(h^2);

    A(N+2*k-3,2*k-1)=A(N+2*k-3,2*k-1)-lambda*0.5*12*dt/(h^3);
    A(N+2*k-1,2*k-3)=A(N+2*k-1,2*k-3)-lambda*0.5*12*dt/(h^3);

    A(N+2*k-3,2*k)=A(N+2*k-3,2*k)+lambda*0.5*6*dt/(h^2);
```

```
    A(N+2*k,2*k-3)=A(N+2*k,2*k-3)+lambda*0.5*6*dt/(h^2);

    A(N+2*k-2,2*k-1)=A(N+2*k-2,2*k-1)-lambda*0.5*6*dt/(h^2);
    A(N+2*k-1,2*k-2)=A(N+2*k-1,2*k-2)-lambda*0.5*6*dt/(h^2);

    A(N+2*k-2,2*k)=A(N+2*k-2,2*k)+lambda*0.5*2*dt/(h);
    A(N+2*k,2*k-2)=A(N+2*k,2*k-2)+lambda*0.5*2*dt/(h);

    A(N+2*k-1,2*k)=A(N+2*k-1,2*k)-lambda*0.5*6*dt/(h^2);
    A(N+2*k,2*k-1)=A(N+2*k,2*k-1)-lambda*0.5*6*dt/(h^2);


    %mass matrix


    A(N+2*k-3,N+2*k-3)=A(N+2*k-3,N+2*k-3)+ mu*h*(13/35);

    A(N+2*k-2,N+2*k-2)=A(N+2*k-2,N+2*k-2)+ mu*h^3*(1/105);

    A(N+2*k-1,N+2*k-1)=A(N+2*k-1,N+2*k-1)+ mu*h*(13/35);

    A(N+2*k,N+2*k)=A(N+2*k,N+2*k)+ mu*h^3*(1/105);

    A(N+2*k-3,N+2*k-2)=A(N+2*k-3,N+2*k-2)+ mu*h^2*(11/210);
    A(N+2*k-2,N+2*k-3)=A(N+2*k-2,N+2*k-3)+ mu*h^2*(11/210);

    A(N+2*k-3,N+2*k-1)=A(N+2*k-3,N+2*k-1)+ mu*h*(9/70);
    A(N+2*k-1,N+2*k-3)=A(N+2*k-1,N+2*k-3)+ mu*h*(9/70);

    A(N+2*k-3,N+2*k)=A(N+2*k-3,N+2*k)+ mu*h^2*(-13/420);
    A(N+2*k,N+2*k-3)=A(N+2*k,N+2*k-3)+ mu*h^2*(-13/420);

    A(N+2*k-2,N+2*k-1)=A(N+2*k-2,N+2*k-1)+ mu*h^2*(13/420);
    A(N+2*k-1,N+2*k-2)=A(N+2*k-1,N+2*k-2)+ mu*h^2*(13/420);

    A(N+2*k-2,N+2*k)=A(N+2*k-2,N+2*k)+ mu*h^3*(-1/140);
    A(N+2*k,N+2*k-2)=A(N+2*k,N+2*k-2)+ mu*h^3*(-1/140);

    A(N+2*k-1,N+2*k)=A(N+2*k-1,N+2*k)+ mu*h^2*(-11/210);
    A(N+2*k,N+2*k-1)=A(N+2*k,N+2*k-1)+ mu*h^2*(-11/210);


end

%stiffness matrix

A(N+1,1)=A(N+1,1)+lambda*0.5*12*dt/(h^3);

A(N+2,2)=A(N+2,2)+lambda*0.5*4*dt/(h);

A(N+1,2)=A(N+1,2)-lambda*0.5*6*dt/(h^2);
```

```
A(N+2,1)=A(N+2,1)-lambda*0.5*6*dt/(h^2);

A(2*N-1,N-1)=A(2*N-1,N-1)+0.5*k_2*dt;

A(2*N,N)=A(2*N,N)+0.5*k_1*dt;


%mass matrix

A(N+1,N+1)=A(N+1,N+1)+ mu*h*(13/35);

A(N+2,N+2)=A(N+2,N+2)+ mu*h^3*(1/105);

A(N+1,N+2)=A(N+1,N+2)- mu*h^2*(11/210);

A(N+2,N+1)=A(N+2,N+1)- mu*h^2*(11/210);

A(2*N-1,2*N-1)=A(2*N-1,2*N-1)+ M + 0.5*dt*d_2;

A(2*N,2*N)=A(2*N,2*N)+ J + 0.5*dt*d_1;

%controller part
for i = 1 : c_dim
     A(2*N+i,2*N+i)=1;
     A(2*N+c_dim+i,2*N+c_dim+i)=1;
end

%matrix B
B=zeros(2*N+2*c_dim,2*N+2*c_dim);

B(1,1)=1; B(2,2)=1;
B(1,N+1)=0.5*dt; B(2,N+2)=0.5*dt;

for k = 2:nn

    B(2*k,2*k)=1;  B(2*k-1,2*k-1)=1;
    B(2*k,N+2*k)=0.5*dt;  B(2*k-1,N+2*k-1)=0.5*dt;

    %%stiffness matrix

    B(N+2*k-3,2*k-3)=B(N+2*k-3,2*k-3)-lambda*0.5*12*dt/(h^3);

    B(N+2*k-2,2*k-2)=B(N+2*k-2,2*k-2)-lambda*0.5*4*dt/(h);

    B(N+2*k-1,2*k-1)=B(N+2*k-1,2*k-1)-lambda*0.5*12*dt/(h^3);

    B(N+2*k,2*k)=B(N+2*k,2*k)-lambda*0.5*4*dt/(h);

    B(N+2*k-3,2*k-2)=B(N+2*k-3,2*k-2)-lambda*0.5*6*dt/(h^2);
    B(N+2*k-2,2*k-3)=B(N+2*k-2,2*k-3)-lambda*0.5*6*dt/(h^2);
```

```
    B(N+2*k-3,2*k-1)=B(N+2*k-3,2*k-1)+lambda*0.5*12*dt/(h^3);
    B(N+2*k-1,2*k-3)=B(N+2*k-1,2*k-3)+lambda*0.5*12*dt/(h^3);

    B(N+2*k-3,2*k)=B(N+2*k-3,2*k)-lambda*0.5*6*dt/(h^2);
    B(N+2*k,2*k-3)=B(N+2*k,2*k-3)-lambda*0.5*6*dt/(h^2);

    B(N+2*k-2,2*k-1)=B(N+2*k-2,2*k-1)+lambda*0.5*6*dt/(h^2);
    B(N+2*k-1,2*k-2)=B(N+2*k-1,2*k-2)+lambda*0.5*6*dt/(h^2);

    B(N+2*k-2,2*k)=B(N+2*k-2,2*k)-lambda*0.5*2*dt/(h);
    B(N+2*k,2*k-2)=B(N+2*k,2*k-2)-lambda*0.5*2*dt/(h);

    B(N+2*k-1,2*k)=B(N+2*k-1,2*k)+lambda*0.5*6*dt/(h^2);
    B(N+2*k,2*k-1)=B(N+2*k,2*k-1)+lambda*0.5*6*dt/(h^2);


    %mass matrix


    B(N+2*k-3,N+2*k-3)=B(N+2*k-3,N+2*k-3)+ mu*h*(13/35);

    B(N+2*k-2,N+2*k-2)=B(N+2*k-2,N+2*k-2)+ mu*h^3*(1/105);

    B(N+2*k-1,N+2*k-1)=B(N+2*k-1,N+2*k-1)+ mu*h*(13/35);

    B(N+2*k,N+2*k)=B(N+2*k,N+2*k)+ mu*h^3*(1/105);

    B(N+2*k-3,N+2*k-2)=B(N+2*k-3,N+2*k-2)+ mu*h^2*(11/210);
    B(N+2*k-2,N+2*k-3)=B(N+2*k-2,N+2*k-3)+ mu*h^2*(11/210);

    B(N+2*k-3,N+2*k-1)=B(N+2*k-3,N+2*k-1)+ mu*h*(9/70);
    B(N+2*k-1,N+2*k-3)=B(N+2*k-1,N+2*k-3)+ mu*h*(9/70);

    B(N+2*k-3,N+2*k)=B(N+2*k-3,N+2*k)+ mu*h^2*(-13/420);
    B(N+2*k,N+2*k-3)=B(N+2*k,N+2*k-3)+ mu*h^2*(-13/420);

    B(N+2*k-2,N+2*k-1)=B(N+2*k-2,N+2*k-1)+ mu*h^2*(13/420);
    B(N+2*k-1,N+2*k-2)=B(N+2*k-1,N+2*k-2)+ mu*h^2*(13/420);

    B(N+2*k-2,N+2*k)=B(N+2*k-2,N+2*k)+ mu*h^3*(-1/140);
    B(N+2*k,N+2*k-2)=B(N+2*k,N+2*k-2)+ mu*h^3*(-1/140);

    B(N+2*k-1,N+2*k)=B(N+2*k-1,N+2*k)+ mu*h^2*(-11/210);
    B(N+2*k,N+2*k-1)=B(N+2*k,N+2*k-1)+ mu*h^2*(-11/210);


end

%stiffness matrix

B(N+1,1)=B(N+1,1)-lambda*0.5*12*dt/(h^3);
```

```
B(N+2,2)=B(N+2,2)-lambda*0.5*4*dt/(h);

B(N+1,2)=B(N+1,2)+lambda*0.5*6*dt/(h^2);
B(N+2,1)=B(N+2,1)+lambda*0.5*6*dt/(h^2);

B(2*N-1,N-1)=B(2*N-1,N-1)-0.5*k_2*dt;

B(2*N,N)=B(2*N,N)-0.5*k_1*dt;

%mass matrix

B(N+1,N+1)=B(N+1,N+1)+ mu*h*(13/35);

B(N+2,N+2)=B(N+2,N+2)+ mu*h^3*(1/105);

B(N+1,N+2)=B(N+1,N+2)- mu*h^2*(11/210);
B(N+2,N+1)=B(N+2,N+1)- mu*h^2*(11/210);

B(2*N-1,2*N-1)=B(2*N-1,2*N-1)+ M - 0.5*dt*d_2;
B(2*N,2*N)=B(2*N,2*N)+ J - 0.5*dt*d_1;

%controller part

for i = 1 : c_dim
    B(2*N+i,2*N+i)=1;
    B(2*N+c_dim+i,2*N+c_dim+i)=1;
end
```

Next, the system is initialized, and the initial state is saved in the corresponding output vectors:

```
z0 = zeros(2*N+2*c_dim,1);

%initialization for u0=0.2*(-3(x/L)^2 + 2(x/L)^3)
for k=1:nn
    z0(N+2*k-1,1)= 0;  %v_0(x_k)
    z0(N+2*k,1)= 0;    %v_0'(x_k)
    z0(2*k,1)=0.2*(-6*(x(k+1)/L)/L + 6*(x(k+1)/L)^2/L); %u_0'(x_k)
    z0(2*k-1,1)=0.2*(-3*(x(k+1)/L)^2 + 2*(x(k+1)/L)^3); %u_0(x_k)

end

for k = 1 : c_dim
    z0(2*N + k)=0.3;
    z0(2*N + c_dim + k)=0.3;
end

pos(1,1)=z0(2*nn-1,1);
ang(1,1)=z0(2*nn,1);
l(1,1) = lyapunov(nn,z0);
```

```
for k=1:nn
    Z(k+1,1)=z0(2*k-1,1);
end
```

A nonlinear system is solved for each time step `k`, for `k = 1, ..., ts`, using Newtons method for cases a) and b) from Subsection 5.3. In the first case, there holds $k_j(x) = 0.1x + x^3$, $d_j(w) = 0.5 + 5\|w\|^3$:

```
for k=1:ts

     w0 = z0;

     f = A*w0 - B*z0;

     for s = 1 : c_dim
       f(2*N+s,1) = f(2*N+s,1) +dt*0.125*(w0(2*N+s,1)+z0(2*N+s,1))^3...
         -0.25*0.5*dt*(w0(2*N+s,1)+z0(2*N+s,1))^2*(w0(2*N,1)+z0(2*N,1));
       f(2*N+c_dim+s,1) = f(2*N+c_dim+s,1) + dt*0.125*...
         (w0(2*N+c_dim+s,1)+z0(2*N+c_dim+s,1))^3 ...
        -0.25*0.5*dt*(w0(2*N+c_dim+s,1)+z0(2*N+c_dim+s,1))^2*...
         (w0(2*N-1,1)+z0(2*N-1,1));
     end


     f(2*N-1,1) = f(2*N-1,1)+ 0.25*dt*(w0(2*N-1,1) + z0(2*N-1,1))*...
             (d_22*norm(w0(2*N+c_dim+1:2*N+2*c_dim,1),2)^2 + ...
                 d_22*norm(z0(2*N+c_dim+1:2*N+2*c_dim,1),2)^2);
     f(2*N,1) = f(2*N,1) + 0.25*dt*(w0(2*N,1) + z0(2*N,1))*...
             (d_11*norm(w0(2*N+1:2*N+c_dim,1),2)^2 +...
                d_11*norm(z0(2*N+1:2*N+c_dim,1),2)^2);

     f(2*N-1,1) = f(2*N-1,1) + 0.25*dt*k_22*...
        (w0(N-1,1)+z0(N-1,1))*(w0(N-1,1)^2+z0(N-1,1)^2);
     f(2*N,1) = f(2*N,1) + 0.25*dt*k_11*(w0(N,1)+z0(N,1))*...
        (w0(N,1)^2+z0(N,1)^2);

     f(2*N-1,1) = f(2*N-1,1)+0.125*dt*sum((w0(2*N+c_dim+1: 2*N+2*c_dim,1)...
      +z0(2*N+c_dim+1: 2*N+2*c_dim,1)).^3);
     f(2*N,1) = f(2*N,1) + 0.125*dt*...
        sum((w0(2*N+1: 2*N+c_dim,1)+z0(2*N+1: 2*N+c_dim,1)).^3);

     it = 0;

     while (norm(f,2) > 10^(-11) && it < 50)

     it = it+1;

      Df = A;
```

```
 Df(2*N-1, 2*N-1) = Df(2*N-1, 2*N-1) + 0.25*dt*...
   (d_22*norm(w0(2*N+c_dim+1:2*N+2*c_dim,1),2)^2 + ...
     d_22*norm(z0(2*N+c_dim+1:2*N+2*c_dim,1),2)^2);

Df(2*N, 2*N) = Df(2*N, 2*N) + 0.25*dt*...
   (d_11*norm(w0(2*N+1:2*N+c_dim,1),2)^2 + ...
     d_11*norm(z0(2*N+1:2*N+c_dim,1),2)^2);

Df(2*N-1, 2*N + c_dim +1 : 2*N + 2*c_dim) = ...
Df(2*N-1, 2*N + c_dim +1 : 2*N + 2*c_dim) + d_22*0.5*dt*(w0(2*N-1,1)...
    + z0(2*N-1,1))* w0(2*N + c_dim +1 : 2*N + 2*c_dim,1)';

Df(2*N, 2*N +1 : 2*N + c_dim) = Df(2*N, 2*N +1 : 2*N + c_dim) + ...
   d_11*0.5*dt*(w0(2*N,1) +z0(2*N,1))* w0(2*N +1 : 2*N + c_dim,1)';

 Df(2*N-1, N-1) = Df(2*N-1, N-1) + 0.25*dt*k_22*...
 (3*w0(N-1,1)^2 + 2*w0(N-1,1)*z0(N-1,1) +z0(N-1,1)^2);

Df(2*N, N) = Df(2*N, N) + 0.25*dt*k_11*...
 (3*w0(N,1)^2 + 2*w0(N,1)*z0(N,1) +z0(N,1)^2);

Df(2*N-1, 2*N+c_dim+1:2*N+2*c_dim) = ...
 Df(2*N-1, 2*N+c_dim+1:2*N+2*c_dim)+0.125*dt*...
 (3*(w0(2*N+c_dim+1:2*N+2*c_dim,1)+z0(2*N+c_dim+1:2*N+2*c_dim,1)).^2)';

Df(2*N, 2*N+1:2*N+c_dim) = Df(2*N, 2*N+1:2*N+c_dim) + 0.125*dt*...
(3*(w0(2*N+1:2*N+c_dim,1)+z0(2*N+1:2*N+c_dim,1)).^2)';

 Df(2*N+c_dim+1:2*N+2*c_dim,2*N-1) = ...
Df(2*N+c_dim+1:2*N+2*c_dim,2*N-1)- 0.5*dt*0.25*...
(w0(2*N+c_dim+1:2*N+2*c_dim,1)+z0(2*N+c_dim+1:2*N+2*c_dim,1)).^2;

Df(2*N+1:2*N+c_dim,2*N) = Df(2*N+1:2*N+c_dim,2*N) - 0.5*dt*0.25*...
  (w0(2*N+1:2*N+c_dim,1)+z0(2*N+1:2*N+c_dim,1)).^2;

for br1 = 1 : c_dim
   Df(2*N+c_dim+br1,2*N+c_dim+br1) = Df(2*N+c_dim+br1,2*N+c_dim+br1)...
       -0.25*dt*(w0(2*N-1,1) + z0(2*N-1,1))*...
       (w0(2*N+c_dim+br1,1)+ z0(2*N+c_dim+br1,1));

   Df(2*N+c_dim+br1,2*N+c_dim+br1) = Df(2*N+c_dim+br1,2*N+c_dim+br1)...
       +dt*0.125*3*(w0(2*N+c_dim+br1,1)+ z0(2*N+c_dim+br1,1))^2;

   Df(2*N+br1,2*N+br1) = Df(2*N+br1,2*N+br1) - 0.25*dt*...
       (w0(2*N,1) + z0(2*N,1))*(w0(2*N+br1,1)+ z0(2*N+br1,1));

   Df(2*N+br1,2*N+br1) = Df(2*N+br1,2*N+br1) + dt*0.125*3*...
     (w0(2*N+br1,1)+ z0(2*N+br1,1))^2;

end
```

```
        w1 = linsolve(Df, Df*w0 - f);

        f = A*w1 - B*z0;

        for s = 1 : c_dim
          f(2*N+s,1) = f(2*N+s,1) + dt*0.125*(w1(2*N+s,1)+z0(2*N+s,1))^3 ...
            -0.25*0.5*dt*(w1(2*N+s,1)+z0(2*N+s,1))^2*...
             (w1(2*N,1)+z0(2*N,1));
          f(2*N+c_dim+s,1) = f(2*N+c_dim +s,1) +dt*0.125*...
           (w1(2*N+c_dim+s,1)+z0(2*N+c_dim+s,1))^3-0.25*0.5*dt*...
           (w1(2*N+c_dim+s,1)+z0(2*N+c_dim+s,1))^2*(w1(2*N-1,1)+z0(2*N-1,1));
        end

        f(2*N-1,1) = f(2*N-1,1)+ 0.25*dt*(w1(2*N-1,1) + z0(2*N-1,1))*...
          (d_22*norm(w1(2*N+c_dim+1:2*N+2*c_dim,1),2)^2+ d_22*...
            norm(z0(2*N+c_dim+1:2*N+2*c_dim,1),2)^2);
        f(2*N,1) = f(2*N,1) + 0.25*dt*(w1(2*N,1) + z0(2*N,1))*...
          (d_11*norm(w1(2*N+1:2*N+c_dim,1),2)^2 + d_11*...
            norm(z0(2*N+1:2*N+c_dim,1),2)^2);

        f(2*N-1,1) = f(2*N-1,1)+ 0.25*dt*k_22*(w1(N-1,1)+z0(N-1,1))*...
           (w1(N-1,1)^2+z0(N-1,1)^2);
        f(2*N,1) = f(2*N,1) + 0.25*dt*k_11*...
           (w1(N,1)+z0(N,1))*(w1(N,1)^2+z0(N,1)^2);

        f(2*N-1,1) = f(2*N-1,1)+ 0.125*dt*...
         sum((w1(2*N+c_dim+1:2*N+2*c_dim,1)+z0(2*N+c_dim+1:2*N+2*c_dim,1)).^3);
        f(2*N,1) = f(2*N,1) + 0.125*dt*...
         sum((w1(2*N+1: 2*N+c_dim,1)+z0(2*N+1: 2*N+c_dim,1)).^3);
        w0 = w1;

        end

        z0=w0;

        l(k+1,1) = lyapunov(nn,z0);

        for j=1 : nn
           Z(j+1,k+1)=z0(2*j-1,1);
        end

        pos(k+1,1)=z0(2*nn-1,1);
        ang(k+1,1)=z0(2*nn,1);

end
```

In the second case, there holds $k_j(x) = 0.1x + \sinh(x)$, $d_j(w) = 0.5 + 5\sinh(\|w\|^2)$:

```
for k=1:ts
```

```
w0 = z0;

f = A*w0 - B*z0;

 for s = 1 : c_dim
   f(2*N+s,1) = f(2*N+s,1) +dt*0.125*(w0(2*N+s,1)+z0(2*N+s,1))^3 ...
     -0.25*0.5*dt*(w0(2*N+s,1)+z0(2*N+s,1))^2*(w0(2*N,1)+z0(2*N,1));
   f(2*N+c_dim+s,1) = f(2*N+c_dim+s,1) +dt*0.125*...
          (w0(2*N+c_dim+s,1)+z0(2*N+c_dim+s,1))^3 ...
         -0.25*0.5*dt*(w0(2*N+c_dim+s,1)+z0(2*N+c_dim+s,1))^2*...
          (w0(2*N-1,1)+z0(2*N-1,1));
end


f(2*N-1,1) = f(2*N-1,1)+ 0.25*dt*(w0(2*N-1,1) + z0(2*N-1,1))*...
            (d_22*sinh(norm(w0(2*N+c_dim+1:2*N+2*c_dim,1),2)^2) ...
            + d_22*sinh(norm(z0(2*N+c_dim+1:2*N+2*c_dim,1),2)^2));
f(2*N,1) = f(2*N,1) + 0.25*dt*(w0(2*N,1) + z0(2*N,1))*...
            (d_11*sinh(norm(w0(2*N+1:2*N+c_dim,1),2)^2)...
             + d_11*sinh(norm(z0(2*N+1:2*N+c_dim,1),2)^2));

if  w0(N-1,1)==z0(N-1,1)
   f(2*N-1,1) = f(2*N-1,1)+ dt*k_22*sinh(w0(N-1,1));
else
   f(2*N-1,1) = f(2*N-1,1)+ dt*k_22*...
       (cosh(w0(N-1,1))-cosh(z0(N-1,1)))/(w0(N-1,1)-z0(N-1,1));
    end

if  w0(N,1)==z0(N,1)
    f(2*N,1) = f(2*N,1)+ dt*k_11*sinh(w0(N,1));
else
    f(2*N,1) = f(2*N,1)+ dt*k_11*...
       (cosh(w0(N,1))-cosh(z0(N,1)))/(w0(N,1)-z0(N,1));
end


f(2*N-1,1) = f(2*N-1,1)+ 0.125*dt*...
  sum((w0(2*N+c_dim+1:2*N+2*c_dim,1)+z0(2*N+c_dim+1:2*N+2*c_dim,1)).^3);
f(2*N,1) = f(2*N,1) + 0.125*dt*...
   sum((w0(2*N+1: 2*N+c_dim,1)+z0(2*N+1: 2*N+c_dim,1)).^3);

it = 0;

while (norm(f,2) > 10^(-12) && it < 50)

it = it+1;

Df = A;

Df(2*N-1, 2*N-1) = Df(2*N-1, 2*N-1) + 0.25*dt*...
  (d_22*sinh(norm(w0(2*N+c_dim+1:2*N+2*c_dim,1),2)^2)+...
```

```
    d_22*sinh(norm(z0(2*N+c_dim+1:2*N+2*c_dim,1),2)^2));

Df(2*N, 2*N) = Df(2*N, 2*N) + 0.25*dt*...
   (d_11*sinh(norm(w0(2*N+1:2*N+c_dim,1),2)^2)+...
    d_11*sinh(norm(z0(2*N+1:2*N+c_dim,1),2)^2));

Df(2*N-1, 2*N + c_dim +1 : 2*N + 2*c_dim) = ...
Df(2*N-1, 2*N + c_dim +1 : 2*N + 2*c_dim) + ...
  d_22*0.5*dt*(w0(2*N-1,1) +z0(2*N-1,1))* ...
 cosh(norm(w0(2*N+c_dim + 1:2*N+2*c_dim,1),2)^2)*...
  w0(2*N + c_dim +1 : 2*N + 2*c_dim,1)';

Df(2*N, 2*N +1 : 2*N + c_dim) = ...
Df(2*N, 2*N +1 : 2*N + c_dim) + d_11*0.5*dt*(w0(2*N,1) +z0(2*N,1))*...
  cosh(norm(w0(2*N+1:2*N+c_dim,1),2)^2)*w0(2*N +1 : 2*N + c_dim,1)';

if  w0(N-1,1)==z0(N-1,1)
   Df(2*N-1, N-1) = Df(2*N-1, N-1) + dt*k_22*0.5*cosh(z0(N-1,1));
else
   Df(2*N-1, N-1) = Df(2*N-1, N-1) + dt*k_22*...
        (sinh(w0(N-1,1))*(w0(N-1,1)-z0(N-1,1))-...
     cosh(w0(N-1,1))+cosh(z0(N-1,1)))/(w0(N-1,1)-z0(N-1,1))^2;
end

if  w0(N,1)==z0(N,1)
  Df(2*N, N) = Df(2*N, N) + dt*k_11*0.5*cosh(z0(N,1));
else
  Df(2*N, N) = Df(2*N, N) + dt*k_11*(sinh(w0(N,1))*(w0(N,1)-z0(N,1))-...
           cosh(w0(N,1))+cosh(z0(N,1)))/(w0(N,1)-z0(N,1))^2;
end

Df(2*N-1,2*N+c_dim+1:2*N+2*c_dim) = Df(2*N-1,2*N+c_dim+1:2*N+2*c_dim)...
   +0.125*dt*(3*(w0(2*N+c_dim+1:2*N+2*c_dim,1)+...
  z0(2*N+c_dim+1:2*N+2*c_dim,1)).^2)';

Df(2*N, 2*N+1:2*N+c_dim) = Df(2*N, 2*N+1:2*N+c_dim) + 0.125*dt*...
   (3*(w0(2*N+1:2*N+c_dim,1)+z0(2*N+1:2*N+c_dim,1)).^2)';

Df(2*N+c_dim+1:2*N+2*c_dim,N-1) = Df(2*N+c_dim+1:2*N+2*c_dim,N-1) -...
           0.25*0.5*dt*(w0(2*N+c_dim+1:2*N+2*c_dim,1)+...
           z0(2*N+c_dim+1:2*N+2*c_dim,1)).^2;

Df(2*N+1:2*N+c_dim,N) = Df(2*N+1:2*N+c_dim,N) - 0.25*0.5*dt*...
      (w0(2*N+1:2*N+c_dim,1)+z0(2*N+1:2*N+c_dim,1)).^2;

for br1 = 1 : c_dim
    Df(2*N+c_dim+br1,2*N+c_dim+br1) = Df(2*N+c_dim+br1,2*N+c_dim+br1)...
        - 0.25*dt*(w0(2*N-1,1) + z0(2*N-1,1))*...
    (w0(2*N+c_dim+br1,1)+ z0(2*N+c_dim+br1,1));

    Df(2*N+c_dim+br1,2*N+c_dim+br1) = Df(2*N+c_dim+br1,2*N+c_dim+br1)...
```

```
        + dt*0.125*3*(w0(2*N+c_dim+br1,1)+ z0(2*N+c_dim+br1,1))^2;

    Df(2*N+br1,2*N+br1) = Df(2*N+br1,2*N+br1) - 0.25*dt*(w0(2*N,1)...
        + z0(2*N,1))*(w0(2*N+br1,1)+ z0(2*N+br1,1));

    Df(2*N+br1,2*N+br1) = Df(2*N+br1,2*N+br1) + dt*0.125*3*...
      (w0(2*N+br1,1)+ z0(2*N+br1,1))^2;

end

 w1 = linsolve(Df, Df*w0 - f);

 f = A*w1 - B*z0;

for s = 1 : c_dim

   f(2*N+s,1) = f(2*N+s,1) +dt*0.125*(w1(2*N+s,1)+z0(2*N+s,1))^3 ...
   -0.25*0.5*dt*(w1(2*N+s,1)+z0(2*N+s,1))^2*(w1(2*N,1)+z0(2*N,1));
     f(2*N+c_dim+s,1) = f(2*N+c_dim +s,1) +dt*0.125*...
    (w1(2*N+c_dim+s,1)+z0(2*N+c_dim+s,1))^3 -0.25*0.5*dt*...
    (w1(2*N+c_dim+s,1)+z0(2*N+c_dim+s,1))^2*(w1(2*N-1,1)+z0(2*N-1,1));

end

f(2*N-1,1) = f(2*N-1,1)+ 0.25*dt*(w1(2*N-1,1) + z0(2*N-1,1))*...
      (d_22*sinh(norm(w1(2*N+c_dim+1:2*N+2*c_dim,1),2)^2)+...
           d_22*sinh(norm(z0(2*N+c_dim+1:2*N+2*c_dim,1),2)^2));
f(2*N,1) = f(2*N,1) + 0.25*dt*(w1(2*N,1) + z0(2*N,1))*...
      (d_11*sinh(norm(w1(2*N+1:2*N+c_dim,1),2)^2)+...
          d_11*sinh(norm(z0(2*N+1:2*N+c_dim,1),2)^2));

if  w0(N-1,1)==z0(N-1,1)
     f(2*N-1,1) = f(2*N-1,1)+ dt*k_22*sinh(w1(N-1,1));
else
     f(2*N-1,1) = f(2*N-1,1)+ dt*k_22*...
           (cosh(w1(N-1,1))-cosh(z0(N-1,1)))/(w1(N-1,1)-z0(N-1,1));
end

if  w0(N,1)==z0(N,1)
    f(2*N,1) = f(2*N,1)+ dt*k_11*sinh(w1(N,1));
else
    f(2*N,1) = f(2*N,1)+ dt*k_11*...
       (cosh(w1(N,1))-cosh(z0(N,1)))/(w1(N,1)-z0(N,1));
end

 f(2*N-1,1) = f(2*N-1,1)+ 0.125*dt*...
 sum((w1(2*N+c_dim+1:2*N+2*c_dim,1)+z0(2*N+c_dim+1:2*N+2*c_dim,1)).^3);
f(2*N,1) = f(2*N,1) + 0.125*dt*sum((w1(2*N+1:2*N+c_dim,1)+...
  z0(2*N+1: 2*N+c_dim,1)).^3);

w0 = w1;
```

```
     end

     z0=w0;

     l(k+1,1) = lyapunov(nn,z0);

     for j=1 : nn
          Z(j+1,k+1)=z0(2*j-1,1);
     end

     pos(k+1,1)=z0(2*nn-1,1);
     ang(k+1,1)=z0(2*nn,1);

end
```

Function lyapunov calculates the value of the Lyapunov functional of the solution, and is given as follows:

```
function n = lyapunov(rnn,z1)

mu = 1; lambda = 1; L = 1;

h = L/rnn; N = 2*rnn;

M = 0.1; J = 0.1;

k_1 = 0.1; k_2 = 0.1; k_11 = 1; k_22 = 1;
n=0;
 %adding integral of u_xx^2
for l=2: rnn
  n = n + 0.5*(z1(2*l-3,1)^2 *12 + z1(2*l-2,1)^2 * 4*h^2 ...
   + z1(2*l-1,1)^2 *12 + z1(2*l,1)^2 *4*h^2 + ...
   z1(2*l-3,1)*z1(2*l-2,1)*2*6*h + z1(2*l-3,1)*z1(2*l-1,1)*2*(-12) + ...
   z1(2*l-3,1)*z1(2*l,1)*2*6*h + z1(2*l-2,1)*z1(2*l-1,1)*2*(-6*h) + ...
   z1(2*l-2,1)*z1(2*l,1)*2*2*h^2 + ...
   z1(2*l-1,1)*z1(2*l,1)*2*(-6*h))*lambda/(h^3);
end
  %adding integral of u_xx^2 on [x0,x1]
n = n + lambda*0.5*(12*z1(1,1)^2 - 2*6*h *z1(1,1)*z1(2,1) + ...
  4*h^2*z1(2,1)^2)/(h^3);

  %adding integral of v^2

for l=2: rnn
  n = n + 0.5*mu*(z1(N+2*l-3,1)^2*156 + z1(N+2*l-2,1)^2*4*h^2 ...
   +z1(N+2*l-1,1)^2*156+z1(N+2*l,1)^2*4*h^2 + ...
   z1(N+2*l-3,1)*z1(N+2*l-2,1)*2*22*h + ...
   z1(N+2*l-3,1)*z1(N+2*l-1,1)*2*54 + ...
   z1(N+2*l-3,1)*z1(N+2*l,1)*2*(-13)*h + ...
   z1(N+2*l-2,1)*z1(N+2*l-1,1)*2*13*h...
   +z1(N+2*l-2,1)*z1(N+2*l,1)*2*(-3)*h^2+...
```

```
   z1(N+2*l-1,1)*z1(N+2*l,1)*2*(-22)*h)*h/420;
end

  %adding integral of v^2 on [x0,x1]
n = n + 0.5*mu*(z1(N+1,1)^2 *156 + z1(N+2,1)^2*4*h^2 + ...
   2 * z1(N+1,1) * z1(N+2,1)*(-22)*h)*h/420;

 %adding 0.5*|zeta|^2
m = size(z1);
n = n + 0.5*norm(z1(2*N+1:m(1),1),2)^2;
n = n + M*0.5*z1(2*N-1,1)^2+ J*0.5*z1(2*N,1)^2;
n = n + k_1*0.5*z1(N,1)^2;
n = n+ k_2*0.5*z1(N-1,1)^2;
```

In order to include the integrals of k_1 and k_2 as seen in (4.19), the cases a) and b) need to be distinguished again. In case when $k_j(x) = 0.1x + x^3$, $j = 1, 2$ the following is added to the Lyapunov function:

```
n = n + k_11*0.5*0.25*z1(N,1)^4;
n = n+ k_22*0.5*0.25*z1(N-1,1)^4;
n = sqrt(n);

end
```

In case when $k_j(x) = 0.1x + \sinh(x)$, the following is added:

```
n = n + k_1*0.5*(cosh(z1(N,1))-1);
n = n+ k_2*0.5*(cosh((N-1,1))-1);
n = sqrt(n);

end
```

Conclusion and outlook

To conclude this thesis, the main results are revised and the next research steps and open questions are discussed.

Conclusion

The main contribution of the thesis is the extension of the existing stability results for the EBB with tip body and dynamic feedback boundary control [40, 47, 18]. It has been demonstrated that although the linear dynamic controller using low-order boundary terms stabilizes the system asymptotically, the system is not exponentially stable. This is the generalization of the result obtained for the static controller in [58]. There the author demonstrates that to obtain the exponential stability, the higher-order boundary terms need to be used.

In order to demonstrate the asymptotic stability of the beam system with nonlinear boundary conditions, it is vital to show the precompactness property of the trajectories. However, none of the methods from the literature [23, 55, 54, 70, 20, 66] can be utilized for the system observed in this thesis, since the nonlinear part of the system operator applied to the trajectory can not be shown to be L^1-integrable in time, the system operator is not dissipative nor is the linear semigroup generated by the linear part of the system operator compact. The novel approach introduced in this work, is based on demonstrating that the norm of the time derivative of the solution is uniformly bounded in time. However, this result has been shown only for classical and not for all mild solutions.

Finally, it had been shown that applying finite element method for discretization in time and Crank-Nicolson method for discretization in time leads to a dissipative, stable numerical method. The dissipativity property is independent of the choice of the finite element space. The method is validated in simulations. The numerical method for EBB system with dynamic control is novel.

Outlook

To complete the stability analysis in Chapters 3 and 4, it would be of interest to extend the asymptotic stability result obtained for the classical to mild solutions. Furthermore, uniqueness of the weak solutions for the EBB systems with a nonlinear spring and damper attached to the beam tip, and a nonlinear dynamic controller, respectively has not been shown in this work. However, since the mild solutions to these systems are unique, it is expected that the weak solutions are unique as well. Therefore, demonstrating uniqueness of weak solutions is a further research assignment.

Another interesting topic for future research is the extension of the obtained numerical method to a general class of hyperbolic systems with passivity based feedback control. In particular, also the extension of the numerical method for EBB with dynamic controller to the case when condition (4.96) does not hold would be of interest.

Finally, the stability results obtained for the EBB with nonlinear spring and damper could be used to extend the research in [7], to analyze the stability of a system consisting of flexible micro-gripper used for DNA manipulation.

Appendices

Appendix A

In order to keep the integrity of the thesis, some of the established results and their proofs have been deferred to this Appendix. The following result was used in Proof of Theorem 2.30.

Theorem A.1. *Let*

$$\tilde{H}_0^2(0,L) := \{u \in H^2(0,L)| \quad u(0) = u_x(0) = 0\}.$$

Then there exists a set of functions $\{w_k\}_{k=1}^{\infty}$ *that is an orthogonal basis of* $\tilde{H}_0^2(0,L)$ *and an orthonormal basis of* $L^2(0,L)$*.*

Proof. Let the operator L be a fourth order differential operator given by:

$$Lu = u_{xxxx}.$$

The following initial value problem is observed:

$$\begin{aligned} Lu(x) &= f(x), \quad x \in (0,L) \\ u(0) &= 0, \\ u_x(0) &= 0, \\ u_{xx}(L) &= 0, \\ u_{xxx}(L) &= 0. \end{aligned}$$

Assuming that $f \in L^2(0,L)$ a weak solution is defined to be $u \in \tilde{H}_0^2(0,L)$ such that

$$\int_0^L u_{xx} w_{xx} \, \mathrm{d}x = \int_0^L f w \, \mathrm{d}x$$

$\forall w \in \tilde{H}_0^2(0,L)$. Since symmetric bilinear form

$$b(v,w) = \int_0^L v_{xx} w_{xx} \, \mathrm{d}x$$

is coercive and bounded on $\tilde{H}_0^2(0,L)$, from Lax-Milgram Lemma it follows that weak formulation has unique solution $u \in \tilde{H}_0^2(0,L)$. Then the following holds

$$u = L^{-1}(f).$$

Operator $L^{-1} : L^2(0,L) \longrightarrow L^2(0,L)$ is obviously linear and bounded. Moreover,

$$\|u\|_{H^2(0,L)} \leq C\|f\|_{L^2(0,L)}$$

and since $\tilde{H}_0^2(0,L)$ is compactly embedded in $L^2(0,L)$ follows that L^{-1} is compact. Finally it is shown that L^{-1} is symmetric on $L^2(0,L)$. Let $f, g \in L^2(0,L)$ and denote

$$\begin{aligned} u &= L^{-1}f, \\ v &= L^{-1}g. \end{aligned}$$

Then

$$\begin{aligned} (L^{-1}f, g)_{L^2(0,L)} &= (u, g)_{L^2(0,L)} &= b(v,u) \\ (f, L^{-1}g)_{L^2(0,L)} &= (f, v)_{L^2(0,L)} &= b(u,v). \end{aligned}$$

Obtaining the symmetric property, it follows that there exists a countable orthonormal basis $\{w_k\}_{k=1}^{\infty}$ of $L^2(0,L)$ consisting of eigenvectors of L^{-1}. Furthermore, these eigenvectors are $\tilde{H}_0^2(0,L)$ functions according to definition of L^{-1} and from the weak formulation, one can see that the basis $\{w_k\}_{k=1}^{\infty}$ is orthogonal as well in $\tilde{H}_0^2(0,L)$ with respect to the inner product $b(.,.)$. □

Proof of Theorem 2.36. First, from (2.134) and (2.135) (written in the style of (2.87)) is obtained:

$$\frac{u^{k+1} - u^k}{\Delta t} = \frac{v^{k+1} + v^k}{2}, \tag{A.1}$$

$$\begin{aligned} &\int_0^L \mu \, \frac{v^{k+1} - v^k}{\Delta t} w_h \, \mathrm{d}x + \int_0^L \Lambda \, \frac{u_{xx}^{k+1} + u_{xx}^k}{2} (w_h)_{xx} \, \mathrm{d}x \\ &\quad + M \frac{v^{k+1}(L) - v^k(L)}{\Delta t} w_h(L) + J \frac{v_x^{k+1}(L) - v_x^k(L)}{\Delta t} (w_h)_x(L) \\ &\quad + k_1 \frac{u_x^{k+1}(L) + u_x^k(L)}{2} (w_h)_x(L) + k_2 \frac{u^{k+1}(L) + u^k(L)}{2} w_h(L) \\ &\quad + d_1 \frac{v_x^{k+1}(L) + v_x^k(L)}{2} (w_h)_x(L) + d_2 \frac{v^{k+1}(L) + v^k(L)}{2} w_h(L) \\ &\quad + c_1 \cdot \frac{\zeta_1^{k+1} + \zeta_1^k}{2} (w_h)_x(L) + c_2 \cdot \frac{\zeta_2^{k+1} + \zeta_2^k}{2} w_h(L) = 0, \end{aligned} \tag{A.2}$$

for all $w_h \in W_h$. Next, equation (A.1) is multiplied by $\mu\,(v^{k+1} - v^k)$, and integrate over $[0,L]$ to obtain

$$\frac{1}{2} \int_0^L \mu \left((v^{k+1})^2 - (v^k)^2 \right) \, \mathrm{d}x = \int_0^L \mu \, \frac{u^{k+1} - u^k}{\Delta t} (v^{k+1} - v^k) \, \mathrm{d}x,$$

and taking $w_h = u^{k+1}$ in (A.2):

$$\begin{aligned}\frac{1}{2}\int_0^L \Lambda(u_{xx}^{k+1})^2\ \mathrm{d}x = & -\frac{1}{2}\int_0^L \Lambda u_{xx}^{k+1}u_{xx}^k\ \mathrm{d}x - \int_0^L \mu\frac{v^{k+1}-v^k}{\Delta t}u^{k+1}\ \mathrm{d}x\\ & -M\frac{v^{k+1}(L)-v^k(L)}{\Delta t}u^{k+1}(L) - J\frac{v_x^{k+1}(L)-v_x^k(L)}{\Delta t}u_x^{k+1}(L)\\ & -k_1\frac{u_x^{k+1}(L)+u_x^k(L)}{2}u_x^{k+1}(L) - k_2\frac{u^{k+1}(L)+u^k(L)}{2}u^{k+1}(L)\\ & -d_1\frac{v_x^{k+1}(L)+v_x^k(L)}{2}u_x^{k+1}(L) - d_2\frac{v^{k+1}(L)+v^k(L)}{2}u^{k+1}(L)\\ & -c_1\cdot\frac{\zeta_1^{k+1}+\zeta_1^k}{2}u_x^{k+1}(L) - c_2\cdot\frac{\zeta_2^{k+1}+\zeta_2^k}{2}u^{k+1}(L).\end{aligned}$$

Next, taking $w_h = u^k$ in (A.2) yields:

$$\begin{aligned}\frac{1}{2}\int_0^L \Lambda(u_{xx}^{k})^2\ \mathrm{d}x = & -\frac{1}{2}\int_0^L \Lambda u_{xx}^{k+1}u_{xx}^k\ \mathrm{d}x - \int_0^L \mu\frac{v^{k+1}-v^k}{\Delta t}u^{k}\ \mathrm{d}x\\ & -M\frac{v^{k+1}(L)-v^k(L)}{\Delta t}u^{k}(L) - J\frac{v_x^{k+1}(L)-v_x^k(L)}{\Delta t}u_x^{k}(L)\\ & -k_1\frac{u_x^{k+1}(L)+u_x^k(L)}{2}u_x^{k}(L) - k_2\frac{u^{k+1}(L)+u^k(L)}{2}u^{k}(L)\\ & -d_1\frac{v_x^{k+1}(L)+v_x^k(L)}{2}u_x^{k}(L) - d_2\frac{v^{k+1}(L)+v^k(L)}{2}u^{k}(L)\\ & -c_1\cdot\frac{\zeta_1^{k+1}+\zeta_1^k}{2}u_x^{k}(L) - c_2\cdot\frac{\zeta_2^{k+1}+\zeta_2^k}{2}u^{k}(L).\end{aligned}$$

This yields for the norm of the time-discrete solution, as defined in (2.133):

$$\begin{aligned}&\|z^{k+1}\|^2 - \|z^k\|^2\\ = & \ M\left(-\frac{v^{k+1}(L)-v^k(L)}{\Delta t}(u^{k+1}(L)-u^k(L)) + \frac{v^{k+1}(L)^2-v^k(L)^2}{2}\right)\\ + & \ J\left(-\frac{v_x^{k+1}(L)-v_x^k(L)}{\Delta t}(u_x^{k+1}(L)-u_x^k(L)) + \frac{v_x^{k+1}(L)^2-v_x^k(L)^2}{2}\right)\\ + & \ \frac{k_1}{2}\left(-\left(u_x^{k+1}(L)+u_x^k(L)\right)(u_x^{k+1}(L)-u_x^k(L)) + u_x^{k+1}(L)^2 - u_x^k(L)^2\right)\\ + & \ \frac{k_2}{2}\left(-\left(u^{k+1}(L)+u^k(L)\right)(u^{k+1}(L)-u^k(L)) + u^{k+1}(L)^2 - u^k(L)^2\right)\\ - & \ \frac{d_1}{2}(v_x^{k+1}(L)+v_x^k(L))(u_x^{k+1}(L)-u_x^k(L))\\ - & \ \frac{d_2}{2}(v^{k+1}(L)+v^k(L))(u^{k+1}(L)-u^k(L))\\ - & \ \frac{1}{2}c_1\cdot(\zeta_1^{k+1}+\zeta_1^k)(u_x^{k+1}(L)-u_x^k(L)) + \frac{1}{2}(\zeta_1^{k+1})^\top P_1\zeta_1^{k+1} - \frac{1}{2}(\zeta_1^k)^\top P_1\zeta_1^k\end{aligned}$$

$$- \frac{1}{2} c_2 \cdot (\zeta_2^{k+1} + \zeta_2^k)(u^{k+1}(L) - u^k(L)) + \frac{1}{2}(\zeta_2^{k+1})^\top P_2 \zeta_2^{k+1} - \frac{1}{2}(\zeta_2^k)^\top P_2 \zeta_2^k.$$

For the first six lines, equation (2.134) is applied, and for the rest $c_j = P_j b_j + q_j \tilde{\delta}_j$ (cf. (1.9)) is used to obtain:

$$\begin{aligned}
\|z^{k+1}\|^2 &= \|z^k\|^2 - \frac{d_1}{\Delta t}\left(u_x^{k+1}(L) - u_x^k(L)\right)^2 - \frac{d_2}{\Delta t}(u^{k+1}(L) - u^k(L))^2 \\
&- \frac{\left(\zeta_1^{k+1} + \zeta_1^k\right)^\top}{2}(P_1 b_1 + q_1 \tilde{\delta}_1)(u_x^{k+1}(L) - u_x^k(L)) \\
&- \frac{\left(\zeta_2^{k+1} + \zeta_2^k\right)^\top}{2}(P_2 b_2 + q_2 \tilde{\delta}_2)(u^{k+1}(L) - u^k(L)) \\
&+ \frac{1}{2}(\zeta_1^{k+1})^\top P_1 \zeta_1^{k+1} - \frac{1}{2}(\zeta_1^k)^\top P_1 \zeta_1^k + \frac{1}{2}(\zeta_2^{k+1})^\top P_2 \zeta_2^{k+1} - \frac{1}{2}(\zeta_2^k)^\top P_2 \zeta_2^k. \quad \text{(A.3)}
\end{aligned}$$

For the second and the third line of (A.3) equations (2.134), (2.136), and (2.137) from the Crank-Nicolson scheme are applied:

$$\begin{aligned}
\|z^{k+1}\|^2 &= \|z^k\|^2 - \frac{d_1}{\Delta t}\left(u_x^{k+1}(L) - u_x^k(L)\right)^2 - \frac{d_2}{\Delta t}(u^{k+1}(L) - u^k(L))^2 \\
&- \frac{\left(\zeta_1^{k+1} + \zeta_1^k\right)^\top}{2} P_1 \left(\zeta_1^{k+1} - \zeta_1^k - \Delta t\, A_1 \frac{\zeta_1^k + \zeta_1^{k+1}}{2}\right) \\
&- \frac{\left(\zeta_1^{k+1} + \zeta_1^k\right)}{2} \cdot q_1 \tilde{\delta}_1 (u_x^{k+1}(L) - u_x^k(L)) \\
&- \frac{\left(\zeta_2^{k+1} + \zeta_2^k\right)^\top}{2} P_2 \left(\zeta_2^{k+1} - \zeta_2^k - \Delta t\, A_2 \frac{\zeta_2^{k+1} + \zeta_2^k}{2}\right) \\
&- \frac{\left(\zeta_2^{k+1} + \zeta_2^k\right)}{2} \cdot q_2 \tilde{\delta}_2 (u^{k+1}(L) - u^k(L)) \\
&+ \frac{1}{2}(\zeta_1^{k+1})^\top P_1 \zeta_1^{k+1} - \frac{1}{2}(\zeta_1^k)^\top P_1 \zeta_1^k + \frac{1}{2}(\zeta_2^{k+1})^\top P_2 \zeta_2^{k+1} - \frac{1}{2}(\zeta_2^k)^\top P_2 \zeta_2^k.
\end{aligned}$$

Since $P_j,\ j = 1, 2$ are symmetric matrices, this yields

$$\begin{aligned}
\|z^{k+1}\|^2 &= \|z^k\|^2 - \frac{d_1}{\Delta t}\left(u_x^{k+1}(L) - u_x^k(L)\right)^2 - \frac{d_2}{\Delta t}(u^{k+1}(L) - u^k(L))^2 \\
&+ \Delta t \frac{\left(\zeta_1^{k+1} + \zeta_1^k\right)^\top}{2} P_1 A_1 \frac{\zeta_1^k + \zeta_1^{k+1}}{2} \\
&- \frac{\left(\zeta_1^{k+1} + \zeta_1^k\right)}{2} \cdot q_1 \tilde{\delta}_1 (u_x^{k+1}(L) - u_x^k(L)) \\
&+ \Delta t \frac{\left(\zeta_2^{k+1} + \zeta_2^k\right)^\top}{2} P_2 A_2 \frac{\zeta_2^{k+1} + \zeta_2^k}{2} \\
&- \frac{\left(\zeta_2^{k+1} + \zeta_2^k\right)}{2} \cdot q_2 \tilde{\delta}_2 (u^{k+1}(L) - u^k(L)),
\end{aligned}$$

which is the claimed result (by using (1.9)). □

Proof of Theorem 2.38. Let $k \in \{0, 1, \ldots, S\}$ be arbitrary. Taylor's Theorem yields $\forall x \in [0, L]$:

$$\frac{\breve{u}(t_{k+1}, x) - \breve{u}(t_k, x)}{\Delta t} = \frac{\breve{u}_t(t_{k+1}, x) + \breve{u}_t(t_k, x)}{2} + \Delta t\, T_1^k(x), \tag{A.4}$$

where

$$\begin{aligned} T_1^k(x) &= \int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{\breve{u}_{ttt}(t, x)}{2\,(\Delta t)^2}(t_{k+1} - t)^2 \ \mathrm{d}t + \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{\breve{u}_{ttt}(t, x)}{2\,(\Delta t)^2}(t_k - t)^2 \ \mathrm{d}t \\ &- \int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{\breve{u}_{ttt}(t, x)}{2\Delta t}(t_{k+1} - t) \ \mathrm{d}t + \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{\breve{u}_{ttt}(t, x)}{2\Delta t}(t_k - t) \ \mathrm{d}t. \end{aligned}$$

From (A.4), it is obtained that:

$$\frac{\varepsilon^{k+1} - \varepsilon^k}{\Delta t} + \Delta t\, T_1^k = \frac{\Phi^{k+1} + \Phi^k}{2}. \tag{A.5}$$

Multiplying (A.5) by $\mu(\Phi^{k+1} - \Phi^k)$ and integrating over $[0, L]$ yields:

$$\begin{aligned} &\int_0^L \mu \frac{\varepsilon^{k+1} - \varepsilon^k}{\Delta t} \left(\Phi^{k+1} - \Phi^k\right) \ \mathrm{d}x \\ &= \frac{1}{2} \int_0^L \mu \left(\Phi^{k+1}\right)^2 \ \mathrm{d}x - \frac{1}{2} \int_0^L \mu \left(\Phi^k\right)^2 \ \mathrm{d}x - \Delta t \int_0^L \mu T_1^k \left(\Phi^{k+1} - \Phi^k\right) \ \mathrm{d}x. \end{aligned} \tag{A.6}$$

Furthermore, from (2.87) with $t = t_{k+\frac{1}{2}}$ and Taylor's Theorem, it follows that $\forall w \in \tilde{H}_0^2(0, L)$:

$$\begin{aligned} &\int_0^L \mu \frac{u_t(t_{k+1}, x) - u_t(t_k, x)}{\Delta t} w \ \mathrm{d}x + \int_0^L \Lambda \frac{u_{xx}(t_{k+1}, x) + u_{xx}(t_k, x)}{2} w_{xx} \ \mathrm{d}x \\ &+ M \frac{u_t(t_{k+1}, L) - u_t(t_k, L)}{\Delta t} w(L) + J \frac{u_{tx}(t_{k+1}, L) - u_{tx}(t_k, L)}{\Delta t} w_x(L) \\ &+ k_1 \frac{u_x(t_{k+1}, L) + u_x(t_k, L)}{2} w_x(L) + k_2 \frac{u(t_{k+1}, L) + u(t_k, L)}{2} w(L) \\ &+ d_1 \frac{u_{tx}(t_{k+1}, L) + u_{tx}(t_k, L)}{2} w_x(L) + d_2 \frac{u_t(t_{k+1}, L) + u_t(t_k, L)}{2} w(L) \\ &+ c_1 \cdot \frac{\zeta_1(t_{k+1}) + \zeta_1(t_k)}{2} w_x(L) + c_2 \cdot \frac{\zeta_2(t_{k+1}) + \zeta_2(t_k)}{2} w(L) = \Delta t\, T_2^k(w), \end{aligned}$$

(A.7)

with the functional $T_2^k \colon \tilde{H}_0^2(0,L) \to \mathbb{R}$ defined as

$$\begin{aligned}
&T_2^k(w) = \\
&\int_0^L \mu\Big(\int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{u_{tttt}(t,x)}{2(\Delta t)^2}\,(t_{k+1}-t)^2\;\,\mathrm{d}t + \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{u_{tttt}(t,x)}{2(\Delta t)^2}\,(t_k-t)^2\;\,\mathrm{d}t\Big) w\;\mathrm{d}x \\
&+ \int_0^L \Lambda\Big(\int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{u_{ttxx}(t,x)}{2\Delta t}\,(t_{k+1}-t)\;\,\mathrm{d}t - \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{u_{ttxx}(t,x)}{2\Delta t}\,(t_k-t)\;\,\mathrm{d}t\Big) w_{xx}\;\mathrm{d}x \\
&+ M\Big(\int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{u_{tttt}(t,L)}{2(\Delta t)^2}\,(t_{k+1}-t)^2\;\,\mathrm{d}t + \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{u_{tttt}(t,L)}{2(\Delta t)^2}\,(t_k-t)^2\;\,\mathrm{d}t\Big) w(L) \\
&+ J\Big(\int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{u_{ttttx}(t,L)}{2(\Delta t)^2}\,(t_{k+1}-t)^2\;\,\mathrm{d}t + \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{u_{ttttx}(t,L)}{2(\Delta t)^2}\,(t_k-t)^2\;\,\mathrm{d}t\Big) w_x(L) \\
&+ k_1\Big(\int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{u_{ttx}(t,L)}{2\Delta t}\,(t_{k+1}-t)\;\,\mathrm{d}t - \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{u_{ttx}(t,L)}{2\Delta t}\,(t_k-t)\;\,\mathrm{d}t\Big) w_x(L) \\
&+ k_2\Big(\int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{u_{tt}(t,L)}{2\Delta t}\,(t_{k+1}-t)\;\,\mathrm{d}t - \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{u_{tt}(t,L)}{2\Delta t}\,(t_k-t)\;\,\mathrm{d}t\Big) w(L) \\
&+ d_1\Big(\int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{u_{tttx}(t,L)}{2\Delta t}\,(t_{k+1}-t)\;\,\mathrm{d}t - \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{u_{tttx}(t,L)}{2\Delta t}\,(t_k-t)\;\,\mathrm{d}t\Big) w_x(L) \\
&+ d_2\Big(\int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{u_{ttt}(t,L)}{2\Delta t}\,(t_{k+1}-t)\;\,\mathrm{d}t - \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{u_{ttt}(t,L)}{2\Delta t}\,(t_k-t)\;\,\mathrm{d}t\Big) w(L) \\
&+ c_1\cdot\Big(\int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{(\zeta_1)_{tt}(t)}{2\Delta t}\,(t_{k+1}-t)\;\,\mathrm{d}t - \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{(\zeta_1)_{tt}(t)}{2\Delta t}\,(t_k-t)\;\,\mathrm{d}t\Big) w_x(L) \\
&+ c_2\cdot\Big(\int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{(\zeta_2)_{tt}(t)}{2\Delta t}\,(t_{k+1}-t)\;\,\mathrm{d}t - \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{(\zeta_2)_{tt}(t)}{2\Delta t}\,(t_k-t)\;\,\mathrm{d}t\Big) w(L).
\end{aligned} \tag{A.8}$$

Now, from (2.135) and (A.7) follows $\forall w_h \in W_h$:

$$\begin{aligned}
&\int_0^L \mu\frac{\Phi^{k+1}-\Phi^k}{\Delta t} w_h\;\mathrm{d}x + \int_0^L \Lambda\frac{\varepsilon_{xx}^{k+1}+\varepsilon_{xx}^k}{2}(w_h)_{xx}\;\mathrm{d}x \\
&+M\frac{\Phi^{k+1}(L)-\Phi^k(L)}{\Delta t}(w_h)(L) + J\frac{\Phi_x^{k+1}(L)-\Phi_x^k(L)}{\Delta t}(w_h)_x(L) \\
&+k_1\frac{\varepsilon_x^{k+1}(L)+\varepsilon_x^k(L)}{2}(w_h)_x(L) + k_2\frac{\varepsilon^{k+1}(L)+\varepsilon^k(L)}{2}w_h(L)
\end{aligned}$$

$$
\begin{aligned}
&+d_1 \frac{\Phi_x^{k+1}(L)+\Phi_x^k(L)}{2}(w_h)_x(L) + d_2 \frac{\Phi^{k+1}(L)+\Phi^k(L)}{2} w_h(L) \\
&+c_1 \cdot \frac{\zeta_{e,1}^{k+1}+\zeta_{e,1}^k}{2}(w_h)_x(L) + c_2 \cdot \frac{\zeta_{e,2}^{k+1}+\zeta_{e,2}^k}{2} w_h(L) \\
&\qquad\qquad = -\Delta t\, T_2^k(w_h) + G_1^k(w_h),
\end{aligned}
\tag{A.9}
$$

where the functional $G_1^k(w_h)$ is given by

$$
\begin{aligned}
G_1^k(w_h) \quad := \quad & \int_0^L \mu \frac{u_t^e(t_{k+1},x)-u_t^e(t_k,x)}{\Delta t} w_h \; \mathrm{d}x \\
& + M \frac{u_t^e(t_{k+1},L)-u_t^e(t_k,L)}{\Delta t} w_h(L) + J \frac{u_{tx}^e(t_{k+1},L)-u_{tx}^e(t_k,L)}{\Delta t}(w_h)_x(L) \\
& + d_1 \frac{u_{tx}^e(t_{k+1},L)+u_{tx}^e(t_k,L)}{2}(w_h)_x(L) + d_2 \frac{u_t^e(t_{k+1},L)+u_t^e(t_k,L)}{2} w_h(L).
\end{aligned}
\tag{A.10}
$$

A Taylor expansion of ζ_j about $t_{k+\frac{1}{2}}$ yields with (2.91):

$$
\begin{aligned}
\frac{\zeta_1(t_{k+1})-\zeta_1(t_k)}{\Delta t} - A_1 \frac{\zeta_1(t_{k+1})+\zeta_1(t_k)}{2} - b_1 \frac{u_{tx}(t_{k+1},L)+u_{tx}(t_k,L)}{2} &= \Delta t\, T_3^k, \\
\frac{\zeta_2(t_{k+1})-\zeta_2(t_k)}{\Delta t} - A_2 \frac{\zeta_2(t_{k+1})+\zeta_2(t_k)}{2} - b_2 \frac{u_t(t_{k+1},L)+u_t(t_k,L)}{2} &= \Delta t\, T_4^k,
\end{aligned}
\tag{A.11}
$$

with

$$
\begin{aligned}
T_3^k \quad = \quad & \int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{(\zeta_1)_{ttt}(t)}{2(\Delta t)^2}\,(t_{k+1}-t)^2 \; \mathrm{d}t + \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{(\zeta_1)_{ttt}(t)}{2(\Delta t)^2}\,(t_k-t)^2 \; \mathrm{d}t \\
& - A_1 \left(\int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{(\zeta_1)_{tt}(t)}{2\Delta t}\,(t_{k+1}-t) \; \mathrm{d}t - \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{(\zeta_1)_{tt}(t)}{2\Delta t}\,(t_k-t) \; \mathrm{d}t \right) \\
& - b_1 \left(\int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{u_{tttx}(t,L)}{2\Delta t}\,(t_{k+1}-t) \; \mathrm{d}t - \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{u_{tttx}(t,L)}{2\Delta t}\,(t_k-t) \; \mathrm{d}t \right),
\end{aligned}
$$

$$
\begin{aligned}
T_4^k \quad = \quad & \int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{(\zeta_2)_{ttt}(t)}{2(\Delta t)^2}\,(t_{k+1}-t)^2 \; \mathrm{d}t + \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{(\zeta_2)_{ttt}(t)}{2(\Delta t)^2}\,(t_k-t)^2 \; \mathrm{d}t \\
& - A_2 \left(\int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{(\zeta_2)_{tt}(t)}{2\Delta t}\,(t_{k+1}-t) \; \mathrm{d}t - \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{(\zeta_2)_{tt}(t)}{2\Delta t}\,(t_k-t) \; \mathrm{d}t \right)
\end{aligned}
$$

$$-b_2\left(\int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{u_{ttt}(t,L)}{2\Delta t}\,(t_{k+1}-t)\ \mathrm{d}t - \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{u_{ttt}(t,L)}{2\Delta t}\,(t_k-t)\ \mathrm{d}t \right).$$

Using (2.136), (2.137), and (A.11), it follows

$$\begin{aligned} \frac{\zeta_{e,1}^{k+1}-\zeta_{e,1}^{k}}{\Delta t} - A_1\frac{\zeta_{e,1}^{k+1}+\zeta_{e,1}^{k}}{2} - b_1\frac{\Phi_x^{k+1}(L)+\Phi_x^{k}(L)}{2} &= -\Delta t\, T_3^k - G_2^k, \\ \frac{\zeta_{e,2}^{k+1}-\zeta_{e,2}^{k}}{\Delta t} - A_2\frac{\zeta_{e,2}^{k+1}+\zeta_{e,2}^{k}}{2} - b_2\frac{\Phi^{k+1}(L)+\Phi^{k}(L)}{2} &= -\Delta t\, T_4^k - G_3^k, \end{aligned} \tag{A.12}$$

with

$$\begin{aligned} G_2^k &= b_1\frac{u_{tx}^e(t_{k+1},L)+u_{tx}^e(t_k,L)}{2}, \\ G_3^k &= b_2\frac{u_t^e(t_{k+1},L)+u_t^e(t_k,L)}{2}. \end{aligned}$$

Due to (A.5), function w_h in equation (A.9) can be taken as $w_h := \Delta t\frac{\Phi^{k+1}+\Phi^k}{2} \in W_h$. Using (A.6) and (A.12), yields:

$$\begin{aligned} \|z_e^{k+1}\|^2 - \|z_e^k\|^2 &= -(\Delta t)^2\frac{1}{2}\int_0^L \Lambda\left(\varepsilon_{xx}^{k+1}+\varepsilon_{xx}^k\right)(T_1^k)_{xx}\,dx + \frac{\Delta t}{2}G_1^k(\Phi^{k+1}+\Phi^k) \\ &- (\Delta t)^2\left(k_1\frac{\varepsilon_x^{k+1}(L)+\varepsilon_x^k(L)}{2}(T_1^k)_x(L) + k_2\frac{\varepsilon^{k+1}(L)+\varepsilon^k(L)}{2}T_1^k(L)\right) \\ &- \frac{\Delta t}{2}\left(q_1\frac{\zeta_{e,1}^{k+1}+\zeta_{e,1}^k}{2} + \tilde{\delta}_1\frac{\Phi_x^{k+1}(L)+\Phi_x^k(L)}{2}\right)^2 \\ &- \Delta t\delta_1\left(\frac{\Phi_x^{k+1}(L)+\Phi_x^k(L)}{2}\right)^2 - \Delta t\frac{\varepsilon_1}{2}\frac{\zeta_{e,1}^{k+1}+\zeta_{e,1}^k}{2}\cdot P_1\frac{\zeta_{e,1}^{k+1}+\zeta_{e,1}^k}{2} \\ &- P_1\frac{\zeta_{e,1}^{k+1}+\zeta_{e,1}^k}{2}\cdot\left((\Delta t)^2T_3^k + \Delta t\, G_2^k\right) \\ &- \frac{\Delta t}{2}\left(q_2\frac{\zeta_{e,2}^{k+1}+\zeta_{e,2}^k}{2} + \tilde{\delta}_2\frac{\Phi^{k+1}(L)+\Phi^k(L)}{2}\right)^2 \\ &- \Delta t\delta_2\left(\frac{\Phi^{k+1}(L)+\Phi^k(L)}{2}\right)^2 - \Delta t\frac{\varepsilon_2}{2}\frac{\zeta_{e,2}^{k+1}+\zeta_{e,2}^k}{2}\cdot P_2\frac{\zeta_{e,2}^{k+1}+\zeta_{e,2}^k}{2} \\ &- P_2\frac{\zeta_{e,2}^{k+1}+\zeta_{e,2}^k}{2}\cdot\left((\Delta t)^2T_4^k + \Delta t\, G_3^k\right) \\ &- \frac{1}{2}(\Delta t)^2T_2^k(\Phi^{k+1}+\Phi^k). \end{aligned}$$

Therefore,

$$\|z_e^{k+1}\|^2 - \|z_e^k\|^2 \le -(\Delta t)^2\frac{1}{2}\int_0^L \Lambda\left(\varepsilon_{xx}^{k+1}+\varepsilon_{xx}^k\right)(T_1^k)_{xx}\ \mathrm{d}x + \frac{\Delta t}{2}G_1^k(\Phi^{k+1}+\Phi^k)$$

$$
\begin{aligned}
&- (\Delta t)^2 \left(k_1 \frac{\varepsilon_x^{k+1}(L) + \varepsilon_x^k(L)}{2} (T_1^k)_x(L) + k_2 \frac{\varepsilon^{k+1}(L) + \varepsilon^k(L)}{2} T_1^k(L) \right) \\
&- P_1 \frac{\zeta_{e,1}^{k+1} + \zeta_{e,1}^k}{2} \cdot \big((\Delta t)^2 T_3^k + \Delta t\, G_2^k \big) \\
&- P_2 \frac{\zeta_{e,2}^{k+1} + \zeta_{e,2}^k}{2} \cdot \big((\Delta t)^2 T_4^k + \Delta t\, G_3^k \big) \\
&- \frac{1}{2} (\Delta t)^2 T_2^k (\Phi^{k+1} + \Phi^k). \qquad \text{(A.13)}
\end{aligned}
$$

Next, from (A.10) follows:

$$
\begin{aligned}
|G_1^k(\Phi^{k+1} + \Phi^k)| \;\le\;& C \left(\| \frac{u_t^e(t_{k+1},x) - u_t^e(t_k,x)}{\Delta t} \|_{L^2}^2 + \|\Phi^{k+1} + \Phi^k\|_{L^2}^2 \right. \\
&+ \left| \frac{u_t^e(t_{k+1},L) - u_t^e(t_k,L)}{\Delta t} \right|^2 + \left| \frac{u_{tx}^e(t_{k+1},L) - u_{tx}^e(t_k,L)}{\Delta t} \right|^2 \\
&+ \left| \frac{u_{tx}^e(t_{k+1},L) + u_{tx}^e(t_k,L)}{2} \right|^2 + \left| \frac{u_t^e(t_{k+1},L) + u_t^e(t_k,L)}{2} \right|^2 \\
&+ \left. |\Phi^{k+1}(L) + \Phi^k(L)|^2 + |\Phi_x^{k+1}(L) + \Phi_x^k(L)|^2 \right) \\
\le\;& C \Big(\|\Phi^{k+1} + \Phi^k\|_{L^2}^2 + |\Phi^{k+1}(L) + \Phi^k(L)|^2 + |\Phi_x^{k+1}(L) + \Phi_x^k(L)|^2 \\
&+ \frac{1}{\Delta t} \int_{t_k}^{t_k+1} \|u_{tt}^e(t)\|_{L^2}^2 + |u_{tt}^e(t,L)|^2 + |u_{ttx}^e(t,L)|^2 \ \mathrm{d}t + \|u_t^e\|_{C([t_k,t_{k+1}];H^2)}^2 \Big). \qquad \text{(A.14)}
\end{aligned}
$$

It can easily be seen that

$$
\|T_1^k\|_{H^2}^2 \le \Delta t \int_{t_k}^{t_{k+1}} \|\breve{u}_{ttt}(t)\|_{H^2}^2 \ \mathrm{d}t \le C\Delta t \int_{t_k}^{t_{k+1}} \|u_{ttt}(t)\|_{H^2}^2 \ \mathrm{d}t, \qquad \text{(A.15)}
$$

$$
\|T_3^k\|^2 \le C\Delta t \int_{t_k}^{t_{k+1}} \|u_{ttt}(t)\|_{H^2}^2 + \|(\zeta_1)_{tt}\|^2 + \|(\zeta_1)_{ttt}\|^2 \ \mathrm{d}t, \qquad \text{(A.16)}
$$

$$
\|T_4^k\|^2 \le C\Delta t \int_{t_k}^{t_{k+1}} \|u_{ttt}(t)\|_{H^1}^2 + \|(\zeta_2)_{tt}\|^2 + \|(\zeta_2)_{ttt}\|^2 \ \mathrm{d}t, \qquad \text{(A.17)}
$$

and

$$
\begin{aligned}
T_2^k(\Phi^k) \;\le\;& C \Big(\|\Phi^k\|_{L^2}^2 + |\Phi^k(L)|^2 + |\Phi_x^k(L)|^2 + \\
&+ \Delta t \int_{t_k}^{t_{k+1}} \|u_{tt}(t)\|_{H^4}^2 + \|u_{ttt}(t)\|_{H^2}^2 + \|u_{tttt}(t)\|_{H^2}^2 \ \mathrm{d}t \\
&+ \Delta t \int_{t_k}^{t_{k+1}} \|(\zeta_1)_{tt}(t)\|^2 + \|(\zeta_2)_{tt}(t)\|^2 \ \mathrm{d}t \Big). \qquad \text{(A.18)}
\end{aligned}
$$

For the above estimate, the second term of $T_2^k(\Phi^k)$ in (A.8) can be rewritten as:

$$\int_0^L \left(\int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{u_{ttxx}(t,x)}{2\Delta t}\,(t_{k+1}-t)\ \mathrm{d}t - \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{u_{ttxx}(t,x)}{2\Delta t}\,(t_k-t)\ \mathrm{d}t \right)\Phi^k_{xx}\ \mathrm{d}x$$
$$= \int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{t_{k+1}-t}{2\Delta t}\left(u_{ttxx}(t,L)\Phi^k_x(L) - u_{ttxxx}(t,L)\Phi^k(L) + \int_0^L u_{ttxxxx}(t,x)\Phi^k\ \mathrm{d}x \right)\ \mathrm{d}t$$
$$- \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{t_k-t}{2\Delta t}\left(u_{ttxx}(t,L)\Phi^k_x(L) - u_{ttxxx}(t,L)\Phi^k(L) + \int_0^L u_{ttxxxx}(t,x)\Phi^k\ \mathrm{d}x \right)\ \mathrm{d}t,$$

using $\Phi^k(0) = \Phi^k_x(0) = 0$, and then the Sobolev embedding Theorem. From (A.13) – (A.18), now follows:

$$\begin{aligned} \|z_e^{k+1}\|^2 - \|z_e^k\|^2 \;\le\; & C\Bigg(\Delta t(\|z_e^{k+1}\|^2 + \|z_e^k\|^2) + \Delta t\|u_t^e\|^2_{C([t_k,t_{k+1}];H^2)} \\ & + \int_{t_k}^{t_{k+1}} \|u^e_{tt}(t)\|^2_{L^2} + |u^e_{tt}(t,L)|^2 + |u^e_{ttx}(t,L)|^2\ \mathrm{d}t \\ & + (\Delta t)^4 \sum_{i=1}^{2} \int_{t_k}^{t_{k+1}} \|(\zeta_i)_{tt}\|^2 + \|(\zeta_i)_{ttt}\|^2\ \mathrm{d}t \\ & + (\Delta t)^4 \int_{t_k}^{t_{k+1}} \|u_{tt}(t)\|^2_{H^4} + \|u_{ttt}(t)\|^2_{H^2} + \|u_{tttt}(t)\|^2_{H^2}\ \mathrm{d}t \Bigg). \end{aligned} \tag{A.19}$$

Let now $m \in \{1,\dots,S\}$. Assuming $\Delta t \le \frac{1}{2C}$ (with C from (A.19)), and summing (A.19) over $k \in \{0,\dots,m\}$, gives:

$$\begin{aligned} \frac{1}{2}\|z_e^{m+1}\|^2 \;\le\; & \frac{3}{2}\|z_e^0\|^2 + C\Bigg(\Delta t \sum_{k=1}^{m} \|z_e^k\|^2 + \|u^e_t\|^2_{C([0,T];H^2)} + \|u^e_{tt}\|^2_{L^2(0,T;H^2)} \\ & + (\Delta t)^4 \Bigg[\sum_{i=1}^{2} \|(\zeta_i)_{tt}(t)\|^2_{L^2(0,T;\mathbb{R}^n)} + \|(\zeta_i)_{ttt}(t)\|^2_{L^2(0,T;\mathbb{R}^n)} \\ & + \|u_{tt}(t)\|^2_{L^2(0,T;H^4)} + \|u_{ttt}(t)\|^2_{L^2(0,T;H^2)} + \|u_{tttt}(t)\|^2_{L^2(0,T;H^2)} \Bigg]\Bigg). \end{aligned} \tag{A.20}$$

Finally, using the discrete-in-time Gronwall inequality and (A.4), it is obtained that:

$$\begin{aligned} \|z_e^{m+1}\|^2 \;\le\; & C\Bigg(\|z_e^0\|^2 + h^4\left(\|u_t\|^2_{C([0,T];H^4)} + \|u_{tt}\|^2_{L^2(0,T;H^4)} \right) \\ & + (\Delta t)^4 \Bigg[\sum_{i=1}^{2} \|(\zeta_i)_{tt}(t)\|^2_{L^2(0,T;\mathbb{R}^n)} + \|(\zeta_i)_{ttt}(t)\|^2_{L^2(0,T;\mathbb{R}^n)} \\ & + \|u_{tt}(t)\|^2_{L^2(0,T;H^4)} + \|u_{ttt}(t)\|^2_{L^2(0,T;H^2)} + \|u_{tttt}(t)\|^2_{L^2(0,T;H^2)} \Bigg]\Bigg). \end{aligned} \tag{A.21}$$

The result now follows from (A.21), (2.138), and the triangle inequality. □

Even though the analysis in Chapter 3 is carried out for real-valued functions u and as a consequence in the real Hilbert space $\mathcal{H}$, the spectral analysis of the occurring linear operators needs to be performed in a complex Hilbert space. This section contains some of those results. In order to perform the spectral analysis of the operator A defined in Section 3.1 with (3.6), the complex Hilbert space $\mathcal{X}$ is introduced by:

$$\mathcal{X} := \{y = [u, v, \xi, \psi]^\top : u \in \tilde{H}_0^2(0, L), v \in L^2(0, L), \xi, \psi \in \mathbb{C}\},$$

equipped with the inner product

$$\langle y_1, y_2 \rangle_{\mathcal{X}} := \frac{\Lambda}{2} \int_0^L (u_1)_{xx} (\overline{u_2})_{xx} \, \mathrm{d}x + \frac{\mu}{2} \int_0^L v_1 \overline{v_2} \, \mathrm{d}x + \frac{1}{2J} \xi_1 \overline{\xi_2} + \frac{1}{2M} \psi_1 \overline{\psi_2}, \quad \forall y_1, y_2 \in \mathcal{H}.$$

For the operator A, the natural continuation to $\mathcal{X}$ is considered, still denoted by A. This continuation still satisfies (3.6), and the domain is now

$$D_{\mathbb{C}}(A) = \{y \in \mathcal{X} : u \in \tilde{H}_0^4(0, L), v \in \tilde{H}_0^2(0, L), \xi = Jv_x(L), \psi = Mv(L)\},$$

where the occurring Sobolev spaces contain also all appropriate complex valued functions.

The following theorem is employed in the proof of Lemma 3.1.

Theorem A.2. *The linear operator A is skew-adjoint and has compact resolvent in $\mathcal{X}$. The spectrum $\sigma(A)$ consists of countably many eigenvalues $\{\lambda_n\}_{n \in \mathbb{Z}}$. They are all isolated and purely imaginary, and each eigenspace has finite dimension. All eigenspaces form a complete orthogonal decomposition of $\mathcal{X}$.*

Proof. It can easily be shown that for all $y_1, y_2 \in D_{\mathbb{C}}(A)$

$$\langle Ay_1, y_2 \rangle_{\mathcal{X}} = \frac{\Lambda}{2} \int_0^L \left[(v_1)_{xx} (\overline{u_2})_{xx} - (u_1)_{xx} (\overline{v_2})_{xx} \right] \mathrm{d}x = -\langle y_1, Ay_2 \rangle_{\mathcal{X}}$$

i.e. A is skew-symmetric. Straightforward calculations, analogous to those in [40], demonstrate that A is invertible and $A^{-1} : \mathcal{H} \to \mathcal{H}$ is compact. So $0 \in \rho(A)$, and due to the corollary of Theorem VII.3.1 in [73] this proves that A is skew-adjoint. Then, according to Theorem III.6.26 in [35] the spectrum $\sigma(A)$ consists of countably many eigenvalues, which are all isolated. The corresponding eigenspaces are finite-dimensional, and the eigenvectors form an orthogonal basis according to Theorem V.2.10 in [35]. □

Proof of Lemma 3.1. From Theorem A.2, it is known that A is skew-adoint in $\mathcal{X}$. Therefore Stone's Theorem may be applied, and $(\mathrm{e}^{tA})_{t \geq 0}$ is a C_0-semigroup of unitary operators in $\mathcal{X}$. Clearly this also holds for the restriction to $\mathcal{H}$. □

Theorem A.3 ([49])**.** *Let H be a Banach space, A a densely defined linear, m-dissipative operator with compact resolvent, and let operator $\mathcal{N} : \mathcal{H} \to \mathcal{H}$ be continuously differentiable. The C_0-semigroup of contractions generated by A shall be denoted by $\{T(t)\}_{t \geq 0}$.*

Assume that for all $y_0 \in \mathcal{H}$, semilinear evolution problem

$$y_t = (A + \mathcal{N})y \tag{A.22}$$

has a global, mild, uniformly bounded solution $y\colon [0,\infty) \to \mathcal{H}$. Then the operator family $\{S(t)\}_{t\geq 0}$ defined by $S(t)y_0 = y(t)$ is a strongly continuous semigroup, and for all $t > 0$, the following holds:

$$\int_0^t S(\tau)y_0 \,\mathrm{d}\tau \in D(\mathcal{A}), \tag{A.23}$$

and

$$S(t)y_0 - y_0 = A\int_0^t S(\tau)y \,\mathrm{d}\tau + \int_0^t \mathcal{N}S(\tau)y_0 \,\mathrm{d}\tau. \tag{A.24}$$

For a more general version of this result, see [62].

Proof. Case $y_0 \in D(\mathcal{A})$ is considered first. According to Theorem B.6 in Appendix B, $S(t)y_0$ is a classical solution of (A.22), and satisfies the integrated mild formulation:

$$S(t)y_0 - y_0 = \int_0^t AS(\tau)y_0 \,\mathrm{d}\tau + \int_0^t \mathcal{N}S(\tau)y_0 \,\mathrm{d}\tau.$$

Since $S(t)y_0 \in C^1(\mathbb{R}^+, \mathcal{H})$ and $\mathcal{N}$ is continuously differentiable, it follows that both $t \mapsto \mathcal{N}S(t)y_0$ and $t \mapsto \mathcal{A}S(t)y_0$ are continuous, so $AS(t)y_0 \in C(\mathbb{R}^+, \mathcal{H})$. Therefore the following may be written:

$$\int_0^t S(\tau)y_0 \,\mathrm{d}\tau = \lim_{N\to\infty}\sum_{j=1}^N \frac{t}{N}S\Big(\frac{jt}{N}\Big)y_0, \quad \int_0^t AS(\tau)y_0 \,\mathrm{d}\tau = \lim_{N\to\infty}\sum_{j=1}^N \frac{t}{N}AS\Big(\frac{jt}{N}\Big)y_0.$$

Due to the linearity of A, for the second sum there holds:

$$\sum_{j=1}^N \frac{t}{N}AS(\frac{jt}{N})y_0 = A\sum_{j=1}^N \frac{t}{N}S(\frac{jt}{N})y_0.$$

The following convergence as $N \to \infty$ holds:

$$\sum_{j=1}^N \frac{t}{N}S\Big(\frac{jt}{N}\Big)y_0 \to \int_0^t S(s)y_0 \,\mathrm{d}s,$$
$$A\sum_{j=1}^N \frac{t}{N}S\Big(\frac{jt}{N}\Big)y_0 \to \int_0^t AS(s)y_0 \,\mathrm{d}s.$$

Since A is a closed linear operator, it is obtained that:

$$\int_0^t S(\tau)y_0 \,\mathrm{d}s \in D(A), \quad A\int_0^t S(\tau)y_0 \,\mathrm{d}\tau = \int_0^t AS(\tau)y_0 \,\mathrm{d}\tau.$$

So there holds (A.23) and (A.24) for $y_0 \in D(\mathcal{A})$.

Let now $y_0 \in \mathcal{H} \setminus D(\mathcal{A})$, and $\{y_{n,0}\} \subset D(\mathcal{A})$ such that $y_{n,0} \to y_0$. For every $T > 0$, there holds $S(t)y_{n,0} \to S(t)y_0 \in C([0,T],\mathcal{H})$. Since furthermore $\mathcal{N}$ is locally Lipschitz continuous, for every $t > 0$ in the limit $n \to \infty$ it is obtained that:

$$(S(t)y_{n,0} - y_{n,0}) \to (S(t)y_0 - y_0),$$
$$\int_0^t \mathcal{N}S(\tau)y_{n,0}\,\mathrm{d}\tau \to \int_0^t \mathcal{N}S(\tau)y_0\,\mathrm{d}\tau.$$

Together with (A.24), for $n \to \infty$ this gives:

$$\int_0^t S(\tau)y_{n,0}\,\mathrm{d}\tau \to \int_0^t S(\tau)y_0\,\mathrm{d}\tau,$$
$$A\int_0^t S(\tau)y_{n,0}\,\mathrm{d}\tau \to S(t)y_0 - y_0 - \int_0^t \mathcal{N}S(\tau)y_0\,\mathrm{d}\tau.$$

Since A is closed, (A.23) and (A.24) can be concluded.

□

Next, the spectral analysis of the linear operator $\mathcal{B}$ defined with (3.48) in Section 3.3 follows. To this end, the Hilbert space $\tilde{\mathcal{X}}$ is introduced

$$\tilde{\mathcal{X}} := \{w = [u, v, \xi]^\top : u \in \tilde{H}_0^2(0,L), v \in L^2(0,L), \xi \in \mathbb{C}\},$$

equipped with the inner product

$$\langle\!\langle w_1, w_2 \rangle\!\rangle_{\tilde{\mathcal{X}}} := \frac{\Lambda}{2}\int_0^L (u_1)_{xx}\overline{(u_2)_{xx}}\,\mathrm{d}x + \frac{\mu}{2}\int_0^L v_1\overline{v_2}\,\mathrm{d}x + \frac{1}{2J}\xi_1\overline{\xi_2}.$$

The continuation of $\mathcal{B}$ to $\tilde{\mathcal{X}}$ is still denoted by $\mathcal{B}$ and given by 3.48, and has the domain

$$D_{\mathbb{C}}(\mathcal{B}) := \{y \in \tilde{\mathcal{X}} \colon u \in \tilde{H}_0^4(0,L), v \in \tilde{H}_0^2(0,L), \xi = Jv_x(L), u_{xxx}(L) = 0\}.$$

The following Proposition A.4 and Corollary A.5 has shown to be essential for the analysis in Section 3.3.

Proposition A.4. *The operator $\mathcal{B}$ is skew-adjoint and has compact resolvent in $\tilde{\mathcal{X}}$. The spectrum $\sigma(\mathcal{B})$ consists entirely of isolated eigenvalues $\{\lambda_n\}_{n\in\mathbb{Z}}$ located on the imaginary axis and have no accumulation point. All eigenspaces are one-dimensional, and Φ_n denotes the normalized eigenfunction associated to λ_n. Thereby Φ_n is given by*

$$\Phi_n = \begin{bmatrix} u_n \\ \lambda_n u_n \\ \lambda_n J(u_n)_x(L), \end{bmatrix}$$

where the real function $u_n \in \tilde{H}_0^4(0,L)$ is the unique (up to normalization) solution of the boundary value problem (3.50). Thereby the Φ_n are normalized by one.

Proof. First, observe that $\overline{D(\mathcal{B})} = \tilde{\mathcal{X}}$ and

$$\begin{aligned}\langle\!\langle \mathcal{B}w, \breve{w}\rangle\!\rangle_{\tilde{\mathcal{X}}} &= \frac{\Lambda}{2}\Big(\int_0^L \breve{v}_{xx}\overline{u}_{xx}\,\mathrm{d}x - \int_0^L \breve{u}_{xx}\overline{v}_{xx}\,\mathrm{d}x\Big)\\ &= -\langle\!\langle w, \mathcal{B}\breve{w}\rangle\!\rangle_{\tilde{\mathcal{X}}}\end{aligned}$$

where the partial integration in space was performed twice. Hence $\mathcal{B}$ is skew-symmetric. The invertibility of $\mathcal{B}$, i.e. $0 \in \rho(\mathcal{B})$, and the compactness of $\mathcal{B}^{-1}$ are shown as in [40], see also the proof of Theorem A.2 above. Now the Corollary of Theorem VII.3.1 in [73] can be applied, which proves that the skew-symmetric operator $\mathcal{B}$ is even skew-adjoint. According to Theorem III.6.26 in [35] the spectrum $\sigma(\mathcal{B})$ consists of countably many eigenvalues $\{\lambda_n\}_{n\in\mathbb{Z}}$, which are all isolated. The corresponding eigenspaces are finite-dimensional, and the eigenvectors form an orthogonal basis according to Theorem V.2.10 in [35]. Since $\mathcal{B}$ is skew-adjoint, it follows that $\sigma(\mathcal{B}) \subset \mathrm{i}\mathbb{R}$.

Let $\Phi_n = [u_n, v_n, \xi_n]^\top \in D(\mathcal{B})$ be an eigenfunction corresponding to λ_n for $n \in \mathbb{Z}$ i.e. $\mathcal{B}\Phi_n = \lambda_n\Phi_n$. Now Φ_n satisfies the eigenvalue equation if and only if u_n solves (3.50). Functions v_n and ξ_n can be determined from u_n via $v_n = \lambda_n u_n$ and $\xi_n = J\lambda_n(u_n)_x(L)$. The system (3.50) has a non-trivial solution if and only if $\lambda_n \in \sigma(\mathcal{B})$. In this case, the general solution $u_n \in \tilde{H}_0^4(0, L)$ of (3.50a) can be written as

$$u_n(x) = C_1[\cosh px - \cos px] + C_2[\sinh px - \sin px], \tag{A.25}$$

where $p = \big(\frac{-\mu\lambda_n^2}{\Lambda}\big)^{\frac{1}{4}} > 0$, and $C_i \in \mathbb{C}$. Thereby, the zero boundary conditions at $x = 0$ are already incorporated. Using the condition $(u_n)_{xxx}(L) = 0$ from (3.50b) yields

$$C_1[\sinh pL - \sin pL] = -C_2[\cosh pL + \cos pL].$$

Clearly, since always $\lambda_n \neq 0$ both coefficients are always nonzero. So C_2 can always uniquely be determined from C_1. Thus, if (3.50) has a non-trivial solution, it is unique up to multiplicity. This shows that all eigenspaces of $\mathcal{B}$ are one-dimensional, spanned by the Φ_n. Finally, (3.50c) can be used to determine the λ_n for which there is a non-trivial solution. □

Corollary A.5. *The eigenfunctions $\{\Phi_n\}_{n\in\mathbb{Z}}$ form an orthonormal basis of $\tilde{\mathcal{X}}$.*

Proof. This is an immediate consequence of Proposition A.4 and Theorem V.2.10 in [35]. □

Proof of Lemma 4.2. Given $\tilde{y} \in \mathcal{H}$, it needs to be demonstrated that there exists a unique $y = [u\, v\, z_1\, z_2\, \xi\, \psi]^\top \in D(A)$ such that $Ay = \tilde{y}$, i.e.

$$\begin{bmatrix} v \\ -\frac{\Lambda}{\mu}u_{xxxx} \\ A_1z_1 + \frac{1}{J}B_1\xi \\ A_2z_2 + \frac{1}{M}B_2\psi \\ -\Lambda u_{xx}(L) - [C_1z_1 + \frac{1}{J}D_1\xi + K_1u_x(L)] \\ \Lambda u_{xxx}(L) - [C_2z_2 + \frac{1}{M}D_2\psi + K_2u(L)] \end{bmatrix} = \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{\xi} \\ \tilde{\psi} \end{bmatrix}. \tag{A.26}$$

From the first line in (A.26) it follows $v = \tilde{u} \in \tilde{H}_0^2(0, L)$. Furthermore $\xi = Jv_x(L)$ and $\psi = Mv(L)$, since $y \in D(A)$. It is assumed that the A_j are invertible, therefore z_1 and z_2 can be determined uniquely from the third and the fourth line in (A.26). the last two lines of (A.26) give

$$-\Lambda u_{xx}(L) - K_1 u_x(L) = \tilde{\xi} + C_1 z_1 + \frac{1}{J} D_1 \xi, \tag{A.27a}$$

$$\Lambda u_{xxx}(L) - K_2 u(L) = \tilde{\psi} + C_2 z_2 + \frac{1}{M} D_2 \psi, \tag{A.27b}$$

where the right hand sides are already determined. As in [40], it can be noted that the following holds:

$$u(x) = -\frac{\mu}{\Lambda} \int_0^x \int_0^{\delta_1} \int_L^{\delta_2} \int_L^{\delta_3} \tilde{v}(\delta_4) \, \mathrm{d}\delta_4 \, \mathrm{d}\delta_3 \, \mathrm{d}\delta_2 \, \mathrm{d}\delta_1 + u_{xx}(L) \frac{x^2}{2} + u_{xxx}(L) \Big(\frac{x^3}{6} - L \frac{x^2}{2} \Big). \tag{A.28}$$

This is the unique function $u \in \tilde{H}_0^4(0, L)$ that satisfies $-\Lambda u_{xxxx} = -\mu \tilde{v}$ and fulfills the boundary conditions $u_{xx}(L), u_{xxx}(L)$. Now (A.28) implies:

$$u(L) = -\frac{\mu}{\Lambda} \int_0^L \int_0^{\delta_1} \int_L^{\delta_2} \int_L^{\delta_3} \tilde{v}(\delta_4) \, \mathrm{d}\delta_4 \, \mathrm{d}\delta_3 \, \mathrm{d}\delta_2 \, \mathrm{d}\delta_1 + u_{xx}(L) \frac{L^2}{2} - u_{xxx}(L) \frac{L^3}{3}, \tag{A.29a}$$

$$u_x(L) = -\frac{\mu}{\Lambda} \int_0^L \int_0^{\delta_1} \int_L^{\delta_2} \tilde{v}(\delta_4) \, \mathrm{d}\delta_3 \, \mathrm{d}\delta_2 \, \mathrm{d}\delta_1 + u_{xx}(L) L - u_{xxx}(L) \frac{L^2}{2}. \tag{A.29b}$$

Inserting (A.29) into (A.27), gives a system matrix with strictly negative determinant, hence $u_{xx}(L)$ and $u_{xxx}(L)$ can be uniquely determined. Inserting this in (A.28), solution u is obtained. Next the compactness of A^{-1} is demonstrated. Due to the compact embedding $\tilde{H}_0^2(0, L) \hookrightarrow C^1([0, L])$ it holds

$$|\xi|, |\psi| \leq C \|v\|_{H^2(0,L)} = \|\tilde{u}\|_{H^2(0,L)}.$$

Since the matrices A_j are invertible, the third and fourth line of (A.26) imply the boundedness of $|z_j|$ in terms of $|\tilde{z}_j|$ and $|\xi|, |\psi|$, i.e. $\|\tilde{u}\|_{H^2(0,L)}$. Due to the continuous embedding $\tilde{H}_0^4(0, L) \hookrightarrow \tilde{H}_0^3(0, L) \hookrightarrow C([0, L])$, there follows:

$$\sup_{x \in [0,L]} \left| \int_0^x \int_0^{\delta_1} \int_L^{\delta_2} \int_L^{\delta_3} \tilde{v}(\delta_4) \, \mathrm{d}\delta_4 \, \mathrm{d}\delta_3 \, \mathrm{d}\delta_2 \, \mathrm{d}\delta_1 \right| \leq C \|\tilde{v}\|_{L^2(0,L)}, \tag{A.30}$$

$$\sup_{x \in [0,L]} \left| \int_0^x \int_0^{\delta_1} \int_L^{\delta_2} \tilde{v}(\delta_4) \, \mathrm{d}\delta_3 \, \mathrm{d}\delta_2 \, \mathrm{d}\delta_1 \right| \leq C \|\tilde{v}\|_{L^2(0,L)}. \tag{A.31}$$

Again, by replacing $u(L)$ and $u_x(L)$ from (A.29) in (A.27), and by using (A.30)-(A.31), it follows that

$$|u_{xx}(L)|, |u_{xxx}(L)| \leq C \|\tilde{v}\|_{L^2(0,L)} + \|\tilde{u}\|_{H^2(0,L)} + \|z_1\| + \|z_2\|.$$

Utilizing this inequality, and from (A.28) it is finally obtained

$$\|u\|_{H^4(0,L)} \le C\|\tilde{v}\|_{L^2(0,L)} + \|\tilde{u}\|_{H^2(0,L)} + \|z_1\| + \|z_2\|.$$

Altogether, it is shown that A^{-1} is bounded from $\mathcal{H}$ to $\tilde{H}_0^4(0,L) \times \tilde{H}_0^2(0,L) \times \mathbb{R}^{n_1+n_2+2}$. Since the latter space is compactly embedded into $\mathcal{H}$, this proves the compactness of A^{-1} in $\mathcal{H}$. □

The definition and the properties of the operator A_p introduced in Subsection 4.1.3 shall be stated and demonstrated in the following. The system (4.32) is the mild formulation of the evolution problem $(y_p)_t = A_p y_p$ with $y_p = [u,v]^\top \in \mathcal{H}_p$. Thereby $\mathcal{H}_p := \tilde{H}_0^2(0,L) \times L^2(0,L)$, and

$$A_p : \begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} v \\ -\frac{\Lambda}{\mu} u_{xxxx} \end{bmatrix},$$

with the domain

$$\begin{aligned} D(A_p) = \{[u,v]^\top \in \mathcal{H}_p \colon & u \in \tilde{H}_0^4(0,L), v \in \tilde{H}_0^2(0,L), \\ & \Lambda u_{xx}(L) + \tilde{K}_1 u_x(L) = 0,\ \Lambda u_{xxx}(L) - \tilde{K}_2 u(L) = 0\}. \end{aligned}$$

The space $\mathcal{H}_p$ is equipped with the following inner product:

$$\langle y_p, \tilde{y}_p \rangle_p := \Lambda \int_0^L u_{xx}\tilde{u}_{xx}\,\mathrm{d}x + \mu \int_0^L v\tilde{v}\,\mathrm{d}x + \tilde{K}_1 u_x(L)\tilde{u}_x(L) + \tilde{K}_2 u(L)\tilde{u}(L). \tag{A.32}$$

The constants $\tilde{K}_1, \tilde{K}_2$ are defined in (4.31). Hence, $D(A_p)$ and the above inner product depend on these constants. However, from the proof of Theorem 4.17, it is known that $u_0(L) = (u_0)_x(L) = 0$. Hence, $\tilde{K}_j = K_j$.

Moreover, operator A_p has the following properties:

Lemma A.6. *The inverse $A_p : \mathcal{H}_p \to D(A_p)$ exists and is a bijection. Furthermore, A_p^{-1} is compact in $\mathcal{H}_p$.*

Proof. The proof is analogous to the proof of Lemma 4.2, see also Section 4.2 in [40]. □

Lemma A.7. *The operator A_p is skew-adjoint.*

Proof. First it is shown that A_p is skew-symmetric, i.e. for all $y, \tilde{y} \in D(A_p)$ there holds $\langle A_p y, \tilde{y} \rangle_p = -\langle y, A_p \tilde{y} \rangle_p$:

$$\begin{aligned} \langle A_p y, \tilde{y} \rangle_p &= \Lambda \int_0^L v_{xx}\tilde{u}_{xx}\,\mathrm{d}x - \Lambda \int_0^L u_{xxxx}\tilde{v}\,\mathrm{d}x + \tilde{K}_1 v_x(L)\tilde{u}_x(L) + \tilde{K}_2 v(L)\tilde{u}(L) \\ &= \Lambda \Bigg(\int_0^L v\tilde{u}_{xxxx}\,\mathrm{d}x + v_x(L)\tilde{u}_{xx}(L) - v(L)\tilde{u}_{xxx}(L) \\ &\quad - \int_0^L u_{xx}\tilde{v}_{xx}\,\mathrm{d}x - u_{xxx}(L)\tilde{v}(L) + u_{xx}(L)\tilde{v}_x(L) \Bigg) \end{aligned}$$

$$+ \tilde{K}_1 v_x(L)\tilde{u}_x(L) + \tilde{K}_2 v(L)\tilde{u}(L).$$

Essential boundary conditions $\Lambda u_{xx}(L) + \tilde{K}_1 u_x(L) = 0$ and $\Lambda u_{xxx}(L) - \tilde{K}_2 u(L) = 0$, as included in $D(A_p)$, imply:

$$\begin{aligned}\langle A_p y, \tilde{y}\rangle_p &= \Lambda \int_0^L v\tilde{u}_{xxx}\,\mathrm{d}x - \tilde{K}_1 v - x(L)\tilde{u}_x(L) - \tilde{K}_2 v(L)\tilde{u}(L) - \Lambda \int_0^L u_{xx}\tilde{v}_{xx}\,\mathrm{d}x \\ &\quad - \tilde{K}_2 u(L)\tilde{v}(L) - \tilde{K}_1 u_x(L)\tilde{v}_x(L) + \tilde{K}_1 v_x(L)\tilde{u}_x(L) + \tilde{K}_2 v(L)\tilde{u}(L) \\ &= -\langle y, A_p\tilde{y}\rangle_p.\end{aligned}$$

Hence A_p is skew-symmetric. Furthermore, due to Lemma A.6 it follows that ran $A_p = \mathcal{H}_p$. Therefore, the Corollary of Theorem VII.3.1 in [73] can be applied, which proves the skew-adjointness of A_p. □

Lemma A.8. *A_p generates a C_0-semigroup of unitary operators in $\mathcal{H}_p$.*

Proof. Since A_p is skew-adjoint, the claim follows from Stone's theorem (see Theorem B.11). □

Appendix B

The standard results from the literature on linear and nonlinear semigroup theory and functional analysis, which were used or referred to in this thesis, are included in this Appendix for completeness. The results are stated in their order of appearance. The following result has been used to demonstrate the existence of the classical solution in Theorem 2.3, Section 2.1:

Theorem B.1 (Theorem 2.64 in [47])**.** *Let $\mathcal{A}$ be a densely defined linear operator in a Banach space X with $\rho(\mathcal{A}) \neq \emptyset$. The Cauchy problem*

$$\begin{aligned} z_t &= \mathcal{A}z, \\ z(0) &= z_0 \in \mathcal{X}, \end{aligned} \tag{B.33}$$

has unique solution for $z_0 \in D(\mathcal{A})$, which is continuously differentiable for $t \geq 0$ if and only if $\mathcal{A}$ generates C_0-semigroup $T(t)$ on X. Furthermore, $z(t) = T(t)z_0$.

The following theorem is used in Section 2.1 in order to justify asymptotic stability of EBB system with linear boundary control.

Theorem B.2 (La Salle's Invariance Principle, Theorem 3.64 in [47])**.** *Let V be a continuous Lyapunov function for a continuous nonlinear semigroup of contractions $T(t)$ on $\mathcal{H}$, and let $\mathcal{E}$ be the largest invariant subset of*

$$\{z \in \mathcal{H} \,|\, \dot{V}(z) = 0\}.$$

If $\gamma(z)$ is precompact, then

$$\lim_{t\to\infty} d(T(t)z, \mathcal{E}) = 0.$$

Here, by invariance of $\mathcal{E}$ under $T(t)$, the property $T(t)\mathcal{E} = \mathcal{E}$, for all $t \geq 0$ is understood.

The following Theorem is utilized for the Theorem 2.10, Section 2.1.

Theorem B.3 (Theorem 3.26 (iii) in [47]). *Let $T(t)$ be a uniformly bounded C_0-semigroup on a Banach space X and let $\mathcal{A}$ be its generator. If $\mathcal{A}$ has a compact resolvent, then $T(t)$ is asymptotically stable if and only if*

$$Re(\lambda) < 0, \quad \forall \lambda \in \sigma(\mathcal{A}).$$

The following result was utilized in Section 2.1 for the proofs of Theorem 2.13 and Lemma 2.23.

Theorem B.4 (Theorem 5.3.1 in [37]). *Suppose that $f, g : U \to \mathbb{C}$ are holomorphic functions on an open set $U \subset \mathbb{C}$. Suppose also that the closed ball $\overline{B}(P, r) \subset U$ and that, for each $\zeta \in \partial D(P, r)$,*

$$|f(\zeta) - g(\zeta)| < |f(\zeta)| + |g(\zeta)|.$$

Then

$$\frac{1}{2\pi} \oint_{\partial D(P,r)} \frac{f'(\zeta)}{f(\zeta)} \, \mathrm{d}\zeta = \frac{1}{2\pi} \oint_{\partial D(P,r)} \frac{g'(\zeta)}{g(\zeta)} \, \mathrm{d}\zeta$$

That is, the number of zeros of f in $D(P, r)$ counting multiplicities equals the number of zeros of g in $D(P, r)$ counting multiplicities.

For the proof of Proposition 3.4 in Section 3.1 and Proposition 4.5 in Subsection 4.1.2, the following result is needed.

Theorem B.5 (Theorem 6.1.4 in [56]). *Assume X is a Banach space. Let $F : [0, \infty) \times X \to X$ be continuous in t for $t \geq 0$ and locally Lipschitz continuous in u, uniformly in t on bounded intervals. If $-A$ is the infinitesimal generator of a C_0-semigroup $T(t)$ on X then for every $u_0 \in X$ there is a $t_{max} \leq \infty$ such that the initial value problem*

$$\begin{aligned} \frac{d}{dt}u(t) + Au(t) &= F(t, u(t)), \quad t \geq 0 \\ u(0) &= u_0, \end{aligned} \tag{B.34}$$

has a unique mild solution on $[0, t_{max})$. Moreover, if $t_{max} < \infty$ then $\lim_{t \nearrow t_{max}} \|u(t)\|_X = \infty$

The following result has been employed in Lemma 3.5 in Section 3.1, and Proposition 4.5 in Subsection 4.1.2.

Theorem B.6 (Theorem 6.1.5 in [56]). *Let $-A$ be the infinitesimal generator of a C_0-semigroup $T(t)$ on X. If $f : [0, T) \times X \to X$ is continuously differentiable from $[0, T] \times X$ into X, then the mild solution of* (B.34) *with $u_0 \in D(A)$ is a classical solution of the initial value problem.*

The following result has been used in Proposition 3.6 in Section 3.1 and in Proposition 4.10 in the Subsection 4.1.2 in order to approximate the mild solutions.

Proposition B.7 (Proposition 4.3.7 in [9]). *Let X be a Banach space, let A be a linear, m-dissipative operator with dense domain and let $F : X \to X$ be Lipschitz continuous on bounded subsets of X. The contraction semigroup generated by A is denoted by $\{T(t)\}_{t\geq 0}$. Furthermore, let $y \in C([0, T(y_0)), X)$ denote the unique solution to mild formulation*

$$y(t) = T(t)y_0 + \int_0^t T(t-\sigma)F(y(\sigma))\,\mathrm{d}\sigma, \tag{B.35}$$

with initial condition $y_0 \in X$. If $\lim_{n\to\infty} y_{n0} = y_0$ and $T < T(y_0)$, then

$$\lim_{n\to\infty} y_n = y$$

in $C([0, T], X)$, where y_n are the solutions to (B.35) *corresponding to the initial data y_{n0}.*

The following result is used in the discussion in Remark 3.14 in Section 3.2, and in the proof of Theorem 4.25 in Subsection 4.1.5.

Theorem B.8 (Theorem 4 in [23]). *Let $-A$ be a maximal monotone operator densely defined on a convex closed subset C of a Hilbert space H. Assume $0 \in \mathcal{R}(A)$ and $(\lambda A+I)^{-1}$ is compact for some $\lambda > 0$. Then for any $u_0 \in C$ and $f \in L^1(\mathbb{R}^+; H)$, the weak solution $u(t)$ of the Cauchy problem*

$$\begin{aligned} u_t(t) + Au(t) &\ni f(t), \\ u(0) &= u_0, \end{aligned}$$

approaches, as $t \to \infty$, a compact subset Ω of a sphere $\{y\colon \|y-a\| = r\}$, $r \leq \|u_0 - a\| + \int_0^\infty \|f(t)\|\,dt$, $a \in A^{-1}(0)$. Furthermore, Ω is minimal, strongly invariant and equi-almost periodic under the semigroup by $-A$ and T, restricted on $\mathrm{CL}\,\mathrm{co}\,\Omega$, is an affine group of isometries.

The following result is used in the Remark 4.4 in Subsection 4.1.1.

Theorem B.9 (Theorem 2.2 in [21]). *Let A be a dissipative subset of $H \times H$. Then A is maximal dissipative if and only if A is hyper-dissipative i.e. if for every $\breve{z} \in H$ there is at least one $(z, w) \in A$ such that $\breve{z} = z - w$.*

In Subsection 4.1.3, the following result was utilized.

Theorem B.10 (Theorem 1.2.4 b) in [56]). *Let $T(t)$ be a C_0-semigroup of linear operators and let the linear operator A be its infinitesimal generator. Then for $x \in X$, $\int_0^t T(s)x\,ds \in D(A)$ and $A\left(\int_0^t T(s)x\,ds\right) = T(t)x - x$.*

The following theorem is used in Lemma A.8, Appendix A.

Theorem B.11 (Theorem II.3.24 (Stone, 1932) in [24]). *Let $(A, D(A))$ be a densely defined operator on a Hilbert space H. Then A generates a unitary group $T(t)$ on H if and only if A is skew-adjoint, i.e., $A^* = -A$.*

The following result is used in the proof of Lemma A.7, Appendix A.

Theorem B.12 (Corollary of Theorem VII.3.1 in [73]). *A symmetric operator T in a Hilbert space X is self-adjoint if $D(T) = X$ or if $R(T) = X$.*

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