Diese Arbeit haben begutachtet

....

Anton Arnold Christian Schmeiser

DISSERTATION

Insights into Evolution Equations – From Fokker-Planck to Euler-Bernoulli

Ausgeführt zum Zwecke der Erlangung des akademischen Grades eines
Doktors der technischen Wissenschaften
unter der Leitung von

Univ.-Prof. Dr. Anton Arnold

E101
Institut für Analysis und Scientific Computing

eingereicht an der Technischen Universität Wien
Fakultät für Mathematik und Geoinformation

von

Dipl.-Ing. Dominik Stürzer
Matrikelnummer: 0425912

Wien, am 22. Oktober 2015

Kurzfassung

Die vorliegende Arbeit untersucht das qualitative Verhalten zweier Familien partieller Differentialgleichungen:

Fokker-Planck Gleichung mit nicht-lokaler Störung. Im ersten Teil der Dissertation werden lineare, nicht-degenerierte Fokker-Planck Gleichungen betrachtet, die durch einen zusätzlichen, nicht-lokalen Operator gestört werden. Dieser ist durch die Faltung mit einer masselosen Funktion gegeben. Solche gestörten Fokker-Planck Gleichungen können als Vereinfachung von Wigner-Fokker-Planck Gleichungen angesehen werden, welche eine kinetische Beschreibung quantenmechanischer Systeme liefern. Primäres Ziel dieser Arbeit ist ein tieferes Verständnis des Langzeitverhaltens der Lösungen der besprochenen Gleichung, inklusive des Nachweises eines bis auf Normierung eindeutigen Stationärzustandes.

Die entsprechende Analysis teilt sich in mehrere Schritte auf. Zunächst ist es günstig, die Gleichung als Evolutionsgleichung in einem geeigneten Hilbertraum aufzufassen. Im Zuge dessen werden für den ungestörten Fokker-Planck Operator Eigenschaften des Spektrums und der generierten Halbgruppe in verschiedenen gewichteten Lebesgue-Räumen diskutiert und schließlich ein exponentielles Gewicht für die weitere Analysis gewählt. Erst dann wird die Störung in die Untersuchungen einbezogen. Es stellt sich heraus, dass die gestörte Fokker-Planck Gleichung stets einen eindeutigen normierten Stationärzustand hat. Dessen Gestalt hängt vom Faltungskern ab. Des Weiteren konvergiert jede normierte Lösung gegen den Stationärzustand, und zwar mit einer exponentiellen Rate. Das Ungewöhnliche dabei ist, dass diese Rate unabhängig von der Wahl des Faltungskerns ist. Dieses Resultat stellt eine signifikante Verbesserung klassischer Störungs-Resultate für stark stetige Halbgruppen dar.

Euler-Bernoulli Balken mit dissipativen, nichtlinearen Randbedingungen. In diesem Teil werden zweierlei Modelle zur Regelung eines schwingenden Balkens auf ihr Langzeitverhalten hin untersucht. Der Balken wird als elastisch angenommen und durch die Euler-Bernoulli Gleichung beschrieben. Ein Ende ist fixiert, und am freien Ende befindet sich eine Nutzlast. Daran ist ein nichtlinearer Regler gekoppelt, der im Wesentlichen eine nichtlineare Funktion aktueller Messgrößen des Balken ist. Es ist von Interesse, das Langzeitverhalten eines solchen, durch diese gekoppelten Gleichungen beschriebenen, schwingenden Balkens zu verstehen. Ziel ist es, die Funktionalität des Reglers und die Stabilität des Systems nachzuweisen, indem man die asymptotische Konvergenz der Auslenkung des Balkens gegen den Ruhezustand beweist.

Dabei wird wie folgt vorgegangen: Das gesamte System wird in eine Evolutionsgleichung in einem geeigneten Banach-Raum umformuliert. Dann wird Existenz, Eindeutigkeit und Regularität von Lösungen diskutiert, unter Verwendung der Theorie von nichtlinear gestörten bzw. inhomogenen, linearen Evolutionsgleichungen. Um sicherzustellen, dass Lösungen überhaupt konvergieren, wird im Anschluss die Präkompaktheit differenzierbarer Lösungen bewiesen. Dazu wird eine neue, von uns eigens dafür entwickelte Methode verwendet. Im letzten Schritt werden mit Hilfe eines Lyapunov-Funktionals mögliche Grenzwerte der Lösungen untersucht. Es stellt sich heraus, dass bis auf wenige Ausnahmen jede klassische Lösung tatsächlich gegen den Ruhezustand konvergiert. In den anderen Fällen nähert sich die Lösung periodischen Orbits an, welche explizit charakterisiert werden.

Abstract

In this thesis we investigate the qualitative behavior of two different types of partial differential equations.

Fokker-Planck equation with a non-local perturbation. In the first part of this thesis we consider linear, non-degenerate Fokker-Planck equations that are perturbed by adding a non-local operator. It is given by a convolution with a massless kernel. Such perturbed Fokker-Planck equations arise as a simplification of Wigner-Fokker-Planck equations, which are a kinetic description of open quantum systems. The primary goal of the underlying work is to thoroughly discuss the long-time behavior of solutions, as well as the existence and uniqueness of stationary solutions.

For the analysis it is useful to view the perturbed Fokker-Planck equation as an evolution equation in an appropriate Hilbert space. First, we discuss the unperturbed Fokker-Planck operator in several weighted Lebesgue spaces, and characterize its spectrum and properties of the generated semigroup. Then we fix a convenient exponential weight function, and consider the perturbed Fokker-Planck operator in the corresponding space. We find that the operator still possesses a zero-eigenfunction, unique up to a normalizing constant. Moreover, any normalized solution of the perturbed equation converges towards this steady state at an exponential rate. It is remarkable that the rate is independent of the particular choice of the perturbation. This represents a significant improvement compared to the standard perturbation results for linear semigroups.

Euler-Bernoulli beam with nonlinear boundary dissipation. We consider two different types of boundary controls for a flexible beam. The beam is assumed to be described by an Euler-Bernoulli equation, where one end is clamped. The free end carries a payload, and is coupled to a nonlinear controller, which depends on certain parameters of the system through nonlinear functions and differential equations. The main question of interest here is the long-time behavior of the vibrating beam. In order to prove the functioning of the controller we demonstrate the asymptotic stability of the system, i.e. that the deflection of the beam asymptotically approaches the resting position.

The first step in the analysis consists of rewriting the system as an evolution equation in an appropriate Banach space. By using the standard theory of perturbed or inhomogeneous linear evolution equations we discuss existence, uniqueness and regularity of solutions. In order to ensure that solutions actually possess a limit we then prove precompactness of classical trajectories. For this purpose we developed a new technique. Finally, with the help of a Lyapunov functional we characterize possible limits of solutions. It turns out that in most cases these limits are zero, and the classical solutions asymptotically tend to zero, which proves the asymptotic stability of the system. In the few remaining cases the solutions approach a time-periodic solution, which we characterize explicitly.

Acknowledgments

I wish to express my deepest gratitude to my advisor, Anton Arnold, for giving me the opportunity to write this thesis. Anton is a great mentor, and I am thankful for our fruitful collaboration and countless scientific discussions.

Furthermore, I am very grateful for the productive collaboration with Maja Miletić, which led to two publications. Also, I want to thank Andreas Kugi for discussions and his input concerning the aspects of control theory.

I also wish to thank Christian Schmeiser and Birgit Schörkhuber for scientific discussions, which helped me to further enhance my understanding of some aspects of this thesis.

In addition, I am grateful to Angelika Manhart and Philipp Fuchs for proofreading parts of this manuscript.

On a more social level, I want to thank Philipp, Birgit, Karli, Mario, Maja, Sabine, Franz, Sofi, Jan-Frederik, Stefan S., Stefan K., Jan, Kirian, Annalisa, and Christian for their wonderful friendship. They all made my stay at the TU a very special one.

A very special thanks goes to my parents Claire and Otto, who have always supported me and to whom I doubtless owe my fascination for science.

Finally, I wish to thank Gini for her support, and our common interest in mathematics.

Contents

Introduction

When working in the field of time-dependent partial differential equations (PDEs), one frequently encounters evolution equations. Here, an evolution equation is a differential equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = f(t, u(t)), \quad t \geq 0, \tag{E}$$

together with an initial condition $u(0) = u_0$. This equation is set in a suitable Banach space X, and u maps from $\mathbb{R}^+$ into X. How are evolution equations related to PDEs? Usually, they occur as a reformulation of a time-dependent PDE, where the time-derivative is isolated on one side, and the remaining terms are put on the other side, which corresponds to f in (E). These terms typically constitute a differential operator in an appropriate Banach space. By interpreting a PDE as an evolution equation, one gains access to the rich analytical toolbox provided by the theory of evolution equations. This is particularly helpful when investigating the qualitative behavior of PDEs, such as existence of solutions and their long-time behavior. In this thesis we use this theory to discuss properties of solutions of two particular types of PDEs.

We start with some basic concepts about evolution equations. (E) is called autonomous if f does not explicitly depend on t. For the rest of this paragraph, we suppose that the evolution equation is autonomous and that for every initial condition u_0 it possesses a solution $u(t)$ in some sense. We then define the family of operators $(T(t))_{t\geq 0}$ by $T(t)u_0 := u(t)$. Since (E) is autonomous, we have the following property: $T(t+s) = T(t)T(s)$. We call this the semigroup property. Together with the obvious fact that $T(0)$ is the identity, this makes $(T(t))_{t\geq 0}$ a commutative semigroup, namely the *semigroup generated by* f. In order to have a sufficiently strong concept of solutions of the evolution equation, one usually requires *strong continuity* of the semigroup, i.e. that the map $t \mapsto T(t)u_0$ is continuous for every $u_0 \in X$. In the case when f is linear, $(T(t))_{t\geq 0}$ is then called a *C_0-semigroup of bounded operators*, and f is the *(infinitesimal) generator* of the semigroup.

If f is linear and autonomous, the generator and the corresponding semigroup are strongly interconnected via the resolvent of the generator. Results such as the Hille-Yosida theorem use specific bounds on the resolvent of the generator to show existence of the generated semigroup, and give an estimate of the operator-norm of the semigroup operators, which depends exponentially on t. Here, the rate is an upper bound for the real part of the spectrum of the generator. Hence, in order to make statements about the asymptotic behavior of solutions of (E), one usually needs a good understanding of the spectrum and the resolvent of f.

The situation becomes more elaborate when f is nonlinear. Then, no general theory exists, and only in special cases one is able to obtain statements about the generated semigroup. One very well-studied situation is when f is autonomous and maximal monotone. Then it generates a semigroup of nonlinear contractions. We will not follow this path in this thesis, and refer the reader to standard references such as [Bar76].

Another setting which is well-understood is when f can be written as the sum of a linear, autonomous generator of a C_0-semigroup, and a nonlinear operator. There are many results regarding the existence, uniqueness and regularity of solutions of (E). Usually, they rely on the regularity or the t-integrability of the nonlinearity. Statements about long-time behavior of the generated nonlinear semigroup are often weaker than in the linear case, and one is rarely able to give an explicit exponential decay rate of the semigroup.

The first equation considered is a linear Fokker-Planck equation with an added perturbation, which is given by a convolution with a massless kernel ϑ. In its most simple form it reads:

$$\frac{\partial u}{\partial t} = \nabla \cdot (\nabla u + \mathbf{x} u) + \vartheta * u. \tag{FP}$$

Here, $\mathbf{x} \in \mathbb{R}^d$ is the spatial variable of u. Clearly, this Fokker-Planck equation is already in the form of an evolution equation. The differential operator on the right hand side is the Fokker-Planck operator, which generates a C_0-semigroup. The convolution is viewed as a (bounded) perturbation of the generator. We call the sum of those two the perturbed Fokker-Planck operator. The equation arises as a simplification of the Wigner-Fokker-Planck equation, which has a similar structure. It is a kinetic description of open quantum systems. It can be used for example to model electron densities in semiconductor devices. Here, we are mainly interested in the existence and uniqueness of stationary solutions. Furthermore, we address the question of exponential convergence of time-dependent solutions towards the steady state.

We view (FP) as an evolution equation in order to apply the theory available for linear semigroups. From this viewpoint, the existence of a steady state of (FP) is equivalent to the existence of a zero eigenfunction of the perturbed Fokker-Planck operator. The exponential convergence of solutions towards the stationary solution is then shown by appropriate estimates of the corresponding semigroup. The biggest part of the underlying analysis is a thorough investigation of the spectrum and the eigenspaces of the unperturbed and the perturbed Fokker-Planck operator in an appropriate Banach space. The problem of finding a suitable space is a crucial and non-trivial part of this work. In this thesis we are able to give extensive and satisfactory answers to the questions posed, mainly with the tools of spectral analysis and the theory of C_0-semigroups. We can show that indeed there exists a stationary solution of (FP) and that it is unique. Furthermore, any time-dependent solution converges towards this steady state at a uniform exponential rate. The corresponding analysis is carried out in Part 1.

The second type of equations discussed in this thesis describes an Euler-Bernoulli beam under the influence of external forces. The Euler-Bernoulli beam is a model for an elastic beam, and describes the time-dependent deflection u, by means of a fourth-order wave equation. In its most simple form it reads $u_{tt} + u_{xxxx} = 0$, where $x \in [0, L]$ is the position variable along the beam, and L is its length. By introducing $v = u_t$, we can reformulate the free Euler-Bernoulli equation as an evolution equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v \\ -u_{xxxx} \end{bmatrix}. \tag{EBB}$$

Then, we additionally assume that the beam is clamped at the left end, and the right end (i.e. the tip) is free. The tip carries a payload, and further forces are assumed to act on it: We investigate the influence of a nonlinear spring and nonlinear friction, as well as of a nonlinear feedback controller. The latter is given by a system of ordinary differential equations which are coupled to the Euler-Bernoulli system via the boundary conditions at the tip. In both cases, the complete system, consisting of the beam equation and the dynamical boundary conditions incorporating the external forces, can we written as an evolution equation by adding additional components to (EBB). In contrast to the Fokker-Planck equation before, this evolution equation is *not* linear any more. The main question of interest here is if the damping or the controller is sufficient to extinguish unwanted vibrations of the beam, and make the system tend towards the resting position. From a mathematical point of view, we investigate which solutions of the nonlinear evolution equation converge to zero as time tends to infinity.

We can split the nonlinear operator which governs the evolution into a linear part, coming from the free beam equation, and a nonlinear perturbation, which stems from the nonlinear boundary control. Thus, we can apply the theory of semilinear evolution equations, which is exhaustively discussed in [Paz83]. The linear part generates a C_0-semigroup, and by assuming sufficient differentiability of the nonlinearity, we can easily show existence and uniqueness of solutions. In order to investigate their long-time behavior, we proceed in two steps. Firstly, we show that every classical trajectory is precompact, hence, every classical solution possesses a limit set, which it approaches as time tends to infinity. For this purpose we developed a new technique. Secondly, we characterize possible limit sets by using a Lyapunov function. Except for certain parameter values we can show with this technique that the only limit is zero, and hence, all classical solutions asymptotically converge to zero. However, we are neither able to make any statement about the rate of decay nor do we know what happens to mild solutions. This is mainly due to the nonlinearity of the considered problem, and the resulting limitations of the corresponding theory of nonlinear evolution equations. The analysis of the Euler-Bernoulli beam can be found in Part 2: Chapter 2 investigates the beam coupled to a nonlinear spring and a nonlinear damper, and Chapter 3 considers a beam with a nonlinear dynamical feedback controller.

To summarize, we see that for both problems presented above the approach of interpreting them as an evolution equation is very useful. Especially when one is interested in the qualitative behavior of solutions, the comparatively abstract theory of semigroups has many successful applications, and is a powerful tool for examining the equations of interest.

Authorship: In Part 1, all results were obtained and written by Dominik Stürzer, under the supervision of Anton Arnold. The results were published in [SA14] and [AAS15]. In Part 2 the contributions are as follows: Chapter 1 was written by Dominik Stürzer, and the results of Chapter 2 are joint work of Maja Miletić, Dominik Stürzer and Anton Arnold. They were published in [MSA15]. In Chapter 3, the modeling of the controller and the introduction are due to Andreas Kugi, and the rest is again joint work of Maja Miletić, Dominik Stürzer and Anton Arnold. The contents of Chapter 3 were accepted for publication, see [MSAK15].

Bibliography

[AAS15] Achleitner F., Arnold A., Stürzer D. Large-time behavior in non-symmetric Fokker-Planck equations. *Riv. Math. Univ. Parma*, **6**, no. 1 (2015), pp. 1–68.

[Bar76] Barbu V. *Nonlinear semigroups and differential equations in Banach spaces*. Editura Academiei Republicii Socialiste România, Bucharest; Noordhoff International Publishing, Leiden, 1976. Translated from the Romanian.

[MSA15] Miletić M., Stürzer D., Arnold A. An Euler-Bernoulli beam with nonlinear damping and a nonlinear spring at the tip. *Discrete Contin. Dyn. Syst. Ser. B*, **20**, no. 9 (2015).

[MSAK15] Miletić M., Stürzer D., Arnold A., Kugi A. Stability of an Euler-Bernoulli beam with a nonlinear dynamic feedback system. *Conditionally accepted in: IEEE Transactions on Automatic Control.*

[Paz83] Pazy A. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer, New York, 1983.

[SA14] Stürzer D., Arnold A. Spectral analysis and long-time behavior of a Fokker-Planck equation with a non-local perturbation. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, **25**, no. 1 (2014), pp. 53–89.

Part 1

Analysis of a perturbed Fokker-Planck equation

CHAPTER 1

Introduction

This work deals with the analysis of the following class of perturbed Fokker-Planck equations:

$$f_t = \nabla \cdot (\mathbf{D}\nabla f + \mathbf{C}\mathbf{x} f) + \Theta f, \tag{1.1}$$

together with an appropriate initial condition at $t = 0$. Here $f = f(t, \mathbf{x})$, and $t \geq 0$ and $\mathbf{x} \in \mathbb{R}^d$, with $d \in \mathbb{N}$. Here, f_t denotes the time derivative of f. The matrices $\mathbf{C}, \mathbf{D} \in \mathbb{R}^{d\times d}$ are symmetric and positive definite. The perturbation is given by a convolution $\Theta f = \vartheta * f$ with respect to $\mathbf{x}$. The convolution kernel ϑ is assumed to be time-independent and with zero mean, i.e. $\int_{\mathbb{R}^d} \vartheta(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0$. Also, it is assumed to satisfy certain regularity conditions, which will be specified in the beginning of the Sections 3.2 and 4.3, respectively.

The above equation is mainly motivated by the quantum-kinetic Wigner-Fokker-Planck equation, describing so-called open quantum systems, see [AGG+12, ALMS04]. It is of the form

$$\begin{aligned} \partial_t u &= \nabla_{\mathbf{x},\mathbf{v}} \cdot (\nabla_{\mathbf{x},\mathbf{v}} u + (\nabla_{\mathbf{x},\mathbf{v}} A + \mathbf{F})u) + \Xi[V]u \\ u|_{t=0} &= u_0, \end{aligned}$$

where $u = u(t, \mathbf{x}, \mathbf{v})$ is the phase-space quasi-probability-density, with $\mathbf{x}, \mathbf{v} \in \mathbb{R}^d$ denoting position and momentum. The given coefficient function $\nabla_{\mathbf{x},\mathbf{v}} A + \mathbf{F}$ is affine in $(\mathbf{x}, \mathbf{v})$ and models the confinement and friction of the system. $\Xi[V]$ is a non-local operator (convolution in $\mathbf{v}$) determined by an external potential $V(\mathbf{x})$. One question of interest for this problem is to show the existence of a unique normalized stationary state, and to prove uniform exponential convergence of the solution to the stationary state. In the case of a quadratic confinement potential with a small perturbation these questions have been answered positively in [AGG+12], see also [AFN08] for an operator-theoretic approach. However, from the physical point of view, the restriction to nearly quadratic potentials seems quite artificial. This raises the question if the results can be extended to a more general family of (confining) potentials. In order to gain insight into what can be expected and what mechanisms are responsible for the actual behavior, we shall consider here (1.1) as a similar, yet simplified model, which still preserves the essential structure. The non-local operator $\Xi[V]$, which is a convolution in $\mathbf{v}$, is replaced by a convolution with kernel ϑ. This represents a first step towards the full analysis.

Other examples of non-local perturbations in Fokker-Planck equations appear e.g. in the linearized vorticity formulation of the 2D Navier-Stokes equations (cf. (12)–(14) in [GW05]) or in electronic transport models (cf. the linearization of equations (1), (6), (7) in [LK05]).

For the unperturbed equation (1.1), i.e. the case $\vartheta = 0$, the natural functional setting is the space $L^2(\mu^{-1})$, with $\mu(\mathbf{x}) = \exp(-|\mathbf{x}|^2/2)$. Here, $\mu/(2\pi)^{d/2}$ is the unique steady state with normalized mass, i.e. $\int_{\mathbb{R}^d} \mu/(2\pi)^{d/2}\,\mathrm{d}\mathbf{x} = 1$, and all solutions to initial conditions with mass one decay towards this state with exponential rate of at least -1, see e.g. [BGM94]. However, if Θ is added, the situation often becomes more complicated. One reason is that many non-local (convolution) operators are unbounded in the space $L^2(\mu^{-1})$. This can be illustrated for the simple example with the convolution kernel $\vartheta = \delta_{-\alpha} - \delta_\alpha$, $\alpha \in \mathbb{R}$, in one dimension. It corresponds to the operator $(\Theta f)(x) = f(x+\alpha) - f(x-\alpha)$, $x \in \mathbb{R}$, which is unbounded in $L^2(\mu^{-1})$. In this case one can show with an eigenfunction expansion that every non-trivial stationary state of (1.1) is *not* even an element of $L^2(\mu^{-1})$. Thus, this space is not suitable for our intended large-time analysis, since it is “too small”. This motivates to consider (1.1) in some larger space $L^2(\omega)$, with a weight ω growing slower than μ^{-1}. Because of the previous discussion we shall choose ω such that a large class of non-local operators becomes bounded. But the new space should not be “too large” either, since we would risk to loose many convenient properties (like the spectral gap) of the unperturbed Fokker-Planck operator. For example, in $L^2(\mathbb{R}^d)$ the spectrum of L is the left half plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \le d/2\}$, cf. [Met01]. It will turn out that $\omega(\mathbf{x}) := \sum_{i=1}^d \cosh \beta x_i$, for $\beta > 0$, is a convenient choice. Moreover, there is a useful characterization of the functions of $L^2(\omega)$ in terms of their Fourier transform, see Proposition 4.5.

Here we focus on the Fokker-Planck operator in exponentially weighted spaces. For L^2-spaces with polynomial weights, the spectrum of L was studied in [GW02]. Furthermore, our results complement the analysis of Metafune [Met01], where a larger class of Ornstein-Uhlenbeck operators is investigated in unweighted L^p-spaces with $p \ge 1$.

Part 1 of this thesis is organized as follows. Chapter 2 introduces the notation and important definitions. Since the analysis in the d-dimensional case is very similar to the one-dimensional case, we first discuss the one-dimensional problem in great detail in Chapter 3, to keep the notation and arguments more concise. In Chapter 4 we then generalize the results to higher dimensions.

The course of action is very similar in the Chapters 3 and 4. First, we discuss the spectral properties of the unperturbed Fokker-Planck operator L in the self-adjoint setting $L^2(\mu^{-1})$, and then in $L^2(\omega)$ (with ω as defined above). We then call it $\mathcal{L}$. We even give a (non-orthogonal) decomposition of $L^2(\omega)$ into eigenspaces of $\mathcal{L}$, and describe the decay rates of the semigroup generated by $\mathcal{L}$ on each subspace complementary to finitely many eigenspaces. Then we investigate the perturbed operator $\mathcal{L} + \Theta$ in $L^2(\omega)$. We show that Θ (under mild regularity assumptions) leaves the spectrum of L unchanged, i.e. $\sigma(\mathcal{L}) = \sigma(\mathcal{L} + \Theta)$, and Θ is an *isospectral deformation*[1] of $\mathcal{L}$. Finally, we are able to show that Θ defines a similarity mapping which transforms $\mathcal{L}$ into $\mathcal{L} + \Theta$, on each of the subspaces complementary to finitely many eigenspaces of $\mathcal{L}$. By applying this similarity to the corresponding resolvents, it becomes clear that the semigroup generated by $\mathcal{L} + \Theta$ has the same decay rate as $\mathcal{L}$ on each of those subspaces. In particular, every solution of (1.1) converges to the stationary solution with an exponential rate, dependent on $\mathbf{C}$ and $\mathbf{D}$.

[1]On a formal level this isospectral property of $\mathcal{L}+\Theta$ can be understood as follows: In the eigenbasis of $\mathcal{L}$, Θ corresponds to a strictly lower triangular (infinite) matrix, and $\mathcal{L}$ to a diagonal (infinite) matrix.

CHAPTER 2

Preliminaries

The aim of this chapter is to present a collection of definitions, concepts and notations used in Part 1.

We use the convention $\mathbb{N} = \{0, 1, 2, \dots\}$, and the (algebraic) shorthand notation $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. For $d \in \mathbb{N}^*$ the elements of $\mathbb{C}^d$ are denoted by bold lowercase letters. Given some $\mathbf{z} \in \mathbb{C}^d$, the i-th component is denoted by z_i, and as a consequence we may write $\mathbf{z} = [z_1, \dots, z_d]^\top$ as a column vector. The complex conjugate of $\mathbf{z}$ is denoted by $\overline{\mathbf{z}} := [\overline{z_1}, \dots, \overline{z_d}]^\top$, i.e. the complex conjugate is taken component-wise. For a multiindex $\mathbf{k} \in \mathbb{N}^d$ we define $\mathbf{z}^{\mathbf{k}} := z_1^{k_1} \cdots z_d^{k_d}$. Given a real number $s > 0$ we define

$$s^{\mathbf{z}} := [s^{z_1}, \dots, s^{z_d}]^\top.$$

For $i \in \{1, \dots, d\}$ the i-th unit vector in $\mathbb{R}^d$ (or $\mathbb{C}^d$) is denoted by $\mathbf{e}_i$. All norms on $\mathbb{C}^d$ are equivalent, for $1 \le p \le \infty$ we define the p-norm by

$$\begin{aligned} |\mathbf{z}|_p &:= \Big(\sum_{i=1}^{d} |z_i|^p \Big)^{\frac{1}{p}}, \quad 1 \le p < \infty, \\ |\mathbf{z}|_\infty &:= \max_{1 \le i \le d} |z_i|. \end{aligned}$$

With respect to the norm $|\cdot|_p$ the ball in $\mathbb{C}^d$ with radius $r > 0$ and center $\mathbf{a} \in \mathbb{C}^d$ is

$$B_r^p(\mathbf{a}) = \{\mathbf{z} \in \mathbb{C}^d : |\mathbf{z} - \mathbf{a}|_p < r\}.$$

The closed ball is then $\overline{B_r^p(\mathbf{a})}$. Its complement in $\mathbb{C}^d$ is denoted by $B_r^p(\mathbf{a})^c := \mathbb{C}^d \setminus B_r^p(\mathbf{a})$. Depending on the context we may consider the open ball in $\mathbb{R}^d$ instead of $\mathbb{C}^d$, the definition is the same.

Matrices are denoted by bold capital letters, and for the unit matrix we write $\mathbf{I}$. For a matrix $\mathbf{M} \in \mathbb{C}^{d \times d}$ and a real number $s > 0$ we define $s^{\mathbf{M}} := \exp(\mathbf{M} \ln s)$, using the matrix exponential. Furthermore, we need the Kronecker delta δ_i^j, which takes the value 1 if $i = j$ and 0 otherwise.

Finally, for $1 \le p \le \infty$ the space $\ell^p(\mathbb{N})$ is the set of all complex sequences for which the norm $|\cdot|_p$ is finite. This norm $|\cdot|_p$ is defined analogously to the finite-dimensional case above.

On a domain $\Omega \subseteq \mathbb{R}^d$ we call a real-valued function $w \in L^\infty_{\mathrm{loc}}(\Omega)$ a *weight function* if there also holds $1/w \in L^\infty_{\mathrm{loc}}(\Omega)$. The corresponding weighted space $L^2(\Omega; w)$ is the set of all measurable functions $f \colon \Omega \to \mathbb{C}$ such that the norm

$$\|f\|_{\Omega;w} := \Big(\int_\Omega |f(\mathbf{x})|^2 w(\mathbf{x}) \,\mathrm{d}\mathbf{x} \Big)^{\frac{1}{2}}$$

is finite. This norm is induced by the inner product

$$\langle f, g\rangle_{\Omega;w} = \int_\Omega f(\mathbf{x})\overline{g(\mathbf{x})}w(\mathbf{x})\,\mathrm{d}\mathbf{x},$$

and it makes $L^2(\Omega; w)$ a Hilbert space, see Theorem 1.3 in [KO84].

Also, we introduce weighted Sobolev spaces. For two weight functions w_0 and w_1 the space $H^1(\Omega; w_0, w_1)$ consists of all functions $f \in L^2(\Omega; w_0)$ whose distributional first order derivatives satisfy $\partial f/\partial x_j \in L^2(\Omega; w_1)$ for all $1 \le j \le n$. We equip the space $H^1(\Omega; w_0, w_1)$ with the norm

$$\|f\|_{\Omega;w_0,w_1} := \left(\|f\|^2_{\Omega;w_0} + \|\nabla f\|^2_{\Omega;w_1}\right)^{\frac{1}{2}},$$

which makes it a Banach space, see Theorem 1.11 in [KO84]. More generally, we define the weighted Sobolev space $W^{1,p}(\Omega; w_0, w_1)$ for $1 \le p \le \infty$, where we use the weighted L^p-norm instead of the L^2-norm. We have $H^1(\Omega; w_0, w_1) = W^{1,2}(\Omega; w_0, w_1)$. If $\Omega = \mathbb{R}^d$ we shall omit the symbol Ω in these notations. We call two sets of weight functions equivalent if the corresponding weighted spaces are the same. In the case where the weight functions are all equivalent to a constant function, we omit the weight function in the notation, e.g. $L^2(\Omega; 1) = L^2(\Omega)$.

For functions $f \in L^1(\mathbb{R}^d)$ we define the Fourier transform of f as

$$\mathcal{F}_{\mathbf{x}\to\boldsymbol{\xi}} f \equiv \mathcal{F}[f](\boldsymbol{\xi}) \equiv \hat{f}(\boldsymbol{\xi}) := \int_{\mathbb{R}^d} f(\mathbf{x})\mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot\boldsymbol{\xi}}\,\mathrm{d}\mathbf{x}.$$

We use the same notation for the natural extension of the Fourier transform to tempered distributions $f \in \mathscr{S}'(\mathbb{R}^d)$. With this scaling we may identify $\hat{f}(\mathbf{0})$ with the *mass* or *mean* of f. The inverse Fourier transform is usually denoted by $\mathcal{F}^{-1}$, such that $f(\mathbf{x}) = \mathcal{F}^{-1}_{\boldsymbol{\xi}\to\mathbf{x}}[\mathcal{F}_{\mathbf{x}\to\boldsymbol{\xi}} f]$ (for $f \in \mathscr{S}'(\mathbb{R}^d)$). For a tempered distribution $f \in \mathscr{S}'(\mathbb{R}^d)$ and a multiindex $\mathbf{k} \in \mathbb{N}^d$ we define

$$\nabla^{\mathbf{k}} f(\mathbf{x}) := \frac{\partial^{|\mathbf{k}|_1} f}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}}(\mathbf{x})$$

as a distributional derivative.

For an analytic function f on a simply connected domain $\Omega \subset \mathbb{C}$ we denote the line integral of f along a (continuous) path from $z_1 \in \Omega$ to $z_2 \in \Omega$ inside of Ω by

$$\int_{z_1\to z_2} f(\zeta)\,\mathrm{d}\zeta.$$

In order to properly define complex powers, we specify a branch of the logarithm. For $\xi \in \mathbb{C}\setminus\{0\}$ we set $\ln\xi := \log|\xi| + \mathrm{i}\arg\xi$, with $\arg\xi \in [-\frac{\pi}{2}, \frac{3\pi}{2})$, and $\log(\cdot)$ is the natural logarithm on $\mathbb{R}^+$. For $\zeta \in \mathbb{C}$ we may then define the complex power $\xi^\zeta := \exp(\zeta\ln(\xi))$.

Furthermore, we present some definitions and properties concerning linear operators and their spectrum. Let $X, \mathcal{X}$ be Hilbert spaces. If X is continuously and densely embedded in $\mathcal{X}$, we write $X \hookrightarrow \mathcal{X}$, and $X \hookrightarrow\hookrightarrow \mathcal{X}$ indicates that the embedding is compact. Given a subset $Y \subset X$, the closure of Y in X is either denoted by $\overline{Y}$ or by $\mathrm{cl}_X Y$. $\mathscr{C}(X)$ denotes the set of all closed operators A in X with dense domain $D(A)$. Given a closable operator A which is densely defined in X, we write $\mathrm{cl}_X A$ for its closure (in the sense of graphs).

The set of all bounded operators $A\colon X \to \mathcal{X}$ is $\mathscr{B}(X, \mathcal{X})$; if $X = \mathcal{X}$ we just write $\mathscr{B}(X)$. The norm $\|\cdot\|_{\mathscr{B}(X)}$ denotes the operator norm. For an operator $A \in \mathscr{C}(X)$ its range is $\operatorname{ran} A$, its null space is $\ker A$, and its algebraic null space is defined by $M(A) := \bigcup_{k \geq 0} \ker A^k$. Note that there always holds $\ker A \subset D(A)$. A closed, linear subspace $Y \subset X$ is said to be *invariant* under $A \in \mathscr{C}(X)$ (or A-*invariant*) iff $D(A) \cap Y$ is dense in Y and $\operatorname{ran} A|_Y \subset Y$, see e.g. [AV03]. For any $\zeta \in \mathbb{C}$ lying in the resolvent set $\rho(A)$ we denote the resolvent by $R_A(\zeta) := (\zeta - A)^{-1}$. In this context ζ stands for $\zeta \cdot I$, and I is the identity operator. The complement of $\rho(A)$ is the spectrum $\sigma(A)$, and $\sigma_p(A)$ is the point spectrum. We define the spectral bound of an operator by $\mathrm{s}(A) := \sup\{\operatorname{Re}\zeta : \zeta \in \sigma(A)\}$. If $A \in \mathscr{C}(X)$ generates a C_0-semigroup of bounded operators, we usually denote it by $(\mathrm{e}^{tA})_{t\geq 0}$.

For an isolated subset $\sigma' \subset \sigma(A)$ the corresponding *spectral projection* $\mathrm{P}_{A,\sigma'}$ is defined via the line integral

$$\mathrm{P}_{A,\sigma'} := \frac{1}{2\pi \mathrm{i}} \oint_\Gamma R_A(\zeta)\,\mathrm{d}\zeta, \tag{2.1}$$

where Γ is a C^1-contour (see Definition B.4 in the Appendix) strictly separating σ' from $\sigma(A) \setminus \sigma'$, with σ' in the inside of Γ and $\sigma(A) \setminus \sigma'$ on the outside. For a discussion of some properties of $\mathrm{P}_{A,\sigma'}$ see the Appendix B.2.

A final remark concerns constants occurring in estimates: Throughout the following chapters, C denotes some positive constant, not necessarily always the same. Dependence on certain parameters will be indicated in brackets, e.g. $C(t)$ for dependence on t.

CHAPTER 3

The one-dimensional Fokker-Planck equation

In this chapter we discuss the perturbed Fokker-Planck equation (1.1) in one dimension, i.e. $d = 1$, and for $\mathbf{C} = \mathbf{D} = 1$:

$$f_t = f'' + xf' + f + \Theta f. \tag{3.1}$$

Here the situation is more transparent than in the general d-dimensional case, and a lot of computations can even be done explicitly. The contents of this chapter has been published in [SA14].

Note that in this chapter we write $\mathbf{x} = x \in \mathbb{R}$, and the derivative of a function $f(x)$ with respect to x is simply denoted by $f'(x)$.

3.1. The Fokker-Planck operator in exponentially weighted L^2-spaces

We begin our analysis by investigating the unperturbed one-dimensional Fokker-Planck operator $Lf := f'' + xf' + f$ in various weighted spaces. The natural space to consider L in is $X := L^2(1/\mu)$ with $\mu(x) := \exp(-x^2/2)$. We use the notation $\|\cdot\|_X$ for the norm and $\langle\cdot,\cdot\rangle_X$ for the inner product. Writing the operator in the form

$$Lf = \left(\left(\frac{f}{\mu}\right)'\mu\right)'$$

shows that $L|_{C_0^\infty(\mathbb{R})}$ is symmetric and dissipative in X. Then, the proper definition of L is obtained by the closure of $L|_{C_0^\infty(\mathbb{R})}$, and this procedure yields its domain $D(L) \subset X$. In the subsequent theorem we summarize some important properties of L in X, see Appendix A.1 for the proof and further references.

Theorem 3.1. *The Fokker-Planck operator L as defined above has the following properties in X:*

(i) *L is self-adjoint and has a compact resolvent.*
(ii) *The spectrum is $\sigma(L) = -\mathbb{N}$, and it consists only of eigenvalues.*
(iii) *For each eigenvalue $-k \in \sigma(L)$ the corresponding eigenspace is one-dimensional, spanned by $\mu_k := \frac{1}{\sqrt{2\pi}} H_k \mu$, where*

$$H_k(x) = \mu(x)^{-1} \frac{\mathrm{d}^k}{\mathrm{d}x^k}\mu(x)$$

is the k-th Hermite polynomial. There holds the recursion relation $\mu_{k+1} = \mu_k'$, for all $k \in \mathbb{N}$.
(iv) *The eigenvectors $\{\mu_k\}_{k\in\mathbb{N}}$ form an orthogonal basis of X.*
(v) *There holds the spectral representation*

$$L = \sum_{k\in\mathbb{N}} -k\Pi_{L,k}, \quad \textit{where} \quad \Pi_{L,k} := \frac{\sqrt{2\pi}}{k!}\mu_k\langle\cdot,\mu_k\rangle_X$$

is the spectral projection onto the k-th eigenspace. In particular, $D(L)$ consists of all elements of X for which the above sum converges in the strong sense.

(vi) *For every $k \in \mathbb{N}^*$ we define $X_k := \operatorname{span}\{\mu_0, \dots, \mu_{k-1}\}^\perp$, and $X_0 := X$. Then, L is decomposed according to $X = X_k \oplus \operatorname{span}\{\mu_0, \dots, \mu_{k-1}\}$.*

(vii) *The operator L generates a C_0-semigroup of contractions on X_k for all $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ the semigroup satisfies the estimate*

$$\left\| \mathrm{e}^{tL}|_{X_k} \right\|_{\mathscr{B}(X_k)} \le \mathrm{e}^{-kt}, \quad t \ge 0.$$

Hence, the Fokker-Planck equation $\partial_t f = Lf$ has a unique stationary solution with normalized mass, given by μ_0. It is a Gaussian with mean one. Its orthogonal complement X_1 consists of all elements of X with zero mass:

$$X_1 = \mu_0^\perp = \Big\{ f \in X : \langle f, \mu \rangle_X = \int_{\mathbb{R}} f(x)\,\mathrm{d}x = 0 \Big\}. \tag{3.2}$$

Furthermore, according to Result (vii) for $k = 1$ we have for every $f \in X$:

$$\| \mathrm{e}^{tL} f - \Pi_{L,0} f \|_X \le \mathrm{e}^{-t} \| f \|_X.$$

In combination with $\Pi_{L,0} f = \mu_0 \int_{\mathbb{R}} f(x)\,\mathrm{d}x$ this shows that any solution of $\partial_t f = Lf$ converges towards an appropriately scaled version of μ_0 with an exponential rate of at least -1 in the X-norm.

Lemma 3.2. *For every $k \in \mathbb{N}$ the space X_k is given by*

$$X_k = \Big\{ f \in X : \int_{\mathbb{R}} x^j f(x)\,\mathrm{d}x = 0, \quad \forall 0 \le j \le k-1 \Big\}. \tag{3.3}$$

Proof. We show this by induction. For $k = 0$ we trivially have $X_0 = X$, and according to (3.2) the equality (3.3) holds true also for $k = 1$. Let us now assume that (3.3) holds true for some $k \in \mathbb{N}$. Then we have, according to the definition,

$$X_{k+1} = X_k \cap \mu_k^\perp.$$

Hence, by using the induction hypothesis, we find that $f \in X_{k+1}$ iff $\int_{\mathbb{R}} x^j f(x)\,\mathrm{d}x = 0$, for all $0 \le j \le k-1$, and $\int_{\mathbb{R}} H_k(x) f(x)\,\mathrm{d}x = 0$. But since $H_k(x)$ is a polynomial of order k, the second condition is equivalent to $\int_{\mathbb{R}} x^k f(x)\,\mathrm{d}x = 0$, by using the induction hypothesis. So, $f \in X_{k+1}$ is equivalent to $\int_{\mathbb{R}} x^j f(x)\,\mathrm{d}x = 0$ for all $0 \le j \le k$. $\square$

In order to analyze the perturbed equation (3.1), we quickly find that X is not appropriate. As an example, for the simple unbounded perturbation $\Theta f(x) := f(x+a) - f(x-a)$, with $a \in \mathbb{R}$, we can explicitly compute the stationary solution f_0 of (3.1) and expand it with respect to the orthogonal basis $\{\mu_k\}_{k \in \mathbb{N}}$ of X. The obtained Fourier coefficients form a divergent sequence, and so $f_0 \notin X$ follows (see Section A.2.1 in the Appendix for the corresponding calculations). Therefore we consider some larger space $L^2(\omega)$ instead of X, with a weight function ω growing more slowly than μ^{-1}. Hence, we choose ω such that Θ becomes a bounded operator in $L^2(\omega)$ for a large family of convolution kernels. For example, one can directly verify that $\Theta f(x) = f(x+a) - f(x-a)$ is bounded in $L^2(\nu)$ if $\nu(x)$ grows at most exponentially as $|x| \to \infty$, and if $\nu(x)$ grows super-exponentially, this Θ becomes unbounded (for a more detailed discussion see Section A.2.2 in the Appendix). This motivates the choice of a weight function $\omega(x)$ which grows at most exponentially as $|x| \to \infty$. At the same time, ω should grow fast enough such that L still has a spectral gap in

$L^2(\omega)$, i.e. there exists some $\alpha < 0$ such that $\{\zeta \in \mathbb{C} : \operatorname{Re}\zeta > \alpha\} \cap \sigma(L) = \{0\}$. If ω is constant, the spectral gap is lost, see [Met01]. These requirements suggest that exponentially growing weights would be good candidates, growing as fast as permissible while still admitting a large class of non-local operators. So, for the rest of this chapter, we choose the weight function $\omega(x) = \cosh \beta x$ for some fixed $\beta > 0$, and work in the corresponding space $\mathcal{X} := L^2(\cosh \beta x)$. As we will see in the following, the space $\mathcal{X}$ is very convenient also for technical purposes, since it can easily be characterized using the Fourier transform.

Proposition 3.3. *There holds $f \in \mathcal{X}$ iff the corresponding Fourier transform $\hat{f}$ possesses an analytic continuation (still denoted by $\hat{f}$) to the open strip $\Omega_{\beta/2} := \{z \in \mathbb{C} : |\operatorname{Im} z| < \beta/2\}$, which satisfies*

$$\sup_{\substack{|b|<\beta/2 \\ b\in\mathbb{R}}} \|\hat{f}(\cdot + \mathrm{i}b)\|_{L^2(\mathbb{R})} < \infty. \tag{3.4}$$

In this case, $\hat{f}$ satisfies

(i) *For $\xi \in \mathbb{R}$ and $|b| < \beta/2$, $\hat{f}$ is explicitly given by $\hat{f}(\xi + \mathrm{i}b) = \mathcal{F}_{x\to\xi}(\mathrm{e}^{bx} f(x))$.*

(ii) *The following function lies in $L^2(\mathbb{R})$:*

$$\xi \mapsto \hat{f}\Big(\xi \pm \mathrm{i}\frac{\beta}{2}\Big) := \mathcal{F}_{x\to\xi}\big(\mathrm{e}^{\pm\frac{\beta}{2}x} f(x)\big), \quad \text{for a.e. } \xi \in \mathbb{R}. \tag{3.5}$$

Moreover, $b \mapsto \hat{f}(\cdot + \mathrm{i}b)$ lies in $C([-\beta/2, \beta/2]; L^2(\mathbb{R}))$. In particular (3.5) is a natural continuation of $\hat{f}$ from $\Omega_{\beta/2}$ to the closure $\overline{\Omega_{\beta/2}}$.

The above proposition can be generalized to higher space dimensions, see Proposition 4.5 in Chapter 4. Hence, for the proof of Proposition 3.3 we refer to Proposition 4.5.

In the following, $\hat{f}$ always denotes the extension of the Fourier transform of $f \in \mathcal{X}$ to $\overline{\Omega_{\beta/2}}$ according to Proposition 3.3 (i)-(ii). Using this convention, we introduce an alternative norm on the space $\mathcal{X}$:

$$|||f|||_\omega^2 := \|\hat{f}(\cdot + \mathrm{i}\beta/2)\|_{L^2(\mathbb{R})}^2 + \|\hat{f}(\cdot - \mathrm{i}\beta/2)\|_{L^2(\mathbb{R})}^2, \tag{3.6}$$

which is equal to $4\pi\|f\|_\omega^2$. Furthermore, we notice that there holds a Poincaré-type inequality in $\mathcal{X}$:

Lemma 3.4 (Poincaré inequality). *The inequality*

$$\|f\|_\omega \le C_\beta \|f'\|_\omega \tag{3.7}$$

holds for all $f \in H^1(\omega, \omega)$, where $C_\beta > 0$ is a constant only depending on β.

Proof. Use $|\widehat{f'}(\xi)| = |\xi \hat{f}(\xi)|$, and $|\xi| \ge \beta/2$ on $|\operatorname{Im}\xi| = \beta/2$. Then apply the norm $|||\cdot|||_\omega$. □

Our next step is to properly define the Fokker-Planck operator in $\mathcal{X}$.

Lemma 3.5. *Let $\zeta \in \mathbb{C}$ with $\operatorname{Re}\zeta \ge 1 + \beta^2/2$, and consider the resolvent equation $(\zeta - L)f = g$ for $f, g \in C_0^\infty(\mathbb{R})$. Then there exists a constant $C > 0$ independent of f, g, such that*

$$\|f\|_\varpi + \|f'\|_\omega \le C\|g\|_\omega, \tag{3.8}$$

where $\varpi(x) = (1 + |x|)\omega(x)$.

Proof. Let us fix $\zeta \in \mathbb{C}$ with $\operatorname{Re}\zeta \geq 1 + \beta^2/2$. Now we consider the resolvent equation $(\zeta - L)f = g$ for $f, g \in C_0^\infty(\mathbb{R})$. Applying $\langle \cdot, f\rangle_\omega$ to both sides yields:

$$\begin{aligned}\int_{\mathbb{R}} \overline{f} g \omega \,\mathrm{d}x &= \int_{\mathbb{R}} \zeta |f|^2 \omega - (f' + xf)' \overline{f} \omega \,\mathrm{d}x \\ &= \int_{\mathbb{R}} |f'|^2 \omega + |f|^2 (x\omega' + \zeta\omega) + f'\overline{f}\omega' + f\overline{f}' x\omega \,\mathrm{d}x.\end{aligned}$$

Next we take the real part:

$$\begin{aligned}\operatorname{Re} \int_{\mathbb{R}} \overline{f} g \omega \,\mathrm{d}x &= \int_{\mathbb{R}} |f'|^2 \omega + |f|^2 (x\omega' + \operatorname{Re}(\zeta)\omega) + \frac{1}{2} |f^2|' (\omega' + x\omega) \,\mathrm{d}x \\ &= \int_{\mathbb{R}} |f'|^2 \omega + \frac{1}{2} |f|^2 \tilde{\omega} \,\mathrm{d}x, \end{aligned} \tag{3.9}$$

with $\tilde{\omega} := -\omega'' + x\omega' + (2\operatorname{Re}\zeta - 1)\omega$. For our choice $\omega(x) = \cosh \beta x$ we obtain $\tilde{\omega}(x) = (2\operatorname{Re}\zeta - 1 - \beta^2)\omega(x) + x\beta \sinh \beta x$. For $\operatorname{Re}\zeta \geq 1 + \beta^2/2$ the function $\tilde{\omega}$ is strictly positive. Thus, $\tilde{\omega}$ is a weight function, and it has the asymptotic behavior $\tilde{\omega}(x) \sim \beta|x|\omega(x)$ as $x \to \pm\infty$. Applying the Cauchy-Schwarz inequality to the left hand side of (3.9) yields

$$\frac{1}{2}\|f\|_{\tilde{\omega}}^2 + \|f'\|_\omega^2 \leq \|f\|_\omega \|g\|_\omega.$$

For the left hand side we use $\omega(x) \leq \tilde{\omega}(x)$ and the Poincaré inequality (3.7) to obtain

$$\frac{1}{2}\|f\|_{\tilde{\omega}} + \frac{1}{C_\beta}\|f'\|_\omega \leq \|g\|_\omega.$$

The result follows, since the weight functions $\tilde{\omega}$ and ϖ define equivalent norms. $\square$

Corollary 3.6. *The operator* $(1 + \beta^2/2 - L)|_{C_0^\infty(\mathbb{R})}$ *is dissipative in* $\mathcal{X}$.

Proof. We use the result (3.9) for $\zeta = 1 + \beta^2/2$. We then estimate the right hand side for $f \in C_0^\infty(\mathbb{R})$:

$$\operatorname{Re} \int_{\mathbb{R}} \overline{f}(\zeta - L) f \,\mathrm{d}x \geq \Big(C_\beta + \frac{1}{2}\Big)\|f\|_\omega^2 \geq 0,$$

where we have used the Poincaré inequality and $\tilde{\omega} \geq \omega$. $\square$

The above results can be used to establish the proper definition of the Fokker-Planck operator in $\mathcal{X}$. To this end we define the distributional Fokker-Planck operator $\mathfrak{L}f := f'' + xf' + f$ for $f \in \mathscr{S}'(\mathbb{R})$.

Lemma 3.7. *The operator* $L|_{C_0^\infty(\mathbb{R})}$ *is closable in* $\mathcal{X}$. *Its closure* $\mathcal{L} := \mathrm{cl}_{\mathcal{X}}\, L|_{C_0^\infty(\mathbb{R})}$ *has the domain of definition* $D(\mathcal{L}) = \{f \in \mathcal{X} : \mathfrak{L}f \in \mathcal{X}\}$. *For* $f \in D(\mathcal{L})$ *we have* $\mathcal{L}f = \mathfrak{L}f$.

The proof is covered by the proof of Lemma 4.9, which is the generalization of Lemma 3.7 for higher-dimensional Fokker-Planck operators.

Corollary 3.8. *The resolvent set* $\rho(\mathcal{L})$ *is non-empty. It contains the half-plane* $\{\zeta \in \mathbb{C} : \operatorname{Re}\zeta \geq 1 + \beta^2/2\}$.

Proof. According to Corollary 3.6 the operator $(1 + \beta^2/2 - \mathcal{L})$ is injective. Since the part $L = \mathcal{L}|_X$ in X generates a C_0-semigroup of contractions in X, we know that $1 + \beta^2/2 \in \rho(L)$, and as a consequence $\operatorname{ran}(1 + \beta^2/2 - \mathcal{L})$ is dense in $\mathcal{X}$. Finally, according to Lemma 3.5 the inverse $(1+\beta^2/2-\mathcal{L})^{-1}$ is bounded in $\mathcal{X}$. Since $\mathcal{L} \in \mathscr{C}(\mathcal{X})$ it follows that $\operatorname{ran}(1 + \beta^2/2 - \mathcal{L}) = \mathcal{X}$, and $1 + \beta^2/2 \in \rho(\mathcal{L}) \neq \emptyset$.

Since we can repeat the above argument for any other $\zeta \in \mathbb{C}$ with $\operatorname{Re}\zeta \geq 1 + \beta^2/2$, we obtain the desired result. $\square$

As it turns out, the resolvent estimate (3.8) is strong enough to prove compactness of the resolvent. To this end we shall use the following corollary of Lemma B.16 in the Appendix (see also Theorem 2.4 in [Opi89]).

Lemma 3.9. *Let $w_0, w_1 \in C(\mathbb{R})$ be weight functions. If $\lim_{|x|\to\infty} w_1(x)/w_0(x) = 0$, then the compact embedding holds:*

$$H^1(w_0, w_1) \hookrightarrow\hookrightarrow L^2(w_1).$$

This compact embedding allows to prove that $R_{\mathcal{L}}(\zeta)$ is compact:

Theorem 3.10. *For any $\zeta \in \rho(\mathcal{L})$ the resolvent operator $R_{\mathcal{L}}(\zeta)$ is compact in $\mathcal{X}$. Furthermore, the spectrum of $\mathcal{L}$ consists entirely of isolated eigenvalues. In particular $\sigma(\mathcal{L}) = \sigma_p(\mathcal{L})$.*

Proof. To begin with, we fix some $\zeta \in \mathbb{C}$ with $\operatorname{Re}\zeta \geq 1 + \beta^2/2$. From Corollary 3.8 we know that $\zeta \in \rho(\mathcal{L})$. According to Lemma 3.5 we have the estimate (3.8), which implies: There exists a constant $C > 0$ such that

$$\|R_{\mathcal{L}}(\zeta)g\|_{\varpi,\omega} \leq C\|g\|_{\omega}, \quad \forall g \in \mathcal{X}.$$

We have used the density of $C_0^\infty(\mathbb{R})$ in $H^1(\varpi, \omega)$, see Lemma B.17. This shows $R_{\mathcal{L}}(\zeta) \in \mathscr{B}(\mathcal{X}, H^1(\varpi, \omega))$. Clearly, there holds the asymptotic behavior $\omega(x)/\varpi(x) \sim 1/|x| \to 0$ as $x \to \pm\infty$. Therefore we can apply Lemma 3.9 for $w_1 = \omega$ and $w_0 = \varpi$, which yields the compact embedding $H^1(\varpi, \omega) \hookrightarrow\hookrightarrow \mathcal{X}$. Thus, the resolvent $R_{\mathcal{L}}(\zeta) \colon \mathcal{X} \to \mathcal{X}$ is compact for $\operatorname{Re}\zeta \geq 1 + \beta^2/2$. Now we apply Theorem III.6.29 in [Kat66], which proves that $R_{\mathcal{L}}(\zeta)$ is compact for all $\zeta \in \rho(\mathcal{L})$, and that $\sigma(\mathcal{L})$ consists entirely of isolated eigenvalues. $\square$

With these preparations we can now characterize the spectrum of $\mathcal{L}$:

Proposition 3.11. *We have $\sigma(\mathcal{L}) = -\mathbb{N}$. Each eigenspace is one-dimensional, and for $k \in \mathbb{N}$ we have $\ker(k + \mathcal{L}) = \operatorname{span}\{\mu_k\}$.*

Proof. We consider the Fourier transform of the eigenvalue equation $(\zeta - \mathcal{L})f = 0$ for $f \in D(\mathcal{L})$. The general solution of the Fourier-transformed equation on the real line reads:

$$\hat{f}(\xi) = C_\pm \mu(\xi)\xi^{-\zeta}, \quad \xi \in \mathbb{R}^\pm. \tag{3.10}$$

For details see the computation in the beginning of the Appendix A.3 for $g = \vartheta = 0$. Since $f \in \mathcal{X}$, $\hat{f}$ has to be analytic in $\Omega_{\beta/2}$, see Proposition 3.3. With the specification of the complex logarithm in Section 3.1 we may extend both parts of $\hat{f}$ from (3.10) analytically to the complex half-planes $\{\xi \in \mathbb{C} : \operatorname{Re}\xi > 0\}$ and $\{\xi \in \mathbb{C} : \operatorname{Re}\xi < 0\}$ respectively. However, if $\zeta \in \mathbb{C}\setminus\mathbb{Z}$, the two extensions do not meet continuously at the imaginary axis, thus $\hat{f}$ is not analytic in $\Omega_{\beta/2}$ (except for the trivial case $C_\pm = 0$). If

$\zeta \in \mathbb{Z}$, we obtain continuity of $\hat{f}$ in $\mathbb{C} \setminus \{0\}$ iff $C_- = C_+$. But for $\zeta \in \mathbb{N}^*$, $\hat{f}$ still has a pole at $\xi = 0$, thus it is not analytic. In the remaining case $\zeta \in -\mathbb{N}$ the function $\hat{f}$ from (3.10) has an analytic extension to $\mathbb{C}$, when we choose $C_- = C_+$. So $f \in \mathcal{X}$ solves the eigenvalue equation for ζ iff $\zeta \in -\mathbb{N}$. And according to (3.10) the eigenspaces are still spanned by the μ_k, $k \in \mathbb{N}$, since $\hat{\mu}_k(\xi) = (\mathrm{i}\xi)^k \mu(\xi)$. □

The main difference to L in X is that the eigenfunctions do not form an orthogonal basis any more. However, we are still able to transfer the concept of the L-invariant subspaces $X_k \subset X$ to $\mathcal{X}$.

Proposition 3.12. *For every $k \in \mathbb{N}$ we have the following facts:*

(i) *We define the subspaces $\mathcal{X}_k := \mathrm{cl}_{\mathcal{X}} X_k$, and $\mathcal{L}$ is decomposed according to $\mathcal{X} = \mathcal{X}_k \oplus \operatorname{span}\{\mu_{k-1}, \dots, \mu_0\}$, see the Appendix B.2.2. There holds*

$$\sigma(\mathcal{L}|_{\mathcal{X}_k}) = -\mathbb{N} \setminus \{0, -1, -2 \dots, -k+1\}.$$

(ii) *For $k \in \mathbb{N}^*$ the spectral projection $\Pi_{\mathcal{L},k}$ of $\mathcal{L}$ associated to the eigenvalue $-k$ satisfies*

$$\ker \Pi_{\mathcal{L},k} = \mathcal{X}_{k+1} \oplus \operatorname{span}\{\mu_{k-1}, \dots, \mu_0\}, \quad \operatorname{ran} \Pi_{\mathcal{L},k} = \operatorname{span}\{\mu_k\}.$$

Moreover, $\ker \Pi_{\mathcal{L},0} = \mathcal{X}_1$ *and* $\operatorname{ran} \Pi_{\mathcal{L},0} = \operatorname{span}\{\mu_0\}$.

Proof. Since $\sigma(L) = \sigma(\mathcal{L}) = -\mathbb{N}$, and $R_L(\zeta) \subset R_{\mathcal{L}}(\zeta)$ for all $\zeta \in \mathbb{C} \setminus (-\mathbb{N})$, we conclude from (2.1) that for any $\sigma' \subset \sigma(\mathcal{L})$ there holds $\Pi_{L,\sigma'} \subset \Pi_{\mathcal{L},\sigma'}$, and they are bounded projections in X and $\mathcal{X}$, respectively. For $\sigma' := \{0, \dots, -k+1\}$, $k \in \mathbb{N}$, we apply Lemma C.1 from the Appendix: This yields

$$\begin{aligned} \operatorname{ran} \Pi_{\mathcal{L},\sigma'} &= \mathrm{cl}_{\mathcal{X}} \operatorname{ran} \Pi_{L,\sigma'} = \mathrm{cl}_{\mathcal{X}} \operatorname{span}\{\mu_0, \dots, \mu_{k-1}\} = \operatorname{span}\{\mu_0, \dots, \mu_{k-1}\}, \\ \ker \Pi_{\mathcal{L},\sigma'} &= \mathrm{cl}_{\mathcal{X}} \ker \Pi_{L,\sigma'} = \mathrm{cl}_{\mathcal{X}} X_k \equiv \mathcal{X}_k. \end{aligned}$$

With this, Result (i) follows from Proposition B.12.

For (ii) we use the same arguments as before, with $\sigma' = \{-k\}$ instead. □

Corollary 3.13. *For every $k \in \mathbb{N}$ we have $M(k + \mathcal{L}) = \ker(k + \mathcal{L}) = \operatorname{span}\{\mu_k\}$, i.e. the algebraic and geometric eigenspaces coincide.*

Proof. According to Proposition B.8 there holds $M(k + \mathcal{L}) \subseteq \operatorname{ran} \Pi_{\mathcal{L},k}$. But also $\operatorname{ran} \Pi_{\mathcal{L},k} = \ker(k + \mathcal{L}) \subseteq M(k + \mathcal{L})$, so the result follows. □

For a better understanding of $\mathcal{L}|_{\mathcal{X}_k}$ see the Appendix B.2 and Proposition B.10 therein.

Next we characterize the subspaces $\mathcal{X}_k$.

Proposition 3.14. *For $k \in \mathbb{N}^*$ the subspace $\mathcal{X}_k$ is explicitly given by*

$$\mathcal{X}_k = \Big\{ f \in \mathcal{X} : \int_{\mathbb{R}} f(x) x^j \, \mathrm{d}x = 0,\ 0 \le j \le k-1 \Big\}. \tag{3.11}$$

Furthermore, there holds

$$\mathcal{X}_k = \big\{ f \in \mathcal{X} : \hat{f}^{(j)}(0) = 0,\ 0 \le j \le k-1 \big\}, \tag{3.12}$$

where $\hat{f}^{(j)}$ denotes the j-th derivative of $\hat{f}$.

Proof. We introduce the functionals $\psi_j \colon f \mapsto \int_{\mathbb{R}} f(x)x^j \,\mathrm{d}x$, $j \in \mathbb{N}$, which are continuous in $\mathcal{X}$. We define $\tilde{\psi}_j := \psi_j|_X$. According to (3.3) we have $f \in \mathcal{X}_k$ iff $\tilde{\psi}_0(f) = \ldots = \tilde{\psi}_{k-1}(f) = 0$. Applying Lemma C.2 from the Appendix shows that $\mathrm{cl}_{\mathcal{X}} X_k = \{f \in \mathcal{X} : \psi_j(f) = 0,\, 0 \le j \le k-1\}$, which is equal to $\mathcal{X}_k$ by definition. This proves (3.11).

The second equality (3.12) immediately follows from (3.11) and

$$\int_{\mathbb{R}} f(x)x^j \,\mathrm{d}x = \mathcal{F}[f(x)x^j](0) = \mathrm{i}^j \hat{f}^{(j)}(0), \quad \forall j \in \mathbb{N}.$$

□

Remark 3.15. The representation (3.11) of the $\mathcal{X}_k$ also holds in polynomially weighted spaces, see [GW02, Appendix A], and Remark 5.7. For other weighted spaces see Remark 5.10.

The final result of this section deals with the analysis of the semigroup $(\mathrm{e}^{t\mathcal{L}})_{t\ge 0}$ generated by $\mathcal{L}$ in $\mathcal{X}$.

Lemma 3.16. *The operator $\mathcal{L}$ is the infinitesimal generator of a C_0-semigroup $(\mathrm{e}^{t\mathcal{L}})_{t\ge 0}$ in $\mathcal{X}$.*

Proof. According to Theorem 1.4.5 in [Paz83], Corollary 3.6 implies that $\mathcal{L} - 1 - \beta^2/2 = \mathrm{cl}_{\mathcal{X}}((L - 1 - \beta^2/2)|_{C_0^\infty(\mathbb{R})})$ is dissipative in $\mathcal{X}$. From Proposition 3.11 we also know that any $\zeta \in \mathbb{C}$ with $\operatorname{Re}\zeta > 0$ lies in $\rho(\mathcal{L})$. So we can apply the Lumer-Phillips Theorem Theorem 1.4.3 in [Paz83] and find that $\mathcal{L}$ generates a C_0-semigroup $(\mathrm{e}^{t\mathcal{L}})_{t\ge 0}$ of bounded operators. □

According to [Met01] the semigroup generated by $\mathcal{L}$ has the following form in $L^2(\mathbb{R})$ (and hence also in $\mathcal{X}$):

$$\mathcal{F}_{x\to\xi}\big[\mathrm{e}^{t\mathcal{L}} f\big] = \exp\Big(-\frac{\xi^2}{2}(1-\mathrm{e}^{-2t})\Big)\hat{f}\big(\xi \mathrm{e}^{-t}\big), \quad t \ge 0. \tag{3.13}$$

Proposition 3.17. *For every $k \in \mathbb{N}$ we have:*

(i) $(\mathrm{e}^{t\mathcal{L}}|_{\mathcal{X}_k})_{t\ge 0}$ *is the C_0-semigroup generated by $\mathcal{L}|_{\mathcal{X}_k}$ on $\mathcal{X}_k$.*

(ii) *There exists some $C_k > 0$ such that*

$$\|\mathrm{e}^{t\mathcal{L}}|_{\mathcal{X}_k}\|_{\mathscr{B}(\mathcal{X}_k)} \le C_k \mathrm{e}^{-kt}, \quad t \ge 0.$$

Proof. According to the results of Proposition 3.12, $\mathcal{L}$ is decomposed according to $\mathcal{X} = \mathcal{X}_k \oplus \operatorname{span}\{\mu_{k-1}, \ldots, \mu_0\}$. Thus we can repeat the proof of Lemma 3.16 in $\mathcal{X}_k$ instead of $\mathcal{X}$. So $\mathcal{L}|_{\mathcal{X}_k}$ generates a C_0-semigroup on $\mathcal{X}_k$, which is given by the family $(\mathrm{e}^{t\mathcal{L}}|_{\mathcal{X}_k})_{t\ge 0}$.

In order to show (ii) we perform the following computations, by making the additional assumption $t \ge 1$:

$$\begin{aligned}\big\|\mathcal{F}[\mathrm{e}^{t\mathcal{L}} f](\cdot \pm \mathrm{i}\beta/2)\big\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \exp\Big(\Big[-\xi^2 + \frac{\beta^2}{4}\Big](1-\mathrm{e}^{-2t})\Big)\Big|\hat{f}\Big(\Big[\xi \pm \mathrm{i}\frac{\beta}{2}\Big]\mathrm{e}^{-t}\Big)\Big|^2 \mathrm{d}\xi \\ &\le \mathrm{e}^{\frac{\beta^2}{4}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{\xi^2}{2}} \Big|\Big[\xi \pm \mathrm{i}\frac{\beta}{2}\Big]\mathrm{e}^{-t}\Big|^{2k} \left|\frac{\hat{f}\Big(\Big[\xi \pm \mathrm{i}\frac{\beta}{2}\Big]\mathrm{e}^{-t}\Big)}{\Big(\Big[\xi \pm \mathrm{i}\frac{\beta}{2}\Big]\mathrm{e}^{-t}\Big)^k}\right|^2 \mathrm{d}\xi \end{aligned} \tag{3.14}$$

Here we have used the inequality $\frac{1}{2} < 1 - \mathrm{e}^{-2t} < 1$ for $t \geq 1$.

$$\begin{aligned}
\left\| \frac{\hat{f}\left(\left[\xi \pm \mathrm{i}\frac{\beta}{2}\right]\mathrm{e}^{-t}\right)}{\left(\left[\xi \pm \mathrm{i}\frac{\beta}{2}\right]\mathrm{e}^{-t}\right)^k} \right\|_{L^\infty(\mathbb{R}_\xi)} &= \left\| \mathcal{F}_{x\to\xi}\left(\exp\left(\frac{\beta}{2}\mathrm{e}^{-t}x\right)\mathcal{F}^{-1}_{\xi\to x}\left[\frac{\hat{f}(\xi)}{\xi^k}\right]\right)\right\|_{L^\infty(\mathbb{R}_\xi)} \\
&\leq \left\| \exp\left(\frac{\beta}{2}\mathrm{e}^{-t}x\right)\mathcal{F}^{-1}_{\xi\to x}\left[\frac{\hat{f}(\xi)}{\xi^k}\right]\right\|_{L^1(\mathbb{R}_x)} \\
&\leq \tilde{C}(t)\left\|\mathcal{F}^{-1}_{\xi\to x}\left[\frac{\hat{f}(\xi)}{\xi^k}\right]\right\|_\omega \\
&\leq C(t)\left\|(\mathrm{i}\partial_x)^k\mathcal{F}^{-1}_{\xi\to x}\left[\frac{\hat{f}(\xi)}{\xi^k}\right]\right\|_\omega = C(t)\|f\|_\omega.
\end{aligned}$$

In the first estimate we have used the continuity of $\mathcal{F}\colon L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$. The constant $\tilde{C}(t)$ is given by

$$\tilde{C}(t) = \int_{\mathbb{R}} \frac{\exp(\beta\mathrm{e}^{-t}x)}{\cosh\beta x}\,\mathrm{d}x,$$

which is uniformly bounded for $t \geq 1$. In the last estimate we have used the Poincaré inequality (3.7). Inserting this result in (3.14) yields for $t \geq 1$:

$$\begin{aligned}
\left\|\mathcal{F}[\mathrm{e}^{t\mathcal{L}}f](\cdot + \mathrm{i}\beta/2)\right\|^2_{L^2(\mathbb{R})} &\leq C\mathrm{e}^{\frac{\beta^2}{4}}\mathrm{e}^{-2kt}\|f\|^2_\omega \int_{\mathbb{R}} \mathrm{e}^{-\frac{\xi^2}{2}}\left|\xi + \mathrm{i}\frac{\beta}{2}\right|^{2k}\mathrm{d}\xi \\
&= C\mathrm{e}^{-2kt}\|f\|^2_\omega.
\end{aligned}$$

Thus there exists a constant $C > 0$ such that $|||\mathrm{e}^{t\mathcal{L}}f|||_\omega \leq C\mathrm{e}^{-kt}|||f|||_\omega$ for all $t \geq 1$ and $f \in \mathcal{X}$. Since $(\mathrm{e}^{t\mathcal{L}})_{t\geq 0}$ is a C_0-semigroup, this family is also uniformly bounded for $t \in [0,1]$, so altogether we get the desired decay estimate for the semigroup in $\mathcal{X}_k$. □

Before we turn to the perturbed Fokker-Planck equation we summarize our results so far:

Theorem 3.18. *Let $\omega(x) := \cosh\beta x$ for any $\beta > 0$. Then the Fokker-Planck operator $L|_{C_0^\infty(\mathbb{R})}$ is closable in $\mathcal{X} = L^2(\omega)$, and its closure $\mathcal{L} = \mathrm{cl}_{\mathcal{X}}\, L|_{C_0^\infty(\mathbb{R})}$ has the following properties:*

(i) *The spectrum satisfies $\sigma(\mathcal{L}) = -\mathbb{N}$, and $\ker(\mathcal{L} + k) = \mathrm{span}\{\mu_k\}$ for every $k \in \mathbb{N}$. The eigenfunctions satisfy the relation $\mu_k = \mu_0^{(k)}$, where $\mu_0^{(k)}$ denotes the k-th derivative of μ_0.*

(ii) *The resolvent $R_{\mathcal{L}}(\zeta)$ is compact in $\mathcal{X}$ for all $\zeta \in \rho(\mathcal{L})$.*

(iii) *For every $k \in \mathbb{N}$ the operator $\mathcal{L}$ is decomposed according to*

$$\mathcal{X} = \mathcal{X}_k \oplus \mathrm{span}\{\mu_{k-1}, \ldots, \mu_0\},$$

where $\mathcal{X}_k := \mathrm{cl}_{\mathcal{X}}\,\mathrm{span}\{\mu_k, \mu_{k+1}, \ldots\}$, and $\mathcal{X}_0 = \mathcal{X}$.

(iv) *The spectral projection $\Pi_{\mathcal{L},k}$ corresponding to the eigenvalue $-k \in -\mathbb{N}$ fulfills $\mathrm{ran}\,\Pi_{\mathcal{L},k} = \mathrm{span}\{\mu_k\}$ and $\ker\Pi_{\mathcal{L},k} = \mathcal{X}_{k+1} \oplus \mathrm{span}\{\mu_{k-1}, \ldots, \mu_0\}$ for $k \in \mathbb{N}$.*

(v) *For every $k \in \mathbb{N}$ the operator $\mathcal{L}|_{\mathcal{X}_k}$ generates a C_0-semigroup on $\mathcal{X}_k$, and there exists a constant $C_k \geq 1$ such that we have the estimate*

$$\left\| \mathrm{e}^{t\mathcal{L}}|_{\mathcal{X}_k} \right\|_{\mathscr{B}(\mathcal{X}_k)} \leq C_k \mathrm{e}^{-kt}, \quad \forall t \geq 0.$$

3.2. Analysis of the perturbed Fokker-Planck operator

So far we have discussed the one-dimensional Fokker-Planck operator $\mathcal{L}$ in $\mathcal{X} = L^2(\omega)$, with $\omega(x) = \cosh \beta x$. In this section we investigate the properties of the perturbed (one-dimensional) operator $\mathcal{L} + \Theta$ in $\mathcal{X}$, and we shall summarize the results in Theorem 3.37. We begin by specifying the assumptions we make on the perturbation Θ.

(C) Conditions on Θ: We assume that $\Theta f = \vartheta * f$ for all $f \in \mathcal{X}$, where $\vartheta \in \mathscr{S}'(\mathbb{R})$ fulfills the following properties in $\Omega_{\beta/2}$ for some $\beta > 0$:

(i) The Fourier transform $\hat{\vartheta}$ can be extended to an analytic function in $\Omega_{\beta/2}$ (also denoted by $\hat{\vartheta}$), and $\hat{\vartheta} \in L^\infty(\Omega_{\beta/2})$.
(ii) It holds $\hat{\vartheta}(0) = 0$, i.e. ϑ has zero mean.
(iii) The mapping $\xi \mapsto \operatorname{Re} \int_0^1 \hat{\vartheta}(\xi s)/s \, \mathrm{d}s$ is essentially bounded in $\Omega_{\beta/2}$.

Remark 3.19. If ϑ satisfies the conditions **(C)(i)-(ii)**, then we find that the mapping $\xi \mapsto \int_0^1 \hat{\vartheta}(\xi s)/s \, \mathrm{d}s$ is analytic in $\Omega_{\beta/2}$. This becomes clear when writing $\hat{\vartheta}(\xi s)/s = \xi \hat{\vartheta}(\xi s)/(\xi s)$, which is analytic for all $s \in (0, 1]$ and can be continuously extended to $\hat{\vartheta}'(0)\xi$ for $s = 0$. The analyticity of $\xi \mapsto \int_0^1 \hat{\vartheta}(\xi s)/s \, \mathrm{d}s$ on $\Omega_{\beta/2}$ then follows from Theorem 4.9.1 in [Det84].

Lemma 3.20. *There holds $\Theta f \in \mathcal{X}$ for all $f \in \mathcal{X}$ iff the condition **(C)(i)** holds.*

Proof. Clearly, $\widehat{\Theta f} = \hat{\vartheta} \hat{f}$ is analytic in $\Omega_{\beta/2}$ for $f \in \mathcal{X}$. According to Proposition 3.3 there holds $\Theta f \in \mathcal{X}$ iff

$$\sup_{|b| < \beta/2} \|(\hat{\vartheta} \hat{f})(\cdot + \mathrm{i}b)\|_{L^2(\mathbb{R})} < \infty, \tag{3.15}$$

where we use $\widehat{\Theta f} = \hat{\vartheta} \hat{f}$. Now we apply Hölder's inequality and find that (3.15) holds for all $f \in \mathcal{X}$ iff ϑ satisfies **(C)(i)**. □

As a consequence of the above lemma and (3.15), for every $f \in \mathcal{X}$ the product $\hat{\vartheta} \hat{f}$ itself is the Fourier transform of an element of $\mathcal{X}$. So we may define $(\hat{\vartheta} \hat{f})(\cdot \pm \mathrm{i}\beta/2) \in L^2(\mathbb{R})$ for $f \in \mathcal{X}$ according to (3.5) whenever ϑ satisfies **(C)(i)**. With this we obtain according to Proposition 3.3:

$$b \mapsto (\hat{\vartheta} \hat{f})(\cdot + \mathrm{i}b) \in C([-\beta/2, \beta/2]; L^2(\mathbb{R})). \tag{3.16}$$

Lemma 3.21. *The convolution Θ is bounded in $\mathcal{X}$ if the condition **(C)(i)** holds.*

Proof. We apply the norm $\|\!| \cdot |\!\|_\omega$ to Θf. We obtain

$$\begin{aligned} \|\!|\Theta f|\!\|_\omega^2 &= \int_{\mathbb{R}} |\hat{\vartheta} \hat{f}(\xi - \mathrm{i}\beta/2)|^2 \, \mathrm{d}\xi + \int_{\mathbb{R}} |\hat{\vartheta} \hat{f}(\xi + \mathrm{i}\beta/2)|^2 \, \mathrm{d}\xi \\ &= \lim_{b \nearrow \beta/2} \Big[\int_{\mathbb{R}} |\hat{\vartheta} \hat{f}(\xi - \mathrm{i}b)|^2 \, \mathrm{d}\xi + \int_{\mathbb{R}} |\hat{\vartheta} \hat{f}(\xi + \mathrm{i}b)|^2 \, \mathrm{d}\xi \Big] \end{aligned}$$

$$
\begin{aligned}
&\le \|\hat\vartheta\|^2_{L^\infty(\Omega_{\beta/2})} \lim_{b\nearrow\beta/2} \Big[\int_{\mathbb{R}} |\hat f(\xi - \mathrm{i}b)|^2 \,\mathrm{d}\xi + \int_{\mathbb{R}} |\hat f(\xi + \mathrm{i}b)|^2 \,\mathrm{d}\xi \Big] \\
&= \|\hat\vartheta\|^2_{L^\infty(\Omega_{\beta/2})} |\!|\!| f |\!|\!|^2_\omega.
\end{aligned}
$$

We have used the fact that both $b \mapsto \hat f(\cdot \pm \mathrm{i}b)$ and $b \mapsto \hat\vartheta \hat f(\cdot \pm \mathrm{i}b)$ lie in $C([-\beta/2,\beta/2]; L^2(\mathbb{R}))$, see Proposition 3.3 (ii) and (3.16). □

Lemma 3.22. *Under the assumption **(C)** there holds $\Theta\colon \mathcal{X}_k \to \mathcal{X}_{k+1} \subset \mathcal{X}_k$ for every $k \in \mathbb{N}$.*

Proof. According to Proposition 3.14, $f \in \mathcal{X}_k$ iff $\xi = 0$ is a zero of $\hat f(\xi)$ of order greater than or equal to k. Because of the assumption $\hat\vartheta(0) = 0$ the Fourier transform $\widehat{\Theta f} = \hat\vartheta \hat f$ has a zero at least of order $k+1$ for $f \in \mathcal{X}_k$, so $\Theta f \in \mathcal{X}_{k+1}$. □

Corollary 3.23. *Let **(C)** hold, and $k \in \mathbb{N}$. Then the space $\mathcal{X}_k$ is an $(\mathcal{L} + \Theta)$-invariant subspace of $\mathcal{X}$.*

Since the conditions **(C)** are not very handy for direct applications, the following lemma gives some criteria that are simpler to verify and sufficient for **(C)**.

Lemma 3.24. *Let $\beta > 0$ and $\omega(x) = \cosh \beta x$, and assume that $\vartheta \in \mathscr{S}'(\mathbb{R})$ fulfills*

(i) *$\hat\vartheta(0) = 0$,*

(ii) *$\vartheta = \vartheta_W + \vartheta_D$ with $\vartheta_W \in W^{1,1}(\omega^{\frac12}, \omega^{\frac12})$ and $\vartheta_D \in D := \operatorname{span}\{\delta_y : y \in \mathbb{R}\}$, where δ_y denotes the delta distribution located at $y \in \mathbb{R}$.*

*Then $\Theta f = \vartheta * f$ satisfies **(C)** for this $\beta > 0$.*

Proof. In general $\hat\vartheta_W(0)$ and $\hat\vartheta_D(0)$ are not zero, so it is convenient to define $\vartheta^*_W := \vartheta_W + M\mu$ and $\vartheta^*_D := \vartheta_D - M\mu$, where $M := \hat\vartheta_D(0)/\sqrt{2\pi}$. Then ϑ^*_W and ϑ^*_D have zero mean, and we have **(C)(ii)** for both ϑ^*_D and ϑ^*_W. Note that still $\vartheta^*_W \in W^{1,1}(\omega^{\frac12}, \omega^{\frac12})$.

Since $\mathcal{F}_{x\to\xi}\delta_y = \mathrm{e}^{-\mathrm{i}\xi y}$ and $\hat\mu(\xi) = \sqrt{2\pi}\mu(\xi)$, it is immediate that ϑ^*_D satisfies **(C)(i)**. In order to see **(C)(iii)** for ϑ^*_D, we note that the integral occurring in this condition can be rewritten as the line integral from 0 to ξ:

$$
\int_{0\to\xi} \frac{\hat\vartheta^*_D(z)}{z} \,\mathrm{d}z
$$

which is path-independent in $\Omega_{\beta/2}$, since $\hat\vartheta^*_D(z)/z$ is analytic in $\Omega_{\beta/2}$ with a removable singularity at $z = 0$. Therefore the integral itself is analytic, and thus uniformly bounded on every compact subset of $\mathbb{C}$. Because of this, it is sufficient to show uniform boundedness of this integral as $|\xi| \to \infty$ in $\Omega_{\beta/2}$. We outline this for the map $\xi \mapsto \mathrm{e}^{-\mathrm{i}y\xi}$ for any fixed $y \in \mathbb{R}$ and $\operatorname{Re}\xi > 1$, the case $\operatorname{Re}\xi < -1$ is analogous. For this we choose the following integration path (note that we may start from $z = 1$, since the integral from 0 to 1 is a constant)

$$
\begin{aligned}
\left| \int_{1\to\xi} \frac{\mathrm{e}^{-\mathrm{i}yz}}{z} \,\mathrm{d}z \right| &\le \left| \int_1^{\operatorname{Re}(\xi)} \frac{\mathrm{e}^{-\mathrm{i}yz}}{z} \,\mathrm{d}z \right| + \left| \int_{\operatorname{Re}(\xi)\to\operatorname{Re}(\xi)+\mathrm{i}\operatorname{Im}(\xi)} \frac{\mathrm{e}^{-\mathrm{i}yz}}{z} \,\mathrm{d}z \right| \\
&\le \left| \int_y^{\operatorname{Re}(\xi)y} \frac{\mathrm{e}^{-\mathrm{i}z}}{z} \,\mathrm{d}z \right| + \frac{\beta}{2} \mathrm{e}^{|y|\beta/2}.
\end{aligned}
$$

The first integral is known to remain uniformly bounded as $\mathrm{Re}(\xi) \to +\infty$. For estimating the second integral we used $\xi \in \Omega_{\beta/2}$ and $\mathrm{Re}\,\xi \geq 1$. Since $\hat{\mu} = \sqrt{2\pi}\mu$ decays sufficiently fast in $\Omega_{\beta/2}$, it is clear that the integral of $\hat{\mu}(z)/z$ from 1 to ξ also remains uniformly bounded as $\xi \to +\infty$. Altogether, we conclude that $\hat{\vartheta}^*_D$ satisfies **(C)(iii)**.

Now we verify the same properties for ϑ^*_W. Since $\vartheta^*_W \in L^1(\omega^{\frac{1}{2}})$, we may extend $\hat{\vartheta}^*_W$ to an analytic function in $\Omega_{\beta/2}$, and $\hat{\vartheta}^*_W(\xi + \mathrm{i}b) = \mathcal{F}_{x\to\xi}[\vartheta^*_W(x)\mathrm{e}^{bx}]$, cf. Proposition XVI.1.3 in [DL92]. The Fourier transform is a continuous map from $L^1(\mathbb{R})$ to $B_0(\mathbb{R})$ (the continuous functions decaying at infinity, equipped with the supremum norm). Therefore, $\vartheta^*_W \in L^1(\omega^{\frac{1}{2}})$ implies

$$\begin{aligned}\|\hat{\vartheta}^*_W\|_{L^\infty(\Omega_{\beta/2})} &= \sup_{|b|<\frac{\beta}{2}} \sup_{\xi\in\mathbb{R}} |\hat{\vartheta}^*_W(\xi + \mathrm{i}b)| \leq \sup_{|b|<\frac{\beta}{2}} \|\vartheta^*_W(x)\mathrm{e}^{bx}\|_{L^1(\mathbb{R})} \\ &\leq \|\vartheta^*_W(x)\mathrm{e}^{\frac{\beta}{2}|x|}\|_{L^1(\mathbb{R})} < \infty.\end{aligned}$$

So **(C)(i)** is satisfied. For **(C)(iii)** it is sufficient to show that for some $C > 0$ and all $\xi \in \Omega_{\beta/2}$ with $|\xi| \geq 1$ there holds $|\hat{\vartheta}^*_W(\xi)| \leq C/|\xi|$, which is fulfilled if $\mathcal{F}(\vartheta^*_W{}') \in L^\infty(\Omega_{\beta/2})$. Analogously to the previous calculations we obtain that this is satisfied if $\vartheta^*_W{}' \in L^1(\omega^{\frac{1}{2}})$. We conclude that ϑ^*_W fulfills **(C)(i)** and **(C)(iii)** if $\vartheta^*_W \in W^{1,1}(\omega^{\frac{1}{2}}, \omega^{\frac{1}{2}})$. □

For the rest of the article, we shall always assume that Θ satisfies the condition **(C)** for some fixed $\beta > 0$, and we choose the weight function $\omega(x) = \cosh \beta x$ with this particular β. The first result about the perturbed Fokker-Planck operator is the following lemma:

Lemma 3.25. *The operator $\mathcal{L} + \Theta$ has compact resolvent in $\mathcal{X}$.*

Proof. According to the Theorems 3.10 and 3.18 we have that $\mathcal{L}$ is the infinitesimal generator of a C_0-semigroup of bounded operators, and has a compact resolvent. Furthermore, Lemma 3.21 shows that Θ is bounded in $\mathcal{X}$. With this, the result directly follows by application of Proposition III.1.12 in [EN00]. □

As a consequence, the spectrum of $\mathcal{L} + \Theta$ in $\mathcal{X}$ is non-empty and consists only of eigenvalues. In order to characterize the entire spectrum, we introduce the following ladder operators[1], namely the *annihilation operator*

$$\alpha^- : \mathcal{X}_1 \to \mathcal{X} : f \mapsto \int_{-\infty}^{x} f(y)\,\mathrm{d}y,$$

and its formal inverse $\alpha^+ : f \mapsto f'$, the *creation operator.*

Lemma 3.26. *The annihilation operator α^- has the following properties:*

(i) *For any $k \in \mathbb{N}^*$ there holds $\alpha^- \in \mathscr{B}(\mathcal{X}_k, \mathcal{X}_{k-1})$.*
(ii) *In $\mathcal{X}_1$ the operators Θ and α^- commute.*
(iii) *Let $f \in \mathcal{X}_1$, $\zeta \in \mathbb{C}$ such that $(\mathcal{L} + \Theta)f = \zeta f$. Then*

$$(\mathcal{L} + \Theta)(\alpha^- f) = (\zeta + 1)(\alpha^- f).$$

[1] One of the best-known applications of ladder operators occurs in the spectral analysis of the quantum harmonic oscillator, see e.g. [Hel02]. There, the ladder operators are maps between neighboring energy levels.

Proof. First we show (i). The boundedness of α^- with respect to the $\mathcal{X}$-norm follows immediately from the Poincaré inequality (3.7). The property $\alpha^-\colon \mathcal{X}_k \to \mathcal{X}_{k-1}$ can be verified as follows. For $f \in \mathcal{X}_1 \cap C_0^\infty(\mathbb{R})$ we find after integration by parts, for all $n \in \mathbb{N}$:

$$\int_{\mathbb{R}} x^n \alpha^- f(x)\,\mathrm{d}x = -\frac{1}{n+1}\int_{\mathbb{R}} x^{n+1} f(x)\,\mathrm{d}x.$$

By using the explicit representation (3.11) of the $\mathcal{X}_k$ this proves that $\alpha^-\colon C_0^\infty(\mathbb{R}) \cap \mathcal{X}_k \to \mathcal{X}_{k-1}$ for all $k \in \mathbb{N}^*$. Since α^- is bounded and $C_0^\infty(\mathbb{R})$ is dense in $\mathcal{X}$, we conclude $\alpha^-\colon \mathcal{X}_k \to \mathcal{X}_{k-1}$.

Property (ii) holds true since Θ is a convolution. For Result (iii) one applies α^- to the equation $(\mathcal{L}+\Theta)f = \zeta f$, and uses the identity $\alpha^-(\mathcal{L}f) = \mathcal{L}(\alpha^- f) - \alpha^- f$ and Property (ii). □

By using the annihilation operator, we are able to prove:

Proposition 3.27. *We have the following spectral properties of $\mathcal{L}+\Theta$ in $\mathcal{X}$:*

(i) $\sigma(\mathcal{L}+\Theta) = -\mathbb{N}$.
(ii) *For each $k \in \mathbb{N}$ the (geometric) eigenspace* $\ker(\mathcal{L}+\Theta+k)$ *is one-dimensional.*
(iii) *For every $k \in \mathbb{N}$ an eigenfunction to the eigenvalue $-k$ is given by f_k, where*

$$f_k = (\alpha^+)^k f_0 = f_0^{(k)}, \quad \text{and} \quad f_0(\xi) = \mathcal{F}^{-1}_{\xi\to x}\Big[\exp\Big(-\frac{\xi^2}{2} + \int_0^1 \frac{\hat\vartheta(\xi s)}{s}\,\mathrm{d}s\Big)\Big]. \tag{3.17}$$

In particular, f_0 is the unique stationary solution with unit mass of the perturbed Fokker-Planck equation (3.1).

Proof. In order to show (i) we first prove that $\bigcap_{k\in\mathbb{N}} \mathcal{X}_k = \{0\}$. According to (3.12) there holds

$$\bigcap_{k\in\mathbb{N}} \mathcal{X}_k = \Big\{ f \in \mathcal{X} : \hat f^{(k)}(0) = 0,\ k \in \mathbb{N}\Big\}.$$

But for $f \in \mathcal{X}$, $\hat f$ is analytic in $\Omega_{\beta/2}$, and the only analytic function with a zero of infinite order is the zero function, which proves the statement.

Thus, for any eigenfunction f of $\mathcal{L}$ to an eigenvalue $\zeta \in \mathbb{C}$, there exists a unique $k \in \mathbb{N}$ such that $f \in \mathcal{X}_k \backslash \mathcal{X}_{k+1}$, which is the minimal $k \in \mathbb{N}$ with the property $\Pi_{\mathcal{L},k} f \neq 0$, see Proposition 3.12. Applying this projection to the eigenvalue equation yields

$$\Pi_{\mathcal{L},k}(\mathcal{L}+\Theta)f = -k\Pi_{\mathcal{L},k} f = \zeta \Pi_{\mathcal{L},k} f,$$

where we used $\Theta f \in \mathcal{X}_{k+1}$ (cf. Lemma 3.22). Hence, the eigenvalue corresponding to f satisfies $\zeta = -k$. Thus $\sigma(\mathcal{L}+\Theta) \subseteq -\mathbb{N}$. If now f_k is an eigenfunction with the eigenvalue $-k$, we can apply k times the continuous operator α^- to f_k, and create eigenfunctions to all eigenvalues $\{-k+1, \dots, 0\}$. So either $\sigma(\mathcal{L}+\Theta) = -\mathbb{N}$ or $\sigma(\mathcal{L}+\Theta) = \{-k_0, \dots, 0\}$ for some $k_0 \in \mathbb{N}$ (note that $\sigma(\mathcal{L}+\Theta) \neq \emptyset$ since the resolvent of $\mathcal{L}+\Theta$ is compact). But the latter scenario is actually not possible, because then the operator $(\mathcal{L}+\Theta)|_{\mathcal{X}_{k_0+1}}$ would have empty spectrum in $\mathcal{X}_{k_0+1}$, which contradicts the fact that it still has a compact resolvent in $\mathcal{X}_{k_0+1}$.

In order to verify (ii) we recall from the first part of the proof that if f is an eigenfunction of $\mathcal{L}+\Theta$ to the eigenvalue $-k$, then $k = \operatorname{argmin}\{\Pi_{\mathcal{L},j} f \neq 0 : j \in \mathbb{N}\}$. In particular,

$$\Pi_{\mathcal{L},k} f \neq 0 \tag{3.18}$$

for such an eigenfunction. Assume that $\dim\ker(\mathcal{L}+\Theta+k)>1$ for some $k\in\mathbb{N}$. Then we may choose two linearly independent eigenfunctions to the eigenvalue $-k$. Since $\dim\operatorname{ran}\Pi_{\mathcal{L},k}=1$ (see Theorem 3.18 (iv)), we can find a linear combination of these two eigenfunctions, yielding an eigenfunction f which satisfies $\Pi_{\mathcal{L},k}f=0$. But this contradicts (3.18) and hence $\dim\ker(\mathcal{L}+\Theta+k)=1$.

For Result (iii) we consider the Fourier transform of the eigenvalue equation $(\mathcal{L}+\Theta)f_k=-kf_k$ for $k\in\mathbb{N}$. This yields the following ordinary differential equation for $\hat{f}_k$:

$$\xi\hat{f}_k'(\xi)=\big(\hat{\vartheta}(\xi)+k-\xi^2\big)\hat{f}_k(\xi).$$

Its general solution reads

$$\hat{f}_k(\xi)=c_k\xi^k q(\xi),\quad \text{with}\quad q(\xi):=\exp\Big(-\frac{\xi^2}{2}+\int_0^1\frac{\hat{\vartheta}(\xi s)}{s}\,\mathrm{d}s\Big),$$

for all $k\in\mathbb{N}$, with $c_k\in\mathbb{C}$. We may now fix $c_k:=\mathrm{i}^k$, which completes the proof. $\square$

Remark 3.28. According to the results of Proposition 3.12 (ii) we may formally write Θ and $\mathcal{L}$ as infinite-dimensional matrices with respect to the eigenfunctions $\{\mu_k\}_{k\in\mathbb{N}}$. Because of the property $\Theta\colon\mathcal{X}_k\to\mathcal{X}_{k+1}$ shown in Lemma 3.22 this representation of Θ is *strictly lower triangular.* Furthermore, because of Theorem 3.18 (iii), $\mathcal{L}$ is formally diagonal. And according to Proposition B.10 $\sigma(\mathcal{L})=\sigma(\mathcal{L}+\Theta)$. This situation resembles the finite-dimensional case, in which adding a strictly triangular matrix does not change the spectrum of a diagonal matrix.

Lemma 3.29. *The spectral projection $\mathcal{P}_k$ of $\mathcal{L}+\Theta$ corresponding to the eigenvalue $-k\in-\mathbb{N}$ fulfills*

$$\operatorname{ran}\mathcal{P}_k=\operatorname{span}\{f_k\},\quad \ker\mathcal{P}_k=\mathcal{X}_{k+1}\oplus\operatorname{span}\{f_{k-1},\dots,f_0\},$$

with the eigenfunctions $f_k,\dots,f_0$ given in (3.17). Therefore, all singularities of the resolvent are of order one, and for all $k\in\mathbb{N}$ there holds $M(\mathcal{L}+\Theta+k)=\ker(\mathcal{L}+\Theta+k)$.

Proof. We define the set $\mathcal{K}_k:=\mathcal{X}_{k+1}\oplus\operatorname{span}\{f_{k-1},\dots,f_0\}$. Due to Proposition 3.12 and Corollary 3.23 it is immediate that $\mathcal{L}+\Theta$ is decomposed according to $\mathcal{K}_k\oplus\operatorname{span}\{f_k\}$. Hence, for any $j\in\mathbb{N}^*$ we have $(k+\mathcal{L}+\Theta)^j\colon\mathcal{K}_k\to\mathcal{K}_k$. Therefore, $\mathcal{K}_k$ does not contain generalized eigenfunctions corresponding to the eigenvalues $\{0,\dots,-k+1\}$. First, this implies $M(\mathcal{L}+\Theta+k)=\ker(\mathcal{L}+\Theta+k)=\operatorname{span}\{f_k\}$, for every $k\in\mathbb{N}$. Second, since $\sigma((\mathcal{L}+\Theta)|_{\mathcal{K}_k})$ still consists only of eigenvalues, $\sigma((\mathcal{L}+\Theta)|_{\mathcal{K}_k})=-\mathbb{N}\setminus\{-k\}$. So we can apply Lemma B.13 from the Appendix, with $Y=\mathcal{K}_k$ and $\zeta_0=-k$ and $N=0$, which yields the desired properties of the corresponding spectral projection.

Since $\dim\operatorname{ran}\mathcal{P}_k=1$ and $M(\mathcal{L}+\Theta+k)=\ker(\mathcal{L}+\Theta+k)$, the singularity of $R_{\mathcal{L}+\Theta}(\zeta)$ at $\zeta=-k$ is a pole of order one, see Proposition B.10 (iv)-(v). $\square$

Having explicitly determined the spectrum of the perturbed Fokker-Planck operator, we now turn to the generated semigroup and the corresponding decay rates. We start with the fact that $\mathcal{L}+\Theta$ generates a C_0-semigroup:

Proposition 3.30. *For each $k\in\mathbb{N}$ the operator $(\mathcal{L}+\Theta)|_{\mathcal{X}_k}$ is the infinitesimal generator of a C_0-semigroup on $\mathcal{X}_k$. The semigroup on $\mathcal{X}$ preserves mass, i.e.*

$$\int_{\mathbb{R}}f(x)\,\mathrm{d}x=\int_{\mathbb{R}}[\mathrm{e}^{t(\mathcal{L}+\Theta)}f](x)\,\mathrm{d}x,\quad \forall t\geq 0,\ f\in\mathcal{X}.$$

Proof. According to Proposition 3.17 $\mathcal{L}|_{\mathcal{X}_k}$ generates the semigroup $(\mathrm{e}^{t\mathcal{L}}|_{\mathcal{X}_k})_{t\geq 0}$ on $\mathcal{X}_k$ for every $k \in \mathbb{N}$. Because of Lemma 3.22 and Lemma 3.21 we have $\Theta|_{\mathcal{X}_k} \in \mathscr{B}(\mathcal{X}_k)$. Now a bounded perturbation of the infinitesimal generator of a C_0-semigroup is again infinitesimal generator, see Theorem III.1.3 in [EN00], and so the first result follows.

To show the conservation of mass we use the decomposition of $(\mathrm{e}^{t(\mathcal{L}+\Theta)})_{t\geq 0}$ according to $\mathcal{P}_0$ (or $\mathcal{X} = \mathcal{X}_1 \oplus \operatorname{span}\{f_0\}$). The space $\mathcal{X}_1$ consists of all functions with zero mean, see (3.11), so the part $\mathcal{P}_0 f$ alone determines the mass of any $f \in \mathcal{X}$. Clearly, $\mathcal{P}_0$ and $(\mathrm{e}^{t(\mathcal{L}+\Theta)})_{t\geq 0}$ commute. Furthermore we have $\mathcal{P}_0 f \in \ker(\mathcal{L}+\Theta)$, and hence $\mathrm{e}^{t(\mathcal{L}+\Theta)}\mathcal{P}_0 f = \mathcal{P}_0 f$ for all $t \geq 0$. So we obtain $\mathcal{P}_0 \mathrm{e}^{t(\mathcal{L}+\Theta)} f = \mathcal{P}_0 f$ for all $f \in \mathcal{X}$, $t \geq 0$, i.e. the semigroup preserves mass. □

Next we investigate the decay rate of $(\mathrm{e}^{t(\mathcal{L}+\Theta)})_{t\geq 0}$ on the subspaces $\mathcal{X}_k$. To this end we define:

$$\hat{\psi}(\xi) := \exp\Big(\int_0^1 \frac{\hat{\vartheta}(\xi s)}{s}\,\mathrm{d}s\Big), \quad \xi \in \Omega_{\beta/2}, \tag{3.19}$$

which is analytic in $\Omega_{\beta/2}$ according to Remark 3.19. With this we introduce the operator $\Psi\colon f \mapsto f * \psi$. As we shall see below, Ψ provides a similarity transformation between $\mathcal{L}$ and $\mathcal{L}+\Theta$.

Lemma 3.31. *The map Ψ has the properties:*

(i) *For every $k \in \mathbb{N}$, the operator $\Psi\colon \mathcal{X}_k \to \mathcal{X}_k$ is a bijection, and the inverse is $\Psi^{-1}\colon f \mapsto f * \mathcal{F}^{-1}[1/\hat{\psi}]$.*

(ii) *$\Psi, \Psi^{-1} \in \mathscr{B}(\mathcal{X})$.*

Proof. We define $\bar{\Psi}\colon f \mapsto f * \mathcal{F}^{-1}[1/\hat{\psi}]$. Because of condition **(C)(iii)** both $\hat{\psi}$ and $1/\hat{\psi}$ are analytic in $\Omega_{\beta/2}$ and lie in $L^\infty(\Omega_{\beta/2})$. Hence, analogously to Lemma 3.20 we find that $\Psi, \bar{\Psi}\colon \mathcal{X} \to \mathcal{X}$, and analogously to the proof of Lemma 3.21 we obtain the boundedness of Ψ and $\bar{\Psi}$.

Let now $f \in \mathcal{X}_k$ for some $k \in \mathbb{N}$. Then $\hat{f}(\xi)$ has a zero of order greater than or equal to k at $\xi = 0$, cf. Proposition 3.14. Since $\hat{\psi}$ and $1/\hat{\psi}$ are analytic in $\Omega_{\beta/2}$, the zero at $\xi = 0$ of $\mathcal{F}_{x\to\xi}\Psi f = \hat{f}(\xi)\hat{\psi}(\xi)$ and of $\mathcal{F}_{x\to\xi}\bar{\Psi} f = \hat{f}(\xi)/\hat{\psi}(\xi)$ is of the same order as of $\hat{f}$. So $\Psi, \bar{\Psi}\colon \mathcal{X}_k \to \mathcal{X}_k$ for all $k \in \mathbb{N}^*$.

It remains to show that indeed $\bar{\Psi} = \Psi^{-1}$. By applying the Fourier transform, we see that $\Psi \circ \bar{\Psi} f = \bar{\Psi} \circ \Psi f = f$ for all $f \in \mathcal{X}$, i.e. $\bar{\Psi} = \Psi^{-1}$, and $\Psi, \Psi^{-1}\colon \mathcal{X}_k \to \mathcal{X}_k$ are bijections for all $k \in \mathbb{N}$. □

The map Ψ plays a crucial role in the analysis of the perturbed Fokker-Planck operator $\mathcal{L}+\Theta$, because it relates the eigenspaces of $\mathcal{L}$ to the eigenspaces of $\mathcal{L}+\Theta$ according to (3.17):

$$f_k = \Psi \mu_k, \quad k \in \mathbb{N}. \tag{3.20}$$

By using this property of Ψ we obtain the following result:

Proposition 3.32. *Let $k \in \mathbb{N}$ and $\zeta \in \mathbb{C}\backslash\{-k, -k-1, \ldots\}$. Then there holds*

$$R_{\mathcal{L}+\Theta}(\zeta)|_{\mathcal{X}_k} = \Psi \circ R_{\mathcal{L}}(\zeta) \circ \Psi^{-1}|_{\mathcal{X}_k}. \tag{3.21}$$

In particular there exists a constant $\tilde{C}_k > 0$ such that

$$\big\|\big(R_{\mathcal{L}+\Theta}(\zeta)|_{\mathcal{X}_k}\big)^n\big\|_{\mathscr{B}(\mathcal{X}_k)} \leq \frac{\tilde{C}_k}{(\operatorname{Re}\zeta + k)^n}, \quad \operatorname{Re}\zeta > -k,\ n \in \mathbb{N}^*. \tag{3.22}$$

Proof. We fix $k \in \mathbb{N}$. Then, for all $j \geq k$ and $\zeta \in \mathbb{C}\backslash\{-k, -k-1, \dots\}$ there holds, because of (3.20):

$$R_{\mathcal{L}}(\zeta)\mu_j = \frac{\mu_j}{\zeta + j} = \Psi^{-1} \circ R_{\mathcal{L}+\Theta}(\zeta) f_j = \Psi^{-1} \circ R_{\mathcal{L}+\Theta}(\zeta) \circ \Psi\mu_j.$$

So we have $R_{\mathcal{L}}(\zeta) = \Psi^{-1} \circ R_{\mathcal{L}+\Theta}(\zeta) \circ \Psi$ in the space $\operatorname{span}\{\mu_j : j \geq k\} \subset X_k$, which is dense in $\mathcal{X}_k$. Then this identity extends to $\mathcal{X}_k$ because of the continuity of the occurring operators.

In order to prove the resolvent estimate (3.22) we use

$$\big(R_{\mathcal{L}+\Theta}(\zeta)|_{\mathcal{X}_k}\big)^n = R_{\mathcal{L}+\Theta}(\zeta)^n|_{\mathcal{X}_k} = \Psi \circ R_{\mathcal{L}}(\zeta)^n \circ \Psi^{-1}|_{\mathcal{X}_k},$$

which follows from (3.21) and Lemma 3.31 (i). Because of $\Psi, \Psi^{-1} \in \mathscr{B}(\mathcal{X}_k)$ we conclude

$$\big\|\big(R_{\mathcal{L}+\Theta}(\zeta)|_{\mathcal{X}_k}\big)^n\big\|_{\mathscr{B}(\mathcal{X}_k)} \leq \|\Psi|_{\mathcal{X}_k}\|_{\mathscr{B}(\mathcal{X}_k)}\big\|\big(R_{\mathcal{L}}(\zeta)|_{\mathcal{X}_k}\big)^n\big\|_{\mathscr{B}(\mathcal{X}_k)}\|\Psi^{-1}|_{\mathcal{X}_k}\|_{\mathscr{B}(\mathcal{X}_k)}. \tag{3.23}$$

We apply the Hille-Yosida Theorem to the estimate of $(\mathrm{e}^{t\mathcal{L}})_{t\geq 0}$ in Theorem 3.18 (v), which shows the following estimate:

$$\big\|\big(R_{\mathcal{L}}(\zeta)|_{\mathcal{X}_k}\big)^n\big\|_{\mathscr{B}(\mathcal{X}_k)} \leq \frac{C_k}{(\operatorname{Re}\zeta + k)^n}, \quad \operatorname{Re}\zeta > -k,\, n \in \mathbb{N}^*.$$

We insert this estimate in (3.23), which shows (3.22), with the constant $\tilde{C}_k = C_k\|\Psi|_{\mathcal{X}_k}\|_{\mathscr{B}(\mathcal{X}_k)} \cdot \|\Psi^{-1}|_{\mathcal{X}_k}\|_{\mathscr{B}(\mathcal{X}_k)}$. ☐

Corollary 3.33. *Let $k \in \mathbb{N}$. Then there exists a constant $\tilde{C}_k > 0$ such that*

$$\big\|\mathrm{e}^{t(\mathcal{L}+\Theta)}|_{\mathcal{X}_k}\big\|_{\mathscr{B}(\mathcal{X}_k)} \leq \tilde{C}_k \mathrm{e}^{-kt}, \quad \forall t \geq 0. \tag{3.24}$$

Proof. The result immediately follows from (3.22) by application of the Hille-Yosida theorem. ☐

Remark 3.34. According to (3.21) the operators $\mathcal{L}$ and $\mathcal{L}+\Theta$ are similar:

$$\mathcal{L} + \Theta = \Psi \circ \mathcal{L} \circ \Psi^{-1}.$$

Now we consider the family of operators $(\mathcal{L}(\tau))_{\tau\in\mathbb{R}} := (\mathcal{L} + \tau\Theta)_{\tau\in\mathbb{R}}$. Clearly, for every $\tau \in \mathbb{R}$ the operators $\mathcal{L}(\tau)$ and $\mathcal{L}(0) = \mathcal{L}$ are similar with the transformation operator $\Psi(\tau)$ defined according to Lemma 3.31 (where we replace ϑ by $\tau\vartheta$ in (3.19)). Therefore, according to [Lax68], there exists a family of operators $(B(\tau))_{\tau\in\mathbb{R}}$ such that $(\mathcal{L}(\tau), B(\tau))$ form a *Lax pair*, i.e. they obey

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{L}(\tau) = [B(\tau), \mathcal{L}(\tau)],$$

where the right hand side denotes the commutator. Since we explicitly know the transformation operator $\Psi(\tau)$ we can compute $B(\tau)$ for all $f \in \mathcal{X}$:

$$Bf := -\Psi(\tau) \circ \frac{\mathrm{d}[\Psi(\tau)]^{-1}}{\mathrm{d}\tau} f = \mathcal{F}^{-1}\Big[\int_0^1 \frac{\hat{\vartheta}(\xi s)}{s}\,\mathrm{d}s\,\hat{f}\Big],$$

which is independent of τ.

Remark 3.35. The above result implies the exponential convergence of any solution of (1.1) towards the (appropriately scaled) stationary state. To see this we choose any $f \in \mathcal{X}$. Then there exists a unique constant $m \in \mathbb{C}$ (the "mass" of f) such that $\mathcal{P}_0 f = mf_0$. So $f - mf_0 = (1 - \mathcal{P}_0)f \in \mathcal{X}_1$, cf. Lemma 3.29, which implies

$e^{t(\mathcal{L}+\Theta)}f - mf_0 = e^{t(\mathcal{L}+\Theta)}(f - mf_0) \in \mathcal{X}_1$ for all $t \geq 0$, because of Proposition 3.30. With (3.24) and $k = 1$ this implies

$$\|e^{t(\mathcal{L}+\Theta)}f - mf_0\|_\omega \leq \tilde{C}_1\|f - mf_0\|_\omega e^{-t}, \quad t \geq 0.$$

Remark 3.36. In the one dimensional case we can explicitly compute the Fourier transform of $R_{\mathcal{L}+\Theta}(\zeta)g$, see Proposition A.10: For any $k \in \mathbb{N}$, $\operatorname{Re}\zeta > -k$, and $g \in \mathcal{X}_k$, the unique solution $f \in \mathcal{X}_k$ of $(\zeta - \mathcal{L} - \Theta)f = g$ satisfies

$$\hat{f}(\xi) = \mathcal{F}_{x\to\xi}[R_{\mathcal{L}+\Theta}(\zeta)g] = \hat{f}_0(\xi)\int_0^1 \frac{\hat{g}(s\xi)}{\hat{f}_0(s\xi)} s^{\zeta-1}\,ds, \quad \xi \in \Omega_{\beta/2},$$

where $s^\zeta = e^{\zeta\log s}$ and $\log(\cdot)$ is the natural logarithm on $\mathbb{R}^+$. For the proper definition of complex powers see Chapter 2.

Now we summarize our results in the final theorem:

Theorem 3.37. *Let $\mathcal{X} = L^2(\omega)$, where $\omega(x) = \cosh\beta x$, for some $\beta > 0$, and let Θ fulfill the condition* **(C)** *for this $\beta > 0$. Then the perturbed operator $\mathcal{L} + \Theta$ has the following properties in $\mathcal{X}$:*

(i) *It has compact resolvent, and $\sigma(\mathcal{L} + \Theta) = \sigma_p(\mathcal{L} + \Theta) = -\mathbb{N}$.*

(ii) *There holds $M(\mathcal{L} + \Theta + k) = \ker(\mathcal{L} + \Theta + k) = \operatorname{span}\{f_k\}$, where f_k is the eigenfunction to the eigenvalue $-k$ given by (3.17). The eigenfunctions are related by $f_k = f_0^{(k)}$.*

(iii) *The spectral projection $\mathcal{P}_k$ corresponding to the eigenvalue $-k \in -\mathbb{N}^*$ fulfills*

$$\operatorname{ran}\mathcal{P}_k = \operatorname{span}\{f_k\}, \quad \ker\mathcal{P}_k = \mathcal{X}_{k+1} \oplus \operatorname{span}\{f_{k-1}, \dots, f_0\},$$

where the $(\mathcal{L}+\Theta)$-invariant spaces $\mathcal{X}_k$ are explicitly given in (3.11). Moreover, $\operatorname{ran}\mathcal{P}_0 = \operatorname{span}\{f_0\}$ and $\ker\mathcal{P}_0 = \mathcal{X}_1$.

(iv) *For every $k \in \mathbb{N}$, the operator $(\mathcal{L} + \Theta)|_{\mathcal{X}_k}$ generates a C_0-semigroup on $\mathcal{X}_k$, denoted by $(e^{t(\mathcal{L}+\Theta)}|_{\mathcal{X}_k})_{t\geq 0}$, which satisfies the estimate*

$$\|e^{t(\mathcal{L}+\Theta)}|_{\mathcal{X}_k}\|_{\mathscr{B}(\mathcal{X}_k)} \leq \tilde{C}_k e^{-kt}, \quad \forall t \geq 0,$$

where the constant $\tilde{C}_k > 0$ is independent of t.

Remark 3.38. Apparently, the particular choice of $\beta > 0$ has no influence on the above results, except possibly for the constants $\tilde{C}_k$. In practice, the constant β may therefore be chosen arbitrarily small, such that Θ satisfies **(C)** for this β.

CHAPTER 4

The perturbed Fokker-Planck equation in higher dimensions

4.1. Introduction

In this chapter we investigate the full d-dimensional perturbed Fokker-Planck equation, introduced in the beginning of the Introduction:

$$f_t = \nabla \cdot (\mathbf{D}\nabla f + \mathbf{C}\mathbf{x} f) + \Theta f. \tag{1.1}$$

The assumptions on the perturbation Θ are specified in the beginning of Section 4.3. The matrices $\mathbf{C}, \mathbf{D} \in \mathbb{R}^{d\times d}$ are symmetric and positive definite. Actually we may assume w.l.o.g. that $\mathbf{D} = \mathbf{I}$ and $\mathbf{C}$ is diagonal. This can be seen from the following considerations.

Since $\mathbf{D}$ is a symmetric and positive definite matrix, we may introduce the new variables $\mathbf{y} := \sqrt{\mathbf{D}}^{-1}\mathbf{x}$, and $g(\mathbf{y}) := f(\mathbf{x})$. The derivatives transform according to $\nabla_{\mathbf{x}} = \sqrt{\mathbf{D}}^{-1}\nabla_{\mathbf{y}}$. Hence we find that the Fokker-Planck operator transforms to

$$\nabla_{\mathbf{x}} \cdot (\mathbf{D}\nabla_{\mathbf{x}} f + \mathbf{C}\mathbf{x} f) = \nabla_{\mathbf{y}} \cdot (\nabla_{\mathbf{y}} g + \tilde{\mathbf{C}}\mathbf{y} g), \tag{4.1}$$

with $\tilde{\mathbf{C}} := \sqrt{\mathbf{D}}^{-1}\mathbf{C}\sqrt{\mathbf{D}}$. Since $\tilde{\mathbf{C}}$ is still symmetric and positive definite, there exists a unitary matrix $\mathbf{U} \in \mathbb{R}^{d\times d}$ such that $\mathbf{U}^\top\tilde{\mathbf{C}}\mathbf{U}$ is diagonal. We introduce the variables $\mathbf{z} = \mathbf{U}^\top\mathbf{y}$ and the function $h(\mathbf{z}) = g(\mathbf{y})$. There holds $\nabla_{\mathbf{y}} = \mathbf{U}\nabla_{\mathbf{z}}$. Analogously to the previous coordinate change we find that the Fokker-Planck operator in (4.1) further transforms to:

$$\nabla_{\mathbf{y}} \cdot (\nabla_{\mathbf{y}} g + \tilde{\mathbf{C}}\mathbf{y} g) = \Delta h(\mathbf{z}) + \mathbf{z}^\top(\mathbf{U}^\top\tilde{\mathbf{C}}\mathbf{U})\nabla_{\mathbf{z}} h(\mathbf{z}) + \mathrm{tr}(\mathbf{U}^\top\tilde{\mathbf{C}}\mathbf{U}) = \nabla_{\mathbf{z}} \cdot (\nabla_{\mathbf{z}} g + \mathbf{U}^\top\tilde{\mathbf{C}}\mathbf{U} g),$$

where we have used that $\mathbf{U}$ is unitary. Note that $\mathbf{U}^\top\tilde{\mathbf{C}}\mathbf{U}$ is diagonal. The right hand side is a Fokker Planck operator in the form of (1.1) with $\mathbf{D} = \mathbf{I}$ and $\mathbf{C}$ diagonal. Hence, w.l.o.g. we shall always assume in the following that our coordinates $\mathbf{x} \in \mathbb{R}^d$ are such that $\mathbf{D} = \mathbf{I}$ and $\mathbf{C}$ is diagonal. We write $\mathbf{C} = \mathrm{diag}(c_1, c_2, \ldots)$, and introduce $\mathbf{c} := [c_1, \ldots, c_d]$ where we fix $0 < c_1 \le c_2 \le \ldots \le c_d$.

The corresponding analysis presented in this chapter is similar to the one of the one-dimensional problem discussed in Chapter 3. However, some of the proofs need different approaches, since for them there is no straightforward generalization to higher dimensions. Up to small modifications, the contents of this chapter has been published in [AAS15].

Concerning notation and general definitions we refer to Chapter 2.

4.2. Analysis of the Fokker-Planck operator

We start by discussing properties of the unperturbed Fokker-Planck equation

$$f_t = \nabla \cdot (\nabla f + \mathbf{C}\mathbf{x} f) = \Delta f + \mathbf{x}^\top \mathbf{C} \nabla f + \operatorname{tr} \mathbf{C} f, \tag{4.2}$$

or the unperturbed Fokker-Planck operator $L := \Delta + \mathbf{x}^\top \mathbf{C}\nabla + \operatorname{tr}\mathbf{C}$, respectively. $\mathbf{C}$ is a positive diagonal matrix. One can check that

$$\mu := \exp(-\tfrac{1}{2}\mathbf{x}^\top \mathbf{C}\mathbf{x})$$

is a steady state, i.e. is a zero eigenfunction of L, at least on a formal level. With this we can alternatively write

$$Lf = \nabla \cdot \Big(\mu \nabla \Big(\frac{f}{\mu}\Big)\Big). \tag{4.3}$$

The next step is to properly define the operator L in an appropriate Hilbert space. The natural (self-adjoint) setting for L is the space $X := L^2(\mu^{-1})$. We use the notation $\langle \cdot, \cdot \rangle_X$ for the inner product. The operator $L|_{C_0^\infty(\mathbb{R}^d)}$ is symmetric and dissipative in X, which is immediately seen from the right hand side representation of L in (4.3). In the following it is helpful to notice that the isometric transformation $\iota\colon X \to L^2(\mathbb{R}^d)\colon f \mapsto f/\sqrt{\mu} \equiv z$ transforms $L|_{C_0^\infty(\mathbb{R}^d)}$ in X to

$$A|_{C_0^\infty(\mathbb{R}^d)} := \big[\Delta - \big(-\tfrac{1}{2}\operatorname{tr}\mathbf{C} + \tfrac{1}{4}|\mathbf{C}\mathbf{x}|_2^2\big)\big]\big|_{C_0^\infty(\mathbb{R}^d)} \equiv [\Delta - V(\mathbf{x})]|_{C_0^\infty(\mathbb{R}^d)} \tag{4.4}$$

in the unweighted space $L^2(\mathbb{R}^d)$, see also Section 2.1 in [AMTU01]. So $L|_{C_0^\infty(\mathbb{R}^d)}$ in X and $A|_{C_0^\infty(\mathbb{R}^d)}$ in $L^2(\mathbb{R}^d)$ are isometrically equivalent, i.e. $L|_{C_0^\infty(\mathbb{R}^d)} = \iota^{-1} \circ A \circ \iota|_{C_0^\infty(\mathbb{R}^d)}$, and most properties of A carry over to L. In particular $A|_{C_0^\infty(\mathbb{R}^d)}$ is symmetric, and since $V(\mathbf{x}) \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and bounded from below, Theorem X.29 from [RS75] implies that $A|_{C_0^\infty(\mathbb{R}^d)}$ is essentially self-adjoint in $L^2(\mathbb{R}^d)$. Thus the closures $L := \operatorname{cl}_X L|_{C_0^\infty(\mathbb{R}^d)}$ and $A := \operatorname{cl}_{L^2(\mathbb{R}^d)} A|_{C_0^\infty(\mathbb{R}^d)}$ are self-adjoint, closed operators in X and $L^2(\mathbb{R}^d)$, respectively. The appropriate domain $D(L)$ is also determined by the closure, but it is not needed explicitly here. According to Theorem 1.4.5 in [Paz83] $L = L^*$ is dissipative in X as well, and then Corollary 1.4.4 in [Paz83] proves that L generates a C_0-semigroup of contractions in X, denoted by $(\mathrm{e}^{tL})_{t\geq 0}$. As a consequence $\sigma(L) \subset (-\infty, 0]$.

Finally we prove the compactness of the resolvent $(\lambda - L)^{-1}$ in X for $\lambda \in \rho(L)$. It is equivalent to the compactness of $(\lambda - A)^{-1}$ in $L^2(\mathbb{R}^d)$, which we demonstrate in the following. According to Theorem XIII.64 in [RS78] it is sufficient to prove the compactness for one $\lambda \in \rho(L)$, and this already implies the compactness for all $\lambda \in \rho(L)$. So we choose $\lambda := 1 + \operatorname{tr}\mathbf{C}$ and consider $(\lambda - A)f = g$ for any $f, g \in C_0^\infty(\mathbb{R}^d)$. We multiply by the complex conjugate $\overline{f}$ and integrate (where V was defined in (4.4)):

$$\int_{\mathbb{R}^d} (\lambda + V(\mathbf{x}))|f|^2 \,\mathrm{d}\mathbf{x} + \|\nabla f\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} g\overline{f}\,\mathrm{d}\mathbf{x}$$

$$\Rightarrow \int_{\mathbb{R}^d} \big(1 + \tfrac{1}{2}\operatorname{tr}\mathbf{C} + \tfrac{1}{4}|\mathbf{C}\mathbf{x}|_2^2\big)|f|^2\,\mathrm{d}\mathbf{x} + \|\nabla f\|^2_{L^2(\mathbb{R}^d)} \leq \tfrac{1}{2}\big(\|f\|^2_{L^2(\mathbb{R}^d)} + \|g\|^2_{L^2(\mathbb{R}^d)}\big)$$

$$\Leftrightarrow \int_{\mathbb{R}^d} \big(2 + \operatorname{tr}\mathbf{C} + \tfrac{1}{2}|\mathbf{C}\mathbf{x}|_2^2\big)|f|^2\,\mathrm{d}\mathbf{x} + 2\|\nabla f\|^2_{L^2(\mathbb{R}^d)} \leq \|g\|^2_{L^2(\mathbb{R}^d)}.$$

So $(\lambda - A)^{-1}\colon L^2(\mathbb{R}^d) \to H^1(1 + |\mathbf{C}\mathbf{x}|_2^2, 1)$ is a bounded operator. Now we can apply Lemma B.16 in the Appendix, which proves the compact embedding

$H^1(1+|\mathbf{C}\mathbf{x}|_2^2, 1) \hookrightarrow\hookrightarrow L^2(\mathbb{R}^d)$. This implies the compactness of the resolvent of A in $L^2(\mathbb{R}^d)$. As a consequence the resolvent of L is compact in X, because of the isometrical equivalence of L and A. Thus, according to Theorem III.6.29 in [Kat66], $\sigma(L)$ consists entirely of eigenvalues, and the corresponding eigenspaces are finite-dimensional. According to Theorem V.2.10 in [Kat66] the eigenspaces form a complete orthogonal family in X, see also Theorem XIII.64 in [RS78]. Additionally, according to Proposition B.11 in the Appendix the self-adjointness of L implies that the algebraic eigenspaces coincide with the geometric eigenspaces.

Next we identify all eigenfunctions of L. To this end we need to introduce the distributional Fokker-Planck operator

$$\mathfrak{L}\colon \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)\colon f \mapsto \Delta f + \mathbf{x}^\top \mathbf{C}\nabla f + \operatorname{tr}\mathbf{C} f.$$

In particular, $\mathfrak{L}$ is then a well-defined linear map from X into $\mathscr{S}'(\mathbb{R}^d)$, defined on the whole space X.

Proposition 4.1. *There holds $\sigma(L) = \{-\mathbf{c}\mathbf{k} : \mathbf{k} \in \mathbb{N}^d\}$. The zero eigenspace of L is one-dimensional and is spanned by $\mu_{\mathbf{0}}(\mathbf{x}) := \det(\mathbf{C}/(2\pi))^{\frac{1}{2}} \exp(-\frac{1}{2}\mathbf{x}^\top \mathbf{C}\mathbf{x})$. More generally, the eigenspace to $\lambda \in \sigma(L)$ is given by*

$$\ker(\lambda - L) = \operatorname{span}\{\mu_{\mathbf{k}} : -\mathbf{c}\mathbf{k} = \lambda\},$$

where $\mu_{\mathbf{k}} := \nabla^{\mathbf{k}}\mu_{\mathbf{0}}$ for all $\mathbf{k} \in \mathbb{N}^d$.

Proof. For any $j \in \{1,\dots,d\}$ and f sufficiently smooth there holds

$$\partial_j(\mathbf{x}^\top \mathbf{C}\nabla f) - c_j\partial_j f = \mathbf{x}^\top \mathbf{C}\nabla(\partial_j f).$$

Therefore we have at least on a formal level that $\mathfrak{L}f = \lambda f$ implies $\mathfrak{L}(\partial_j f) = (\lambda - c_j)(\partial_j f)$. Applying this argument repeatedly to $\mu_{\mathbf{0}}$ it is clear that for every $\mathbf{k} \in \mathbb{N}^d$ the function $\mu_{\mathbf{k}} = \nabla^{\mathbf{k}}\mu_{\mathbf{0}}$ is a formal eigenfunction of L to the eigenvalue $-\mathbf{c}\mathbf{k}$. In the following we verify that the family $\{\mu_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}^d}$ lies in $D(L)$, and that it already spans all the eigenspaces.

First we note that for every $\mathbf{k} \in \mathbb{N}^d$

$$\mu_{\mathbf{k}}(\mathbf{x}) = \mu_{\mathbf{0}}(\mathbf{x}) \prod_{j=1}^d H_{k_j}^{c_j}(x_j). \tag{4.5}$$

For given $k \in \mathbb{N}$ and $a > 0$ the “rescaled” Hermite polynomial H_k^a of order k is defined by

$$H_k^a(x) = \mu\big(a^{\frac{1}{2}}x\big)^{-1} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \mu\big(a^{\frac{1}{2}}x\big),$$

with $\mu(x) = \exp(-x^2/2)$ being the standard Gaussian. Comparing this with the definition for “standard” Hermite polynomials H_k discussed in (B.1) in the Appendix, we easily find the relation

$$H_k^a(x) = a^{\frac{k}{2}} H_k\big(a^{\frac{1}{2}}x\big).$$

Using this relation, we obtain from the orthogonality property (B.4) for every $a > 0$ and $k \neq j$:

$$\int_{\mathbb{R}} H_k^a(x) H_j^a(x) \mu\big(a^{\frac{1}{2}}x\big)\,\mathrm{d}x = 0.$$

Using this relation and Fubini's Theorem it follows immediately from (4.5) that the family $\{\mu_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}^d}$ is an orthogonal family in X.

Next we prove that the $\mu_{\mathbf{k}}$ indeed lie in $D(L)$. To this end let $\mathbf{k} \in \mathbb{N}^d$, and we introduce a $\varphi \in C_0^\infty(\mathbb{R}^d)$ that fulfills $\varphi(\mathbf{x}) = 1$ for all $|\mathbf{x}|_2 \leq 1$ and $\varphi(\mathbf{x}) = 0$ for $|\mathbf{x}|_2 \geq 2$. For $n \in \mathbb{N}^*$ we then define $\varphi_n(\mathbf{x}) := \varphi(\mathbf{x}/n)$. Then $\varphi_n \mu_{\mathbf{k}} \in C_0^\infty(\mathbb{R}^d)$, for all $n \in \mathbb{N}^*$. Because of the super-exponential decay of $\mu_{\mathbf{k}}(\mathbf{x})$ as $|\mathbf{x}|_2 \to \infty$ it is easily verified that $\varphi_n \mu_{\mathbf{k}} \to \mu_{\mathbf{k}}$ and $L(\varphi_n \mu_{\mathbf{k}}) \to \mathfrak{L}\mu_{\mathbf{k}}$ in X as $n \to \infty$. Since L is defined as the closure of $L|_{C_0^\infty(\mathbb{R}^d)}$ this proves that all $\mu_{\mathbf{k}}$ lie in $D(L)$ and they satisfy $L\mu_{\mathbf{k}} = \mathfrak{L}\mu_{\mathbf{k}}$, and thus they are eigenfunctions of L.

Finally we show that the $\{\mu_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}^d}$ span all eigenfunctions. To this end we show that the orthogonal complement $\{\mu_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\}^\perp$ in X consists just of the zero function. Assume that there exists some non-zero $f \in \{\mu_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\}^\perp$. By induction we can show that the orthogonality of f implies

$$\int_{\mathbb{R}^d} f(\mathbf{x})\mathbf{x}^{\mathbf{k}}\,\mathrm{d}\mathbf{x} = 0, \quad \forall \mathbf{k} \in \mathbb{N}^d,$$

see Lemma 4.3 below for a similar argument. This is equivalent to saying that for all $\mathbf{k} \in \mathbb{N}^d$ there holds $\nabla^{\mathbf{k}}\hat{f}(\mathbf{0}) = 0$, i.e. $\hat{f}$ has a zero of infinite order at $\boldsymbol{\xi} = \mathbf{0}$. As a consequence of the Paley-Wiener theorems, see the Theorems IX.13 and IX.14 in [RS75], the Fourier transform of every $f \in X$ is analytic in $\mathbb{C}^d$. But the zero function is the only function that is analytic in $\mathbb{C}^d$ with a zero of infinite order at the origin, so $f(\mathbf{x}) \equiv 0$. Hence, $\{\mu_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}^d}$ is an orthogonal basis of X, and the set $\operatorname{span}\{\mu_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\}$ has zero orthogonal complement.

Thus, every eigenfunction $f \in X$ has a unique representation

$$f = \sum_{\mathbf{k}\in\mathbb{N}^d} a_{\mathbf{k}}\mu_{\mathbf{k}},$$

with coefficients $a_{\mathbf{k}} \in \mathbb{C}$ for $\mathbf{k} \in \mathbb{N}^d$. Inserting this in the eigenvalue equation $Lf = \lambda f$, with $\lambda \in \mathbb{R}$, yields that $a_{\mathbf{k}}$ is non-zero only if $-\mathbf{ck} = \lambda$. Since all $c_j > 0$, for $j \in \{1, \ldots, d\}$, there always are only finitely many $\mathbf{k} \in \mathbb{N}^d$ for which this is possible. Hence, $f \in \operatorname{span}\{\mu_{\mathbf{k}} : -\mathbf{ck} = \lambda\}$. □

From the previous proof we also get this result:

Corollary 4.2. *The family of eigenfunctions $\{\mu_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}^d}$ is an orthogonal basis of X.*

As a consequence we define the orthogonal projection $\Pi_{L,\mathbf{k}}$ corresponding to $\mu_{\mathbf{k}}$ for every $\mathbf{k} \in \mathbb{N}^d$ by

$$\Pi_{L,\mathbf{k}} := \langle \cdot, \mu_{\mathbf{k}} \rangle_X \frac{\mu_{\mathbf{k}}}{\|\mu_{\mathbf{k}}\|_X^2}.$$

With this, the spectral projection corresponding to an eigenvalue $\lambda = -\mathbf{ck}$ is given by the orthogonal sum

$$\Pi_{L,\lambda} := \sum_{\substack{\mathbf{k}\in\mathbb{N}^d \\ -\mathbf{ck}=\lambda}} \Pi_{L,\mathbf{k}}. \tag{4.6}$$

In order to see that this indeed is the spectral projection corresponding to λ we refer to Proposition B.11 in the Appendix, which confirms that the spectral projections of L must be orthogonal, and the ranges coincide with the geometric eigenspaces.

Finally we introduce a convenient family of subspaces of X. For every $k \in \mathbb{N}$ we define

$$X_k := \operatorname{span}\{\mu_{\mathbf{k}} : |\mathbf{k}|_1 \leq k-1\}^{\perp}. \tag{4.7}$$

Lemma 4.3. *Let $k \in \mathbb{N}$. There holds $f \in X_k$ iff*

$$\int_{\mathbb{R}^d} f(\mathbf{x})\mathbf{x}^{\mathbf{k}}\,\mathrm{d}\mathbf{x} = 0, \quad \forall |\mathbf{k}|_1 \leq k-1. \tag{4.8}$$

Proof. For this we rely on the representation (4.5) of the $\mu_{\mathbf{k}}$. The result is then shown by induction. Clearly, we have $X_0 = X$ and for $k=1$ we obtain

$$X_1 = \mu_{\mathbf{0}}^{\perp} = \Big\{ f \in X : \int_{\mathbb{R}^d} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \Big\}.$$

Let us assume now that (4.7) holds for some $k \in \mathbb{N}^*$. According to (4.5) we have

$$X_{k+1} = \Big\{ f \in X_k : \int_{\mathbb{R}^d} f(\mathbf{x}) \prod_{j=1}^d H_{k_j}^{c_j}(x_j)\,\mathrm{d}\mathbf{x} = 0, \, \forall |\mathbf{k}|_1 = k \Big\}.$$

For $f \in X_k$ we conclude from the induction hypothesis that for $|\mathbf{k}|_1 = k$

$$0 = \int_{\mathbb{R}^d} f(\mathbf{x}) \prod_{j=1}^d H_{k_j}^{c_j}(x_j)\,\mathrm{d}\mathbf{x} = (-\mathbf{c})^{\mathbf{k}} \int_{\mathbb{R}^d} f(\mathbf{x})\mathbf{x}^{\mathbf{k}}\,\mathrm{d}\mathbf{x},$$

where $(-\mathbf{c})^{\mathbf{k}} \neq 0$ is the leading coefficient of the (Hermite) polynomial in the integral. All other parts of the integral vanish because of the induction hypothesis. Since this holds for all $|\mathbf{k}|_1 = k$ this proves the desired condition for $f \in X_{k+1}$. ☐

We conclude with a few remarks about the semigroup generated by L (see page 32 for the existence). Since the eigenfunctions of L, $\{\mu_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}^d}$, form an orthogonal basis we find the spectral decomposition in X

$$\mathrm{e}^{tL} = \sum_{\mathbf{k}\in\mathbb{N}^d} \mathrm{e}^{-\mathbf{ck}t}\Pi_{L,\mathbf{k}} = \sum_{\lambda\in\sigma(L)} \mathrm{e}^{\lambda t}\Pi_{L,\lambda}, \quad \forall t \geq 0.$$

Hence, for the restriction $(\mathrm{e}^{tL}|_{X_k})_{t\geq 0}$ of the semigroup to X_k for $k \in \mathbb{N}$ we find the representation

$$\mathrm{e}^{tL}|_{X_k} = \sum_{\substack{\mathbf{k}\in\mathbb{N}^d \\ |\mathbf{k}|_1 \geq k}} \mathrm{e}^{-\mathbf{ck}t}\Pi_{L,\mathbf{k}}, \quad \forall t \geq 0.$$

For this reason it is clear that $\|\mathrm{e}^{tL}|_{X_k}\|_{\mathscr{B}(X_k)} = \mathrm{e}^{-kc_1 t}$ holds for all $t \geq 0$, since $-c_1 k = \max\{-\mathbf{ck} : |\mathbf{k}|_1 \geq k\}$.

We now summarize the main properties of L in X in the following theorem.

Theorem 4.4 (L in X). *The Fokker-Planck operator L in X has the following properties:*

(i) *The operator $L := \mathrm{cl}_X\, L|_{C_0^\infty(\mathbb{R}^d)}$ is self-adjoint and has a compact resolvent.*
(ii) *The spectrum consists entirely of isolated eigenvalues and is given by*

$$\sigma(L) = \{-\mathbf{ck} : \mathbf{k} \in \mathbb{N}^d\}.$$

(iii) *The zero eigenfunction is* $\mu_{\mathbf{0}}(\mathbf{x}) = \det(\mathbf{C}/(2\pi))^{\frac{1}{2}} \exp(-\frac{1}{2}\mathbf{x}^\top \mathbf{C}\mathbf{x})$*, and for every* $\mathbf{k} \in \mathbb{N}^d$ *the function*

$$\mu_{\mathbf{k}}(\mathbf{x}) = \nabla^{\mathbf{k}} \mu_{\mathbf{0}}(\mathbf{x})$$

is an eigenfunction to the eigenvalue $-\mathbf{ck}$.

(iv) *For every* $\lambda \in \sigma(L)$ *we have*

$$\ker(\lambda - L) = \operatorname{span}\{\mu_{\mathbf{k}} : \lambda = -\mathbf{ck}\}.$$

(v) *The family of eigenfunctions* $\{\mu_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}^d}$ *is an orthogonal basis of* X.

(vi) L *generates a* C_0*-semigroup of contractions in* X*, and on every subspace* X_k*, for* $k \in \mathbb{N}$*, there holds*

$$\|e^{tL}|_{X_k}\|_{\mathscr{B}(X_k)} = \mathrm{e}^{-kc_1 t},$$

where c_1 *is the minimal entry of* $\mathbf{c}$.

For the analysis of the perturbed equation (1.1) the space X is not convenient, see the discussion for the one-dimensional Fokker-Planck equation on page 16. Hence, we investigate $L+\Theta$ in a weighted L^2-space with a weaker weight function ω. A convenient choice is

$$\omega(\mathbf{x}) := \sum_{i=1}^{d} \cosh \beta x_i, \tag{4.9}$$

with $\beta > 0$. The corresponding weighted space is $\mathcal{X} := L^2(\omega)$. The natural norm and the inner product in $\mathcal{X}$ are denoted by $\|\cdot\|_\omega$ and $\langle\cdot,\cdot\rangle_\omega$, respectively. Next we introduce a characterization of functions in $\mathcal{X}$ with the help of the Fourier transform.

Proposition 4.5. *There holds* $f \in \mathcal{X}$ *if and only if its Fourier transform* $\hat{f}$ *possesses an analytic continuation (still denoted by* $\hat{f}$*) to the open set* $\Omega_{\beta/2} := \{\mathbf{z} \in \mathbb{C}^d : |\operatorname{Im}\mathbf{z}|_1 < \beta/2\}$*, with the property*

$$\sup_{\substack{\mathbf{b}\in\mathbb{R}^d\\ |\mathbf{b}|_1<\beta/2}} \|\hat{f}(\cdot + \mathrm{i}\mathbf{b})\|_{L^2(\mathbb{R}^d)} < \infty. \tag{4.10}$$

In this case we have:

(i) *For every* $\mathbf{b} \in \mathbb{R}^d$ *with* $|\mathbf{b}|_1 < \beta/2$ *there holds*

$$\hat{f}(\boldsymbol{\xi} + \mathrm{i}\mathbf{b}) = \mathcal{F}[f(\mathbf{x})\exp(\mathbf{b}\cdot\mathbf{x})](\boldsymbol{\xi}), \quad \forall \boldsymbol{\xi} \in \mathbb{R}^d. \tag{4.11}$$

(ii) *For every* $|\mathbf{b}|_1 = \beta/2$ *we define* $\hat{f}(\boldsymbol{\xi} + \mathrm{i}\mathbf{b}) := \mathcal{F}[f(\mathbf{x})\exp(\mathbf{b}\cdot\mathbf{x})](\boldsymbol{\xi})$*, which lies in* $L^2(\mathbb{R}^d_{\boldsymbol{\xi}})$*. With this there holds* $\mathbf{b} \mapsto \hat{f}(\cdot + \mathrm{i}\mathbf{b}) \in C(\overline{B^1_{\beta/2}(\mathbf{0})}; L^2(\mathbb{R}^d))$.

We prove this proposition by mimicking the proof of Theorem IX.13 in [RS75]. The proof consists of several lemmata, they are collected in the Appendix C.3. Often we use the following norm, which is equivalent to $\|\cdot\|_\omega$:

$$|||f|||_\omega^2 := \sum_{\ell=1}^{d} \left(\left\| \hat{f}\Big(\cdot + \mathrm{i}\frac{\beta}{2}\mathbf{e}_\ell\Big) \right\|^2_{L^2(\mathbb{R}^d)} + \left\| \hat{f}\Big(\cdot - \mathrm{i}\frac{\beta}{2}\mathbf{e}_\ell\Big) \right\|^2_{L^2(\mathbb{R}^d)} \right). \tag{4.12}$$

Lemma 4.6 (Poincaré inequality). *There exists a constant* $C_p > 0$ *such that for every* $f \in H^1(\omega,\omega)$ *there holds*

$$\|f\|_\omega \le C_p \|\nabla f\|_\omega. \tag{4.13}$$

Proof. For this we use the norm $\|\!|\cdot|\!\|_\omega$. We compute

$$\begin{aligned}\|\!|\nabla f|\!\|_\omega^2 &= \sum_{j=1}^d \sum_{\ell=1}^d \Big(\big\| \big(\xi_j + \mathrm{i}\tfrac{\beta}{2}\delta_{j\ell}\big)\hat{f}\big(\boldsymbol{\xi} + \mathrm{i}\tfrac{\beta}{2}\mathbf{e}_\ell\big)\big\|^2_{L^2(\mathbb{R}^d_{\boldsymbol{\xi}})} \\ &\qquad\qquad + \big\| \big(\xi_j - \mathrm{i}\tfrac{\beta}{2}\delta_{j\ell}\big)\hat{f}\big(\boldsymbol{\xi} - \mathrm{i}\tfrac{\beta}{2}\mathbf{e}_\ell\big)\big\|^2_{L^2(\mathbb{R}^d_{\boldsymbol{\xi}})}\Big) \\ &\geq \sum_{\ell=1}^d \Big(\big\| \big(\xi_\ell + \mathrm{i}\tfrac{\beta}{2}\big)\hat{f}\big(\boldsymbol{\xi} + \mathrm{i}\tfrac{\beta}{2}\mathbf{e}_\ell\big)\big\|^2_{L^2(\mathbb{R}^d_{\boldsymbol{\xi}})} + \big\| \big(\xi_\ell - \mathrm{i}\tfrac{\beta}{2}\big)\hat{f}\big(\boldsymbol{\xi} - \mathrm{i}\tfrac{\beta}{2}\mathbf{e}_\ell\big)\big\|^2_{L^2(\mathbb{R}^d_{\boldsymbol{\xi}})}\Big) \\ &\geq \big(\tfrac{\beta}{2}\big)^2 \sum_{\ell=1}^d \Big(\big\| \hat{f}\big(\boldsymbol{\xi} + \mathrm{i}\tfrac{\beta}{2}\mathbf{e}_\ell\big)\big\|^2_{L^2(\mathbb{R}^d_{\boldsymbol{\xi}})} + \big\| \hat{f}\big(\boldsymbol{\xi} - \mathrm{i}\tfrac{\beta}{2}\mathbf{e}_\ell\big)\big\|^2_{L^2(\mathbb{R}^d_{\boldsymbol{\xi}})}\Big) = \big(\tfrac{\beta}{2}\big)^2 \|\!|f|\!\|_\omega^2.\end{aligned}$$

This proves the Poincaré inequality with the constant $C_p = \frac{\beta}{2}$. ☐

Lemma 4.7. *Let* $\operatorname{Re}\zeta \geq \frac{1}{2}(1+\beta^2+\operatorname{tr}\mathbf{C})$*, and* $f, g \in C_0^\infty(\mathbb{R}^d)$ *such that* $(\zeta - L)f = g$*. Then there exists a constant* $C > 0$ *independent of* f, g *such that*

$$\|f\|_\varpi + \|\nabla f\|_\omega \leq C\|g\|_\omega. \tag{4.14}$$

Here $\varpi(\mathbf{x}) := (1 + |\mathbf{x}|_2)\omega(\mathbf{x})$.

Proof. We apply $\langle \cdot, f\rangle_\omega$ to $(\zeta - L)f = g$, and compute

$$\begin{aligned}\operatorname{Re}\int_{\mathbb{R}^d} g\overline{f}\omega\,\mathrm{d}\mathbf{x} &= \operatorname{Re}\int_{\mathbb{R}^d} \big(\zeta f - \nabla\cdot(\nabla f + f\mathbf{C}\mathbf{x})\big)\overline{f}\omega\,\mathrm{d}\mathbf{x} \\ &= \operatorname{Re}\zeta\int_{\mathbb{R}^d} |f|^2\omega\,\mathrm{d}\mathbf{x} + \operatorname{Re}\int_{\mathbb{R}^d}(\nabla f + f\mathbf{C}\mathbf{x})\cdot(\omega\nabla\overline{f} + \overline{f}\nabla\omega)\,\mathrm{d}\mathbf{x} \\ &= \|\nabla f\|_\omega^2 + \frac{1}{2}\int_{\mathbb{R}^d}|f|^2(2\operatorname{Re}\zeta\omega - \Delta\omega - \omega\operatorname{tr}\mathbf{C} + \mathbf{x}^\top\mathbf{C}\nabla\omega)\,\mathrm{d}\mathbf{x} \\ &= \|\nabla f\|_\omega^2 + \frac{1}{2}\int_{\mathbb{R}^d}|f|^2\nu\,\mathrm{d}\mathbf{x}. \end{aligned}\tag{4.15}$$

We temporarily define $\nu(\mathbf{x}) := 2\operatorname{Re}\zeta\omega - \Delta\omega - \omega\operatorname{tr}\mathbf{C} + \mathbf{x}^\top\mathbf{C}\nabla\omega$. We observe that $\Delta\omega = \beta^2\omega$ and $\mathbf{x}^\top\mathbf{C}\nabla\omega = \beta\sum_{i=1}^d c_i x_i \sinh\beta x_i \geq 0$ for all $\mathbf{x} \in \mathbb{R}^d$. So if $\operatorname{Re}\zeta \geq \frac{1}{2}(1 + \beta^2 + \operatorname{tr}\mathbf{C})$, the function $\nu(\mathbf{x})$ is a weight function with $\nu(\mathbf{x}) \geq \omega(\mathbf{x})$ on $\mathbb{R}^d$. Next we apply the Cauchy-Schwarz inequality to the left hand side of (4.15), which yields

$$\|\nabla f\|_\omega^2 + \frac{1}{2}\|f\|_\nu^2 \leq \|f\|_\omega\|g\|_\omega.$$

We now use the Poincaré inequality on the first term and $\nu(\mathbf{x}) \geq \omega(\mathbf{x})$ on the second term, and divide by $\|f\|_\omega$:

$$\|\nabla f\|_\omega + \|f\|_\nu \leq C\|g\|_\omega.$$

Finally we observe that for any fixed $\operatorname{Re}\zeta \geq \frac{1}{2}(1 + \beta^2 + \operatorname{tr}\mathbf{C})$ there is a constant $C > 0$ such that $\nu(\mathbf{x}) \geq C\varpi(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^d$. Hence, ν and ϖ are equivalent weight functions. This concludes the proof. ☐

Before we properly define the Fokker-Planck operator as a closed operator in $\mathcal{X}$, we need the lemma below. It determines all formal eigenfunction of the Fokker-Planck operator, i.e. the eigenfunctions of the *distributional* Fokker-Planck operator $\mathfrak{L}$ in $\mathcal{X}$ (see the definition of $\mathfrak{L}$ before Proposition 4.1). As a consequence of the following lemma it will be straightforward to determine the spectrum of the Fokker-Planck operator in $\mathcal{X}$.

Lemma 4.8. *The distributional Fokker-Planck operator $\mathfrak{L}$ satisfies the eigenvalue equation $\mathfrak{L}f = \zeta f$ for some $\zeta \in \mathbb{C}$ and some $f \in \mathcal{X} \setminus \{0\}$ iff $\zeta \in \{-\mathbf{ck} : \mathbf{k} \in \mathbb{N}^n\}$. In this case, there holds $f \in \operatorname{span}\{\mu_{\mathbf{k}} : -\mathbf{ck} = \zeta\}$.*

Proof. Since the functions $\mu_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}^d$, lie in $\mathcal{X}$ it is clear that they are also eigenfunctions of $\mathfrak{L}$. In order to show that they already span all eigenspaces we consider the Fourier transform of $(\zeta - \mathfrak{L})f = 0$ for $\zeta \in \mathbb{C}$, which reads

$$(\zeta + \boldsymbol{\xi}^2)\hat{f} + \boldsymbol{\xi}^\top \mathbf{C} \nabla \hat{f} = 0. \tag{4.16}$$

We are now looking for $f \in \mathcal{X}$ and $\zeta \in \mathbb{C}$ satisfying this (eigenvalue) equation. In particular we are interested in solutions $\hat{f}$ which are analytic in $\Omega_{\beta/2}$. Expecting f to be a linear combination of the $\mu_{\mathbf{k}} \equiv \nabla^{\mathbf{k}}\mu_{\mathbf{0}}$, we make the ansatz $\hat{f} = p\hat{\mu}_{\mathbf{0}}$, with p analytic in $\Omega_{\beta/2}$. This is admissible (and not restrictive) since $\hat{\mu}_{\mathbf{0}}$ is nonzero and analytic in $\Omega_{\beta/2}$. We know that $\hat{\mu}_{\mathbf{0}}$ satisfies the zero eigenvalue equation $\boldsymbol{\xi}^2\hat{\mu}_{\mathbf{0}} + \boldsymbol{\xi}^\top \mathbf{C}\nabla\hat{\mu}_{\mathbf{0}} = 0$, so after inserting $\hat{f} = p\hat{\mu}_{\mathbf{0}}$ in (4.16) we obtain the following equation for p:

$$\boldsymbol{\xi}^\top \mathbf{C} \nabla p = -\zeta p. \tag{4.17}$$

Next we introduce the (unique) solution $\boldsymbol{\xi}(t)$ of the ordinary differential equation $\dot{\boldsymbol{\xi}} = \mathbf{C}\boldsymbol{\xi}$ with $\boldsymbol{\xi}(0) = \boldsymbol{\xi}_0 \in \mathbb{C}^d$. It is verified by application of the chain rule that for any such curve

$$\frac{\mathrm{d}}{\mathrm{d}t} p(\boldsymbol{\xi}(t)) = \boldsymbol{\xi}(t)^\top \mathbf{C} \nabla p(\boldsymbol{\xi}(t)).$$

This means that any solution of (4.17) fulfills the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} p(\boldsymbol{\xi}(t)) = -\zeta p(\boldsymbol{\xi}(t))$$

along these curves, i.e. $p(\boldsymbol{\xi}(t)) = p(\boldsymbol{\xi}_0)\mathrm{e}^{-\zeta t}$, for all $\boldsymbol{\xi}_0 \in \mathbb{C}^d$. We further use $\boldsymbol{\xi}(t) = \boldsymbol{\xi}_0 \mathrm{e}^{t\mathbf{C}}$. Introducing $s = \mathrm{e}^t$ (with $s \in \mathbb{R}^+$) we obtain (see the introduction concerning the notation for matrix powers)

$$\boldsymbol{\xi}(\ln s) = \boldsymbol{\xi}_0 s^{\mathbf{C}}, \quad p(\boldsymbol{\xi}_0 s^{\mathbf{C}}) = p(\boldsymbol{\xi}_0) s^{-\zeta}. \tag{4.18}$$

Now p needs to be analytic in $\Omega_{\beta/2}$, so (4.18) implies that $\operatorname{Re}\zeta \leq 0$, otherwise p would have a singularity in the origin $\boldsymbol{\xi} = \mathbf{0}$, a contradiction. By induction we deduce from (4.17) that for all $\mathbf{k} \in \mathbb{N}^d$

$$\boldsymbol{\xi}^\top \mathbf{C} \nabla\big(\nabla^{\mathbf{k}} p\big) = -(\zeta + \mathbf{ck})\nabla^{\mathbf{k}} p.$$

Since all derivatives $\nabla^{\mathbf{k}} p$ need to be analytic in $\Omega_{\beta/2}$ as well, an analogous argument to above proves that either $\operatorname{Re}\zeta \leq -\mathbf{ck}$ for all $\mathbf{k} \in \mathbb{N}^d$ (which is impossible since $\mathbf{C} > 0$) or $\nabla^{\mathbf{k}} p \equiv 0$ for some $\mathbf{k} \in \mathbb{N}$. So p has to be a polynomial, and we can make the ansatz

$$p(\boldsymbol{\xi}) = \sum_{\mathbf{k} \in \mathbb{N}^d} p_{\mathbf{k}} \boldsymbol{\xi}^{\mathbf{k}},$$

with $p_{\mathbf{k}} \in \mathbb{C}$, $\mathbf{k} \in \mathbb{N}^d$, and $p_{\mathbf{k}} = 0$ for almost all $\mathbf{k} \in \mathbb{N}^d$. We now insert this in (4.17) and obtain

$$\sum_{\mathbf{k} \in \mathbb{N}^d} (\mathbf{ck}) p_{\mathbf{k}} \boldsymbol{\xi}^{\mathbf{k}} = -\zeta \sum_{\mathbf{k} \in \mathbb{N}^d} p_{\mathbf{k}} \boldsymbol{\xi}^{\mathbf{k}}.$$

This holds true for all $\boldsymbol{\xi} \in \Omega_{\beta/2}$ iff $\zeta = -\mathbf{ck}$ for all $\mathbf{k} \in \mathbb{N}^d$ for which $p_{\mathbf{k}} \neq 0$. So the (formal) eigenvalues of $\mathfrak{L}$ in $\mathcal{X}$ are given by the set $\{-\mathbf{ck} : \mathbf{k} \in \mathbb{N}^d\}$.

From the above analysis we conclude

$$\hat{f}(\boldsymbol{\xi}) = \hat{\mu}_{\mathbf{0}}(\boldsymbol{\xi}) \sum_{-\mathbf{ck}=\zeta} p_{\mathbf{k}} \boldsymbol{\xi}^{\mathbf{k}}.$$

Now recall from Theorem 4.4 (iii) that $\hat{\mu}_{\mathbf{k}} = \mathrm{i}^{|\mathbf{k}|_1} \boldsymbol{\xi}^{\mathbf{k}} \hat{\mu}_{\mathbf{0}}$ holds for all $\mathbf{k} \in \mathbb{N}^d$. Hence, $f \in \operatorname{span}\{\mu_{\mathbf{k}} : -\mathbf{ck} = \zeta\}$. So we conclude that the eigenspaces of $\mathfrak{L}$ in $\mathcal{X}$ are precisely spanned by the $\mu_{\mathbf{k}}$. ☐

Now we can properly define the Fokker-Planck operator in the space $\mathcal{X}$.

Lemma 4.9. *The operator $L|_{C_0^\infty(\mathbb{R}^d)}$ is closable in $\mathcal{X}$, and $\mathcal{L} := \operatorname{cl}_{\mathcal{X}} L|_{C_0^\infty(\mathbb{R}^d)}$. The domain is $D(\mathcal{L}) = \{f \in \mathcal{X} : \mathfrak{L}f \in \mathcal{X}\}$, and for $f \in D(\mathcal{L})$ we have $\mathcal{L}f = \mathfrak{L}f$.*

Proof. According to (4.15) we have that $(L - \zeta)|_{C_0^\infty(\mathbb{R}^d)}$ is dissipative in $\mathcal{X}$ if $\operatorname{Re}\zeta \geq \frac{1}{2}(1 + \beta^2 + \operatorname{tr}\mathbf{C})$. This implies (cf. Theorem 1.4.5 (c) in [Paz83]) that $(L - \zeta)|_{C_0^\infty(\mathbb{R}^d)}$ and consequently also $L|_{C_0^\infty(\mathbb{R}^d)}$ is closable in $\mathcal{L}$.

Now we define $\mathcal{L} := \operatorname{cl}_{\mathcal{X}} L|_{C_0^\infty}$. The domain $D(\mathcal{L})$ consists of all $f \in \mathcal{X}$ for which there exists some $g \in \mathcal{X}$ and a sequence $(f_n)_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ such that

$$\begin{cases} \lim\limits_{n\to\infty} \|f_n - f\|_\omega = 0, \\ \lim\limits_{n\to\infty} \|Lf_n - g\|_\omega = 0. \end{cases} \tag{4.19}$$

This also implies that $((\zeta - L)f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{X}$. Thus, according to (4.14), $(\nabla f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{X}$. So altogether, $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^1(\omega,\omega)$, which converges to some $\tilde{f} \in H^1(\omega,\omega)$. But since we already know that $f_n \to f$ in $\mathcal{X}$, this implies that even $f \in H^1(\omega,\omega)$. Next we temporarily introduce the weight $\omega_2(\mathbf{x}) := \omega(\frac{\mathbf{x}}{2})$ and the corresponding weighted space $\mathcal{X}_2 := L^2(\omega_2)$. Since $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^1(\omega,\omega)$, $(\mathbf{x}^\top \mathbf{C}\nabla f_n + \operatorname{tr}\mathbf{C} f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{X}_2$. Together with the second line of (4.19) this implies that $(\Delta f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{X}_2$. Applying the Fourier transform and the norm (4.12) we have that, for every $\ell \in \{1,\dots,d\}$, the two sequences

$$\left(\left|\boldsymbol{\xi} \pm \mathrm{i}\frac{\beta}{4}\mathbf{e}_\ell\right|^2 \hat{f}_n\left(\boldsymbol{\xi} \pm \mathrm{i}\frac{\beta}{4}\mathbf{e}_\ell\right)\right)_{n\in\mathbb{N}}$$

are Cauchy sequences in $L^2(\mathbb{R}^d_{\boldsymbol{\xi}})$. But we also know that $\hat{f}_n(\boldsymbol{\xi} \pm \mathrm{i}\frac{\beta}{4}\mathbf{e}_\ell)$ converges to $\hat{f}(\boldsymbol{\xi} \pm \mathrm{i}\frac{\beta}{4}\mathbf{e}_\ell)$ in $L^2(\mathbb{R}^d_{\boldsymbol{\xi}})$, for every $\ell \in \{1,\dots,d\}$. By identifying the limits it becomes clear that $\Delta f_n \to \Delta f$ in $\mathcal{X}_2$, and $\Delta f \in \mathcal{X}_2$, and altogether $Lf_n \to \mathfrak{L}f$ in $\mathcal{X}_2$. According to (4.19) we can make the identification $\mathfrak{L}f = g$ in $\mathcal{X}_2$, and since $g \in \mathcal{X}$, we conclude that $Lf_n \to \mathfrak{L}f$ in $\mathcal{X}$. This proves the inclusion $D(\mathcal{L}) \subseteq \{f \in \mathcal{X} : \mathfrak{L}f \in \mathcal{X}\}$, and $\mathcal{L}f = \mathfrak{L}f$ for all $f \in D(\mathcal{L})$.

Finally we prove that this inclusion indeed is an equality. First we note that $D(L) \subset D(\mathcal{L})$ since $L = \operatorname{cl}_X L|_{C_0^\infty(\mathbb{R}^d)}$ and $X \hookrightarrow \mathcal{X}$. So we have $L \subset \mathcal{L}$ in the sense of graphs. Let us then take $\zeta > 0$ so large that the estimate (4.14) holds. As we have mentioned in the beginning of this proof the operator $(\mathcal{L} - \zeta)|_{C_0^\infty(\mathbb{R}^d)}$ is dissipative in $\mathcal{X}$, and from Theorem 1.4.5 in [Paz83] it follows that $\mathcal{L} - \zeta$ is also dissipative. In particular it is injective and thus invertible. So $(\zeta - \mathcal{L})^{-1}$ exists. Now according to Theorem 4.4,

$\zeta - L\colon D(L) \to X$ is a bijection, so $\operatorname{ran}(\zeta - \mathcal{L}) \supset X$, which is dense in $\mathcal{X}$. Because of this and the estimate (4.14), $(\zeta - \mathcal{L})^{-1}$ is a densely defined, bounded operator in $\mathcal{X}$. But by definition $(\zeta - \mathcal{L})^{-1}$ is already closed, so $\operatorname{ran}(\zeta - \mathcal{L}) = \mathcal{X}$ and $\zeta \in \rho(\mathcal{L})$ (and thus $\rho(\mathcal{L}) \neq \emptyset$). We take now this particular $\zeta \in \rho(\mathcal{L})$, and assume there exists some $f^* \in \mathcal{X} \setminus D(\mathcal{L})$ such that $\mathfrak{L}f^* \in \mathcal{X}$. Hence also $(\zeta - \mathfrak{L})f^* \in \mathcal{X}$. Since $\zeta \in \rho(\mathcal{L})$ we have $(\zeta - \mathcal{L})^{-1}(\zeta - \mathfrak{L})f^* \in D(\mathcal{L})$, so the assumption is equivalent to saying that there is some $f^\sharp := (\zeta - \mathcal{L})^{-1}(\zeta - \mathfrak{L})f^* - f^* \in \mathcal{X} \setminus D(\mathcal{L})$ such that $(\zeta - \mathfrak{L})f^\sharp = 0$. But according to Lemma 4.8 we know that $\zeta \in \rho(\mathcal{L})$ cannot be an eigenvalue of $\mathfrak{L}$ in $\mathcal{X}$, since $\zeta > 0$. So $f^\sharp = 0$, and $D(\mathcal{L}) = \{f \in \mathcal{X} : \mathfrak{L}f \in \mathcal{X}\}$. □

Lemma 4.10. *For any $\zeta \in \rho(\mathcal{L})$ the resolvent $R_{\mathcal{L}}(\zeta)$ is compact in $\mathcal{X}$.*

Proof. We choose $\zeta > 0$ large enough, so we can apply Lemma 4.7, which proves that $(\zeta - \mathcal{L})^{-1}$ is an element of $\mathscr{B}(\mathcal{X}, H^1(\varpi, \omega))$. Note that this requires the density of $C_0^\infty(\mathbb{R}^d)$ in $\mathcal{X}$, see Lemma B.17 in the Appendix. It remains to show that $H^1(\varpi, \omega)$ is compactly embedded in $\mathcal{X}$. According to the definition of ϖ in Lemma 4.7 it is clear that for all $n \in \mathbb{N}$ there holds

$$\sup_{|\mathbf{x}|_2 > n} \frac{\omega(\mathbf{x})}{\varpi(\mathbf{x})} = \frac{1}{1+n},$$

which tends to zero as $n \to \infty$. Thus we can apply Lemma B.16 in the Appendix, which proves the compact embedding $H^1(\varpi, \omega) \hookrightarrow\hookrightarrow \mathcal{X}$. Since $R_{\mathcal{L}}(\zeta) \in \mathscr{B}(\mathcal{X}, H^1(\varpi, \omega))$ this implies that $R_{\mathcal{L}}(\zeta)\colon \mathcal{X} \to \mathcal{X}$ is compact. Finally we remark that according to Theorem III.6.29 in [Kat66] the compactness of $R_{\mathcal{L}}(\zeta)$ follows for all other $\zeta \in \rho(\mathcal{L})$. □

Corollary 4.11. *The spectrum $\sigma(\mathcal{L})$ consists entirely of eigenvalues, and $\sigma(\mathcal{L}) = \{-\mathbf{ck} : \mathbf{k} \in \mathbb{N}^d\}$. For every $\lambda \in \sigma(\mathcal{L})$ the (geometric) eigenspace is given by*

$$\ker(\lambda - \mathcal{L}) = \operatorname{span}\{\mu_{\mathbf{k}} : -\mathbf{ck} = \lambda\}.$$

Proof. We apply Theorem III.6.29 in [Kat66] which proves again that $\sigma(\mathcal{L})$ consists entirely of eigenvalues, and the corresponding eigenspaces are finite-dimensional. According to Lemma 4.9 the eigenfunctions of $\mathcal{L}$ in $D(\mathcal{L})$ are precisely the (formal) eigenfunctions of $\mathfrak{L}$ in $\mathcal{X}$. And the eigenspaces of $\mathfrak{L}$ are then given by Lemma 4.8, they are spanned by the $\{\mu_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{N}^d}$. □

We introduce the closed subspaces $\mathcal{X}_k := \operatorname{cl}_{\mathcal{X}} X_k \subset \mathcal{X}$ for $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ the space X_k was defined as a subspace of X by (4.7).

Lemma 4.12. *For every $k \in \mathbb{N}$ there holds*

$$\mathcal{X}_k = \Big\{ f \in \mathcal{X} : \int_{\mathbb{R}^d} f(\mathbf{x})\mathbf{x}^{\mathbf{k}} \,\mathrm{d}\mathbf{x} = 0, \quad \forall \mathbf{k} \in \mathbb{N}^d \text{ with } |\mathbf{k}|_1 \le k-1 \Big\}. \tag{4.20}$$

Proof. We start from the characterization of the X_k in Lemma 4.3, and apply Lemma C.2. For every $\mathbf{k} \in \mathbb{N}^d$ we define the functional

$$\eta_{\mathbf{k}}\colon \mathcal{X} \to \mathbb{C}\colon f \mapsto \int_{\mathbb{R}^d} f(\mathbf{x})\mathbf{x}^{\mathbf{k}} \,\mathrm{d}\mathbf{x}.$$

We first prove the continuity of the $\eta_{\mathbf{k}}$. For $\mathbf{k} \in \mathbb{N}^d$ and $f \in \mathcal{X}$ we have

$$\Big| \int_{\mathbb{R}^d} f(\mathbf{x})\mathbf{x}^{\mathbf{k}} \,\mathrm{d}\mathbf{x} \Big| \le \int_{\mathbb{R}^d} |f(\mathbf{x})\omega(\mathbf{x})^{1/2}| \cdot \Big| \frac{\mathbf{x}^{\mathbf{k}}}{\omega(\mathbf{x})^{1/2}} \Big| \,\mathrm{d}\mathbf{x}$$

$$\le \|f\|_\omega \cdot \Big(\int_{\mathbb{R}^d} \frac{\mathbf{x}^{2\mathbf{k}}}{\omega(\mathbf{x})} \,\mathrm{d}\mathbf{x} \Big)^{\frac{1}{2}},$$

where we applied the Cauchy-Schwarz inequality in the second step. Since ω grows exponentially in every direction it is clear that the remaining integral on the right hand side is finite for every $\mathbf{k} \in \mathbb{N}^d$, i.e. the $\eta_\mathbf{k}$ are bounded linear functionals in $\mathcal{X}$. Furthermore we verify that the family $\{\eta_\mathbf{k} : \mathbf{k} \in \mathbb{N}^d\}$ is linearly independent. If the family would be linearly dependent, there would exist a polynomial $p(\mathbf{x}) \not\equiv 0$ such that

$$\int_{\mathbb{R}^d} f(\mathbf{x}) p(\mathbf{x}) \,\mathrm{d}\mathbf{x} = 0, \quad \forall f \in \mathcal{X}.$$

Since $C_0^\infty(\mathbb{R}^d) \subset \mathcal{X}$, this is impossible, due to the fundamental lemma of calculus of variations. Now we have verified the assumptions of Lemma C.2 in the Appendix. Since

$$X_k = \bigcap_{|\mathbf{k}|_1 \le k-1} \ker \eta_\mathbf{k}|_X,$$

we conclude that

$$\mathcal{X}_k := \mathrm{cl}_\mathcal{X} X_k = \bigcap_{|\mathbf{k}|_1 \le k-1} \ker \eta_\mathbf{k}.$$

The intersection on the right is exactly the set (4.20). □

Corollary 4.13. *For $k \in \mathbb{N}$ there holds the identity*

$$\mathcal{X}_k = \big\{ f \in \mathcal{X} : \nabla^\mathbf{k} \hat{f}(\mathbf{0}) = 0, \quad \forall |\mathbf{k}|_1 \le k-1 \big\}. \tag{4.21}$$

Proof. This follows immediately from the fact that for $f \in \mathcal{X}$ and any $\mathbf{k} \in \mathbb{N}^d$

$$\int_{\mathbb{R}^d} \mathbf{x}^\mathbf{k} f(\mathbf{x}) \,\mathrm{d}\mathbf{x} = \mathcal{F}[\mathbf{x}^\mathbf{k} f(\mathbf{x})](\mathbf{0}) = \mathrm{i}^{|\mathbf{k}|_1} \nabla^\mathbf{k} \hat{f}(\mathbf{0}).$$

We use this in (4.20) and the result follows. □

At every $\lambda \in \sigma(\mathcal{L})$ the resolvent map $\zeta \mapsto R_\mathcal{L}(\zeta)$ has an isolated singularity. We denote the corresponding spectral projection of $\mathcal{L}$ by $\Pi_{\mathcal{L},\lambda}$, which is explicitly given by (2.1) for $\sigma' = \{\lambda\}$. In particular $\Pi_{\mathcal{L},\lambda} = \mathrm{cl}_\mathcal{X} \Pi_{L,\lambda}$:

Proposition 4.14. *For every $k \in \mathbb{N}$ we have the following facts:*

(i) *There holds $\mathcal{X} = \mathcal{X}_k \oplus \mathrm{span}\{\mu_\mathbf{k} : |\mathbf{k}|_1 \le k-1\}$, and the intersection contains only zero.*
(ii) *$\mathcal{L}$ is decomposed according to $\mathcal{X} = \mathcal{X}_k \oplus \mathrm{span}\{\mu_\mathbf{k} : |\mathbf{k}|_1 \le k-1\}$, and on $\mathcal{X}_k$ we have $\sigma(\mathcal{L}|_{\mathcal{X}_k}) = \{-\mathbf{ck} : |\mathbf{k}|_1 \ge k\}$.*

Proof. *Step 1 (representation of X_k)*: In X there holds for any fixed $k \in \mathbb{N}$

$$X_k^\perp = \mathrm{span}\, \big\{ \mu_\mathbf{k} : |\mathbf{k}|_1 \le k-1 \big\}, \tag{4.22}$$

and for every $\lambda \in \sigma(L)$ we have for the corresponding spectral projection

$$\mathrm{ran}\, \Pi_{L,\lambda} = \mathrm{span}\, \big\{ \mu_\mathbf{k} : -\mathbf{ck} = \lambda \big\}, \tag{4.23a}$$

$$\ker \Pi_{L,\lambda} = \mathrm{span}\, \big\{ \mu_\mathbf{k} : -\mathbf{ck} \ne \lambda \big\}. \tag{4.23b}$$

For a given $k \in \mathbb{N}$ we define the set

$$\sigma_k := \{-\mathbf{ck} : |\mathbf{k}|_1 \le k-1\} \subset \mathbb{R}_0^-,$$

which is the set of all eigenvalues which "contribute" to $X_k^\perp$ (note that there may be $\mathbf{k} \in \mathbb{N}^d$ such that $-\mathbf{ck} \in \sigma_k$ but $|\mathbf{k}|_1 \geq k$). From (4.23a) we conclude that

$$\bigcup_{\lambda \in \sigma_k} \operatorname{ran} \Pi_{L,\lambda} \supset X_k^\perp .$$

Taking the orthogonal complement of this relation yields:

$$\bigcap_{\lambda \in \sigma_k} \ker \Pi_{L,\lambda} \subset X_k. \tag{4.24}$$

Next we investigate which eigenfunctions $\mu_{\mathbf{k}}$ need to be added to the left hand side of (4.24) such that the corresponding span equals X_k. First we observe that, according to (4.23), there holds $\mu_{\mathbf{k}} \in \big(\bigcap_{\lambda \in \sigma_k} \ker \Pi_{L,\lambda}\big)^\perp$ iff $\mu_{\mathbf{k}} \in \operatorname{ran} \Pi_{L,\lambda}$ for some $\lambda \in \sigma_k$. This is also equivalent to the condition $-\mathbf{ck} \in \sigma_k$. To complement the left hand side of (4.24), we also require $\mu_{\mathbf{k}} \in X_k$, which gives the constraint $|\mathbf{k}|_1 \geq k$, see (4.22). Hence, we conclude that

$$X_k = \Big(\bigcap_{\lambda \in \sigma_k} \ker \Pi_{L,\lambda} \Big) \oplus_\perp \operatorname{span}\{\mu_{\mathbf{k}} : -\mathbf{ck} \in \sigma_k \wedge |\mathbf{k}|_1 \geq k\}. \tag{4.25}$$

Step 2 (spectral projections in $\mathcal{X}$): For $\zeta \in \rho(\mathcal{L})$ we have $R_L(\zeta) \subset R_{\mathcal{L}}(\zeta)$ (in the sense of graphs), and as a consequence the spectral projection for $\lambda \in \sigma(\mathcal{L})$ satisfies $\Pi_{L,\lambda} \subset \Pi_{\mathcal{L},\lambda}$, see (2.1). Furthermore, both $\Pi_{L,\lambda}$ and $\Pi_{\mathcal{L},\lambda}$ are bounded projections in X and $\mathcal{X}$, respectively. According to Lemma C.1 in the appendix there holds

$$\ker \Pi_{\mathcal{L},\lambda} = \operatorname{cl}_{\mathcal{X}} \ker \Pi_{L,\lambda} \quad \text{and} \quad \operatorname{ran} \Pi_{\mathcal{L},\lambda} = \operatorname{cl}_{\mathcal{X}} \operatorname{ran} \Pi_{L,\lambda}. \tag{4.26}$$

Since the projections are bounded we have $\mathcal{X} = \ker \Pi_{\mathcal{L},\lambda} \oplus \operatorname{ran} \Pi_{\mathcal{L},\lambda}$, and both components of the direct sum are closed subspaces of $\mathcal{X}$, see Section III.3.4 in [Kat66].

Step 3 (decomposition of $\mathcal{X}$): Following the arguments of Step 2 we obtain, by applying the closure in $\mathcal{X}$ to (4.25):

$$\mathcal{X}_k = \Big(\bigcap_{\lambda \in \sigma_k} \ker \Pi_{\mathcal{L},\lambda} \Big) \oplus \operatorname{span}\{\mu_{\mathbf{k}} : -\mathbf{ck} \in \sigma_k \wedge |\mathbf{k}|_1 \geq k\}. \tag{4.27}$$

Notice that σ_k is finite. The sum is still a direct sum, since every $\mu_{\mathbf{k}}$ in the "span-term" of the right hand side lies in the range of some $\Pi_{\mathcal{L},\lambda}$ with $\lambda \in \sigma_k$. Altogether this implies that $\mathcal{X}_k$ is a closed subspace of $\mathcal{X}$ such that

$$\mathcal{X} = \mathcal{X}_k \oplus \operatorname{span}\{\mu_{\mathbf{k}} : |\mathbf{k}|_1 \leq k-1\}, \tag{4.28}$$

and the two components are closed and disjoint subspaces of $\mathcal{X}$.

Step 4 (decomposition of $\mathcal{L}$): $\mathcal{L}$ can always be decomposed according to its spectral projections. Since we can write each subspace $\mathcal{X}_k$ as (4.27), it becomes clear that $\mathcal{L}$ can also be decomposed according to (4.28).

Concerning the spectrum of $\mathcal{L}$ in $\mathcal{X}_k$ we recall that $\sigma(\mathcal{L}|_{\ker \Pi_{\mathcal{L},\lambda}}) = \sigma(\mathcal{L}) \setminus \{\lambda\}$. Thus, we obtain from (4.27) that $\sigma(\mathcal{L}|_{\mathcal{X}_k}) = \{-\mathbf{ck} : |\mathbf{k}|_1 \geq k\}$. □

Corollary 4.15. *For every $\lambda \in \sigma(\mathcal{L})$ the algebraic eigenspace coincides with the geometric eigenspace.*

Proof. Since $\dim \operatorname{ran} \Pi_{L,\lambda} < \infty$ for every $\lambda \in \sigma(L)$, we conclude from (4.26) that $\operatorname{ran} \Pi_{\mathcal{L},\lambda} = \operatorname{span}\{\mu_{\mathbf{k}} : -\mathbf{ck} = \lambda\}$. Therefore, $M(\lambda - \mathcal{L}) \subseteq \operatorname{span}\{\mu_{\mathbf{k}} : -\mathbf{ck} = \lambda\}$, according to (B.7). On the other hand we know that

$$\operatorname{span}\{\mu_{\mathbf{k}} : -\mathbf{ck} = \lambda\} = \ker(\lambda - \mathcal{L}) \subseteq M(\lambda - \mathcal{L}).$$

□

After having established the spectrum-related subspaces $\mathcal{X}_k$, we now turn to the semigroup which is generated by $\mathcal{L}$.

Lemma 4.16. *The Fokker-Planck operator $\mathcal{L}$ generates a C_0-semigroup of bounded operators in $\mathcal{X}$, which is denoted by $(\mathrm{e}^{t\mathcal{L}})_{t\geq 0}$.*

Proof. From (4.15) in the proof of Lemma 4.7 we find that for $\zeta = \frac{1}{2}(1 + \beta^2 + \operatorname{tr} \mathbf{C})$ the operator $(\mathcal{L} - \zeta)|_{C_0^\infty(\mathbb{R}^d)}$, and thus $\mathcal{L} - \zeta$, is dissipative. Furthermore, every $\lambda > \zeta$ lies in $\rho(\mathcal{L} - \zeta)$, see Corollary 4.11. So we may apply the Lumer-Phillips Theorem (cf. Theorem 1.4.3 in [Paz83]) which proves that $\mathcal{L} - \zeta$ generates a C_0-semigroup of contractions, thus $\mathcal{L}$ generates a C_0-semigroup of bounded operators in $\mathcal{X}$. □

According to equation (1.2) in [Met01] the semigroup operators $\mathrm{e}^{t\mathcal{L}}$ for $t > 0$ are explicitly given by

$$(\mathrm{e}^{t\mathcal{L}} f)(\mathbf{x}) = (4\pi)^{-n/2} \det \mathbf{Q}_t^{-1/2} \mathrm{e}^{t \operatorname{tr} \mathbf{C}} \int_{\mathbb{R}^d} \exp\Big(- \frac{1}{4} \mathbf{y}^\top \mathbf{Q}_t^{-1} \mathbf{y}\Big) f(\mathrm{e}^{t\mathbf{C}}\mathbf{x} - \mathbf{y})\, \mathrm{d}\mathbf{y}, \quad (4.29)$$

where $\mathbf{Q}_t = (2\mathbf{C})^{-1}(\mathrm{e}^{2t\mathbf{C}} - \mathbf{I})$. We can equivalently use the following representation in Fourier space, which is useful for the subsequent analysis.

Lemma 4.17. *For $f \in \mathcal{X}$ and $t \geq 0$ there holds*

$$\mathcal{F}[\mathrm{e}^{t\mathcal{L}} f](\boldsymbol{\xi}) = \exp\big(- \boldsymbol{\xi}^\top [(2\mathbf{C})^{-1}(\mathbf{I} - \mathrm{e}^{-2t\mathbf{C}})]\boldsymbol{\xi}\big) \cdot \hat{f}(\mathrm{e}^{-t\mathbf{C}}\boldsymbol{\xi}). \quad (4.30)$$

Proof. If $t = 0$ the identity is obviously fulfilled, so we assume $t > 0$ in the following. For $f \in \mathcal{X}$ (4.29) is well defined, and we can write it as

$$(\mathrm{e}^{t\mathcal{L}} f)(\mathbf{x}) = (4\pi)^{-n/2} \det \mathbf{Q}_t^{-1/2} \mathrm{e}^{t \operatorname{tr} \mathbf{C}} (\varphi * f)(\mathrm{e}^{t\mathbf{C}}\mathbf{x}),$$

where $\varphi(\mathbf{x}) = \exp(-\frac{1}{4}\mathbf{y}^\top \mathbf{Q}_t^{-1}\mathbf{y})$. Using the fact that $\mathbf{Q}_t$ is diagonal we immediately obtain that $\hat{\varphi}(\boldsymbol{\xi}) = (\det 4\pi \mathbf{Q}_t)^{1/2} \exp(-\boldsymbol{\xi}^\top \mathbf{Q}_t \boldsymbol{\xi})$. With this we can write the Fourier transform of (4.29) as

$$\begin{aligned}\mathcal{F}[\mathrm{e}^{t\mathcal{L}} f](\boldsymbol{\xi}) &= (4\pi)^{-n/2} \det \mathbf{Q}_t^{-1/2} \mathcal{F}[\varphi * f](\mathrm{e}^{-t\mathbf{C}}\boldsymbol{\xi}) \\ &= \exp\big(- \boldsymbol{\xi}^\top [(2\mathbf{C})^{-1}(\mathbf{I} - \mathrm{e}^{-2t\mathbf{C}})]\boldsymbol{\xi}\big) \hat{f}(\mathrm{e}^{-t\mathbf{C}}\boldsymbol{\xi}).\end{aligned}$$

So (4.29) and (4.30) are equivalent for all $f \in \mathcal{X}$. □

For the rest of this article we mostly use the representation (4.30) of $(\mathrm{e}^{t\mathcal{L}})_{t\geq 0}$. Since $\mathcal{X}_k$ is $\mathcal{L}$-invariant for all $k \in \mathbb{N}$, it is immediate that $\mathcal{L}|_{\mathcal{X}_k}$ generates a semigroup on every $\mathcal{X}_k$. In the next step we investigate the long-time behavior of $(\mathrm{e}^{t\mathcal{L}}|_{\mathcal{X}_k})_{t\geq 0}$ on the subspaces $\mathcal{X}_k$.

Proposition 4.18. *For every $k \in \mathbb{N}$ there exists a constant $C_k > 0$ such that there holds*

$$\|\mathrm{e}^{t\mathcal{L}}|_{\mathcal{X}_k}\|_{\mathscr{B}(\mathcal{X}_k)} \leq C_k \mathrm{e}^{-tkc_1}, \quad \forall t \geq 0, \quad (4.31)$$

where c_1 is the smallest entry of $\mathbf{c}$.

Proof. We fix $k \in \mathbb{N}$ and take any $f \in \mathcal{X}_k$. Then we estimate $\mathrm{e}^{t\mathcal{L}} f$ using the $\|\!|\cdot|\!\|_\omega$-norm. (4.30) implies for any $\ell \in \{1,\dots,d\}$ and for $t \geq 1$:

$$\begin{aligned}
\Big\| \mathcal{F}[\mathrm{e}^{t\mathcal{L}} f]\Big(\boldsymbol{\xi} + \mathrm{i}\frac{\beta}{2}\mathbf{e}_\ell\Big)\Big\|^2_{L^2(\mathbb{R}^d_{\boldsymbol{\xi}})} &= \int_{\mathbb{R}^d} \Big| \exp\Big[-\Big(\boldsymbol{\xi} + \mathrm{i}\frac{\beta}{2}\mathbf{e}_\ell\Big)^\top [\mathbf{C}^{-1}(\mathbf{I} - \mathrm{e}^{-2t\mathbf{C}})]\Big(\boldsymbol{\xi} + \mathrm{i}\frac{\beta}{2}\mathbf{e}_\ell\Big)\Big]\Big| \\
&\qquad \cdot \Big| \hat{f}\Big(\mathrm{e}^{-t\mathbf{C}}\Big(\boldsymbol{\xi} + \mathrm{i}\frac{\beta}{2}\mathbf{e}_\ell\Big)\Big)\Big|^2 \,\mathrm{d}\boldsymbol{\xi} \\
&\leq C \int_{\mathbb{R}^d} \exp\big(-\boldsymbol{\xi}^\top [\mathbf{C}^{-1}(\mathbf{I} - \mathrm{e}^{-2t\mathbf{C}})]\boldsymbol{\xi}\big) \\
&\qquad \cdot \Big| \hat{f}\Big(\mathrm{e}^{-t\mathbf{C}}\Big(\boldsymbol{\xi} + \mathrm{i}\frac{\beta}{2}\mathbf{e}_\ell\Big)\Big)\Big|^2 \,\mathrm{d}\boldsymbol{\xi} \\
&\leq C \int_{\mathbb{R}^d} \exp\big(-|\boldsymbol{\xi}|_2^2 \gamma_{\mathbf{C}}\big) \Big| \hat{f}\Big(\mathrm{e}^{-t\mathbf{C}}\Big(\boldsymbol{\xi} + \mathrm{i}\frac{\beta}{2}\mathbf{e}_\ell\Big)\Big)\Big|^2 \,\mathrm{d}\boldsymbol{\xi}, \quad (4.32)
\end{aligned}$$

where $\gamma_{\mathbf{C}} = (1 - \mathrm{e}^{-2c_1})/c_1 > 0$. Now $\hat{f}$ is analytic, and since $f \in \mathcal{X}_k$ we get from (4.21) that $\hat{f}(\boldsymbol{\xi}) = \mathcal{O}(|\boldsymbol{\xi}|_2^k)$ as $|\boldsymbol{\xi}|_2 \to 0$ (use the inequality between the geometric and the quadratic mean). In particular, there exists some $C > 0$ and $r > 0$ such that $|\hat{f}(\boldsymbol{\xi})| \leq C|\boldsymbol{\xi}|_2^k$ for all $|\boldsymbol{\xi}|_2 < r$. Let us now consider all $\boldsymbol{\xi} \in \mathbb{R}^d$ such that $|\mathrm{e}^{-t\mathbf{C}}(\boldsymbol{\xi} + \mathrm{i}\frac{\beta}{2}\mathbf{e}_\ell)|_2 < r$ for some $t > 0$. This is equivalent to $|\mathrm{e}^{-t\mathbf{C}}\boldsymbol{\xi}|_2^2 < r^2 - \frac{\beta^2}{4}\mathrm{e}^{-2tc_\ell}$. Choose now $t_0 > 0$ such that $\frac{\beta^2}{4}\mathrm{e}^{-2t_0 c_\ell} \leq \frac{r^2}{2}$. Then, for any $t > t_0$, the previous inequality is certainly satisfied if $\mathrm{e}^{-2tc_1}|\boldsymbol{\xi}|_2^2 < \frac{r^2}{2}$, or equivalently $|\boldsymbol{\xi}|_2 < r\mathrm{e}^{tc_1}/2 =: R_t$. We use this to continue the estimate of (4.32), now for $t \geq \max\{t_0, 1\}$:

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathrm{e}^{-|\boldsymbol{\xi}|_2^2\gamma_{\mathbf{C}}} \Big| \hat{f}\Big(\mathrm{e}^{-t\mathbf{C}}\Big(\boldsymbol{\xi} + \mathrm{i}\frac{\beta}{2}\mathbf{e}_\ell\Big)\Big)\Big|^2 \,\mathrm{d}\boldsymbol{\xi} \\
&\qquad \leq C \int_{|\boldsymbol{\xi}|_2 < R_t} \mathrm{e}^{-|\boldsymbol{\xi}|_2^2\gamma_{\mathbf{C}}} \Big| \mathrm{e}^{-t\mathbf{C}}\Big(\boldsymbol{\xi} + \mathrm{i}\frac{\beta}{2}\mathbf{e}_\ell\Big)\Big|_2^{2k} \,\mathrm{d}\boldsymbol{\xi} \\
&\qquad\quad + \int_{|\boldsymbol{\xi}|_2 > R_t} \mathrm{e}^{-|\boldsymbol{\xi}|_2^2\gamma_{\mathbf{C}}} \Big| \hat{f}\Big(\mathrm{e}^{-t\mathbf{C}}\Big(\boldsymbol{\xi} + \mathrm{i}\frac{\beta}{2}\mathbf{e}_\ell\Big)\Big)\Big|^2 \,\mathrm{d}\boldsymbol{\xi} \\
&\qquad \leq C\mathrm{e}^{-2kc_1 t} \int_{|\boldsymbol{\xi}|_2 < R_t} \mathrm{e}^{-|\boldsymbol{\xi}|_2^2\gamma_{\mathbf{C}}} \Big| \boldsymbol{\xi} + \mathrm{i}\frac{\beta}{2}\mathbf{e}_\ell\Big|_2^{2k} \,\mathrm{d}\boldsymbol{\xi} + C \int_{|\boldsymbol{\xi}|_2 > R_t} \mathrm{e}^{-|\boldsymbol{\xi}|_2^2\gamma_{\mathbf{C}}} \,\mathrm{d}\boldsymbol{\xi} \\
&\qquad \leq C\mathrm{e}^{-2kc_1 t} \int_{\mathbb{R}^d} \mathrm{e}^{-|\boldsymbol{\xi}|_2^2\gamma_{\mathbf{C}}} \Big| \boldsymbol{\xi} + \mathrm{i}\frac{\beta}{2}\mathbf{e}_\ell\Big|_2^{2k} \,\mathrm{d}\boldsymbol{\xi} + C \int_{|\boldsymbol{\xi}|_2 > R_t} \mathrm{e}^{-|\boldsymbol{\xi}|_2^2\gamma_{\mathbf{C}}} \,\mathrm{d}\boldsymbol{\xi} \\
&\qquad = C\mathrm{e}^{-2kc_1 t} + C \int_{|\boldsymbol{\xi}|_2 > R_t} \mathrm{e}^{-|\boldsymbol{\xi}|_2^2\gamma_{\mathbf{C}}} \,\mathrm{d}\boldsymbol{\xi}.
\end{aligned}$$

In the above calculations we have used the behavior of $\hat{f}$ around the origin in the first integral, and the uniform boundedness of $\hat{f}$ shown in Lemma C.4 in order to estimate the second integral. Next we verify that the remaining integral decays super-exponentially as $t \to \infty$. We easily find that for $t > 0$ (and thus R_t) sufficiently large, by using spherical coordinates (S_d is the surface of the unit sphere in $\mathbb{R}^d$):

$$\begin{aligned}
\int_{|\boldsymbol{\xi}|_2 > R_t} \mathrm{e}^{-|\boldsymbol{\xi}|_2^2\gamma_{\mathbf{C}}} \,\mathrm{d}\boldsymbol{\xi} &= S_d \int_{\rho > R_t} \exp(-\rho^2\gamma_{\mathbf{C}})\rho^{d-1} \,\mathrm{d}\rho \leq S_d \int_{\rho > R_t} \exp(-\rho\gamma_{\mathbf{C}})\rho^{d-1} \,\mathrm{d}\rho \\
&\leq S_d \int_{\rho > R_t} \exp(-\rho\gamma_{\mathbf{C}}/2) \,\mathrm{d}\rho = \frac{2S_d}{\gamma_{\mathbf{C}}} \exp\Big(-\frac{r\gamma_{\mathbf{C}}}{4}\mathrm{e}^{tc_1}\Big).
\end{aligned}$$

This decays super-exponentially, and we conclude that there exists some $C > 0$ such that for all $t > 0$ sufficiently large there holds

$$\int_{\mathbb{R}^d} \mathrm{e}^{-|\boldsymbol{\xi}|_2^2 \gamma_{\mathbf{C}}} \left| \hat{f}\left(\mathrm{e}^{-t\mathbf{C}}\left(\boldsymbol{\xi} + \mathrm{i}\frac{\beta}{2}\mathbf{e}_\ell\right)\right)\right|^2 \mathrm{d}\boldsymbol{\xi} \leq C\mathrm{e}^{-2kc_1 t}. \tag{4.33}$$

Since $(\mathrm{e}^{t\mathcal{L}})_{t\geq 0}$ is uniformly bounded the above estimate (4.33) holds true for all $t \geq 0$ for an appropriately large constant $C > 0$. We get the estimate (4.33) for every term in the norm $\|\mathrm{e}^{t\mathcal{L}} f\|_\omega$, so we conclude: For every $f \in \mathcal{X}_k$ there exists some $C > 0$ such that we have for all $t \geq 0$:

$$\|\mathrm{e}^{t\mathcal{L}} f\|_\omega \leq C\mathrm{e}^{-kc_1 t}.$$

This implies for every $f \in \mathcal{X}_k$ and $\varepsilon > 0$

$$\int_0^\infty \|\mathrm{e}^{t(\mathcal{L}+kc_1-\varepsilon)} f\|_\omega \,\mathrm{d}t < \infty.$$

We now apply the Theorem of Datko and Pazy, see Theorem V.1.8 in [EN00], for $p = 1$, which yields that for every $\varepsilon > 0$ there is a constant $C_{k,\varepsilon} > 0$ such that

$$\|\mathrm{e}^{t\mathcal{L}}|_{\mathcal{X}_k}\|_{\mathscr{B}(\mathcal{X}_k)} \leq C_{k,\varepsilon}\mathrm{e}^{-t(kc_1-\varepsilon)}. \tag{4.34}$$

In order to prove that the decay rate is indeed sharp (i.e. we can choose $\varepsilon = 0$), we start with the observation that according to Proposition 4.14 we can write $\mathcal{X}_k = \mathcal{X}_{k+1} \oplus \operatorname{span}\{\mu_{\mathbf{k}} : |\mathbf{k}|_1 = k\}$, and both spaces are invariant under the semigroup $(\mathrm{e}^{t\mathcal{L}})_{t\geq 0}$. According to (4.34) the semigroup on $\mathcal{X}_{k+1}$ decays with an exponential rate of $-(k+1)c_1 + \varepsilon$. For each eigenfunction $\mu_{\mathbf{k}}$ we know the exact rate $-\mathbf{ck}$, so on $\operatorname{span}\{\mu_{\mathbf{k}} : |\mathbf{k}|_1 = k\}$ the semigroup $(\mathrm{e}^{t\mathcal{L}})_{t\geq 0}$ decays with the exponential rate

$$-\min_{|\mathbf{k}|_1 = k} \mathbf{ck} = -kc_1.$$

Combining those two decay estimates, we conclude that on $\mathcal{X}_k$ the estimate (4.34) indeed holds true for $\varepsilon = 0$, which is exactly the desired inequality (4.31). □

Remark 4.19. In the above proof one can alternatively use the Taylor expansion of $\hat{f}$ with remainder in Lagrange form. Then the proof is carried out in a similar form, however one can avoid the splitting of the integral coming from the norm into two parts, as well as the application of the Theorem of Datko and Pazy. Hence, the analysis becomes more straight-forward. That alternative proof can be found in [AAS15], after Proposition 4.3.

To conclude this section, we summarize the results obtained in this section.

Theorem 4.20. *Let $\omega(\mathbf{x}) = \sum_{i=1}^d \cosh \beta x_i$ for any $\beta > 0$. The Fokker-Planck operator $L|_{C_0^\infty(\mathbb{R}^d)}$ is closable in $\mathcal{X} := L^2(\omega)$, we write $\mathcal{L} = \operatorname{cl}_{\mathcal{X}} L|_{C_0^\infty(\mathbb{R}^d)}$. In $\mathcal{X}$ the operator $\mathcal{L}$ has the following properties:*

(i) *The resolvent of $\mathcal{L}$ is compact, and $\sigma(\mathcal{L})$ consists entirely of isolated eigenvalues.*

(ii) *The spectrum of $\mathcal{L}$ is given by*

$$\sigma(\mathcal{L}) = \{-\mathbf{ck} : \mathbf{k} \in \mathbb{N}^d\},$$

where $\mathbf{c}$ is the line vector containing the diagonal entries of $\mathbf{C}$.

(iii) *For every $\lambda \in \sigma(\mathcal{L})$ the corresponding eigenspace of $\mathcal{L}$ is given by*

$$\operatorname{span}\{\mu_{\mathbf{k}} : -\mathbf{c}\mathbf{k} = \lambda, \mathbf{k} \in \mathbb{N}^d\},$$

where the functions $\mu_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}^d$, are defined in Theorem 4.4.

(iv) *For every $k \in \mathbb{N}$ the subspaces $\mathcal{X}_k \subseteq \mathcal{X}$ (see (4.20)) are $\mathcal{L}$-invariant, and $\sigma(\mathcal{L}|_{\mathcal{X}_k}) = \{-\mathbf{c}\mathbf{k} : |\mathbf{k}|_1 \geq k\}$.*

(v) *For every $k \in \mathbb{N}$ the operator $\mathcal{L}|_{\mathcal{X}_k}$ generates a C_0-semigroup $(\mathrm{e}^{t\mathcal{L}}|_{\mathcal{X}_k})_{t\geq 0}$ on $\mathcal{X}_k$, and there exists a constant C_k such that*

$$\|\mathrm{e}^{t\mathcal{L}}|_{\mathcal{X}_k}\|_{\mathscr{B}(\mathcal{X}_k)} \leq C_k \mathrm{e}^{-tkc_1}, \quad \forall t \geq 0. \tag{4.35}$$

4.3. The perturbed Fokker-Planck operator

Having established $\mathcal{L}$ in $\mathcal{X}$, we now turn to the investigation of the properties of the perturbed operator $\mathcal{L} + \Theta$. We make the following assumptions on Θ:

(C) Conditions on Θ: We assume that $\Theta f := \vartheta * f$ for all $f \in \mathcal{X}$, and the convolution kernel ϑ has the following properties:

(i) The Fourier transform $\hat{\vartheta}$ can be extended to an analytic function in $\Omega_{\beta/2}$ (also denoted by $\hat{\vartheta}$), and $\hat{\vartheta} \in L^\infty(\Omega_{\beta/2})$.

(ii) There holds $\hat{\vartheta}(\mathbf{0}) = 0$, i.e. ϑ is massless.

(iii) The function

$$\boldsymbol{\xi} \mapsto \int_0^1 \frac{1}{s} \hat{\vartheta}(\boldsymbol{\xi}^\top s^{\mathbf{C}}) \, \mathrm{d}s$$

is analytic in $\Omega_{\beta/2}$, and its real part lies in $L^\infty(\Omega_{\beta/2})$.

Lemma 4.21. *Let the assumptions **(C)** hold. Then, for every $k \in \mathbb{N}$ there holds $\Theta \in \mathscr{B}(\mathcal{X}_k, \mathcal{X}_{k+1})$.*

Proof. *Step 1: Boundedness of* Θ: Because of **(C)(i)** we have for every $f \in \mathcal{X}$ that $\mathcal{F}[\Theta f] = \hat{\vartheta}\hat{f}$ is analytic in $\Omega_{\beta/2}$, and since f satisfies (4.10) we find

$$\sup_{\substack{|\mathbf{b}|_1 < \beta/2 \\ \mathbf{b} \in \mathbb{R}^d}} \|\hat{\vartheta}\hat{f}(\cdot + \mathrm{i}\mathbf{b})\|_{L^2(\mathbb{R}^d)} < \infty.$$

So according to Proposition 4.5 Θ maps $\mathcal{X}$ into $\mathcal{X}$. In order to show the boundedness of Θ we use the norm $|\!|\!| \cdot |\!|\!|_\omega$, see (4.12). We start with the following computation, where $\ell \in \{1, \ldots, d\}$:

$$\begin{aligned} \int_{\mathbb{R}^d} \left|(\hat{\vartheta}\hat{f})\Big(\boldsymbol{\xi} \pm \mathrm{i}\frac{\beta}{2}\mathbf{e}_\ell\Big)\right|^2 \mathrm{d}\boldsymbol{\xi} &= \lim_{b \nearrow \beta/2} \int_{\mathbb{R}^d} \left|(\hat{\vartheta}\hat{f})(\boldsymbol{\xi} \pm \mathrm{i}b\mathbf{e}_\ell)\right|^2 \mathrm{d}\boldsymbol{\xi} \\ &\leq \|\hat{\vartheta}\|^2_{L^\infty(\Omega_{\beta/2})} \lim_{b \nearrow \beta/2} \int_{\mathbb{R}^d} \left|\hat{f}(\boldsymbol{\xi} \pm \mathrm{i}b\mathbf{e}_\ell)\right|^2 \mathrm{d}\boldsymbol{\xi} \\ &= \|\hat{\vartheta}\|^2_{L^\infty(\Omega_{\beta/2})} \int_{\mathbb{R}^d} \left|\hat{f}\Big(\boldsymbol{\xi} \pm \mathrm{i}\frac{\beta}{2}\mathbf{e}_\ell\Big)\right|^2 \mathrm{d}\boldsymbol{\xi}. \end{aligned}$$

Here we have used (ii) in Proposition 4.5. Note that $\hat{\vartheta}\hat{f}$ is the Fourier-transform of an element of $\mathcal{X}$, and thus we may evaluate it at the boundary of $\Omega_{\beta/2}$ in the sense of L^2-functions. We can repeat the above estimate for every $\ell \in \{1, \ldots, d\}$ and conclude from (4.12) that Θ is bounded in $\mathcal{X}$ with a norm proportional to $\|\hat{\vartheta}\|_{L^\infty(\Omega_{\beta/2})}$.

Step 2: $\Theta\colon \mathcal{X}_k \to \mathcal{X}_{k+1}$: According to (4.21) f lies in $\mathcal{X}_k$ iff $\hat{f}$ has a zero of order greater than or equal to k at the origin. Now, because of **(C)(ii)**, $\hat{\vartheta}\hat{f}$ has a zero of order greater than or equal to $k+1$ at the origin. Together with the first part of this proof this shows that $\Theta f \in \mathcal{X}_{k+1}$. □

Corollary 4.22. *If Θ satisfies **(C)** then for every $k \in \mathbb{N}$ the space $\mathcal{X}_k$ is invariant under $\mathcal{L} + \Theta$.*

Proof. This is an immediate consequence of Proposition 4.14 and Lemma 4.21 above. □

Throughout the rest of this article we always assume that Θ is such that the conditions **(C)** are satisfied in $\mathcal{X}$ for some $\beta > 0$. Now we fix this β and consider $\mathcal{X}$ with the according weight function $\omega(\mathbf{x}) = \sum_{j=1}^{d} \cosh \beta x_j$.

Lemma 4.23. *The spectrum $\sigma(\mathcal{L} + \Theta)$ consists entirely of isolated eigenvalues.*

Proof. First we show that the resolvent of $\mathcal{L} + \Theta$ is compact in $\mathcal{X}$. According to Theorem 4.20 $\mathcal{L}$ generates a C_0-semigroup of bounded operators in $\mathcal{X}$ and has a compact resolvent. According to Lemma 4.21 Θ is a bounded operator. Thus we can apply Proposition III.1.12 in [EN00], which proves that $R_{\mathcal{L}+\Theta}(\zeta)$ is compact in $\mathcal{X}$ for every $\zeta \in \rho(\mathcal{L} + \Theta)$. It now remains to apply the Theorem III.6.29 in [Kat66], which proves that $\sigma(\mathcal{L} + \Theta)$ consists entirely of isolated eigenvalues. □

In order to characterize the spectrum of $\mathcal{L} + \Theta$ and the corresponding semigroup we introduce the operator $\Psi\colon \mathcal{X} \to \mathcal{X}\colon f \mapsto f * \psi$. The function ψ is defined via the Fourier transform by

$$\hat{\psi}(\boldsymbol{\xi}) := \exp\Big(\int_0^1 \frac{1}{s}\hat{\vartheta}(\boldsymbol{\xi}^\top s^{\mathbf{C}})\,\mathrm{d}s\Big).$$

Lemma 4.24. *Ψ satisfies the following properties in $\mathcal{X}$:*

(i) *For every $k \in \mathbb{N}$ the operator $\Psi\colon \mathcal{X}_k \to \mathcal{X}_k$ is a bijection.*
(ii) *Both Ψ and its inverse Ψ^{-1} are bounded, and $\Psi^{-1} f = \mathcal{F}^{-1}[\hat{f}/\hat{\psi}]$ for all $f \in \mathcal{X}$.*

Proof. For the moment we define the operator $\bar{\Psi} f := \mathcal{F}^{-1}[\hat{f}/\hat{\psi}]$ for all $f \in \mathcal{X}$, and show in the following that it is the inverse of Ψ. To begin with we note that because of the condition **(C)(iii)** both $\hat{\psi}$ and $1/\hat{\psi}$ are analytic and uniformly bounded in $\Omega_{\beta/2}$. Thus it follows analogously to the proof of Lemma 4.21 that both Ψ and $\bar{\Psi}$ are bounded operators in $\mathcal{X}$.

Since $\hat{\psi}$ and $1/\hat{\psi}$ both do not have any zeros in $\Omega_{\beta/2}$ (especially at $\boldsymbol{\xi} = \mathbf{0}$) it follows from the characterization (4.21) of the space $\mathcal{X}_k$ that Ψ and $\bar{\Psi}$ map $\mathcal{X}_k$ into itself for every $k \in \mathbb{N}$.

Finally we observe that for every $f \in \mathcal{X}$ there holds $\Psi\bar{\Psi} f = \bar{\Psi}\Psi = f$, which finally proves that $\bar{\Psi} = \Psi^{-1}$. □

Proposition 4.25. *There holds*

(i) $\sigma(\mathcal{L} + \Theta) = \sigma(\mathcal{L})$.
(ii) *For every $\mathbf{k} \in \mathbb{N}^d$ the function $f_{\mathbf{k}} := \Psi\mu_{\mathbf{k}}$ is an eigenfunction of $\mathcal{L} + \Theta$ to the eigenvalue $-\mathbf{ck}$. Furthermore, for every $\lambda \in \sigma(\mathcal{L} + \Theta)$*

$$\ker(\lambda - \mathcal{L} - \Theta) = \operatorname{span}\{f_{\mathbf{k}} : -\mathbf{ck} = \lambda\}.$$

(iii) *The eigenfunctions satisfy $f_{\mathbf{k}} = \nabla^{\mathbf{k}} f_{\mathbf{0}}$ for all $\mathbf{k} \in \mathbb{N}^d$.*

Proof. From Lemma 4.23 we know that the spectrum of $\mathcal{L}+\Theta$ consists entirely of eigenvalues. So, in order to determine the spectrum we look for $\lambda \in \mathbb{C}$ and non-trivial solutions $f \in \mathcal{X}$ of $(\lambda-\mathcal{L}-\Theta)f = 0$. After applying the Fourier transform this equation reads

$$(\lambda + \boldsymbol{\xi}\cdot\boldsymbol{\xi})\hat{f} + \boldsymbol{\xi}^\top \mathbf{C}\nabla \hat{f} = \hat{\vartheta}\hat{f}.$$

We now make the (non-restrictive) ansatz $\hat{f} = \hat{p}\hat{\psi}$. Note that because of **(C)(iii)** and $\hat{\psi} \neq 0$ in $\Omega_{\beta/2}$, the requirement $f \in \mathcal{X}$ implies that $\hat{p}$ is analytic in $\Omega_{\beta/2}$. A short calculations shows that $\hat{\psi}\hat{\vartheta} = \boldsymbol{\xi}^\top\mathbf{C}\nabla\hat{\psi}$. Using this, we obtain the following equation for $\hat{p}$:

$$(\lambda + \boldsymbol{\xi}\cdot\boldsymbol{\xi})\hat{p} + \boldsymbol{\xi}^\top \mathbf{C}\nabla \hat{p} = 0.$$

We find that this is exactly the equation (4.16). In the proof of Lemma 4.8 we have shown that $0 \not\equiv p \in \mathcal{X}$ is a solution iff $\lambda \in \{-\mathbf{ck} : \mathbf{k} \in \mathbb{N}^d\}$. And for a fixed $\lambda \in \mathbb{C}$, $p \in \operatorname{span}\{\mu_{\mathbf{k}} : -\mathbf{ck} = \lambda\}$. □

Corollary 4.26. *Ψ is a similarity map between $\mathcal{L}$ and $\mathcal{L}+\Theta$.*

Proof. According to Proposition 4.25 the eigenfunctions of $\mathcal{L}$ and $\mathcal{L}+\Theta$ are related by $f_{\mathbf{k}} = \Psi\mu_{\mathbf{k}}$, for every $\mathbf{k} \in \mathbb{N}^d$. So we find for every $\zeta \notin \sigma(\mathcal{L})$ and $\mathbf{k} \in \mathbb{N}^d$ that the resolvents satisfy

$$R_{\mathcal{L}}(\zeta)\mu_{\mathbf{k}} = \frac{1}{\zeta + \mathbf{ck}}\mu_{\mathbf{k}} = \Psi^{-1}\frac{1}{\zeta+\mathbf{ck}} f_{\mathbf{k}} = \Psi^{-1}R_{\mathcal{L}+\Theta}(\zeta)\Psi\mu_{\mathbf{k}}.$$

Since $\operatorname{span}\{\mu_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\} \subset \mathcal{X}$ is dense and all occurring operators are bounded, we conclude the following operator equality in $\mathcal{X}$:

$$\Psi R_{\mathcal{L}}(\zeta)\Psi^{-1} = R_{\mathcal{L}+\Theta}(\zeta). \tag{4.36}$$

Taking the inverse shows the similarity $\mathcal{L}+\Theta = \Psi\mathcal{L}\Psi^{-1}$. □

Proposition 4.27. *On every space $\mathcal{X}_k$, with $k \in \mathbb{N}$, the operator $\mathcal{L}+\Theta$ generates a C_0-semigroup of bounded operators $(\mathrm{e}^{t(\mathcal{L}+\Theta)}|_{\mathcal{X}_k})_{t\geq 0}$. For every $k \in \mathbb{N}$ there exists some constant $\tilde{C}_k > 0$ such that*

$$\|\mathrm{e}^{t(\mathcal{L}+\Theta)}|_{\mathcal{X}_k}\|_{\mathscr{B}(\mathcal{X}_k)} \leq \tilde{C}_k \mathrm{e}^{-tkc_1}, \quad \forall t \geq 0.$$

Proof. For this we show the similarity of $\mathcal{L}$ and $\mathcal{L}+\Theta$ shown in Corollary 4.26. Fix any $k \in \mathbb{N}$. According to Corollary 4.22 and Lemma 4.24 the identity (4.36) holds also in $\mathcal{X}_k$, and $R_{\mathcal{L}+\Theta}(\zeta)$ is a bounded operator in $\mathcal{X}_k$. Now we apply the Hille-Yosida Theorem to the decay estimate presented in Theorem 4.20 (v). It shows that for all $n \in \mathbb{N}^*$ and $\operatorname{Re}\zeta > -kc_1$ there holds

$$\|R_{\mathcal{L}}(\zeta)^n|_{\mathcal{X}_k}\|_{\mathscr{B}(\mathcal{X}_k)} \leq \frac{C_k}{(\operatorname{Re}\zeta + kc_1)^n},$$

where $C_k > 0$ is the same constant as in (4.35). Applying this resolvent estimate to (4.36) yields that for all $n \in \mathbb{N}$ and $\operatorname{Re}\zeta > -kc_1$ we have

$$\|R_{\mathcal{L}+\Theta}(\zeta)^n|_{\mathcal{X}_k}\|_{\mathscr{B}(\mathcal{X}_k)} \leq \frac{C_k\|\Psi\|_{\mathscr{B}(\mathcal{X}_k)}\|\Psi^{-1}\|_{\mathscr{B}(\mathcal{X}_k)}}{(\operatorname{Re}\zeta + kc_1)^n}.$$

Applying the Hille-Yosida Theorem again implies that $\mathcal{L}+\Theta$ generates a C_0-semigroup of bounded operators, and that it satisfies the following estimate:

$$\|\mathrm{e}^{t(\mathcal{L}+\Theta)}|_{\mathcal{X}_k}\|_{\mathscr{B}(\mathcal{X}_k)} \leq \tilde{C}_k \mathrm{e}^{-tkc_1},$$

where $0 < \tilde{C}_k \leq C_k \|\Psi\|_{\mathscr{B}(\mathcal{X}_k)} \|\Psi^{-1}\|_{\mathscr{B}(\mathcal{X}_k)}$. □

We conclude this section by summarizing the main results.

Theorem 4.28. *Under the conditions **(C)** on Θ, the perturbed Fokker-Planck operator $\mathcal{L}+\Theta$ has the following properties in $\mathcal{X}$:*

(i) $\sigma(\mathcal{L}+\Theta) = \sigma(\mathcal{L}) = \{-\mathbf{ck} : \mathbf{k} \in \mathbb{N}^d\}$, *i.e. $\mathcal{L}+\Theta$ is an* isospectral deformation *of $\mathcal{L}$.*

(ii) *The functions $f_{\mathbf{k}} = \Psi\mu_{\mathbf{k}}$ are eigenfunctions of $\mathcal{L}+\Theta$ for all $\mathbf{k} \in \mathbb{N}^d$. For every $\lambda \in \sigma(\mathcal{L}+\Theta)$ the corresponding eigenspace is given by*

$$\ker(\lambda - (\mathcal{L}+\Theta)) = \operatorname{span}\{f_{\mathbf{k}} : -\mathbf{ck} = \lambda\}.$$

(iii) *For every $k \in \mathbb{N}$, $(\mathcal{L}+\Theta)|_{\mathcal{X}_k}$ generates a C_0-semigroup $(\mathrm{e}^{t(\mathcal{L}+\Theta)}|_{\mathcal{X}_k})_{t\geq 0}$ on $\mathcal{X}_k$, and there exists a constant $\tilde{C}_k > 0$ such that*

$$\left\|\mathrm{e}^{t(\mathcal{L}+\Theta)}|_{\mathcal{X}_k}\right\|_{\mathscr{B}(\mathcal{X}_k)} \leq \tilde{C}_k \mathrm{e}^{-c_1 kt}, \quad \forall t \geq 0.$$

In particular, this theorem implies exponential convergence of the solutions of the perturbed Fokker-Planck equation towards the stationary solution:

Corollary 4.29. *Let an initial condition $\varphi \in \mathcal{X}$ at $t = 0$ be given, and let $f(t) := \mathrm{e}^{t(\mathcal{L}+\Theta)}\varphi$ be the corresponding solution of the perturbed Fokker-Planck equation* (1.1)*. Set $m := \int_{\mathbb{R}^d} \varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} \in \mathbb{C}$ (the "mass" of φ). Then there exists a constant $C > 0$ such that*

$$\|f(t) - mf_{\mathbf{0}}\|_\omega \leq C\|\varphi - mf_{\mathbf{0}}\|_\omega \mathrm{e}^{-tc_1}, \quad \forall t \geq 0,$$

i.e. $f(t)$ converges exponentially to $mf_{\mathbf{0}}$, with a rate independent of Θ.

Proof. Since $f_{\mathbf{0}}$ is the unique normalized zero eigenfunction of $\mathcal{L}+\Theta$ we obtain:

$$f(t) - mf_{\mathbf{0}} = \mathrm{e}^{t(\mathcal{L}+\Theta)}(\varphi - mf_{\mathbf{0}}).$$

Since $\varphi - mf_{\mathbf{0}}$ has zero mean, it follows from Lemma 4.20 that it lies in $\mathcal{X}_1$. But $(\mathrm{e}^{t(\mathcal{L}+\Theta)})_{t\geq 0}$ decays exponentially on $\mathcal{X}_1$ with rate $-c_1$, see Theorem 4.28 (iii). So we get for all $t \geq 0$:

$$\|f(t) - mf_{\mathbf{0}}\|_\omega = \|\mathrm{e}^{t(\mathcal{L}+\Theta)}(\varphi - mf_{\mathbf{0}})\|_\omega \leq \tilde{C}_1\|(\varphi - mf_{\mathbf{0}})\|_\omega\, \mathrm{e}^{-tc_1}.$$

□

Remark 4.30. Note that $\mathcal{L}+\Theta$ is neither self-adjoint in X nor in $\mathcal{X}$. But the fact that $\mathcal{L}+\Theta$ is similar to $\mathcal{L}$, see Corollary 4.26, and that the similarity map Ψ is a bounded bijection with bounded inverse, suggests that $\mathcal{L}+\Theta$ is self-adjoint in an appropriate space. To demonstrate this we introduce the inner product

$$\langle f, g\rangle_{\mathfrak{X}} := \int_{\mathbb{R}^d} \frac{1}{\mu}\Psi^{-1}f \cdot \overline{\Psi^{-1}g}\,\mathrm{d}\mathbf{x},$$

and the corresponding norm $\|\cdot\|_{\mathfrak{X}}$. The associated space $\mathfrak{X}$ is the set of all functions such that $\|\cdot\|_{\mathfrak{X}}$ is finite. This is indeed a Hilbert space, and Ψ is an isometry between X and $\mathfrak{X}$. Using (4.36) we see the self-adjointness of $L+\Theta$ in $\mathfrak{X}$:

$$\begin{aligned}\langle (L+\Theta)f, g\rangle_{\mathfrak{X}} &= \langle \Psi \circ L \circ \Psi^{-1} f, g\rangle_{\mathfrak{X}} \\ &= \langle L(\Psi^{-1}f), \Psi^{-1}g\rangle_X = \langle \Psi^{-1}f, L(\Psi^{-1}g)\rangle_X \\ &= \langle f, (L+\Theta)g\rangle_{\mathfrak{X}},\end{aligned}$$

where we have used the self-adjointness of L in X. In $\mathfrak{X}$ the eigenfunctions $f_{\mathbf{k}}$ of $L+\Theta$ are orthogonal again (like the functions $\mu_{\mathbf{k}}$ in X). Altogether, we conclude that L in H and $L+\Theta$ in $\mathfrak{X}$ are isometrically equivalent via the map Ψ. Hence, $L+\Theta$ inherits most properties of L. However, we point out that discovering the map Ψ, without the preceding analysis, is a non-trivial issue.

The Hilbert space $\mathfrak{X}$ is difficult to be characterized explicitly. In particular, it is usually not possible to describe $\mathfrak{X}$ as a weighted L^2-space. A simple calculation shows that $\mathfrak{X} = L^2(\nu)$ for some weight function ν only if for all $f \in C_0^\infty(\mathbb{R}^d)$ there holds

$$\nu = \frac{1}{f} \cdot \Big(\frac{1}{\psi}\Big) * \Big(\frac{(1/\psi) * f}{\mu}\Big).$$

But in general this function ν will not be independent of f.

CHAPTER 5

The Fokker-Planck operator in other weighted spaces

The results of the previous chapters can be extended to L^2-spaces with more general weight functions, namely (5.12). We give an analysis of the unperturbed Fokker-Planck operator (4.3) in the corresponding L^2-spaces, which is based on a technique presented in [GMM10]. To this end we need to introduce further notation and definitions.

Let X be a Hilbert space. For $A \in \mathscr{C}(X)$ and $B \in \mathscr{B}(X)$, we define $A+B := A|_{D(A)} + B|_{D(A)}$, which is closed with the domain $D(A+B) := D(A)$. It is convenient to define the open half plane $\Delta_a := \{z \in \mathbb{C} : \operatorname{Re} z > a\}$ for $a \in \mathbb{R}$. When we introduce, for fixed $N \in \mathbb{N}$, constants $\zeta_0, \dots, \zeta_{N-1} \in \mathbb{C}$, we use the convention $\{\zeta_0, \dots, \zeta_{N-1}\} := \emptyset$ whenever $N = 0$. For a closed operator A we make the following definitions:

Condition 5.1. Let $a \in \mathbb{R}$ and $N \in \mathbb{N}$ be given, and consider the (possibly empty) finite set $Z := \{\zeta_0, \dots, \zeta_{N-1}\} \subset \Delta_a$. We say that $A \in \mathscr{C}(X)$ fulfills the

- *weak localization of the spectrum* iff $\sigma(A) \cap \Delta_a = Z$. For a, Z given, we denote the set of all operators $A \in \mathscr{C}(X)$ satisfying this condition by $\mathrm{WLS}_X(a, Z)$.
- *strong localization of the spectrum* iff $A \in \mathrm{WLS}_X(a, Z)$ and furthermore the range of the spectral projection corresponding to the set Z is finite-dimensional. The set of all such operators $A \in \mathscr{C}(X)$ is denoted by $\mathrm{SLS}_X(a, Z)$.

Whenever the choice of X is clear from the context, we shall omit the subscript X in the notation, i.e. $\mathrm{SLS}_X(a, Z) \equiv \mathrm{SLS}(a, Z)$ and $\mathrm{WLS}_X(a, Z) \equiv \mathrm{WLS}(a, Z)$.

Remark 5.2. Note that if $A \in \mathrm{SLS}(a, Z)$ for some $a \in \mathbb{R}$ and a finite set $Z \subset \mathbb{C}$, then $Z \subset \sigma_p(A)$, see Proposition B.10 (iv).

Condition 5.3. Consider two Hilbert spaces $X \hookrightarrow \mathcal{X}$, a constant $a \in \mathbb{R}$, and a finite set $Z := \{\zeta_0, \dots, \zeta_{N-1}\} \subset \Delta_a$, with $N \in \mathbb{N}$. We define the set $\mathrm{DCP}_{X,\mathcal{X}}(a, Z)$ as the set of all operators $\mathcal{A} \in \mathscr{C}(\mathcal{X})$ with the following properties: There exist operators $\mathcal{B} \in \mathscr{B}(\mathcal{X}, X)$ and $\mathcal{S} \in \mathscr{C}(\mathcal{X})$ with $D(\mathcal{A}) = D(\mathcal{S})$, such that

(i) $\mathcal{A} = \mathcal{B} + \mathcal{S}$,
(ii) $\sigma(\mathcal{S}) \cap \Delta_a \subseteq Z$,
(iii) there is some $\lambda \in \Delta_a \setminus Z$ such that $\lambda - \mathcal{A}$ is injective in $\mathcal{X}$.

The operators in $\mathrm{DCP}_{X,\mathcal{X}}(a, Z)$ are said to satisfy the *decomposition property.*

The following theorem is the key ingredient to finding an appropriate weight ω:

Theorem 5.4. *Let $X \hookrightarrow \mathcal{X}$ be two Hilbert spaces, and $Z := \{\zeta_0, \dots, \zeta_{N-1}\} \subset \Delta_a$ for some $N \in \mathbb{N}$, with $a \in \mathbb{R}$ fixed. Consider an operator $A \in \mathscr{C}(X)$ with the following properties in X:*

(i) *For all $0 \le j \le N-1$ there holds $\zeta_j \in \sigma(A)$ and $\dim M(\zeta_j - A) < \infty$.*

(ii) *A can be decomposed according to the orthogonal projection onto*

$$Y_N := [M(\zeta_0 - A) \oplus \cdots \oplus M(\zeta_{N-1} - A)]^{\perp}.$$

(iii) *$A - a$ is dissipative in Y_N.*
(iv) *$\operatorname{ran}(A - b)|_{Y_N} = Y_N$ for some $b > a$.*

Furthermore, assume in $\mathcal{X}$:

(v) *There exist $\mathcal{B} \in \mathscr{B}(\mathcal{X}, X)$ and an operator $\mathcal{S} \in \mathscr{C}(\mathcal{X})$ with the property that $\mathcal{S} - a$ is dissipative, and there holds the decomposition $A \subset \mathcal{B} + \mathcal{S}$ (in the sense of graphs).*

Then we have:

1. *A generates a C_0-semigroup both on X and Y_N. On Y_N the semigroup satisfies the following estimate: $\|\mathrm{e}^{tA}|_{Y_N}\|_{\mathscr{B}(Y_N)} \leq \mathrm{e}^{at}$.*
2. *We have $A \in \mathrm{SLS}_X(a, Z)$, and $Y_N = \ker(\Pi_{A,0} + \cdots + \Pi_{A,N-1})$, where $\Pi_{A,j}$ denotes the spectral projection of A corresponding to ζ_j.*
3. *A is closable in $\mathcal{X}$, its closure $\mathcal{A} := \mathrm{cl}_{\mathcal{X}} A$ also satisfies $\mathcal{A} \in \mathrm{SLS}_{\mathcal{X}}(a, Z)$. The algebraic eigenspaces satisfy $M(\zeta_j - \mathcal{A}) = M(\zeta_j - A) \subset X$, for every $0 \leq j \leq N-1$.*
4. *For $0 \leq j \leq N-1$, the spectral projection $\Pi_{\mathcal{A},j}$ of $\mathcal{A}$ corresponding to $\zeta_j \in \sigma(\mathcal{A})$ equals the closure of $\Pi_{A,j}$. In particular, $\operatorname{ran}\Pi_{\mathcal{A},j} = \operatorname{ran}\Pi_{A,j} = M(\zeta_j - A)$, and*
$$\ker \Pi_{\mathcal{A},j} = \mathcal{Y}_N \oplus \Big(\bigoplus_{\ell \neq j} M(\zeta_\ell - A)\Big),$$
where $\mathcal{Y}_N := \mathrm{cl}_{\mathcal{X}} Y_N$. $\mathcal{A}$ is decomposed according to $\sum_{j=0}^{N-1} \Pi_{\mathcal{A},j}$.
5. *$\mathcal{A}$ generates a C_0-semigroup both on $\mathcal{X}$ and $\mathcal{Y}_N$, and for any $a' > a$ there exists some $C_{a'} \geq 1$ such that*
$$\|\mathrm{e}^{t\mathcal{A}}|_{\mathcal{Y}_N}\|_{\mathscr{B}(\mathcal{Y}_N)} \leq C_{a'} \mathrm{e}^{a't}, \quad \forall t \geq 0.$$

This result and the corresponding proof is based on Corollary 4.2 in [GMM10], and [AGG+12], where a simpler version of that theory was applied to the Wigner-Fokker-Planck operator.

Proof. We start by proving Result 1. According to Assumption (ii) we have $A \in \mathscr{C}(Y_N)$. Because of the Conditions (iii) and (iv) the operator $A - a$ fulfills the requirements of the Lumer-Phillips Theorem in Y_N. Hence $A - a$ generates a C_0-semigroup of contractions on Y_N. Equivalently, A generates a C_0-semigroup $(\mathrm{e}^{tA}|_{Y_N})_{t\geq 0}$ on Y_N which satisfies the decay estimate given in Result 1. The orthogonal complement of Y_N in X, $M(\zeta_0 - A) \oplus \cdots \oplus M(\zeta_{N-1} - A)$, is a finite-dimensional, A-invariant space. So A is a bounded operator on this space, and clearly generates a semigroup $(\mathrm{e}^{tA}|_{Y_N^\perp})_{t\geq 0}$ on $Y_N^\perp$. Since A is decomposed according to $X = Y_N \oplus Y_N^\perp$(see the discussion after Proposition B.9 in the Appendix), we conclude that A generates a C_0-semigroup of bounded operators $(\mathrm{e}^{tA})_{t\geq 0}$ on $X = Y_N \oplus Y_N^\perp$, where $(\mathrm{e}^{tA})_{t\geq 0} = (\mathrm{e}^{tA}|_{Y_N})_{t\geq 0} \otimes (\mathrm{e}^{tA}|_{Y_N^\perp})_{t\geq 0}$.

We now show Result 2. Because of semigroup decay estimate shown in Result 1 above we know that $\sigma(A|_{Y_N}) \cap \Delta_a = \emptyset$. On $Y_N^\perp$ we have $\sigma(A|_{Y_N^\perp}) = \sigma_p(A|_{Y_N^\perp}) = Z$. Combining these results we obtain $\sigma(A) \cap \Delta_a = Z$. This proves that $A \in \mathrm{WLS}_X(a, Z)$. Next we apply Lemma B.13 with $Y = Y_N$, which shows that for every $0 \leq j \leq N-1$ we have $\operatorname{ran}\Pi_{A,j} = M(\zeta_j - A)$, where $\Pi_{A,j}$ is the spectral projection corresponding to ζ_j.

Since the algebraic eigenspaces $M(\zeta_j - A)$, for $0 \le j \le N-1$, are finite-dimensional, this proves $A \in \mathrm{SLS}_X(a, Z)$, according to Proposition B.10 (iv). Furthermore, we obtain that $Y_N = \ker(\Pi_{A,0} + \cdots + \Pi_{A,N-1})$.

For Result 3 we adapt the proof of Corollary 4.2 in [GMM10]. We begin by showing that $\mathcal{S}$ from Assumption (v) generates a C_0-semigroup on $\mathcal{X}$. From Assumption (v) we find that $\mathcal{B}|_X \in \mathscr{B}(X)$, and since A generates a C_0-semigroup on X, see Result 1, the operator $\mathcal{S}|_{D(A)} = A - \mathcal{B}|_{D(A)}$ also generates a C_0-semigroup $(\mathrm{e}^{t\mathcal{S}|_{D(A)}})_{t\ge 0}$ on X. For $f \in D(A)$ the map $t \mapsto \mathrm{e}^{t\mathcal{S}|_{D(A)}} f$ lies in $C^1((0,\infty); X)$. Because of $X \in \mathcal{X}$ we find that this map also lies in $C^1((0,\infty); \mathcal{X})$. Since $\mathcal{S} - a$ is dissipative in $\mathcal{X}$, we obtain for all $t > 0$ and $f \in D(A)$:

$$\frac{\mathrm{d}}{\mathrm{d}t} \big\| \mathrm{e}^{t\mathcal{S}|_{D(A)}} f \big\|_{\mathcal{X}}^2 = 2 \operatorname{Re} \langle \mathcal{S} \mathrm{e}^{t\mathcal{S}|_{D(A)}} f, \mathrm{e}^{t\mathcal{S}|_{D(A)}} f \rangle_{\mathcal{X}} \le 2a \big\| \mathrm{e}^{t\mathcal{S}|_{D(A)}} f \big\|_{\mathcal{X}}^2 .$$

The Grönwall inequality implies for all $f \in D(A)$:

$$\big\| \mathrm{e}^{t\mathcal{S}|_{D(A)}} f \big\|_{\mathcal{X}} \le \mathrm{e}^{at} \| f \|_{\mathcal{X}}, \quad \forall t \ge 0. \tag{5.1}$$

Because of $X \hookrightarrow \mathcal{X}$ we have $D(A) \subset \mathcal{X}$ dense. So we may consider the closure $T(t) := \mathrm{cl}_{\mathcal{X}}\, \mathrm{e}^{t\mathcal{S}|_{D(A)}}$, which is a bounded operator on $\mathcal{X}$ for every $t \ge 0$ with the norm e^{at}, and we get an estimate of the form (5.1) for all $f \in \mathcal{X}$. We then apply Lemma C.5 (with $A = \mathcal{S}|_{D(A)}$), which proves that $(T(t))_{t\ge 0}$ is a C_0-semigroup on $\mathcal{X}$, and its generator is $\mathcal{S} := \mathrm{cl}_{\mathcal{X}}\, \mathcal{S}|_{D(A)}$ (here we possibly redefine $\mathcal{S}$). This (new) $\mathcal{S}$ still fulfills Assumption (v). Together with $\mathcal{B} \in \mathscr{B}(\mathcal{X})$ this implies that $\mathcal{A} := \mathrm{cl}_{\mathcal{X}}\, A \equiv \mathcal{B} + \mathcal{S}$ is the generator of a C_0-semigroup $(\mathrm{e}^{t\mathcal{A}})_{t\ge 0}$. With this choice of $\mathcal{B}, \mathcal{S}$ there holds $\mathcal{A} \in \mathrm{DCP}_{X,\mathcal{X}}(a, Z)$. To see this, we verify the conditions in Definition 5.3. According to the assumptions, Condition (i) from Definition 5.3 is clearly verified. From the estimate (5.1) for the semigroup generated by $\mathcal{S}$ on $\mathcal{X}$ we deduce for the spectral bound $\mathrm{s}(\mathcal{S}) \le a$. Hence, $\sigma(\mathcal{S}) \cap \Delta_a = \emptyset$. This shows that Definition 5.3 (ii) is fulfilled. Finally, since $\mathcal{A}$ generates a C_0-semigroup, there holds $\rho(A) \supset \Delta_{a'}$ for some $a' > a$. So we can easily find some $\lambda \in \Delta_a$ that fulfills Definition 5.3 (iii).

From Result 2 we know that $A \in \mathrm{WLS}_X(a, Z)$. And we have just shown that $\mathcal{A} \in \mathrm{DCP}_{X,\mathcal{X}}(a, Z)$. So the requirements of Theorem B.15 are fulfilled. We conclude that $\mathcal{A} \in \mathrm{SLS}_{\mathcal{X}}(a, Z)$, and $M(\zeta_j - \mathcal{A}) = M(\zeta_j - A)$ for all $0 \le j \le N-1$. This confirms Result 3.

For Result 4 we first characterize $\Pi_{A,j}$ in X. We have already demonstrated in the proof of Result 2 that $\operatorname{ran} \Pi_{A,j} = M(\zeta_j - A)$ for $0 \le j \le N-1$, and

$$\ker \Pi_{A,j} = Y_N \oplus \Big(\bigoplus_{\ell \ne j} M(\zeta_\ell - A) \Big), \quad 0 \le j \le N-1.$$

There holds $A \in \mathrm{SLS}_X(a, Z)$ and $\mathcal{A} \in \mathrm{SLS}_{\mathcal{X}}(a, Z)$. We denote the spectral projection of A and $\mathcal{A}$ corresponding to the set Z by $\Pi_{A,Z}$ and $\Pi_{\mathcal{A},Z}$, respectively. Similar to the proof of Proposition 3.12 we observe that $\Pi_{A,Z} \subset \Pi_{\mathcal{A},Z}$. By applying Lemma C.1 we conclude that

$$\mathcal{Y}_N := \mathrm{cl}_{\mathcal{X}}\, Y_N = \ker \Pi_{\mathcal{A},Z}, \qquad \bigoplus_{j=0}^{N-1} M(\zeta_j - A) = \operatorname{ran} \Pi_{\mathcal{A},Z}.$$

Note that $\dim M(\zeta_j - A) < \infty$ for all $0 \le j \le N-1$. Since $\Pi_{\mathcal{A},Z} \in \mathscr{B}(\mathcal{X})$, this shows that we can write (see the discussion in Section C.2 in the Appendix)

$$\mathcal{X} = \mathcal{Y}_N \oplus \Big(\bigoplus_{j=0}^{N-1} M(\zeta_j - A) \Big). \tag{5.2}$$

By mimicking the above argument with $\Pi_{\mathcal{A},j}$ instead of $\Pi_{\mathcal{A},Z}$ and by using the fact that $\dim M(\zeta_j - A) = \dim \operatorname{ran} \Pi_{A,j} < \infty$ for all $0 \le j \le N-1$ we obtain that $\operatorname{ran} \Pi_{\mathcal{A},j} = M(\zeta_j - \mathcal{A}) = M(\zeta_j - A)$, for all $0 \le j \le N-1$. Applying this result in (5.2) proves Result 4.

Finally, we show Result 5. From the end of the proof of Result 3 we know that we can apply Theorem B.15. It yields the following representation of the resolvent:

$$R_{\mathcal{A}}(\zeta) = (\zeta - \mathcal{S})^{-1} + R_A(\zeta)\mathcal{B}(\zeta - \mathcal{S})^{-1}, \quad \forall \zeta \in \Delta_a \backslash Z. \tag{5.3}$$

Our goal is to show a uniform estimate of $R_{\mathcal{A}}(\zeta)$ by using (5.3). The operators of the semigroup $(T(t))_{t\ge0}$ generated by $\mathcal{S}$ satisfy $\|T(t)\|_{\mathscr{B}(\mathcal{X})} \le \mathrm{e}^{at}$ for all $t \ge 0$, according to (5.1). So we may apply the Hille-Yosida theorem to get the resolvent estimate

$$\|(\zeta - \mathcal{S})^{-1}\|_{\mathscr{B}(\mathcal{X})} \le \frac{1}{\operatorname{Re}\zeta - a}, \quad \operatorname{Re}\zeta > a. \tag{5.4}$$

By combining the growth bound of $(\mathrm{e}^{tA})_{t\ge0}$ shown in Result 1 and Result 2 we may apply Lemma B.14 from the Appendix. This yields the following estimate, which holds true (uniformly) for all $\zeta \in \Delta_a \setminus Z$:

$$\|R_A(\zeta)\|_{\mathscr{B}(X)} \le C \max\left\{ \frac{1}{\operatorname{Re}\zeta - a}, \frac{|\zeta - \zeta_0|^{d_0-1} + 1}{|\zeta - \zeta_0|^{d_0}}, \dots, \frac{|\zeta - \zeta_{N-1}|^{d_{N-1}-1} + 1}{|\zeta - \zeta_{N-1}|^{d_{N-1}}} \right\}. \tag{5.5}$$

Here, d_j is the order of the pole of $R_{\mathcal{A}}(\zeta)$ at $\zeta = \zeta_j$. By inserting (5.4) and (5.5) into (5.3) we get the following estimate, which is uniform in $\zeta \in \Delta_a \setminus Z$:

$$\begin{aligned} \|R_{\mathcal{A}}(\zeta)\|_{\mathscr{B}(\mathcal{X})} \le \frac{1}{\operatorname{Re}\zeta - a}\Bigg[1 + C \max\Bigg\{ &\frac{1}{\operatorname{Re}\zeta - a}, \frac{|\zeta - \zeta_0|^{d_0-1} + 1}{|\zeta - \zeta_0|^{d_0}}, \dots \\ &\dots, \frac{|\zeta - \zeta_{N-1}|^{d_{N-1}-1} + 1}{|\zeta - \zeta_{N-1}|^{d_{N-1}}} \Bigg\}\Bigg], \end{aligned} \tag{5.6}$$

where we absorbed $\|\mathcal{B}\|_{\mathscr{B}(\mathcal{X},X)}$ in the constant C. Now we follow an idea from the proof of Proposition 4.8 in [AGG+12]: Because of Result 4 just shown above, $\zeta \mapsto R_{\mathcal{A}}(\zeta)|_{\mathcal{Y}_N}$ is analytic in Δ_a, so it has no singularities. Hence, it is uniformly bounded on every compact subset of Δ_a. Now we fix a compact set $\Omega_c \subset \Delta_a$, such that $Z \subset \Omega_c^\circ$. So $R_{\mathcal{A}}(\zeta)|_{\mathcal{Y}_N}$ is uniformly bounded in Ω_c. In the "complement" $\Delta_a \backslash \Omega_c$ we may apply (5.6). Since we are away from the poles of $R_{\mathcal{A}}(\zeta)$ in Δ_a, it implies that $R_{\mathcal{A}}(\zeta)$ (and therefore also $R_{\mathcal{A}}(\zeta)|_{\mathcal{Y}_N}$) remains uniformly bounded on $\Delta_{a'} \setminus \Omega_c$ for any $a' > a$. Combining these estimates we get

$$\sup_{\operatorname{Re}\zeta > a'} \|R_{\mathcal{A}}(\zeta)|_{\mathcal{Y}_N}\|_{\mathscr{B}(\mathcal{Y}_N)} < \infty, \quad \forall a' > a.$$

Then, by applying the Gearhart-Prüss-Greiner theorem (see Theorem V.1.11 in [EN00]) we obtain Result 5. $\square$

Next we apply this theorem to the d-dimensional Fokker-Planck operator L (see Theorem 4.4 for its properties in $L^2(1/\mu)$). Our aim is to find a weight function ω such that the closure of L in $L^2(\omega)$ still satisfies most of the spectral properties of L in X. In order to do so we are looking for operators $\mathcal{B}, \mathcal{S}$, such that the requirements of Theorem 5.4 are fulfilled for *any* $a \in \mathbb{R}$. The following lemma yields the appropriate condition on the weight function. We use the notation from Theorem 4.4.

Lemma 5.5. *Let $a < 0$, and fix a weight function $\omega(\mathbf{x}) \in C^2(\mathbb{R}^d)$ that satisfies the conditions*

$$\exists R > 0 : \forall |\mathbf{x}|_2 > R : \quad \Delta\omega(\mathbf{x}) - \mathbf{x}^\top \mathbf{C} \nabla\omega(\mathbf{x}) + \omega(\mathbf{x}) \operatorname{tr} \mathbf{C} \le 2a\omega(\mathbf{x}), \tag{5.7a}$$

$$\exists W_2 > W_1 > 0 : \forall \mathbf{x} \in \mathbb{R}^d : \quad W_1 \le \omega(\mathbf{x}) \le W_2/\mu(\mathbf{x}). \tag{5.7b}$$

Then, the Fokker-Planck operator L is closable in $\mathcal{X} := L^2(\omega)$, and the closure $\mathcal{L} := \operatorname{cl}_{\mathcal{X}} L|_{C_0^\infty(\mathbb{R}^d)}$ has the following properties in $\mathcal{X}$:

(i) *The spectrum satisfies $\sigma(\mathcal{L}) \cap \Delta_a = \sigma(L) \cap \Delta_a = \{-\mathbf{ck} : \mathbf{k} \in \mathbb{N}^d \wedge -\mathbf{ck} > a\}$. We write $\sigma(\mathcal{L}) \cap \Delta_a =: \{\zeta_0, \dots, \zeta_{N-1}\}$, where we assume $0 = \zeta_0 > \zeta_1 > \dots > \zeta_{N-1}$ and $N \in \mathbb{N}^*$. We have $M(\mathcal{L} - \zeta_j) = \ker(\mathcal{L} - \zeta_j) = \operatorname{span}\{\mu_{\mathbf{k}} : -\mathbf{ck} = \zeta_j\}$ for every $0 \le j \le N-1$.*

(ii) *For every $1 \le j \le N$ we define*

$$\mathcal{Y}_j := \operatorname{cl}_{\mathcal{X}} \operatorname{span}\{\mu_{\mathbf{k}} : -\mathbf{ck} \ge \zeta_{j-1}\}^{\perp_X},$$

and $\mathcal{Y}_0 := \mathcal{X}$. $\mathcal{L}$ is decomposed according to $\mathcal{X} = \mathcal{Y}_j \oplus \operatorname{span}\{\mu_{\mathbf{k}} : -\mathbf{ck} \ge \zeta_{j-1}\}$.

(iii) *The spectral projection $\Pi_{\mathcal{L},j}$ corresponding to the eigenvalue ζ_j fulfills*

$$\operatorname{ran} \Pi_{\mathcal{L},j} = \operatorname{span}\{\mu_{\mathbf{k}} : -\mathbf{ck} = \zeta_j\}, \quad \ker \Pi_{\mathcal{L},j} = \mathcal{Y}_{j+1} \oplus \operatorname{span}\{\mu_{\mathbf{k}} : -\mathbf{ck} > \zeta_j\}.$$

(iv) *For every $0 \le j \le N-1$, the operator $\mathcal{L}$ generates a C_0-semigroup on $\mathcal{Y}_j$, and there exists a constant $C_j \ge 1$ such that we have the estimate*

$$\left\| e^{t\mathcal{L}}|_{\mathcal{Y}_j} \right\|_{\mathscr{B}(\mathcal{Y}_j)} \le C_j e^{\zeta_j t}, \quad \forall t \ge 0.$$

Proof. Our aim is to apply Theorem 5.4 for $A = L$. We fix $a < 0$, some weight function ω that satisfies (5.7), and define $\mathcal{X} := L^2(\omega)$. Because of the condition (5.7b) there always holds $X \hookrightarrow \mathcal{X}$.

Verification of the requirements of Theorem 5.4: In order to satisfy Condition (i) in Theorem 5.4 we define $Z := \{\zeta_0, \dots, \zeta_{N-1}\} \equiv \sigma(L) \cap \Delta_a = \{-\mathbf{ck} : \mathbf{k} \in \mathbb{N}^d \wedge -\mathbf{ck} > a\}$, see Theorem 4.4. Since L is self-adjoint in X, Proposition B.11 shows that for every ζ_j the corresponding geometric eigenspace of L in X coincides with the corresponding algebraic eigenspace. And Theorem 4.4 (iv) shows that the eigenspaces corresponding to the ζ_j are all finite-dimensional. Hence condition (i) in Theorem 5.4 is satisfied.

The orthogonal complement Y_N of the eigenspaces of L in X corresponding to Z is just

$$Y_N = \operatorname{cl}_X \operatorname{span}\{\mu_{\mathbf{k}} : -\mathbf{ck} \le a\}.$$

Because the eigenfunctions $\{\mu_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{N}^d}$ are an orthogonal basis of X, it is clear that L is decomposed according to

$$X = Y_N \oplus \Big(\bigoplus_{j=0}^{N-1} M(\zeta_j - L) \Big)$$

Hence, Theorem 5.4 (ii) is verified. By writing $(L-a)|_{Y_N}$ as a Fourier series with respect to $\{\mu_{\mathbf{k}} : -\mathbf{ck} \le a\}$ we immediately see that $(L-a)|_{Y_N}$ is dissipative in Y_N. This shows that Condition (iii) in Theorem 5.4 is fulfilled. Since $(L-a)|_{Y_N}$ is self-adjoint in X, see Theorem 4.4, we can apply the Lumer-Phillips Theorem for self-adjoint operators (see Corollary 1.4.4 in [Paz83]), which proves that $(L-a)|_{Y_N}$ generates a contraction semigroup in Y_N. This proves $\Delta_0 \subset \rho((L-a)|_{Y_N})$, and in particular $b \in \rho((L-a)|_{Y_N})$ for all $b > a$. This confirms Condition (iv) in Theorem 5.4.

In order to demonstrate that also the condition (v) is fulfilled, we make the following calculation for $f \in C_0^\infty(\mathbb{R}^d)$:

$$\begin{aligned}
\operatorname{Re}\langle Lf, f\rangle_\omega &= -\operatorname{Re}\int_{\mathbb{R}^d} (\nabla f + \mathbf{C}\mathbf{x}f)\cdot(\omega\nabla\overline{f} + \overline{f}\nabla\omega)\,\mathrm{d}\mathbf{x} \\
&= -\|\nabla f\|_\omega^2 - \operatorname{Re}\int_{\mathbb{R}^d} |f|^2 \mathbf{x}^\top \mathbf{C}\nabla\omega\,\mathrm{d}\mathbf{x} - \operatorname{Re}\int_{\mathbb{R}^d} \overline{f}\nabla f\cdot(\mathbf{C}\mathbf{x}\omega + \nabla\omega)\,\mathrm{d}\mathbf{x} \\
&= -\|\nabla f\|_\omega^2 + \frac{1}{2}\int_{\mathbb{R}^d} |f|^2(\Delta\omega - \mathbf{x}^\top\mathbf{C}\nabla\omega + \omega\operatorname{tr}\mathbf{C})\,\mathrm{d}\mathbf{x}. \qquad (5.8)
\end{aligned}$$

Now we want to find some $\mathcal{B} \in \mathscr{B}(\mathcal{X}, X)$ such that $(L-\mathcal{B}-a)|_{C_0^\infty(\mathbb{R}^d)}$ is dissipative in $\mathcal{X}$. According to (5.8) this is equivalent to

$$-\|\nabla f\|_\omega^2 + \frac{1}{2}\int_{\mathbb{R}^d} |f|^2(\Delta\omega - \mathbf{x}^\top\mathbf{C}\nabla\omega + \omega\operatorname{tr}\mathbf{C})\,\mathrm{d}\mathbf{x} \le a\|f\|_\omega^2 + \langle\mathcal{B}f, f\rangle_\omega, \qquad (5.9)$$

for all $f \in C_0^\infty(\mathbb{R}^d)$. We use this to construct a convenient operator $\mathcal{B}$: Let $R > 0$ be the constant from (5.7a). For functions $f \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp} f \subset B_{R+1}^2(\mathbf{0})^c$ we shall allow $\mathcal{B}f \equiv 0$, since then (5.9) is fulfilled, because of (5.7a). This motivates us to set $\mathcal{B}f := b\cdot\eta_R\cdot f$, with $b > 0$, where $\eta_R \in C_0^\infty(\mathbb{R}^d;[0,1])$, $\operatorname{supp}\eta_R \subset B_{R+1}^2(\mathbf{0})$ and $\eta_R|_{B_R^2(\mathbf{0})} \equiv 1$. It remains to determine $b > 0$. Since $\omega \in C^2(\mathbb{R}^d)$ there holds

$$\sup_{\mathbf{x}\in B_R^2(\mathbf{0})} \frac{1}{2}(\Delta\omega - \mathbf{x}^\top\mathbf{C}\nabla\omega + \omega\operatorname{tr}\mathbf{C} - 2a) =: M < \infty.$$

Because of (5.7b), the inequality (5.9) is fulfilled if $M \le W_1 b$. Hence, we define $b := \max\{0, \frac{M}{W_1}\}$. With this choice of $\mathcal{B}$ the operator $(L-\mathcal{B}-a)|_{C_0^\infty(\mathbb{R}^d)}$ is dissipative in $\mathcal{X}$. According to (5.7b) the norms $\|\cdot\|_X$ and $\|\cdot\|_{\mathcal{X}}$ are equivalent on $\{f \in C_0^\infty(\mathbb{R}^d) : \operatorname{supp} f \subset B_{R+1}^2(\mathbf{0})\}$, and so $\mathcal{B} \in \mathscr{B}(\mathcal{X}, X)$ follows immediately. With this we may define $\mathcal{S}|_{C_0^\infty(\mathbb{R}^d)} := (L-\mathcal{B})|_{C_0^\infty(\mathbb{R}^d)}$. According to (5.9) the operator $(\mathcal{S}-a)|_{C_0^\infty(\mathbb{R}^d)}$ is dissipative in $\mathcal{X}$, hence it is closable. So both $\mathcal{S}|_{C_0^\infty(\mathbb{R}^d)}$ and $L|_{C_0^\infty(\mathbb{R}^d)} \equiv (\mathcal{B}+\mathcal{S})|_{C_0^\infty(\mathbb{R}^d)}$ are closable in $\mathcal{X}$ as well, and we may define $\mathcal{S} := \operatorname{cl}_{\mathcal{X}} \mathcal{S}|_{C_0^\infty(\mathbb{R}^d)}$ and $\mathcal{L} := \operatorname{cl}_{\mathcal{X}} L|_{C_0^\infty(\mathbb{R}^d)}$. Obviously $L \subset \mathcal{L}$, and so $L \subset \mathcal{B}+\mathcal{S}$, and Requirement (v) of Theorem 5.4 is fulfilled.

Application of Theorem 5.4: In the previous paragraph of this proof we have already seen that $\mathcal{L} = \operatorname{cl}_{\mathcal{X}} L|_{C_0^\infty(\mathbb{R}^d)}$ exists in $\mathscr{C}(\mathcal{X})$. Result 3 of Theorem 5.4 immediately proves Result (i).

Next we show the Results (ii) and (iii). Since the family $\{\mu_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}^d}$ is an orthogonal basis of X, we easily find

$$\operatorname{ran}\Pi_{L,j} = \operatorname{span}\{\mu_{\mathbf{k}} : -\mathbf{ck} = \zeta_j\}, \quad \ker\Pi_{L,j} = \operatorname{span}\{\mu_{\mathbf{k}} : -\mathbf{ck} = \zeta_j\}^\perp,$$

where we have already used Result (i). From Result 4 of Theorem 5.4 we know that for every $0 \le j \le N-1$ there holds $\Pi_{\mathcal{L},j} = \mathrm{cl}_{\mathcal{X}}\, \Pi_{L,j}$. By applying Lemma C.1 we obtain

$$\operatorname{ran} \Pi_{\mathcal{L},j} = \operatorname{span}\{\mu_{\mathbf{k}} : -\mathbf{ck} = \zeta_j\}, \quad \ker \Pi_{\mathcal{L},j} = \mathrm{cl}_{\mathcal{X}} \left(\operatorname{span}\{\mu_{\mathbf{k}} : -\mathbf{ck} = \zeta_j\}^{\perp_X} \right). \tag{5.10}$$

Next we introduce the spaces

$$\mathcal{Y}_j := \mathrm{cl}_{\mathcal{X}} \left(\{\mu_{\mathbf{k}} : -\mathbf{ck} > \zeta_j\}^{\perp_X} \right), \quad 0 \le j \le N-1.$$

Since the spectral projections satisfy $\Pi_{\mathcal{L},i}\Pi_{\mathcal{L},j} = 0$ for $i \neq j$ (see Section B.2.2 in the Appendix) we conclude with the help of (5.10) that

$$\mathcal{X} = \mathcal{Y}_j \oplus M(\mathcal{L} - \zeta_{j-1}) \oplus \cdots \oplus M(L - \zeta_0) \equiv \mathcal{Y}_j \oplus \operatorname{span}\{\mu_{\mathbf{k}} : -\mathbf{ck} > \zeta_j\}, \tag{5.11}$$

for all $0 \le j \le N-1$. Furthermore, we see that $\mathcal{Y}_j$ is the kernel of the spectral projection $\Pi_{\mathcal{L},Z}$ corresponding to the set $Z := \{\zeta_0, \dots, \zeta_{j-1}\} \subset \sigma(\mathcal{L})$. Hence it is clear that $\mathcal{L}$ can be decomposed according to (5.11). $\square$

Remark 5.6. The condition (5.7) on the weight function ω is not the only one which leads to the results of Lemma 5.5. In [GMM10, Section 5.1], a different condition denoted by **(FP3)** is presented. According to this, we could replace (5.7a) by

$$\exists R > 0 : \forall |\mathbf{x}|_2 > R : \quad \omega(\mathbf{x}) L\left(\frac{1}{\omega(\mathbf{x})}\right) \le 2a. \tag{5.7a'}$$

In order to see that the results of Lemma 5.5 still hold true if ω satisfies (5.7a') instead of (5.7a), we only need to repeat the calculations below (5.8). We start similar to (5.7a), with $f \in C_0^\infty(\mathbb{R}^d)$ (in order to keep notation more clear, we define $\nu := 1/\omega$):

$$\begin{aligned}
\operatorname{Re}\langle Lf, f\rangle_\omega &= -\operatorname{Re} \int_{\mathbb{R}^d} \mu \nabla\Big(\frac{f}{\mu}\Big) \cdot \nabla\Big(\overline{\frac{f}{\nu}}\Big) \,\mathrm{d}\mathbf{x} = -\operatorname{Re} \int_{\mathbb{R}^d} \mu \nabla\Big(\frac{f}{\nu} \cdot \frac{\nu}{\mu}\Big) \cdot \nabla\Big(\overline{\frac{f}{\nu}}\Big) \,\mathrm{d}\mathbf{x} \\
&= -\operatorname{Re} \int_{\mathbb{R}^d} \mu \nabla\Big(\overline{\frac{f}{\nu}}\Big) \cdot \left[\frac{\nu}{\mu} \nabla\Big(\frac{f}{\nu}\Big) + \frac{f}{\nu} \nabla\Big(\frac{\nu}{\mu}\Big)\right] \mathrm{d}\mathbf{x} \\
&= -\int_{\mathbb{R}^d} \Big|\nabla\Big(\frac{f}{\nu}\Big)\Big|^2 \nu \,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d} \frac{|f|^2}{2\nu^2} \nabla\Big(\mu \nabla\Big(\frac{\nu}{\mu}\Big)\Big) \,\mathrm{d}\mathbf{x} \\
&= -\int_{\mathbb{R}^d} \Big|\nabla\Big(\frac{f}{\nu}\Big)\Big|^2 \nu \,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d} \frac{|f|^2}{2\nu^2} L\nu \,\mathrm{d}\mathbf{x}.
\end{aligned}$$

Analogously to the proof of Lemma 5.5 we therefore need to find an operator $\mathcal{B} \in \mathscr{B}(\mathcal{X}, X)$ such that the following inequality holds for all $f \in C_0^\infty(\mathbb{R}^d)$:

$$-\int_{\mathbb{R}^d} \Big|\nabla\Big(\frac{f}{\nu}\Big)\Big|^2 \nu \,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d} \frac{|f|^2}{2\nu^2} L\nu \,\mathrm{d}\mathbf{x} \le a\|f\|_\omega^2 + \langle \mathcal{B}f, f\rangle_\omega.$$

Given (5.7a') we can construct $\mathcal{B}$ as in the proof of Lemma 5.5 and proceed analogously from there on.

Remark 5.7. In the one-dimensional case ($d = 1$), and for $a < 0$ given, the polynomial weight function $\omega(x) = 1 + x^{2(-\lfloor a \rfloor + 1)}$ fulfills (5.7), thus the conclusions of Lemma 5.5 apply to the Fokker-Planck operator $\mathcal{L}$ in $L^2(\omega)$ for $N = -\lfloor a \rfloor$. (Here, $\lfloor \cdot \rfloor$ denotes the floor function.) Therefore, the spectrum of $\mathcal{L}$ in the half plane Δ_a is given by $\{0, -1, \dots, -N+1\}$. These results are complemented in [GW02, Appendix A], where an exhaustive analysis of the spectral properties of $\mathcal{L}$ in polynomially weighted spaces is given, and the complete spectrum is determined. For the one-dimensional case see also [GR98].

In the following we turn our attention to $\mathcal{L}$ in exponentially weighted spaces. In this case we are able to completely characterize the spectrum, and deduce the following result:

Theorem 5.8. *Choose constants $\beta_1, \dots, \beta_d > 0$ and $\gamma_1, \dots, \gamma_d \in (0,2]$. Furthermore, whenever $\gamma_j = 2$ for some $j \in \{1, \dots, d\}$, we additionally require $\beta_j < c_j/2$. Define the weight function $\omega(\mathbf{x})$ either by*

$$\omega(\mathbf{x}) := \sum_{j=1}^{d} \cosh\big(\beta_j |x_j|^{\gamma_j}\big), \quad \text{or by} \quad \omega(\mathbf{x}) := \prod_{j=1}^{d} \cosh\big(\beta_j |x_j|^{\gamma_j}\big). \tag{5.12}$$

Then we have the following results in $\mathcal{X} := L^2(\omega)$:

(i) *$L|_{C_0^\infty(\mathbb{R}^d)}$ is closable in $\mathcal{X}$, and we define $\mathcal{L} := \mathrm{cl}_{\mathcal{X}}\, L|_{C_0^\infty(\mathbb{R}^d)}$.*

(ii) *The spectrum satisfies $\sigma(\mathcal{L}) = \{-\mathbf{ck} : \mathbf{k} \in \mathbb{N}^d\}$, and for every $\lambda \in \sigma(\mathcal{L})$ the corresponding eigenspace satisfies*

$$M(\lambda - \mathcal{L}) = \ker(\zeta - \mathcal{L}) = \mathrm{span}\{\mu_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d \wedge -\mathbf{ck} = \lambda\}.$$

Here, the functions $\mu_{\mathbf{k}}$, for $\mathbf{k} \in \mathbb{N}^d$, are defined in Theorem 4.4. The sequence of (sorted) eigenvalues is denoted by $(\zeta_j)_{j\in\mathbb{N}}$. We define them by $0 = \zeta_0 < \zeta_1 < \zeta_2 < \dots$, and $\{\zeta_j : j \in \mathbb{N}\} = \sigma(\mathcal{L})$.

(iii) *We define $\mathcal{Y}_0 := \mathcal{X}$, and for every $N \in \mathbb{N}^*$*

$$\mathcal{Y}_N := \mathrm{cl}_X\, \mathrm{span}\{\mu_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d \wedge -\mathbf{ck} \le \zeta_N\}.$$

The finite-dimensional space $\mathrm{span}\{\mu_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d \wedge -\mathbf{ck} \ge \zeta_{N-1}\}$ is a complement, and $\mathcal{L}$ is decomposed according to

$$\mathcal{X} = \mathcal{Y}_N \oplus \mathrm{span}\{\mu_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d \wedge -\mathbf{ck} \ge \zeta_{N-1}\}.$$

(iv) *For every $j \in \mathbb{N}$, the spectral projection $\Pi_{\mathcal{L},\zeta_j}$ of $\mathcal{L}$ corresponding to $\zeta_j \in \sigma(\mathcal{L})$ fulfills*

$$\begin{aligned} \mathrm{ran}\,\Pi_{\mathcal{L},\zeta_j} &= \mathrm{span}\{\mu_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d \wedge -\mathbf{ck} = \zeta_j\}, \\ \ker\Pi_{\mathcal{L},\zeta_j} &= \mathcal{Y}_{j+1} \oplus \mathrm{span}\{\mu_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d \wedge -\mathbf{ck} \ge \zeta_{j-1}\}. \end{aligned}$$

(v) *For every $j \in \mathbb{N}$ the operator $\mathcal{L}|_{\mathcal{Y}_j}$ generates a C_0-semigroup on $\mathcal{Y}_j$:*

$$(\mathrm{e}^{t\mathcal{L}|_{\mathcal{Y}_j}})_{t\ge0} \equiv (\mathrm{e}^{t\mathcal{L}}|_{\mathcal{Y}_j})_{t\ge0}.$$

There exists a constant $C_j \ge 1$ such that the semigroup satisfies the estimate

$$\big\|\mathrm{e}^{t\mathcal{L}}|_{\mathcal{Y}_j}\big\|_{\mathscr{B}(\mathcal{Y}_j)} \le C_j \mathrm{e}^{-\zeta_j t}, \quad \forall t \ge 0.$$

Proof. This is a direct application of Lemma 5.5. Given a weight function ω from (5.12), we only need to verify that for every $a < 0$ the conditions (5.7) are verified. We start with

$$\omega(\mathbf{x}) = \sum_{j=1}^{d} \cosh\big(\beta_j |x_j|^{\gamma_j}\big).$$

Since $\mathbf{C}$ is diagonal, the condition (5.7a) decouples. Thus, it is sufficient to show that for every index $j \in \{1, \dots, d\}$ the following one-dimensional condition is verified (with $x := x_j$, $\beta := \beta_j$, $\gamma := \gamma_j$ and $c := c_j$):

$$\lim_{|x|\to\infty} \frac{(\cosh(\beta|x|^\gamma))'' - cx(\cosh(\beta|x|^\gamma))' + c\cosh(\beta|x|^\gamma)}{\cosh(\beta|x|^\gamma)} = -\infty.$$

Note that the lack of differentiability of ω around the origin is not an issue, since we are only interested in the behavior as $|x| \to \infty$. Since the expression in the limit is symmetric in x, we can omit the modulus everywhere. With this, we compute

$$\begin{aligned}
&\lim_{x\to\infty} \frac{(\cosh(\beta x^\gamma))'' - cx(\cosh(\beta x^\gamma))' + c\cosh(\beta x^\gamma)}{\cosh(\beta x^\gamma)} \\
&\quad = \lim_{x\to\infty} \frac{(\exp(\beta x^\gamma))'' - cx(\exp(\beta x^\gamma))' + c\exp(\beta x^\gamma)}{\exp(\beta x^\gamma)} \qquad (5.13)\\
&\quad = \lim_{x\to\infty} \frac{[\beta\gamma(\gamma-1)x^{\gamma-2} + (\beta\gamma x^{\gamma-1})^2]\exp(\beta x^\gamma) - c\beta\gamma x^\gamma \exp(\beta x^\gamma) + c\exp(\beta x^\gamma)}{\exp(\beta x^\gamma)} \\
&\quad = \lim_{x\to\infty} \big(- c\beta\gamma x^\gamma + \beta^2\gamma^2 x^{2\gamma-2} + \beta\gamma(\gamma-1)x^{\gamma-2} + c\big).
\end{aligned}$$

The result of this limit depends on the highest power of x. Provided $\beta, \gamma, c > 0$, the limit can be $-\infty$ only if

$$\gamma \geq 2\gamma - 2 \quad \Leftrightarrow \quad \gamma \leq 2.$$

If $\gamma < 2$, the limit is always $-\infty$. If $\gamma = 2$, the limit is $-\infty$ iff

$$-c\beta\gamma + \beta^2\gamma^2 < 0 \quad \Leftrightarrow \quad \beta < \frac{c}{2}.$$

Under these conditions ω fulfills the condition (5.7) for all $a < 0$, so it is now straightforward to conclude the results of Theorem 5.8 from Lemma 5.5.

For the other weight function

$$\omega(\mathbf{x}) = \prod_{j=1}^d \cosh\left(\beta_j |x_j|^{\gamma_j}\right),$$

we proceed similarly to above. We find

$$\begin{aligned}
&\lim_{\mathbf{x}\to\infty} \frac{\Delta\omega(\mathbf{x}) - \mathbf{x}^\top \mathbf{C}\nabla\omega(\mathbf{x}) + \omega(\mathbf{x})\operatorname{tr}\mathbf{C}}{\omega(\mathbf{x})} \\
&\quad = \lim_{|\mathbf{x}|_2\to\infty} \sum_{j=1}^d \frac{(\exp(\beta_j x_j^{\gamma_j}))'' - c_j x_j(\exp(\beta_j x_j^{\gamma_j}))' + c_j\exp(\beta_j x_j^{\gamma_j})}{\exp(\beta_j x_j^{\gamma_j})}.
\end{aligned}$$

This limit is $-\infty$ iff for every $j \in \{1, \dots, d\}$:

$$\lim_{x_j\to\infty} \frac{(\exp(\beta_j x_j^{\gamma_j}))'' - c_j x_j(\exp(\beta_j x_j^{\gamma_j}))' + c_j\exp(\beta_j x_j^{\gamma_j})}{\exp(\beta_j x_j^{\gamma_j})} = -\infty.$$

Now, for every $j \in \{1, \dots, d\}$ this limit coincides with (5.13). Hence, we obtain the same conditions on β_j and γ_j. From there on, we can proceed analogously to the first part of the proof, and obtain the desired results of Theorem 5.8 also for the second weight function in (5.12). □

Remark 5.9. For $\gamma_j = 0$, for all $j \in \{1, \dots, d\}$, in Theorem 5.8 (i.e. $\mathcal{X} = L^2(\mathbb{R}^d)$) the situation changes completely. From Theorem 4.4 in [Met01] we know: In $L^2(\mathbb{R}^d)$ there holds $\sigma(\mathcal{L}) = \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta \leq \frac{\operatorname{tr}\mathbf{C}}{2}\}$ and every $\zeta \in \mathbb{C}$ with $\operatorname{Re}\zeta < \frac{\operatorname{tr}\mathbf{C}}{2}$ is an eigenvalue.

Remark 5.10. The representation of the $\mathcal{X}_k$ given in Proposition 3.14 still holds true for the weight functions chosen in Theorem 5.8, since the proofs are completely analogous.

CHAPTER 6

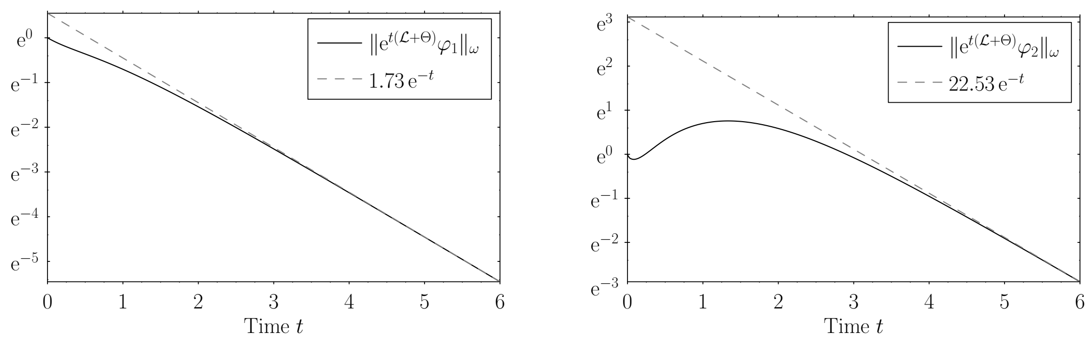
Simulation results

In this section we shall illustrate numerically the exponential convergence of solutions of the one-dimensional perturbed Fokker-Planck equation (3.1) towards the stationary solution f_0. Here we choose $\vartheta := \varepsilon(\delta_{-\alpha} - \delta_\alpha)$, so $\Theta f(x) = \varepsilon(f(x+\alpha) - f(x-\alpha))$, for some $\varepsilon, \alpha \in \mathbb{R}$. The eigenfunctions f_k of the evolution operator $\mathcal{L} + \Theta$ can be obtained by an inverse Fourier transform, with $\hat{f}_k$ explicitly given in (3.17). Assume that the initial condition φ is a (finite) linear combination of the f_k:

$$\varphi = \sum_{j=1}^{n} a_j f_{k_j}.$$

Then, the solution to (3.1) reads explicitly

$$f(t,x) = \mathrm{e}^{t(\mathcal{L}+\Theta)}\Big[\sum_{j=1}^{n} a_j f_{k_j}\Big] = \sum_{j=1}^{n} a_j \mathrm{e}^{-k_j t} f_{k_j}, \quad \forall t \geq 0.$$



(a) Initial condition $\varphi_1 = (f_1 - 1.32 f_2)/\|f_1 - 1.32 f_2\|_\omega$.

(b) Initial condition $\varphi_2 = (\chi_{[-4,0]} - \chi_{[0,4]})/\|\chi_{[-4,0]} - \chi_{[0,4]}\|_\omega \in \mathcal{X}_1$.

Figure 1. Evolution of the norm $\|\cdot\|_\omega$ of solutions of the perturbed equation for different initial conditions φ. First published in [SA14].

In the simulation we use a mass conserving Crank-Nicolson finite difference scheme for (3.1). It is employed on the spatial interval $[-25, 25]$ (with 1500 gridpoints) along with zero-flux boundary conditions. Moreover, we choose $\alpha = \varepsilon = 2$ and $\beta = 1$, i.e. $\mathcal{X} = L^2(\cosh x)$.

The following numerical results verify the decaying behavior of solutions of (3.1), and yield an estimate to the constants $\tilde{C}_k$ from Theorem 3.37. First we consider the initial condition $\varphi_1 = (f_1 - 1.32 f_2)/\|f_1 - 1.32 f_2\|_\omega$. For the corresponding solution we plot $\|f(t,\cdot)\|_\omega$ in Figure 1(a). Since the sequence $(f_k)_{k\in\mathbb{N}}$ is not orthogonal in $\mathcal{X}$, the initial decay rate is here smaller than the individual decay rate of f_1 (i.e. -1). But after some time, the f_1-term becomes dominant, and the decay rate approaches -1. For large times, the norm behaves approximately like $1.73\,\mathrm{e}^{-t}$, so we have the lower bound $\tilde{C}_1 \geq 1.73$.

As a second example we choose the initial condition (at $t = 0$)

$$\varphi_2 = (\chi_{[-4,0]} - \chi_{[0,4]})/\|\chi_{[-4,0]} - \chi_{[0,4]}\|_\omega.$$

It lies in $\mathcal{X}_1$ since it is massless. The evolution of $\|f(t,\cdot)\|_\omega$ is displayed in Figure 1(b). Here, the norm even increases initially. Only after some time, the norm begins to decay with a rate tending to -1. For large times t, the norm behaves approximately like $22.53\,\mathrm{e}^{-t}$, which shows $\tilde{C}_1 \geq 22.53$.

APPENDIX A

Further results

A.1. The one-dimensional Fokker-Planck operator in the self-adjoint setting

The aim of this section is to discuss some basic properties of the one-dimensional Fokker-Planck operator L in X. Here we use the notation from Section 3.1.

Most of the results of Theorem 3.1 are covered by [Met01, BGM94, HN05]. Furthermore, it can be easily verified that L in X is isometrically equivalent to the (dimensionless) quantum harmonic oscillator Hamiltonian $A = -\Delta - 1/2 + x^2/4$ in $L^2(\mathbb{R})$. This is a well-understood operator, so we can transfer many results from A to L. An extensive reference concerning properties of A is [Par10]. The compactness of the resolvent then follows from Theorem XIII.67 in [RS78]. For some particular properties of the spectral projections one might also consider Section V.3.5 in [Kat66].

However, for sake of completeness we also rigorously discuss the main properties of L in X in the following, and give an independent proof of Theorem 3.1. To start with, we consider the (still formal) Fokker-Planck operator $Lf := f'' + xf' + f = ((f/\mu)'\mu)'$ in the space $X = L^2(1/\mu)$. Introducing the isometry $\iota\colon X \to L^2(\mathbb{R})\colon f \mapsto f/\sqrt{\mu}$ we see that $L|_{C_0^\infty(\mathbb{R})}$ in X is equivalent to the following harmonic oscillator Hamiltonian in $L^2(\mathbb{R})$:

$$A|_{C_0^\infty(\mathbb{R})} = \iota \circ L|_{C_0^\infty(\mathbb{R})} \circ \iota^{-1} = \Big(\Delta + \Big[\frac{1}{2} - \frac{x^2}{4}\Big]\Big)\Big|_{C_0^\infty(\mathbb{R})}.$$

We first investigate A in $L^2(\mathbb{R})$, which is mostly a summary of the standard results concerning the quantum harmonic oscillator.

Proposition A.1. *The operator $A|_{C_0^\infty(\mathbb{R})}$ is closable in $L^2(\mathbb{R})$. The closure $A := \mathrm{cl}_{L^2(\mathbb{R})} A|_{C_0^\infty(\mathbb{R})}$ is dissipative in $L^2(\mathbb{R})$.*

Proof. For $z \in C_0^\infty(\mathbb{R})$ we compute

$$\begin{aligned}\langle A|_{C_0^\infty} z, z\rangle_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \mu^{-1/2} L(z\mu^{1/2})\overline{z}\,\mathrm{d}x \\ &= -\int_{\mathbb{R}} \mu\big|\big(z\mu^{-1/2}\big)'\big|^2\,\mathrm{d}x \le 0.\end{aligned}$$

So $A|_{C_0^\infty(\mathbb{R})}$ is dissipative. Therefore it is also closable in $L^2(\mathbb{R})$ and the closure is also dissipative, see Theorem 1.4.5 in [Paz83]. □

Next we investigate A in $L^2(\mathbb{R})$ more closely.

Proposition A.2. *The operator A has the following properties in $L^2(\mathbb{R})$:*

(i) *A is self-adjoint and generates a C_0-semigroup of contractions.*
(ii) *For any $\zeta \in \rho(A)$ the resolvent $R_A(\zeta)$ is compact.*

Proof. The self-adjointness of A immediately follows from the Faris-Lavine theorem, see Theorem X.38 in [RS75]. Together with Proposition A.1 we obtain that $A = A^*$ is dissipative. So, according to Corollary 1.4.4 in [Paz83] A generates a C_0-semigroup of contractions.

The compactness of the resolvent is a direct consequence of Theorem XIII.67 in [RS78]. $\square$

Corollary A.3. *For A we have in $L^2(\mathbb{R})$:*

(i) *The spectrum $\sigma(A)$ consists entirely of isolated points, which are all eigenvalues.*

(ii) *For every eigenvalue the algebraic eigenspace coincides with the geometric eigenspace, which has finite dimension. The eigenspaces form a complete orthogonal family in $L^2(\mathbb{R})$.*

Proof. The statement (i) is a straightforward application of Theorem III.6.29 in [Kat66]. Result (ii) follows from the spectral theorem for normal, compact operators, cf. Theorem V.2.10 in [Kat66], and the discussion in Section V.3.5 in [Kat66]. $\square$

Proposition A.4. *There holds $\sigma(A) = -\mathbb{N}$. For every $k \in \mathbb{N}$ the eigenspace of A corresponding to the eigenvalue $-k$ is one-dimensional and spanned by the function $H_k \mu^{1/2}$, where H_k is the k-th Hermite polynomial.*

Proof. According to Corollary A.3 it is sufficient to look for eigenvalues in order to compute the spectrum $\sigma(A)$. Thus, we want to find all $\lambda \in \mathbb{R}$ and $z \in L^2(\mathbb{R})$ such that $(\lambda - A)z = 0$. For the eigenfunction z we make the ansatz $z = \mu^{1/2}p$. Inserting this in the eigenvalue equation and canceling the factor $\mu^{1/2}$ we end up with the following equation for p:

$$p'' - xp' = \lambda p. \tag{A.1}$$

It is well-known that the solutions of this equation are the Hermite polynomials. From the discussion in Section VIII.2.7.4 in [DL90] and Section V.10.4 in [CH53] we find that (A.1) has a non-trivial solution in $L^2(\mathbb{R})$ iff $\lambda \in -\mathbb{N}$. For every $k \in \mathbb{N}$ the solution of (A.1) for $\lambda = -k$ is a polynomial, and it is a multiple of the Hermite polynomial H_k. $\square$

See Appendix B.1 for some properties of the Hermite polynomials H_k, see also [DL90].

With the isometrical equivalence of A and L via ι as discussed in the beginning of this section it is immediate that all properties derived for A in $L^2(\mathbb{R})$ can be transferred one-to-one to L in X. The only modification concerns the eigenfunctions, the eigenfunctions of L are (up to a normalization constant) given by the functions μH_k for $k \in \mathbb{N}$.

A.2. Discussion of a particular Θ in X

In the following we use the notation from Section 3.1, with $\mu = \exp(-x^2/2)$ and $X = L^2(1/\mu)$.

The reason why the analysis of the perturbed Fokker-Planck operator $L + \Theta$ is not carried out in X is because X is 'too small' and does not even contain the stationary solution of $L + \Theta$. In Subsection A.2.1 we demonstrate this for a particular choice of Θ. We suspect that the unboundedness of most nonlocal operators in X is the reason for this inconvenient behavior. For this reason, in Subsection A.2.2, we discuss the boundedness of Θ in weighted L^2-spaces with a weight which grows more slowly.

A.2.1. Nonexistence of a stationary solution in X. In the following we demonstrate that for the particular perturbation $\Theta_a f(x) := f(x+a) - f(x-a)$, with $a > 0$, the equation $(L + \Theta_a)f = 0$ only has the trivial solution $f \equiv 0$ in X. Note that $\Theta_a f$ satisfies condition **(C)**, see page 23. Our aim is to express both Θf and the candidate for the stationary solution f_0 in terms of the eigenfunction basis $\{\mu_k\}_{k\in\mathbb{N}}$ of L.

According to Theorem 3.1 we the family $\{\mu_k\}_{k\in\mathbb{N}}$ is an orthogonal basis in X, and there holds $\mu_k = (2\pi)^{-1/2} H_k \mu$, for all $k \in \mathbb{N}$, with $\mu(x) = \exp(-x^2/2)$. From Lemma B.2 we obtain

$$\|\mu_k\|_X^2 = (2\pi)^{-\frac{1}{2}} k!\,, \quad \forall k \in \mathbb{N}.$$

With this we can define the normalized eigenfunctions

$$\mu_k^\circ := (2\pi)^{-1/4}(k!)^{-1/2} H_k \mu, \quad \forall k \in \mathbb{N}. \tag{A.2}$$

The family $\{\mu_k^\circ\}_{k\in\mathbb{N}}$ is an orthonormal basis of X. We proceed with two technical lemmata.

Lemma A.5. *For every $m \in \mathbb{N}$ there holds*

$$\int_{\mathbb{R}} \mu(x) x^m \,\mathrm{d}x = \begin{cases} \frac{m!}{2^{m/2}(m/2)!}\sqrt{2\pi}, & m \text{ even},\\ 0, & m \text{ odd}. \end{cases} \tag{A.3}$$

Proof. For any $m \in \mathbb{N}$ we obtain after integration by parts

$$\int_{\mathbb{R}} x^m \mu(x)\,\mathrm{d}x = \frac{1}{m+1}\int_{\mathbb{R}} x^{m+2}\mu(x)\,\mathrm{d}x$$
$$\Leftrightarrow \int_{\mathbb{R}} x^{m+2}\mu(x)\,\mathrm{d}x = \frac{(m+2)(m+1)}{2((m+2)/2)}\int_{\mathbb{R}} x^m \mu(x)\,\mathrm{d}x.$$

This recursion relation combined with the initial values

$$\int_{\mathbb{R}} \mu(x)\,\mathrm{d}x = \sqrt{2\pi}, \quad \int_{\mathbb{R}} x\mu(x)\,\mathrm{d}x = 0$$

yields the desired result. □

Lemma A.6. *For $k, \ell \in \mathbb{N}$ we have*

$$\int_{\mathbb{R}} H_k(x)\mu(x)x^\ell \,\mathrm{d}x = \begin{cases} (-1)^k\sqrt{2\pi}\frac{\ell!}{2^{(\ell-k)/2}((\ell-k)/2)!}, & (\ell-k) \in 2\mathbb{N},\\ 0, & \text{else}. \end{cases}$$

Proof. Integration by parts proves that for every $j \le \min(k, \ell)$ there holds

$$\int_{\mathbb{R}} H_k(x)\mu(x)x^\ell \,\mathrm{d}x = (-1)^j \frac{\ell!}{(\ell-j)!}\int_{\mathbb{R}} \mu(x)^{(k-j)} x^{\ell-j}\,\mathrm{d}x.$$

If $k \le \ell$, we set $j := k$, and use the result (A.3). If $\ell < k$, we set $j := \ell$, and find immediately that the above integral becomes zero. □

With the help of the above lemmata we can now express Θ_a in terms of the family $\{\mu_k^\circ\}_{k\in\mathbb{N}}$. The following proposition provides the key ingredient for determining the coefficients for the Fourier series.

Proposition A.7. *For $k, j \in \mathbb{N}$ there holds*

$$\langle \Theta_a \mu_k^\circ, \mu_j^\circ \rangle_X = \begin{cases} 2\Big(\dfrac{j!}{k!}\Big)^{\frac{1}{2}} \dfrac{a^{j-k}}{(j-k)!}, & \text{if } \frac{j-k-1}{2} \in \mathbb{N}, \\ 0, & \text{else.} \end{cases} \tag{A.4}$$

For the following proof it is helpful to introduce the function

$$\sigma(m) := \begin{cases} 1, & m \in 2\mathbb{N}, \\ 0, & \text{else.} \end{cases}$$

Proof. As a preparatory step we begin with the following computations, using the explicit form of the H_k in (B.5):

$$\begin{aligned} H_j(x \pm a) &= \sum_{i=0}^{[j/2]} (-1)^{i+j} \frac{j!}{i!(j-2i)!2^i} (x \pm a)^{j-2i} \\ &= \sum_{i=0}^{[j/2]} \sum_{\ell=0}^{j-2i} (-1)^{i+j} \frac{j! a^{j-2i-\ell}}{i!\ell!(j-2i-\ell)!2^i} x^\ell \\ \Rightarrow H_j(x-a) - H_j(x+a) &= \sum_{i=0}^{[j/2]} \sum_{\ell=0}^{j-2i} (-1)^{1+i+j} \frac{j! a^{j-2i-\ell} \sigma(1+j-2i-\ell)}{i!\ell!(j-2i-\ell)!2^{i-1}} x^\ell. \end{aligned} \tag{A.5}$$

With this result we compute

$$\begin{aligned} \langle \Theta_a \mu_k, \mu_j \rangle_X &= \frac{1}{2\pi} \int_{\mathbb{R}} \big(H_k(x+a)\mu(x+a) - H_k(x-a)\mu(x-a) \big) H_j(x) \, \mathrm{d}x \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} H_k(x)\mu(x) \big(H_j(x-a) - H_j(x+a) \big) \, \mathrm{d}x \\ &= \frac{1}{2\pi} \sum_{i=0}^{[j/2]} \sum_{\ell=0}^{j-2i} (-1)^{1+i+j} \frac{j! a^{j-2i-\ell} \sigma(1+j-2i-\ell)}{i!\ell!(j-2i-\ell)!2^{i-1}} \int_{\mathbb{R}} H_k(x)\mu(x) x^\ell \, \mathrm{d}x. \end{aligned}$$

Now we evaluate the occurring integral with the help of Lemma A.6:

$$\ldots = \frac{1}{\sqrt{2\pi}} \sum_{i=0}^{[j/2]} \sum_{\ell=0}^{j-2i} (-1)^{1+i+j+k} \frac{j! a^{j-2i-\ell} \sigma(1+j-2i-\ell)\sigma(\ell-k)}{i!(j-2i-\ell)!\big(\frac{\ell-k}{2}\big)!2^{i-1+(\ell-k)/2}}.$$

Next we reverse the order of summation in the second sum, i.e. we perform the index shift $\ell \to j - 2i - \ell$:

$$\begin{aligned} \ldots &= \frac{1}{\sqrt{2\pi}} \sum_{i=0}^{[j/2]} \sum_{\ell=0}^{j-2i} (-1)^{1+i+j+k} \frac{j! a^\ell \sigma(1+\ell)\sigma(j-2i-\ell-k)}{i!\ell!\big(\frac{j-2i-\ell-k}{2}\big)!2^{(j-\ell-k)/2-1}} \\ &= \frac{1}{\sqrt{2\pi}} \sum_{i=0}^{[j/2]} \sum_{\ell=0}^{j-2i-k} (-1)^{1+i+j+k} \frac{j! a^\ell \sigma(1+\ell)\sigma(j-2i-\ell-k)}{i!\ell!\big(\frac{j-2i-\ell-k}{2}\big)!2^{(j-\ell-k)/2-1}}. \end{aligned}$$

Now we interchange the order of summation, and we obtain:

$$\begin{aligned}
\ldots &= \frac{1}{\sqrt{2\pi}} \sum_{\ell=0}^{j-k} \sum_{i=0}^{[\frac{j-\ell-k}{2}]} (-1)^{1+i+j+k} \frac{j! a^\ell \sigma(1+\ell)\sigma(j-2i-\ell-k)}{i!\ell!\left(\frac{j-2i-\ell-k}{2}\right)! 2^{(j-\ell-k)/2-1}} \\
&= \frac{1}{\sqrt{2\pi}} \sum_{\ell=0}^{j-k} (-1)^{1+j+k} \frac{j! a^\ell \sigma(1+\ell)\sigma(j-\ell-k)}{\ell!\left(\frac{j-\ell-k}{2}\right)! 2^{(j-\ell-k)/2-1}} \sum_{i=0}^{\frac{j-\ell-k}{2}} (-1)^i \binom{\frac{j-\ell-k}{2}}{i} \\
&= \frac{1}{\sqrt{2\pi}} \sum_{\ell=0}^{j-k} (-1)^{1+j+k} \frac{j! a^\ell \sigma(1+\ell)\sigma(j-\ell-k)}{\ell!\left(\frac{j-\ell-k}{2}\right)! 2^{(j-\ell-k)/2-1}} (1-1)^{(j-\ell-k)/2} \\
&= (-1)^{1+j-k} \sigma(j-k-1) \sqrt{\frac{2}{\pi}} \cdot \frac{j! a^{j-k}}{(j-k)!}.
\end{aligned}$$

In the last step we have used the convention that empty sums are treated as zero. Using the definition of the normalized eigenfunctions from (A.2), we conclude

$$\langle \Theta_a \mu_k^\circ, \mu_j^\circ \rangle_X = \left(\frac{2\pi}{k!j!}\right)^{\frac{1}{2}} \langle \Theta_a \mu_k, \mu_j \rangle_X,$$

which completes the proof. □

It is a non-trivial question whether Θ_a can be defined as a closed operator in X or not. However, we do not address this question here. Our aim is rather to show that on a formal level the equation $(L + \Theta_a)f = 0$ only has the trivial solution in X.

Lemma A.8. *Formally, the equation $(L + \Theta_a)f = 0$ only has the trivial solution $f = 0$ in X.*

Proof. We fix some $f \in X$ that satisfies $(L + \Theta_a)f = 0$. It then has the unique representation

$$f = \sum_{j=0}^{\infty} f_j \mu_j^\circ,$$

where $f_j := \langle f, \mu_j^\circ \rangle_X$ (they should not be confused with the eigenfunctions from Section 3.2). Our goal is to show that the sequence $(f_j)_{j \in \mathbb{N}}$ is unbounded in X.

Step 1 (equations for the f_j): For all $k, \ell \in \mathbb{N}$ we define $\vartheta_{k\ell} := \langle \Theta_a \mu_k^\circ, \mu_\ell^\circ \rangle_X$, which can be evaluated according to Proposition A.7. With respect to the orthonormal basis $\{\mu_k^\circ\}_{k \in \mathbb{N}}$ we obtain:

$$\begin{aligned}
(L + \Theta_a)f &= \sum_{k=0}^{\infty} -k f_k \mu_k^\circ + \sum_{k=0}^{\infty} f_k \Theta_a \mu_k^\circ \\
&= \sum_{k=0}^{\infty} -k f_k \mu_k^\circ + \sum_{k=0}^{\infty} f_k \sum_{\ell=k+1}^{\infty} \vartheta_{k\ell} \mu_\ell^\circ \\
&= \sum_{\ell=1}^{\infty} \Big[-\ell f_\ell + \sum_{k=0}^{\ell-1} f_k \vartheta_{k\ell} \Big] \mu_\ell^\circ. \qquad \text{(A.6)}
\end{aligned}$$

Here we have used that according to Proposition A.7 $\vartheta_{k\ell} = 0$ if $\ell \le k$. We require (A.6) to be zero, hence the sequence $(f_j)_{j\in\mathbb{N}}$ needs to satisfy the system

$$-\ell f_\ell + \sum_{k=0}^{\ell-1} f_k \vartheta_{k\ell}, \quad \forall \ell \in \mathbb{N}^*. \tag{A.7}$$

This system is linear, infinite-dimensional and strictly lower triangular. Note that due to (A.4) and $a > 0$ we have $\vartheta_{jk} \ge 0$ for all $0 \le j \le k-1$.

Step 2 (estimate of the f_j): Without loss of generality we may choose $f_0 = 1$. The f_j for $j > 0$ are determined inductively by (A.7), i.e. by solving the following system:

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ \vdots \end{pmatrix} = \begin{pmatrix} \vartheta_{0,1} & & & & & \\ 0 & \frac{1}{2}\vartheta_{1,2} & & & 0 & \\ \frac{1}{3}\vartheta_{0,3} & 0 & \frac{1}{3}\vartheta_{2,3} & & & \\ 0 & \frac{1}{4}\vartheta_{1,4} & 0 & \frac{1}{4}\vartheta_{3,4} & & \\ \frac{1}{5}\vartheta_{0,5} & 0 & \frac{1}{5}\vartheta_{2,5} & 0 & \frac{1}{5}\vartheta_{4,5} & \\ 0 & \ddots & & \ddots & & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ \vdots \end{pmatrix}.$$

This system defines a recursion for the coefficients. Since all occurring entries are non-negative, we can use this to obtain the following estimate (use Proposition A.7 again):

$$f_{3(k+1)} \ge \frac{1}{3(k+1)} \vartheta_{3k,3(k+1)} f_{3k}, \quad \forall k \in \mathbb{N}. \tag{A.8}$$

Furthermore, we find for all $k \in \mathbb{N}$ from the explicit representation of the $\vartheta_{k\ell}$ in (A.4):

$$\vartheta_{3k,3(k+1)} = 2\Big(\frac{(3k+3)!}{(3k)!}\Big)^{\frac{1}{2}} \frac{a^3}{3!} = \frac{a^3}{3}\big((3k+3)(3k+2)(3k+1)\big)^{\frac{1}{2}} \ge a^3 3^{\frac{1}{2}} k^{\frac{3}{2}}.$$

We apply this estimate in (A.8) and conclude for all $k \ge 1$:

$$f_{3(k+1)} \ge \frac{a^3 k^{\frac{3}{2}}}{3^{\frac{1}{2}}(k+1)} f_{3k} \ge \frac{a^3}{2\cdot 3^{\frac{1}{2}}} k^{\frac{1}{2}} f_{3k}.$$

Since the factor $k^{1/2}$ becomes arbitrarily large as $k \to \infty$, and $f_0 = 1$, it follows that the subsequence of coefficients $(f_{3k})_{k\in\mathbb{N}}$ grows super-exponentially. Hence, $(f_k)_{k\in\mathbb{N}} \notin \ell^2(\mathbb{N})$, and $f \notin X$. □

A.2.2. Boundedness of Θ in weighted spaces. Throughout this section $\nu(x)$ denotes a weight function which is symmetric. As in the previous section, $\Theta_a f(x) := f(x+a) - f(x-a)$, with $a > 0$.

Lemma A.9. *The operator Θ_a is bounded in $L^2(\nu)$ if the logarithm of the weight function $\ln \nu(x)$ is uniformly Lipschitz continuous on $\mathbb{R}$.*

Proof. Since $C_0^\infty(\mathbb{R}) \subset L^2(\nu)$ is dense, cf. Lemma B.18, it suffices to prove the boundedness of $\Theta_a|_{C_0^\infty(\mathbb{R})}$. To this end, let $f \in C_0^\infty(\mathbb{R})$. We immediately find

$$\|\Theta_a f\|_\nu^2 \le 2\big(\|f(\cdot + a)\|_\nu^2 + \|f(\cdot - a)\|_\nu^2\big).$$

Thus, since $\nu(x)$ is required to be symmetric and we may always take $f(-x)$ instead of $f(x)$, it is sufficient to estimate $\|f(\cdot+a)\|_\nu$. For this we obtain

$$\begin{aligned}\|f(\cdot+a)\|_\nu &= \int_{\mathbb{R}} |f(x)|^2\nu(x-a)\,\mathrm{d}x = \int_{\mathbb{R}} |f(x)|^2\nu(x)\cdot\frac{\nu(x-a)}{\nu(x)}\,\mathrm{d}x\\ &\le \|f\|_\nu^2\cdot\Big\|\frac{\nu(x-a)}{\nu(x)}\Big\|_{L^\infty(\mathbb{R})},\end{aligned}$$

where we have used Hölder's inequality in the last step. Thus, Θ_a is bounded in X if $\|\nu(x-a)/\nu(x)\|_{L^\infty(\mathbb{R})}$ is finite. The latter holds true iff there exists a constant $C \ge 1$ (given by $C = \|\nu(x-a)/\nu(x)\|_{L^\infty(\mathbb{R})}$) such that

$$C^{-1}\nu(x) \le \nu(x+a) \le C\nu(x), \quad \forall x \in \mathbb{R}.$$

We now make the ansatz $\nu(x) = \exp(\varphi(x))$. Taking the logarithm in the above inequalities we obtain

$$\varphi(x) - \ln C \le \varphi(x+a) \le \varphi(x) + \ln C \quad \Leftrightarrow \quad \Big|\frac{\varphi(x+a)-\varphi(x)}{a}\Big| \le \frac{\ln C}{a}.$$

In particular, this is satisfied if $\varphi(x)$ is uniformly Lipschitz continuous in $\mathbb{R}$. □

As consequence of the above lemma we find that Θ_a is bounded in $L^2(\nu)$ if $\nu(x)$ asymptotically grows or decreases at most exponentially as $|x| \to \infty$. From the above proof and the sharpness of Hölder's inequality it is furthermore clear that the operator Θ_a becomes unbounded in $L^2(\nu)$ as soon as $\|\nu(x-a)/\nu(x)\|_{L^\infty(\mathbb{R})}$ is unbounded. And it is easily verified that the latter is the case for every weight $\nu(x)$ that grows super-exponentially.

A.3. Fourier transform of the resolvent

This section deals with the explicit computation of the Fourier transform of the resolvent $R_{\mathcal{L}+\Theta}(\zeta)$ of the (one-dimensional) perturbed Fokker-Planck operator $\mathcal{L}+\Theta$ in $\mathcal{X}$ (with $\mathcal{X}$ defined on page 17), where Θ fulfills the condition **(C)** (see page 23). We begin by considering the resolvent equation

$$(\zeta - \mathcal{L} - \Theta)f = g$$

on $\mathbb{R}$, where we assume $\operatorname{Re}\zeta > -k$ and $f, g \in \mathcal{X}_k$ for some $k \in \mathbb{N}$ (see Proposition 3.12 for the definition of the $\mathcal{X}_k$). We apply the Fourier transform, which yields the following differential equation:

$$\xi\Big[\hat f'(\xi) + \Big(\xi + \frac{\zeta - \hat\vartheta(\xi)}{\xi}\Big)\hat f(\xi)\Big] = \hat g(\xi).$$

By defining $\tilde f := \hat f/\hat f_0$ and $\tilde g := \hat g/\hat f_0$, where f_0 is the zero eigenfunction given in (3.17), we obtain the equivalent equation

$$\xi\tilde f'(\xi) + \zeta\tilde f(\xi) = \tilde g(\xi). \tag{A.9}$$

The general solution for $\xi \in \mathbb{R}^\pm$ reads

$$\tilde f(\xi) = \int_0^1 \tilde g(\xi s)s^{\zeta-1}\,\mathrm{d}s + C_\pm\xi^{-\zeta} =: I(\xi) + C_\pm\xi^{-\zeta}, \tag{A.10}$$

where the $C_\pm \in \mathbb{C}$ are integration constants to be determined.

First we shall show that the integral $I(\xi)$ is an analytic function on $\Omega_{\beta/2}$: If $g \in \mathcal{X}_k$, then $\tilde{g}$ is analytic in $\Omega_{\beta/2}$ and has a zero at $\xi = 0$ of order larger than or equal to k, see (3.12). Therefore, for any fixed $\zeta \in \mathbb{C}$ with $\operatorname{Re}\zeta > -k$,

$$\tilde{g}(\xi s)s^{\zeta-1} = \frac{\tilde{g}(\xi s)}{s^k}s^{\zeta+k-1}, \quad s \in (0,1],$$

is locally integrable at $s = 0$, and $I(\xi)$ is well defined for all $\xi \in \Omega_{\beta/2}$. To see that I is actually analytic, we define $I_\varepsilon(\xi) := \int_\varepsilon^1 G_k(\xi, s)s^{\zeta+k-1}\,\mathrm{d}s$ for $\varepsilon \in [0,1)$, where

$$G_k(\xi,s) := \begin{cases} \dfrac{\tilde{g}(\xi s)}{s^k}, & s \in (0,1], \\ \dfrac{\tilde{g}^{(k)}(0)\xi^k}{k!}, & s = 0, \end{cases}$$

for $\xi \in \Omega_{\beta/2}$. The function $G_k(\cdot, s)$ is analytic in $\Omega_{\beta/2}$ for all (fixed) $s \in [0,1]$, and G_k is continuous in $\Omega_{\beta/2} \times [0,1]$. According to [Det84, Theorem 4.9.1], the functions $I_\varepsilon(\xi)$ are analytic in $\Omega_{\beta/2}$ for all $\varepsilon \in (0,1)$. Now we show that $(I_\varepsilon)_{\varepsilon\in(0,1)}$ converges normally to I in $\Omega_{\beta/2}$ as $\varepsilon \to 0$: Let $K \subset \Omega_{\beta/2}$ be compact. Then we have

$$\begin{aligned} \sup_{\substack{\xi\in K\\ s\in[0,1]}} |G_k(\xi,s)| \le \sup_{\substack{\xi\in K_0\\ s\in[0,1]}} |G_k(\xi,s)| &= \sup_{\substack{\xi\in K_0\setminus\{0\}\\ s\in(0,1]}} \left|\frac{\tilde{g}(\xi s)}{(\xi s)^k}\xi^k\right| \\ &\le \sup_{\xi\in K_0\setminus\{0\}} \left|\frac{\tilde{g}(\xi)}{\xi^k}\right| \cdot \sup_{\xi\in K_0} |\xi^k| =: C_K < \infty, \end{aligned} \tag{A.11}$$

since $\tilde{g}(\xi)/\xi^k$ is analytic in $\Omega_{\beta/2}$ (the singularity at $\xi = 0$ is removable). Here, K_0 is an appropriate star-shaped (with center $\boldsymbol{\xi} = \mathbf{0}$), compact set with $\{0\} \cup K \subseteq K_0 \subset \Omega_{\beta/2}$, and $C_K > 0$ is a constant dependent on K. With (A.11) we obtain the following estimate for $\xi \in K$ and $0 < \varepsilon \le 1$:

$$|I(\xi) - I_\varepsilon(\xi)| = \left|\int_0^\varepsilon G_k(\xi,s)s^{\zeta+k-1}\,\mathrm{d}s\right| \le C_K \frac{\varepsilon^{\operatorname{Re}\zeta+k}}{\operatorname{Re}\zeta+k}.$$

Since $\operatorname{Re}\zeta + k > 0$, this shows the normal convergence of the analytic functions I_ε towards I. According to [Det84, Theorem 4.2.3] this implies that $I(\xi)$ is analytic in $\Omega_{\beta/2}$.

Now it remains to determine the constants $C_\pm$ in (A.10). If we require $f \in \mathcal{X}_k$, it is necessary that $\tilde{f}$ is analytic in $\Omega_{\beta/2}$ and has a zero of order greater than or equal to k at $\xi = 0$. As already shown, $I(\xi)$ is analytic in $\Omega_{\beta/2}$. Furthermore, for $g \in \mathcal{X}_k$ and all $s \in [0,1]$, $\xi \mapsto G_k(\xi, s)$ has a zero of order greater than or equal to k at $\xi = 0$. Therefore $I(\xi) = \int_0^1 G_k(\xi,s)s^{\zeta+k-1}\,\mathrm{d}s$ has the same property, so $\mathcal{F}^{-1}I \in \mathcal{X}_k$. Thus, it is sufficient to consider the term $C_\pm\xi^{-\zeta}$. If $\zeta \notin -\mathbb{N}$, then $\xi^{-\zeta}$ is not analytic in $\Omega_{\beta/2}$ anyway, hence $C_+ = C_- = 0$. If $\zeta \in \{-k+1, \dots, 0\}$ for $g \in \mathcal{X}_k$ (note that we assume $\operatorname{Re}\zeta > -k$), $\xi^{-\zeta}$ is analytic, and we obtain $C_+ = C_-$ because we require continuity of the solution. But the order of the zero of $\xi^{-\zeta}$ is at most $k-1$. Since we need a zero of at least order k, we again obtain $C_+ = C_- = 0$. The conclusion of the above analysis is summarized in the following proposition:

Proposition A.10. *Let $g \in \mathcal{X}_k$ for some $k \in \mathbb{N}$, and $\operatorname{Re}\zeta > -k$. Then the unique $f \in \mathcal{X}_k$ with $f = R_{\mathcal{L}+\Theta}(\zeta)g$ satisfies*

$$\hat{f}(\xi) = \hat{f}_0(\xi) \int_0^1 \frac{\hat{g}(s\xi)}{\hat{f}_0(s\xi)} s^{\zeta-1} \, \mathrm{d}s, \quad \xi \in \Omega_{\beta/2}.$$

APPENDIX B

Background

B.1. Hermite polynomials

Here we discuss some properties of the Hermite polynomials H_k, for $k \in \mathbb{N}$, which are introduced in Theorem 3.1, see also [DL90] for further reading. In this thesis we consider Hermite polynomials defined by

$$H_k(x) := \frac{1}{\mu(x)} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \mu(x), \quad k \in \mathbb{N}, \tag{B.1}$$

with $\mu(x) = \exp(-x^2/2)$ and $x \in \mathbb{R}$.

Lemma B.1. *The Hermite polynomials $\{H_j\}_{j\in\mathbb{N}}$ satisfy the following relations, for all $k \in \mathbb{N}^*$ and $x \in \mathbb{R}$:*

$$H_{k+1} = -xH_k - kH_{k-1}, \tag{B.2}$$

$$H_k' = -kH_{k-1}. \tag{B.3}$$

Proof. For every $k \in \mathbb{N}^*$ there holds $\mu^{(k+1)} = -(x\mu^{(k)} + k\mu^{(k-1)})$. This can easily be shown by induction. Dividing this equation by μ yields (B.2). Furthermore, from (B.1) we obtain by differentiation:

$$H_k' = xH_k + H_{k+1}, \quad \forall k \in \mathbb{N}^*.$$

In this equation we now eliminate H_{k+1} by using (B.2), which finally yields (B.3). $\square$

Lemma B.2. *For $k, j \in \mathbb{N}$ there holds*

$$\int_{\mathbb{R}} H_k(x)H_j(x)\mu(x)\,\mathrm{d}x = \begin{cases} k!\sqrt{2\pi}, & \textit{if } j = k, \\ 0, & \textit{if } j \neq k. \end{cases} \tag{B.4}$$

Proof. We assume without loss of generality that $k \geq j$. Then, we use (B.1) for H_j, integrate by parts j times, and then use (B.3):

$$\begin{aligned} \int_{\mathbb{R}} H_k(x)H_j(x)\mu(x)\,\mathrm{d}x &= \int_{\mathbb{R}} H_k(x)\mu^{(j)}(x)\,\mathrm{d}x \\ &= \frac{k!}{(k-j)!} \int_{\mathbb{R}} H_{k-j}\mu(x)\,\mathrm{d}x \\ &= \frac{k!}{(k-j)!} \int_{\mathbb{R}} \mu^{(k-j)}(x)\,\mathrm{d}x. \end{aligned}$$

Since all derivatives of $\mu(x)$ lie in the Schwartz space $\mathscr{S}(\mathbb{R})$ the above integral vanishes for $k > j$. For $k = j$ we use $\int_{\mathbb{R}} \mu(x)\,\mathrm{d}x = \sqrt{2\pi}$, which completes the proof. $\square$

Lemma B.3. *For every $k \in \mathbb{N}$ the Hermite polynomial H_k has the following explicit representation:*

$$H_k(x) = \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^{j+k} \frac{k!}{j!(k-2j)!2^j} x^{k-2j}. \tag{B.5}$$

Here $\lfloor \cdot \rfloor$ is the floor function.

Proof. We prove this by induction. From (B.1) we obtain that $H_0(x) = 1$ and $H_1(x) = -x$. This shows that (B.5) holds true for $k = 0$ and $k = 1$. We assume now that for some $k \in \mathbb{N}^*$ both H_k and H_{k-1} satisfy (B.5). Using the recursion relation (B.2) we now derive a representation for H_{k+1}:

$$\begin{aligned} H_{k+1}(x) = &\sum_{j=0}^{\lfloor (k-1)/2 \rfloor} (-1)^{j+k} \frac{k!}{j!(k-1-2j)!2^j} x^{k-1-2j} \\ &+ \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^{j+k+1} \frac{k!}{j!(k-2j)!2^j} x^{k+1-2j} \\ = &\sum_{j=1}^{\lfloor (k+1)/2 \rfloor} (-1)^{j-1+k} \frac{k!}{(j-1)!(k+1-2j)!2^{j-1}} x^{k+1-2j} \\ &+ \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^{j+k+1} \frac{k!}{j!(k-2j)!2^j} x^{k+1-2j} \\ = &\sum_{j=1}^{\lfloor k/2 \rfloor} \left((-1)^{j+k+1} \frac{(k+1)!}{j!(k+1-2j)!2^j} x^{k+1-2j} \right) \\ &+ (-x)^{k+1} + \frac{1-(-1)^k}{2} \cdot (-1)^{(k+1)/2} \cdot \frac{k!}{\left(\frac{k-1}{2}\right)!\, 2^{(k-1)/2}}. \end{aligned}$$

In the last step, we collected the two sums into one. Note that the last term only occurs if k is odd. Now we verify that the two terms remaining outside of the sum can be absorbed as extra summands of the sum in the last line. For $j = 0$ we have

$$\left. \left((-1)^{j+k+1} \frac{(k+1)!}{j!(k+1-2j)!2^j} x^{k+1-2j} \right) \right|_{j=0} = (-x)^{k+1},$$

and for $j = (k+1)/2$ we find (only if k is odd)

$$\left. \left((-1)^{j+k+1} \frac{(k+1)!}{j!(k+1-2j)!2^j} x^{k+1-2j} \right) \right|_{j=\frac{k+1}{2}} = (-1)^{(k+1)/2} \cdot \frac{k!}{\left(\frac{k-1}{2}\right)!\, 2^{(k-1)/2}}.$$

We conclude that

$$H_{k+1}(x) = \sum_{j=1}^{\lfloor (k+1)/2 \rfloor} (-1)^{j+k+1} \frac{(k+1)!}{j!(k+1-2j)!2^j} x^{k+1-2j},$$

which is of the form (B.5) and thus completes the induction step. $\square$

B.2. Spectral projections

The aim of this section is to illustrate some basic properties of spectral projections, and to discuss some valuable results. Throughout this section, X is a Hilbert space, and $A \in \mathscr{C}(X)$ is a closed operator.

B.2.1. Holomorphic functions. For a closed operator A the map $\zeta \mapsto R_A(\zeta)$ is actually a holomorphic function on the resolvent set $\rho(A)$. Usually, a holomorphic function is a differentiable map from a domain in $\mathbb{C}$ into $\mathbb{C}$. However, this concept can be generalized to maps from a domain in $\mathbb{C}$ into a Banach space X. We give here a brief introduction to Banach space-valued holomorphic functions. The following material is taken mostly from the first chapter of [GL09], see also Chapter V.3 in [Yos80] and [Heu92] for further reference.

In the following, X is a Banach space, and $\Omega \subset \mathbb{C}$ is an simply connected[1], open domain. A function $f\colon \Omega \to X$ is called *holomorphic* or *complexly differentiable* of for every $z \in \Omega$ the limit exists:

$$f'(z) = \lim_{w \to z} \frac{f(z) - f(w)}{z - w}.$$

The function f' is the derivative of f.

Definition B.4 (C^1-contour). A curve $\gamma\colon [0,1] \to \Omega$ is called *C^1-contour* if the following conditions are satisfied:

- $\gamma'(t) \neq 0$ for all $0 \leq t \leq 1$.
- $\gamma(t) \neq \gamma(s)$ for all $0 \leq t, s < 1$ with $t \neq s$.
- $\gamma(0) = \gamma(1)$ and $\gamma'(0) = \gamma'(1)$.
- γ has positive orientation.

Theorem B.5 (Cauchy's Integral Theorem). *Let $f\colon \Omega \to X$ be holomorphic, and let γ be a C^1-contour in Ω. Then*

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

In particular, this implies the path-independence of line integrals of holomorphic functions.

Theorem B.6 (Cauchy's Integral Formula). *Let $f\colon \Omega \to X$ be holomorphic, and let $U \subset \Omega$ be an open set such that its boundary ∂U is a C^1-contour. Then, for every $w \in U$ there holds*

$$f(w) = \frac{1}{2\pi \mathrm{i}} \int_{\partial U} \frac{f(z)}{z - w}\,\mathrm{d}z.$$

Moreover, for the n-th derivative ($n \in \mathbb{N}$) there holds

$$f^{(n)}(w) = \frac{n!}{2\pi \mathrm{i}} \int_{\partial U} \frac{f(z)}{(z - w)^{n+1}}\,\mathrm{d}z.$$

[1]Informally, this means that Ω is connected and has no "holes".

Theorem B.7. *A function $f\colon \Omega \to X$ is holomorphic iff for every $w \in \Omega$ there holds the power series representation*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-w)^n, \quad \forall |z-w| < r_w,$$

with $r_w > 0$ sufficiently small. Then, the coefficients are given by $a_n = f^{(n)}(w)/n!$ for all $n \in \mathbb{N}$.

Finally, we briefly discuss holomorphic functions with *singularities*. To this end, let $w \in \Omega$, and $\Omega' := \Omega \setminus \{w\}$. The definition of holomorphicity carries over to Ω'. For a holomorphic function $f\colon \Omega' \to X$ the point w is called *singularity* of f. For $r_w > 0$ sufficiently small, there holds the *Laurent series* representation of f:

$$f(z) = \sum_{n\in\mathbb{Z}} a_n (z-w)^n, \quad 0 < |z-w| < r_w.$$

The coefficients a_n are again given by

$$a_n = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(z)}{(z-w)^{n+1}} \,\mathrm{d}z, \quad \forall n \in \mathbb{Z},$$

where γ is a sufficiently small C^1-contour in Ω around w.

B.2.2. Spectral projections. We review some properties of spectral projections and resolvents, see Chapters V.9–10 in [TL80], Chapter VIII.8 in [Yos80], Section VIII.1.4 in [DL90] and Sections III.6.4–5 in [Kat66].

For a closed linear operator A the map $\zeta \mapsto R_A(\zeta)$ is a holomorphic function from each connected subset of $\rho(A)$ into the Banach space $\mathscr{B}(X)$, see Theorem 1 in Section VIII.2 in [Yos80]. In the following let $\lambda \in \sigma(A)$ be an isolated point of the spectrum. Note that it does not necessarily have to be an eigenvalue. Then the map $\zeta \mapsto R_A(\zeta)$ has a singularity at the point λ, and $R_A(\zeta)$ has the following *Laurent series* representation

$$R_A(\zeta) = \sum_{i=-\infty}^{\infty} (\zeta-\lambda)^i \Lambda_i, \quad \text{with } \Lambda_i := \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{R_A(z)}{(z-\lambda)^{i+1}} \,\mathrm{d}z. \tag{B.6}$$

This holds for all $0 < |\zeta - \lambda| < r$, and $r > 0$ is such that $\overline{B_r^2(\lambda)} \cap \sigma(A) = \{\lambda\}$. The path Γ is a C^1-contour around λ, staying in the interior of $B_r(\lambda)$. For the operators Λ_i we have the following properties, see Theorems VIII.8.1 and VIII.8.2 in [Yos80]:

Lemma B.8. *For every $i \in \mathbb{Z}$ there holds $\Lambda_i \in \mathscr{B}(X)$, and*

(i) $\Lambda_i = (-1)^i \Lambda_0^{i+1}$ *for all $i \in \mathbb{N}^*$,*
(ii) $\Lambda_{-i-j+1} = \Lambda_{-i}\Lambda_{-j}$ *for all $i, j \in \mathbb{N}^*$,*
(iii) $\Lambda_{i+1} = -R_A(\lambda)\Lambda_i$ *for all $i \in \mathbb{Z} \setminus \{-1, 0\}$,*
(iv) $\Lambda_i \Lambda_j = 0$ *for all $i \geq 0$ and $j \leq -1$.*

Note that (ii) implies that Λ_{-1} is a projection (see Section C.2 for more facts about projections). Indeed, Λ_{-1} is the *spectral projection* $\mathrm{P}_{A,\lambda}$ defined in (2.1), and according to Lemma B.8 it is a bounded operator. In the following we focus on the spectral projection $\mathrm{P}_{A,\lambda} \equiv \Lambda_{-1}$ and discuss some of its properties. First we need the following definitions: The singularity of $R_A(\zeta)$ at $\zeta = \lambda$ is called a *pole* of order $m \in \mathbb{N}^*$ if

$\Lambda_{-m} \neq 0$ and $\Lambda_{-m-1} = 0$. Note that this implies that $\Lambda_i = 0$ for all $i \leq -m-1$, because of Lemma B.8 (iii).

Given an operator A, the *ascent of* A is the smallest $k \in \mathbb{N}$ such that $\ker A^k = \ker A^{k+1}$, analogously, the *descent of* A is the smallest $k \in \mathbb{N}$ such that $\operatorname{ran} A^k = \operatorname{ran} A^{k+1}$. For the following result see the lines (10-8) and (10-9) in Section V.10 of [TL80], and Theorems V.10.1 and V.10.2 in [TL80], and Theorem VIII.8.3 in [Yos80]:

Proposition B.9. *For every $j \in \mathbb{N}^*$ we have*

$$\begin{aligned} \operatorname{ran}(\lambda - A)^j &\supseteq \ker \mathrm{P}_{A,\lambda}, \\ \ker(\lambda - A)^j &\subseteq \operatorname{ran} \mathrm{P}_{A,\lambda}. \end{aligned} \tag{B.7}$$

The isolated singularity λ is a pole of $R_A(\zeta)$ iff $\lambda - A$ has finite ascent and finite descent. In this case the order $m \in \mathbb{N}^*$ of the pole coincides with the ascent and the descent of $\lambda - A$, and

$$\operatorname{ran}(\lambda - A)^j = \ker \mathrm{P}_{A,\lambda}, \quad \ker(\lambda - A)^j = \operatorname{ran} \mathrm{P}_{A,\lambda}, \quad \forall j \geq m.$$

Note that this implies that if λ is a pole of order one of $R_A(\zeta)$ then the algebraic and the geometric eigenspace corresponding to λ coincide, and it is given by $\operatorname{ran} \mathrm{P}_{A,\lambda}$.

The following can be found in Section III.5.6 of [Kat66]. See also Section C.2 for more properties of projections. Let there hold $X = X_1 \oplus X_2$ for two closed subspaces X_1, X_2, this defines a unique bounded projection P with $\operatorname{ran} P = X_1$ and $\ker P = X_2$. An operator A is said to be *decomposed* according to $X = X_1 \oplus X_2$ (or P, equivalently) if

$$PD(A) \subset D(A), \quad AX_1 \subset X_1, \quad AX_2 \subset X_2.$$

Note that the first condition implies that $D(A) \cap X_j$ is dense in X_j for $j = 1, 2$. Hence, $X = X_1 \oplus X_2$ is a decomposition of X into two A-invariant subspaces. The *part* of A in X_j is the restriction $A|_{X_j}$, which is a closed operator in the Banach space X_j, with the domain $D(A|_{X_j}) = D(A) \cap X_j$, for $j = 1, 2$. Note that if A can be decomposed according to P, then also the resolvent $R_A(\zeta)$ and the semigroup $(\mathrm{e}^{tA})_{t \geq 0}$ are decomposed according to P.

Now we can formulate some important results on spectral projections. The following facts can be found in the Sections III.6.4–5 in [Kat66].

Proposition B.10. *Given an operator $A \in \mathscr{C}(X)$ and the spectral projection $\mathrm{P}_{A,\lambda}$, corresponding to isolated point $\lambda \in \sigma(A)$, we have:*

(i) *A can be decomposed according to $\mathrm{P}_{A,\lambda}$.*
(ii) *The spectra of the parts of A are given by $\sigma(A|_{\ker \mathrm{P}_{A,\lambda}}) = \sigma(A) \backslash \{\lambda\}$ and $\sigma(A|_{\operatorname{ran} \mathrm{P}_{A,\lambda}}) = \{\lambda\}$.*
(iii) *$A|_{\operatorname{ran} \mathrm{P}_{A,\lambda}} \in \mathscr{B}(\operatorname{ran} \mathrm{P}_{A,\lambda})$.*
(iv) *If $\dim \operatorname{ran} \mathrm{P}_{A,\lambda} < \infty$, then $(\lambda - A)|_{\operatorname{ran} \mathrm{P}_{A,\lambda}}$ is nilpotent, λ is a pole of $R_A(\zeta)$, and $\lambda \in \sigma_p(A)$.*
(v) *If λ is a pole of $R_A(\zeta)$, then $M(\lambda - A) = \ker(\lambda - A)$ iff the pole is of order one.*

In the case where $\dim \operatorname{ran} \mathrm{P}_{A,\lambda} < \infty$, this dimension is called the *(algebraic) multiplicity* if the eigenvalue λ. The following result is from Section V.3.5 in [Kat66]:

Proposition B.11. *Let A be self-adjoint, and $\lambda \in \sigma(A)$ is an isolated point of the spectrum. Then $\lambda \in \sigma_p(A)$, and the algebraic eigenspace coincides with the (geometric) eigenspace. Furthermore, λ is a pole of $R_A(\zeta)$ of order 1, and the corresponding spectral projection is orthogonal.*

Spectral projections can be defined not only for isolated points of the spectrum. More generally, we can define a spectral projection in the case where $\sigma(A)$ contains a bounded set σ' that can be separated from the rest $\sigma'' := \sigma(A) \setminus \sigma'$ of the spectrum by a C^1-contour, which contains σ' in its interior. The spectral projection $\mathrm{P}_{A,\sigma'}$ is then again defined by the formula (2.1), where Γ is now the C^1-contour separating σ' from σ''. The following generalization of Proposition B.10 is a collection of results from Section III.6.4 in [Kat66]:

Proposition B.12. *Let $\sigma' \subset \sigma(A)$ be a set as defined above. Then the following holds:*

(i) *A is decomposed according to $\mathrm{P}_{A,\sigma'}$.*
(ii) *The spectra of the two parts of A are $\sigma(A|_{\operatorname{ran}\mathrm{P}_{A,\sigma'}}) = \sigma'$ and $\sigma(A|_{\ker\mathrm{P}_{A,\sigma'}}) = \sigma''$.*
(iii) *$A|_{\operatorname{ran}\mathrm{P}_{A,\sigma'}} \in \mathscr{B}(\operatorname{ran}\mathrm{P}_{A,\sigma'})$.*

As a consequence, the restriction of the resolvent $R_A(\zeta)|_{\operatorname{ran}\mathrm{P}_{A,\sigma'}}$ coincides with the resolvent $R_{A'}(\zeta)$ in $\operatorname{ran}\mathrm{P}_{A,\sigma'}$, and it is holomorphic on $\mathbb{C} \setminus \sigma'$. Here $A' := A|_{\operatorname{ran}\mathrm{P}_{A,\sigma'}}$ is the part of A in $\operatorname{ran}\mathrm{P}_{A,\sigma'}$. An analogous result holds for σ'' instead of σ'.

Finally, assume that $\sigma(A)$ can be separated into finitely many bounded parts $\sigma_1, \dots, \sigma_J$, with $J \in \mathbb{N}^*$, and one possibly unbounded part σ_0. According to Section III.6.4 in [Kat66] there holds: If every σ_j for $1 \le j \le J$ can be separated from the rest of $\sigma(A)$ by a C^1-contour lying in $\rho(A)$, we can define the spectral projection P_{A,σ_j} according to (2.1) with $\Gamma = \Gamma_j$, for every $1 \le j \le J$. Every P_{A,σ_j} possesses the properties described in Proposition B.12. Furthermore, there holds $\mathrm{P}_{A,\sigma_i}\mathrm{P}_{A,\sigma_j} = 0$ whenever $i \neq j$. Hence, A can be decomposed according to

$$X = \operatorname{ran}\mathrm{P}_{A,\sigma_1} \oplus \cdots \oplus \operatorname{ran}\mathrm{P}_{A,\sigma_J} \oplus \ker(\mathrm{P}_{A,\sigma_1} + \cdots + \mathrm{P}_{A,\sigma_J}).$$

Lemma B.13. *Let $A \in \mathscr{C}(X)$. For some $N \in \mathbb{N}$ let $Z := \{\zeta_0, \dots, \zeta_N\} \subset \sigma(A)$ be a set of isolated eigenvalues of A. Furthermore, for every $0 \le k \le N$ there holds $\dim M(\zeta_k - A) < \infty$. Assume that there exists a closed subspace $Y \subset X$, such that*

(i) *We can write $X = Y \oplus M(\zeta_0 - A) \oplus \ldots \oplus M(\zeta_N - A)$. This defines a bounded projection Π with $\ker \Pi = Y$ and $\operatorname{ran}\Pi = M(\zeta_0 - A) \oplus \ldots \oplus M(\zeta_N - A)$.*
(ii) *A is decomposed according to Π, and $\sigma(A|_Y) \cap Z = \emptyset$.*

Then Π is the spectral projection of A corresponding to the set Z.

Proof. According to the assumptions there holds $\sigma(A|_Y) = \sigma(A) \setminus \{\zeta_0, \dots, \zeta_N\}$, and therefore the map $\zeta \mapsto R_A(\zeta)|_Y$ is analytic in $\rho(A) \cup \{\zeta_0, \dots, \zeta_N\}$. According to the definition (2.1) of spectral projections this implies that $\Pi_{A,k} Y \equiv \{0\}$ for every $\Pi_{A,k}$, since $R_A(\zeta)|_Y$ is holomorphic in a neighborhood of ζ_k, and hence the contour integral (2.1) vanishes due to Cauchy's Theorem. Therefore $Y \subseteq \ker \Pi_A$. On the other hand we have $M(\zeta_k - A) \subseteq \operatorname{ran}\Pi_{A,k}$ for all $0 \le k \le N$, according to Proposition B.9. From (i) we conclude that the inclusions have to be equalities, otherwise $\ker\Pi_A \cap \operatorname{ran}\Pi_A \neq \{0\}$, which is impossible. ◻

B.3. Space enlargement results

In the following we use the notation and definitions from Chapter 5. We give an explicit bound for the resolvent in the case that A generates a C_0-semigroup and $R_A(\zeta)$ has a finite number of poles in Δ_a. For this we use the Conditions 5.1 and 5.3.

Lemma B.14. *Let $A \in \mathrm{SLS}_X(a, Z)$ for some $a \in \mathbb{R}$ and $Z := \{\zeta_0, \dots, \zeta_{N-1}\} \subset \Delta_a$, with $N \in \mathbb{N}$. Furthermore, let A generate a C_0-semigroup $(\mathrm{e}^{tA})_{t\geq 0}$ satisfying the growth bound*

$$\left\|\mathrm{e}^{tA}\big(I - \Pi_{A,Z}\big)\right\|_{\mathscr{B}(X)} \leq C_a \mathrm{e}^{at}, \quad \forall t \geq 0, \tag{B.8}$$

for some constant $C_a \geq 1$, where $\Pi_{A,Z}$ denotes the spectral projection of A corresponding to Z. Then there exists a constant $C > 0$ such that for all $\zeta \in \Delta_a \backslash \{\zeta_0, \dots, \zeta_{N-1}\}$:

$$\|R_A(\zeta)\|_{\mathscr{B}(X)} \leq C \max\left\{\frac{C_a}{\operatorname{Re}\zeta - a}, \frac{|\zeta - \zeta_0|^{d_0 - 1} + 1}{|\zeta - \zeta_0|^{d_0}}, \dots, \frac{|\zeta - \zeta_{N-1}|^{d_{N-1}-1} + 1}{|\zeta - \zeta_{N-1}|^{d_{N-1}}}\right\}, \tag{B.9}$$

where d_j is the order of the pole of $R_A(\zeta)$ at ζ_j.

Proof. Let $\zeta \in \Delta_a \setminus Z$. Since $A \in \mathrm{SLS}_X(a, Z)$, we have due to Proposition B.10 that $M(\zeta_j) = \operatorname{ran}\Pi_{A,j}$ for all $j \in \{0, \dots, N-1\}$, where $\Pi_{A,j}$ is the spectral projection of A corresponding to $\zeta_j \in \sigma(A)$. Hence, we obtain the decomposition of A according to $X = Y \oplus M(\zeta_0 - A) \oplus \ldots \oplus M(\zeta_{N-1} - A)$, where $Y := \ker \Pi_{A,Z}$. First we consider $R_A(\zeta)$ on each subspace individually. On Y we have, due to (B.8) and the Hille-Yosida theorem, the estimate

$$\|R_A(\zeta)|_Y\|_{\mathscr{B}(Y)} \leq \frac{C_a}{\operatorname{Re}\zeta - a}, \quad \forall \operatorname{Re}\zeta > a. \tag{B.10}$$

Next we consider $R_A(\zeta)$ on $M(\zeta_j - A) = \operatorname{ran}\Pi_{A,j}$ for a fixed $0 \leq j \leq N-1$. Now the coefficients Λ_i in (B.6) (with $\lambda := \zeta_j$) satisfy the relation $\Lambda_n \Pi_{A,j} = 0$ for all $n \in \mathbb{N}$, which can be seen by using $\Lambda_{-1} = \Pi_{A,j}$ in Lemma B.8 (iv). Therefore we have because of (B.6)

$$R_A(\zeta)\Pi_{A,j} = \sum_{n=1}^{d_j} \frac{\Lambda_{-n}\Pi_{A,j}}{(\zeta - \zeta_j)^n}, \quad \forall \zeta \neq \zeta_j. \tag{B.11}$$

Since the operators Λ_{-n} are all bounded, we deduce for every $0 \leq j \leq N-1$ the estimate

$$\begin{aligned}\|R_A(\zeta)|_{\operatorname{ran}\Pi_{A,j}}\|_{\mathscr{B}(\operatorname{ran}\Pi_{A,j})} &\leq \sum_{n=1}^{d_j} \frac{\|\Lambda_{-n}\|_{\mathscr{B}(X)}}{|\zeta - \zeta_j|^n} \leq D_j \left(\frac{1}{|\zeta - \zeta_j|} + \frac{1}{|\zeta - \zeta_j|^{d_j}}\right) \\ &= D_j \frac{|\zeta - \zeta_j|^{d_j - 1} + 1}{|\zeta - \zeta_j|^{d_j}},\end{aligned}$$

where $D_j > 0$ is a constant. Using the unique decomposition of elements of X according to $X = Y \oplus \operatorname{ran}\Pi_{A,0} \oplus \cdots \oplus \operatorname{ran}\Pi_{A,N-1}$, we may estimate the norm $\|R_A(\zeta)\|_{\mathscr{B}(X)}$ by the sum of the norms of $R_A(\zeta)$ on each of the subspaces of the decomposition of X (for more details on this see Section C.2 in the Appendix). Combining (B.10) and (B.11) yields the desired estimate. $\square$

Now we turn to the actual theory of enlarging the space X. The following theorem is an extension of Theorem 2.1 in [GMM10]. Here we cover the more general case in which the isolated points $\zeta_0, \dots, \zeta_{N-1}$ of the spectrum are not necessarily eigenvalues of A.

Theorem B.15. *Let $X \hookrightarrow \mathcal{X}$ be two Hilbert spaces, $A \in \mathscr{C}(X)$ and $\mathcal{A} := \mathrm{cl}_{\mathcal{X}} A$. Assume there exists some $a \in \mathbb{R}$ and a finite set $Z := \{\zeta_0, \dots, \zeta_{N-1}\} \subset \Delta_a$ for some $N \in \mathbb{N}$, such that $\mathcal{A} \in \mathrm{DCP}_{X,\mathcal{X}}(a, Z)$. Then there holds:*

(i) *If $A \in \mathrm{WLS}_X(a, Z)$ then $\mathcal{A} \in \mathrm{WLS}_{\mathcal{X}}(a, Z)$.*

(ii) *Let $A \in \mathrm{WLS}_X(a, Z)$. For any $\zeta \in \Delta_a \setminus Z$ we have the representation*

$$R_{\mathcal{A}}(\zeta) = (\zeta - \mathcal{S})^{-1} + R_A(\zeta)\mathcal{B}(\zeta - \mathcal{S})^{-1}, \tag{B.12}$$

which implies the estimate

$$\|R_{\mathcal{A}}(\zeta)\|_{\mathscr{B}(\mathcal{X})} \leq \|(\zeta - \mathcal{S})^{-1}\|_{\mathscr{B}(\mathcal{X})} + \|R_A(\zeta)\|_{\mathscr{B}(X)}\|\mathcal{B}(\zeta - \mathcal{S})^{-1}\|_{\mathscr{B}(\mathcal{X},X)}.$$

(iii) *If $A \in \mathrm{SLS}_X(a, Z)$ then $\mathcal{A} \in \mathrm{SLS}_{\mathcal{X}}(a, Z)$. The corresponding algebraic eigenspaces coincide, i.e. $M(\zeta_k - A) = M(\zeta_k - \mathcal{A})$ for all $k \in \{0, \dots, N - 1\}$.*

Parts of the following proof are similar to the proof of Theorem 2.1 in [GMM10].

Proof. We start by proving (ii) and the inclusion $Z \subseteq \sigma(\mathcal{A}) \cap \Delta_a$ from (i). For $\zeta \in \Delta_a \setminus Z$ we define the operator

$$U(\zeta) := (\zeta - \mathcal{S})^{-1} + R_A(\zeta)\mathcal{B}(\zeta - \mathcal{S})^{-1},$$

where $\mathcal{B}, \mathcal{S}$ are the operators decomposing $\mathcal{A} \in \mathrm{DCP}_{X,\mathcal{X}}(a, Z)$ according to Condition 5.3. According to the assumptions there holds $U(\zeta) \in \mathscr{B}(\mathcal{X})$. Next we show that $U(\zeta)$ is a right inverse of $\zeta - \mathcal{A}$, where we use $\mathcal{A} = \mathcal{B} + \mathcal{S}$:

$$\begin{aligned}(\zeta - \mathcal{A})U(\zeta) &= (\zeta - \mathcal{B} - \mathcal{S})(\zeta - \mathcal{S})^{-1} + (\zeta - \mathcal{A})R_A(\zeta)\mathcal{B}(\zeta - \mathcal{S})^{-1} \\ &= \mathrm{Id}_{\mathcal{X}} - \mathcal{B}(\zeta - \mathcal{S})^{-1} + \mathcal{B}(\zeta - \mathcal{S})^{-1} = \mathrm{Id}_{\mathcal{X}}.\end{aligned}$$

This proves that $\zeta - \mathcal{A}$ is surjective in $\mathcal{X}$. Now, according to Condition 5.3 (iii) there exists some $\lambda \in \Delta_a \setminus Z$ such that $\lambda - \mathcal{A}$ is injective in $\mathcal{X}$. Hence $\lambda \in \rho(\mathcal{A})$, and $U(\lambda) = R_{\mathcal{A}}(\lambda)$.

We fix now any $\zeta \in \Delta_a \setminus Z$, and show that $\zeta \in \rho(\mathcal{A})$. To this end we choose a rectifiable path $\gamma : [0, 1] \to \Delta \setminus Z$ with $\gamma(0) = \lambda$ and $\gamma(1) = \zeta$. Because of Condition 5.3 (ii) the map $t \mapsto \|(\gamma(t) - \mathcal{S})^{-1}\|_{\mathscr{B}(\mathcal{X})}$ is continuous on $[0, 1]$, and therefore uniformly bounded on $[0, 1]$. Analogously we obtain the uniform boundedness of $t \mapsto \|R_A(\gamma(t))\|_{\mathscr{B}(X)}$ on $[0, 1]$. Combining these proves that

$$\sup_{t\in[0,1]} \|U(\gamma(t))\|_{\mathscr{B}(\mathcal{X})} < \infty \quad \Leftrightarrow \quad \inf_{t\in[0,1]} \|U(\gamma(t))\|^{-1}_{\mathscr{B}(\mathcal{X})} =: \delta > 0. \tag{B.13}$$

Now, if $\lambda \in \rho(\mathcal{A})$ then also $\lambda' \in \rho(\mathcal{A})$ for all $|\lambda' - \lambda| < \|R_{\mathcal{A}}(\lambda)\|^{-1}_{\mathscr{B}(\mathcal{X})}$ (see the proof of Theorem VIII.2.1 in [Yos80]), and for these λ' we have $U(\lambda') = R_{\mathcal{A}}(\lambda')$. Together with observation (B.13) this proves that a δ-neighborhood of the set $\gamma([0, 1])$ lies in $\rho(\mathcal{A})$. In particular, $\gamma(1) \equiv \zeta \in \rho(A)$ and $U(\zeta) = R_{\mathcal{A}}(\zeta)$. Since $\zeta \in \Delta_a \setminus Z$ was arbitrary, this proves $\Delta_a \setminus Z \subset \rho(\mathcal{A})$ and the validity of equation (B.12) for all $\zeta \in \Delta_a \setminus Z$. The remaining estimate from (ii) follows directly from (B.12).

We proceed in completing the proof of (i) by showing that indeed $\zeta_j \in \sigma(\mathcal{A})$ for all $0 \leq j \leq N - 1$. The corresponding proof in [GMM10] covers only the case $\zeta_j \in \sigma_p(A)$. Hence, we provide here an independent demonstration. We fix ζ_j, with $0 \leq j \leq N - 1$. The corresponding spectral projection of A (in X) is denoted by $\Pi_{A,j}$, and analogously we write $\Pi_{\mathcal{A},j}$ for the spectral projection of $\mathcal{A}$. Since for all $\zeta \in \Delta_a \setminus Z$ there holds $R_A(\zeta) \subset R_{\mathcal{A}}(\zeta)$ (in the sense of graphs), we conclude that every pole of $R_A(\zeta)$ in $\Delta_a \setminus Z$ is a non-removable singularity of $R_{\mathcal{A}}(\zeta)$. Hence $Z \subseteq \sigma(\mathcal{A})$.

In order to prove (iii) we use the previously shown inclusion $\Pi_{A,j} \subset \Pi_{\mathcal{A},j}$, which follows from (2.1) and the previously shown inclusion $R_A(\zeta) \subset R_{\mathcal{A}}(\zeta)$. Note that $\Pi_{A,j}$ and $\Pi_{\mathcal{A},j}$ are bounded in X and $\mathcal{X}$, respectively. Hence, we can apply Lemma C.1 from the Appendix, which shows that $\operatorname{ran}\Pi_{\mathcal{A},j} = \operatorname{cl}_{\mathcal{X}} \operatorname{ran}\Pi_{A,j}$. If $A \in \mathrm{SLS}_X(a, Z)$, then $\operatorname{ran}\Pi_{A,j}$ is already closed (since it is finite-dimensional), and so $\operatorname{ran}\Pi_{\mathcal{A},j} = \operatorname{ran}\Pi_{A,j}$. As a consequence of Proposition B.10 (iv) and Proposition B.9 the equality of the (algebraic) eigenspaces follows. □

B.4. Lebesgue and Sobolev spaces

B.4.1. Compact Sobolev embeddings. On $\Omega = \mathbb{R}^d$ it is possible to find compact embeddings of weighted Sobolev spaces into weighted L^2-spaces if certain conditions on the weight functions are satisfied. Here, we need the following Corollary of Theorem 2.4 in [Opi89]:

Lemma B.16. *Let v, w be two weight functions on $\Omega = \mathbb{R}^d$. Assume further that*

$$\lim_{r\to\infty} \operatorname*{ess\,sup}_{\mathbf{x}\in B_r^2(\mathbf{0})^c} \frac{w(\mathbf{x})}{v(\mathbf{x})} = 0. \tag{B.14}$$

Then there holds the compact embedding $H^1(v, w) \hookrightarrow\hookrightarrow L^2(w)$.

Proof. by definition the weights are locally (essentially) uniformly bounded from below. So we have $H^1(B_r^2(\mathbf{0}); v, w) = H^1(B_r^2(\mathbf{0}))$ and $L^2(B_r^2(\mathbf{0}); w) = L^2(B_r^2(\mathbf{0}))$, for all $r > 0$. According to the Rellich-Kondrachov Theorem (cf. Theorem 6.2 in [AF03]) we know that the compact embedding $H^1(B_r^2(\mathbf{0})) \hookrightarrow\hookrightarrow L^2(B_r^2(\mathbf{0}))$ holds for all $r > 0$. In order to verify the assumptions of Theorem 2.4 in [Opi89] we evaluate for $\|u\|_{v,w} \leq 1$:

$$\begin{aligned}\|u\|^2_{B_r^2(\mathbf{0})^c;w} &\leq \int_{B_r^2(\mathbf{0})^c} |u(\mathbf{x})|^2 v(\mathbf{x}) \operatorname*{ess\,sup}_{\mathbf{y}\in B_r^2(\mathbf{0})^c} \frac{w(\mathbf{y})}{v(\mathbf{y})}\,\mathrm{d}\mathbf{x} \\ &\leq \operatorname*{ess\,sup}_{\mathbf{y}\in B_r^2(\mathbf{0})^c} \frac{w(\mathbf{y})}{v(\mathbf{y})}.\end{aligned}$$

According to the assumption (B.14) the right hand side tends to zero as $r \to \infty$. Thus the condition (2.13) in [Opi89] is satisfied, and this proves the desired embedding. □

B.4.2. Density of test functions. Usually we initially consider operators only on $C_0^\infty(\mathbb{R}^d)$, and then conclude results by closure or continuous continuation. However, for this we need the density of $C_0^\infty(\mathbb{R}^d)$ in the respective weighted spaces. This is not a trivial question, as for example $C_0^\infty((0,1))$ is not dense in $H^1(0,1)$. The first result concerns weighted Sobolev spaces, for details see Theorem 14.4 in [GO91].

Lemma B.17. *Let w_0, w_1 be weight functions on $\mathbb{R}^d$ that are radially symmetric, so there exist weight functions v_0, v_1 on $\mathbb{R}^+$ such that $w_j(\mathbf{x}) = v_j(|\mathbf{x}|_2)$ for all $\mathbf{x} \in \mathbb{R}^d$ and $j \in \{0, 1\}$. Suppose there exists some $C > 0$ and $s_0 > 0$ such that*

$$v_0(s) \geq C v_1(s) s^{-2}, \quad \forall s > s_0. \tag{B.15}$$

Then the inclusion $C_0^\infty(\mathbb{R}^d) \subset H^1(w_0, w_1)$ is dense.

For weighted L^2-spaces the situation is more straightforward:

Lemma B.18. *Let ν be a weight function on $\mathbb{R}^d$. Then $C_0^\infty(\mathbb{R}^d)$ is dense in $L^2(\nu)$.*

Proof. This result follows from the fact that $C_0^\infty(\mathbb{R}^d)$ is dense in the unweighted space $L^2(\mathbb{R}^d)$, see Theorem 2.19 in [AF03]. Choose any $f \in L^2(\nu)$. Then $f\nu^{1/2} \in L^2(\mathbb{R}^d)$. Clearly there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ with the property that $\|f_n - f\nu^{1/2}\|_{L^2(\mathbb{R}^d)} < \frac{1}{n}$ for all $n \in \mathbb{N}$. Now for every $n \in \mathbb{N}$ the function $f_n\nu^{-1/2}$ lies in $L^2(\mathbb{R}^d)$, so there exists a sequence $(g_m^n)_{m\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ such that $g_m^n \to f_n\nu^{-1/2}$ in $L^2(\mathbb{R}^d)$ as $m \to \infty$. Since for every $n \in \mathbb{N}$ the support $\operatorname{supp} f_n$ is compact and $\nu^{-1/2}$ is (essentially) uniformly bounded on that support, we conclude that also $g_m^n\nu^{1/2} \to f_n$ in $L^2(\mathbb{R}^d)$ as $m \to \infty$. In particular we can chose those sequences in a way that $\|g_m^n\nu^{1/2} - f_n\|_{L^2(\mathbb{R}^d)} < \frac{1}{m}$ for all $m, n \in \mathbb{N}$. So we conclude by applying the triangle inequality that for all $n \in \mathbb{N}$ (set $m = n$)

$$\|g_n^n\nu^{1/2} - f\nu^{1/2}\|_{L^2(\mathbb{R}^d)} \leq \|g_n^n\nu^{1/2} - f_n\|_{L^2(\mathbb{R}^d)} + \|f_n - f\nu^{1/2}\|_{L^2(\mathbb{R}^d)} < \frac{2}{n}.$$

So $(g_n^n)_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ is a sequence converging to f in $L^2(\nu)$. □

APPENDIX C

Deferred proofs and technicalities

C.1. Technical lemmata

Lemma C.1. *Consider two Hilbert spaces* $X \hookrightarrow \mathcal{X}$*, and a projection* $\mathrm{P}_{\mathcal{X}} \in \mathscr{B}(\mathcal{X})$*, such that* $\mathrm{P}_X := \mathrm{P}_{\mathcal{X}}|_X \in \mathscr{B}(X)$*. Then* $\operatorname{ran} \mathrm{P}_{\mathcal{X}} = \mathrm{cl}_{\mathcal{X}} \operatorname{ran} \mathrm{P}_X$ *and* $\ker \mathrm{P}_{\mathcal{X}} = \mathrm{cl}_{\mathcal{X}} \ker \mathrm{P}_X$*.*

Proof. We give here the proof of the equality of the ranges, the other identity can be shown analogously, using the complementary projections instead. On the one hand we have $\operatorname{ran} \mathrm{P}_X \subseteq \operatorname{ran} \mathrm{P}_{\mathcal{X}}$. Since $\operatorname{ran} \mathrm{P}_{\mathcal{X}}$ is closed in $\mathcal{X}$ because of the boundedness of $\mathrm{P}_{\mathcal{X}}$, this implies that $\mathrm{cl}_{\mathcal{X}} \operatorname{ran} \mathrm{P}_X \subseteq \operatorname{ran} \mathrm{P}_{\mathcal{X}}$. On the other hand $\mathrm{P}_{\mathcal{X}} = \mathrm{cl}_{\mathcal{X}} \mathrm{P}_X$, which implies $\operatorname{ran} \mathrm{P}_{\mathcal{X}} \subseteq \mathrm{cl}_{\mathcal{X}} \operatorname{ran} \mathrm{P}_X$. □

Lemma C.2. *Let* $X \hookrightarrow \mathcal{X}$ *be Hilbert spaces, and* $\psi_0, \dots, \psi_{k-1} \in \mathscr{B}(\mathcal{X}, \mathbb{C})$*,* $k \in \mathbb{N}^*$*, be linearly independent functionals. Then* $\tilde{\psi}_j := \psi_j|_X \in \mathscr{B}(X, \mathbb{C})$ *for all* $0 \le j \le k-1$*, and*

$$\bigcap_{j=0}^{k-1} \ker \psi_j = \mathrm{cl}_{\mathcal{X}} \bigcap_{j=0}^{k-1} \ker \tilde{\psi}_j. \tag{C.1}$$

Proof. Since the intersections in (C.1) are finite, it suffices to consider the case $k = 1$, and we omit the index j in the following. The boundedness of $\tilde{\psi}$ follows immediately from the continuous embedding $X \hookrightarrow \mathcal{X}$. In order to prove that $\ker \psi = \mathrm{cl}_{\mathcal{X}} \ker \tilde{\psi}$, we use the fact that according to the Riesz representation theorem there exists a unique $x \in X$ such that $\tilde{\psi}(\cdot) = \langle \cdot, x \rangle_X$, where $\langle \cdot, \cdot \rangle_X$ denotes the inner product in X. We use this $x \in X$ to define the orthogonal projection $\tilde{P} \in \mathscr{B}(X)$ by

$$\tilde{P} \colon y \mapsto \frac{x}{\|x\|_X^2} \langle y, x \rangle_X = \frac{x}{\|x\|_X^2} \tilde{\psi}(y).$$

Since $\psi \in \mathscr{B}(\mathcal{X}, \mathbb{C})$, it becomes clear from this representation that $\tilde{P}$ has a unique continuation to a bounded projection $P \in \mathscr{B}(\mathcal{X})$, which is then given by

$$P \colon y \mapsto \frac{x}{\|x\|_X^2} \psi(y).$$

From this representations it is clear that

$$\ker P = \ker \psi, \quad \ker \tilde{P} = \ker \tilde{\psi}. \tag{C.2}$$

Since $P|_X = \tilde{P}$, we can apply Lemma C.1, which shows that $\ker P = \mathrm{cl}_{\mathcal{X}} \ker \tilde{P}$. By combining this with (C.2) we conclude that $\ker \psi = \mathrm{cl}_{\mathcal{X}} \ker \tilde{\psi}$. □

Lemma C.3. *Let* $\mathcal{X} = L^2(\omega)$ *with* ω *defined in* (4.9)*. Then, for every* $f \in \mathcal{X}$ *there exists a constant* $C > 0$ *such that*

$$|\nabla^{\mathbf{k}} \hat{f}(\mathbf{0})| \le C \|f\|_\omega, \quad \mathbf{k} \in \mathbb{N}^d.$$

Proof. We have

$$|\nabla^{\mathbf{k}}\hat{f}(\mathbf{0})| \le \|\nabla^{\mathbf{k}}\hat{f}\|_{L^\infty(\mathbb{R}^d)} = \|\mathcal{F}[\mathbf{x}^{\mathbf{k}}f(\mathbf{x})]\|_{L^\infty(\mathbb{R}^d)} \le \|\mathbf{x}^{\mathbf{k}}f(\mathbf{x})\|_{L^1(\mathbb{R}^d)}$$
$$= \int_{\mathbb{R}^d} |f(\mathbf{x})|\omega(\mathbf{x})^{1/2}\cdot\left|\mathbf{x}^{\mathbf{k}}\omega(\mathbf{x})^{-1/2}\right|\,\mathrm{d}\mathbf{x} \le \|f\|_\omega\Big(\int_{\mathbb{R}^d}\mathbf{x}^{2\mathbf{k}}\omega(\mathbf{x})^{-1}\,\mathrm{d}\mathbf{x}\Big)^{1/2}.$$

Since $\omega(\mathbf{x})^{-1}$ decays exponentially as $|\mathbf{x}|_2\to\infty$ the integral on the right hand side is finite for every $\mathbf{k}\in\mathbb{N}^d$. □

Lemma C.4. *Let $\mathcal{X}$ be as in Lemma C.3, and consider $f\in\mathcal{X}$. Then for every $0<\beta'<\beta$ there holds*

$$\sup_{\mathbf{z}\in\Omega_{\beta'/2}} |\hat{f}(\mathbf{z})| < \infty.$$

Proof. Using Result (i) in Proposition 4.5 and the continuity of the Fourier transform from $L^1(\mathbb{R}^d)$ into $L^\infty(\mathbb{R}^d)$, we shall show that $\|f(\mathbf{x})\mathrm{e}^{\mathbf{b}\cdot\mathbf{x}}\|_{L^1(\mathbb{R}^d_{\mathbf{x}})}$ is uniformly bounded for $|\mathbf{b}|_1<\beta'/2$. So we compute

$$\left\|f(\mathbf{x})\mathrm{e}^{\mathbf{b}\cdot\mathbf{x}}\right\|_{L^1(\mathbb{R}^d_{\mathbf{x}})} = \int_{\mathbb{R}^d}|f(\mathbf{x})|\mathrm{e}^{\mathbf{b}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{x} \le \|f\|_\omega\Big(\int_{\mathbb{R}^d}\frac{\mathrm{e}^{2\mathbf{b}\cdot\mathbf{x}}}{\omega(\mathbf{x})}\,\mathrm{d}\mathbf{x}\Big)^{\frac{1}{2}}.$$

In the integral on the right hand side we apply the (discrete) Hölder inequality in the exponent, and obtain:

$$\int_{\mathbb{R}^d}\frac{\mathrm{e}^{2\mathbf{b}\cdot\mathbf{x}}}{\omega(\mathbf{x})}\,\mathrm{d}\mathbf{x} \le \int_{\mathbb{R}^d}\frac{\mathrm{e}^{\beta'|\mathbf{x}|_\infty}}{\omega(\mathbf{x})}\,\mathrm{d}\mathbf{x} \le 2\int_{\mathbb{R}^d}\frac{\omega\big(\frac{\beta'}{\beta}\mathbf{x}\big)}{\omega(\mathbf{x})}\,\mathrm{d}\mathbf{x}<\infty,$$

which now is a bound independent of $\mathbf{b}$. □

Lemma C.5. *Let $X\hookrightarrow\mathcal{X}$ be two Banach spaces, and $A\in\mathscr{C}(X)$ is the generator of a C_0-semigroup $(S(t))_{t\ge0}$ on X. Assume that for every $t\ge0$ the closure $T(t):=\operatorname{cl}_{\mathcal{X}}S(t)$ exists. Furthermore, assume that the operators $T(t)$ are bounded in $\mathcal{X}$, and that there exist constants $M,\omega>0$ such that*

$$\|T(t)\|_{\mathscr{B}(\mathcal{X})}\le M\mathrm{e}^{\omega t},\quad \forall t\ge0. \tag{C.3}$$

Then there holds:

(i) *The family $(T(t))_{t\ge0}$ is a C_0-semigroup of bounded operators on $\mathcal{X}$.*
(ii) *The generator $\mathcal{A}$ of $(T(t))_{t\ge0}$ fulfills $\mathcal{A}=\operatorname{cl}_{\mathcal{X}}A$.*

Proof. For the proof of (i) it only remains to verify the C_0-property of $(T(t))_{t\ge0}$. To this end we fix $f\in\mathcal{X}$ and choose an arbitrary sequence $(f_n)_{n\in\mathbb{N}}\subset X$ with $f_n\to f$ in $\mathcal{X}$. By using the continuous embedding $X\hookrightarrow\mathcal{X}$ and (C.3) we compute

$$\|T(t)f-f\|_{\mathcal{X}} \le \|T(t)(f-f_n)\|_{\mathcal{X}}+\|S(t)f_n-f_n\|_{\mathcal{X}}+\|f_n-f\|_{\mathcal{X}}$$
$$\le (M\mathrm{e}^{\omega t}+1)\|f-f_n\|_{\mathcal{X}}+C\|S(t)f_n-f_n\|_X.$$

Here $C>0$ is the constant from the continuous embedding $X\hookrightarrow\mathcal{X}$. We now take the limit $t\searrow0$ in the above estimate, and obtain

$$\lim_{t\searrow0}\|T(t)f-f\|_{\mathcal{X}}\le(M+1)\|f-f_n\|_{\mathcal{X}},\quad\forall n\in\mathbb{N}.$$

Here we have used the strong continuity of $(S(t))_{t\ge0}$ in X. Since $n\in\mathbb{N}$ is arbitrary, we may take the limit $n\to\infty$ on the right hand side, which then converges to zero. This proves the strong continuity of $(T(t))_{t\ge0}$ in $\mathcal{X}$.

Now we show that the generator $\mathcal{A}$ of $(T(t))_{t\geq 0}$ indeed satisfies $\mathcal{A} = \mathrm{cl}_{\mathcal{X}} A$. Because of $X \hookrightarrow \mathcal{X}$ there clearly holds $A \subset \mathcal{A}$. Since A and $\mathcal{A}$ both generate a C_0-semigroup, we can find some real number $\lambda > \max\{\mathrm{s}(A), \mathrm{s}(\mathcal{A})\}$, where $\mathrm{s}(\cdot)$ denotes the spectral bound of an operator (see Chapter 2). Clearly there holds $\lambda \in \rho(A) \cap \rho(\mathcal{A})$. Because of $A \subset \mathcal{A}$ we also have for the resolvents $R_A(\lambda) \subset R_{\mathcal{A}}(\lambda)$. Since $R_{\mathcal{A}}(\lambda) \in \mathscr{B}(\mathcal{X})$ this implies $R_{\mathcal{A}}(\lambda) = \mathrm{cl}_{\mathcal{X}} R_A(\lambda)$. But this is equivalent to $\mathcal{A} = \mathrm{cl}_{\mathcal{X}} A$. $\square$

C.2. Bounded projections

In this section we briefly discuss bounded projections. In the following X is a Banach space.

A *bounded projection* is an operator $P \in \mathscr{B}(X)$ that satisfies $P^2 = P$. According to the discussion in Section III.3.4 in [Kat66], for a bounded projection P both $\operatorname{ran} P$ and $\ker P$ are closed linear subspaces of X, and

$$X = \ker P \oplus \operatorname{ran} P.$$

Conversely, assume we have $X = M \oplus N$ with closed linear subspaces M and N. Since the sum is direct, for every $x \in X$ there are unique $m \in M$ and $n \in N$ such that $x = m + n$. Hence, we may define a linear map on X via $Px := m$. This P satisfies $P^2 = P$. Clearly, P is also closed: Consider a Cauchy sequence $(x_j)_{j\in\mathbb{N}} \subset X$ such that $(Px_j)_{j\in\mathbb{N}} \subset M$ is a also Cauchy sequence. Hence, both $(Px_j)_{j\in\mathbb{N}}$ and $(x_j - Px_j)_{j\in\mathbb{N}}$ are Cauchy sequences, and their limit lies in M and N, respectively. So we get

$$\lim_{j\to\infty} Px_j + \lim_{j\to\infty} (x_j - Px_j) = \lim_{j\to\infty} x_j.$$

Hence, $\lim_{j\to\infty} Px_j = P \lim_{j\to\infty} x_j$, which is the component of $\lim_{j\to\infty} x_j$ in M by definition of P. Therefore, P is closed. Since P is also defined on all X, we can apply the Closed Graph Theorem, which proves that indeed $P \in \mathscr{B}(X)$.

The boundedness of the projection P defined by $X = M \oplus N$ has a useful consequence. If P is a bounded projection, then also $I - P$ is a bounded projection. The boundedness of these projections implies that there exists a constant $C > 0$ such that for all $x \in X \setminus \{0\}$

$$\frac{\|Px\| + \|(I-P)x\|}{\|x\|} \leq C.$$

This is equivalent to $\|m\| + \|n\| \leq C\|m + n\|$ for all $m \in M$ and $n \in N$. On the other hand, the triangle inequality also shows $\|m+n\| \leq \|m\|+\|n\|$. Hence, $\|Px\|+\|(I-P)x\|$ is an equivalent norm to $\|x\|$ for $x \in X$.

C.3. Proof of Proposition 4.5

The following is a collection of results that in combination lead to the proof of Proposition 4.5. Throughout this section, $\mathcal{X}$ is the weighted L^2-space defined on page 36, with the weight function (4.9). The proof orientates itself on Exercise 76 (proof of Theorem IX.14) of Chapter IX in [RS75].

Lemma C.6. *For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ there holds $f \in \mathcal{X}$ iff for all $\mathbf{b} \in \mathbb{R}^d$ with $|\mathbf{b}|_1 \leq \beta/2$ there holds*

$$\int_{\mathbb{R}^d} \left|f(\mathbf{x})\mathrm{e}^{\mathbf{x}\cdot\mathbf{b}}\right|^2 \mathrm{d}\mathbf{x} < \infty. \tag{C.4}$$

Proof. Let $f \in \mathcal{X}$. We observe that for every $\mathbf{x} \in \mathbb{R}^d$

$$2\omega(\mathbf{x}) \geq \mathrm{e}^{\beta|\mathbf{x}|_\infty} = \sup_{\substack{\mathbf{b}\in\mathbb{R}^d \\ |\mathbf{b}|_1 \leq \beta}} \mathrm{e}^{\mathbf{b}\cdot\mathbf{x}}.$$

Here, we have used Hölder's inequality. So we have the estimate $2\omega(\mathbf{x}) \geq \mathrm{e}^{\mathbf{b}\cdot\mathbf{x}}$ for all $\mathbf{b} \in \mathbb{R}^d$ with $|\mathbf{b}|_1 \leq \beta$, and all $\mathbf{x} \in \mathbb{R}^d$. Hence, we can estimate the integral in (C.4) by the $\|f\|_\omega^2$, which is finite.

Let now (C.4) hold for some $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. We conclude that for every $\ell \in \{1, \ldots, d\}$ there holds

$$\int_{\mathbb{R}^d} |f(\mathbf{x})|^2 \mathrm{e}^{\pm\beta x_\ell} \,\mathrm{d}\mathbf{x} < \infty.$$

Summing over all $\ell \in \{1, \ldots, d\}$ yields $2\|f\|_\omega^2$, and so $f \in \mathcal{X}$. $\square$

Lemma C.7. *If $f \in \mathcal{X}$ then $\hat{f}$ is analytic in $\Omega_{\beta/2}$ (with the notation of Proposition 4.5). For every $\mathbf{b} \in \mathbb{R}^d$ with $|\mathbf{b}|_1 < \beta/2$ there holds* (4.11)*, i.e.:*

$$\hat{f}(\boldsymbol{\xi} + \mathrm{i}\mathbf{b}) = \mathcal{F}[f(\mathbf{x})\exp(\mathbf{b}\cdot\mathbf{x})](\boldsymbol{\xi}), \quad \boldsymbol{\xi} \in \mathbb{R}^d.$$

Proof. Inspired by (4.11) we define the following function for $\boldsymbol{\xi} \in \mathbb{R}^d$ and for all $\mathbf{b} \in \mathbb{R}^d$ for which the following integral exists:

$$F(\boldsymbol{\xi} + \mathrm{i}\mathbf{b}) := \int_{\mathbb{R}^d} f(\mathbf{x}) \exp\big(-\mathrm{i}\mathbf{x}\cdot(\boldsymbol{\xi} + \mathrm{i}\mathbf{b})\big) \,\mathrm{d}\mathbf{x}.$$

In the following we show that this indeed coincides with the analytic continuation of $\hat{f}$. First we investigate the definition of F for given $\mathbf{b} \in \mathbb{R}^d$. This integral is defined in the classical sense iff $f(\mathbf{x})\exp\big(\mathbf{x}\cdot\mathbf{b}\big) \in L^1(\mathbb{R}^d)$. Since $f \in \mathcal{X}$, this holds true whenever $\exp(2\mathbf{b}\cdot\mathbf{x})$ is uniformly bounded by $\omega(\varepsilon\mathbf{x})$ for some $\varepsilon \in (0, 1)$. Clearly, there holds for all $\mathbf{x} \in \mathbb{R}^d$

$$\frac{\omega(\varepsilon\mathbf{x})}{\exp(\varepsilon\beta|\mathbf{x}|_\infty)} \in (\tfrac{1}{2}, d].$$

Thus, $\exp(2\mathbf{b}\cdot\mathbf{x})$ can be uniformly bounded by $\omega(\varepsilon\mathbf{x})$ iff $\mathbf{b}\cdot\mathbf{x} \leq \varepsilon\frac{\beta}{2}|\mathbf{x}|_\infty$, for all $\mathbf{x} \in \mathbb{R}^d$. Using Hölder's inequality we find that this is true iff $|\mathbf{b}|_1 \leq \varepsilon\frac{\beta}{2}$. We conclude that F is defined whenever $|\mathbf{b}|_1 < \beta/2$.

We now show that F is analytic in $\Omega_{\beta/2}$. Formally we get for every $j \in \{1, \ldots, d\}$ and suitable $\mathbf{z} \in \mathbb{C}^d$:

$$\frac{\partial}{\partial z_j} F(\mathbf{z}) = \int_{\mathbb{R}^d} f(\mathbf{x}) \exp(-\mathrm{i}\mathbf{x}\cdot\mathbf{z})(-\mathrm{i}x_j) \,\mathrm{d}\mathbf{x}. \tag{C.5}$$

This proves the analyticity (i.e. complex differentiability), if we can justify the interchange of the differentiation and the integration. For $\mathbf{z} \in \Omega_{\beta/2}$ there exists some $\varepsilon > 0$ such that $B^1_\varepsilon(\mathbf{z}) \subset \Omega_{\beta/2}$. So we can find some $0 < \beta' < \beta$ and $C \geq 1$ such that we have for all $\mathbf{z}' \in B^1_\varepsilon(\mathbf{z})$ and all $\mathbf{x} \in \mathbb{R}^d$:

$$|\exp(-\mathrm{i}\mathbf{x}\cdot\mathbf{z}')x_j| \leq C\exp\big(\tfrac{\beta'}{2}|\mathbf{x}|_\infty\big),$$

again using the Hölder inequality. Therefore, for all $\mathbf{z}' \in B^1_\varepsilon(\mathbf{z})$ the integrand

$$f(\mathbf{x})\exp(-\mathrm{i}\mathbf{x}\cdot\mathbf{z}')(-\mathrm{i}x_j)$$

is uniformly bounded by $|f(\mathbf{x})\exp(\frac{\beta'}{2}|\mathbf{x}|_\infty)|$, which lies in $L^1(\mathbb{R}^d)$. So we can apply the Theorem of Dominated Convergence, which proves (C.5). Therefore F is the analytic continuation of $\hat{f}$ to $\Omega_{\beta/2}$, and this continuation $\hat{f}$ indeed satisfies the identity (4.11). $\square$

Corollary C.8. *If $f \in C_0^\infty(\mathbb{R}^d)$, then the Fourier transform $\hat{f}$ is analytic on $\mathbb{C}^d$, and* (4.11) *holds true for all $\mathbf{b} \in \mathbb{R}^d$.*

Lemma C.9. *For $f \in \mathcal{X}$ we define for every $|\mathbf{b}|_1 = \beta/2$ a continuation of the Fourier transform by $\hat{f}(\boldsymbol{\xi} + \mathrm{i}\mathbf{b}) := \mathcal{F}[f(\mathbf{x})\exp(\mathbf{b}\cdot\mathbf{x})](\boldsymbol{\xi})$, which lies in $L^2(\mathbb{R}^d)$. There holds $\mathbf{b} \mapsto \hat{f}(\cdot + \mathrm{i}\mathbf{b}) \in C(\overline{B^1_{\beta/2}(\mathbf{0})}; L^2(\mathbb{R}^d))$.*

Proof. We fix $f \in \mathcal{X}$ and consider $\mathbf{b}_0, \mathbf{b} \in \overline{B^1_{\beta/2}(\mathbf{0})}$. According to Lemma C.7 we need to show that

$$\lim_{\substack{\mathbf{b}\to\mathbf{b}_0 \\ \mathbf{b}\in\overline{B^1_{\beta/2}(\mathbf{0})}}} \int_{\mathbb{R}^d} \big|f(\mathbf{x})^2\big(\exp(2\mathbf{b}_0\cdot\mathbf{x}) - \exp(2\mathbf{b}\cdot\mathbf{x})\big)\big|\,\mathrm{d}\mathbf{x} = 0. \tag{C.6}$$

Clearly there holds $\exp(2\mathbf{b}_0\cdot\mathbf{x}) \le 2\omega(\mathbf{x})$ and $\exp(2\mathbf{b}\cdot\mathbf{x}) \le 2\omega(\mathbf{x})$ uniformly for $\mathbf{x} \in \mathbb{R}^d$. So for every $\varepsilon > 0$ we can find some $R > 0$ such that the following holds uniformly for all $\mathbf{b} \in \overline{B^1_{\beta/2}(\mathbf{0})}$:

$$\int_{B^2_R(\mathbf{0})^c} \big|f(\mathbf{x})^2\big(\exp(2\mathbf{b}_0\cdot\mathbf{x}) - \exp(2\mathbf{b}\cdot\mathbf{x})\big)\big|\,\mathrm{d}\mathbf{x} < \varepsilon.$$

For the remaining part it is clear that

$$\lim_{\substack{\mathbf{b}\to\mathbf{b}_0 \\ \mathbf{b}\in\overline{B^1_{\beta/2}(\mathbf{0})}}} \int_{B^2_R(\mathbf{0})} \big|f(\mathbf{x})^2\big(\exp(2\mathbf{b}_0\cdot\mathbf{x}) - \exp(2\mathbf{b}\cdot\mathbf{x})\big)\big|\,\mathrm{d}\mathbf{x} = 0.$$

Since $\varepsilon > 0$ was arbitrary the validity of (C.6) follows. $\square$

Lemma C.10. *Let $g \in C_0^\infty(\mathbb{R}^d)$ and $f \in L^2(\mathbb{R}^d)$ such that the Fourier transform $\hat{f}$ has an analytic continuation to $\Omega_{\beta/2}$. Furthermore, assume that* (4.10) *holds. Then the following equality holds true for all $|\mathbf{b}|_1 < \beta/2$:*

$$(2\pi)^d \int_{\mathbb{R}^d} \overline{g(\mathbf{x})} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^d} \overline{\hat{g}(\boldsymbol{\xi} - \mathrm{i}\mathbf{b})}\hat{f}(\boldsymbol{\xi} + \mathrm{i}\mathbf{b})\,\mathrm{d}\boldsymbol{\xi}. \tag{C.7}$$

Proof. Fix $\mathbf{b} \in B^1_{\beta/2}(\mathbf{0})$. Using Corollary C.8 and the representation formula (4.11) we immediately find that

$$\overline{\hat{g}(\boldsymbol{\xi} - \mathrm{i}\mathbf{b})} = \hat{\overline{g}}(-\boldsymbol{\xi} - \mathrm{i}\mathbf{b}). \tag{C.8}$$

Since $g \in C_0^\infty(\mathbb{R}^d)$ we can apply the Paley-Wiener Theorem IX.11 from [RS75], which proves that $\lim_{|\operatorname{Re}\boldsymbol{\xi}|_2\to\infty} \hat{g}(\boldsymbol{\xi} + \mathrm{i}\mathbf{b}) = 0$, and this limit is uniform in $\mathbf{b} \in \mathbb{R}^d$. In combination with (4.10) and by using (C.8) we can shift the domain of integration on the right hand side of (C.7) from $\mathbb{R}^d$ to $\mathbb{R}^d - \mathrm{i}\mathbf{b}$, without changing the value of the integral according to Cauchy's Integral Theorem:

$$\int_{\mathbb{R}^d} \overline{\hat{g}(\boldsymbol{\xi} - \mathrm{i}\mathbf{b})}\hat{f}(\boldsymbol{\xi} + \mathrm{i}\mathbf{b})\,\mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^d} \overline{\hat{g}(\boldsymbol{\xi})}\hat{f}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}.$$

According to Parseval's formula (see [DL88]) the right hand side coincides with $(2\pi)^d \int_{\mathbb{R}^d} \overline{g(\mathbf{x})} f(\mathbf{x})\,\mathrm{d}\mathbf{x}$. □

Proposition C.11. *Assume that $\hat{f}$ is analytic in $\Omega_{\beta/2}$ and it satisfies* (4.10)*. Then $f \in \mathcal{X}$.*

Proof. Choose some $g \in C_0^\infty(\mathbb{R}^d)$. For every $|\mathbf{b}|_1 < \beta/2$ we introduce the function $g_{\mathbf{b}}(\mathbf{x}) := g(\mathbf{x})\mathrm{e}^{\mathbf{b}\cdot\mathbf{x}} \in C_0^\infty(\mathbb{R})$. Inserting this in (C.7) and using Corollary C.8 for g yields

$$(2\pi)^d \int_{\mathbb{R}^d} \overline{g(\mathbf{x})}\mathrm{e}^{\mathbf{b}\cdot\mathbf{x}} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^d} \overline{\hat{g}(\boldsymbol{\xi})}\hat{f}(\boldsymbol{\xi} + \mathrm{i}\mathbf{b})\,\mathrm{d}\boldsymbol{\xi}.$$

Since $g \in C_0^\infty(\mathbb{R}^d)$ was arbitrary, and both $C_0^\infty(\mathbb{R}^d)$ and $\mathcal{F}[C_0^\infty(\mathbb{R}^d)]$ are dense in $L^2(\mathbb{R}^d)$, this proves that

$$\hat{f}(\boldsymbol{\xi} + \mathrm{i}\mathbf{b}) = \mathrm{e}^{\mathbf{b}\cdot\mathbf{x}} f(\mathbf{x}),$$

and $\mathrm{e}^{\mathbf{b}\cdot\mathbf{x}} f(\mathbf{x}) \in L^2(\mathbb{R}^d)$. This holds for all $|\mathbf{b}|_1 < \beta/2$. But together with (4.10) these intermediate results imply that $\mathrm{e}^{\mathbf{b}\cdot\mathbf{x}} f(\mathbf{x}) \in L^2(\mathbb{R}^d)$ for all $|\mathbf{b}|_1 \le \beta/2$. And now we can apply Lemma C.6, which finally proves $f \in \mathcal{X}$. □

Bibliography

[AAS15] Achleitner F., Arnold A., Stürzer D. Large-time behavior in non-symmetric Fokker-Planck equations. *Riv. Math. Univ. Parma*, **6**, no. 1 (2015), pp. 1–68.

[AF03] Adams R.A., Fournier J.J.F. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Second edition. Elsevier/Academic Press, Amsterdam, 2003.

[AFN08] Arnold A., Fagnola F., Neumann L. Quantum Fokker-Planck models: the Lindblad and Wigner approaches. In *Quantum probability and related topics*, volume 23 of *QP–PQ: Quantum Probab. White Noise Anal.* World Sci. Publ., Hackensack, NJ, 2008, pp. 23–48.

[AGG$^+$12] Arnold A., Gamba I.M., Gualdani M.P., Mischler S., Mouhot C., Sparber C. The Wigner-Fokker-Planck equation: stationary states and large time behavior. *Math. Models Methods Appl. Sci.*, **22**, no. 11 (2012), pp. 1250034, 31.

[ALMS04] Arnold A., López J.L., Markowich P.A., Soler J. An analysis of quantum Fokker-Planck models: a Wigner function approach. *Rev. Mat. Iberoamericana*, **20**, no. 3 (2004), pp. 771–814.

[AMTU01] Arnold A., Markowich P., Toscani G., Unterreiter A. On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations. *Comm. Partial Differential Equations*, **26**, no. 1-2 (2001), pp. 43–100.

[AV03] Albrecht E., Vasilescu F.H. Invariant subspaces for some families of unbounded subnormal operators. *Glasg. Math. J.*, **45**, no. 1 (2003), pp. 53–67.

[BGM94] Bakry D., Gill R.D., Molchanov S.A. *Lectures on probability theory*, volume 1581 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1994.

[CH53] Courant R., Hilbert D. *Methods of mathematical physics. Vol. I.* Interscience Publishers, Inc., New York, N.Y., 1953.

[Det84] Dettman J.W. *Applied complex variables.* Dover Publications Inc., New York, 1984.

[DL88] Dautray R., Lions J.L. *Mathematical analysis and numerical methods for science and technology. Vol. 2.* Springer-Verlag, Berlin, 1988.

[DL90] Dautray R., Lions J.L. *Mathematical analysis and numerical methods for science and technology. Vol. 3.* Springer-Verlag, Berlin, 1990.

[DL92] Dautray R., Lions J.L. *Mathematical analysis and numerical methods for science and technology. Vol. 5.* Springer-Verlag, Berlin, 1992.

[EN00] Engel K.J., Nagel R. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[GL09] Gohberg I., Leiterer J. *Holomorphic operator functions of one variable and applications*, volume 192 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2009.

[GMM10] Gualdani M.P., Mischler S., Mouhot C. Factorization for non-symmetric operators and exponential H-theorem. *Preprint:* ***http://arxiv.org/abs/1006.5523v1***.

[GO91] Gurka P., Opic B. Continuous and compact imbeddings of weighted Sobolev spaces III. *Czechoslovak Mathematical Journal*, **41**, no. 2 (1991), pp. 317–341.

[GR98] Gallay T., Raugel. Scaling variables and asymptotic expansions in damped wave equations. *Journal of Differential Equations*, **150** (1998), pp. 42–97.

[GW02] Gallay T., Wayne C.E. Invariant manifolds and the long-time asymptotics of the navier-stokes and vorticity equations on $\mathbb{R}^2$. *Archive for Rational Mechanics and Analysis*, **163**, no. 3 (2002), pp. 209–258.

[GW05] Gallay T., Wayne C.E. Global stability of vortex solutions of the two-dimensional Navier-Stokes equation. *Comm. Math. Phys.*, **255**, no. 1 (2005), pp. 97–129.

[Hel02] Helffer B. *Semiclassical analysis, Witten Laplacians, and statistical mechanics*, volume 1 of *Series in Partial Differential Equations and Applications*. World Scientific Publishing Co. Inc., 2002.

[Heu92] Heuser H. *Funktionalanalysis*. Third edition. Mathematische Leitfäden. [Mathematical Textbooks], B. G. Teubner, Stuttgart, 1992.

[HN05] Helffer B., Nier F. *Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians*, volume 1862 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2005.

[Kat66] Kato T. *Perturbation theory for linear operators*, volume 132 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, New York, 1966.

[KO84] Kufner A., Opic B. How to define reasonably weighted Sobolev spaces. *Comment. Math. Univ. Carolin.*, **25**, no. 3 (1984), pp. 537–554.

[Lax68] Lax P.D. Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math.*, **21** (1968), pp. 467–490.

[LK05] Lee S.H., Kang K.G. Numerical analysis of electronic transport characteristics in dielectrics irradiated by ultrashort pulsed laser using the nonlocal Fokker-Planck equation. *Numerical Heat Transfer, Part A*, **48**, no. 1 (2005), pp. 59–76.

[Met01] Metafune G. L^p-spectrum of Ornstein-Uhlenbeck operators. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, **30**, no. 1 (2001), pp. 97–124.

[Opi89] Opic B. Necessary and sufficient conditions for imbeddings in weighted Sobolev spaces. *Časopis Pěst. Mat.*, **114**, no. 4 (1989), pp. 343–355.

[Par10] Parmeggiani A. *Spectral theory of non-commutative harmonic oscillators: an introduction*, volume 1992 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2010.

[Paz83] Pazy A. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer, New York, 1983.

[RS75] Reed M., Simon B. *Methods of modern mathematical physics II: Fourier analysis, self-adjointness*. Academic Press, New York, 1975.

[RS78] Reed M., Simon B. *Methods of modern mathematical physics IV: Analysis of operators*. Academic Press, New York, 1978.

[SA14] Stürzer D., Arnold A. Spectral analysis and long-time behavior of a Fokker-Planck equation with a non-local perturbation. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, **25**, no. 1 (2014), pp. 53–89.

[TL80] Taylor A.E., Lay D.C. *Introduction to functional analysis*. Second edition. John Wiley & Sons, New York-Chichester-Brisbane, 1980.

[Yos80] Yosida K. *Functional analysis*, volume 123 of *Grundlehren der Mathematischen Wissenschaften*. Sixth edition. Springer-Verlag, Berlin, 1980.

Part 2

An Euler-Bernoulli beam with nonlinear boundary dissipation

CHAPTER 1

Preliminaries & some results on semilinear evolution equations

In this chapter we discuss some properties of semilinear evolution equations in a Hilbert space. Those results will be needed in the Chapters 2 and 3.

1.1. Preliminaries

For a function $u(t,x)$, $t \geq 0$, $x \in [0,L]$ for some $L > 0$, we use the notation u_t or $\dot{u}$ for the derivative with respect to the time variable t, and we write u' for the x-derivative. Higher order x-derivatives are denoted by roman superscripts. Whenever it is clear from the context, we omit the time variable in the notation and write for example $u(L) \equiv u(t,L)$ and $u(0) \equiv u(t,0)$. If not stated otherwise all functions occurring in this part are considered to be real valued, and only real valued solutions of equations are sought. Therefore, in addition to the Hilbert spaces $L^2(0,L)$ and $H^n(0,L)$, which are understood to consist of complex valued functions, we also need

$$L^2_{\mathbb{R}}(0,L) := \{f \in L^2(0,L) : \; f\colon [0,L] \to \mathbb{R}\},$$

and analogously we define $H^n_{\mathbb{R}}(0,L)$, for $n \in \mathbb{N} \setminus \{0\}$. Additionally, we define for $n \in \mathbb{N} \setminus \{0,1\}$:

$$\tilde{H}^n_{0,\mathbb{R}}(0,L) := \{f \in H^n_{\mathbb{R}}(0,L) : f(0) = f'(0) = 0\}. \tag{1.1}$$

A linear operator A is a closed, linear map $A : \mathcal{X} \to \mathcal{X}$, where $\mathcal{X}$ is a suitable real or complex Banach space, and the domain $D(A)$ needs to be dense in $\mathcal{X}$. The range of A is $\operatorname{ran} A \subset \mathcal{X}$. A closed linear subset X of a Hilbert space $\mathcal{X}$ is called A-invariant if $X \cap D(A) \subset X$ is dense and $\operatorname{ran} A|_X \subset X$.

Throughout this part C denotes a positive constant, not necessarily always the same.

1.2. Semilinear evolution equations

Throughout Chapter 1, we make the following assumptions: $\mathcal{H}$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and the norm $\| \cdot \|_{\mathcal{H}}$. A is always a linear operator with domain $D(A)$, and $\mathcal{N}\colon \mathcal{H} \to \mathcal{H}$ is a (possibly nonlinear) operator. Furthermore, we always assume that the following is fulfilled:

Assumption 1.1. For the operators A and $\mathcal{N}$ we require:

(i) A generates a C_0-semigroup of contractions on $\mathcal{H}$, denoted by $(\mathrm{e}^{tA})_{t\geq 0}$.
(ii) $\mathcal{N}$ is differentiable in $\mathcal{H}$ and Lipschitz continuous on bounded sets (we refer to this as *local Lipschitz continuity*).

We define the nonlinear operator $\mathcal{A} := A + \mathcal{N}$ on the domain $D(\mathcal{A}) := D(A)$. In the following we are interested in evolution equations in $\mathcal{H}$ of the following form:

$$\begin{aligned} \frac{\mathrm{d}y(t)}{\mathrm{d}t} &= Ay(t) + \mathcal{N}y(t), \qquad t > 0, \\ y(0) &= y_0 \in \mathcal{H}. \end{aligned} \tag{1.2}$$

For evolution equations in the form of (1.2) we distinguish between two different kinds of solutions. Let $T > 0$ be fixed. A function $y \colon [0,T) \to \mathcal{H}$ is a

- *classical solution* of (1.2) on $[0,T)$ if $y \in C([0,T);\mathcal{H})$ and $y \in C^1((0,T);\mathcal{H})$, $y(t) \in D(A)$ for all $t \in (0,T)$, and $y(t)$ satisfies (1.2) on $[0,T)$.
- *mild solution* of (1.2) on $[0,T)$ if $y \in C([0,T);\mathcal{H})$ and y satisfies the *Duhamel formula*

$$y(t) = \mathrm{e}^{tA}y_0 + \int_0^t \mathrm{e}^{(t-s)A}\mathcal{N}y(s)\,\mathrm{d}s, \quad 0 < t < T. \tag{1.3}$$

A classical solution is always a mild solution too. If we can choose $T = \infty$ in those definitions, the solution $y(t)$ is called *global* in time. A first existence result is due to Theorem 6.1.4 in [Paz83]:

Proposition 1.2. *For every $y_0 \in \mathcal{H}$ there exists a unique mild solution $y(t)$ on $[0, T_{\max}(y_0))$ of (1.2), where $T_{\max}(y_0)$ is the maximal time interval for which the solution exists. If $T_{\max}(y_0) < \infty$, then a blow-up occurs, i.e.*

$$\lim_{t \nearrow T_{\max}(y_0)} \|y(t)\|_{\mathcal{H}} = \infty.$$

Proof. Since we assume that $\mathcal{N} \colon \mathcal{H} \to \mathcal{H}$ is locally Lipschitz continuous, the result directly follows from Theorem 6.1.4 in [Paz83]. □

Lemma 1.3. *If $y_0 \in D(A)$, then the corresponding mild solution $y(t)$ of (1.2) is classical.*

Proof. This follows immediately from Theorem 6.1.5 in [Paz83], since $\mathcal{N}$ is differentiable. □

Throughout this chapter, we additionally assume the existence of a function V with the following properties:

Assumption 1.4. There exists a continuous function $V \colon \mathcal{H} \to [0,\infty)$ such that

(i) For every $y_0 \in D(A)$ and the corresponding classical solution $y(t)$ of (1.2) there holds

$$t \mapsto V(y(t)) \quad \text{is monotonically decreasing on } [0, T_{\max}(y_0)).$$

Here, $T_{\max}(y_0)$ is the bound introduced in Proposition 1.2.

(ii) For every sequence $(y_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ there holds

$$\limsup_{n\to\infty} V(y_n) = \infty \quad \Leftrightarrow \quad \limsup_{n\to\infty} \|y_n\|_{\mathcal{H}} = \infty.$$

This allows us to show that every classical solution $y(t)$, corresponding to some initial condition $y_0 \in D(A)$, is global in time, i.e. $T_{\max}(y_0) = \infty$.

Lemma 1.5. *Let $y_0 \in D(A)$ and let $y(t)$ be the corresponding classical solution of (1.2). Then $T_{\max}(y_0) = \infty$, i.e. $y(t)$ is global in time.*

Proof. According to Assumption 1.4 (i), V is monotonically decreasing along $y(t)$. In particular, $V(y(t)) \le V(y_0) < \infty$ for all $t \in [0, T_{\max}(y_0))$. Then, Assumption 1.4 (ii) implies that $\|y(t)\|_{\mathcal{H}}$ does not blow up as $t \nearrow T_{\max}(y_0)$. According to Proposition 1.2 this is only possible if $T_{\max}(y_0) = \infty$. Hence, $y(t)$ is global. □

Finally, we show that $V(y(t))$ is monotonically decreasing also for (non-classical) mild solutions $y(t)$. The key ingredient is the following corollary of Proposition 4.3.7 in [CH98]:

Proposition 1.6. *Let $y_0 \in \mathcal{H}$ and $(y_{n,0})_{n\in\mathbb{N}} \subset D(A)$ such that $y_{n,0} \to y_0$ in $\mathcal{H}$. Denote the corresponding (mild) solutions of* (1.2) *by $y(t)$ and $y_n(t)$, respectively. Then, for every $0 < T < T_{\max}(y_0)$, there holds $y_n \to y$ in $C([0,T]; \mathcal{H})$.*

With this we are able to prove the following theorem:

Theorem 1.7. *For every $y_0 \in \mathcal{H}$ the corresponding mild solution $y(t)$ of* (1.2) *is global in time, i.e. $T_{\max}(y_0) = \infty$. Furthermore, $t \mapsto V(y(t))$ is monotonically decreasing, and $t \mapsto \|y(t)\|_{\mathcal{H}}$ is uniformly bounded on $\mathbb{R}^+$.*

Proof. According to Lemmata 1.3 and 1.5, and Assumption 1.4 it remains to show the result for $y_0 \in \mathcal{H} \setminus D(A)$. To this end we choose a sequence of classical solutions $(y_n)_{n\in\mathbb{N}}$ as in Proposition 1.6. Since $V\colon \mathcal{H} \to \mathbb{R}$ is continuous, we conclude from Proposition 1.6 that for all $t \in [0, T_{\max}(y_0))$:

$$V(y(t)) = \lim_{n\to\infty} V(y_n(t)). \tag{1.4}$$

According to Assumption 1.4 (i) we have for every $n \in \mathbb{N}$:

$$V(y_n(t_1)) \ge V(y_n(t_2)), \quad 0 \le t_1 \le t_2.$$

Taking the limit $n \to \infty$ yields, with (1.4):

$$V(y(t_1) \ge V(y(t_2)), \quad 0 \le t_1 \le t_2 < T_{\max}(y_0),$$

hence $t \mapsto V(y(t))$ is monotonically decreasing on $[0, T_{\max}(y_0))$. Then, Assumption 1.4 (ii) implies that $\|y(t)\|_{\mathcal{H}}$ remains uniformly bounded as $t \nearrow T_{\max}(y_0)$, so now blow-up occurs. According to Proposition 1.2 this shows that $y(t)$ is global in time, and $T_{\max}(y_0) = \infty$. □

We now introduce the family of (nonlinear) operators $(S_{\mathcal{A}}(t))_{t\ge 0}$: For every $y_0 \in \mathcal{H}$ let $y(t)$ be the mild solution of (1.2). We then define

$$S_{\mathcal{A}}(t)y_0 := y(t), \quad t \ge 0. \tag{1.5}$$

Sometimes we shall use the shorthand notation $S_{\mathcal{A}} \equiv (S_{\mathcal{A}}(t))_{t\ge0}$. Note that there holds $S_{\mathcal{A}}(0)y_0 = y_0$ for all $y_0 \in \mathcal{H}$.

Lemma 1.8. *The family $(S_{\mathcal{A}}(t))_{t\ge0}$ is a C_0-semigroup of nonlinear operators in $\mathcal{H}$.*

Proof. This is a direct consequence of Theorem 9.3.2 in [CH98]. □

Definition 1.9 (Lyapunov function). A continuous function $V\colon \mathcal{H} \to \mathbb{R}$ is called *Lyapunov function* for $S_{\mathcal{A}}$ (or (1.2), equivalently) if V is monotonically decreasing along every mild solution of (1.2).

Corollary 1.10. *Any function V satisfying Assumption 1.4 is a Lyapunov function for $S_{\mathcal{A}}$.*

Definition 1.11. Given an initial value $y_0 \in \mathcal{H}$, we define the following generalized time derivative of $t \mapsto V(S_{\mathcal{A}}(t)y_0)$:

$$\dot{V}(y_0) := \limsup_{t \searrow 0} \frac{V(S_{\mathcal{A}}(t)y_0) - V(y_0)}{t},$$

which may take the value $-\infty$.

1.3. Regularity of the integrated semigroup

This section addresses the question of regularity of $\int_0^t S_{\mathcal{A}}(s)y_0 \,\mathrm{d}s$. We still use the definition of $A, \mathcal{N}$ and $\mathcal{H}$ from Section 1.2. To this end we start with the following general result:

Proposition 1.12. *Let B the infinitesimal generator of a linear C_0-semigroup in a Banach space X, and $N : \mathbb{R}^+ \times X \to X$ is a nonlinear operator. We assume that the initial value problem*

$$\dot{x}(t) = Bx(t) + N(t, x(t)), \quad t > 0, \tag{1.6a}$$

$$x(0) = x_0 \in X, \tag{1.6b}$$

has a unique global mild solution $x(t)$ for every $x_0 \in X$. Furthermore we assume the following:

(i) *If $x_0 \in D(B)$, then the solution $x(t)$ is classical.*
(ii) *Let $x_0 \in X$, and $(x_{n,0})_{n\in\mathbb{N}} \subset D(B)$, and denote the corresponding solutions of (1.6) by $x(t)$ and $x_n(t)$, respectively. If $x_{n,0} \to x_0$ in X, then $x_n \to x$ in $C([0,T]; X)$ for every $T > 0$.*
(iii) *The map $t \mapsto N(t, x(t))$ is continuous on $\mathbb{R}^+$ for every mild solution $x(t)$.*
(iv) *For every $T > 0$ the map $N(t, \cdot) : X \to X$ is uniformly continuous on bounded sets in X, uniformly in $t \in [0, T]$.*

Then for every $x_0 \in X$ and the corresponding mild solution $x(t)$ of (1.6) there holds $\int_0^t x(s) \,\mathrm{d}s \in D(B)$ for all $t > 0$, and

$$x(t) - x_0 = B \int_0^t x(s) \,\mathrm{d}s + \int_0^t N(t, x(s)) \,\mathrm{d}s. \tag{1.7}$$

Proof. We first consider $x_0 \in D(B)$. Then $x(t)$ is a classical solution of (1.6), and satisfies the integrated equation:

$$x(t) - x_0 = \int_0^t Bx(s) \,\mathrm{d}s + \int_0^t N(s, x(s)) \,\mathrm{d}s. \tag{1.8}$$

The crucial part of the following proof is to justify of the interchange of B and the integral. Since $x(t) \in C^1(\mathbb{R}^+; X)$ and because of the assumption (iii), we find that both $t \mapsto N(t, x(t))$ and $t \mapsto \dot{x}(t) = Bx(t) + N(t, x(t))$ are continuous. This implies that $Bx(t) \in C(\mathbb{R}^+; X)$. Therefore the following integrals exist, and we can write them as Riemann sums, for any $t > 0$:

$$\int_0^t x(s) \,\mathrm{d}s = \lim_{K\to\infty} \sum_{k=1}^{K} \frac{t}{K} x\Big(\frac{kt}{K}\Big),$$

$$\int_0^t Bx(s)\,\mathrm{d}s = \lim_{K\to\infty}\sum_{k=1}^{K}\frac{t}{K}Bx\Big(\frac{kt}{K}\Big) = \lim_{K\to\infty} B\sum_{k=1}^{K}\frac{t}{K}x\Big(\frac{kt}{K}\Big).$$

We used the linearity of B in the last step. Since B is an infinitesimal generator, it is closed. Thus we obtain from the above that

$$\int_0^t x(s)\,\mathrm{d}s \in D(B), \quad B\int_0^t x(s)\,\mathrm{d}s = \int_0^t Bx(s)\,\mathrm{d}s. \tag{1.9}$$

Inserting this in (1.8) yields the desired result for any $t > 0$.

Consider now $x_0 \in X \setminus D(B)$. Due to the prerequisite (ii) there is a sequence of classical solutions $(x_n(t))_{n\in\mathbb{N}}$ such that $x_n \to x$ in $C([0,T];X)$ as $n \to \infty$, for any $T > 0$. In combination with with the assumption (iv) this implies the convergence of $N(t, x_n(t)) \to N(t, x(t))$ in $C([0,T];X)$. Hence we find for every $t > 0$:

$$\lim_{n\to\infty}(x_n(t) - x_{n,0}) = x(t) - x_0,$$
$$\lim_{n\to\infty}\int_0^t N(s, x_n(s))\,\mathrm{d}s = \int_0^t N(s, x(s))\,\mathrm{d}s.$$

Consider (1.7) for every x_n and apply the above limits. We obtain

$$\lim_{n\to\infty} B\int_0^t x_n(s)\,\mathrm{d}s = x(t) - x_0 - \int_0^t N(s, x(s))\,\mathrm{d}s.$$

But since $x_n \to x$ in $C([0,T];X)$ there also holds

$$\lim_{n\to\infty}\int_0^t x_n(s)\,\mathrm{d}s = \int_0^t x(s)\,\mathrm{d}s.$$

Since B is closed, these last two limits prove (1.9) for $x_0 \in X \setminus D(B)$, which concludes the proof analogously to the first part. □

In our particular situation, the above result yields the following:

Lemma 1.13. *Let $A, \mathcal{N}$ satisfy Assumption 1.1, and let there be a function V which fulfills Assumption 1.4. Then, for every $y_0 \in \mathcal{H}$ the following holds, for all $t > 0$:*

$$\int_0^t S_{\mathcal{A}}(s)y_0\,\mathrm{d}s \in D(A), \tag{1.10}$$

and

$$S_{\mathcal{A}}(t)y_0 - y_0 = A\int_0^t S_{\mathcal{A}}(s)y_0\,\mathrm{d}s + \int_0^t \mathcal{N}S_{\mathcal{A}}(s)y_0\,\mathrm{d}s. \tag{1.11}$$

Proof. We need to verify that the requirements of Proposition 1.12 are fulfilled, with $B = A$ and $N(t,\cdot) = \mathcal{N}(\cdot)$. According to Theorem 1.7 and Proposition 1.2, (1.2) has a unique global mild solution $y(t)$ for every $y_0 \in \mathcal{H}$. Lemma 1.3 ensures that $y(t)$ is classical if $y_0 \in D(A)$. This verifies (i). Assumption (ii) is ensured by Proposition 1.6. Since, according to Assumption 1.1 (ii), $\mathcal{N}\colon \mathcal{H} \to \mathcal{H}$ is differentiable, it is also continuous. Furthermore, any mild solution $y(t)$ is continuous too, so $t \mapsto \mathcal{N}(y(t))$ is continuous, and the condition (iii) is fulfilled. Finally, condition (iv) is satisfied, since we assume $\mathcal{N}$ to be locally Lipschitz continuous. Hence, we can apply Proposition 1.12, which concludes the proof. □

For $\mathcal{N} = 0$, i.e. the linear case, (1.10) and (1.11) are standard results, see for example Theorem 1.2.4 in [Paz83].

1.4. ω-limit & asymptotic stability

The aim of this part of this thesis is to investigate the long-time behavior of certain nonlinear evolution equations. Here, we give a brief overview of the concepts used here. In the following we still assume the definitions from Section 1.2.

Definition 1.14 (Asymptotic stability). We say that the problem (1.2) or the corresponding semigroup $S_{\mathcal{A}}$, respectively, is *asymptotically stable* if for every $y_0 \in \mathcal{H}$ there holds

$$\lim_{t\to\infty} \|S_{\mathcal{A}}(t)y_0\|_{\mathcal{H}} = 0.$$

Definition 1.15 (Trajectory). Given an initial condition $y_0 \in \mathcal{H}$ for (1.2), the corresponding *trajectory* $\gamma(y_0)$ is defined by

$$\gamma(y_0) := \bigcup_{t\geq 0} S_{\mathcal{A}}(t)y_0.$$

It is the set of all values that $y(t) \equiv S_{\mathcal{A}}(t)y_0 \in \mathcal{H}$ takes for $t \in [0, \infty)$.

Definition 1.16 (ω-limit set). Let $y_0 \in \mathcal{H}$ be an initial condition of (1.2). Then the *ω-limit set* $\omega(y_0)$ for y_0 is defined by

$$\omega(y_0) := \big\{y \in \mathcal{H} : \exists (t_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+ \text{ such that } \lim_{n\to\infty} t_n = \infty \wedge \lim_{n\to\infty} S_{\mathcal{A}}(t_n)y_0 = y\big\}.$$

We collect some results about ω-limits from Chapter 9.1 in [CH98]:

Lemma 1.17. *Let $y_0 \in \mathcal{H}$. There holds:*

(i) *$\omega(y_0)$ is $S_{\mathcal{A}}$-invariant, i.e. $S_{\mathcal{A}}(t)\omega(y_0) \subseteq \omega(y_0)$ for all $t > 0$.*
(ii) *If $\gamma(y_0)$ is precompact, then $\omega(y_0) \neq \emptyset$.*
(iii) *$\lim_{t\to\infty} d(S_{\mathcal{A}}(t)y_0, \omega(y_0)) = 0$, where $d(\cdot,\cdot)$ denotes the minimal distance in the $\mathcal{H}$-norm.*

Note that by first showing that $\gamma(y_0)$ is precompact for all $y_0 \in \mathcal{H}$ (which ensures $\omega(y_0) \neq \emptyset$), and then proving that $\omega(y_0) = \{\mathbf{0}\}$, Lemma 1.17 yields the asymptotic stability of $S_{\mathcal{A}}$. However, in general it is difficult to determine $\omega(y_0)$. To this end, the Lyapunov function V is helpful. Since for every $y_0 \in \mathcal{H}$ the map $t \mapsto V(S_{\mathcal{A}}(t))$ is monotonically decreasing (cf. Corollary 1.10), and it is bounded from below by zero, the following limit exists:

$$\nu(y_0) := \lim_{t\to\infty} V(S_{\mathcal{A}}(t)y_0) \geq 0. \tag{1.12}$$

Note that it even exists if $\omega(y_0) = \emptyset$.

Lemma 1.18. *Suppose $\omega(y_0) \neq \emptyset$. Then there holds*

$$V(\omega(y_0)) = \{\nu(y_0)\},$$

i.e. V is constant on $\omega(y_0)$. In particular $\dot{V}(y) = 0$ for all $y \in \omega(y_0)$.

Proof. Let $y \in \omega(y_0)$. There exists a sequence $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$, with $t_n \to \infty$, such that $\lim_{n\to\infty} S_{\mathcal{A}}(t_n)y_0 = y$. Since V is continuous,

$$\lim_{n\to\infty} V(S_{\mathcal{A}}(t_n)y_0) = V(y).$$

According to (1.12) we have $V(y) = \nu(y_0)$, and the result follows. $\square$

This is the intrinsic motivation for the following famous result (cf. Theorem 3.64 in [LGM99]):

Theorem 1.19 (LaSalle's Invariance Principle). *Let $\Omega \subset \mathcal{H}$ be the largest $S_{\mathcal{A}}$-invariant subset of*

$$\{y \in \mathcal{H} : \dot{V}(y) = 0\}.$$

If $y_0 \in \mathcal{H}$ and $\gamma(y_0)$ is precompact, then

$$\lim_{t\to\infty} d(S_{\mathcal{A}}(t)y_0, \Omega) = 0,$$

where $d(\cdot,\cdot)$ was defined in Lemma 1.17.

CHAPTER 2

An Euler-Bernoulli beam with nonlinear damping and a nonlinear spring at the tip

2.1. Introduction

This chapter considers an Euler-Bernoulli beam where one end is clamped, and the free end holds a rigid tip mass. Models of this form play a fundamental role in many mechanical systems and thus occur in multiple applications such as flexible robot arms, helicopter rotor blades, spacecraft antennae, airplane wings, high-rise buildings, etc. An important issue is the suppression of vibrations, since undesired oscillations can reduce the performance of the system, or worse, result in damage to the structure. For this reason, the Euler-Bernoulli beam is often coupled with a boundary control, which acts on the tip and is used to dissipate the vibration. Frequently, the boundary control is realized as a suspension system, consisting typically of springs and dampers.

In the last four decades, considerable attention has been paid to the stability analysis of such systems in the literature, see e.g. [LM88, Guo02a, GW06, MA15]. Most results deal with the situation in which the control is linear, thus obtaining linear boundary conditions. In general, the respective stability analysis uses results from linear functional analysis.

However, the generalization to nonlinear boundary conditions is not straightforward in most cases, since they often require a model-dependent analysis. To the author's knowledge the only beam models considered in the literature with nonlinearities at the boundary do not comprise a rigid body attached to the tip (see for example [CC99, CP24, CP94, CdN98]). A main goal of this chapter is to increase the analytical understanding of nonlinear beam models.

In this chapter we investigate an Euler-Bernoulli beam which is clamped at one end (see Figure 1). The free end holds a tip mass, whose mass m and moment of inertia J are both positive. The controller acting on the tip consists of a spring and a damper, both nonlinear. On the one hand this model is rather simple, but it still exhibits an interesting mechanical behavior (asymptotic stability vs. the existence of periodic orbits, depending on the value of $J > 0$). On the other hand, its mathematical analysis requires a new strategy that deviates from existing techniques for nonlinear models. The analysis presented in this chapter can easily be extended to more complex models of this kind, see Chapter 3.

Our beam model satisfies a linear PDE with high order nonlinear boundary conditions. In order to make the system accessible for analysis it is a common strategy to rewrite it as a nonlinear evolution equation in an appropriate (infinite-dimensional) Hilbert space $\mathcal{H}$, and use the total energy of the system as a Lyapunov functional, see [CLW13, Gra09, KT05, Mor01, VZLGM09]. In general, proving that every solution

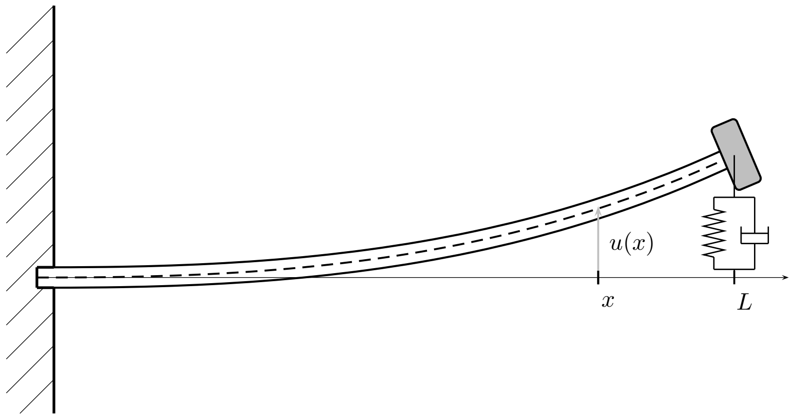


Figure 1. Clamped beam with tip mass, coupled to a spring and damper (both nonlinear). Source: [MSA15].

tends to zero as time goes to infinity consists of two steps, namely showing the precompactness of the trajectories and proving that the only possible limit is the zero solution, see LaSalle's Invariance Principle (Theorem 1.19). In the linear case, verifying the precompactness is straightforward by showing that the resolvent of the system operator is compact, see Remark 2.16 in Section 2.4. For the nonlinear case, the inspection of the precompactness property is more complex. In some situations, it can be shown that the (nonlinear) system operator is dissipative (in the sense of [CP69]) and has compact resolvent, see for example [CP94]. However, in our case, this operator is *not* dissipative, and does not generate a semigroup of nonlinear contractions, see Remark 2.11 in Section 2.3.

A very common point of view for proving asymptotic stability of nonlinear evolution equations is to consider the system as a quasi-linear evolution equation. For this situation, the most commonly used criteria for the precompactness of trajectories can be found in [DS73, Paz81, Paz75, Web79], and further generalizations in [Cou02, TV03]. They all split the system operator into the sum of two operators $A + \mathcal{N}$ (A being its linear, and $\mathcal{N}$ its nonlinear part) and infer precompactness under the following conditions. In [DS73], for example, A is required to be m-dissipative and $\mathcal{N}$ applied to a trajectory is L^1 in time. In [Paz75] the requirement on $\mathcal{N}$ is loosened by just assuming uniform local integrability of $\mathcal{N}$ applied to a trajectory. However, the linear semigroup $(\mathrm{e}^{tA})_{t\geq 0}$ needs additionally to be compact in order to still ensure precompactness. Finally, in [Web79] $\mathcal{N}$ just needs to map into a compact set, but A needs to generate an exponentially stable linear C_0-semigroup. These strategies have been successfully applied in the literature to the Euler-Bernoulli beam without tip payload and with nonlinear boundary control: In [CP24] the precompactness of the trajectories follows directly from the m-dissipativity of the system operator, and in [CC99] from the L^1-integrability of the nonlinearity.

In contrast to the mentioned literature, the nonlinear boundary conditions considered in this chapter do not fall into any of these sets of assumptions. In our case A is m-dissipative, but the generated semigroup is neither compact nor exponentially stable. On the other hand, $\mathcal{N}$ apparently does not satisfy strong assumptions either, for it is compact, but we can not guarantee L^1-integrability. Thus the properties of our system operator are too weak in order to apply the mentioned standard results. However, we are still able to prove precompactness of the trajectories in a novel way. Hence, this chapter provides a new strategy for such evolution equations.

In this chapter we show that, for the Euler-Bernoulli beam with tip mass, coupled to a nonlinear spring and a nonlinear damper, all trajectories that are C^1 in time are precompact. Furthermore, for almost all values of the moment of inertia $J > 0$ the trajectories tend to zero as time goes to infinity. Interestingly we find that, for countably many values of J, the trajectories tend to a time-periodic solution (see our main result, Theorem 2.21, for the detailed formulation). For given initial conditions we are able to characterize this asymptotic limit explicitly, including its phase. Such periodic limiting orbit appears when the (linear) beam equation has an eigenfunction with a node at the free end (i.e. some $u_n(L) = 0$). Then, the controller at the tip is inactive for all time.

This chapter is organized as follows. In Section 2.2 the equations of motion are derived for the system consisting of the Euler-Bernoulli beam with tip mass, connected to a nonlinear spring and damper. Next, it is shown that the energy functional is an appropriate Lyapunov function for the system. Section 2.3 is concerned with the formulation of the problem in an appropriate functional analytical setting and the investigation of existence and uniqueness of the corresponding solutions. In Section 2.4 we prove precompactness of the trajectories for all initial conditions lying in a dense subset of the underlying Hilbert space. Section 2.5 deals with the characterization of possible ω-limit sets, proving that any classical solution tends either to zero or to a periodic solution, depending on the prescribed value J.

The contents of Chapter 2 has been published in [MSA15].

2.2. Preliminaries and derivation of the model

For the derivation of the model we follow [GZH11] and [KT05], where we require that the beam satisfies the Euler-Bernoulli assumption. We assume that the beam has uniform mass per length $\rho > 0$ and length L. The beam is parametrized with $x \in [0, L]$, and is described by its deviation $u(t, x)$ from the horizontal (as depicted in Figure 1). The constant bending stiffness is $\Lambda > 0$, and the tension is assumed to be zero. At the tip of the beam there is a payload of mass $m > 0$, which has the moment of inertia $J > 0$. We neglect friction of any kind. Only two external forces are assumed to act on the beam, both on the tip, perpendicular to the resting position $u \equiv 0$. The first comes from a nonlinear spring attached to the tip, producing the restoring force $-k_1(u(t, L))$. The second force is due to a nonlinear damping, and is given by $-k_2(u_t(t, L))$. Throughout the rest of Chapter 2 we shall make the following assumptions on the two nonlinearities:

Assumption 2.1. We assume $k_1, k_2 \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$, and

$$\int_0^z k_1(s)\,\mathrm{d}s \geq 0, \quad \forall z \in \mathbb{R}, \tag{2.1}$$

$$k_2'(z) \geq 0, \; k_2(0) = 0, \quad \forall z \in \mathbb{R}. \tag{2.2}$$

Furthermore, we require that

$$|k_2(z)| \geq Kz^2, \quad \forall z \in (-\delta, \delta), \tag{2.3}$$

for some positive constant $K > 0$ and $\delta > 0$ small.

Notice that (2.1) implies $k_1(0) = 0$, and (2.2) combined with (2.3) implies that $k_2(z) = 0$ iff $z = 0$.

Remark 2.2. The assumption $k_1, k_2 \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ is particularly needed in the proof of Lemma 2.15, which specifically requires the L^∞-boundedness of k_1'' and k_2'' on bounded sets.

In many situations the damping k_2 will originate from the drag produced by the flow of a fluid around the immersed tip. Hence, the dependence of the drag k_2 on the velocity of the tip will, in general, either be linear (*Stokes drag*; for low Reynolds numbers), or quadratic (*drag equation*; for high Reynolds numbers with turbulence behind the object), see [Bat00]. From this point of view, condition (2.3) is not restrictive.

The equations of motion can be derived according to Hamilton's principle, see [GZH11]. Hence, they are the Euler-Lagrange equations corresponding to the action functional. In our model the kinetic energy E_k and the potential (strain) energy E_p are

$$E_k = \frac{\rho}{2}\int_0^L u_t^2 \,\mathrm{d}x + \frac{m}{2}u_t(L)^2 + \frac{J}{2}u_t'(L)^2, \qquad E_p = \frac{\Lambda}{2}\int_0^L (u'')^2 \,\mathrm{d}x. \tag{2.4}$$

Additionally, we have the virtual work δW coming from the external forces:

$$\delta W = -k_1(u(L))\delta u(L) - k_2(u_t(L))\delta u(L).$$

Taking into account the boundary conditions $u(0) = u'(0) = 0$ of the clamped end we find that, according to Hamilton's principle, u satisfies:

$$\rho u_{tt}(t,x) + \Lambda u^{\mathrm{IV}}(t,x) = 0, \tag{2.5a}$$

$$u(t,0) = u'(t,0) = 0, \tag{2.5b}$$

$$-\Lambda u'''(t,L) + m u_{tt}(t,L) = -k_1(u(t,L)) - k_2(u_t(t,L)), \tag{2.5c}$$

$$\Lambda u''(t,L) + J u_{tt}'(t,L) = 0, \tag{2.5d}$$

where $(t,x) \in (0,\infty) \times (0,L)$. For the rest of this chapter we investigate the existence and asymptotic behavior of solutions u of the system (2.5).

Finally we derive a candidate for a Lyapunov function (see Definition 1.9): The total energy of the system is a natural candidate, since it will decrease in time because of the damping. The total energy is given by $E_{\mathrm{tot}} = E_k + E_p + E_s$, where $E_s := \int_0^{u(L)} k_1(s)\,\mathrm{d}s$ represents the potential energy stored in the nonlinear spring, and E_k and E_p are given in (2.4). Now (2.1) ensures that E_s always stays non-negative. For sufficiently regular solutions u of (2.5) we compute the time derivative of the total energy, using the Euler-Lagrange equations (2.5):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{tot}} = \Lambda\int_0^L u'' u_t'' \,\mathrm{d}x + \rho\int_0^L u_{tt}u_t \,\mathrm{d}x + m u_{tt}(L)u_t(L) + J u_{tt}'(L)u_t'(L) \\ + k_1(u(L))u_t(L)\end{aligned}$$

$$
\begin{aligned}
&= \Lambda \int_0^L u^{\mathrm{IV}} u_t \, \mathrm{d}x + \Lambda u'' u_t' \big|_0^L - \Lambda u''' u_t \big|_0^L + \rho \int_0^L u_{tt} u_t \, \mathrm{d}x \\
&\quad + m u_{tt}(L) u_t(L) + J u_{tt}'(L) u_t'(L) + k_1(u(L)) u_t(L) \\
&= \Lambda u''(L) u_t'(L) - \Lambda u'''(L) u_t(L) + m u_{tt}(L) u_t(L) + J u_{tt}'(L) u_t'(L) \\
&\quad + k_1(u(L)) u_t(L) \\
&= -k_2(u_t(L)) u_t(L).
\end{aligned} \tag{2.6}
$$

According to (2.2) this last line is always non-positive. This also confirms our previous claim that the friction, k_2, is responsible for the energy dissipation in (2.5). So E_{tot} is a candidate for a Lyapunov function:

$$
V(u) := \frac{\Lambda}{2} \int_0^L (u'')^2 \, \mathrm{d}x + \frac{\rho}{2} \int_0^L u_t^2 \, \mathrm{d}x + \frac{m}{2} u_t(L)^2 + \frac{J}{2} u_t'(L)^2 + \int_0^{u(L)} k_1(s) \, \mathrm{d}s, \tag{2.7}
$$

and it is non-negative. According to (2.6) its derivative along classical solutions of (2.5) satisfies:

$$
\frac{\mathrm{d}}{\mathrm{d}t} V(u(t)) = -k_2(u_t(t,L)) u_t(t,L) \le 0. \tag{2.8}
$$

2.3. Formulation as an evolution equation

The aim of this section is to show that for sufficiently regular initial conditions $u(0,x) = u_0(x)$ and $u_t(0,x) = v_0(x)$ the system (2.5) has a unique (mild) solution $u(t,x)$. First, we introduce the standard setting for the Euler-Bernoulli beam with a tip payload (see [KT05], [MA15]). To this end we define the following real Hilbert space:

$$
\mathcal{H} := \{ y = [u, v, \xi, \psi]^\top : u \in \tilde{H}^2_{0,\mathbb{R}}(0,L), v \in L^2_{\mathbb{R}}(0,L), \xi, \psi \in \mathbb{R} \}, \tag{2.9}
$$

where $\tilde{H}^n_{0,\mathbb{R}}(0,L)$ is defined in (1.1). The space $\mathcal{H}$ is equipped with the inner product

$$
\langle y_1, y_2 \rangle_{\mathcal{H}} := \frac{\Lambda}{2} \int_0^L u_1'' u_2'' \, \mathrm{d}x + \frac{\rho}{2} \int_0^L v_1 v_2 \, \mathrm{d}x + \frac{1}{2J} \xi_1 \xi_2 + \frac{1}{2m} \psi_1 \psi_2, \quad \forall y_1, y_2 \in \mathcal{H}.
$$

We next consider the following linear operator on $\mathcal{H}$:

$$
A : y \mapsto \begin{bmatrix} v \\ -\frac{\Lambda}{\rho} u^{\mathrm{IV}} \\ -\Lambda u''(L) \\ \Lambda u'''(L) \end{bmatrix} \tag{2.10}
$$

on the dense domain

$$
D(A) := \{ y \in \mathcal{H} \colon u \in \tilde{H}^4_{0,\mathbb{R}}(0,L), v \in \tilde{H}^2_{0,\mathbb{R}}(0,L), \xi = J v'(L), \psi = m v(L) \}. \tag{2.11}
$$

Furthermore, we define the bounded nonlinear operator $\mathcal{N}$ on $\mathcal{H}$:

$$
\mathcal{N} : y \mapsto \begin{bmatrix} 0 \\ 0 \\ 0 \\ -k_1(u(L)) - k_2(\frac{\psi}{m}) \end{bmatrix}. \tag{2.12}
$$

For the rest of Chapter 2 we use the above definitions of A, $\mathcal{N}$ and $\mathcal{H}$. Finally, we introduce the nonlinear operator $\mathcal{A} := A + \mathcal{N}$ on the domain $D(\mathcal{A}) = D(A)$. With this notation the system (2.5) can be written formally as the following nonlinear evolution equation in $\mathcal{H}$:

$$y_t = \mathcal{A}y, \tag{2.13a}$$

$$y(0) = y_0, \tag{2.13b}$$

for some initial condition $y_0 \in \mathcal{H}$. For some properties and results of nonlinear evolution equations we refer to Section 1.2.

Lemma 2.3. *A generates a C_0-semigroup $(e^{tA})_{t\geq 0}$ of unitary operators in $\mathcal{H}$.*

The proof of Lemma 2.3 is included in the proof of Lemma A.2 in the Appendix. The latter is an extension of Lemma 2.3 to the complex analogue of $\mathcal{H}$.

Lemma 2.4. *The operator $\mathcal{N}\colon \mathcal{H} \to \mathcal{H}$ is continuously differentiable.*

Proof. First, we note that the derivative of $\mathcal{N}$ at y is the following linear operator:

$$\mathcal{N}'(y) : h \mapsto \begin{bmatrix} 0 \\ 0 \\ 0 \\ -k_1'(u(L))u_h(L) - k_2'(\frac{\psi}{m})\psi_h \end{bmatrix},$$

where $h = [u_h, v_h, \xi_h, \psi_h]^\top$. By using the continuous embedding $H^2(0,L) \hookrightarrow C([0,L])$ for u, and the Assumption 2.1 on the k_j, we immediately verify that $\mathcal{N}'(y)$ is indeed the Fréchet derivative of $\mathcal{N}$ at $y \in \mathcal{H}$. Since, according to Assumption 2.1, the functions k_j lie in $W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$, it is clear that $\mathcal{N}'$ is a continuous map from $\mathcal{H}$ into $\mathscr{B}(\mathcal{H})$. $\square$

Corollary 2.5. *The operators A and $\mathcal{N}$ fulfill Assumption 1.1.*

Following the arguments from Chapter 1 we conclude that (2.13) has a unique mild solution for every $y_0 \in \mathcal{H}$, and if $y_0 \in D(A)$, this solution is classical. Now we can look for a suitable Lyapunov function on $\mathcal{H}$. In the previous section we obtained a candidate by (2.7), which was defined along classical solutions $u(t)$ of (2.5). For $y = [u, v, \xi, \psi]^\top \in D(\mathcal{A})$ this can equivalently be written as

$$V(y) = \frac{\Lambda}{2}\int_0^L (u'')^2 \,\mathrm{d}x + \frac{\rho}{2}\int_0^L v^2 \,\mathrm{d}x + \frac{1}{2m}\psi^2 + \frac{1}{2J}\xi^2 + \int_0^{u(L)} k_1(s)\,\mathrm{d}s. \tag{2.14}$$

This is then defined for all $y \in \mathcal{H}$, and according to (2.8) there holds

$$\frac{\mathrm{d}}{\mathrm{d}t}V(y(t)) = -k_2\Big(\frac{\psi(t)}{m}\Big)\frac{\psi(t)}{m} \leq 0, \quad \forall 0 < t < T_{\max}(y_0), \tag{2.15}$$

along all classical solutions $y(t)$. Throughout the rest of Chapter 2, V denotes the function (2.14). The following two results are easily verified:

Lemma 2.6. *The function $V\colon \mathcal{H} \to \mathbb{R}$ is continuous with respect to the norm $\|\cdot\|_{\mathcal{H}}$.*

Proof. We observe that $V(y) = \|y\|^2_{\mathcal{H}} + \int_0^{u(L)} k_1(s)\,\mathrm{d}s$. Hence, it remains to show continuity for the integral term. Let $y \in \mathcal{H}$, and $(y_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ with $y_n \to y$ in $\mathcal{H}$. In particular, for the first component we have $u_n \to u$ in $H^2(0,L)$. Because of the continuous embedding $H^2(0,L) \hookrightarrow C([0,L])$, this implies $u_n(L) \to u(L)$. Since k_1 is integrable, this proves

$$\lim_{n\to\infty}\int_0^{u_n(L)} k_1(s)\,\mathrm{d}s = \int_0^{u(L)} k_1(s)\,\mathrm{d}s,$$

and consequently, $\lim_{n\to\infty} V(y_n) = V(y)$. □

Lemma 2.7. *For every sequence $(y_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ there holds*

$$\limsup_{n\to\infty} V(y_n) = \infty \quad \Leftrightarrow \quad \limsup_{n\to\infty} \|y_n\|_{\mathcal{H}} = \infty.$$

Proof. We argue similarly to the proof of Lemma 2.6. Again, we use $V(y) = \|y\|_{\mathcal{H}}^2 + \int_0^{u(L)} k_1(s)\,\mathrm{d}s$. According to Assumption 2.1, the integral on the right hand side is always non-negative. Hence, if $(\|y_n\|_{\mathcal{H}})_{n\in\mathbb{N}}$ is unbounded, then also $(V(y_n))_{n\in\mathbb{N}}$ is unbounded. Suppose now that $(\|y_n\|_{\mathcal{H}})_{n\in\mathbb{N}}$ is bounded. Hence, also $(\|u_n\|_{H^2(0,L)})_{n\in\mathbb{N}}$ is bounded, and because of the continuous embedding $H^2(0,L) \hookrightarrow C([0,L])$, also $(|u_n(L)|)_{n\in\mathbb{N}}$ is uniformly bounded. According to Assumption 2.1, $k_1 \in C(\mathbb{R})$, so

$$\Big(\int_0^{u_n(L)} k_1(s)\,\mathrm{d}s\Big)_{n\in\mathbb{N}}$$

stays uniformly bounded. Hence, $(V(y_n))_{n\in\mathbb{N}}$ is bounded. □

Corollary 2.8. *The function V satisfies Assumption 1.4.*

With the Corollaries 2.5 and 2.8 at hand, we can apply all results from Chapter 1 to the evolution equation (2.13) of this chapter (with the same notation). This establishes the following existence result:

Theorem 2.9. *For every $y_0 \in \mathcal{H}$ the initial value problem* (2.13) *has a unique global mild solution $y(t)$, which is classical for $y_0 \in D(\mathcal{A})$. Moreover, the function V is a non-negative Lyapunov function, and for every mild solution $y(t)$ the norm $\|y(t)\|_{\mathcal{H}}$ is uniformly bounded for $t \geq 0$.*

Corollary 2.10. *All results from the Sections 1.3 and 1.4 apply to the system* (2.13) *and the Lyapunov function V from* (2.14).

Then, as in (1.5) we introduce the nonlinear semigroup $S_{\mathcal{A}} \equiv (S_{\mathcal{A}}(t))_{t\geq 0}$, generated by $\mathcal{A}$.

Remark 2.11. If we assume that k_1 is linear, i.e. $k_1(u(L)) = K_1 \cdot u(L)$ for some $K_1 > 0$, and Assumption 2.1 holds, then the (still) nonlinear operator $\mathcal{A}$ is dissipative in $\mathcal{H}$, with respect to the modified inner product

$$\langle y_1, y_2\rangle_{\mathcal{H},2} := \frac{\Lambda}{2}\int_0^L u_1'' u_2''\,\mathrm{d}x + \frac{\rho}{2}\int_0^L v_1 v_2\,\mathrm{d}x + \frac{\xi_1\xi_2}{2J} + \frac{\psi_1\psi_2}{2m} + \frac{K_1\Lambda}{2} u_1(L)u_2(L).$$

Then, $\mathcal{A}$ even generates a semigroup of nonlinear contractions (cf. [CP69]). In this case, the asymptotic stability of the semigroup is shown more easily, see Remark 2.16. However, we assume k_1 to be nonlinear, and we cannot find a formulation of (2.5) such that the system operator becomes dissipative.

In the remaining part of this chapter we investigate the asymptotic stability of the nonlinear semigroup $S_{\mathcal{A}}$. As it turns out, the semigroup is asymptotically stable "in most cases", i.e. for all but countably many values of the parameter J. For these exceptional values of J, there exist non-trivial solutions which are periodic in time and do not decay towards zero, see Lemma 2.18 in Section 2.5. The main result of this chapter is stated in Theorem 2.21. The underlying ideas for the corresponding analysis are presented in Section 1.4.

2.4. Precompactness of the trajectories

In this section we investigate the precompactness of the trajectories of (2.13). First, we prove the precompactness of trajectories that are twice differentiable (in time), and then extend this result to all classical solutions. The main result is Lemma 2.15 below. Note that we use the assumptions and results presented in the previous section. Furthermore, according to Corollary 2.10 we can use the results from the Sections 1.3 and 1.4.

Lemma 2.12. *Let $y_0 \in D(\mathcal{A}^2)$ and let $y(t)$ be the corresponding solution of* (2.13)*. Then $y \in C^2(\mathbb{R}^+; \mathcal{H})$ and $y_t(t) \in D(\mathcal{A})$ for all $t > 0$.*

Proof. First, notice that if $y \in C^2(\mathbb{R}^+; \mathcal{H})$ then $\tilde{y} := y_t$ would satisfy

$$\tilde{y}_t = A\tilde{y} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -k_1'(u(L))\frac{\psi}{m} - k_2'(\frac{\psi}{m})\frac{\tilde{\psi}}{m} \end{bmatrix}. \tag{2.16}$$

However, for the moment we only know that $y \in C^1(\mathbb{R}^+; \mathcal{H})$, see Theorem 2.9. Motivated by (2.16) we define the following functions for this particular classical solution $y(t)$:

$$F(t) := -k_1'(u(t,L))\frac{\psi(t)}{m},$$
$$G(t,z) := -k_2'\Big(\frac{\psi(t)}{m}\Big)\frac{\chi}{m} \equiv g(t)\chi,$$

where $z = [U, V, \zeta, \chi]^\top \in \mathcal{H}$. Since $y(t)$ is a classical solution, and k_1, $k_2 \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ due to Assumption 2.1, both $F(t)$ and $g(t)$ lie in $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$. Consequently, the operator $\tilde{\mathcal{N}}\colon [0,T] \times \mathcal{H} \to \mathcal{H}$, defined by $\tilde{\mathcal{N}}(t,z) := [0,0,0,F(t) + G(t,z)]^\top$, is Lipschitz continuous in both variables, for every $T > 0$. In the following we consider the (non-autonomous) initial value problem

$$z_t = Az + \tilde{\mathcal{N}}(t,z), \tag{2.17a}$$
$$z(0) = z_0 \in \mathcal{H}. \tag{2.17b}$$

We apply Theorem 6.1.2 in [Paz83] which proves that there exists a unique global mild solution $z(t)$ of (2.17) for every $z_0 \in \mathcal{H}$. If we even require $z_0 \in D(A)$, then we can apply Theorem 6.1.6 in [Paz83], which proves that $z(t)$ is a classical solution[1] of (2.17).

We next show that for the given classical solution $y(t)$ the function $y_t(t)$ is a mild solution of (2.17) for $z_0 = \mathcal{A}y_0$. Clearly, $y(t)$ satisfies the Duhamel formula (1.3), and differentiation with respect to t yields

$$y_t(t) = \mathrm{e}^{tA}Ay_0 + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathrm{e}^{(t-s)A}\mathcal{N}y(s)\,\mathrm{d}s. \tag{2.18}$$

[1]Actually, Theorem 6.1.6 only states that $z(t)$ is a strong solution. However, the corresponding proof in [Paz83] shows that $z(t)$ is actually a classical solution. See Section A.2 in the Appendix for more details.

According to the proof of Corollary 4.2.5 in [Paz83] there holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathrm{e}^{(t-s)A}\mathcal{N}y(s)\,\mathrm{d}s = \mathrm{e}^{tA}\mathcal{N}y_0 + \int_0^t \mathrm{e}^{(t-s)A}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{N}y(s)\,\mathrm{d}s.$$

Inserting this in (2.18) proves that $y_t(t)$ fulfills the Duhamel formula for (2.17), and as a consequence $y_t(t)$ is the unique mild solution of (2.17) to the initial condition $z_0 = \mathcal{A}y_0$. But from the first part of the proof we know that this mild solution $z(t) = y_t(t)$ is a classical solution of (2.17) if $\mathcal{A}y_0 \in D(\mathcal{A})$, i.e. $y_0 \in D(\mathcal{A}^2)$. So $y_t \in C^1(\mathbb{R}^+;\mathcal{H})$ and $y \in C^2(\mathbb{R}^+;\mathcal{H})$. □

Remark 2.13. In the situation where the evolution equation is linear and autonomous, i.e. $\mathcal{N} = 0$ in our case, the above result is straightforward. If $y_0 \in D(A^2)$, then we have according to Section II.5.a in [EN00] that $y(t) \in D(A^2)$ for all $t \geq 0$. Therefore $y_t(t) = Ay(t) \in D(A)$, and so $y_{tt} = Ay_t = A^2y$, and $y_{tt} \in C(\mathbb{R}^+)$. Here it is crucial that the time derivative and the operator A on the right hand side commute. This does not hold in the nonlinear situation any more, which makes the proof more complicated. According to Section II.5.a in [EN00] the density of $D(A^2)$ in $\mathcal{H}$ is also immediate. In our case $D(\mathcal{A}^2)$ is a nonlinear subset of $\mathcal{H}$, see (2.19)–(2.24), so we need to check the density separately.

For the proof Lemma 2.15 the density of $D(\mathcal{A}^2) \subset D(\mathcal{A})$ is needed. But instead we even show the stronger property that $\mathcal{A}|_{D(\mathcal{A}^2)} \subset \mathcal{A}|_{D(\mathcal{A})}$ is dense in the product topology of $\mathcal{H}\times\mathcal{H}$. Note that in contrast to $D(\mathcal{A})$ the domain $D(\mathcal{A}^2)$ will be nonlinear.

Lemma 2.14. *For every $y \in D(\mathcal{A})$ there exists a sequence $(y_n)_{n\in\mathbb{N}}$ in $D(\mathcal{A}^2)$ such that $\lim_{n\to\infty} y_n = y$ and $\lim_{n\to\infty} \mathcal{A}y_n = \mathcal{A}y$ in $\mathcal{H}$.*

Proof. First we characterize $D(\mathcal{A}^2)$. We use that $y \in D(\mathcal{A}^2)$ if and only if $y \in D(\mathcal{A})$ and $\mathcal{A}y \in D(\mathcal{A})$, or equivalently

$$v \in \tilde{H}^4_{0,\mathbb{R}}(0,L), \tag{2.19}$$

$$u \in \tilde{H}^6_{0,\mathbb{R}}(0,L) \wedge u^{\mathrm{IV}}(0) = u^{\mathrm{V}}(0) = 0, \tag{2.20}$$

$$\xi = Jv'(L), \tag{2.21}$$

$$\psi = mv(L), \tag{2.22}$$

$$u''(L) = \frac{J}{\rho}u^{\mathrm{V}}(L), \tag{2.23}$$

$$\Lambda u'''(L) - k_1(u(L)) - k_2\Big(\frac{\psi}{m}\Big) = -\frac{m\Lambda}{\rho}u^{\mathrm{IV}}(L). \tag{2.24}$$

It suffices to show that for an arbitrary $y \in D(\mathcal{A})$ we can construct $(y_n)_{n\in\mathbb{N}} \subset D(\mathcal{A}^2)$ such that $y_n = [u_n,\, v_n,\, \xi_n,\, \psi_n]^\top$ converges to y in the space $H^4(0,L) \times H^2(0,L) \times \mathbb{R}^2$. Since $\tilde{C}_0^\infty(0,L) := \{f \in C^\infty(0,L)\colon f^{(k)}(0) = 0, \forall k \in \mathbb{N}\cup\{0\}\}$ is dense in $\tilde{H}^2_0(0,L)$ (see Theorem 3.17 in [AF03]), there exists a sequence $(v_n)_{n\in\mathbb{N}} \subset \tilde{C}_0^\infty(0,L)$ such that $\lim_{n\to\infty} v_n = v$ in $H^2(0,L)$. Clearly $v_n \in \tilde{H}^4_{0,\mathbb{R}}(0,L)$ for all $n \in \mathbb{N}$. Defining $\xi_n := Jv_n'(L)$ and $\psi_n := mv_n(L)$ ensures that y_n satisfies (2.21) and (2.22). Moreover, the Sobolev embedding $H^2(0,L) \hookrightarrow C^1([0,L])$ implies that $\lim_{n\to\infty} \xi_n = \xi$ and $\lim_{n\to\infty} \psi_n = \psi$.

As a final step, we construct a sequence $(u_n)_{n\in\mathbb{N}} \subset C^\infty(0,L)$ such that u_n satisfies (2.20), (2.23), and (2.24) for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} u_n = u$ in $H^4(0,L)$. For this purpose we first introduce the following polynomial, for every $n \in \mathbb{N}$:

$$h_n(x) := h_{2,n}x^2 + h_{3,n}x^3 + h_{6,n}x^6 + h_{7,n}x^7 + h_{8,n}x^8 + h_{9,n}x^9 + h_{10,n}x^{10} + h_{11,n}x^{11}.$$

We next determine the coefficients $h_{2,n},\dots,h_{11,n} \in \mathbb{R}$ such that $h_n(x)$ satisfies appropriate boundary conditions, for every $n \in \mathbb{N}$. In the following, we fix $n \in \mathbb{N}$ arbitrary. From the definition of h_n it is already immediate that

$$h_n(0) = h_n'(0) = h_n^{\mathrm{IV}}(0) = h_n^{\mathrm{V}}(0) = 0. \tag{2.25}$$

Then we set $h_{2,n} = \frac{u''(0)}{2}$ and $h_{3,n} = \frac{u'''(0)}{6}$, which ensures

$$h_n''(0) = u''(0),\ \ h_n'''(0) = u'''(0). \tag{2.26}$$

Furthermore, we require

$$h_n^{(k)}(L) = u^{(k)}(L), \qquad k \in \{0,1,2,3\}, \tag{2.27}$$

which reads equivalently[2]:

$$r_1 = h_{n,6} + h_{n,7}L + h_{n,8}L^2 + h_{n,9}L^3 + h_{n,10}L^4 + h_{n,11}L^5 \tag{2.28a}$$

$$r_2 = 6h_{n,6} + 7h_{n,7}L + 8h_{n,8}L^2 + 9h_{n,9}L^3 + 10h_{n,10}L^4 + 11h_{n,11}L^5 \tag{2.28b}$$

$$\begin{aligned} r_3 = {}& 6^{\underline{2}}h_{n,6} + 7^{\underline{2}}h_{n,7}L + 8^{\underline{2}}h_{n,8}L^2 + 9^{\underline{2}}h_{n,9}L^3 \\ &+ 10^{\underline{2}}h_{n,10}L^4 + 11^{\underline{2}}h_{n,11}L^5 \end{aligned} \tag{2.28c}$$

$$\begin{aligned} r_4 = {}& 6^{\underline{3}}h_{n,6} + 7^{\underline{3}}h_{n,7}L + 8^{\underline{3}}h_{n,8}L^2 + 9^{\underline{3}}h_{n,9}L^3 \\ &+ 10^{\underline{3}}h_{n,10}L^4 + 11^{\underline{3}}h_{n,11}L^5, \end{aligned} \tag{2.28d}$$

where

$$r_1 = \frac{u(L)}{L^6} - \frac{u''(0)}{2L^4} - \frac{u'''(0)}{6L^3},\quad r_2 = \frac{u'(L)}{L^5} - \frac{u''(0)}{L^4} - \frac{u'''(0)}{2L^3},$$
$$r_3 = \frac{u''(L)}{L^4} - \frac{u''(0)}{L^4} - \frac{u'''(0)}{L^3},\quad r_4 = \frac{u'''(L)}{L^3} - \frac{u'''(0)}{L^3}.$$

Finally the two additional conditions are imposed on h_n:

$$\frac{m\Lambda}{\rho}h_n^{\mathrm{IV}}(L) = -\Lambda u'''(L) + k_1(u(L)) + k_2\Big(\frac{\psi_n}{m}\Big), \tag{2.29}$$

$$\frac{J}{\rho}h_n^{\mathrm{V}}(L) = u''(L). \tag{2.30}$$

The equations (2.29) and (2.30) are equivalent to:

$$6^{\underline{4}}h_{n,6} + 7^{\underline{4}}h_{n,7}L + 8^{\underline{4}}h_{n,8}L^2 + 9^{\underline{4}}h_{n,9}L^3 + 10^{\underline{4}}h_{n,10}L^4 + 11^{\underline{4}}h_{n,11}L^5 = r_5, \tag{2.31a}$$

$$6^{\underline{5}}h_{n,6} + 7^{\underline{5}}h_{n,7}L + 8^{\underline{5}}h_{n,8}L^2 + 9^{\underline{5}}h_{n,9}L^3 + 10^{\underline{5}}h_{n,10}L^4 + 11^{\underline{5}}h_{n,11}L^5 = r_6, \tag{2.31b}$$

with

$$r_5 = \rho\frac{-\Lambda u'''(L) + k_1(u(L)) + k_2(\frac{\psi_n}{m})}{\Lambda m L^2},\quad r_6 = \frac{\rho u''(L)}{JL}.$$

[2]The coefficient $k^{\underline{l}}$ (the Pochhammer symbol, see [GKP89]) for $k,l \in \mathbb{N}$, $l \le k$ is defined by $k^{\underline{l}} := k\cdot(k-1)\cdots(k-l+1)$.

The linear system consisting of (2.28) and (2.31) has a strictly positive determinant. Hence, this system possesses a unique solution $h_{6,n}, \dots, h_{11,n}$. Combining these coefficients with $h_{2,n}$ and $h_{3,n}$ defined before (2.26), we have obtained a polynomial $h_n(x)$ which satisfies (2.25)–(2.27), (2.29) and (2.30).

Consequently, (2.25), (2.26), and (2.28) imply that $u - h_n \in H_0^4(0, L)$, for all $n \in \mathbb{N}$. Since $C_0^\infty(0, L)$ is dense in $H_0^4(0, L)$, there exists a sequence $(\tilde{u}_n)_{n\in\mathbb{N}} \subset C_0^\infty(0, L)$ such that $\|\tilde{u}_n - (u - h_n)\|_{H^4} < \frac{1}{n}$, $\forall n \in \mathbb{N}$. Now defining $u_n = \tilde{u}_n + h_n$, gives $\lim_{n\to\infty} u_n = u$ in $H^4(0, L)$. Obviously u_n satisfies (2.20) for all $n \in \mathbb{N}$. And (2.29) and (2.30) yield (2.23) and (2.24) for u_n as well.

Hence, we have constructed a sequence $([u_n, v_n, \xi_n, \psi_n]^\top)_{n\in\mathbb{N}} \subset D(\mathcal{A}^2)$ that converges to $y \in D(\mathcal{A})$ in the $H^4(0, L) \times H^2(0, L) \times \mathbb{R}^2$-norm, and the statement follows. $\square$

Lemma 2.15. *For every $y_0 \in D(\mathcal{A})$ the trajectory $\gamma(y_0)$ of* (2.13) *is precompact.*

Proof. We fix $y_0 \in D(\mathcal{A})$ and show that the trajectory of the corresponding solution $y(t)$ of (2.13) is precompact in $\mathcal{H}$. According to Theorem 2.9, the solution $y(t)$ is classical. Because of the compact embeddings $H^4(0, L) \hookrightarrow\hookrightarrow H^2(0, L) \hookrightarrow\hookrightarrow L^2(0, L)$ it is sufficient to show that

$$\sup_{t>0} \|\mathcal{A}y(t)\|_{\mathcal{H}} < \infty.$$

Since we have $y_t = \mathcal{A}y$ for classical solutions, it is equivalent to show that $\|y_t(t)\|_{\mathcal{H}}$ is uniformly bounded for $t > 0$.

Step 1: In the first part of this proof we assume that $y_0 \in D(\mathcal{A}^2)$, and show the precompactness of the corresponding C^2-trajectory. According to Lemma 2.12 the time derivative $y_t(t)$ of the corresponding solution of (2.13) is a classical solution of the system (2.13) differentiated in time once. Hence, it fulfills the following system, which is obtained by differentiating (2.5):

$$\rho u_{ttt} + \Lambda u_t^{\mathrm{IV}} = 0, \tag{2.32a}$$

$$u_t(t, 0) = 0, \tag{2.32b}$$

$$u_t'(t, 0) = 0, \tag{2.32c}$$

$$m u_{ttt}(L) - \Lambda u_t'''(L) + k_1'(u(L))u_t(L) + k_2'(u_t(L))u_{tt}(L) = 0, \tag{2.32d}$$

$$J u_{ttt}'(L) + \Lambda u_t''(L) = 0. \tag{2.32e}$$

We now evaluate the time derivative of $V(y_t)$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} V(y_t) &= \Lambda \int_0^L u_{tt}'' u_t'' \,\mathrm{d}x + \rho \int_0^L u_{ttt} u_{tt} \,\mathrm{d}x + J u_{ttt}'(L) u_{tt}'(L) \\
&\quad + m u_{ttt}(L) u_{tt}(L) + k_1(u_t(L)) u_{tt}(L) \\
&= u_{tt}(L)\big(m u_{ttt}(L) - \Lambda u_t'''(L) + k_1(u_t(L))\big) \\
&\quad + u_{tt}'(L)\big(\Lambda u_t''(L) + J u_{ttt}'(L)\big) \\
&= u_{tt}(L)\big[k_1(u_t(L)) - k_1'(u(L))u_t(L) - k_2'(u_t(L))u_{tt}(L)\big],
\end{aligned} \tag{2.33}$$

where we have performed partial integration in x twice and used (2.32b)–(2.32e). According to (2.2) in Assumption 2.1 we have

$$-k_2'(u_t(L))u_{tt}(L)^2 \le 0, \qquad \forall t \ge 0,$$

so after integration of (2.33) in time we obtain

$$V(y_t(t)) \le V(y_t(0)) + \int_0^t u_{tt}(\tau, L)\left[k_1(u_t(\tau, L)) - k_1'(u(\tau, L))u_t(\tau, L)\right] \mathrm{d}\tau. \tag{2.34}$$

For the first integral on the right hand side we compute

$$\begin{aligned}\int_0^t u_{tt}(\tau, L)k_1(u_t(\tau, L))\,\mathrm{d}\tau &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^{u_t(\tau,L)} k_1(s)\,\mathrm{d}s\,\mathrm{d}\tau \\ &= \int_0^{u_t(t,L)} k_1(s)\,\mathrm{d}s - \int_0^{u_t(0,L)} k_1(s)\,\mathrm{d}s.\end{aligned} \tag{2.35}$$

Hence, we find that it is uniformly bounded since $u_t(t, L) = \frac{\psi(t)}{m}$ is uniformly bounded, see Theorem 2.9. For the remaining term in (2.34) we obtain

$$\begin{aligned}&\int_0^t u_{tt}(\tau, L)k_1'(u(\tau, L))u_t(\tau, L)\,\mathrm{d}\tau = \int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{(u_t(\tau, L))^2}{2}\right) k_1'(u(\tau, L))\,\mathrm{d}\tau \\ &\quad = \frac{u_t(t, L)^2}{2}k_1'(u(t, L)) - \frac{u_t(0, L)^2}{2}k_1'(u(0, L)) - \int_0^t \frac{u_t(\tau, L)^3}{2}k_1''(u(\tau, L))\,\mathrm{d}\tau.\end{aligned} \tag{2.36}$$

According to the Sobolev embedding $H^2(0, L) \hookrightarrow C([0, L])$ we have the estimate $|u(t, L)| \le C\|u\|_{H^2} \le C\|y\|_{\mathcal{H}}$. Therefore $k_1''(u(t, L))$ is also essentially uniformly bounded for $t \in [0, \infty)$, cf. Assumption 2.1. In combination with the previously shown uniform boundedness of $u_t(t, L)$ we find that the first two terms in (2.36) are uniformly bounded, and for the remaining integral we get

$$\Big|\int_0^t \frac{u_t(\tau, L)^3}{2}k_1''(u(\tau, L))\,\mathrm{d}\tau\Big| \le C\int_0^t |u_t(\tau, L)|^3\,\mathrm{d}\tau.$$

Because of (2.3) in Assumption 2.1, and considering that $u_t(t, L)$ is uniformly bounded for $t \in \mathbb{R}$, there exists a positive constant $C > 0$ such that $|k_2(u_t(t, L))| \ge Cu_t(t, L)^2$ for all $t \ge 0$. This yields

$$\int_0^\infty |u_t(t, L)|^3\,\mathrm{d}t \le C\int_0^\infty k_2(u_t(t, L))u_t(t, L)\,\mathrm{d}t,$$

and since $\frac{\mathrm{d}}{\mathrm{d}t}V(y(t)) = -k_2(u_t(t, L))u_t(t, L)$ is integrable on $(0, \infty)$, we obtain $u_t(\cdot, L) \in L^3(\mathbb{R}^+)$.

Therefore, all terms in (2.36) are uniformly bounded for $t \ge 0$. In combination with the uniform boundedness of (2.35) this shows in (2.34) that $V(y_t(t)) \in L^\infty(\mathbb{R}^+)$, and therefore $t \mapsto \|y_t(t)\|_{\mathcal{H}}$ is uniformly bounded, see Lemma 2.7. Hence, $\gamma(y_0)$ is precompact. Moreover, notice that actually

$$\sup_{t\ge 0}\|y_t(t)\|_{\mathcal{H}} \le \tilde{C}(\|y_0\|_{\mathcal{H}}, \|y_t(0)\|_{\mathcal{H}}), \tag{2.37}$$

where the constant $\tilde{C}$ depends continuously on $\|y_0\|_{\mathcal{H}}$ and $\|y_t(0)\|_{\mathcal{H}}$.

Step 2: For the second part of the proof we consider $y_0 \in D(\mathcal{A})$, and deduce the precompactness of $\gamma(y_0)$ from the results of Step 1. According to Lemma 2.14 there exists a sequence $(y_{n,0})_{n\in\mathbb{N}} \subset D(\mathcal{A}^2)$ such that

$$\lim_{n\to\infty} y_{n,0} = y_0 \quad \text{and} \quad \lim_{n\to\infty} \mathcal{A}y_{n,0} = \mathcal{A}y_0. \tag{2.38}$$

For the approximating solutions $y_n(t) := S_{\mathcal{A}}(t)y_{n,0}$ we have $(y_n)_t(0) = \mathcal{A}y_{n,0}$ for all $n \in \mathbb{N}$, and (2.38) thus implies

$$\lim_{n\to\infty} (y_n)_t(0) = \mathcal{A}y_0 \quad \text{in } \mathcal{H}. \tag{2.39}$$

Hence (2.38) and (2.39) imply that both $(y_{n,0})_{n\in\mathbb{N}}$ and $((y_n)_t(0))_{n\in\mathbb{N}}$ are bounded in $\mathcal{H}$. Together with (2.37) this yields that

$$\sup_{\substack{t\geq 0\\ n\in\mathbb{N}}} \|(y_n)_t(t)\|_{\mathcal{H}} < \infty,$$

i.e. $((y_n)_t)_{n\in\mathbb{N}}$ is a bounded sequence in $L^\infty(\mathbb{R}^+;\mathcal{H})$. Now the Banach-Alaoglu Theorem, see Theorem I.3.15 in [Rud91], shows that there exists a $w \in L^\infty(\mathbb{R}^+;\mathcal{H})$ and a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ such that for $k \to \infty$:

$$(y_{n_k})_t \overset{*}{\rightharpoonup} w \quad \text{in} \quad L^\infty(\mathbb{R}^+;\mathcal{H}).$$

So, for arbitrary $z \in \mathcal{H}$ and $t \geq 0$ we have

$$\lim_{k\to\infty} \int_0^t \langle (y_{n_k})_t(\tau), z\rangle_{\mathcal{H}} \,\mathrm{d}\tau = \int_0^t \langle w(\tau), z\rangle_{\mathcal{H}} \,\mathrm{d}\tau,$$

which is equivalent to

$$\lim_{k\to\infty} \langle y_{n_k}(t) - y_{n_k}(0), z\rangle_{\mathcal{H}} = \Big\langle \int_0^t w(\tau)\,\mathrm{d}\tau, z\Big\rangle_{\mathcal{H}}.$$

Since $y_n(t)$ converges to $y(t)$ strongly in $L^\infty((0,T);\mathcal{H})$ for every $T > 0$, we conclude from the above

$$\langle y(t) - y(0), z\rangle_{\mathcal{H}} = \Big\langle \int_0^t w(\tau)\,\mathrm{d}\tau, z\Big\rangle_{\mathcal{H}}.$$

Now, owing to $z \in \mathcal{H}$ being arbitrary, it follows that for all $t > 0$

$$y(t) - y(0) = \int_0^t w(\tau)\,\mathrm{d}\tau. \tag{2.40}$$

Since $y \in C^1(\mathbb{R}^+;\mathcal{H})$, we can take the time derivative of (2.40), and obtain $y_t \equiv w$. This implies $y_t \in L^\infty(\mathbb{R}^+;\mathcal{H})$, i.e. $\|y_t(\cdot)\|_{\mathcal{H}}$ is uniformly bounded, which proves the precompactness of $\gamma(y_0)$. $\square$

Remark 2.16. In the linear case, i.e. $\mathcal{N} = 0$, the proof of the trajectory precompactness is much simpler: For classical solutions $y(t)$ we have $Ay(t) = A\mathrm{e}^{tA}y_0 = \mathrm{e}^{tA}Ay_0$, so $Ay(t)$ is uniformly bounded. Since A^{-1} is compact, this proves the precompactness for classical solutions. Since $(\mathrm{e}^{tA})_{t\geq 0}$ is a contraction semigroup, any mild solution can be approximated uniformly by classical solutions, and the precompactness follows also for mild solutions.

In the case when k_1 is linear (i.e. $k_1(u(L)) = K_1 \cdot u(L)$), and k_1, k_2 satisfy Assumption 2.1, the precompactness property of the trajectories can also be verified easily: If we incorporate the k_1-term and the linear part of k_2 into A, this operator still generates a (nonlinear) contraction semigroup (see also Remark 2.11), with respect to the modified inner product

$$\langle y_1, y_2\rangle_{\mathcal{H},2} := \frac{\Lambda}{2}\int_0^L u_1'' u_2''\,\mathrm{d}x + \frac{\rho}{2}\int_0^L v_1 v_2\,\mathrm{d}x + \frac{\xi_1\xi_2}{2J} + \frac{\psi_1\psi_2}{2m} + \frac{K_1\Lambda}{2}u_1(L)u_2(L).$$

Furthermore, A it is invertible and has a compact resolvent in $\mathcal{H}$. For the remaining nonlinear term we can show $\mathcal{N}(y(\cdot)) \in L^1(\mathbb{R}^+;\mathcal{H})$, using (2.15) and the integrability of $\dot{V}(y)$ according to (1.12). Then the requirements of Theorem 4 in the article by Dafermos and Slemrod [DS73] are fulfilled, and the precompactness of the trajectories for all mild solutions follows. See also Section 3.5 below for an application of this approach.

2.5. ω-limit set and asymptotic stability

In the following Ω will always be the set defined in Theorem 1.19. Notations and results from Chapters 1 and 2 are still assumed to hold.

Proposition 2.17. *For all $y = [u, v, \xi, \psi]^\top \in \Omega$ there holds $\psi = 0$, $u(L) = 0$.*

Proof. For a fixed $y_0 \in \Omega$ let $y(t) = S_{\mathcal{A}}(t)y_0$, which lies in Ω entirely, since Ω is $S_{\mathcal{A}}$-invariant. By definition we have $V(y(t)) = \nu(y_0)$ for all $t \geq 0$. First we show that

$$\psi(t) = 0, \qquad \forall t \geq 0. \tag{2.41}$$

In the case when $y_0 \in \Omega \cap D(\mathcal{A})$, (2.41) follows easily since (2.15) implies for the corresponding classical solution

$$\dot{V}(y(t)) = 0 \quad \Leftrightarrow \quad \psi(t) = 0.$$

We next investigate the case when $y_0 \in \Omega \setminus D(\mathcal{A})$. Then there exists a sequence $(y_{n,0})_{n\in\mathbb{N}} \subset D(\mathcal{A})$ such that $\lim_{n\to\infty} y_{n,0} = y_0$ in $\mathcal{H}$. We can apply Proposition 1.6, which implies $y_n(t) \to y(t)$ in $C([0,T];\mathcal{H})$ for any $T > 0$, where $y_n(t) = S_{\mathcal{A}}(t)y_{n,0}$. Since V is Lipschitz continuous on bounded sets in $\mathcal{H}$, $(V(y_n(t)))_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];\mathbb{R})$. Furthermore, we also have the convergence

$$\psi_n(t) \to \psi(t) \quad \text{in} \quad C([0,T];\mathbb{R}). \tag{2.42}$$

By applying this in (2.15) we obtain that

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t} V(y_n(t))\Big)_{n\in\mathbb{N}}$$

is a Cauchy sequence in $C([0,T];\mathbb{R})$. We conclude that $(V(y_n(t)))_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^1([0,T];\mathbb{R})$. So there exists a unique $w \in C^1([0,T];\mathbb{R})$ such that

$$V(y_n(t)) \to w(t) \quad \text{in} \quad C^1([0,T];\mathbb{R}). \tag{2.43}$$

On the other hand, we know that $V(y_n(t)) \to V(y(t)) = \nu(y_0)$ for every $t \geq 0$, and hence $w(t) \equiv \nu(y_0)$. This, combined with (2.43), implies $\dot{V}(y_n(t)) = -k_2(\frac{\psi_n}{m})\frac{\psi_n}{m} \to 0$ uniformly on $[0,T]$. With (2.42) this now yields (2.41) and in particular $\psi(0) = 0$.

We now show that $u(t, L) = 0$ for all $t \geq 0$. From (2.41) and the last line of (1.10) it follows that

$$m\left(\int_0^t v(s)\,\mathrm{d}s\right)\bigg|_{x=L} = \int_0^t \psi(s)\,\mathrm{d}s = 0.$$

Using this, the first component of (1.11) implies

$$0 = \left(\int_0^t v(s)\,\mathrm{d}s\right)\bigg|_{x=L} = u(t,L) - u(0,L).$$

Therefore $u(t,L)$ is constant along $y(t) \subset \Omega$, which implies

$$\int_0^t u(s,L)\,\mathrm{d}s = u_0(L)t, \quad t \geq 0. \tag{2.44}$$

Furthermore, since $\sup_{t>0} \|y(t)\|_{\mathcal{H}} < \infty$, we find that $\sup_{t>0} \|v(t)\|_{L^2(0,L)} < \infty$. Therefore, the second component of (1.11) implies

$$\sup_{t\geq 0} \Big\| \Big(\int_0^t u(s)\,\mathrm{d}s \Big)^{\mathrm{IV}} \Big\|_{L^2(0,L)} < \infty. \tag{2.45}$$

We next apply the following Gagliardo-Nirenberg inequality (cf. [Nir59]), which guarantees the existence of $C > 0$ such that there holds for all $t \geq 0$:

$$\Big\| \int_0^t u(s)\,\mathrm{d}s \Big\|_{L^\infty(0,L)} \leq C \Big\| \Big(\int_0^t u(s)\,\mathrm{d}s \Big)^{\mathrm{IV}} \Big\|_{L^2(0,L)}^{\frac{1}{8}} \Big\| \int_0^t u(s)\,\mathrm{d}s \Big\|_{L^2(0,L)}^{\frac{7}{8}}. \tag{2.46}$$

The first factor on the right hand side is uniformly bounded due to (2.45). For the second factor we observe that, according to Theorem 2.9, $t \mapsto \|u(t)\|_{L^2(0,L)}$ is uniformly bounded, and therefore $t \mapsto \| \int_0^t u(s)\,\mathrm{d}s\|_{L^2(0,L)}$ grows at most linearly. Altogether this implies in (2.46) that $t \mapsto \int_0^t u(s,L)\,\mathrm{d}s$ grows at most like $t^{\frac{7}{8}}$. But this contradicts (2.44) unless $u_0(L) = 0$. This shows that $u(t,L) = 0$ for all $t \geq 0$. $\square$

This result allows to represent any trajectory $\gamma(y_0) \subset \Omega$ as a solution to a simpler linear system characterizing Ω. By inserting the result of Proposition 2.17 in the equation (1.11) we find that any mild solution $y(t)$ of (2.13) with $y \subset \Omega$, satisfies the following system for $t > 0$:

$$u(t) - u(0) = \int_0^t v(s)\,\mathrm{d}s, \tag{2.47a}$$

$$v(t) - v(0) = -\frac{\Lambda}{\rho} \left(\int_0^t u(s)\,\mathrm{d}s \right)^{\mathrm{IV}}, \tag{2.47b}$$

$$\xi(t) - \xi(0) = -\Lambda \left(\int_0^t u(s)\,\mathrm{d}s \right)'' \Bigg|_{x=L}, \tag{2.47c}$$

$$0 = \left(\int_0^t u(s)\,\mathrm{d}s \right)''' \Bigg|_{x=L}, \tag{2.47d}$$

and with the additional boundary condition $u(t,L) = 0$. We will show that this system is overdetermined. To this end we first investigate the system (2.47) without the condition $u(t,L) = 0$, and only incorporate it later.

The system (2.47a)–(2.47c) can be interpreted as the mild formulation of a linear evolution equation in a Hilbert space $\tilde{\mathcal{H}}$:

$$w_t = \mathcal{B}w, \tag{2.48}$$

with $w = [u,v,\xi]^\top \in \tilde{\mathcal{H}}$. Here, $\tilde{\mathcal{H}}$ is the Hilbert space

$$\tilde{\mathcal{H}} := \{ w = [u,v,\xi]^\top : u \in \tilde{H}^2_{0,\mathbb{R}}(0,L), v \in L^2_{\mathbb{R}}(0,L), \xi \in \mathbb{R} \}, \tag{2.49}$$

and $\mathcal{B}$ is the following linear operator in $\tilde{\mathcal{H}}$:

$$\mathcal{B}\begin{bmatrix} u \\ v \\ \xi \end{bmatrix} = \begin{bmatrix} v \\ -\frac{\Lambda}{\rho}u^{\mathrm{IV}} \\ -\Lambda u''(L) \end{bmatrix}, \tag{2.50}$$

with the domain

$$D(\mathcal{B}) := \{w \in \tilde{\mathcal{H}} \colon u \in \tilde{H}^4_{0,\mathbb{R}}(0,L), v \in \tilde{H}^2_{0,\mathbb{R}}(0,L), \xi = Jv'(L), u'''(L) = 0\},$$

which incorporates the condition (2.47d). The space $\tilde{\mathcal{H}}$ is equipped with the inner product

$$\langle\!\langle w_1, w_2 \rangle\!\rangle := \frac{\Lambda}{2}\int_0^L u_1'' u_2'' \,\mathrm{d}x + \frac{\rho}{2}\int_0^L v_1 v_2 \,\mathrm{d}x + \frac{1}{2J}\xi_1\xi_2.$$

The corresponding norm is $\|\cdot\|_{\tilde{\mathcal{H}}}$. According to Proposition A.3 the operator $\mathcal{B}$ is skew-adjoint (in $\tilde{\mathcal{X}}$, i.e. the complexification of $\tilde{\mathcal{H}}$, see the Appendix A.1) and generates a C_0-group of unitary operators in $\tilde{\mathcal{X}}$, and $\tilde{\mathcal{H}}$, respectively. The eigenvalues $\{\mu_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ are purely imaginary, and come in complex conjugated pairs, i.e. $\mu_{-n} = \overline{\mu_n}$ for all $n \in \mathbb{N}$. Zero is not an eigenvalue, since $\mathcal{B}$ is invertible, see [KT05]. The corresponding eigenfunctions $\{\Phi_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ form an orthonormal basis of $\tilde{\mathcal{X}}$. They are given by

$$\Phi_n = \begin{bmatrix} u_n \\ \mu_n u_n \\ \mu_n J u_n'(L) \end{bmatrix}, \tag{2.51}$$

where u_n is the unique real-valued solution of

$$\rho\mu_n^2 u_n + \Lambda u_n^{\mathrm{IV}} = 0, \tag{2.52a}$$
$$u_n'''(L) = 0, \tag{2.52b}$$
$$J\mu_n^2 u_n'(L) + \Lambda u_n''(L) = 0, \tag{2.52c}$$

where u_n is normalized such that $\|\Phi_n\|_{\tilde{\mathcal{H}}} = 1$. Note that $\mu_n^2 < 0$. The fact that the eigenvalues come in purely imaginary, conjugated pairs implies that $\Phi_{-n} = \overline{\Phi_n}$, and $u_{-n} = u_n$. For the complete spectral analysis of $\mathcal{B}$ see Proposition A.3 in the Appendix. For notational simplicity we include the index $n = 0$ in the following by setting $\mu_0 := 0$ and $\Phi_0 := \mathbf{0}$ and $u_0 := 0$.

We note that a trajectory $\gamma(y_0) \subset \Omega$ (more precisely, the first three components) satisfies the (reduced) linear system (2.48) and the boundary condition $u(t,L) = 0$, for all $t \geq 0$. Now it will turn out that, for almost all values of $J > 0$, the equation (2.48) plus the boundary condition $u(t,L) = 0$ only has the trivial solution, and so $\Omega = \{\mathbf{0}\}$. For a countable set $J > 0$, however, it admits non-trivial solutions. They correspond to eigenfunctions of (2.52) having a node at $x = L$.

Lemma 2.18. *For $\mu_n \neq 0$, there exists a non-trivial solution u_n of the system* (2.52) *that additionally satisfies $u_n(L) = 0$ iff*

$$J \in \mathscr{J} := \Big\{\rho\left(\frac{L}{\ell\pi}\right)^3 \frac{(-1)^\ell + \cosh \ell\pi}{\sinh \ell\pi} : \ell \in \mathbb{N}\Big\}. \tag{2.53}$$

In this case, this nontrivial solution $u_n(= u_{-n})$ is unique up to normalization and $\mu_n^2 = -\frac{\Lambda}{\rho}\left(\frac{\ell\pi}{L}\right)^4$. We shall denote the index of this particular eigenfunction by $n = n^*(\ell) > 0$.

Proof. We omit the index n in (2.52) in the following. The general solution $\varphi \in \tilde{H}^4_{0,\mathbb{R}}(0,L)$ of (2.52a) is of the form

$$\varphi(x) = C_1[\cosh px - \cos px] + C_2[\sinh px - \sin px], \tag{2.54}$$

with $p = \left(\frac{-\rho\mu^2}{\Lambda}\right)^{\frac{1}{4}} > 0$ and $C_1, C_2 \in \mathbb{R}$. The boundary conditions (2.52b) and (2.52c) are now equivalent to the following two equations for C_1 and C_2:

$$0 = C_1(\sinh pL - \sin pL) + C_2(\cosh pL + \cos pL), \tag{2.55}$$

and

$$\begin{aligned} 0 = C_1 &\left[J\mu^2(\sinh pL + \sin pL) + p\Lambda(\cosh pL + \cos pL)\right] \\ &+ C_2\left[J\mu^2(\cosh pL - \cos pL) + p\Lambda(\sinh pL + \sin pL)\right]. \end{aligned} \tag{2.56}$$

Furthermore, the additional condition $\varphi(L) = 0$ reads

$$C_1(\cosh pL - \cos pL) + C_2(\sinh pL - \sin pL) = 0. \tag{2.57}$$

First we use (2.55) and (2.57) to determine the constants C_1 and C_2. In order for φ to be non-zero the determinant of the linear system formed by (2.55) and (2.57) needs to vanish, i.e.

$$\begin{aligned} 0 &= (\sinh pL - \sin pL)^2 - (\cosh pL - \cos pL)(\cosh pL + \cos pL) \\ &= -2\sinh pL \sin pL. \end{aligned}$$

Since $pL > 0$, this is true iff $p = \frac{\ell\pi}{L}$ for some $\ell \in \mathbb{N}$. Hence $\mu^2 = -\frac{\Lambda}{\rho}\left(\frac{\ell\pi}{L}\right)^4$. Now (2.55) gives $C_2 = -C_1\frac{\sinh \ell\pi}{\cosh \ell\pi + (-1)^\ell}$. Now we investigate in which situation also the third condition (2.56) is fulfilled. Multiplying (2.56) by $\frac{(-1)^\ell \cosh \ell\pi + 1}{2C_1}$, we get

$$-J\frac{\Lambda}{\rho}\left(\frac{\ell\pi}{L}\right)^4 \sinh \ell\pi + \frac{\ell\pi\Lambda}{L}[\cosh \ell\pi + (-1)^\ell] = 0,$$

and equivalently

$$J = \rho\left(\frac{L}{\ell\pi}\right)^3 \frac{\cosh \ell\pi + (-1)^\ell}{\sinh \ell\pi}.$$

In this case, the eigenfunction φ is given by (up to normalization)

$$\varphi(x) = \left(\cosh\frac{\ell\pi x}{L} - \cos\frac{\ell\pi x}{L}\right) - \frac{\sinh \ell\pi}{\cosh \ell\pi + (-1)^\ell}\left(\sinh\frac{\ell\pi x}{L} - \sin\frac{\ell\pi x}{L}\right). \tag{2.58}$$

□

Hence, (2.48) plus the boundary condition $u(t, L) = 0$ has only the trivial solution iff $J \in \mathscr{J}$, where $\mathscr{J}$ is the countable set from (2.53).

Theorem 2.19. *Let Ω be the set introduced in Theorem 1.19, and $\mathscr{J}$ from* (2.53).

(i) *If $J \notin \mathscr{J}$, then $\Omega = \{\mathbf{0}\}$.*

(ii) *If* $J \in \mathscr{J}$, *then*

$$\Omega = \operatorname{span}_{\mathbb{R}}\{[u_{n^*}, 0, 0, 0]^\top, [0, u_{n^*}, Ju'_{n^*}(L), 0]^\top\}.$$

Here, u_{n^*} *is a non-trivial solution of* (2.52), *for the given* J, *see Lemma 2.18.*

Proof. This proof closely follows the argumentation in [CP24]. Our aim is to characterize the solutions of (2.47), together with the extra condition $u(t, L) = 0$. According to Proposition A.3 in the Appendix we can write the mild solution of the linear evolution equation (2.48) with the initial condition $w_0 \in \tilde{\mathcal{H}}$ as

$$w(t) = e^{t\mathcal{B}} w_0 = \sum_{n \in \mathbb{Z}} \langle\!\langle w_0, \Phi_n \rangle\!\rangle_{\tilde{\mathcal{X}}} e^{\mu_n t} \Phi_n, \tag{2.59}$$

where $\{\mu_n\}_{n\in\mathbb{Z}}$ are the eigenvalues of $\mathcal{B}$, which are purely imaginary, and the Φ_n are the corresponding normalized eigenfunctions[3], see Proposition A.3. Here, $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\tilde{\mathcal{X}}}$ is the inner product in $\tilde{\mathcal{X}}$, see the Appendix A.1. We define $c_n := \langle\!\langle w_0, \Phi_n \rangle\!\rangle_{\tilde{\mathcal{X}}}$ for all $n \in \mathbb{Z}$. Because of the orthonormality of the eigenfunctions $\{\Phi_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ and the fact that $\{\mu_n\}_{n\in\mathbb{Z}} \subset \mathrm{i}\mathbb{R}$ we have for any $N \in \mathbb{N}$:

$$\Big\| \sum_{|n|\geq N} c_n e^{\mu_n t} \Phi_n \Big\|_{\tilde{\mathcal{X}}}^2 = \sum_{|n|\geq N} |c_n|^2. \tag{2.60}$$

Furthermore, Parseval's identity yields $\sum_{n\in\mathbb{Z}} |c_n|^2 = \|w_0\|_{\tilde{\mathcal{X}}}^2$. As a consequence the right hand side in (2.60) tends to zero as $N \to \infty$, uniformly in $t \geq 0$. So, for every $\varepsilon > 0$ there exists some $N > 0$ such that

$$\sup_{t\geq 0} \Big\| \sum_{|n|\geq N} c_n e^{\mu_n t} \Phi_n \Big\|_{\tilde{\mathcal{X}}} < \varepsilon, \quad \forall t \geq 0. \tag{2.61}$$

The first component of the series (2.59) converges in $H^2(0, L)$ and therefore also in $C([0, L])$. Thus we have

$$u(t, L) = \sum_{n\in\mathbb{Z}} c_n \mathrm{e}^{\mu_n t} u_n(L), \quad \forall t \geq 0. \tag{2.62}$$

Using this representation formula we now investigate those $u(t)$ that satisfy $u(t, L) = 0$ for all times. We immediately find for every $N \in \mathbb{N}$:

$$\begin{aligned} \Big| \sum_{|n|\geq N} c_n \mathrm{e}^{\mu_n t} u_n(L) \Big| &\leq C \Big\| \sum_{|n|\geq N} c_n \mathrm{e}^{\mu_n t} u_n \Big\|_{H^2(0,L)} \\ &\leq C \Big\| \sum_{|n|\geq N} c_n \mathrm{e}^{\mu_n t} \Phi_n \Big\|_{\tilde{\mathcal{X}}}. \end{aligned}$$

According to (2.61) this implies that, for every $\varepsilon > 0$, we can find an $N \in \mathbb{N}$ large enough such that

$$\sup_{t\geq 0} \Big| \sum_{n=-N}^{N} c_n \mathrm{e}^{\mu_n t} u_n(L) \Big| < \varepsilon, \quad \forall t \geq 0, \tag{2.63}$$

[3] Note that if $w_0 \in \mathcal{H}$, i.e. w_0 is real valued, then the series always maps into $\mathcal{H}$ again.

provided that $u(t,L)=0$. We fix now some $k\in\mathbb{Z}$ and $\varepsilon>0$, and select $N\in\mathbb{N}$ so large that $|k|<N$ and (2.63) is satisfied. Then we multiply the finite sum by $\mathrm{e}^{-\mu_k t}$ and integrate over $[0,T]$:

$$\frac{1}{T}\int_0^T \sum_{n=-N}^{N} c_n \mathrm{e}^{\mu_n t} u_n(L)\mathrm{e}^{-\mu_k t}\,\mathrm{d}t = \sum_{n=-N}^{N} c_n u_n(L)\frac{1}{T}\int_0^T \mathrm{e}^{(\mu_n-\mu_k)t}\,\mathrm{d}t.$$

According to (2.63) this expression still has modulus less than ε. Now we let $T\to\infty$. Since all eigenvalues μ_n of $\mathcal{B}$ are distinct (see Proposition A.3), all terms in the integral vanish except for the term where $n=k$, and we obtain

$$|c_k\, u_k(L)|<\varepsilon.$$

Since ε was arbitrary, we conclude

$$c_k\, u_k(L)=0,\quad \forall k\in\mathbb{Z}. \tag{2.64}$$

Now we need to distinguish between two situations: Either $J\notin\mathscr{J}$ or $J\in\mathscr{J}$.

(i) $J\notin\mathscr{J}$: We have, according to Lemma 2.18, that $u_n(L)\neq 0$ for all $n\in\mathbb{Z}$. Then (2.64) implies that $c_k=0$ for all $k\in\mathbb{Z}$, and consequently $w_0=w(t)\equiv\mathbf{0}$ for all $t>0$. Therefore $\Omega=\{\mathbf{0}\}$.

(ii) $J\in\mathscr{J}$: Now we consider $J=J_\ell\in\mathscr{J}$. According to Lemma 2.18 we have $u_k(L)=0$ iff $k\neq\pm n^*(\ell)$. So we get from (2.64) that

$$c_k=0,\quad \forall k\in\mathbb{Z}\setminus\{\pm n^*(\ell)\}, \tag{2.65a}$$

$$c_{n^*}\in\mathbb{C}\quad\text{arbitrary}, \tag{2.65b}$$

and $c_{-n^*}=\overline{c_{n^*}}$. This, combined with $\psi=0$ in Ω (cf. Proposition 2.17) proves in this case that

$$\Omega=\operatorname{Re}\operatorname{span}_{\mathbb{C}}\left\{[\Phi_{-n*}^\top,0]^\top,[\Phi_{n*}^\top,0]^\top\right\}=\operatorname{span}_{\mathbb{R}}\left\{[u_{n^*},0,0,0]^\top,[0,u_{n^*},Ju_{n^*}'(L),0]^\top\right\}.$$

□

Remark 2.20. An alternative approach is to consider the system (2.47a)–(2.47c) together with $u(t,L)=0$, momentarily ignoring (2.47d). The system (2.48) is then defined in

$$\tilde{\mathcal{H}}_1:=\{w\in\tilde{\mathcal{H}}:u(L)=0\}$$

instead of $\tilde{\mathcal{H}}$, and $\mathcal{B}$ has a different domain:

$$D_1(\mathcal{B}):=\{w\in\tilde{\mathcal{H}}_1:u\in\tilde{H}^4_{0,\mathbb{R}}(0,L),v\in\tilde{H}^2_{0,\mathbb{R}}(0,L),\xi=Jv'(L),v(L)=0\}.$$

Analogously to the Proposition A.3 one finds that the operator $(\mathcal{B},D_1(\mathcal{B}))$ is again skew-adjoint, generates a C_0-semigroup of unitary operators, and its eigenfunctions form an orthogonal basis. The eigenfunctions are still of the form (2.51), but the corresponding functions u_n now solve (2.52), with (2.52b) replaced by $u_n(L)=0$. Hence, we again get the representation (2.54) for u_n. However, in order to determine the constants, here we use $u_n(L)=0$ instead of $u_n'''(L)=0$ (so, the u_n are in general different to the ones used in the proof above). With these u_n we have again (2.62), and only there we apply the remaining condition (2.47d).

In the case $J\in\mathscr{J}$ we have seen (in Theorem 2.19) that $\Omega=\operatorname{Re}\operatorname{span}\{[\Phi_{\pm n^*}^\top,0]^\top\}$. From the definition of the $\Phi_{\pm n^*}$ we find that they are precisely the (two) common eigenfunctions of $(\mathcal{B},D(\mathcal{B}))$ and $(\mathcal{B},D_1(\mathcal{B}))$. We conclude that, in order to determine the ω-limit set, the two approaches using either $(\mathcal{B},D(\mathcal{B}))$ or $(\mathcal{B},D_1(\mathcal{B}))$ are equivalent.

They only differ in the order in which the boundary conditions $u'''(L) = 0$ and $u(L) = 0$ are applied.

Now we have all the prerequisites to prove our main result:

Theorem 2.21. *Assume that Assumption 2.1 holds for the nonlinearities* k_1, k_2. *Let* $y_0 \in D(\mathcal{A})$, *and* $y(t)$ *be the corresponding classical solution of* (2.13), *and let* $\mathscr{J}$ *be the set from* (2.53). *Then there holds the following:*

(i) *If* $J \notin \mathscr{J}$, *then*
$$\lim_{t\to\infty} \|y(t)\|_{\mathcal{H}} = 0.$$

(ii) *If* $J \in \mathscr{J}$, *then* $y(t)$ *approaches (with respect to* $\|\cdot\|_{\mathcal{H}}$*) the time-periodic solution corresponding to the initial condition* $\Pi^* y_0$ *as* $t \to \infty$. *Here,* Π^* *is the orthogonal projection from* $\mathcal{H}$ *onto* Ω *(with* Ω *from Theorem 2.19 (ii)), and it is given by*
$$\Pi^* y = \begin{bmatrix} \Lambda\langle u'', u''_{n^*}\rangle_{L^2} u_{n^*} \\ |\mu_{n^*}|^2 \big(\rho\langle v, u_{n^*}\rangle_{L^2} + \xi u'_{n^*}(L)\big) u_{n^*} \\ J|\mu_{n^*}|^2 \big(\rho\langle v, u_{n^*}\rangle_{L^2} + \xi u'_{n^*}(L)\big) u'_{n^*}(L) \\ 0 \end{bmatrix}, \tag{2.66}$$
where $\langle\cdot,\cdot\rangle_{L^2}$ *denotes the standard inner product on* $L^2(0, L)$.

Proof. Case (i): According to Lemma 2.15, the trajectory $\gamma(y_0)$ is precompact. According to Theorem 2.19 we further have $\Omega = \{\mathbf{0}\}$. So we can apply LaSalle's Invariance Principle (Theorem 1.19), which proves that $\lim_{t\to\infty} \|y(t)\|_{\mathcal{H}} = 0$.

Case (ii): We consider $n^*(\ell)$ as in Lemma 2.18. According to the end of the proof of Theorem 2.19 the ω-limit set is the (complex) span of the two vectors $\Psi_{\pm n^*} = [\Phi_{\pm n^*}^\top, 0]^\top$, where $\Phi_{-n^*} = \overline{\Phi_{n^*}}$. Since $\Phi_{\pm n^*} \in D(\mathcal{B})$ we know that $u'''_{n^*}(L) = 0$, and so the $\Psi_{\pm n^*}$ are eigenvectors of A to the eigenvalues $\pm\mu_{n^*}$. We may now define the orthogonal projection (first in $\mathcal{X}$, see the Appendix A.1):
$$\Pi^* := \langle\cdot, \Psi_{-n^*}\rangle_{\mathcal{X}} \Psi_{-n^*} + \langle\cdot, \Psi_{n^*}\rangle_{\mathcal{X}} \Psi_{n^*}.$$
According to Proposition A.1 the eigenvectors of A form an orthogonal basis of $\mathcal{X}$, so Π^* commutes with A, and $\mathcal{X} = \ker \Pi^* \oplus \operatorname{ran} \Pi^*$ is an orthogonal, A-invariant decomposition of $\mathcal{X}$. In the following we work with the restriction of Π^* to $\mathcal{H}$, and keep the same notation. The explicit representation of Π^* is given by (2.66).

In the next step we show that Π^* commutes with the nonlinearity $\mathcal{N}$. Since the first component u_{n^*} of Ψ_{n^*} satisfies $u_{n^*}(L) = 0$, it is clear that $\mathcal{N}\Psi_{\pm n^*} = \mathbf{0}$ and thus $\mathcal{N}\Pi^* = \mathbf{0}$. Let now $y \in \mathcal{X}$. Then
$$\mathcal{N} y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -k_1(u(L)) - k_2(\frac{\psi}{m}) \end{bmatrix},$$
and so $\Pi^* \mathcal{N} y = \mathbf{0}$.

As a consequence, the decomposition $\mathcal{H} = \ker \Pi^* \oplus \operatorname{ran} \Pi^*$ is invariant under the nonlinear semigroup $S_{\mathcal{A}}$ generated by $\mathcal{A}$. The trajectories of $S_{\mathcal{A}}|_{\ker \Pi^*}$ lying in $D(\mathcal{A})$ are still precompact. We know from Theorem 2.19 that any ω-limit set of $S_{\mathcal{A}}|_{\ker \Pi^*}$ (as a restriction of $S_{\mathcal{A}}$) has to be a subset of $\operatorname{ran} \Pi^*$. But on the other hand any trajectory and limit of $S_{\mathcal{A}}|_{\ker \Pi^*}$ has to lie within $\ker \Pi^*$, which is orthogonal to $\operatorname{ran} \Pi^*$. Thus the only possible ω-limit set for $S_{\mathcal{A}}|_{\ker \Pi^*}$ is $\{\mathbf{0}\} = \operatorname{ran} \Pi^* \cap \ker \Pi^*$. And therefore $S_{\mathcal{A}}(t)y_0$ approaches $S_{\mathcal{A}}(t)\Pi^* y_0$ as $t \to \infty$. □

Remark 2.22. The asymptotic limit described in Theorem 2.21 can be computed explicitly. If $J = J_\ell \in \mathscr{J}$ for some $\ell \in \mathbb{N}$, it follows from (2.59), (2.65) and Lemma 2.18 that all real non-decaying solutions u_p of (2.5) are given by

$$u_p(t,x) = T(t) u_{n^*}(x), \tag{2.67}$$

where

$$T(t) = a \cos \sqrt{\frac{\Lambda}{\rho}} \left(\frac{\ell\pi}{L}\right)^2 t + b \sin \sqrt{\frac{\Lambda}{\rho}} \left(\frac{\ell\pi}{L}\right)^2 t, \quad a, b \in \mathbb{R},$$

and u_{n^*} is given by (2.58). In particular, it follows from Theorem 2.21 that for a given initial condition y_0 the solution u of (2.5) approaches the solution u_p given in (2.67) in the $\mathcal{H}$-norm, with the coefficients a and b determined by (use (2.59) for this):

$$a := \Lambda \langle u_0'', u_{n^*}'' \rangle_{L^2(0,L)},$$

and

$$b := \sqrt{\frac{\Lambda}{\rho}} \left(\frac{\ell\pi}{L}\right)^2 \big(\rho \langle v_0, u_{n^*} \rangle_{L^2(0,L)} + \xi_0 u_{n^*}'(L)\big).$$

Remark 2.23. As already mentioned in Remark 2.16 it is the nonlinear term $k_1(u(L))$, representing the spring in the model, that prevents the nonlinear operator $\mathcal{A}$ from being dissipative. As a consequence the semigroup $S_{\mathcal{A}}$ is not contractive and therefore it is not possible to extend the precompactness of the classical trajectories to the trajectories of the mild solutions using a density argument (for this see the proof of Theorem 3.65 in [LGM99]). From the physical point of view one might expect that (at least for $J \notin \mathscr{J}$) also the mild solutions tend to zero, which is motivated by the observation that the total energy is dissipated whenever a trajectory does not lie in $\Omega = \{\mathbf{0}\}$, i.e. for almost all times the system looses energy due to friction. However, from the mathematical point of view we have no information if the trajectory is precompact for a non-classical solution. Hence, it is not clear whether the trajectory converges at all as $t \to \infty$.

CHAPTER 3

An Euler-Bernoulli beam with a nonlinear dynamical feedback system

3.1. Introduction

Let us consider a linear homogeneous Euler-Bernoulli beam, clamped at one end and with tip mass at the other free end. The state of the beam at time t is described by its transverse deflection $u(t,x)$ from the zero-state, where $x \in [0,L]$ is the longitudinal coordinate of the beam, see Figure 1. The well known PDE for the motion of the beam reads as

$$\rho u_{tt}(t,x) + \Lambda u^{\mathrm{IV}}(t,x) = 0, \tag{3.1}$$

with the mass per unit length ρ and the flexural rigidity Λ. The boundary conditions for the clamped end at $x = 0$ are given by

$$u(t,0) = u'(t,0) = 0, \tag{3.2}$$

and for the free end at $x = L$, we have

$$J u'_{tt}(t,L) + \Lambda u''(t,L) = -\tau_e, \tag{3.3a}$$

$$M u_{tt}(t,L) - \Lambda u'''(t,L) = -f_e, \tag{3.3b}$$

where J and M denote the mass moment of inertia and the mass of the tip mass, respectively, and $-\tau_e$ and $-f_e$ describe the external torque and force acting on the tip mass. Here and in the following, the notation u_t is used for the derivative with respect to the time variable t, and u' for the x-derivative.

In literature, there exists a number of contributions dealing with the design of boundary controllers to stabilize this type of system. To mention but a few, in [LM88] the asymptotic stability was shown using semigroup formulation and applying the La Salle Invariance principle. To obtain stronger, exponential stability, frequency domain criteria [Mor01], Riesz basis property [Guo02b], [GW06] or energy multiplier methods [Rao95], [CM98] were employed. In contrast to these works, which are mainly based on linear static and dynamic boundary controllers, this chapter is concerned with the interaction of the Euler-Bernoulli beam (3.1)–(3.3) with a dynamic nonlinear feedback system. In particular, it is assumed that this feedback system generates a reaction torque $\tau_e = \tau_{e,1} + \tau_{e,2}$ and a reaction force $f_e = f_{e,1} + f_{e,2}$, respectively. The reaction torque and force is composed of the response of a nonlinear spring-damper system

$$\tau_{e,1} = d_1(u'_t(t,L)) + k_1(u'(t,L)), \tag{3.4a}$$

$$f_{e,1} = d_2(u_t(t,L)) + k_2(u(t,L)) \tag{3.4b}$$

and the response of a nonlinear finite-dimensional system with state $z_j \in \mathbb{R}^{n_j}$, for $j = 1,2$,

$$(z_1)_t = a_1(z_1) + b_1(z_1)u'_t(t,L), \tag{3.5a}$$

$$\tau_{e,2} = c_1(z_1), \tag{3.5b}$$

and

$$(z_2)_t = a_2(z_2) + b_2(z_2)u_t(t, L), \tag{3.6a}$$

$$f_{e,2} = c_2(z_2), \tag{3.6b}$$

which constitutes a strictly passive map from the time derivative of the tip angle $u_t'(t, L)$ to the reaction torque $\tau_{e,2}$ and from the velocity of the tip position $u_t(t, L)$ to the reaction force $f_{e,2}$, respectively. The functions a_j, b_j, c_j, d_j, and k_j, for $j = 1, 2$, as well as their mathematical properties will be specified in detail in the next section. Note that (3.4)–(3.6) represents a nonlinear dynamic boundary controller for the Euler-Bernoulli beam (3.1)–(3.3), a nonlinear dynamic environment, or a combination of both. The system (3.1)–(3.6) may be interpreted as a feedback interconnected system with the lossless Euler Bernoulli beam (3.1)–(3.3) in the forward path and the passive spring-damper system (3.4) as well as the strictly passive system (3.5), (3.6) in the feedback path, see Figure 2. It is well known that the feedback interconnection of passive systems preserves the passivity, see, e.g., [vdS00]. This fact is often exploited in the controller design, see for example [OvdSME02] and [OASKH08] for the finite-dimensional case. However, in the infinite-dimensional case the analysis is typically confined to linear systems, see, e.g., [LGM99], [KT05], [VZLGM09], or very recently [RLGMZ14]. Thus, with this work we want to take a first step towards an extension of the state of the art to the nonlinear case by still considering a linear PDE but allowing for a nonlinear ODE at the boundary.

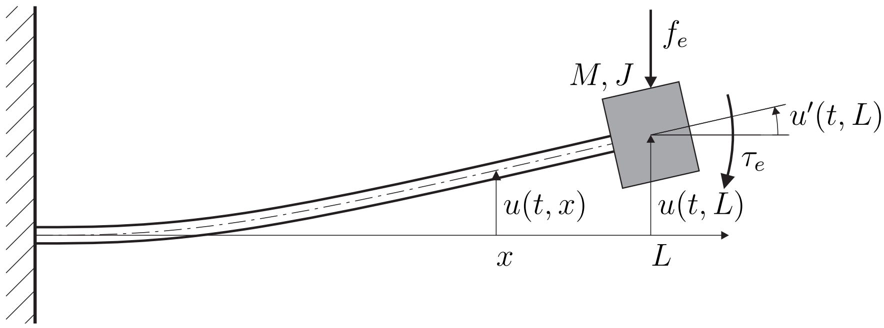


Figure 1. Euler-Bernoulli beam with tip mass. Source: [MSAK15].

The goal of this chapter is to prove the global-in-time wellposedness and, most of all, the asymptotic stability of the feedback interconnected system (3.1)–(3.6) according to Figure 2. For both aspects, we have to deviate from the strategy employed in the analogous linear model (introduced and analyzed in [KT05, MA15]): In the linear case, the generator of the evolution semigroup is dissipative, which readily yields large-time solutions. The nonlinear semigroup for (3.1)–(3.6) is *not* dissipative (in the sense of [CP69]). Hence, standard semigroup theory will first only yield local-in-time solutions, and the construction of an appropriate Lyapunov functional for (3.1)–(3.6) then shows their global existence.

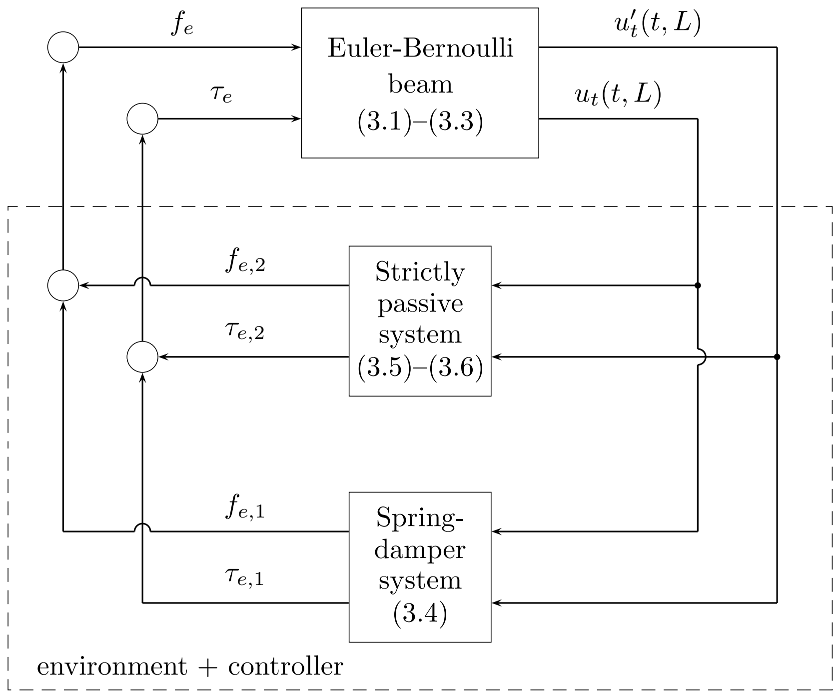


Figure 2. Interconnection of the Euler-Bernoulli beam system to a passive spring-damper system and a strictly passive system in the feedback path. Source: [MSAK15].

Asymptotic stability of the linear counterpart model is based on the precompactness of the trajectories, which can be obtained from the compactness of the resolvent for the generator. For (3.1)–(3.6), we shall follow a strategy devised for a simpler systems in [MSA15] (it consists of a beam with a nonlinear spring and damper at the free end). For the precompactness of the trajectories of (3.1)–(3.6), we shall here prove uniform C^1–bounds (w.r.t. time) on the solution, combined with compact Sobolev embeddings.

Note that the beam in (3.1)–(3.6) (and in its linear counterpart) is undamped. Damping of the complete feedback system is only introduced via the damper of (3.4) and the strictly passive systems (3.5), (3.6). This motivates that the linear model from [MA15] is asymptotically stable, but *not* exponentially stable. Hence, exponential stability also cannot be expected for our nonlinear system (3.1)–(3.6). Of course, exponential stability could be enforced by including damping terms into (3.1) (either a viscous damping of the form $+\alpha u_t$ or a Kelvin-Voigt damping of the form $+\alpha u_t^{\mathrm{IV}}$). While viscous damping would lead to a simple extension of the subsequent analysis, the higher order derivatives in the Kelvin-Voigt damping would require a rather different mathematical setup. Hence, we shall not elaborate on such dampers here.

The contents of Chapter 3 has been accepted for publication in [MSAK15].

3.2. Preliminaries

In the following sections, we will give a rigorous mathematical analysis of the feedback interconnected system (3.1)–(3.6) according to Figure 2. For this, the assumptions on the parameters and functions appearing in (3.1)–(3.6) have to be specified. First of all, let us assume that the mass per unit length ρ, the flexural rigidity Λ, the mass moment of inertia J, and the mass M of the tip mass are constant and positive. For the spring-damper system (3.4) we make the following assumptions for $j = 1, 2$:

(A1) There holds $d_j \in C^2(\mathbb{R}; \mathbb{R})$, and

$$d_j(0) = 0, \tag{3.7a}$$

$$d_j'(s) \geq 0, \quad \forall s \in \mathbb{R}, \tag{3.7b}$$

$$d_j'(0) > 0. \tag{3.7c}$$

Note that this implies $d_j(s)s > 0$ for all $s \neq 0$.

(A2) We have $k_j \in C^2(\mathbb{R}; \mathbb{R})$, with $k_j'(0) > 0$ and

$$V_{k_j}(s) := \int_0^s k_j(\sigma)\,\mathrm{d}\sigma > 0, \quad \forall s \in \mathbb{R} \setminus \{0\}. \tag{3.8}$$

Based on the assumptions (A1)–(A2), it can be easily shown that the spring-damper system (3.4) is strictly passive from the inputs $u_t'(t, L)$ and $u_t(t, L)$ to the outputs $\tau_{e,1}$ and $f_{e,1}$, respectively, with the positive definite storage functions V_{k_j}, $j = 1, 2$, according to (3.8).

As a further consequence, we find uniquely determined constants $D_j, K_j > 0$ and functions $\delta_j, \kappa_j \in C^2(\mathbb{R}; \mathbb{R})$ with $\delta_j(s) = \mathcal{O}(s^2)$ and $\kappa_j(s) = \mathcal{O}(s^2)$ for $s \to 0$ such that

$$d_j(s) = D_j s + \delta_j(s), \quad \forall s \in \mathbb{R}, \tag{3.9}$$

$$k_j(s) = K_j s + \kappa_j(s), \quad \forall s \in \mathbb{R}. \tag{3.10}$$

Hence, $D_j s$ is the linearization of $d_j(s)$, and $K_j s$ is the linearization of $k_j(s)$ around $s = 0$.

(A3) Furthermore, we assume that there exist (storage) functions $V_j \colon \mathbb{R}^{n_j} \to \mathbb{R}$, for $j = 1, 2$, such that for all $z_j \in \mathbb{R}^{n_j}$:

$$V_j(0) = 0, \quad V_j(z_j) \geq 0, \tag{3.11}$$

$$\nabla V_j(z_j) \cdot a_j(z_j) < 0, \quad (z_j \neq 0), \tag{3.12}$$

$$\nabla V_j(z_j) \cdot b_j(z_j) = c_j(z_j). \tag{3.13}$$

According to the Kalman-Yakubovich-Popov (KYP) lemma for nonlinear systems with affine input, see Lemma 4.4 in [LBEM00], this implies the strict passivity of the systems (3.5) and (3.6).

For the mathematical analysis we furthermore require for $j = 1, 2$:

(A4) $V_j \in C^2(\mathbb{R}^{n_j}; \mathbb{R})$ and

$$\lim_{|z_j| \to \infty} V_j(z_j) = \infty, \tag{3.14}$$

$$P_j := \operatorname{Hess}(V_j)(0) > 0. \tag{3.15}$$

(A5) a_j, $b_j \in C^2(\mathbb{R}^{n_j}; \mathbb{R}^{n_j})$ and

$$a_j(0) = 0, \tag{3.16a}$$

$$\det J_{a_j}(0) \neq 0, \tag{3.16b}$$

where $J_{a_j}(0)$ is the Jacobian of a_j at $z_j = 0$.

(A6) $c_j \in C^2(\mathbb{R}^{n_j};\mathbb{R})$, and

$$c_j(0) = 0.$$

The assumptions (A3)–(A6) imply the following, for $j = 1, 2$:

- There exists a unique *regular* matrix $A_j \in \mathbb{R}^{n_j \times n_j}$ and a function $\alpha_j \in C^2(\mathbb{R}^{n_j};\mathbb{R}^{n_j})$ such that for all $z_j \in \mathbb{R}^{n_j}$

$$a_j(z_j) = A_j z_j + \alpha_j(z_j), \tag{3.17a}$$
$$|\alpha_j(z_j)| = \mathcal{O}(|z_j|^2) \quad \text{as } z_j \to 0, \tag{3.17b}$$

  hence $A_j z_j$ is the linearization of $a_j(z_j)$ around the origin. By using the first order Taylor expansion of ∇V_j around the origin we conclude from (3.12) and (3.11) that

$$z_j^\top (P_j A_j) z_j \leq 0, \quad \forall z_j \in \mathbb{R}^{n_j}, \tag{3.18}$$

  and from (3.15) and (3.16) we find

$$|\nabla V_j(z_j) \cdot a_j(z_j)| \geq C|z_j|^2 \quad \text{as } z_j \to 0, \tag{3.19}$$

  for some positive constant C.
- There exists a unique vector $B_j \in \mathbb{R}^{n_j}$ and a function $\beta_j \in C^2(\mathbb{R}^{n_j};\mathbb{R}^{n_j})$ such that for all $z_j \in \mathbb{R}^{n_j}$

$$b_j(z_j) = B_j + \beta_j(z_j), \tag{3.20a}$$
$$\beta_j(z_j) = \mathcal{O}(|z_j|) \quad \text{as } z_j \to 0. \tag{3.20b}$$

- There exists a unique vector $C_j \in \mathbb{R}^{n_j}$ and a function $\gamma_j \in C^2(\mathbb{R}^{n_j};\mathbb{R})$ such that for all $z_j \in \mathbb{R}^{n_j}$

$$c_j(z_j) = C_j \cdot z_j + \gamma_j(z_j), \tag{3.21a}$$
$$|\gamma_j(z_j)| = \mathcal{O}(|z_j|^2) \quad \text{as } z_j \to 0. \tag{3.21b}$$

  Note that (3.13) implies

$$P_j B_j = C_j. \tag{3.21c}$$

Remark 3.1. For this chapter it would even be possible to only make the weaker assumptions $a_j, b_j \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^{n_j};\mathbb{R}^{n_j})$, $c_j \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^{n_j};\mathbb{R})$ and $d_j, k_j \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$, for $j = 1, 2$. In particular, the local behavior of the functions α_j, β_j, γ_j, δ_j and κ_j around the origin also stays the same, which can be seen by using the integral form of the remainder in the respective Taylor expansions.

The only change concerns the proof of Lemma 3.19. There, not Theorem 6.1.5 but Theorem 6.1.6 from [Paz83] needs to be applied, see the proof of Lemma 2.12 in the previous chapter for an analogous proof.

Throughout the rest of this chapter we alway assume that the conditions (A1)–(A6) are satisfied.

3.3. Formulation as an evolution equation

The system (3.1)–(3.6) is reformulated as an evolution equation in the real Hilbert space

$$\mathcal{H} = \{y = [u, v, z_1, z_2, \xi, \psi]^\top : u \in \tilde{H}^2_{0,\mathbb{R}}(0, L), v \in L^2_{\mathbb{R}}(0, L), z_j \in \mathbb{R}^{n_j}, \xi, \psi \in \mathbb{R}\},$$

see (1.1) for the definition of the function spaces. The inner product is defined by

$$\begin{aligned}\langle y, \tilde{y}\rangle_{\mathcal{H}} = {}&\Lambda \int_0^L u''\tilde{u}''\,\mathrm{d}x + \rho \int_0^L v\tilde{v}\,\mathrm{d}x + \frac{1}{J}\xi\tilde{\xi} + \frac{1}{M}\psi\tilde{\psi} \\ &+ K_1 u'(L)\tilde{u}'(L) + K_2 u(L)\tilde{u}(L) + z_1^\top P_1 \tilde{z}_1 + z_2^\top P_2 \tilde{z}_2,\end{aligned} \tag{3.22}$$

where the positive definite matrices P_j are due to (3.15). For the following, the operator

$$\mathcal{A} : \begin{bmatrix} u \\ v \\ z_1 \\ z_2 \\ \xi \\ \psi \end{bmatrix} \mapsto \begin{bmatrix} v \\ -\frac{\Lambda}{\rho} u^{\mathrm{IV}} \\ a_1(z_1) + \frac{1}{J} b_1(z_1)\xi \\ a_2(z_2) + \frac{1}{M} b_2(z_2)\psi \\ -\Lambda u''(L) - [c_1(z_1) + d_1(\frac{\xi}{J}) + k_1(u'(L))] \\ \Lambda u'''(L) - [c_2(z_2) + d_2(\frac{\psi}{M}) + k_2(u(L))] \end{bmatrix}$$

is introduced on the domain

$$D(\mathcal{A}) = \{y \in \mathcal{H} : u \in \tilde{H}^4_{0,\mathbb{R}}(0, L), v \in \tilde{H}^2_{0,\mathbb{R}}(0, L), \xi = Jv'(L), \psi = Mv(L)\}. \tag{3.23}$$

Based on the formulation of the coefficient functions, the operator $\mathcal{A}$ is decomposed into a linear and a nonlinear part:

Linear part: The linear part is denoted by A, which is the linearization of $\mathcal{A}$ around the origin:

$$A : \begin{bmatrix} u \\ v \\ z_1 \\ z_2 \\ \xi \\ \psi \end{bmatrix} \mapsto \begin{bmatrix} v \\ -\frac{\Lambda}{\rho} u^{\mathrm{IV}} \\ A_1 z_1 + \frac{1}{J} B_1 \xi \\ A_2 z_2 + \frac{1}{M} B_2 \psi \\ -\Lambda u''(L) - [C_1 z_1 + \frac{1}{J} D_1 \xi + K_1 u'(L)] \\ \Lambda u'''(L) - [C_2 z_2 + \frac{1}{M} D_2 \psi + K_2 u(L)] \end{bmatrix},$$

and the domain is $D(A) = D(\mathcal{A})$.

Nonlinear part: The nonlinear part $\mathcal{N}$ is defined as the following continuous operator on all of $\mathcal{H}$ as

$$\mathcal{N} : \begin{bmatrix} u \\ v \\ z_1 \\ z_2 \\ \xi \\ \psi \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \\ \alpha_1(z_1) + \frac{1}{J}\beta_1(z_1)\xi \\ \alpha_2(z_2) + \frac{1}{M}\beta_2(z_2)\psi \\ -\gamma_1(z_1) - \delta_1(\frac{\xi}{J}) - \kappa_1(u'(L)) \\ -\gamma_2(z_2) - \delta_2(\frac{\psi}{M}) - \kappa_2(u(L)) \end{bmatrix}.$$

Note that on $D(A)$ we have $\mathcal{A} = A + \mathcal{N}$.

For the rest of this chapter we use the above definitions of A, $\mathcal{N}$, $\mathcal{A}$ and $\mathcal{H}$. In the following, we are interested in solutions of the following initial value problem in $\mathcal{H}$:

$$y_t(t) = \mathcal{A}y(t) \equiv Ay(t) + \mathcal{N}y(t), \tag{3.24a}$$

$$y(0) = y_0 \in \mathcal{H}. \tag{3.24b}$$

Lemma 3.2. *The linear operator A with domain $D(A)$ generates a C_0-semigroup of contractions on $\mathcal{H}$, denoted by $(\mathrm{e}^{tA})_{t\geq 0}$.*

Proof. This result has been shown in Section 4.2 in [KT05], we briefly sketch the main steps of the proof. A short calculation yields for $y \in D(A)$, using (3.18):

$$\langle Ay, y\rangle_{\mathcal{H}} = z_1^\top (P_1 A_1) z_1 + z_2^\top (P_2 A_2) z_2 - D_1 |v'(L)|^2 - D_2 |v(L)|^2 \leq 0.$$

Hence the operator A is dissipative in $\mathcal{H}$ with respect to the inner product (3.22). Furthermore, the inverse A^{-1} exists and is bounded (even compact). Now the statement immediately follows from the Lumer-Phillips theorem. $\square$

Remark 3.3. Since A is the infinitesimal generator of a C_0-semigroup of contractions, $\operatorname{ran}(\lambda - A) = \mathcal{H}$ for all $\lambda > 0$. In particular $\operatorname{ran}(I - A) = \mathcal{H}$. So A is *maximal dissipative* according to Theorem 2.2 in [CP69].

Lemma 3.4. *The nonlinear operator $\mathcal{N} \colon \mathcal{H} \to \mathcal{H}$ is continuously differentiable and locally Lipschitz continuous.*

Proof. By assumption, the functions $\alpha_j, \beta_j, \gamma_j, \delta_j$ and κ_j are continuously differentiable, so $\mathcal{N} : \mathcal{H} \to \mathcal{H}$ is also continuously differentiable. Analogously, one finds the local Lipschitz continuity of $\mathcal{N}$. $\square$

Corollary 3.5. *The operators A and $\mathcal{N}$ satisfy Assumption 1.1.*

According to the results in Section 1.2, the evolution equation (3.24) has a unique mild solution for every $y_0 \in \mathcal{H}$, which is classical whenever $y_0 \in D(A)$.

Next we introduce the functional $V : \mathcal{H} \to \mathbb{R}$, given by

$$\begin{aligned} V(y) := \frac{\Lambda}{2}\int_0^L (u'')^2 \,\mathrm{d}x + \frac{\rho}{2}\int_0^L v^2 \,\mathrm{d}x + \frac{\xi^2}{2J} + \frac{\psi^2}{2M} \\ + \underbrace{\int_0^{u'(L)} k_1(s)\,\mathrm{d}s}_{V_{k_1}(u'(L))} + \underbrace{\int_0^{u(L)} k_2(s)\,\mathrm{d}s}_{V_{k_2}(u(L))} + V_1(z_1) + V_2(z_2). \end{aligned} \tag{3.25}$$

Note that the first integral terms in $V(y)$ correspond to the strain energy and kinetic energy of the Euler-Bernoulli beam, the next two summands are the translational and rotational part of the kinetic energy of the tip mass, V_{k_j}, $j = 1, 2$, is the potential energy stored in the nonlinear spring elements, see (3.4) and (3.8), and V_j, $j = 1, 2$, are the non-negative storage functions of the strictly passive systems (3.5) and (3.6), respectively. Obviously $V(y) \geq 0$ for all $y \in \mathcal{H}$. Thus, $V(y)$ is exactly the sum of the storage functions of the lossless Euler-Bernoulli beam (3.1)–(3.3), the nonlinear spring-damper systems (3.4) and the strictly passive nonlinear dynamic feedback systems (3.5) and (3.6), cf. Figure 2. In the following, we show that V qualifies as a Lyapunov function for the system (3.24).

Lemma 3.6. *The function V is continuous in $\mathcal{H}$.*

Proof. The continuity of the terms in V except for the k_j-terms is immediate. Due to the continuous embedding $H^2(0, L) \hookrightarrow C^1([0, L])$ the continuity of the remaining k_j-terms follows as well. $\square$

Lemma 3.7. *Under the assumption* (3.14) *we have for any sequence* $(y_n)_{n\in\mathbb{N}} \subset \mathcal{H}$*:*

$$\sup_{n\in\mathbb{N}} V(y_n) < \infty \quad \Leftrightarrow \quad \sup_{n\in\mathbb{N}} \|y_n\|_{\mathcal{H}} < \infty.$$

Proof. It suffices to notice that $(V_j(z_{j,n}))_{n\in\mathbb{N}}$ is unbounded iff $(z_{j,n})_{n\in\mathbb{N}}$ is unbounded. □

Lemma 3.8. *For* $y_0 \in D(\mathcal{A})$ *we have* $\dot{V}(y_0) \leq 0$ *(see the Definition 1.11), i.e.* V *is monotonically decreasing along classical solutions.*

Proof. For $y_0 \in D(\mathcal{A})$ the corresponding solution $y(t)$ of (3.24) is classical, and therefore has a continuous right derivative on $[0, T_{\max}(y_0))$. So we can directly compute

$$\begin{aligned}\dot{V}(y_0) &= \frac{\mathrm{d}^+}{\mathrm{d}t} V(y(t))\big|_{t=0} \\ &= a_1(z_1) \cdot \nabla V_1(z_1) + a_2(z_2) \cdot \nabla V_2(z_2) - d_1(v'(L))v'(L) - d_2(v(L))v(L).\end{aligned}$$

Here, we have used (3.13). The non-positivity of the generalized time derivative of V can be directly concluded from (3.12) and (3.7). This concludes the proof. □

Corollary 3.9. *The function* V *satisfies Assumption 1.4.*

According to the Corollaries 3.5 and 3.9 we can apply all the results from Section 1.2, and obtain the following existence result:

Theorem 3.10. *For every* $y_0 \in \mathcal{H}$*, the evolution equation* (3.24) *has a global mild solution, which is unique. If* $y_0 \in D(A)$*, then the solution is classical. The function* V *is a Lyapunov function for* (3.24)*. Every mild solution is uniformly bounded in* $\mathcal{H}$*.*

Corollary 3.11. *All results from Chapter 1 apply to the system* (3.24) *and the Lyapunov function* V*.*

Then, as in (1.5) we introduce the nonlinear semigroup $S_{\mathcal{A}} \equiv (S_{\mathcal{A}}(t))_{t\geq 0}$, generated by $\mathcal{A}$. We keep this notation for the rest of this chapter.

Remark 3.12. Since (3.14) is only needed to show that no blow-up of the solution occurs, we may replace it by the weaker assumption

$$\lim_{|z_j|\to\infty} V_j(z_j) > V(y_0), \tag{3.14'}$$

depending on the initial condition y_0 for the problem (3.24). We argue as follows: According to Theorem 3.10 the function $t \mapsto V(y(t))$ is non-increasing (this is independent of (3.14')), which ensures that no blow-up can occur in any component of $y(t)$ except for z_j. If now $z_1(t)$ or $z_2(t)$ would blow-up, we would get $\lim_{t\to\infty} V(y(t)) > V(y_0)$ according to (3.14'). So $V(y(t))$ could not be monotonically decreasing, which is a contradiction. So (3.14') is sufficient to show that no blow-up occurs and that the solution is global in time.

3.4. ω-limit set

In this section we investigate possible ω-limit sets of $S_{\mathcal{A}}$. However, non-emptiness of the ω-limit sets will only be discussed in the subsequent sections. In the following, Ω will always be the set defined in Theorem 1.19. Notations and results from the previous sections are still assumed to hold. Furthermore, note that according to Corollary 3.11 we can use all the results from Chapter 1.

Lemma 3.13. *Let $y \in \Omega$. Then $y = [u, v, 0, 0, 0, 0]^\top$.*

Proof. First, let $y \in \Omega \cap D(\mathcal{A})$. We know from Lemma 3.8 and the corresponding proof that:

$$\dot{V}(y) = a_1(z_1) \cdot \nabla V_1(z_1) + a_2(z_2) \cdot \nabla V_2(z_2) - d_1(v'(L))v'(L) - d_2(v(L))v(L). \quad (3.26)$$

Since $y \in \Omega$, (3.26) is required to be zero, and according to (3.12) and (3.7) this holds iff $\xi = \psi = z_1 = z_2 = 0$.

Now let $y \in \Omega \setminus D(\mathcal{A})$. Then there exists a sequence $(y_n)_{n\in\mathbb{N}} \subset D(\mathcal{A})$ such that $y_n \to y$ as $n \to \infty$. According to Proposition 1.6 we have $S_\mathcal{A}(t)y_n \to S_\mathcal{A}(t)y$ uniformly on $[0, T]$ for any $T > 0$. Therefore, we have also for the components of the corresponding solutions (with $S_\mathcal{A}(t)y_n \equiv [u_n(t), v_n(t), z_{1,n}(t), z_{2,n}(t), \xi_n(t), \psi_n(t)]^\top$)

$$\begin{aligned} z_{j,n}(t) &\to z_j(t), \quad \text{in } C([0,T];\mathbb{R}^{n_j}), && (3.27)\\ Mv_n(t,L) &\to \psi(t), \quad \text{in } C([0,T];\mathbb{R}), && (3.28)\\ Jv_n'(t,L) &\to \xi(t), \quad \text{in } C([0,T];\mathbb{R}). && (3.29)\end{aligned}$$

Combined with (3.26) this implies that

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t} V(S_\mathcal{A}(t)y_n)\Big)_{n\in\mathbb{N}}$$

is a Cauchy sequence in $C([0,T];\mathbb{R})$. Since V is locally Lipschitz continuous in $\mathcal{H}$, we also have that $(V(S_\mathcal{A}(t)y_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];\mathbb{R})$, so altogether $(V(S_\mathcal{A}(t)y_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^1([0,T];\mathbb{R})$. So there exists a unique $h(t) \in C^1([0,T];\mathbb{R})$ such that

$$V(S_\mathcal{A}(t)y_n) \to h(t) \quad \text{in } C^1([0,T];\mathbb{R}). \quad (3.30)$$

On the other hand we know that $\lim_{n\to\infty} V(S_\mathcal{A}(t)y_n) = V(S_\mathcal{A}(t)y) = V(y) = \nu(y)$ for every $t \geq 0$ (see (1.12)), and hence $h(t) \equiv \nu(y)$. According to (3.30) this implies $\frac{\mathrm{d}}{\mathrm{d}t}V(S_\mathcal{A}(t)y_n) \to 0$ uniformly on $[0,T]$. By using (3.26) for every y_n this now yields that in (3.27)–(3.29) we obtain the limits $z_j(t) = \xi(t) = \psi(t) = 0$. So $S_\mathcal{A}(t)y$ has to be of the form $S_\mathcal{A}(t)y = [u(t), v(t), 0, 0, 0, 0]^\top$. ☐

Theorem 3.14. *There holds $\Omega = \{\mathbf{0}\}$. In particular, for every $y_0 \in \mathcal{H}$ either $\omega(y_0) = \emptyset$ or $\omega(y_0) = \{\mathbf{0}\}$.*

Proof. Take a fixed $y_0 \in \Omega$, and let $y(t)$ be the corresponding mild solution of (3.24). Clearly, $\gamma(y_0) \subset \Omega$, and according to Lemma 3.13 we have $y(t) = [u(t), v(t), 0, 0, 0, 0]^\top$ for appropriate functions $u(t)$ and $v(t)$.

Step 1 (linear system for u(t), v(t)): First we note that, according to (1.10) and the first line of (1.11), there holds for all $t > 0$:

$$\begin{aligned} 0 &= \int_0^t \psi(s)\,\mathrm{d}s = M\int_0^t v(s,L)\,\mathrm{d}s = M(u(t,L) - u_0(L)),\\ 0 &= \int_0^t \xi(s)\,\mathrm{d}s = J\Big(\int_0^t v(s,x)\,\mathrm{d}s\Big)'\Big|_{x=L} = J(u'(t,L) - u_0'(L)).\end{aligned}$$

Thus $u(t, L)$ and $u'(t, L)$ are constant in time. According to (1.11) the (projected) mild solution $y_p(t) = [u(t), v(t)]^\top$ satisfies the following system (i.e. the first, second, fifth, and sixth component of (1.11)):

$$u(t) - u_0 = \int_0^t v(s,x)\,\mathrm{d}s, \tag{3.31a}$$

$$v(t) - v_0 = -\frac{\Lambda}{\rho}\Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)^{\mathrm{IV}}, \tag{3.31b}$$

$$0 = \Lambda\Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)''\Big|_{x=L} + K_1\cdot\Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)'\Big|_{x=L} + \int_0^t \kappa_1(u'(s,L))\,\mathrm{d}s, \tag{3.31c}$$

$$0 = -\Lambda\Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)'''\Big|_{x=L} + K_2\cdot\Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)\Big|_{x=L} + \int_0^t \kappa_2(u(s,L))\,\mathrm{d}s. \tag{3.31d}$$

Since y is a mild solution there holds $u \in C(\mathbb{R}^+; \tilde{H}_0^2(0,L))$. Hence, we can interchange the integration and differentiation in the second term of (3.31c). Using the fact that $u'(t,L)$ is constant, we have for $u_0'(L) \neq 0$ and $t > 0$:

$$\int_0^t \kappa_1(u'(s,L))\,\mathrm{d}s = t\kappa_1(u_0'(L)) = \frac{\kappa_1(u_0'(L))}{u_0'(L)}\Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)'\Big|_{x=L}.$$

Analogously, since $u(t,L)$ is constant, we can rewrite the last integral in (3.31d) in a similar fashion. Then, we define the constants (since $\kappa_j(0) = 0$):

$$\begin{aligned}
\tilde{K}_1 &:= K_1 + \frac{\kappa_1(u_0'(L))}{u_0'(L)}, \quad \text{if } u_0'(L) \neq 0, \quad \text{else } \tilde{K}_1 := K_1,\\
\tilde{K}_2 &:= K_2 + \frac{\kappa_2(u_0(L))}{u_0(L)}, \quad \text{if } u_0(L) \neq 0, \quad \text{else } \tilde{K}_2 := K_2.
\end{aligned} \tag{3.32}$$

With this we may rewrite (3.31) as

$$u(t) - u_0 = \int_0^t v(s)\,\mathrm{d}s, \tag{3.33a}$$

$$v(t) - v_0 = -\frac{\Lambda}{\rho}\Big(\int_0^t u(s)\,\mathrm{d}s\Big)^{\mathrm{IV}}, \tag{3.33b}$$

$$0 = \Lambda\Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)''\Big|_{x=L} + \tilde{K}_1\Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)'\Big|_{x=L}, \tag{3.33c}$$

$$0 = -\Lambda\Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)'''\Big|_{x=L} + \tilde{K}_2\int_0^t u(s,x)\,\mathrm{d}s\Big|_{x=L}, \tag{3.33d}$$

making this system linear. Thus, the projected vector $y_p(t) = [u(t), v(t)]^\top$ is the unique mild solution of

$$(y_p)_t = A_p y_p, \tag{3.34a}$$

$$y_p(0) = [u_0, v_0]^\top, \tag{3.34b}$$

with the operator

$$A_p : \begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} v \\ -\frac{\Lambda}{\rho}u^{\mathrm{IV}} \end{bmatrix},$$

in the Hilbert space $\mathcal{H}_p := \tilde{H}^2_{0,\mathbb{R}} \times L^2_{\mathbb{R}}$. The equations (3.33c) and (3.33d) are incorporated into the domain:

$$D(A_p) = \{[u,v]^\top \in \mathcal{H}_p \colon u \in \tilde{H}^4_{0,\mathbb{R}}(0,L), v \in \tilde{H}^2_{0,\mathbb{R}}(0,L), \\ \Lambda u''(L) + \tilde{K}_1 u'(L) = 0, \Lambda u'''(L) - \tilde{K}_2 u(L) = 0\}.$$

For further details on the operator A_p in the space $\mathcal{H}_p$ see the Appendix B.1.

Step 2 (proof of $u(t,L) = u'(t,L) = 0$): We now investigate solutions of the projected problem (3.34) with the additional property that $u(t,L)$ and $u'(t,L)$ are constant in time. Since the semigroup $(\mathrm{e}^{tA_p})_{t\geq 0}$ is unitary in $\mathcal{H}_p$ (see the Appendix B.1), we know in particular that $\|v(t)\|_{L^2} \leq C = \frac{1}{\sqrt{\rho}}\|y_p(0)\|_{\mathcal{H}_p}$ for all $t \geq 0$ (see (B.1) for the definition of $\|\cdot\|_p$). Applying the L^2-norm to (3.33b) this yields

$$\sup_{t\geq 0} \Big\| \Big(\int_0^t u(s)\,\mathrm{d}s \Big)^{\mathrm{IV}} \Big\|_{L^2(0,L)} < \infty. \tag{3.35}$$

Next we apply the following Gagliardo-Nirenberg inequalities (cf. [Nir59]), which guarantee the existence of a constant $C > 0$ such that there holds for all $t \geq 0$:

$$\begin{aligned} \Big\| \int_0^t u(s)\,\mathrm{d}s \Big\|_{L^\infty(0,L)} &\leq C \Big\| \Big(\int_0^t u(s)\,\mathrm{d}s \Big)^{\mathrm{IV}} \Big\|^{\frac{1}{8}}_{L^2(0,L)} \cdot \Big\| \int_0^t u(s)\,\mathrm{d}s \Big\|^{\frac{7}{8}}_{L^2(0,L)}, \\ \Big\| \int_0^t u'(s)\,\mathrm{d}s \Big\|_{L^\infty(0,L)} &\leq C \Big\| \Big(\int_0^t u(s)\,\mathrm{d}s \Big)^{\mathrm{IV}} \Big\|^{\frac{3}{8}}_{L^2(0,L)} \cdot \Big\| \int_0^t u(s)\,\mathrm{d}s \Big\|^{\frac{5}{8}}_{L^2(0,L)}. \end{aligned} \tag{3.36}$$

The first factor on the right hand side in both inequalities is uniformly bounded (with respect to t) due to (3.35). For the second factor we observe that, according to Theorem 3.10, $t \mapsto \|u(t)\|_{L^2(0,L)}$ is uniformly bounded, and therefore $t \mapsto \|\int_0^t u(s)\,\mathrm{d}s\|_{L^2(0,L)}$ grows at most linearly. Hence, (3.36) implies that $t \mapsto \|\int_0^t u(s,L)\,\mathrm{d}s\|_{L^\infty(0,L)}$ grows at most like $t^{\frac{7}{8}}$ and $t \mapsto \|\int_0^t u'(s,L)\,\mathrm{d}s\|_{L^\infty(0,L)}$ at most like $t^{\frac{5}{8}}$ as $t \to \infty$. But this contradicts the fact that $u(t,L)$ and $u'(t,L)$ are constant, unless $u_0(L) = u_0'(L) = 0$. This shows that $u(t,L) = u'(t,L) = 0$ for all $t \geq 0$.

Step 3 (Holmgren's Theorem): By iterated t-integration we shall now construct C^4-solutions of (3.34a), for which we can apply the Holmgren Uniqueness Theorem [Joh82, Section 3.5]. So we define

$$y_1(t) \equiv [u_1(t), v_1(t)]^\top := \int_0^t y_p(s)\,\mathrm{d}s + A_p^{-1}[u_0, v_0]^\top.$$

Due to Theorem 1.2.4 in [Paz83] and Lemma B.1 we have $y_1(t) \in D(A_p)$ for all $t \geq 0$. So y_1 is a classical solution of (3.34a) to the initial condition $y_1(0) = A_p^{-1}[u_0, v_0]^\top$. Furthermore, because of $u(t,L) = u'(t,L) = 0$, again $u_1(t,L), u_1'(t,L)$ are constant in time. Completely analogous to the previous step we can show again that $u_1(t,L) = u_1'(t,L) = 0$.

Next we shall construct solutions of higher regularity. We iterate the previous step and define recursively $y_n(t) \equiv [u_n(t), v_n(t)]^\top := \int_0^t y_{n-1}(s)\,\mathrm{d}s + A_p^{-n}[u_0, v_0]^\top$, which solves (3.34a) classically with the initial condition $y_n(0) = A_p^{-n}[u_0, v_0]^\top$. Again we have $u_n(t,L) = u_n'(t,L) = 0$. Furthermore, by definition we have on the one hand $A_p y_n(t) = y_{n-1}(t)$. And on the other hand $A_p[u_n, v_n]^\top = [v_n, -\Lambda/\rho\, u_n^{\mathrm{IV}}]^\top$, so we can show inductively that $y_n \in C(\mathbb{R}^+; \tilde{H}_0^{2n+2}(0,L) \times \tilde{H}_0^{2n}(0,L))$. Now we consider the solution u_n for $n \geq 2$. It satisfies the following partial differential equation with boundary conditions:

$$(u_n)_{tt} = -\frac{\Lambda}{\rho} u_n^{\mathrm{IV}}, \tag{3.37a}$$

$$[u_n(0,x), (u_n)_t(0,x)]^\top = A_p^{-n}[u_0, v_0]^\top, \tag{3.37b}$$

$$u_n(t,0) = u_n'(t,0) = 0, \tag{3.37c}$$

$$u_n(t,L) = \ldots = u_n'''(t,L) = 0. \tag{3.37d}$$

By using $u_n \in C(\mathbb{R}^+; \tilde{H}_0^{2n+2}(0,L))$ and $(u_n)_t = v_n \in C(\mathbb{R}^+; \tilde{H}_0^{2n}(0,L))$ and (3.37a), we obtain the following properties for the mixed fourth order space-time derivatives of u_n:

$$\begin{aligned}
u_n^{\mathrm{IV}} &\in C(\mathbb{R}^+; \tilde{H}_0^{2n-2}(0,L)),\\
(u_n)_t''' &\in C(\mathbb{R}^+; \tilde{H}_0^{2n-3}(0,L)),\\
(u_n)_{tt}'' &= -\frac{\Lambda}{\rho} u_n^{\mathrm{VI}} \in C(\mathbb{R}^+; \tilde{H}_0^{2n-4}(0,L)),\\
(u_n)_{ttt}' &= -\frac{\Lambda}{\rho} v_n^{\mathrm{V}} \in C(\mathbb{R}^+; \tilde{H}_0^{2n-5}(0,L)),\\
(u_n)_{tttt} &= \frac{\Lambda^2}{\rho^2} u_n^{\mathrm{VIII}} \in C(\mathbb{R}^+; \tilde{H}_0^{2n-6}(0,L)).
\end{aligned}$$

So for $n \geq 4$, all mixed derivatives of u_n of order four lie in $C(\mathbb{R}^+; \tilde{H}_0^2(0,L)) \subset C(\mathbb{R}^+ \times [0,L])$. Thus $u_n(t,x)$ is a C^4-solution of (3.37).

Now we can apply the Holmgren Uniqueness Theorem [Joh82, Section 3.5] on the strip $\mathbb{R}^+ \times (0,L)$. Due to (3.37d) all partial derivatives up to order 3 of u_4 vanish on the line $\mathbb{R}^+ \times \{L\}$, and furthermore $u_4(t,0) = u_4'(t,0) = 0$. Therefore, Holmgren's Uniqueness Theorem implies that $u_4 = 0$ has to hold everywhere in this strip. (See also the proof of Lemma 3 in [LM88] for a similar result – but without a detailed proof.) Therefore $A_p^{-4}[u_0, v_0]^\top = 0$ has to hold, and since A_p^{-1} is injective, this yields $[u_0, v_0]^\top = 0$. Since $y_p(t) = \mathrm{e}^{tA_p}[u_0, v_0]^\top$, we conclude that $u(t) = v(t) = 0$ for all $t \geq 0$. □

As a consequence we obtain convergence to zero for trajectories with $\omega(y_0) \neq \emptyset$:

Corollary 3.15. *If $\omega(y_0) \neq \emptyset$ for some $y_0 \in \mathcal{H}$, then*

$$\lim_{t\to\infty} \|S_{\mathcal{A}}(t) y_0\|_{\mathcal{H}} = 0.$$

Proof. This follows directly from LaSalle's Invariance Principle (Theorem 1.19) and the fact $\Omega = \{\mathbf{0}\}$ shown in Theorem 3.14 above. □

3.5. Asymptotic stability – linear k_j

In the case where the k_j are linear, we are able to show precompactness for all trajectories, even for the mild, non-classical solutions. This will yield that the ω-limit set $\omega(y_0)$ is always non-empty, and hence the asymptotic stability of the nonlinear semigroup will follow from Corollary 3.15. Notations and results from the previous sections, and Sections 1.3 and 1.4, are still assumed to hold.

Lemma 3.16. *Let $y_0 \in \mathcal{H}$, and $y(t)$ be the corresponding mild solution of* (3.24). *Let $\kappa_j = 0$. Then $\mathcal{N}y(t) \in L^1(\mathbb{R}^+;\mathcal{H})$.*

Proof. First, let us assume that $y_0 \in D(\mathcal{A})$, so $y(t)$ is a classical solution. We know from Theorem 3.10 that $V(y(t))$ is non-increasing. By integrating (3.26) with respect to time we obtain

$$\begin{aligned} V(y(T)) - V(y_0) &= \int_0^T \Big[- d_1\left(\frac{\xi}{J}\right)\frac{\xi}{J} - d_2\left(\frac{\psi}{M}\right)\frac{\psi}{M} \\ &\qquad + a_1(z_1)\cdot\nabla V_1(z_1) + a_2(z_2)\cdot\nabla V_2(z_2)\Big]\,\mathrm{d}t \\ &=: I_T(y_0), \end{aligned} \tag{3.38}$$

where all terms on the right hand side consist of elements of the vector $y(t)$, thus depend on t. If we let $T\to\infty$, we know that $V(y(T))\to\nu(y_0)$, i.e. the limit exists and the integral $I_\infty(y_0)$ is finite.

Now we consider $y_0\in\mathcal{H}$, and $y(t)$ is the corresponding mild solution of (3.24). Let $(y_{0,n})_{n\in\mathbb{N}}\subset D(\mathcal{A})$ be a sequence with $y_{0,n}\to y_0$. According to Proposition 1.6 the corresponding classical solutions $y_n(t)$ converge to $y(t)$ in $C([0,T];\mathcal{H})$ for all $T>0$. Therefore the right hand side of (3.38) converges for all $T>0$, namely $I_T(y_{0,n})\to I_T(y_0)$. Due to continuity of V, also $V(y_n(T)) - V(y_{0,n}) \to V(y(T)) - V(y_0)$ as $n\to\infty$. Thus, (3.38) also holds for mild solutions for any $T>0$. Since $V(y(T))\to\nu(y_0)\in[0,V(y_0)]$ as $T\to\infty$, the integral $I_\infty(y_0)$ is finite.

Now we know that for any (mild) solution $y(t)$ the integral $I_\infty(y_0)$ from (3.38) is finite. Since all the terms in the integrand of (3.38) are non-positive (see (3.7) and (3.12)), we conclude with (3.19) and (3.7) that

$$z_j(t), \psi(t), \xi(t) \in L^2(\mathbb{R}^+). \tag{3.39}$$

Under the assumptions we made in Section 3.2 for the functions occurring in the nonlinear operator $\mathcal{N}$, the properties (3.39) immediately imply $\mathcal{N}y(t)\in L^1(\mathbb{R}^+;\mathcal{H})$. □

Remark 3.17. To obtain $\mathcal{N}y(t) \in L^1(\mathbb{R}^+;\mathcal{H})$ in the above proof, we used in (3.38) that the nonlinear damping functions d_j include a non-vanishing linear part (i.e. $D_j > 0$). The same assumption will also be needed in Step 3 of the proof of Lemma 3.20 below. However, in the nonlinear spring-damper system of [MSA15], a locally quadratic growth of the damper law was sufficient. From a practical point of view, this is not restrictive at all.

We note that (3.38) does not give any control on $u(t,L)$ and $u'(t,L)$ (in the sense of (3.39)). Hence, the linearity assumption $\kappa_j = 0$ was crucial for the above proof. Otherwise, the κ_j-terms in $\mathcal{N}y(t)$ could not be controlled.

Theorem 3.18. *If $\kappa_j = 0$ for $j = 1,2$, there holds $\lim_{t\to\infty}\|S_\mathcal{A}(t)y_0\|_\mathcal{H} = 0$ for every $y_0\in\mathcal{H}$, i.e. the semigroup $S_\mathcal{A}$ is asymptotically stable.*

Proof. According to Remark 3.3 the linear part A of $\mathcal{A}$ is a maximal dissipative operator on $\mathcal{H}$. Clearly $A\mathbf{0}=\mathbf{0}$, and as seen in the proof of Lemma 3.2, A^{-1} exists and is compact. Since A generates a C_0-semigroup of contractions, $(\lambda - A)^{-1}$ exists and is compact for all $\lambda>0$. Finally, according to Lemma 3.16, we have $t\mapsto\mathcal{N}y(t)\in L^1(\mathbb{R}^+;\mathcal{H})$. Due to these facts, we can apply Theorem 4 in [DS73] with $f(t) := \mathcal{N}y(t)$. This shows that the ω-limit set $\omega(y_0)$ is non-empty (in fact the trajectory $\gamma(y_0)$ is precompact).

Thus, due to Corollary 3.15 and Theorem 3.14, we conclude $\omega(y_0) = \{\mathbf{0}\}$ and that the entire solution $y(t)$ converges to zero. □

3.6. Asymptotic stability – nonlinear k_j

According to Corollary 3.15, any trajectory with a non-empty ω-limit set already is asymptotically stable. Thus, in order to complete the discussion we show in this section that (at least) any classical trajectory possesses a non-empty ω-limit. We do this by proving that every classical trajectory is precompact. To this end we follow a strategy introduced in [MSA15]. Throughout this section, notations and results from the previous sections, and Chapter 1, are still assumed to hold.

We begin with the following preparatory result (which would be obvious for linear semigroups):

Lemma 3.19. *Let $y(t)$ be a solution of* (3.24) *with the initial condition $y_0 \in D(\mathcal{A}^2) := \{y \in D(\mathcal{A}) : \mathcal{A}y \in D(\mathcal{A})\}$. Then $y \in C^2(\mathbb{R}^+;\mathcal{H})$ and $y_t(t) \in D(\mathcal{A})$ for all $t > 0$.*

Proof. If we already knew that $y \in C^2(\mathbb{R}^+;\mathcal{H})$, it would follow that $\tilde{y} := y_t$ satisfies

$$\tilde{y}_t = A\tilde{y} + \begin{bmatrix} 0 \\ 0 \\ \alpha_1'(z_1)\tilde{z_1} + \frac{1}{J}[\beta_1'(z_1)\tilde{z}_1\xi + \beta_1(z_1)\tilde{\xi}] \\ \alpha_2'(z_2)\tilde{z}_2 + \frac{1}{M}[\beta_2'(z_2)\tilde{z}_2\psi + \beta_2(z_2)\tilde{\psi}] \\ -\gamma_1'(z_1)\tilde{z}_1 - \frac{1}{J}\delta_1'\big(\frac{\xi}{J}\big)\tilde{\xi} - \kappa_1'(u'(L))\tilde{u}'(L) \\ -\gamma_2'(z_2)\tilde{z}_2 - \frac{1}{M}\delta_2'\big(\frac{\psi}{M}\big)\tilde{\psi} - \kappa_2'(u(L))\tilde{u}(L) \end{bmatrix}. \tag{3.40}$$

However, at this point we only know that $y(t) \in C^1(\mathbb{R}^+;\mathcal{H})$, see Theorem 3.10. Motivated by (3.40) we define the following functions for this fixed solution $y(t) = [u, v, z_1, z_2, \xi, \psi]^\top(t)$ (and we omit t in the definitions):

$$\begin{aligned} G_1(t,Z) &:= \alpha_1'(z_1)\zeta_1 + \frac{1}{J}[\beta_1'(z_1)\zeta_1\xi + \beta_1(z_1)\Xi], \\ G_2(t,Z) &:= \alpha_2'(z_2)\zeta_2 + \frac{1}{M}[\beta_2'(z_2)\zeta_2\psi + \beta_2(z_2)\Psi], \\ G_3(t,Z) &:= -\gamma_1'(z_1)\zeta_1 - \frac{1}{J}\delta_1'\Big(\frac{\xi}{J}\Big)\Xi - \kappa_1'(u'(L))U'(L), \\ G_4(t,Z) &:= -\gamma_2'(z_2)\zeta_2 - \frac{1}{M}\delta_2'\Big(\frac{\psi}{M}\Big)\Psi - \kappa_2'(u(L))U(L), \end{aligned}$$

where $Z := [U, V, \zeta_1, \zeta_2, \Xi, \Psi]^\top \in \mathcal{H}$. Since $y(t)$ is a classical solution and the occurring coefficient functions are differentiable, it follows that $t \mapsto G_j(t, Z)$ is continuously differentiable for all $j = 1, \ldots, 4$. As a consequence the operator $\tilde{\mathcal{N}} : \mathbb{R}_0^+ \times \mathcal{H} \to \mathcal{H}$ defined by

$$\tilde{\mathcal{N}}(t,Z) := [0, 0, G_1(t,Z), G_2(t,Z), G_3(t,Z), G_4(t,Z)]^\top,$$

is also is differentiable with respect to t for every $Z \in \mathcal{H}$. Furthermore, $\tilde{\mathcal{N}}$ is Lipschitz continuous with respect to $Z \in \mathcal{H}$, uniformly in $t \in [0, T]$ for every $T > 0$. Now the linear, non-autonomous initial value problem

$$Z_t = AZ + \tilde{\mathcal{N}}(t,Z), \tag{3.41a}$$

$$Z(0) = Z_0 \in \mathcal{H}, \tag{3.41b}$$

is considered. According to Theorem 6.1.2 in [Paz83] there exists a unique global mild solution $Z(t)$ of (3.41) for every $Z_0 \in \mathcal{H}$. If $Z_0 \in D(A)$ this solution is classical, see Theorem 6.1.5 in [Paz83].

Our next aim is to show that for the classical solution $y(t)$ fixed in the beginning, the (continuous) function $y_t(t)$ is indeed a mild solution of (3.41) for $Z_0 = \mathcal{A}y_0$: Since $y(t)$ satisfies the Duhamel formula (1.3) and is differentiable, we obtain after differentiating with respect to t

$$y_t(t) = \mathrm{e}^{tA} A y_0 + \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \mathrm{e}^{(t-s)A} \mathcal{N} y(s) \, \mathrm{d}s. \tag{3.42}$$

According to the proof of Corollary 4.2.5 in [Paz83] the following statement holds true

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \mathrm{e}^{(t-s)A} \mathcal{N} y(s) \, \mathrm{d}s = \mathrm{e}^{tA} \mathcal{N} y_0 + \int_0^t \mathrm{e}^{(t-s)A} \frac{\mathrm{d}}{\mathrm{d}s} \mathcal{N} y(s) \, \mathrm{d}s. \tag{3.43}$$

Inserting (3.43) in (3.42) yields that $y_t(t)$ fulfills the Duhamel formula for (3.41). As a consequence $y_t(t)$ is the unique mild solution of (3.41) to the initial condition $Z_0 = \mathcal{A}y_0$. Moreover, we know that this mild solution $Z(t) = y_t(t)$ is a classical solution of (3.41) if $\mathcal{A}y_0 \in D(\mathcal{A})$, i.e. $y_0 \in D(\mathcal{A}^2)$. Hence $y_t \in C^1(\mathbb{R}^+; \mathcal{H})$ and $y \in C^2(\mathbb{R}^+; \mathcal{H})$. □

Lemma 3.20. *The trajectory $\gamma(y_0)$ is precompact in $\mathcal{H}$ for $y_0 \in D(\mathcal{A}^2)$. Moreover, there exists a constant $C > 0$ such that*

$$\|y_t(t)\|_{\mathcal{H}} \leq C, \qquad \forall t \geq 0, \tag{3.44}$$

where C depends continuously on $\|y_0\|_{\mathcal{H}}$ and $\|y_t(0)\|_{\mathcal{H}}$.

Proof. In order to prove precompactness of the trajectory, it suffices to show that

$$\sup_{t>0} \|\mathcal{A}y(t)\|_{\mathcal{H}} < \infty,$$

due to the compact embeddings $H^4(0,L) \hookrightarrow\hookrightarrow H^2(0,L) \hookrightarrow\hookrightarrow L^2(0,L)$. However, this is equivalent to showing that y_t is uniformly bounded in $\mathcal{H}$ for $t > 0$, since $y_t = \mathcal{A}y$. Again, this is equivalent to

$$\begin{aligned} V(y_t) = {} & \frac{\rho}{2} \int_0^L u_{tt}^2 \, \mathrm{d}x + \frac{\Lambda}{2} \int_0^L \left(u_t''\right)^2 \mathrm{d}x + \frac{J}{2} \left(u_{tt}'(L)\right)^2 + \frac{M}{2} \left(u_{tt}(L)\right)^2 \\ & + \int_0^{u_t'(L)} k_1(s) \, \mathrm{d}s + \int_0^{u_t(L)} k_2(s) \, \mathrm{d}s + V_1((z_1)_t) + V_2((z_2)_t) \end{aligned}$$

being uniformly bounded, see Lemma 3.7. Since $y(t)$ is a classical solution, we have the following equalities

$$u_t(L) = \frac{\psi}{M}, \quad u_t'(L) = \frac{\xi}{J}.$$

According to Theorem 3.10 those terms are always uniformly bounded. Moreover, due to regularity of the functions a_j, b_j and Theorem 3.10 we see from (3.5a) and (3.6a) that $(z_j)_t \in L^\infty(\mathbb{R}^+)$ for $j = 1, 2$. Therefore, the boundedness of $V(y_t)$ is equivalent to the boundedness of the functional

$$\tilde{V}(y_t) := \frac{\rho}{2} \int_0^L u_{tt}^2 \, \mathrm{d}x + \frac{\Lambda}{2} \int_0^L \left(u_t''\right)^2 \mathrm{d}x + \frac{J}{2} \left(u_{tt}'(L)\right)^2 + \frac{M}{2} \left(u_{tt}(L)\right)^2.$$

Hence, our aim is to derive a system of equations satisfied by $y_t(t)$, and then to show that $\tilde{V}(y_t)$ is uniformly bounded.

Step 1 (Time derivative of the system): According to Lemma 3.19, there holds $y(t) \in C^2(\mathbb{R}^+; \mathcal{H})$. Differentiating (3.1)–(3.3) with respect to time hence shows that y_t is the classical solution to the following system:

$$\rho u_{ttt} + \Lambda u_t^{\mathrm{IV}} = 0, \tag{3.45a}$$

$$u_t(t,0) = u_t'(t,0) = 0, \tag{3.45b}$$

$$\Lambda u_t''(t,L) + J u_{ttt}'(t,L) + (\tau_e)_t(t) = 0, \tag{3.45c}$$

$$-\Lambda u_t'''(t,L) + M u_{ttt}(t,L) + (f_e)_t(t) = 0, \tag{3.45d}$$

where:

$$\begin{aligned} \tau_e &:= c_1(z_1) + d_1(u_t'(L)) + k_1(u'(L)), \\ f_e &:= c_2(z_2) + d_2(u_t(L)) + k_2(u(L)). \end{aligned} \tag{3.46}$$

Therefore, from (3.46) it follows:

$$\begin{aligned} (\tau_e)_t &= \nabla c_1(z_1) \cdot (z_1)_t + d_1'(u_t'(L)) u_{tt}'(L) + k_1'(u'(L)) u_t'(L), \\ (f_e)_t &= \nabla c_2(z_2) \cdot (z_2)_t + d_2'(u_t(L)) u_{tt}(L) + k_2'(u(L)) u_t(L), \end{aligned} \tag{3.47}$$

and from (3.5a) and (3.6a), we obtain

$$(z_1)_{tt} = [J_{a_1}(z_1) + u_t'(L) J_{b_1}(z_1)](z_1)_t + b_1(z_1) u_{tt}'(L), \tag{3.48a}$$

$$(z_2)_{tt} = [J_{a_2}(z_2) + u_t(L) J_{b_2}(z_2)](z_2)_t + b_2(z_2) u_{tt}(L), \tag{3.48b}$$

where J_{a_j}, J_{b_j} denote the Jacobian matrices of the functions a_j and b_j, respectively.

Step 2 (Time derivative of $\tilde{V}(y_t)$): We obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\tilde{V}(y_t) &= \rho \int_0^L u_{ttt} u_{tt} \,\mathrm{d}x + \Lambda \int_0^L u_{tt}'' u_t'' \,\mathrm{d}x + J u_{ttt}'(L) u_{tt}'(L) + M u_{ttt}(L) u_{tt}(L) \\ &= u_{tt}(L)\big(M u_{ttt}(L) - \Lambda u_t'''(L)\big) + u_{tt}'(L)\big(\Lambda u_t''(L) + J u_{ttt}'(L)\big) \\ &= -u_{tt}(L)\big((z_2)_t^\top \nabla c_2(z_2) + k_2'(u(L)) u_t(L)\big) \\ &\quad - u_{tt}'(L)\big((z_1)_t^\top \nabla c_1(z_1) + k_1'(u'(L)) u_t'(L)\big) \\ &\quad - d_2'(u_t(L))\big(u_{tt}(L)\big)^2 - d_1'(u_t'(L))\big(u_{tt}'(L)\big)^2, \end{aligned} \tag{3.49}$$

where we have performed partial integration in x twice, and then used (3.45) and (3.47). Integrating (3.49) on the time interval $[0,t]$, for some arbitrary $t \in \mathbb{R}^+$, we get with (3.7b)

$$\tilde{V}(y_t(t)) \le \tilde{V}(y_t(0)) + I_1(t) + I_2(t), \tag{3.50}$$

where

$$\begin{aligned} I_1(t) &:= -\int_0^t u_{tt}'(L)\Big((z_1)_t^\top \nabla c_1(z_1) + k_1'(u'(L)) u_t'(L)\Big) \,\mathrm{d}s, \\ I_2(t) &:= -\int_0^t u_{tt}(L)\Big((z_2)_t^\top \nabla c_2(z_2) + k_2'(u(L)) u_t(L)\Big) \,\mathrm{d}s. \end{aligned}$$

Step 3 (Boundedness of I_1 and I_2): Next, we show uniform boundedness for each component of I_2 by using partial integration in time:

$$-\int_0^t u_{tt}(L)k_2'(u(L))u_t(L)\,\mathrm{d}s = -\frac{1}{2}\left(u_t(t,L)\right)^2 k_2'(u_t(t,L)) + \frac{1}{2}\left(u_t(0,L)\right)^2 k_2'(u_t(0,L))$$
$$+\frac{1}{2}\int_0^t u_t(L)^3 k_2''(u(L))\,\mathrm{d}s \le C, \quad \forall t \ge 0.$$

Further, it holds that

$$\int_0^t u_{tt}(L)(z_2)_t^\top \nabla c_2(z_2)\,\mathrm{d}s = u_t(t,L)(z_2)_t(t)^\top \nabla c_2(z_2(t)) - u_t(0,L)(z_2)_t(0)^\top \nabla c_2(z_2(0))$$
$$-\int_0^t u_t(L)[(z_2)_t^\top H_{c_2}(z_2)(z_2)_t + (z_2)_{tt}^\top \nabla c_2(z_2)]\,\mathrm{d}s.$$

Here, H_{c_2} denotes the Hessian of the function c_2. Since $c_2 \in C^2(\mathbb{R}^{n_2})$, it follows that

$$\int_0^t |u_t(L)(z_2)_t^\top H_{c_2}(z_2)(z_2)_t|\,\mathrm{d}s \le C\int_0^t |(z_2)_t|^2\,\mathrm{d}s,$$

and (with (3.48)):

$$\begin{aligned}
\int_0^t u_t(L)(z_2)_{tt}^\top \nabla c_2(z_2)\,\mathrm{d}s &= \int_0^t u_t(L)[J_{a_2}(z_2)(z_2)_t + u_t(L)J_{b_2}(z_2)(z_2)_t]^\top \nabla c_2(z_2)\,\mathrm{d}s \\
&\quad + \int_0^t \nabla c_2(z_2)^\top b_2(z_2)u_{tt}(L)u_t(L)\,\mathrm{d}s \\
&= \int_0^t u_t(L)[J_{a_2}(z_2)(z_2)_t + u_t(L)J_{b_2}(z_2)(z_2)_t]^\top \nabla c_2(z_2)\,\mathrm{d}s \\
&\quad + \frac{1}{2}\nabla c_2(z_2(t))^\top b_2(z_2(t))u_t(t,L)^2 \\
&\quad - \frac{1}{2}\nabla c_2(z_2(0))^\top b_2(z_2(0))u_t(0,L)^2 \\
&\quad - \frac{1}{2}\int_0^t u_t(L)^2(z_2)_t^\top \left[J_{b_2}(z_2)^\top \nabla c_2(z_2) + H_{c_2}(z_2)b_2(z_2)\right]\mathrm{d}s \\
&\le C\int_0^t |u_t(L)|^2 + |(z_2)_t|^2\,\mathrm{d}s + \frac{1}{2}\nabla c_2(z_2(t))^\top b_2(z_2(t))u_t(t,L)^2 \\
&\quad - \frac{1}{2}\nabla c_2(z_2(0))^\top b_2(z_2(0))u_t(0,L)^2.
\end{aligned}$$

For the estimate of the second integral we have used the uniform boundedness of $(z_2)_t$, see the discussion before Step 1 of this proof. The uniform boundedness of I_1 follows analogously. Hence, $\tilde{V}(y_t(t))$ is uniformly bounded in time. Furthermore it can be seen that all the positive constants C appearing in the above calculations depend continuously on the initial conditions. This concludes the proof. □

In order to extend this result to all classical solutions, we need the following density argument.

Lemma 3.21. *For any $y \in D(\mathcal{A})$ there is a sequence $(y_n)_{n\in\mathbb{N}}$ in $D(\mathcal{A}^2)$ such that $\lim_{n\to\infty} y_n = y$ and $\lim_{n\to\infty} \mathcal{A}y_n = \mathcal{A}y$.*

Proof. Let an arbitrary $y \in D(\mathcal{A})$ be fixed. Notice that it suffices to show that there exists a sequence $(y_n)_{n\in\mathbb{N}}$ with $y_n = [u_n, v_n, z_{1n}, z_{2n}, \xi_n, \psi_n]^\top$ in $D(\mathcal{A}^2)$ such that $\lim_{n\to\infty} y_n = y$ in the space $H^4(0,L) \times H^2(0,L) \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R} \times \mathbb{R}$. The set $D(\mathcal{A}^2) = \{y \in D(\mathcal{A}) : \mathcal{A}y \in D(\mathcal{A})\}$ is equivalent to

$$v \in \tilde{H}_0^4(0,L), \tag{3.51}$$

$$u \in \tilde{H}_0^6(0,L) \wedge u^{\mathrm{IV}}(0) = u^{\mathrm{V}}(0) = 0, \tag{3.52}$$

$$\xi = Jv'(L), \tag{3.53}$$

$$\psi = Mv(L), \tag{3.54}$$

$$\Lambda u''(L) + [c_1(z_1) + d_1\Big(\frac{\xi}{J}\Big) + k_1(u'(L))] = \frac{\Lambda J}{\rho} u^{\mathrm{V}}(L), \tag{3.55}$$

$$-\Lambda u'''(L) + [c_2(z_2) + d_2\Big(\frac{\psi}{M}\Big) + k_2(u(L))] = \frac{\Lambda M}{\rho} u^{\mathrm{IV}}(L). \tag{3.56}$$

Since $\tilde{C}_0^\infty(0,L) := \{f \in C^\infty([0,L]) \colon f^{(k)}(0) = 0, \forall k \in \mathbb{N}_0\}$ is dense in $\tilde{H}_0^2(0,L)$ (see Theorem 3.17 in [AF03]), there exists a sequence $(v_n)_{n\in\mathbb{N}} \subset \tilde{C}_0^\infty(0,L)$ such that $\lim_{n\to\infty} v_n = v$ in $H^2(0,L)$. Also, v_n satisfies (3.51), for all $n \in \mathbb{N}$. Defining $\xi_n := Jv_n'(L)$ and $\psi_n := Mv_n(L)$ ensures that y_n satisfies (3.53) and (3.54). Moreover, the Sobolev embedding $H^2(0,L) \hookrightarrow C^1([0,L])$ implies that $\lim_{n\to\infty} \xi_n = \xi$ and $\lim_{n\to\infty} \psi_n = \psi$ as well. Next, let $z_{1n} := z_1$ and $z_{2n} := z_2$ for all $n \in \mathbb{N}$.

Finally, the sequence $(u_n)_{n\in\mathbb{N}} \subset C^\infty([0,L])$ will be constructed such that u_n satisfies (3.52), (3.55), and (3.56) for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} u_n = u$ in $H^4(0,L)$. To this end we introduce an auxiliary sequence of polynomial functions

$$h_n(x) := h_{2,n}x^2 + h_{3,n}x^3 + h_{6,n}x^6 + h_{7,n}x^7 + h_{8,n}x^8 + h_{9,n}x^9 + h_{10,n}x^{10} + h_{11,n}x^{11},$$

for all $n \in \mathbb{N}$, where $h_{2,n}, \dots, h_{11,n} \in \mathbb{R}$ are to be determined. It immediately follows that

$$h_n(0) = h_n'(0) = h_n^{\mathrm{IV}}(0) = h_n^{\mathrm{V}}(0) = 0. \tag{3.57}$$

Let $h_{2,n} = \frac{u''(0)}{2}$ and $h_{3,n} = \frac{u'''(0)}{6}$, which is equivalent to

$$h_n''(0) = u''(0),\ h_n'''(0) = u'''(0). \tag{3.58}$$

Further conditions are imposed on h_n:

$$h_n^{(k)}(L) = u^{(k)}(L), \qquad k \in \{0,1,2,3\}.$$

This can equivalently be written in terms of coefficients[1]:

$$h_{6,n} + h_{7,n}L + h_{8,n}L^2 + h_{9,n}L^3 + h_{10,n}L^4 + h_{11,n}L^5 = r_1, \tag{3.59a}$$

$$6h_{6,n} + 7h_{7,n}L + 8h_{8,n}L^2 + 9h_{9,n}L^3 + 10h_{10,n}L^4 + 11h_{11,n}L^5 = r_2, \tag{3.59b}$$

$$6^{\underline{2}}h_{6,n} + 7^{\underline{2}}h_{7,n}L + 8^{\underline{2}}h_{8,n}L^2 + 9^{\underline{2}}h_{9,n}L^3 + 10^{\underline{2}}h_{10,n}L^4 + 11^{\underline{2}}h_{11,n}L^5 = r_3 \tag{3.59c}$$

$$6^{\underline{3}}h_{6,n} + 7^{\underline{3}}h_{7,n}L + 8^{\underline{3}}h_{8,n}L^2 + 9^{\underline{3}}h_{9,n}L^3 + 10^{\underline{3}}h_{10,n}L^4 + 11^{\underline{3}}h_{11,n}L^5 = r_4, \tag{3.59d}$$

with

$$r_1 = \frac{u(L)}{L^6} - \frac{u''(0)}{2L^4} - \frac{u'''(0)}{6L^3}, \quad r_2 = \frac{u'(L)}{L^5} - \frac{u''(0)}{L^4} - \frac{u'''(0)}{2L^3},$$

[1] The coefficient $k^{\underline{l}}$ (the Pochhammer symbol, see [GKP89]) for $k, l \in \mathbb{N}$, $l \le k$ is defined by $k^{\underline{l}} := k \cdot (k-1) \cdots (k-l+1)$.

$$r_3 = \frac{u''(L)}{L^4} - \frac{u''(0)}{L^4} - \frac{u'''(0)}{L^3}, \quad r_4 = \frac{u'''(L)}{L^3} - \frac{u'''(0)}{L^3}.$$

We further require that h_n satisfies:

$$\frac{\Lambda M}{\rho} h_n^{\mathrm{IV}}(L) = -\Lambda u'''(L) + \Big[c_2(z_2) + d_2\Big(\frac{\psi_n}{M}\Big) + k_2(u(L))\Big] =: r_5, \tag{3.60}$$

$$\frac{\Lambda J}{\rho} h_n^{\mathrm{V}}(L) = \Lambda u''(L) + \Big[c_1(z_1) + d_1\Big(\frac{\xi_n}{J}\Big) + k_1(u'(L))\Big] := r_6, \tag{3.61}$$

where (3.60) and (3.61) are equivalent to:

$$6^{\underline{4}} h_{6,n} + 7^{\underline{4}} h_{7,n} L + 8^{\underline{4}} h_{8,n} L^2 + 9^{\underline{4}} h_{9,n} L^3 + 10^{\underline{4}} h_{10,n} L^4 + 11^{\underline{4}} h_{11,n} L^5 = \frac{r_5 \rho}{\Lambda M L^2}, \tag{3.62a}$$

$$6^{\underline{5}} h_{6,n} + 7^{\underline{5}} h_{7,n} L + 8^{\underline{5}} h_{8,n} L^2 + 9^{\underline{5}} h_{9,n} L^3 + 10^{\underline{5}} h_{10,n} L^4 + 11^{\underline{2}} h_{11,n} L^5 = \frac{r_6 \rho}{\Lambda J L}. \tag{3.62b}$$

Such h_n exists and is unique, due to the fact that linear system (3.59) and (3.62) has strictly positive determinant. Consequently, (3.57), (3.58), and (3.59) imply that $u - h_n \in H_0^4(0, L)$, for all $n \in \mathbb{N}$. Since $C_0^\infty(0, L)$ is dense in $H_0^4(0, L)$, there exists a sequence $(\tilde{u}_n)_{n\in\mathbb{N}} \subset C_0^\infty(0, L)$ such that $\|\tilde{u}_n - (u - h_n)\|_{H^4} < \frac{1}{n}$, for all $n \in \mathbb{N}$. Now defining $u_n := \tilde{u}_n + h_n$, gives $\lim_{n\to\infty} u_n = u$ in $H^4(0, L)$. Obviously u_n satisfies (3.52) for all $n \in \mathbb{N}$. Also, due to (3.60) and (3.61), u_n satisfies (3.55) and (3.56), as well. Hence, the statement follows. ☐

Theorem 3.22. *For all $y_0 \in D(\mathcal{A})$ the trajectory $\gamma(y_0)$ is precompact in $\mathcal{H}$.*

Proof. Let $y_0 \in D(\mathcal{A})$ be chosen arbitrarily, and let $(y_{n,0})_{n\in\mathbb{N}} \subset D(\mathcal{A}^2)$ be an approximating sequence as in Lemma 3.21. Then there holds in $\mathcal{H}$:

$$\lim_{n\to\infty} \mathcal{A} y_{n,0} = \mathcal{A} y_0. \tag{3.63}$$

For an arbitrary $T > 0$, and by applying Proposition 1.6 it follows that the approximating solutions $y_n(t)$ converge to $y(t)$ in $C([0,T]; \mathcal{H})$, where $y_n(t)$ and $y(t)$ are the solutions corresponding to the initial conditions $y_{n,0}$ and y_0, respectively. Since $y_n(t) \in C^1(\mathbb{R}^+; \mathcal{H})$ and solves (3.24), for all $n \in \mathbb{N}$, (3.63) yields

$$\lim_{n\to\infty} (y_n)_t(0) = \mathcal{A} y_0 \ \text{ in } \ \mathcal{H}. \tag{3.64}$$

Hence, (3.44) and (3.64) imply that there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$:

$$\sup_{t\geq 0} \|(y_n)_t(t)\|_{\mathcal{H}} \leq C(\|y_0\|_{\mathcal{H}}, \|\mathcal{A} y_0\|_{\mathcal{H}}),$$

where the constant C does not depend on n. From here it follows that $(y_n)_t$ is bounded in $L^\infty(\mathbb{R}^+; \mathcal{H})$. Hence, the Banach-Alaoglu Theorem (see Theorem I.3.15 in [Rud91]) implies that there exists $w \in L^\infty(\mathbb{R}^+; \mathcal{H})$ and a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ such that

$$(y_{n_k})_t \overset{*}{\rightharpoonup} w \ \text{ in } \ L^\infty(\mathbb{R}^+; \mathcal{H}).$$

For arbitrary $z \in \mathcal{H}$ and $t \geq 0$ there holds

$$\lim_{k\to\infty} \int_0^t \langle (y_{n_k})_t(\tau), z\rangle_{\mathcal{H}} \,\mathrm{d}\tau = \int_0^t \langle w(\tau), z\rangle_{\mathcal{H}} \,\mathrm{d}\tau,$$

which is equivalent to

$$\lim_{k\to\infty} \langle y_{n_k}(t) - y_{n_k}(0), z\rangle_{\mathcal{H}} = \Big\langle \int_0^t w(\tau)\,\mathrm{d}\tau, z\Big\rangle_{\mathcal{H}}.$$

Since $\lim_{n\to\infty} y_n(\tau) = y(\tau)$ (in $\mathcal{H}$) for all $\tau \in [0,\infty)$, it follows that

$$\langle y(t) - y(0), z\rangle_{\mathcal{H}} = \Big\langle \int_0^t w(\tau)\,\mathrm{d}\tau, z\Big\rangle_{\mathcal{H}}.$$

Since $z \in \mathcal{H}$ was arbitrary, we obtain

$$y(t) - y(0) = \int_0^t w(\tau)\,\mathrm{d}\tau, \quad \forall t \geq 0. \tag{3.65}$$

Due to continuous differentiability of y, the time derivative of (3.65) can be taken, which yields $y_t \equiv w$. This implies $y_t \in L^\infty(\mathbb{R}^+;\mathcal{H})$, i.e. $\|y_t(\cdot)\|_{\mathcal{H}}$ is uniformly bounded, which proves the theorem. □

Now it is straightforward to prove the main result of this chapter:

Theorem 3.23. *For every $y_0 \in D(\mathcal{A})$ there holds $\lim_{t\to\infty} \|S_{\mathcal{A}}(t)y_0\|_{\mathcal{H}} = 0$.*

Proof. According to Theorem 3.22 all classical trajectories are precompact. Furthermore, we have $\Omega = \{\mathbf{0}\}$, according to Theorem 3.14. Hence, we can apply LaSalle's Invariance Principle, cf. Theorem 1.19, which proves that every classical solution of (3.24) converges to 0 in $\mathcal{H}$ as $t \to \infty$. □

APPENDIX A

Deferred results from Chapter 2

A.1. Functional analytical results

Even though most of the analysis in Chapter 2 is carried out for real-valued functions u and as a consequence in the real Hilbert space $\mathcal{H}$ (see (2.9)), the spectral analysis of the occurring linear operators needs to be performed in a complex Hilbert space. This section contains some of those results. For the spectral analysis of the operator A, defined in (2.10), we introduce the complex Hilbert space

$$\mathcal{X} := \{y = [u, v, \xi, \psi]^\top : u \in \tilde{H}_0^2(0, L), v \in L^2(0, L), \xi, \psi \in \mathbb{C}\},$$

where $\tilde{H}_0^n$ is the complex version of the space $\tilde{H}_{0,\mathbb{R}}^n$ introduced in (1.1). $\mathcal{X}$ is equipped with the inner product

$$\langle y_1, y_2 \rangle_{\mathcal{X}} := \frac{\Lambda}{2} \int_0^L u_1'' \overline{u_2''} \,\mathrm{d}x + \frac{\rho}{2} \int_0^L v_1 \overline{v_2} \,\mathrm{d}x + \frac{1}{2J} \xi_1 \overline{\xi_2} + \frac{1}{2m} \psi_1 \overline{\psi_2}, \quad \forall y_1, y_2 \in \mathcal{X}.$$

For the operator A given by (2.10) we consider the natural continuation to $\mathcal{X}$, still denoted by A. This continuation still is of the form (2.10), and the domain is

$$D_{\mathbb{C}}(A) = \{y \in \mathcal{X} : u \in \tilde{H}_0^4(0, L), v \in \tilde{H}_0^2(0, L), \xi = Jv'(L), \psi = mv(L)\},$$

where the occurring Sobolev spaces now consist of complex valued functions.

Proposition A.1. *The linear operator A from* (2.10) *is skew-adjoint and has compact resolvent in $\mathcal{X}$. The spectrum $\sigma(A)$ consists of countably many eigenvalues $\{\lambda_n\}_{n\in\mathbb{Z}}$. They are all isolated and purely imaginary, and each eigenspace has finite dimension. The eigenvectors are an orthogonal basis of $\mathcal{X}$.*

Proof. It can easily be shown that for all $y_1, y_2 \in D_{\mathbb{C}}(A)$

$$\langle Ay_1, y_2 \rangle_{\mathcal{X}} = \frac{\Lambda}{2} \int_0^L v_1'' \overline{u_2''} - u_1'' \overline{v_2''} \,\mathrm{d}x = -\langle y_1, Ay_2 \rangle_{\mathcal{X}},$$

i.e. A is skew-symmetric. Straightforward calculations, analogous to those in [KT05], demonstrate that A is invertible and $A^{-1} : \mathcal{X} \to \mathcal{X}$ is even compact. So $0 \in \rho(A)$, and according to the corollary of Theorem VII.3.1 in [Yos80] this proves that A is skew-adjoint. Then, according to Theorem III.6.26 in [Kat66] the spectrum $\sigma(A)$ consists of countably many eigenvalues, which are all isolated. The corresponding eigenspaces are finite-dimensional, and the eigenvectors form an orthogonal basis according to Theorem V.2.10 in [Kat66]. □

The following is an extension of Lemma 2.3 from $\mathcal{H}$ to $\mathcal{X}$.

Lemma A.2. *The linear operator A generates a C_0-semigroup $(e^{tA})_{t\geq 0}$ of unitary operators on $\mathcal{X}$.*

Proof. From Proposition A.1 we know that A is skew-adjoint in $\mathcal{X}$. Then, we apply Stone's Theorem (see Theorem 3.24 in [EN00]), which proves that $(\mathrm{e}^{tA})_{t\geq 0}$ is a C_0-semigroup of unitary operators in $\mathcal{X}$. □

In the following, we turn to the spectral analysis of $\mathcal{B}$, defined in (2.50). To this end we introduce the Hilbert space

$$\tilde{\mathcal{X}} := \{w = [u, v, \xi]^\top : u \in \tilde{H}_0^2(0, L), v \in L^2(0, L), \xi \in \mathbb{C}\},$$

equipped with the inner product

$$\langle\langle w_1, w_2\rangle\rangle_{\tilde{\mathcal{X}}} := \frac{\Lambda}{2}\int_0^L u_1''\overline{u_2}''\,\mathrm{d}x + \frac{\rho}{2}\int_0^L v_1\overline{v_2}\,\mathrm{d}x + \frac{1}{2J}\xi_1\overline{\xi_2}.$$

$\tilde{\mathcal{X}}$ is the complex analogue to $\tilde{\mathcal{H}}$ from (2.49). The continuation of $\mathcal{B}$ to $\tilde{\mathcal{X}}$ is still denoted by $\mathcal{B}$ and given by (2.50), and has the domain

$$D_{\mathbb{C}}(\mathcal{B}) := \{y \in \tilde{\mathcal{X}} \colon u \in \tilde{H}_0^4(0, L), v \in \tilde{H}_0^2(0, L), \xi = Jv'(L), u'''(L) = 0\}.$$

Proposition A.3. *The operator $\mathcal{B}$ is skew-adjoint and has compact resolvent in $\tilde{\mathcal{X}}$. The spectrum $\sigma(\mathcal{B})$ consists entirely of isolated eigenvalues $\{\mu_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ located on the imaginary axis, and they have no accumulation point. All eigenspaces are one-dimensional, and the corresponding eigenfunctions form an orthogonal basis of $\tilde{\mathcal{X}}$. For every $n \in \mathbb{Z}\setminus\{0\}$ the normalized eigenfunction associated to μ_n is given by*

$$\Phi_n = \begin{bmatrix} u_n \\ \mu_n u_n \\ \mu_n J u_n'(L)\end{bmatrix}$$

where the real function $u_n \in \tilde{H}_0^4(0, L)$ is the unique (up to normalization) solution of the boundary value problem (2.52). Here, u_n is scaled such that $\|\Phi_n\|_{\tilde{\mathcal{X}}} = 1$.

Proof. Analogously to the proof of Proposition A.1 we can show that $0 \in \rho(\mathcal{B})$, that $\mathcal{B}^{-1}$ is compact in $\tilde{\mathcal{X}}$ and that $\mathcal{B}$ is skew-adjoint. According to Theorem III.6.26 in [Kat66] the spectrum $\sigma(\mathcal{B})$ consists of countably many eigenvalues $\{\mu_n\}_{n\in\mathbb{Z}\setminus\{0\}}$, which are all isolated. They come in complex conjugated pairs, i.e. $\mu_{-n} = \overline{\mu_n}$. The corresponding eigenspaces are finite-dimensional, and the eigenvectors form an orthogonal basis according to Theorem V.2.10 in [Kat66]. Since $\mathcal{B}$ is skew-adjoint we have $\sigma(\mathcal{B}) \subset \mathrm{i}\mathbb{R}$. Finally, the fact that the family $\{\Phi_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ is an orthogonal basis of $\tilde{\mathcal{X}}$ follows immediately from the application of Theorem V.2.10 in [Kat66].

Let $\Phi_n = [u_n, v_n, \xi_n]^\top \in D_{\mathbb{C}}(\mathcal{B})$ be an eigenfunction corresponding to μ_n for $n \in \mathbb{Z}\setminus\{0\}$, i.e. $\mathcal{B}\Phi_n = \mu_n\Phi_n$. First, it is necessary that u_n solves (2.52). The v_n and ξ_n can be determined from u_n via $v_n = \mu_n u_n$ and $\xi_n = J\mu_n u_n'(L)$. The system (2.52) has a non-trivial solution iff $\mu_n \in \sigma(\mathcal{B})$ (note that we have already shown that $0 \notin \sigma(\mathcal{B})$, i.e. we may assume $\mu_n \neq 0$). In this case we get the general solution $u_n \in \tilde{H}_0^4(0, L)$ of (2.52a) as

$$u_n(x) = C_1[\cosh px - \cos px] + C_2[\sinh px - \sin px], \tag{A.1}$$

where $p = \left(\frac{-\rho\mu_n^2}{\Lambda}\right)^{\frac{1}{4}} > 0$, and $C_i \in \mathbb{R}$ (see also the proof of Lemma 2.18). Here, we already incorporated the zero boundary conditions at $x = 0$. Using the condition $u_n'''(L) = 0$ from (2.52b) yields

$$C_1[\sinh pL - \sin pL] = -C_2[\cosh pL + \cos pL].$$

Since $p \neq 0$ due to $\mu_n \neq 0$, both sides are always nonzero. So C_2 can always be determined uniquely from C_1 via this equation. Thus, if (2.52) has a non-trivial solution, it is unique up to multiplicity. This shows that all eigenspaces of $\mathcal{B}$ are one-dimensional, spanned by the Φ_n. Finally, (2.52c) can be used to determine the μ_n for which there is a non-trivial solution, i.e. $\mu_n \in \sigma(\mathcal{B})$. $\square$

A.2. The differentiated system

Here we discuss the system (2.17) arising in the proof of Lemma 2.12:

$$\begin{aligned} \dot{z}(t) &= Az(t) + \tilde{\mathcal{N}}(t, z(t)), \\ z(0) &= z_0. \end{aligned} \tag{2.17}$$

The nonlinearity is given by

$$\tilde{\mathcal{N}}(t, z) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ F(t) + g(t)\chi \end{bmatrix},$$

and $z = [U, V, \zeta, \chi]^\top \in \mathcal{H}$. The time-dependent functions satisfy $F, g \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+)$. We now fix $T > 0$, and consider (2.17) for $t \in [0, T]$. Clearly, $\tilde{\mathcal{N}} : [0, T] \times \mathcal{H} \to \mathcal{H}$ is Lipschitz continuous (in both variables).

According to Theorem 6.1.2 in [Paz83], (2.17) has a unique mild solution $z(t)$ on $[0, T]$ for every initial condition $z_0 \in \mathcal{H}$. In the following we fix $z_0 \in D(A)$. Then we can apply Theorem 6.1.6 in [Paz83], which shows that this mild solution $z(t)$ is even a strong solution, and according to the proof of Theorem 6.1.6 in [Paz83] $z(t)$ is also Lipschitz continuous on $[0, T]$. In particular, the component $\chi(t)$ of $z(t)$ is Lipschitz continuous. For this given $\chi(t)$, corresponding to the fixed $z_0 \in D(A)$, we define the function:

$$f(t) := \begin{bmatrix} 0 \\ 0 \\ 0 \\ F(t) + g(t)\chi(t) \end{bmatrix}.$$

Clearly, this is now a Lipschitz continuous function on $[0, T]$. We reformulate (2.17) as the following equivalent initial value problem on $[0, T]$:

$$\dot{z}(t) = Az(t) + f(t), \tag{A.2a}$$

$$z(0) = z_0 \in D(A), \tag{A.2b}$$

with z_0 is the initial condition fixed before. The strong solution $z(t)$ of (2.17) obtained before is clearly also a strong solution of (A.2). Furthermore, this system has a unique mild solution according to Theorem 6.1.2 in [Paz83]. Now, since f is Lipschitz continuous and $\mathcal{H}$ is a Hilbert space (and hence reflexive), we can apply Corollary 4.2.11 in [Paz83], which shows that (A.2) has a unique classical solution. So, the strong solution $z(t)$ is even a classical solution.

Remark A.4. It turns out that the proof of Theorem 6.1.6 in [Paz83] proves an even stronger result than Theorem 6.1.6: In the formulation of Theorem 6.1.6, the expression "strong solution" can be replaced by "classical solution". This stems from the application of Corollary 4.2.11 in [Paz83] in the proof, which yields a classical solution, and not just a strong solution. The reasoning is essentially the same as the one given above, before this remark.

APPENDIX B

Deferred results from Chapter 3

B.1. The operator A_p

The system (3.33) is the mild formulation of the equation $(y_p)_t = A_p y_p$ with $y_p = [u, v]^\top \in \mathcal{H}_p$. The corresponding space is defined by $\mathcal{H}_p := \tilde{H}_0^2(0, L) \times L^2(0, L)$, and

$$A_p : \begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} v \\ -\frac{\Lambda}{\rho} u^{\mathrm{IV}} \end{bmatrix},$$

with the domain

$$\begin{aligned} D(A_p) = \{[u, v]^\top \in \mathcal{H}_p \colon u \in \tilde{H}_0^4(0, L), v \in \tilde{H}_0^2(0, L), \\ \Lambda u''(L) + \tilde{K}_1 u'(L) = 0, \Lambda u'''(L) - \tilde{K}_2 u(L) = 0\}. \end{aligned}$$

The space $\mathcal{H}_p$ is equipped with the following inner product:

$$\langle y_p, \tilde{y}_p \rangle_p := \Lambda \int_0^L u'' \tilde{u}'' \, \mathrm{d}x + \rho \int_0^L v \tilde{v} \, \mathrm{d}x + \tilde{K}_1 u'(L) \tilde{u}'(L) + \tilde{K}_2 u(L) \tilde{u}(L). \tag{B.1}$$

The constants $\tilde{K}_1, \tilde{K}_2$ are defined in (3.32) and depend, at first glance, on the fixed $y_0 \in \Omega$ in the proof of Theorem 3.14. Hence, $D(\mathcal{A}_p)$ and the above inner product also depend on y_0. But this does not cause any problems. Anyhow, Step 2 in the proof of Theorem 3.14 shows that $u_0(L) = u_0'(L) = 0$. Hence, $\tilde{K}_j = K_j$.

We have the following results:

Lemma B.1. *The operator $A_p^{-1} : \mathcal{H}_p \to D(A_p)$ exists and is a bijection. Furthermore, A_p^{-1} is compact in $\mathcal{H}_p$.*

Proof. The proof is analogous to the proof of Lemma 3.2, see also Section 4.2 in [KT05]. □

Lemma B.2. *The operator A_p is skew-adjoint.*

Proof. First we show that A_p is skew-symmetric, i.e. for all $y, \tilde{y} \in D(A_p)$ there holds $\langle A_p y, \tilde{y} \rangle_p = -\langle y, A_p \tilde{y} \rangle_p$:

$$\begin{aligned} \langle A_p y, \tilde{y} \rangle_p &= \Lambda \int_0^L v'' \tilde{u}'' \, \mathrm{d}x - \Lambda \int_0^L u^{\mathrm{IV}} \tilde{v} \, \mathrm{d}x + \tilde{K}_1 v'(L) \tilde{u}'(L) + \tilde{K}_2 v(L) \tilde{u}(L) \\ &= \Lambda \Big(\int_0^L v \tilde{u}^{\mathrm{IV}} \, \mathrm{d}x + v'(L) \tilde{u}''(L) - v(L) \tilde{u}'''(L) - \int_0^L u'' \tilde{v}'' \, \mathrm{d}x \Big) \\ &\quad - \Lambda u'''(L) \tilde{v}(L) + \Lambda u''(L) \tilde{v}'(L) + \tilde{K}_1 v'(L) \tilde{u}'(L) + \tilde{K}_2 v(L) \tilde{u}(L). \end{aligned}$$

Using the boundary conditions $\Lambda u''(L) + \tilde{K}_1 u'(L) = 0$ and $\Lambda u'''(L) - \tilde{K}_2 u(L) = 0$ from $D(A_p)$ we obtain:

$$\begin{aligned}\langle A_p y, \tilde{y}\rangle_p = \Lambda \int_0^L v\tilde{u}^{\mathrm{IV}}\,\mathrm{d}x - \tilde{K}_1 v'(L)\tilde{u}'(L) - \tilde{K}_2 v(L)\tilde{u}(L) - \Lambda \int_0^L u''\tilde{v}''\,\mathrm{d}x \\ - \tilde{K}_2 u(L)\tilde{v}(L) - \tilde{K}_1 u'(L)\tilde{v}'(L) + \tilde{K}_1 v'(L)\tilde{u}'(L) + \tilde{K}_2 v(L)\tilde{u}(L) \\ = -\langle y, A_p\tilde{y}\rangle_p.\end{aligned}$$

So A_p is skew-symmetric. Furthermore, due to Lemma B.1 we know that ran $A_p = \mathcal{H}_p$. So we can apply the Corollary of Theorem VII.3.1 in [Yos80], which proves the skew-adjointness of A_p. □

Lemma B.3. *A_p generates a C_0-semigroup of unitary operators in $\mathcal{H}_p$.*

Proof. Since A_p is skew-adjoint, this follows from Stone's theorem [EN00, Theorem II.3.24]. □

Bibliography

[AF03] ADAMS R.A., FOURNIER J.J.F. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Second edition. Elsevier/Academic Press, Amsterdam, 2003.

[Bat00] BATCHELOR G.K. *An introduction to fluid dynamics*. Cambridge University Press, 2000. Cambridge Books Online.

[CC99] CHENTOUF B., COUCHOURON J.F. Nonlinear feedback stabilization of a rotating body-beam without damping. *ESAIM: Control, Optimisation and Calculus of Variations*, **4** (1999), pp. 515–535.

[CdN98] CORON J.M., D'ANDREA NOVEL B. Stabilization of a rotating body beam without damping. *Automatic Control, IEEE Transactions on*, **43**, no. 5 (1998), pp. 608–618.

[CH98] CAZENAVE T., HARAUX A. *An introduction to semilinear evolution equations*, volume 13 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, New York, 1998.

[CLW13] CHUESHOV I., LASIECKA I., WEBSTER J.T. Evolution semigroups in supersonic flow-plate interactions. *J. Differential Equations*, **254**, no. 4.

[CM98] CONRAD F., MORGÜL Ö. On the stabilization of a flexible beam with a tip mass. *SIAM Journal on Control and Optimization*, **36**, no. 6 (1998), pp. 1962–1986.

[Cou02] COUCHOURON J.F. Compactness theorems for abstract evolution problems. *Journal of Evolution Equations*, **2**, no. 2 (2002), pp. 151–175.

[CP69] CRANDALL M.G., PAZY A. Semi-groups of nonlinear contractions and dissipative sets. *J. Functional Analysis*, **3** (1969), pp. 376–418.

[CP94] CONRAD F., PIERRE M. Stabilization of second order evolution equations by unbounded nonlinear feedback. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **11**, no. 5 (1994), pp. 485–515.

[CP24] CONRAD F., PIERRE M. Stabilization of Euler-Bernoulli beam by nonlinear boundary feedback. Research Report RR-1235, 1990. `hal.inria.fr/inria-00075324`.

[DS73] DAFERMOS C.M., SLEMROD M. Asymptotic behavior of nonlinear contraction semi-groups. *J. Functional Analysis*, **13** (1973), pp. 97–106.

[EN00] ENGEL K.J., NAGEL R. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[GKP89] GRAHAM R.L., KNUTH D.E., PATASHNIK O. *Concrete Mathematics*. Addison-Wesley, Massachusetts, 1989.

[Gra09] GRABOWSKI P. The motion planning problem and exponential stabilisation of a heavy chain. part i. *International Journal of Control*, **82**, no. 8 (2009), pp. 1539–1563.

[Guo02a] GUO B. On the boundary control of a hybrid system with variable coefficients. *Journal of Optimization Theory and Applications*, **114**, no. 2 (2002), pp. 373–395.

[Guo02b] GUO B.Z. Riesz basis property and exponential stability of controlled Euler-Bernoulli beam equations with variable coefficients. *SIAM Journal on Control and Optimization*, **40**, no. 6 (2002), pp. 1905–1923.

[GW06] GUO B.Z., WANG J.M. Riesz basis generation of abstract second-order partial differential equation systems with general non-separated boundary conditions. *Numerical Functional Analysis and Optimization*, **27**, no. 3-4 (2006), pp. 291–328.

[GZH11] GE S.S., ZHANG S., HE W. Modeling and control of an Euler-Bernoulli beam under unknown spatiotemporally varying disturbance. In *American Control Conference (ACC), 2011*. IEEE, 2011, pp. 2988–2993.

[Joh82] JOHN F. *Partial differential equations*, volume 1 of *Applied Mathematical Sciences*. Fourth edition. Springer, New York, 1982.

[Kat66] KATO T. *Perturbation theory for linear operators*, volume 132 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, New York, 1966.

[KT05] KUGI A., THULL D. Infinite-dimensional decoupling control of the tip position and the tip angle of a composite piezoelectric beam with tip mass. In *Control and Observer Design for Nonlinear Finite and Infinite Dimensional Systems*, volume 322 of *Lecture Notes in Control and Information Science*. Springer, Berlin Heidelberg, 2005, pp. 351–368.

[LBEM00] LOZANO R., BROGLIATO B., EGELAND O., MASCHKE B. *Dissipative Systems Analysis and Control*. Springer, London, 2000.

[LGM99] LUO Z.H., GUO B.Z., MORGÜL Ö. *Stability and stabilization of infinite dimensional systems with applications*. Communications and Control Engineering Series, Springer, London, 1999.

[LM88] LITTMAN W., MARKUS L. Stabilization of a hybrid system of elasticity by feedback boundary damping. *Ann. Mat. Pura Appl. (4)*, **152** (1988), pp. 281–330.

[MA15] MILETIĆ M., ARNOLD A. A piezoelectric Euler-Bernoulli beam with dynamic boundary control: Stability and dissipative FEM. *Acta Applicandae Mathematicae*, **138** (2015), pp. 241–277.

[Mor01] MORGÜL Ö. Stabilization and disturbance rejection for the beam equation. *IEEE Transactions on Automatic Control*, **46**, no. 12 (2001), pp. 1913–1918.

[MSA15] MILETIĆ M., STÜRZER D., ARNOLD A. An Euler-Bernoulli beam with nonlinear damping and a nonlinear spring at the tip. *Discrete Contin. Dyn. Syst. Ser. B*, **20**, no. 9 (2015).

[MSAK15] MILETIĆ M., STÜRZER D., ARNOLD A., KUGI A. Stability of an Euler-Bernoulli beam with a nonlinear dynamic feedback system. *Conditionally accepted in: IEEE Transactions on Automatic Control*.

[Nir59] NIRENBERG L. On elliptic partial differential equations. *Ann. Scuola Norm. Sup. Pisa (3)*, **13** (1959), pp. 115–162.

[OASKH08] OTT C., ALBU-SCHAFFER A., KUGI A., HIRZINGER G. On the passivity-based impedance control of flexible joint robots. *IEEE Transactions on Robotics*, **24**, no. 2 (2008), pp. 416–429.

[OvdSME02] ORTEGA R., VAN DER SCHAFT A., MASCHKE B., ESCOBAR G. Interconnection and damping assignment passivity-based control of port-controlled Hamiltonian systems. *Automatica*, **38**, no. 4 (2002), pp. 585–596.

[Paz75] PAZY A. A class of semi-linear equations of evolution. *Israel J. Math.*, **20** (1975), pp. 23–36.

[Paz81] PAZY A. The Lyapunov method for semigroups of nonlinear contractions in Banach spaces. *J. Analyse Math.*, **40** (1981), pp. 239–262 (1982).

[Paz83] PAZY A. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer, New York, 1983.

[Rao95] RAO B. Uniform stabilization of a hybrid system of elasticity. *SIAM Journal on Control and Optimization*, **33**, no. 2 (1995), pp. 440–454.

[RLGMZ14] RAMIREZ H., LE GORREC Y., MACCHELLI A., ZWART H. Exponential stabilization of boundary controlled port-hamiltonian systems with dynamic feedback. *IEEE Transactions on Automatic Control*, **59**, no. 10 (2014), pp. 2849–2855.

[Rud91] RUDIN W. *Functional analysis*. Second edition. International series in pure and applied mathematics, McGraw-Hill, New York, 1991.

[TV03] THIEME H.R., VRABIE I.I. Relatively compact orbits and compact attractors for a class of nonlinear evolution equations. *J. Dynam. Differential Equations*, **15**, no. 4 (2003), pp. 731–750.

[vdS00] VAN DER SCHAFT A. *L2-gain and passivity techniques in nonlinear control*. Second edition. Communications and Control Engineering, Springer, London, 2000.

[VZLGM09] VILLEGAS J.A., ZWART H., LE GORREC Y., MASCHKE B. Exponential stability of a class of boundary control systems. *IEEE Transactions on Automatic Control*, **54**, no. 1 (2009), pp. 142–147.

[Web79] Webb G.F. Compactness of bounded trajectories of dynamical systems in infinite-dimensional spaces. *Proc. Roy. Soc. Edinburgh Sect. A*, **84**, no. 1-2 (1979), pp. 19–33.

[Yos80] Yosida K. *Functional analysis*, volume 123 of *Grundlehren der Mathematischen Wissenschaften*. Sixth edition. Springer-Verlag, Berlin, 1980.

Wiedner Hauptstraße 8
1040 Wien, Austria
☎ +43 1 58801 10176
✉ dominik.stuerzer@tuwien.ac.at

Dominik Stürzer

Personal Information

Citizenship Austrian

Education

Jul 2010 - **PhD student in Applied Mathematics**, *Vienna University of Technology*.
Supervisor: Prof. Anton Arnold.

Nov 2009 **Master's degree in Technical Mathematics (Dipl.-Ing.)**, *Vienna University of Technology*.
Graduation with highest distinction.

Jan 2008 - Jun 2008 **Exchange semester (Erasmus)**, *Université de la Méditerranée, Aix-Marseille II*, Marseille, France.

2004 - 2009 **Student of Technical Mathematics**, *Vienna University of Technology*.

Publications & Preprints

[1] M. Miletić, D. Stürzer, A. Arnold, A. Kugi. Stability of an Euler-Bernoulli beam with a nonlinear dynamic feedback system. Conditionally accepted in *IEEE Transactions on Automatic Control*. Preprint: arxiv.org/abs/1505.07576.

[2] M. Miletić, D. Stürzer, A. Arnold. An Euler-Bernoulli beam with nonlinear damping and a nonlinear spring at the tip. *Discrete and Continuous Dynamical System - B 20*, 9 (2015). Preprint: arxiv.org/abs/1411.7946.

[3] F. Achleitner, A. Arnold, D. Stürzer. Large-time behavior in non-symmetric Fokker-Planck equations. *Rivista di Matematica della Università di Parma 6*, 1 (2015) Preprint: arxiv.org/abs/1506.02470.

[4] D. Stürzer, A. Arnold. Spectral analysis and long-time behaviour of a Fokker-Planck equation with a nonlocal perturbation. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 25*, 1 (2014), 53–89. See also: arxiv.org/abs/1207.5686.

[5] D. Stürzer, A. Arnold, A. Kugi. Stability of a Closed-Loop Control System – Applied to a Gantry Crane with Heavy Chains. *Proceedings of the Junior Scientist Conference*, (2010).

[6] D. Stürzer. Stability of a closed-loop control system. *Diploma Thesis*, Vienna University of Technology (2009).

[7] D. Stürzer. Perfect Numbers. *Die Wurzel*, 8 (2007).

[8] D. Stürzer. Geometrical Analysis of the Length of Dawn. *Plus Lucis*, 2 (2004).

Current Research Topics

Control theory	Asymptotic stability of mechanical systems coupled to a feedback control. Particular systems: Euler-Bernoulli beam, gantry crane.
Fokker-Planck equations	Existence and uniqueness of solutions, asymptotic behavior.
Wigner equation	Existence and uniqueness of solutions of the stationary Wigner equation.

Conferences & Workshops (Short Selection)

Sep 2015 **Workshop *Nonlocal Nonlinear Partial Differential Equations and Applications***, Anacapri, Italy.
Talk: *Fokker-Planck equations with nonlocal perturbations*

Oct 2014 **Workshop *Dispersive equations with nonlocal dispersion – III***, *WPI Vienna*, Vienna, Austria.
Talk: *Spectral Analysis and Long-Time Behavior of a Linear Fokker-Planck Equation with a Non-Local Perturbation*

Jun 2014 **Summer School *Methods & Models of Kinetic Theory***, *Porto Ercole/Politecnico di Torino*, Italy.
Poster: *Spectral Analysis and Long-Time Behavior of a Fokker-Planck Equation with a Non-Local Perturbation*

Jun 2014 **Workshop *Advances in Nonlinear PDEs: Analysis, Numerics, Stochastics, Applications***, *Vienna University of Technology/University of Vienna*, Vienna, Austria.
Poster: *Spectral Analysis and Asymptotic Behavior of a Fokker-Planck Equation with Non-Local Perturbations*

Apr 2014 **100th European Study Group with Industry (ESGI100)**, *University of Oxford*, Oxford, UK.
Modeling topic: *Saturation in Liquid/Gas Coalescence, Pall Corp.*

Apr 2014 **6th UK Graduate Modeling Camp**, *University of Oxford*, Oxford, UK.
Modeling topic: *Melting behavior of a debris covered glacier*

Dec 2013 **Workshop *Classical and Quantum Mechanical Models of Many-Particle Systems***, *Mathematisches Forschungsinstitut Oberwolfach*, Oberwolfach, Germany.

Jun 2013 **DK Summer Camp in Weißensee**, *University of Vienna/Weißensee*, Weißensee, Austria.
Talk: *Analysis of a Fokker-Planck Equation with a Non-Local Perturbation*

May 2013 **Workshop *Theory and Numerics of Kinetic Equations***, *Saarland University*, Saarbrücken, Germany.
Talk: *Spectral Analysis and Long-Time Behavior of a Fokker-Planck Equation with a Non-Local Perturbation.*

Apr 2010 **Conference *Junior Scientist Conference***, *Vienna University of Technology*, Vienna, Austria.
Poster: *Stability of a Closed Loop Control System - Applied to a Gantry Crane with Heavy Chains* (topic of my diploma thesis).

Feb 2004 **Student Workshop *Applied Mathematics***, *Johannes Kepler University*, Linz, Austria.
Modeling and optimization of a level-luffing crane.

Teaching Experience (Exercise Classes)

2010 – 2013 **Analysis 1–3 for Technical Mathematics, Functional Analysis**, *Vienna University of Technology*.

2008 **Partial Differential Equations**, *Vienna University of Technology*.
Winter term.

2007 & 2008 **Analysis 1 for Physicists**, *Vienna University of Technology*.
Winter terms.

2006 & 2007 **Mathematics 1 for Computer Scientists**, *Vienna University of Technology*.
Winter terms.

Scholarships & Prizes

2006 & 2007 **2 Excellence Scholarships ("Leistungsstipendium")**, *Faculty of Mathematics and Geoinformation, Vienna University of Technology*.

2003 & 2004 **Austrian Mathematical Olympiad, Austrian Physics Olympiad**.
2004: 1st place in the West-Austrian Mathematics Olympiad, 2nd place in the Regional Physics Olympiad

Further Experience

Jan 2010 - Jun 2010 **Military Scientific Expert ('MilWEx')**, *NBC-Defense School*, Korneuburg, Austria.
Scientific position during the obligatory military service.

Languages

German Native

French Native

English Professional proficiency

Vienna, October 12, 2015