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Entropy Method and Large Time Behavior of the Vorticity Equation

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durch

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Chapter 1

Introduction

The aim of this thesis is to discuss the long time asymptotics of solutions of the vorticity equation in two and three dimensions.
This is of interest, because the vorticity equation is connected to the well-known Navier-Stokes equations. These equations are used to describe the velocity and pressure of certain flows and therefore, the study of long time asymptotics of these equations is relevant for practical applications. Beside the obvious usage in fluid dynamics, the Navier-Stokes equations are for example also used in image processing to automatically fill missing parts of images using the remaining information around. However, in this thesis the focal point is the theoretical analysis and some mathematical consequences of the large time behavior will be mentioned as well.

The Navier-Stokes equations for an incompressible, homogeneous and viscous fluid are given as

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \Delta \mathbf{u} \tag{1.1}$$

$$\operatorname{div}(\mathbf{u}) = 0, \tag{1.2}$$

where $\mathbf{u} = \mathbf{u}(x,t) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is the velocity and $p = p(x,t) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is the pressure and the viscosity has been rescaled to 1.

In our further observations we are much more interested in studying the vorticity equation rather than the Navier-Stokes equations. The vorticity equation can be easily derived from equation (1.1) by taking the curl of the equation above. Setting $\omega = \operatorname{rot}(\mathbf{u}) = \partial_{x_1} u_2 - \partial_{x_2} u_1$ in $\mathbb{R}^2$ the equations above become the vorticity equation in two dimensions

$$\omega_t + (\mathbf{u} \cdot \nabla)\omega = \Delta\omega. \tag{1.3}$$

Studying the vorticity equation instead of the equation for the velocity has a lot of advantages. In fact, the pressure has been eliminated and we just have a scalar nonlinear equation instead of a coupled system of three nonlinear equations.

A particular solution of the equations $\operatorname{div}(\mathbf{u}) = 0$, $\operatorname{rot}(\mathbf{u}) = \omega$ using Green's-Function can be obtained as

$$\mathbf{u}(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(y) dy, \quad x \in \mathbb{R}^2,$$

which is also known as the Biot-Savart law. So equation (1.3) is a nonlinear second order partial differential equation with a somewhat quadratic nonlinearity.

In the three dimensional case the curl is a vector and setting $\boldsymbol{\omega} = \mathrm{rot}(\mathbf{u})$ leads to the vorticity equation in three dimensions

$$\boldsymbol{\omega}_t = \Delta\boldsymbol{\omega} - (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} + (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} \tag{1.4}$$

with $\mathrm{div}(\boldsymbol{\omega}) = 0$. Just as in the two dimensional case the velocity field can be reconstructed from the vorticity by

$$\mathbf{u}(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(x-y)\times\boldsymbol{\omega}(y)}{|x-y|^3}dy, \quad x\in\mathbb{R}^3.$$

The difference to the two dimensional case is that in this case we have a system of three equations instead of a scalar equation and we get an additional nonlinear term.

In this thesis we will discuss the large time behavior, in particular the convergence to a steady state, of solutions to the equations (1.3), (1.4) using an entropy approach. Such entropy methods to compute the large time behavior of solutions of partial differential equations have become more and more popular in recent years. The main idea is to find a Ljapunov functional for the equation (the 'entropy' functional) and compute the second derivative of this functional along the trajectories. For some well known equations (e.g. Fokker-Planck type equations) some estimates on the second derivative lead to an inequality involving the second derivative and the first derivative of the Ljapunov functional. Then integration leads to exponential decay in the entropy and afterwards a so called Csiszár-Kullback inequality leads to decay in L^1. In fact, this has been closely studied in [AMTU] for a large class of Fokker-Planck equations and a non-symmetric extension has been discussed in [ACJ]. Further entropy methods for some nonlinear Fokker-Planck equations have been analyzed in [CJMTU].
Such equations are of interest in our case, since we will transform the vorticity equation into a nonlinear Fokker-Planck equation and study the large time behavior of the transformed equation, for which we can use some functional inequalities, which are stated in the mentioned papers. Nonetheless, our transformed equation is not covered by the analysis made in [ACJ] or [CJMTU] since some crucial constraints are not satisfied.

A big advantage of entropy methods over spectral methods, which can be used to study large time behavior as well (for example for the heat equation), is that they work well for some nonlinear equations and since the vorticity equation is nonlinear, this fits well for our studies.

The large time behavior of the vorticity equation has been studied in various papers with a lot of different techniques. For example in [GW1], [GW3] an approach using invariant manifolds has been made for both the two dimensional and the three dimensional case. In [GW2] some Ljapunov functionals have been used similar to the observations in the second chapter of this thesis. One should mention the interesting paper [Rod] as well, where large time behavior for a non-homogeneous fluid has been studied, as well as the paper [Rou], where some two dimensional results have been used for the vorticity equation on the three dimensional layer $\mathbb{R}^2\times(0,1)$.

Our study starts with solutions of the two dimensional vorticity equation, which do not change sign. For those solutions we use a specific Ljapunov functional, which leads to a case similar as in [AMTU] and so we get exponential decay to a steady state. In chapter 3 we will allow solutions that change sign. Due to some difficulties, which arise because of the mixed sign of the solutions, we have not found general results for those solutions concerning convergence rates in the literature as well as in our studies and therefore we decided to make some additional assumptions. In particular, we either have to study solutions of certain structure or solutions with small initial data. For those solutions we either get similar results as in chapter 2 or some reduced convergence rates.

The fourth chapter is devoted to the study of some spectral properties of the linear Fokker-Planck operator in order to show that some particular solutions converge even faster if the solutions do not have a component in the first eigenspaces of the linear Fokker-Planck operator. The interesting part of this analysis is that for these solutions the vorticity equation behaves just like a linear equation.

Finally, the three dimensional case is studied in the last chapter. Since the vorticity equation in three dimensions comes from the Navier-Stokes equations in $\mathbb{R}^3$, finding global results is quite difficult. As it is well known, in three dimensions the existence of a unique global smooth solution is yet unknown and one of the Millennium Prize Problems of the Clay Mathematics Institute. Nonetheless, we try to generalize our results from the two dimensional case for solutions that fulfill some additional assumptions. Finally, we want to present some other results we found in literature concerning large time asymptotics in three dimensions, which are not derived with entropy methods.

Chapter 2

Entropy method and large time behavior for positive solutions in two dimensions

In this chapter we want to establish some basic theorems concerning existence and a maximum principle as well as some useful estimates, which will be used throughout the whole article. Further, we want to study the large time behavior of solutions with nonnegative or nonpositive initial conditions using a Ljapunov functional, the logarithmic relative entropy.

At first we cite an existence theorem for the vorticity equation. Existence and uniqueness theorems for the Navier-Stokes equations and, respectively, the vorticity equation in two dimensions are well known. The sources [Cot], [GMO], [B-A, Bre] all have proven such theorems under various assumptions and we will state one of those theorems.

Theorem 2.1. *Let the initial velocity* $\mathbf{u}_0$ *fulfill* $\sup_{\lambda>0} \lambda \left(meas\{x : |\mathbf{u}_0(x)| > \lambda\} \right)^{\frac{1}{2}} < \infty$ *or in other words if* $\mathbf{u}_0$ *lies in the Lorentz space* $L^{2,\infty}(\mathbb{R}^2)$ *and* $div(\mathbf{u}_0) = 0$. *Further assume that the initial vorticity* ω_0 *is a finite measure. Then the vorticity equation* (1.3) *has a global-in-time solution* ω, *which satisfies*

1. $\omega : [0, \infty) \to \mathcal{M}$ *is bounded and continuous under the weak topology.*
2. *The corresponding velocity field* $\mathbf{u} : [0, \infty) \to L^{2,\infty}(\mathbb{R}^2)$ *is bounded and continuous under the weak* topology.*
3. *The solution can be represented as*

$$\omega(x,t) = \int_{\mathbb{R}^2} \Gamma(x,t;y)\omega_0(y)dy$$

with a continuous function $\Gamma(x,t;y)$ *satisfying* $\int_{\mathbb{R}^2} \Gamma(x,t;y)dy = \int_{\mathbb{R}^2} \Gamma(x,t;y)dx = 1$ *for all* $t > 0$. *Moreover there exist constants* $C_1, C_2, C_3, C_4 > 0$ *such that*

$$C_3 \frac{1}{t} exp\left(-C_4 \frac{|x-y|^2}{t}\right) \geq \Gamma(x,t;y) \geq C_1 \frac{1}{t} exp\left(-C_2 \frac{|x-y|^2}{t}\right)$$

holds, where the constants only depend on a bound for $\|\omega_0\|_{\mathcal{M}}$.

Further, the solution is unique, if ω_0 is a continuous measure.

Proof: The stated theorem is a combination of the Theorems 4.2, 4.3, 4.5 in [GMO]. $\square$

To study the large time behavior of solutions of equation (1.3) it is more convenient to transform the equation to a nonlinear Fokker-Planck equation, because entropy methods for linear Fokker-Planck equations are well known. Most important, they have the big advantage that for those equations a non-trivial steady state exists instead of a time dependent asymptotic state and entropy methods can be used to show convergence to this steady state.

Using the scaling variables as in [GW1]

$$\xi = \frac{x}{\sqrt{1+t}}, \quad \tau = \ln(1+t) \tag{2.1}$$

and setting

$$\begin{aligned} \omega(x,t) &= \frac{1}{1+t} w\left(\frac{x}{\sqrt{1+t}}, \ln(1+t)\right) \\ \mathbf{u}(x,t) &= \frac{1}{\sqrt{1+t}} \mathbf{v}\left(\frac{x}{\sqrt{1+t}}, \ln(1+t)\right), \end{aligned}$$

a simple computation shows that $w = w(\xi, \tau)$ satisfies the equation

$$w_\tau = \Delta w + \frac{1}{2}(\xi \cdot \nabla) w + w - (\mathbf{v} \cdot \nabla) w \tag{2.2}$$

with the velocity field

$$\mathbf{v}(\xi, \tau) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(\xi - y)^\perp}{|\xi - y|^2} w(y, \tau) dy, \quad \xi \in \mathbb{R}^2.$$

We will sometimes write $\mathbf{v}[w](\xi)$ to emphasize to which vorticity w the velocity field $\mathbf{v}$ is corresponding. The linear part of equation (2.2) is a Fokker-Planck equation with the quadratic potential $\frac{1}{4}|\xi|^2$.

For sake of simple notation we will write x, t for the transformed variables ξ, τ. Further equation (2.2) can be written in divergence form as

$$w_t = \operatorname{div}\left(\nabla w + \frac{1}{2} x w - \mathbf{v} w\right), \tag{2.3}$$

since $\operatorname{div}(\mathbf{v}) = 0$. This immediately leads to conservation of mass

$$\int_{\mathbb{R}^2} w(t,x) dx = \int_{\mathbb{R}^2} w_0(x) dx.$$

A normalized steady state of equation (2.3) can be found as

$$G(x) = \frac{1}{4\pi} e^{-\frac{1}{4}|x|^2},$$

which is a steady state of the linear Fokker-Planck equation satisfying $\mathbf{v}[G] \cdot \nabla G = 0$. Since every cG is a steady state and the equation conserves mass, we write $w_\infty(x) = \int_{\mathbb{R}^2} w_0(y)dyG(x)$ as the steady state associated to the initial condition w_0.
An appropriate function space we will use later on is the weighted L^2-space with the weight G^{-1}, which we will denote by $L^2(G^{-1})$.

The study of the large time behavior of linear Fokker-Planck equations using an entropy approach is made in detail in [AMTU]. One should note, that the non-symmetric part of the equation, which is here only the nonlinearity, does not satisfy the crucial constraint formulated in [ACJ], namely that in general $\operatorname{div}(\mathbf{v}w_\infty) \neq 0$, so the results there cannot be used directly. Nonetheless, we will show that in our case this constraint is not needed to prove the same results.

The following lemma states a maximum principle for equation (2.3).

Lemma 2.2. *Suppose $w_0 \in L^1(\mathbb{R}^2)$ and that $w_0 \geq 0$, then $w \equiv 0$ or $w(x,t) > 0$ for all $x, t > 0$.*

Proof: For the Navier-Stokes equations, the assumptions of Theorem 3.10 in [PW] are fulfilled, if we assume w_0 to be in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$, because of the conservation of mass and the boundedness of the solution. Therefore, the result holds for the solution of the Navier-Stokes equations and also after scaling for the Fokker-Planck equation. Since the solution depends continuously on the initial data in $L^1(\mathbb{R}^2)$, the theorem holds for initial conditions in $L^1(\mathbb{R}^2)$ too. □

Definition 2.3. Let J be $\mathbb{R}$ or $\mathbb{R}^+$ and $\psi \in C(\overline{J}) \cap C^4(J)$ with $\psi \geq 0, \psi(1) = 0, \psi'' > 0$ in J and $\left(\frac{1}{\psi''}\right)'' \leq 0$. Further let $w_1 \in L^1(\mathbb{R}^n)$ and $w_2 \in L^1_+(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} w_1 dx = \int_{\mathbb{R}^n} w_2 dx = 1$ and $\frac{w_1}{w_2} \in J$. Then

$$e_\psi(w_1|w_2) := \int_{\mathbb{R}^2} \psi\left(\frac{w_1}{w_2}\right) w_2 dx \geq 0$$

is called an admissible relative entropy of w_1 with respect to w_2 with generating function ψ.

The following two inequalities will be very useful for our analysis and hold for both $n = 2, 3$.

Lemma 2.4 (Generalized Csiszár-Kullback inequality). *For all admissible entropies e_ψ and $w_1, w_2 \in L^1(\mathbb{R}^n)$ satisfying the assumptions of the definition above we have*

$$\frac{1}{2}\|w_1 - w_2\|^2_{L^1(\mathbb{R}^n)} \leq \frac{1}{\psi''(1)} e_\psi(w_1|w_2). \tag{2.4}$$

Proof: A proof can be found in [UAMT]. □

Lemma 2.5 (Convex Sobolev inequalities). *Let ψ be an admissible entropy generating function and $w \in L^1(\mathbb{R}^n)$, $w_\infty \in L^1_+(\mathbb{R}^n)$ with $\frac{w}{w_\infty} \in J$. Then the inequality*

$$\int_{\mathbb{R}^n} \psi\left(\frac{w}{w_\infty}\right) w_\infty dx \leq \int_{\mathbb{R}^n} \psi''\left(\frac{w}{w_\infty}\right) \left|\nabla \frac{w}{w_\infty}\right|^2 w_\infty dx \tag{2.5}$$

holds.

Proof: Such inequalities can be obtained by various ways. The idea we want to sketch is that these inequalities are a byproduct of the entropy method and are deduced by computing the second derivative of an admissible entropy.
In [AMTU] it is shown that for a linear Fokker-Planck equation with the quadratic potential $\frac{1}{4}|x|^2$ (in fact in the paper it is shown for a much larger class of Fokker-Planck equations) we have

$$\frac{d^2}{dt^2} e_\psi(w(t)|w_\infty) = -\frac{d}{dt} e_\psi(w(t)|w_\infty) + r_\psi(w(t))$$

with $r_\psi(w(t)) \geq 0$. So integrating with respect to t leads to the inequality

$$e_\psi(w(t)|w_\infty) \leq -\frac{d}{dt} e_\psi(w(t)|w_\infty) = \int_{\mathbb{R}^n} \psi''\left(\frac{w}{w_\infty}\right)\left|\nabla \frac{w}{w_\infty}\right|^2 w_\infty dx,$$

where the last equality follows using integration by parts.
A different proof of a particular convex Sobolev inequality can be found in chapter 4. □

Remark 2.6. In the following the convex Sobolev inequality is only used for two particular entropies, the logarithmic and quadratic entropy. The logarithmic entropy is generated by the convex function $\psi_1(\sigma) = \sigma \ln \sigma - \sigma + 1$ and the quadratic entropy by $\psi_2(\sigma) = (\sigma - 1)^2$. Obviously, we can only use the logarithmic entropy for (almost everywhere) strictly positive solutions (provided by Lemma 2.2), whereas the quadratic entropy can be used for solutions that change sign.

For functions $w, w_\infty \in L^1_+(\mathbb{R}^n)$ with $\int w_\infty dx = \int w dx$ the logarithmic Sobolev inequality reads as

$$\int_{\mathbb{R}^n} w \ln \frac{w}{w_\infty} dx \leq \int_{\mathbb{R}^n} w \left|\nabla \ln \frac{w}{w_\infty}\right|^2 dx.$$

For the quadratic entropy we get the Poincaré-type inequality

$$\int_{\mathbb{R}^n} (w - w_\infty)^2 \frac{1}{w_\infty} dx \leq 2 \int_{\mathbb{R}^n} \left|\nabla w + \frac{1}{2} x w\right|^2 \frac{1}{w_\infty} dx.$$

For solutions with zero mass this inequality can be written as

$$\int_{\mathbb{R}^n} \frac{w^2}{G} dx \leq 2 \int_{\mathbb{R}^n} \left|\nabla w + \frac{1}{2} x w\right|^2 \frac{1}{G} dx.$$

Further it should be noted, that the stated convex Sobolev inequalities are sharp, which was proved in [AMTU] too.

Remark 2.7. The space $L^2(G^{-1})$ can be embedded in $L^p(\mathbb{R}^2)$ with $p \in [1, 2)$ because of

$$\begin{aligned}\left(\int_{\mathbb{R}^2} |w|^p \, dx\right)^{\frac{1}{p}} &= \left(\int_{\mathbb{R}^2} |w|^p \, |G|^{-\frac{p}{2}} \, |G|^{\frac{p}{2}} \, dx\right)^{\frac{1}{p}} \leq \|w\|_{L^2(G^{-1})} \, \|G\|^{\frac{1}{2}}_{L^{\frac{p}{2-p}}(\mathbb{R}^2)} \\ &= \|w\|_{L^2(G^{-1})} \left(\frac{2-p}{p}\right)^{\frac{2-p}{2}} (4\pi)^{\frac{1-p}{2}},\end{aligned}$$

and for $p = 2$ using the $L^\infty(\mathbb{R}^2)$-norm which leads to the constant $\frac{1}{4\pi}$. Due to that and the inequality

$$|w_1 \ln w_1 - w_2 \ln w_2| \le C \left(|w_1 - w_2|^{\frac{1}{2}} + |w_1 - w_2| \, (w_1^{\frac{1}{2}} + w_2^{\frac{1}{2}}) \right),$$

we have that the logarithmic entropy is continuous in $L^2(G^{-1})$.

The idea of the following theorem, which is stated in [GW2] in a quite similar way, is to show that the logarithmic entropy is a Ljapunov functional for the nonlinear equation (2.3) and then to use the logarithmic Sobolev inequality to show convergence to a steady state in the entropy. Afterwards the generalized Csiszár-Kullback inequality yields L^1-decay.

Theorem 2.8. *Let $e_{\psi_1}(w_0|w_\infty) < \infty$ and $w_0 \ge 0$ almost everywhere, then a solution w of equation* (2.3) *with initial data w_0 satisfies*

$$\|w(t) - w_\infty\|_{L^1(\mathbb{R}^2)} \le \sqrt{2e_{\psi_1}(w(t)|w_\infty)} \le e^{-\frac{t}{2}} \sqrt{2e_{\psi_1}(w_0|w_\infty)}.$$

Proof: Let

$$e_{\psi_1}(w(t)|w_\infty) = \int_{\mathbb{R}^2} w(x,t) \ln \left(\frac{w(x,t)}{w_\infty(x)} \right) dx$$

be the relative logarithmic entropy with respect to the steady state w_∞. Since the case $w \equiv 0$ is not interesting, we have that $w_0 \ge 0$ implies $w > 0$ because of Lemma 2.2, and therefore the entropy is well defined for our solutions.

At next we want to differentiate the logarithmic entropy with respect to t for all $t > 0$. To prove that the entropy is differentiable we at first assume that the initial condition is in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$. Then we indeed have a classical solution $w \in C^1([0,\infty), \mathcal{S}(\mathbb{R}^2))$ (compare [B-A] Theorem A and [GW2] Remark 2.4), which should be understood in the way that the map $t \mapsto p_{\alpha,m}(w(t))$ is continuously differentiable for all seminorms $p_{\alpha,m}(\phi) = \sup_{x \in \mathbb{R}^2} (1+|x|^m) \, |D^\alpha \phi(x)|$. Further we have the Gaussian lower bound of the fundamental solution in Theorem 2.1 that states after rescaling and because of $w_0 \ge 0$ that

$$\begin{aligned} w(x,t) \;&\ge\; \frac{C_1}{1-e^{-t}} \int_{\mathbb{R}^2} \exp\left(-C_2 \frac{\left|x - ye^{-t/2}\right|^2}{2(1-e^{-t})} \right) w_0(y) dy \\ &\ge\; \frac{C_1}{1-e^{-t}} \exp\left(-C_2 \frac{|x|^2}{1-e^{-t}} \right) \int_{\mathbb{R}^2} \exp\left(-C_2 \frac{|y|^2}{1-e^{-t}} \right) w_0(y) dy. \end{aligned}$$

The Gaussian upper bound of the same theorem directly leads to the estimate $w(x,t) \le C_3 \frac{1}{1-e^{-t}}$. So for a fixed t_0 and a neighborhood $[t_1, t_2]$ with $0 < t_1 < t_0 < t_2$ we have that

$$\tilde{C}_3(t_1) \ge \ln\left(w(x)\right) \ge -C(t_2) \, |x|^2 + \tilde{C}_1.$$

Therefore, we obtain the bound

$$\left| 1 + \ln\left(\frac{w}{w_\infty} \right) \right| \le 1 + C_4(1 + |x|^2) + C_5 \frac{|x|^2}{4} \le C_6(1 + |x|^2).$$

Since we have $w \in C^1([0,\infty), S(\mathbb{R}^2))$ it holds that $\sup_{t\in[t_1,t_2]}\sup_{x\in\mathbb{R}^2}|w_t q(x)| < \infty$ for all polynomials q in two variables. This leads to

$$\begin{aligned}\left|w_t(x,t)\left(\ln\left(\frac{w(x,t)}{w_\infty(x)}\right)+1\right)\right| &\le \sup_{t\in[t_1,t_2]}\left|w_t C_6(1+|x|^2)(1+|x|^4)\right|\frac{1}{1+|x|^4}\\ &\le \frac{C}{1+|x|^4}.\end{aligned}$$

So we have found an integrable majorant which is independent of t and the dominated convergence theorem allows us to swap integration and differentiation. Finally, we arrive at

$$\frac{d}{dt}e_{\psi_1}(w(t)|w_\infty) = \int_{\mathbb{R}^2} w_t(x,t)\left(\ln\left(\frac{w(x,t)}{w_\infty(x)}\right)+1\right)dx.$$

Using equation (2.3) in its divergence form we get that this is equal to

$$\int_{\mathbb{R}^2}\operatorname{div}\left(\nabla w + \frac{1}{2}xw - \mathbf{v}w\right)\left(\ln\left(\frac{w}{w_\infty}\right)+1\right)dx.$$

Now using integration by parts, the first part becomes to

$$\begin{aligned}\int_{\mathbb{R}^2}\operatorname{div}\left(\nabla w+\frac{1}{2}xw\right)\left(\ln\left(\frac{w}{w_\infty}\right)+1\right)dx &= -\int_{\mathbb{R}^2}\frac{w_\infty}{w}\nabla\left(\frac{w}{w_\infty}\right)\left(\nabla w+\frac{1}{2}xw\right)dx\\ &= -\int\frac{1}{w}\left|\nabla w+\frac{1}{2}xw\right|^2 dx =: I_{\psi_1}(w(t)|w_\infty)\le 0.\end{aligned}$$

One should note here that no boundary terms appear since our solution decays fast for $|x|\to\infty$ and all integrations are justified, if we additionally assume that $|I_{\psi_1}(w_0|w_\infty)|<\infty$, because we will see later that the entropy dissipation is also decreasing in time. Integration by parts in the other term from above and using $\operatorname{div}(\mathbf{v})=0$ leads to

$$\int_{\mathbb{R}^2}\operatorname{div}(\mathbf{v}w)\left(\ln\left(\frac{w}{w_\infty}\right)+1\right)dx = -\int_{\mathbb{R}^2}\mathbf{v}\cdot\nabla w+\frac{1}{2}x\cdot\mathbf{v}w dx = -\frac{1}{2}\int_{\mathbb{R}^2}x\cdot\mathbf{v}w dx.$$

Using Fubini's Theorem we obtain

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}-y\cdot\frac{(x-y)^\perp}{|x-y|^2}w(y)w(x)dxdy = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}x\cdot\frac{(x-y)^\perp}{|x-y|^2}w(y)w(x)dydx,$$

and hence we have

$$\begin{aligned}\int_{\mathbb{R}^2}x\cdot\mathbf{v}(x)w(x)dx &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}x\cdot\frac{(x-y)^\perp}{|x-y|^2}w(y)w(x)dydx\\ &= \frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(x-y)\cdot\frac{(x-y)^\perp}{|x-y|^2}w(y)w(x)dydx = 0.\end{aligned}$$

Therefore, we get

$$\frac{d}{dt}e_{\psi_1}(w(t)|w_\infty) = I_{\psi_1}(w(t)|w_\infty), \tag{2.6}$$

which shows that the derivative of the logarithmic entropy does not depend on the nonlinear term of the equation (2.3). So we are in fact in the same setting as in [AMTU] for non-symmetric Fokker-Planck equations, where the logarithmic Sobolev inequality has been derived by differentiating $I_{\psi_1}(w(t)|w_\infty)$. At this point we could use the logarithmic Sobolev inequality, but then it would remain to prove that $|I_{\psi_1}(w(t)|w_\infty)| < \infty$ for all $t > 0$. So we decided to compute this manually using some results of [AMTU] Lemma 2.13, which stated

$$\begin{aligned}\frac{d}{dt}\left|I_{\psi_1}(w(t)|w_\infty)\right| &= -\int_{\mathbb{R}^2}\frac{1}{w^2}\left|\nabla w+\frac{1}{2}xw\right|^2 w_t dx\\ &\quad +2\int_{\mathbb{R}^2}\frac{1}{w}\left(\nabla w+\frac{1}{2}xw\right)\cdot\left(\frac{1}{2}xw_t+\nabla w_t\right)dx\\ &\leq -\left|I_{\psi_1}(w(t)|w_\infty)\right|,\end{aligned}$$

if one differentiates along the trajectories of the linear equation, which directly leads to the logarithmic Sobolev inequality after integration. We will show that this holds for the trajectories of the nonlinear equation as well. At first, we should mention that the differentiability and the following integrations by parts can be justified just as for the entropy $e_{\psi_1}(w(t)|w_\infty)$ before. So we have for the nonlinear equation after integration by parts in the last term

$$\begin{aligned}\frac{d}{dt}\left|I_{\psi_1}(w(t)|w_\infty)\right| &\leq -\left|I_{\psi_1}(w(t)|w_\infty)\right|+\int_{\mathbb{R}^2}\frac{1}{w^2}\left|\nabla w+\frac{1}{2}xw\right|^2\mathbf{v}\cdot\nabla w dx\\ &\quad -2\int_{\mathbb{R}^2}\frac{1}{w}\left(\nabla w+\frac{1}{2}xw\right)\cdot\frac{1}{2}x\mathbf{v}\cdot\nabla w+\frac{1}{w^2}\nabla w\cdot\left(\nabla w+\frac{1}{2}xw\right)\mathbf{v}\cdot\nabla w dx\\ &\quad +2\int_{\mathbb{R}^2}\frac{1}{w}\left(\Delta w+\frac{1}{2}x\cdot\nabla w+w\right)\mathbf{v}\cdot\nabla w dx.\end{aligned}$$

Using $\int_{\mathbb{R}^2}|x|^2\,\mathbf{v}\cdot\nabla w dx = -2\int_{\mathbb{R}^2}x\cdot\mathbf{v}w dx = 0$, which was proved above as well as $\int_{\mathbb{R}^2}\mathbf{v}\cdot\nabla w dx = 0$, the remaining terms are

$$\frac{d}{dt}\left|I_{\psi_1}(w(t)|w_\infty)\right| \leq -\left|I_{\psi_1}(w(t)|w_\infty)\right| - \int_{\mathbb{R}^2}\frac{|\nabla w|^2}{w^2}\mathbf{v}\cdot\nabla w + 2\frac{\Delta w}{w}\mathbf{v}\cdot\nabla w dx. \tag{2.7}$$

Again integration by parts shows that the last two terms are equal and therefore, we have shown the exponential decay of the entropy dissipation using Gronwall's-Lemma

$$\left|I_{\psi_1}(w(t)|w_\infty)\right| \leq e^{-t}\left|I_{\psi_1}(w_0|w_\infty)\right|.$$

Since $I_{\psi_1} \leq 0$ the inequality (2.7) is equivalent to $\frac{d}{dt}I_{\psi_1}(w(t)|w_\infty) \geq -I_{\psi_1}(w(t)|w_\infty)$ and integrating from t to ∞ leads to

$$\frac{d}{dt}e_{\psi_1}(w(t)|w_\infty) \leq -e_{\psi_1}(w(t)|w_\infty).$$

Now integrating from 0 to t leads to exponential decay in the entropy

$$e_{\psi_1}(w(t)|w_\infty) \leq e^{-t}e_{\psi_1}(w_0|w_\infty)$$

and the Csiszár-Kullback inequality (2.4) yields

$$\|w(t)-w_\infty\|_{L^1(\mathbb{R}^2)} \leq \sqrt{2e_{\psi_1}(w(t)|w_\infty)} \leq e^{-\frac{1}{2}t}\sqrt{2e_{\psi_1}(w_0|w_\infty)},$$

which proves the theorem for initial conditions w_0 which are in the dense subset $\{w \in \mathcal{S}(\mathbb{R}^2) : |I_{\psi_1}(w|w_\infty)| < \infty\}$.

For the general case we use a density argument similar to [AMTU] Theorem 2.16.

At first, we approximate $w_0 \in \{w \in L^1_+(\mathbb{R}^2) : e_{\psi_1}(w|w_\infty) < \infty\}$ by normalized $w_N \in L^1_+(\mathbb{R}^2)$ which also are in $L^2(G^{-1})$ defined by $w_N(x) := \alpha_N w_0(x)\chi_{w_0/w_\infty \le N}(x)$, where α_N are normalization constants satisfying $\alpha_N \to 1$ for $N \to \infty$. Obviously, these approximations fulfill $w_N \in L^2(G^{-1})$ and $\|w_0 - w_N\|_{L^1(\mathbb{R}^2)} \to 0$ for $N \to \infty$. Since $\alpha_N \to 1$, it is easy to find an integrable majorant because of our assumptions on w_0. Therefore, the dominated convergence theorem leads to convergence in the entropy

$$e_{\psi_1}(w_N|w_\infty) \to e_{\psi_1}(w_0|w_\infty).$$

Now we approximate each w_N by C^∞-functions with compact support $w_{N,M} \in C_0^\infty(\mathbb{R}^2)$. Since

$$C_0^\infty(\mathbb{R}^2) \subset \mathcal{S}(\mathbb{R}^2) \cap \{w \in L^2(G^{-1}) : |I_{\psi_1}(w|w_\infty)| < \infty\},$$

where the first inclusion is obvious and the second has been established in [AMTU]. Since $C_0^\infty(\mathbb{R}^2)$ is a dense subset of $L^2(G^{-1})$, we can approximate w_N by $w_{N,M}$ in $L^2(G^{-1})$ with $e_{\psi_1}(w_{N,M}|w_\infty) \to e_{\psi_1}(w_N|w_\infty)$ for $M \to \infty$ because of Remark 2.7. Now the diagonal sequence $w_{N,M(N)}$ converges to w_0 in $L^1(\mathbb{R}^2)$ as well as in e_{ψ_1}. So for $w_{N,M(N)}$ all the assumptions of the first part of the theorem hold and the inequality

$$e_{\psi_1}(w_{N,M(N)}(t)|w_\infty) \le e^{-t} e_{\psi_1}(w_{N,M(N)}|w_\infty)$$

follows, where $w_{N,M(N)}(t)$ is the solution with initial data $w_{N,M(N)}$. Using this and the Dunford-Pettis Theorem, one obtains weak convergence of $w_{N,M(N)}(t) \to w(t)$ in L^1 and the weak lower semi-continuity of the entropy finally leads to

$$e_{\psi_1}(w(t)|w_\infty) \le \liminf_{N\to\infty} e_{\psi_1}(w_{N,M(N)}(t)|w_\infty) \le e^{-t} \liminf_{N\to\infty} e_{\psi_1}(w_{N,M(N)}|w_\infty) = e^{-t} e_{\psi_1}(w_0|w_\infty),$$

which proves the theorem for all initial conditions with $e_{\psi_1}(w_0|w_\infty) < \infty$. □

Remark 2.9. The statement above can also be proved for nonpositive solutions since then $-w$ is nonnegative and solves the equation

$$w_t = \operatorname{div}\left(\nabla w + \frac{1}{2} x w\right) + \mathbf{v} \cdot \nabla w.$$

Using the logarithmic entropy for that equation now leads to the same result, since the sign of $\mathbf{v} \cdot \nabla w$ does not influence the calculation.

Since we are more interested in the large time behavior for solutions of the vorticity equation rather than the Fokker-Planck equation, the following corollary states the matching result for the vorticity equation.

Corollary 2.10. *Let $e_{\psi_1}(\omega_0|w_\infty) < \infty$ and $\omega_0 \ge 0$. Then the estimate*

$$\left\| \omega(\cdot, t) - \frac{1}{1+t} w_\infty \left(\frac{\cdot}{\sqrt{1+t}} \right) \right\|_{L^1(\mathbb{R}^2)} \le C \frac{1}{(1+t)^{\frac{1}{2}}}$$

holds.

Proof: Undoing the scaling transformation from the beginning of the chapter, one gets for the inequality in the last theorem that

$$\begin{aligned}\left\|\omega(\cdot,t)-\frac{1}{1+t}w_\infty\left(\frac{\cdot}{\sqrt{1+t}}\right)\right\|_{L^1(\mathbb{R}^2)} &= \int_{\mathbb{R}^2}\frac{1}{1+t}\left|w\left(\frac{x}{\sqrt{1+t}},\ln(1+t)\right)-w_\infty\left(\frac{x}{\sqrt{1+t}}\right)\right|dx\\ &= \|w(\ln(1+t))-w_\infty\|_{L^1(\mathbb{R}^2)}\leq Ce^{-\frac{1}{2}\ln(1+t)}\\ &= C\frac{1}{(1+t)^{\frac{1}{2}}}.\end{aligned}$$

□

A consequence of this corollary is that the function $\frac{1}{1+t}w_\infty\left(\frac{x}{\sqrt{1+t}}\right)$ is the only self similar solution of the Navier-Stokes equations in two dimensions that satisfies our assumptions.
Further, we also get that there is no solution of the stationary Navier-Stokes equations beside the steady state zero, since every such solution has to converge to our time dependent asymptotic profile.
Since the result above shows stability of the asymptotic profile $\frac{1}{1+t}G\left(\frac{x}{\sqrt{1+t}}\right)\int_{\mathbb{R}^2}w_0dx$ independent of the size of $\int_{\mathbb{R}^2}w_0dx$, which can be interpreted as the Reynolds number of our flow, we get in contrary to Poseuille flows for example, that our flow is stable even for large Reynolds numbers.

Chapter 3

Solutions that change sign

The condition $w_0 \geq 0$ in the previous chapter of course was necessary, since we took the logarithm of the solution, but seems to be a condition that just arises from our method of proof. In the following we try to use a different entropy, the quadratic entropy, which can be used also for nonpositive solutions. The problem with this approach is that in contrary to the logarithmic entropy terms of the nonlinear part of the equation appear. Therefore, we have not found a general result concerning studying large time behavior with entropy methods for solutions that change sign. In the following, we will analyze some particular solutions, where the nonlinear terms do not appear either. Further we shall prove a similar result for solutions with small initial data.

At first, we want to sketch an argument made in [GW2] for dealing with solutions that change sign using the L^1-norm as Ljapunov functional. If we define the functions w^+ and w^- as the solutions of the equations

$$w_t^{\pm} = \Delta w^{\pm} + \frac{1}{2} x \cdot \nabla w^{\pm} + w^{\pm} - \mathbf{v}[w] \cdot \nabla w^{\pm}$$

with the initial data $w^+(0) = \max(w_0(x), 0)$ and $w^-(0) = -\min(w_0(x), 0)$. Now the maximum principle stated in the previous chapter shows that $w^+(x,t) > 0$ and $w^-(x,t) > 0$, and we have the decomposition in positive and negative part $w = w^+ - w^-$. Further, the equations above conserve mass and so we have

$$\int_{\mathbb{R}^2} |w(x,t)|\, dx \leq \int_{\mathbb{R}^2} w^+(x,t) + w^-(x,t) dx = \int_{\mathbb{R}^2} w^+(x,0) + w^-(x,0) dx = \int_{\mathbb{R}^2} |w_0|\, dx,$$

and we have shown that the L^1-norm is a Ljapunov functional even for solutions that change sign. Using this and the relative compactness of the trajectory in a polynomial weighted L^2-space, the LaSalle invariance principle shows convergence to the steady state in this L^2-space for all solutions with initial conditions in this space.
The advantage of this method is, that no additional assumptions on the solution have been made, but since we are interested in explicit rates of convergence too, this result is a little bit unsatisfying. Therefore, we shall make a different approach, where we have to make some additional assumptions on the solution, but in exchange get explicit rates of convergence.

Let

$$e_{\psi_2}(w(t)|w_\infty) := \int_{\mathbb{R}^2} (w(t) - w_\infty)^2 \frac{1}{G} dx$$

be the quadratic entropy multiplied by $\int w_0 dx$ (due to the otherwise unnecessary problems arising from solutions with zero mass), which is nothing else than the square of the weighted L^2-norm of $w - w_\infty$ with weight G^{-1}.

Again we want to differentiate the entropy with respect to t. This time, showing that we indeed can change integration and differentiation is a lot easier, since the quadratic entropy is a norm. Using the mean value theorem, one can easily show that for $w \in C^1([0,\infty), L^2(G^{-1}))$ we have

$$\frac{d}{dt} \|w - w_\infty\|^2_{L^2(G^{-1})} = 2\,(w_t, w - w_\infty)_{L^2(G^{-1})}\,,$$

which holds for all Hilbert-spaces, not only $L^2(G^{-1})$.
Now differentiating with respect to t and using the equation (2.3) as before leads to

$$\frac{d}{dt} e_{\psi_2}(w(t)|w_\infty) = 2 \int_{\mathbb{R}^2} w \operatorname{div}\left(\nabla w + \frac{1}{2} x w\right) \frac{1}{G} - w\mathbf{v} \cdot \nabla w \frac{1}{G} dx$$

and after integration by parts in the first part we get

$$-2 \int_{\mathbb{R}^2} \left|\nabla w + \frac{1}{2} x w\right|^2 \frac{1}{G} dx - 2 \int_{\mathbb{R}^2} w\mathbf{v} \cdot \nabla w \frac{1}{G} dx.$$

In contrary to the logarithmic entropy the second integral does not vanish this time, since

$$-2 \int_{\mathbb{R}^2} w\mathbf{v} \cdot \nabla w \frac{1}{G} dx = \frac{1}{2} \int_{\mathbb{R}^2} w^2 \mathbf{v} \cdot x \frac{1}{G} dx.$$

Therefore, we have to restrict ourselves to some particular solutions for which the integral vanishes, which are for example radially symmetric solutions.

The following lemma shows that equation (2.3) preserves this property of the initial condition, which of course is needed for our analysis.

Lemma 3.1. *Suppose w_0 to be radially symmetric. Then $w(t)$ is radially symmetric for all $t > 0$.*

Proof: We show that equation (2.3) is invariant under rotations. Let A be an orthogonal matrix and $y = A^T x$ and set $\tilde{w}(y) = w(x)$. It is well known that the Laplacian is invariant under rotations. The same holds for $x \cdot \nabla w$ since $\nabla_y \tilde{w}(y) = A^T \nabla_x w(x)$. The nonlinearity satisfies

$$\mathbf{v}[\tilde{w}](A^T x) = \int_{\mathbb{R}^2} \frac{\left(A^T x - y\right)^\perp}{\left|A^T x - y\right|^2} \tilde{w}(y) dy = A \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} w(y) dy = A\mathbf{v}[w](x),$$

so $\nabla w \cdot \mathbf{v}$ is also invariant under rotations. The uniqueness result of Theorem 3.1 now shows that $w(x) = w(Ax)$ for all x and all rotations A if the initial condition is radially symmetric, so the solution has to be radially symmetric for all $t > 0$.

□

Remark 3.2. For radially symmetric solutions the integral

$$\int_{\mathbb{R}^2} w^2 \mathbf{v} \cdot x \frac{1}{G} dx$$

vanishes, since the integrand is zero, because the vectors $\mathbf{v}(x) = \nabla^\perp \int_{\mathbb{R}^2} \ln|x-y| \, w(y) dy$ and $xw^2 \frac{1}{G}$ are orthogonal to each other.

One should mention here, that the integral vanishes for odd functions w as well, since then the velocity field is even and the integrand is odd. Nevertheless, this is not really useful since solutions with odd initial conditions do not stay odd in general, which can be seen by decomposing the solution into odd and even parts or using the rotation symmetry of our equation. Further, the integral vanishes for functions that are even in one variable as well, but the only solutions for which this property is preserved are radially symmetric solutions, so this does not give additional information as well.

Corollary 3.3. *Suppose w_0 to be a radially symmetric function and let $e_{\psi_2}(w_0|w_\infty) < \infty$. Then we have*

$$\|w(t) - w_\infty\|_{L^p(\mathbb{R}^2)} \le e^{-\frac{t}{2}} C(p) \sqrt{e_{\psi_2}(w_0|w_\infty)}$$

for all $p \in [1,2]$, where $C(p) = \left(\frac{2-p}{p}\right)^{\frac{2-p}{2}} (4\pi)^{\frac{1-p}{2}}$ for $p \in [1,2)$ and $C(p) = (4\pi)^{-\frac{1}{2}}$ for $p = 2$.

Proof: Lemma 3.1 shows that for radially symmetric initial conditions the solution still is radially symmetric. The remark afterwards shows that the nonlinear part does not appear in the entropy for these solutions. So altogether we have for those solutions

$$\frac{d}{dt} \|w(t) - w_\infty\|^2_{L^2(G^{-1})} = -2 \int_{\mathbb{R}^2} \left| \nabla w + \frac{1}{2} x w \right|^2 \frac{1}{G} dx =: I_{\psi_2}(w|w_\infty) \le 0,$$

if the initial condition additionally is in $\mathcal{S}(\mathbb{R}^2)$ and satisfies $|I_{\psi_2}(w_0|w_\infty)| < \infty$ just like in Theorem 2.8. Again, exactly as in Theorem 2.8, integration by parts and using $\int_{\mathbb{R}^2} w^2 \mathbf{v} \cdot x \frac{1}{G} dx = 0$ shows that the entropy dissipation decays exponentially. Using the convex Sobolev inequality as stated in Lemma 2.5 we get

$$-2 \int_{\mathbb{R}^2} \left| \nabla w + \frac{1}{2} x w \right|^2 \frac{1}{G} dx \le - \|w - w_\infty\|^2_{L^2(G^{-1})} .$$

Integrating this inequality leads to

$$\|w - w_\infty\|^2_{L^2(G^{-1})} \le \|w_0 - w_\infty\|^2_{L^2(G^{-1})} e^{-t}$$

for smooth initial conditions. The same density argument as in Theorem 2.8 (one can ignore the first approximation step, since the quadratic entropy obviously is continuous in $L^2(G^{-1})$) can be used to show the convergence in the entropy for initial conditions with $e_{\psi_2}(w_0|w_\infty) < \infty$.

Instead of the Csiszár-Kullback inequality for the quadratic entropy one can use that the space $L^2(G^{-1})$ is embedded into $L^p(\mathbb{R}^2)$ for $p \in [1,2]$ as stated in Remark 2.7 with the embedding constant $C(p)$. Putting these two inequalities together proves the corollary. $\square$

Again this shows like Corollary 2.10 that for solutions of the vorticity equation with radial symmetric initial conditions the estimate

$$\left\|\omega(\cdot,t) - \frac{1}{1+t} w_\infty\left(\frac{\cdot}{\sqrt{1+t}}\right)\right\|_{L^p(\mathbb{R}^2)} \leq C \frac{1}{(1+t)^{\frac{1}{2}+1-\frac{1}{p}}}$$

holds for every $p \in [1,2]$.

The following lemma collects some useful estimates of the vorticity and the velocity field, which are stated in a quite similar way in [GW1].

Lemma 3.4. *Assume $w_0 \in L^2(G^{-1})$.*

1. *For every $p \in [1,\infty]$, there exists a constant $C_p > 0$ such that every solution of equation (2.3) satisfies*

$$\|w(t)\|_{L^p(\mathbb{R}^2)} \leq C_p \frac{\|w_0\|_{L^1(\mathbb{R}^2)}}{(1-e^{-t})^{(1-\frac{1}{p})}}.$$

2. *For all $p \in [1,2]$ there exists a constant C_p such that*

$$\|w(t)\|_{L^p(\mathbb{R}^2)} \leq C_p,$$

   with $C_p = C_p(p, \|w_0\|_{L^2(G^{-1})})$ depending only on p and $\|w_0\|_{L^2(G^{-1})}$.

3. *Fix $0 < \kappa < 1$. Then the velocity field is bounded for every $t > 0$ by*

$$\|\mathbf{v}\|^2_{L^\infty(\mathbb{R}^2)} \leq C(1+t^{-\kappa}),$$

   with $C = C(\kappa, \|w_0\|_{L^2(G^{-1})})$ and C satisfying $C(\|w_0\|_{L^2(G^{-1})}) \to 0$ if $\|w_0\|_{L^2(G^{-1})} \to 0$.

Proof:

1. A proof can be found in [Cot] Theorem 1 and Lemma 4.1, where the inequality has been proven even for initial conditions which are bounded measures.
   An important particular case is that for $p = 1$ we have as described above that

$$\int_{\mathbb{R}^2} |w(x,t)|\, dx \leq \int_{\mathbb{R}^2} |w_0(x)|\, dx.$$

2. For $p \in [1,2]$ Hölder's inequality shows

$$\begin{aligned}\int_{\mathbb{R}^2} |w|^p\, dx &= \int_{\mathbb{R}^2} |w|^p\,(1+|x|^2)^p(1+|x|^2)^{-p} dx \\ &\leq \left(\int |w|^2\,(1+|x|^2)^2 dx\right)^{\frac{p}{2}} \left(\int (1+|x|^2)^{-\frac{2p}{2-p}} dx\right)^{\frac{2-p}{2}}.\end{aligned}$$

   The second integral can be easily computed, and so we have

$$\|w\|_{L^p(\mathbb{R}^2)} \leq \left(\int |w|^2\,(1+|x|^2)^2 dx\right)^{\frac{1}{2}} \left(\frac{2-p}{3p-2}\right)^{\frac{2-p}{2p}}.$$

In Theorem 3.2 in [GW1] it is shown that the weighted L^2-norm on the right side is bounded by a constant C independent of t depending only on the weighted L^2-norm of the initial condition. Obviously, we can replace this weighted L^2-norm of the initial condition with the weighted $L^2(G^{-1})$-norm, since we will work in this weighted space afterwards. Further, the cited theorem shows $C \to 0$ if $\|w_0\|_{L^2(G^{-1})} \to 0$.

3. We connect some estimates made in [GW1] Lemma 2.1, Theorem 3.2 and compute some constants more explicitly.
Fix $1 \le p < 2 < q \le \infty$ and let $\alpha \in (0,1)$ be defined by $\frac{1}{2} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$. At first, we have for all $R > 0$

$$\begin{aligned} |\mathbf{v}(x)| &\le \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{1}{|x-y|}\,|w(y)|\,dy \\ &= \frac{1}{2\pi}\int_{|y|\le R} \frac{1}{|y|}\,|w(x-y)|\,dy + \frac{1}{2\pi}\int_{|y|\ge R} \frac{1}{|y|}\,|w(x-y)|\,dy. \end{aligned}$$

Using Hölder's inequality we get

$$|\mathbf{v}(x)| \le \left(\frac{q-1}{q-2}\right)^{\frac{q-1}{q}} R^{1-\frac{2}{q}}\,\|w\|_{L^q(\mathbb{R}^2)} + \left(\frac{p-1}{2-p}\right)^{\frac{p-1}{p}} R^{1-\frac{2}{p}}\,\|w\|_{L^p(\mathbb{R}^2)}\,.$$

Choosing $R = \left(\frac{\|w\|_{L^p(\mathbb{R}^2)}}{\|w\|_{L^q(\mathbb{R}^2)}}\right)^{\beta}$ with $\beta = \frac{\alpha}{1-2/q} = \frac{1-\alpha}{2/p-1}$ leads to

$$|\mathbf{v}(x)| \le \left[\left(\frac{q-1}{q-2}\right)^{\frac{q-1}{q}} + \left(\frac{p-1}{2-p}\right)^{\frac{p-1}{p}}\right] \|w\|_{L^p(\mathbb{R}^2)}^{\alpha}\,\|w\|_{L^q(\mathbb{R}^2)}^{1-\alpha}\,.$$

Using the estimate of part 1 of the lemma for q and the estimate of part 2 for p we get

$$|\mathbf{v}(x)| \le C(p,q)C_p^{\alpha}C_q^{1-\alpha}\,\|w_0\|_{L^1(\mathbb{R}^2)}^{1-\alpha}\,\frac{1}{(1-e^{-t})^{(1-\frac{1}{q})(1-\alpha)}}.$$

The clue of this estimate is that instead of using the estimate of part 1 both times, this leaves us with an exponent of the term involving t, which is independent of p. Therefore, we can choose $\alpha = 1 - \frac{\kappa}{2(1-\frac{1}{q})}$ to get $\kappa = 2(1-\frac{1}{q})(1-\alpha)$ (and q such that the constant is minimal, since p is defined by fixing α and q). This leads to

$$\|\mathbf{v}\|_{L^\infty(\mathbb{R}^2)}^2 \le C\frac{1}{(1-e^{-t})^{\kappa}} \le C(1+t^{-\kappa})$$

and the constant C is sufficiently small if $\|w_0\|_{L^2(G^{-1})}$ is small enough.

□

Remark 3.5. So far, we have only obtained convergence in L^p for $p \in [1,2]$, since in this case we have the embedding of the weighted L^2-space in those spaces. Using the lemma above we can make an interpolation argument to get a result for L^p-spaces with $p > 2$. The first estimate of the lemma above for $p=\infty$ translates for the vorticity ω into $\|\omega\|_{L^\infty(\mathbb{R}^2)} \le \frac{C}{t}$. Now using this and the estimate we have in L^1, we get that

$$\begin{aligned}
\int_{\mathbb{R}^2} \left|\omega(x,t) - \frac{1}{1+t} w_\infty\left(\frac{x}{\sqrt{1+t}}\right)\right|^p dx &\le \left\|\omega(t) - \frac{1}{1+t} w_\infty\left(\frac{\cdot}{\sqrt{1+t}}\right)\right\|_{L^1(\mathbb{R}^2)} \cdot \\
&\quad \left\|\omega(t) - \frac{1}{1+t} w_\infty\left(\frac{\cdot}{\sqrt{1+t}}\right)\right\|_{L^\infty(\mathbb{R}^2)}^{p-1} \\
&\le \left\|\omega(t) - \frac{1}{1+t} w_\infty\left(\frac{\cdot}{\sqrt{1+t}}\right)\right\|_{L^1(\mathbb{R}^2)} \cdot \\
&\quad \left(\|\omega\|_{L^\infty(\mathbb{R}^2)} + \left\|\frac{1}{1+t} w_\infty\left(\frac{\cdot}{\sqrt{1+t}}\right)\right\|_{L^\infty(\mathbb{R}^2)} \right)^{p-1} \\
&\le \frac{C_1}{(1+t)^{\frac{1}{2}}} \left(\frac{C_2}{t} + \frac{C_3}{1+t}\right)^{p-1}.
\end{aligned}$$

Taking the p-root we get

$$\left\|\omega(t) - \frac{1}{1+t} w_\infty\left(\frac{\cdot}{\sqrt{1+t}}\right)\right\|_{L^p(\mathbb{R}^2)} \le \frac{C}{t^{1-\frac{1}{2p}}},$$

which shows convergence for all $p \in (2,\infty)$ as well. For $p=\infty$ studying the integral equation satisfied by ω, which is

$$\omega(t) = e^{t\Delta}\omega_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta}(\mathbf{u}(s)\omega(s))ds,$$

and using some estimates on the semigroup $e^{t\Delta}$, which can be found for example in [GW1] Appendix A, leads to the same result as above.

The following theorem states convergence of any solution with a slightly reduced rate, if we assume that the $L^2(G^{-1})$-norm of the initial condition is sufficiently small.

Theorem 3.6. *Fix $0<\mu<1$, $\delta>0$. Then for every solution of equation* (2.3) *with $\|w_0\|_{L^2(G^{-1})}$ small enough we have*

$$\|w(t) - w_\infty\|_{L^p(\mathbb{R}^2)} \le e^{-t(\frac{1}{2}-\delta)+\tilde{C}t^\mu} C(p) \|w_0 - w_\infty\|_{L^2(G^{-1})},$$

where $p \in [1,2]$ and the constant $\tilde{C}$ is sufficiently small, provided the $L^2(G^{-1})$-norm of the initial condition is small enough and $C(p) = \left(\frac{2-p}{p}\right)^{\frac{2-p}{2}} (4\pi)^{\frac{1-p}{2}}$ for $p \in [1,2)$ and $C(p) = (4\pi)^{-\frac{1}{2}}$ for $p=2$.

Proof: As in the theorem above we have for sufficiently smooth solutions that

$$\begin{aligned}
\frac{d}{dt}\|w(t) - w_\infty\|_{L^2(G^{-1})}^2 &= -2\int_{\mathbb{R}^2} \left|\nabla w + \frac{1}{2}xw\right|^2 \frac{1}{G}dx - 2\int_{\mathbb{R}^2} w\mathbf{v}\cdot\nabla w \frac{1}{G}dx \\
&=: I_{\psi_2}(w|w_\infty) - r_{\psi_2}(w|w_\infty).
\end{aligned}$$

The goal of the following computations is to get a useful estimate on the nonlinear term. Because of $\int_{\mathbb{R}^2} w\mathbf{v}x dx = 0$ we have

$$2\int_{\mathbb{R}^2} w\mathbf{v}\cdot\nabla w \frac{1}{G}dx = -2\int_{\mathbb{R}^2} w\mathbf{v}\cdot\nabla \frac{w}{G}dx = -2\int_{\mathbb{R}^2}(w-w_\infty)\mathbf{v}\cdot\nabla\frac{w}{G}dx.$$

Using Young's inequality we get

$$\begin{aligned} 2\int_{\mathbb{R}^2}(w-w_\infty)\mathbf{v}\cdot\nabla\frac{w}{G}dx &= 2\int_{\mathbb{R}^2}(w-w_\infty)\mathbf{v}\cdot\nabla\left(\frac{w}{G}\right)G^{-\frac{1}{2}}G^{\frac{1}{2}}dx \\ &\leq \frac{1}{2\varepsilon}\int_{\mathbb{R}^2}(w-w_\infty)^2|\mathbf{v}|^2\frac{1}{G}dx + 2\varepsilon\int_{\mathbb{R}^2}\left|\nabla w+\frac{1}{2}xw\right|^2\frac{1}{G}dx. \end{aligned}$$

The lemma above shows that the square of the L^∞-norm of the velocity field can be estimated with $C(1+t^{-\kappa})$ for every fixed κ. Using this bound we get

$$2\int_{\mathbb{R}^2} w\mathbf{v}\cdot\nabla w\frac{1}{G}dx \leq \frac{1}{2\varepsilon}C(1+t^{-\kappa})\int_{\mathbb{R}^2}(w-w_\infty)^2\frac{1}{G}dx + 2\varepsilon\int_{\mathbb{R}^2}\left|\nabla w+\frac{1}{2}xw\right|^2\frac{1}{G}dx. \tag{3.1}$$

Now with the convex Sobolev inequality (assuming that $\varepsilon < 1$) we obtain

$$\begin{aligned} \frac{d}{dt}\|w(t)-w_\infty\|^2_{L^2(G^{-1})} &\leq (-2+2\varepsilon)\int_{\mathbb{R}^2}\left|\nabla w+\frac{1}{2}xw\right|^2\frac{1}{G}dx \\ &\quad +\frac{1}{2\varepsilon}C(1+t^{-\kappa})\int_{\mathbb{R}^2}(w-w_\infty)^2\frac{1}{G}dx \\ &\leq -(1-\varepsilon)\int_{\mathbb{R}^2}(w-w_\infty)^2\frac{1}{G}dx + \frac{1}{2\varepsilon}C(1+t^{-\kappa})\int_{\mathbb{R}^2}(w-w_\infty)^2\frac{1}{G}dx \\ &= \left(-1+\varepsilon+\frac{1}{2\varepsilon}C(1+t^{-\kappa})\right)\int_{\mathbb{R}^2}(w-w_\infty)^2\frac{1}{G}dx. \end{aligned}$$

Since $C = C(\kappa, \|w_0\|_{L^2(G^{-1})})$ is sufficiently small, if $\|w_0\|_{L^2(G^{-1})}$ is small enough, we can choose $\varepsilon < 1$ such that $\varepsilon + \frac{1}{2\varepsilon}C = 2\delta$. Setting $\tilde{C} = \frac{C}{4\varepsilon}$ and using Gronwall's Lemma for the inequality above we get

$$\|w(t)-w_\infty\|^2_{L^2(G^{-1})} \leq e^{(-1+2\delta)t+2\tilde{C}t^{1-\kappa}}\|w_0-w_\infty\|^2_{L^2(G^{-1})}.$$

Substituting $\mu = 1-\kappa$ and taking the root now shows convergence in the quadratic entropy for sufficiently regular initial conditions. In fact, because of estimate (3.1) it is enough to assume $|I_{\psi_2}(w|w_\infty)| < \infty$, since then for every $t > 0$ we have that $r_{\psi_2}(w|w_\infty)$ is bounded. It remains to show that in this case $I_{\psi_2}(w|w_\infty)$ is still finite for all $t > 0$ if it is at $t = 0$. To prove that, we differentiate with respect to t and get after integration by parts

$$\begin{aligned} \frac{d}{dt}|I_{\psi_2}(w|w_\infty)| &= 4\int_{\mathbb{R}^2}\left(\nabla w+\frac{1}{2}xw\right)\left(\nabla w_t+\frac{1}{2}xw_t\right)\frac{1}{G}dx \\ &= I_{\psi_2}(w|w_\infty) + 4\int_{\mathbb{R}^2} w\mathbf{v}\cdot\nabla w\frac{1}{G}dx + \int_{\mathbb{R}^2}|\nabla w|^2\mathbf{v}x\frac{1}{G}dx. \end{aligned}$$

In [GW1] Proposition B.1 it has been proven in a similar way to part 3 of Lemma 3.4 that $\|x\mathbf{v}\| \leq C(\|bw\|_{L^p(\mathbb{R}^2)} + \|bw\|_{L^q(\mathbb{R}^2)})$ for $b = (1+|x|^2)$ and $p < 2$ and $q > 2$, since $\|x\mathbf{v}\| \leq C\|b\mathbf{v}\|$. If our initial condition is in $\mathcal{S}(\mathbb{R}^2)$ then the solution satisfies $w \in C^1([0,\infty),\mathcal{S}(\mathbb{R}^2))$, as we stated in the proof of Theorem 2.8. So we can bound $\|bw\|_{L^p(\mathbb{R}^2)} + \|bw\|_{L^r(\mathbb{R}^2)}$ by a continuous, integrable function $\eta(t)$ on $[0,t]$. So altogether we get

$$\begin{aligned}
\frac{d}{dt}|I_{\psi_2}(w|w_\infty)| &\leq I_{\psi_2}(w|w_\infty) + \|x\mathbf{v}\|_{L^\infty(\mathbb{R}^2)} \left(\int_{\mathbb{R}^2} |\nabla w|^2 \frac{1}{G} dx + \int_{\mathbb{R}^2} |w|^2 \frac{1}{G} dx \right) \\
&\leq I_{\psi_2}(w|w_\infty) + \eta(t) \int_{\mathbb{R}^2} \left| \nabla w + \frac{1}{2} x w \right|^2 \frac{1}{G} dx + 2\eta(t) \int_{\mathbb{R}^2} |w|^2 \frac{1}{G} dx \\
&\leq (1-\eta(t)) I_{\psi_2}(w|w_\infty) + \gamma(t)
\end{aligned}$$

with an integrable function $\gamma(t)$ on $[0,t]$, since we already know the bound on the $L^2(G^{-1})$-norm of the solution, and so Gronwall's Lemma shows that we indeed have $|I_{\psi_2}(w|w_\infty)| < \infty$ for all $t > 0$. A density argument just as in Theorem 2.8 leads to the result for initial conditions with $e_{\psi_2}(w_0|w_\infty) < \infty$.

Again the embedding $L^2(G^{-1}) \hookrightarrow L^p(\mathbb{R}^2)$ for $p \in [1,2]$ shows $L^p(\mathbb{R}^2)$-convergence to the stationary solution. $\square$

Obviously, the estimate above is not sharp, since for some particular solutions we have the sharper estimate of Corollary 3.3 and Theorem 2.8.
Since for $t \to \infty$ the dominating term on the right side of the inequality is $e^{(-\frac{1}{2}+\delta)t}$, we have shown that solutions with small initial data converge for large t almost at the same rate as solutions that do not change sign and therefore, we essentially have extended the entropy method to the case of arbitrary solutions. The only crucial constraint is that the initial condition has to be small enough, which can be explained by the fact that, since we have a somewhat quadratic nonlinearity, this nonlinearity is not important for small solutions. Nevertheless, this constraint seems to be unnatural, since for solutions that do not change sign no such constraint is needed. Unfortunately, we have not been able to remove this constraint since scaling or multiplying does not work.

Corollary 3.7. *Under the assumptions of the previous theorem, the solutions of the vorticity equation satisfy*

$$\left\| \omega(t) - \frac{1}{1+t} w_\infty \left(\frac{\cdot}{\sqrt{1+t}} \right) \right\|_{L^p(\mathbb{R}^2)} \leq \frac{1}{(1+t)^{\frac{1}{2}+1-\frac{1}{p}-\delta}} \hat{C}(1+t)^{\kappa\tilde{C}} \|w_0 - w_\infty\|_{L^2(G^{-1})}.$$

Throughout our previous analysis for both solutions that change sign or do not change sign we have made the assumption that the relative entropy of the initial condition is finite. This of course is necessary, since we use the entropy functionals as Ljapunov functionals, and can not be removed due to our method of proof. Nonetheless, it is fairly easy, using a density argument, to reduce the assumptions such that the initial condition only has to stay in a polynomially weighted L^2-space.

But if we want to reduce this assumption any further, we would have to change our method of proof. One such method is described in the paper [Car], where the invariance of the vorticity equation under the scaling $w_\lambda(x,t) = \lambda^2\omega(\lambda x, \lambda t^2)$ as well as some estimates on w_λ have been used to show that

$$\lim_{t\to\infty} t^{1-\frac{1}{p}} \left\| \omega(t) - \frac{1}{1+t} w_\infty \left(\frac{\cdot}{\sqrt{1+t}} \right) \right\|_{L^p(\mathbb{R}^2)} = 0,$$

which holds even for initial conditions that are finite Radon measures and satisfy that $\left| \int_{\mathbb{R}^2} w_0 dx \right|$ is sufficiently small. But again this result does not give an explicit estimate like one gets with entropy methods.

Chapter 4

Higher order asymptotics

The aim of this chapter is to prove that the estimates made in the previous chapters can be improved, if some additional assumptions are made. In fact, in some cases the rate of convergence can be linked to the spectral subspaces of the linear operator, in which the initial condition lies, just as for the linear equation.

At first, we compute the spectrum of the linear operator (compare [GW1] appendix A and [Rod]).

Lemma 4.1. *The spectrum of the operator* $Lw := \Delta w + \frac{1}{2}x \cdot \nabla w + w$ *in* $L^2(G^{-1})$ *only consists of the eigenvalues* $\{-\frac{k}{2} : k \in \mathbb{N}_0\}$.

Proof: The operator L is symmetric and nonpositive in $L^2(G^{-1})$ because of

$$(Lw,u)_{L^2(G^{-1})} = \int_{\mathbb{R}^2} \operatorname{div}\left(\nabla w + \frac{1}{2}xw\right) u \frac{1}{G} dx = -\int_{\mathbb{R}^2} \left(\nabla w + \frac{1}{2}xw\right)\left(\nabla u + \frac{1}{2}xu\right)\frac{1}{G}dx.$$

Therefore, the spectrum of L is real and nonpositive.

To show that $-\frac{k}{2}$ is an eigenvalue it is much easier to study the operator L in Fourier space. Using the standard calculation rules for the Fourier transformation we have

$$\widehat{Lu}(p) = -\left(|p|^2 + \frac{1}{2}p \cdot \nabla\right)\hat{u}(p).$$

For all multiindices $\alpha \in \mathbb{N}_0^2$ with $|\alpha| = \alpha_1 + \alpha_2 = k$ the function

$$\hat{\phi}_\alpha(p) = i^{|\alpha|} p_1^{\alpha_1} p_2^{\alpha_2} e^{-|p|^2}$$

solves $\widehat{L\phi_\alpha} = -\frac{k}{2}\hat{\phi}_\alpha$. Especially, it follows that $\hat{\phi}_0(p) = e^{-|p|^2}$ is an eigenfunction to the eigenvalue 0 for the Fourier-transformed equation. So we have that $\phi_0(x) = e^{-\frac{|x|^2}{4}}$ is an eigenfunction for the eigenvalue 0 of the original equation. Using the standard calculation rules for the Fourier transformation we get that

$$\phi_\alpha(x) = \partial^\alpha \phi_0(x)$$

is an eigenfunction to the eigenvalue $-\frac{k}{2}$ and the multiplicity of the eigenvalue is at least $k+1$.

One should note here, that these eigenfunctions are nothing else than the Hermite-polynomials in two variables multiplied with the eigenfunction of the eigenvalue 0.

So we have that $\sigma(L) \supset \{-\frac{k}{2} : k \in \mathbb{N}\}$. To see that the spectrum indeed cannot be larger, one can study the transformed operator $\mathcal{L} := G^{-\frac{1}{2}}(-L)G^{\frac{1}{2}}$, which can be easily computed as $\mathcal{L} = -\Delta + \frac{|x|^2}{16} - \frac{1}{2}$. This is nothing else than the Hamiltonian of the harmonic oscillator, of which the eigenvalues are well known to be $\sigma(L) = \{\frac{k}{2} : k \in \mathbb{N}\}$, as for example stated in [Rod]. $\square$

Remark 4.2. Since the eigenfunctions are the Hermite-polynomials multiplied with the Gaussian G they form an orthogonal basis of the space $L^2(G^{-1})$. Therefore, we can analyze an eigenfunction expansion of our solution, which could lead to higher convergence orders.

One should note here that eigenfunctions associated to eigenvalues $\frac{-2k-1}{2}$ are odd functions and the eigenfunctions associated to the eigenvalues $-k$ are even and we can choose a convenient basis of this subspace such that one of those eigenfunctions is radially symmetric.

The following observations are based on the paper [BBDE], where higher asymptotics for the heat equation are studied. In fact, we try to generalize these methods for some particular solutions of our nonlinear equation.

The following lemma proves a higher order convex Sobolev inequality for the quadratic entropy.

Lemma 4.3. *Assume $\int_{\mathbb{R}^2} w dx = \int_{\mathbb{R}^2} w_\infty dx$ and $\int_{\mathbb{R}^2} w \phi_\alpha \frac{1}{G} dx = 0$ for all eigenfunctions ϕ_α of the linear operator L with $0 < |\alpha| < n$. Then the inequality*

$$\|w - w_\infty\|^2_{L^2(G^{-1})} \leq \frac{2}{n} \int_{\mathbb{R}^2} \left|\nabla w + \frac{1}{2} x w\right|^2 \frac{1}{G} dx$$

holds with the optimal constant $\frac{2}{n}$.

Proof: The following proof is based on a classical variational argument. In fact, we want to minimize the functional

$$F(w) = \int_{\mathbb{R}^2} \left|\nabla w + \frac{1}{2} x w\right|^2 \frac{1}{G} dx$$

under the conditions $G(w) = \int_{\mathbb{R}^2} (w - w_\infty)^2 \frac{1}{G} dx = 1$, $H(w) = \int_{\mathbb{R}^2} w dx = \int_{\mathbb{R}^2} w_\infty dx$. Obviously, the functional is bounded below by 0 and is - as well as the constraints - convex, so we indeed have a minimizer.
The minimizer can be computed (assuming it is in $C^2(\mathbb{R}^2)$, which is fulfilled in our case) using the Euler-Lagrange equation for the functional $F + \lambda G + \mu H$ which is

$$\Delta w + \frac{1}{2} x \cdot w + w = \lambda(w - w_\infty) + \mu G.$$

A homogeneous solution of this equation is an eigenfunction of the linear operator with the eigenvalue λ and a particular solution is $\frac{\lambda\alpha - \mu}{\lambda} G$ with $\alpha = \int_{\mathbb{R}^2} w_0 dx$. Inserting this into the functional F we get that the value of the functional is $-\lambda$.

Since the eigenfunctions of the linear operator are orthogonal, the condition $\int_{\mathbb{R}^2} w\phi_\alpha \frac{1}{G} dx = 0$ for all $0 < |\alpha| < n$ is fulfilled for all w, which are eigenfunctions associated to eigenvalues $\lambda \leq -\frac{n}{2}$. Obviously, the minimal value of the functional is obtained by choosing the eigenfunction associated to the eigenvalue $-\frac{n}{2}$, which proves the lemma. □

This lemma provides us with a direct improvement of the rates stated in Corollary 3.3 and Theorem 3.6 if we assume that the spectral subspaces with $\int_{\mathbb{R}^2} w_0\phi_\alpha \frac{1}{G} dx = 0$ are invariant under the nonlinear evolution. In the following we will show that for the solutions discussed in Lemma 3.1 this is in fact true - as well as for the first spectral subspace, which was stated in [GW1] Theorem 4.5.

Lemma 4.4. *Assume $w_0 \in L^2(G^{-1})$ and that one of the following two statements holds.*

1. *Let w_0 be radially symmetric and satisfy $\int_{\mathbb{R}^2} w_0(x)\phi_\alpha \frac{1}{G} dx = 0$ for a radially symmetric eigenfunction ϕ_α associated to a fixed eigenvalue $-\frac{|\alpha|}{2}$.*
2. *Let $\int_{\mathbb{R}^2} w_0(x)\phi_\alpha \frac{1}{G} dx = 0$ for an eigenfunction ϕ_α with $|\alpha| = 1$.*

Then we have $\int_{\mathbb{R}^2} w(x,t)\phi_\alpha \frac{1}{G} dx = 0$ for all $t > 0$.

Proof:

1. The proof is straightforward differentiating the $L^2(G^{-1})$-inner product using that the linear operator is symmetric. Let λ be the eigenvalue associated to the eigenfunction ϕ_α then
$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^2} w\phi_\alpha \frac{1}{G} dx &= \int_{\mathbb{R}^2} Lw\phi_\alpha \frac{1}{G} dx - \int_{\mathbb{R}^2} \mathbf{v}\cdot\nabla w\phi_\alpha \frac{1}{G} dx \\ &= \lambda \int_{\mathbb{R}^2} w\phi_\alpha \frac{1}{G} dx - \int_{\mathbb{R}^2} \mathbf{v}\cdot\nabla w\phi_\alpha \frac{1}{G} dx \end{aligned} \tag{4.1}$$
holds. Again, this formal computation can be easily justified as in chapter 3. If w_0 is radially symmetric, we get that w is radially symmetric for all times $t > 0$. Since we assumed that ϕ_α is radially symmetric as well, the integral vanishes in this case too. Now integration with respect to t leads to
$$\int_{\mathbb{R}^2} w\phi_\alpha \frac{1}{G} dx = e^{\lambda t}\int_{\mathbb{R}^2} w_0\phi_\alpha \frac{1}{G} dx = 0.$$
2. A basis of the subspace associated to the eigenvalue $-\frac{1}{2}$ is x_1G and x_2G and inserting this in equation (4.1) leads to
$$\frac{d}{dt}\int_{\mathbb{R}^2} wx_i dx = \int_{\mathbb{R}^2} Lwx_i dx - \int_{\mathbb{R}^2} \mathbf{v}\cdot\nabla wx_i dx.$$
Since the linear operator L is in divergence form, we get that
$$\int_{\mathbb{R}^2} Lwx_i dx = -\int_{\mathbb{R}^2} \partial_{x_i} w + \frac{1}{2}x_i w dx = -\frac{1}{2}\int_{\mathbb{R}^2} wx_i dx,$$
because w is decreasing fast for $|x| \to \infty$. Because of $w = \text{rot}\mathbf{v}$ we have for $i = 1$
$$\begin{aligned} \int_{\mathbb{R}^2} \mathbf{v}\cdot\nabla wx_1 dx &= -\int_{\mathbb{R}^2} v_1 w dx = -\int_{\mathbb{R}^2} v_1\left(\partial_{x_1}v_2 - \partial_{x_2}v_1\right) dx \\ &= \int_{\mathbb{R}^2} \partial_{x_1}v_1 v_2 dx = -\int_{\mathbb{R}^2} \partial_{x_2}v_2 v_2 dx = 0 \end{aligned}$$

and obviously, the same holds for $i = 2$. So altogether we get

$$\frac{d}{dt}\int_{\mathbb{R}^2} wx_i dx = -\frac{1}{2}\int_{\mathbb{R}^2} wx_i dx$$

for all solutions w and therefore, the subspace created by $\int_{\mathbb{R}^2} w_0(x)\phi_\alpha \frac{1}{G} dx = 0$ for $|\alpha| = 1$ not only stays invariant by the nonlinear evolution, but also does not appear in the asymptotics for $t \to \infty$.

□

One should note here, that in terms of large time behavior the vorticity equation behaves like a linear equation for radially symmetric solutions, which is nothing special, since then we have $\mathbf{v} \cdot \nabla w = 0$ and therefore, equation (2.3) reduces to the linear Fokker-Planck equation with a quadratic potential. The interesting part of the lemma above is that, due to the particular form of the nonlinearity, we see some linear effects in the lower eigenspaces too.

Remark 4.5. The lemma above shows that if the first moments are zero for the initial condition, then the first moments are zero for all times. Since

$$\int_{\mathbb{R}^2} x_i w_0(x - x_0)dx = \int_{\mathbb{R}^2} (x_i + (x_0)_i)w_0(x)dx$$

for $i = 1, 2$, we can choose a vector $x_0 \in \mathbb{R}^2$ such that for the translated initial condition $w_0(x - x_0)$ the first moments vanish and we get a higher order of convergence for this initial condition. Since the vorticity equation is invariant under translations we can always make such a translation.
However, this does not work for the nonlinear Fokker-Planck equation since the equation is not invariant under translations and a translation also shifts the first moments.

Remark 4.6. In the previous lemma we have shown that no nonlinear terms appear in the evolution of the eigenspace corresponding to the eigenvalue $-\frac{1}{2}$. Obviously, for higher eigenspaces this is not true, but for the next eigenspace a similar result can be deduced. A basis for the eigenspace corresponding to the eigenvalue -1 consists of the functions $\Phi_1 = (|x|^2 - 4)G(x)$, $\Phi_2 = (x_1^2 - x_2^2)G(x)$ and $\Phi_3 = x_1 x_2 G(x)$. One should note that we choose the basis such that the first eigenfunction is radially symmetric. Now inserting Φ_1 into the second integral in equation (4.1) we get

$$\int_{\mathbb{R}^2} \mathbf{v} \cdot \nabla w \Phi_1 \frac{1}{G} dx = -2\int_{\mathbb{R}^2} w\mathbf{v}x dx = 0,$$

where the last equality has been computed in chapter 2. So if Φ_1 does not appear in the eigenfunction expansion of the initial condition, it will not appear for any $t > 0$ either. For Φ_2 and Φ_3 this is not true.
In higher eigenspaces no such result can hold, because for every polynomial p we have using $w = \mathrm{rot}(\mathbf{v})$ that

$$\int_{\mathbb{R}^2} \mathbf{v} \cdot \nabla w p dx = \int_{\mathbb{R}^2} v_1 v_2(\partial_{x_1}^2 p - \partial_{x_2}^2 p) - (v_1^2 - v_2^2)\partial_{x_1}\partial_{x_2} p dx$$

and the equations $\partial_{x_1}^2 p - \partial_{x_2}^2 p = 0$, $\partial_{x_1}\partial_{x_2} p = 0$ cannot be solved for any polynomial in two variables with degree greater than 2.

In [GW1] an improved convergence rate is obtained by linearizing the equation around the stationary solution and subtracting some terms of the eigenfunction expansion. In fact, this has been done up to the eigenvalue $\lambda = -1$, since it becomes more and more technical the more eigenfunctions you subtract. The advantage of our approach is that for some particular solutions using the invariance of the spectral subspaces and in particular the resulting improved convex Sobolev inequality we have an improved convergence result for large eigenvalues too.

Corollary 4.7. *Let w_0 be radially symmetric with $w_0 \in L^2(G^{-1})$ and $\int_{\mathbb{R}^2} w_0 \phi_\alpha \frac{1}{G} dx = 0$ for all multiindices α with $0 < |\alpha| < n$, then we have*

$$\|w(t) - w_\infty\|_{L^p(\mathbb{R}^2)} \leq e^{-\frac{n}{2}t} C(p) \sqrt{e_{\psi_2}(w_0|w_\infty)}$$

for all $p \in [1,2]$, where $C(p) = \left(\frac{2-p}{p}\right)^{\frac{2-p}{2}} (4\pi)^{\frac{1-p}{2}}$ for $p \in [1,2)$ and $C(p) = (4\pi)^{-\frac{1}{2}}$ for $p = 2$.

Proof: If w_0 is radially symmetric, then w is radially symmetric for all times and therefore, we can choose a convenient basis for the eigenspaces such that the eigenfunction expansion of w consists only of radially symmetric eigenfunctions. Because of that and Lemma 4.4 we have that the condition $\int_{\mathbb{R}^2} w(t)\phi_\alpha \frac{1}{G} dx = 0$ for all multiindices α with $0 < |\alpha| < n$ holds and so the assumptions of Lemma 4.3 hold for all $t > 0$ and we have an improved convex Sobolev inequality for our solutions.

As in Corollary 3.3 we have for these solutions that

$$\frac{d}{dt} \|w - w_\infty\|^2_{L^2(G^{-1})} = -2 \int_{\mathbb{R}^2} \left|\nabla w + \frac{1}{2} x w\right|^2 \frac{1}{G} dx,$$

and now the improved convex Sobolev inequality shows

$$-2 \int_{\mathbb{R}^2} \left|\nabla w + \frac{1}{2} x w\right|^2 \frac{1}{G} dx \leq -n \|w - w_\infty\|^2_{L^2(G^{-1})}.$$

Integrating, taking the square root and using the embedding $L^2(G^{-1}) \hookrightarrow L^p(\mathbb{R}^2)$ for $p \in [1,2]$ as before proves the corollary. □

Corollary 4.8. *Under the assumptions of the previous theorem we have for solutions of the vorticity equation*

$$\left\|\omega(\cdot,t) - \frac{1}{1+t} w_\infty \left(\frac{\cdot}{\sqrt{1+t}}\right)\right\|_{L^p(\mathbb{R}^2)} \leq C \frac{1}{(1+t)^{\frac{n}{2}+1-\frac{1}{p}}}.$$

An improved result of Theorem 3.6 holds as well, which can be proven in exactly the same way as Corollary 4.6.

Corollary 4.9. *Let the assumptions of Theorem 3.6 be fulfilled and assume that the solution w satisfies $\int_{\mathbb{R}^2} w \phi_\alpha \frac{1}{G} dx = 0$ for all multiindices α with $0 < |\alpha| < n$ and all $t \geq 0$. Then the estimate*

$$\|w(t) - w_\infty\|_{L^p(\mathbb{R}^2)} \leq e^{-t(\frac{n}{2} - \delta) + \tilde{C} t^\mu} C(p) \|w_0 - w_\infty\|_{L^2(G^{-1})},$$

holds, where all constants have the same meaning as in Theorem 3.6.

For the vorticity equation the right side of the inequality above is replaced by $\frac{C}{(1+t)^{\frac{n}{2}+1-\frac{1}{p}-\gamma}}$.

Similar as in [BBDE] we deduce an improved logarithmic Sobolev inequality from the improved convex Sobolev inequality from Lemma 4.3. Therefore, we define the continuous, nonnegative and decreasing function $h : \mathbb{R}^+ \to \mathbb{R}$ with $h(0) = 1$, $h(1) = \frac{1}{2}$ and $h(s) := \frac{s \ln s-(s-1)}{(s-1)^2}$ on $\mathbb{R}^+\backslash\{0, 1\}$ and the functional on $L^\infty(\mathbb{R}^2)$

$$\mathcal{H}(z) := \|z\|_{L^\infty(\mathbb{R}^2)} \sup_{x\in\mathbb{R}^2} h(z(x)).$$

Lemma 4.10. *Assume that $w \geq 0$ and $\frac{w}{w_\infty} \in L^\infty(\mathbb{R}^2)$ and that $\int_{\mathbb{R}^2} w dx = \int_{\mathbb{R}^2} w_\infty dx$ and $\int_{\mathbb{R}^2} w\phi_\alpha \frac{1}{G} dx = 0$ for all eigenfunctions of the linear operator L with $0 < |\alpha| < n$. Then the inequality*

$$\int_{\mathbb{R}^2} w \ln \left(\frac{w}{w_\infty}\right) dx \leq \frac{2\mathcal{H}\left(\frac{w}{w_\infty}\right)}{n} \int_{\mathbb{R}^2} \left|\nabla \frac{w}{w_\infty}\right|^2 \frac{w_\infty^2}{w} dx \tag{4.2}$$

with the optimal constant $\frac{2}{n}$ holds.

Proof: Using the definition of $\mathcal{H}$ and the conservation of mass we get with $u := \frac{w}{w_\infty}$

$$\int_{\mathbb{R}^2} w \ln \left(\frac{w}{w_\infty}\right) dx \leq \frac{\mathcal{H}(u)}{\|u\|_{L^\infty(\mathbb{R}^2)}} \int_{\mathbb{R}^2} (w - w_\infty)^2 \frac{1}{w_\infty} dx.$$

Now the improved convex Sobolev inequality from Lemma 4.3 leads to

$$\begin{aligned}\frac{\mathcal{H}(u)}{\|u\|_{L^\infty(\mathbb{R}^2)}} \int_{\mathbb{R}^2} (w - w_\infty)^2 \frac{1}{w_\infty} dx &\leq \frac{2}{n} \frac{\mathcal{H}(u)}{\|u\|_{L^\infty(\mathbb{R}^2)}} \int_{\mathbb{R}^2} \left|\nabla \frac{w}{w_\infty}\right|^2 w_\infty dx \\ &\leq \frac{2\mathcal{H}(u)}{n} \int_{\mathbb{R}^2} \left|\nabla \frac{w}{w_\infty}\right|^2 \frac{w_\infty^2}{w} dx.\end{aligned}$$

Proving the optimality of the equation is quite technical and we refer to [BBDE]. □

Now we can improve the convergence estimate of Theorem 2.8 if we make some additional assumptions.

Corollary 4.11. *Assume $w_0 \geq 0$, $e_{\psi_1}(w_0|w_\infty) < \infty$ and $\left\|\frac{w(t)}{w_\infty}\right\|_{L^\infty(\mathbb{R}^2)} \leq C$ for all $t \geq 0$ and $\int_{\mathbb{R}^2} w(t)\phi_\alpha \frac{1}{G} dx = 0$ for all $t > 0$ for all eigenfunctions of the linear operator L with $0 < |\alpha| < n$. Then the estimate*

$$\|w(t) - w_\infty\|_{L^1(\mathbb{R}^2)} \leq e^{-\frac{n}{2}Ct} \sqrt{e_{\psi_1}(w_0|w_\infty)}$$

holds.

Proof: Since $h \leq 1$ and with our assumption on the L^∞-norm, the functional $\mathcal{H}$ satisfies

$$\mathcal{H}\left(\frac{w(t)}{w_\infty}\right) \leq C$$

for all $t \geq 0$, which can be inserted into equation (4.2). Further equation (2.6) stated

$$\frac{d}{dt} e_{\psi_1}(w(t)|w_\infty) = I_{\psi_1}(w(t)|w_\infty).$$

The improved logarithmic Sobolev inequality now leads to

$$\frac{d}{dt} e_{\psi_1}(w(t)|w_\infty) = I_{\psi_1}(w(t)|w_\infty) \leq -nC e_{\psi_1}(w(t)|w_\infty)$$

and integration proves the corollary. □

Remark 4.12. One should note, that the assumption that the functional $\mathcal{H}$ is bounded for all $t \geq 0$ is not very restrictive, since our solutions decay to w_∞ anyways and for example it can be shown that for radially symmetric solutions this functional decays in time.

Chapter 5

Large time asymptotics in three dimensions

In this chapter we try to adapt some results from the two dimensional case for the vorticity equation in three dimensions. As it is well known, studying the Navier-Stokes equations in three dimensions is much more difficult than in two dimensions. Therefore, we again have to make additional assumptions on our solutions.

As described in the introduction, the global existence of a smooth solution of the Navier-Stokes equations in three dimensions is unknown. Nonetheless, the following theorem from [GW3] states existence for solutions of the vorticity equation with small initial data as well as a result concerning large time behavior in certain L^p-norms. The idea of the proof is to make some estimates on the integral equation and an approximation argument afterwards. While the steps are not too difficult, we decided to omit the proof since the methods do not coincide with the ideas of the previous chapters.

Theorem 5.1. *For all initial data $\boldsymbol{\omega}_0 \in L^{3/2}(\mathbb{R}^3)$ with $\|\boldsymbol{\omega}_0\|_{L^{3/2}(\mathbb{R}^3)} \leq \varepsilon$ for a sufficiently small $\varepsilon > 0$ and div$(\boldsymbol{\omega}_0) = 0$ there exists an unique solution $\boldsymbol{\omega} \in C([0,\infty), L^{3/2}(\mathbb{R}^3)) \cap C((0,\infty), L^\infty(\mathbb{R}^3))$ of equation (1.4) with $\boldsymbol{\omega}(0) = \boldsymbol{\omega}_0$. Further, for all $p \in [\frac{3}{2}, \infty]$ there exists a constant $C_p > 0$ such that the inequality*

$$\|\boldsymbol{\omega}(t)\|_{L^p(\mathbb{R}^2)} \leq C_p \frac{\|\boldsymbol{\omega}_0\|_{L^{3/2}(\mathbb{R}^3)}}{t^{1-\frac{3}{2p}}} \tag{5.1}$$

holds for all $t > 0$. The corresponding velocity field $\mathbf{u}$ is in $L^q(\mathbb{R}^3)$ for $q \in [3, \infty]$ and satisfies

$$\|\mathbf{u}(t)\|_{L^q(\mathbb{R}^3)} \leq C_q \frac{\|\boldsymbol{\omega}_0\|_{L^{3/2}(\mathbb{R}^3)}}{t^{\frac{1}{2}-\frac{3}{2q}}} \tag{5.2}$$

for all $t > 0$.

Proof: Compare [GW3] Theorem 2.2 and the paper of Kato [Kat]. □

The three dimensional case can be seen as a direct generalization of the two dimensional case, since if we have an initial condition of the form $(0, 0, (\omega_3)_0(x_1, x_2))^T$, we have that the solution satisfies $\boldsymbol{\omega}(x) = (0, 0, \omega_3(x_1, x_2))$ because $\text{div}(\boldsymbol{\omega}) = 0$ leads to $\partial_{x_3}\omega_3 = 0$. So the velocity field given by the three dimensional Biot-Savart law has the form

$$\begin{aligned}
\mathbf{u}(x) &= -\frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}} \frac{(x-y)\times\boldsymbol{\omega}(y)}{|x-y|^3} dy_3 d(y_1,y_2) \\
&= -\frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}} \frac{1}{|x-y|^3} dy_3 \begin{pmatrix} (x_2-y_2)w_3(y_1,y_2) \\ (-x_1+y_1)w_3(y_1,y_2) \\ 0 \end{pmatrix} d(y_1,y_2) \\
&= \frac{1}{4\pi}\int_{\mathbb{R}^2} \frac{2}{(x_1-y_1)^2+(x_2-y_2)^2} \begin{pmatrix} (x_2-y_2)w_3(y_1,y_2) \\ (-x_1+y_1)w_3(y_1,y_2) \\ 0 \end{pmatrix} d(y_1,y_2),
\end{aligned}$$

where the first two components are exactly the Biot-Savart law in the two dimensional case. Now the first two equations of the three dimensional system are obviously satisfied, since all terms are zero and the third equation becomes the vorticity equation in two dimensions.

In the following we shall transform the vorticity equation in three dimensions into a nonlinear Fokker-Planck equation. In fact, one can make the same transformation as we made in the two dimensional case, but the resulting equation does not conserve mass. Since this is rather unsatisfying in the context of entropy methods we will make a different transformation that ensures conservation of mass. The scaling variables

$$\xi = \frac{3}{\sqrt{6}}\frac{x}{\sqrt{1+t}}, \quad \tau = \frac{3}{2}\ln(1+t)$$

and setting

$$\begin{aligned}
\boldsymbol{\omega}(x,t) &= \frac{1}{(1+t)^{3/2}}\mathbf{w}(\xi,\tau), \\
\mathbf{u}(x,t) &= \frac{\sqrt{6}}{3}\frac{1}{1+t}\mathbf{v}(\xi,\tau)
\end{aligned}$$

lead to the equations

$$\mathbf{w}_\tau = \Delta\mathbf{w} + \frac{1}{3}\xi\cdot\nabla\mathbf{w} + \mathbf{w} - e^{-\frac{\tau}{2}}\frac{2}{3}(\mathbf{v}\cdot\nabla)\mathbf{w} + e^{-\frac{\tau}{2}}\frac{2}{3}(\mathbf{w}\cdot\nabla)\mathbf{v} \tag{5.3}$$

and $\text{div}(\mathbf{w}) = 0$. These equations are a coupled system of nonlinear Fokker-Planck equations with the quadratic potential $\frac{1}{6}|x|^2$. Again we will write x, t for the variables ξ, τ. A time independent solution of the linear system is a constant vector in $\mathbb{R}^3$ multiplied with the Gaussian $G := \frac{1}{(6\pi)^{3/2}}e^{-\frac{|x|^2}{6}}$ for $x \in \mathbb{R}^3$. But in contrary to the two dimensional case we have that this Gaussian is not a stationary solution of the equation (5.3), because the term $(\mathbf{w}\cdot\nabla)\mathbf{v}$ does not vanish. Nonetheless, this seems to be a reasonable asymptotic state, because the nonlinear terms are multiplied with the decaying term $e^{-\frac{t}{2}}$. Considering the conservation of mass, we therefore hope that under some reasonable conditions our solutions converge to $\mathbf{w}_\infty(x) := \left(\int_{\mathbb{R}^3} w_0^1 dx G(x), \int_{\mathbb{R}^3} w_0^2 dx G(x), \int_{\mathbb{R}^3} w_0^3 dx G(x)\right)^T$.

In the two dimensional case we used the logarithmic entropy as a Ljapunov functional for solutions that do not change sign. This can not be done here, since due to the appearance of the second nonlinear term the logarithmic entropy is no Ljapunov functional any more. Therefore, we will use the quadratic entropy similar to Theorem 3.6. This can be done in the three dimensional case as well, because we can bound both nonlinear terms in a similar way as in the two dimensional case. In this case the assumption that the initial data has to be small is not that unnatural, since we indeed only know existence for solutions with small initial data.

Theorem 5.2. *Fix $t_0 > 0$, $\delta > 0$. Then for every solution of equation* (5.3) *with $\|\mathbf{w}_0\|_{L^{3/2}(\mathbb{R}^3)}$ small enough and $\|\mathbf{w}(t_0)\|_{L^2(G^{-1})} < \infty$ we have*

$$\|\mathbf{w}(t) - \mathbf{w}_\infty\|_{L^p(\mathbb{R}^3)} \leq e^{-\frac{1}{12}t} C(p) \sqrt{e^{-\frac{5}{6}t+2\delta t} K \|\mathbf{w}(t_0) - \mathbf{w}_\infty\|^2_{L^2(G^{-1})} + \hat{K} \sum_{i=1}^{3} \left(\int_{\mathbb{R}^3} w_0^i dx \right)^2}$$

for $t > t_0$, where $p \in [1, 2]$ and the constants $K = K\left(t_0, \|\mathbf{w}_0\|_{L^{3/2}(\mathbb{R}^3)}\right)$, $\hat{K} = \hat{K}\left(t_0, \|\mathbf{w}_0\|_{L^{3/2}(\mathbb{R}^3)}\right)$ satisfy $K, \hat{K} \to 0$, if $\|\mathbf{w}_0\|_{L^{3/2}(\mathbb{R}^3)} \to 0$ or if $t_0 \to \infty$, and $C(p)$ is the embedding constant of $L^2(G^{-1})$ in $L^p(\mathbb{R}^3)$.

Proof: Differentiating the weighted L^2-norm in t and using that the right hand side of equation (5.3) is in divergence form leads to

$$\frac{d}{dt} \int_{\mathbb{R}^3} \frac{|\mathbf{w}(t) - \mathbf{w}_\infty|^2}{G} dx = \frac{d}{dt} \int_{\mathbb{R}^3} \frac{w_1^2 + w_2^2 + w_3^2}{G} dx = 2 \int_{\mathbb{R}^3} (w_1 \partial_t w_1 + w_2 \partial_t w_2 + w_3 \partial_t w_3) \frac{1}{G} dx,$$

where the differentiability of the weighted L^2-norm as well as all integrations by parts can be justified with the same arguments as in Corollary 3.3. Using equation (5.3) we get

$$2 \int_{\mathbb{R}^3} w_i \partial_t w_i \frac{1}{G} dx = 2 \int_{\mathbb{R}^3} w_i \left(\operatorname{div} \left(\nabla w_i + \frac{1}{3} x w_i \right) - \frac{2}{3} e^{-\frac{t}{2}} (\mathbf{v} \cdot \nabla) w_i + \frac{2}{3} e^{-\frac{t}{2}} (\mathbf{w} \cdot \nabla) v_i \right) \frac{1}{G} dx$$

and integration by parts in the first term leads to

$$2 \int_{\mathbb{R}^3} w_i \operatorname{div} \left(\nabla w_i + \frac{1}{3} x w_i \right) \frac{1}{G} dx = -2 \int_{\mathbb{R}^3} \frac{\left|\nabla w_i + \frac{1}{3} x w_i\right|^2}{G} dx.$$

So we get altogether

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} \frac{|\mathbf{w}|^2}{G} dx &= \sum_{i=1}^{3} -2 \int_{\mathbb{R}^3} \frac{\left|\nabla w_i + \frac{1}{3} x w_i\right|^2}{G} dx \\ &\quad - \frac{4}{3} e^{-\frac{t}{2}} \int_{\mathbb{R}^3} (\mathbf{v} \cdot \nabla) \mathbf{w} \cdot \mathbf{w} \frac{1}{G} dx + \frac{4}{3} e^{-\frac{t}{2}} \int_{\mathbb{R}^3} (\mathbf{w} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} \frac{1}{G} dx. \end{aligned}$$

Since

$$-2 \int_{\mathbb{R}^3} (\mathbf{v} \cdot \nabla) \mathbf{w} \cdot \mathbf{w} \frac{1}{G} dx = \frac{1}{3} \int_{\mathbb{R}^3} |\mathbf{w}|^2 \mathbf{v} \cdot x \frac{1}{G} dx$$

and

$$2 \int_{\mathbb{R}^3} (\mathbf{w} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} \frac{1}{G} dx = -\frac{1}{3} \int_{\mathbb{R}^3} |\mathbf{w}|^2 \mathbf{v} \cdot x \frac{1}{G} dx - \frac{1}{3} \sum_{i=1}^{3} \int_{\mathbb{R}^3} \mathbf{w} \cdot \nabla w_i v_i \frac{1}{G} dx,$$

the only remaining nonlinear parts are the integrals in the last sum. Those terms can be estimated in a similar way to the two dimensional case using Young's inequality and so we get

$$\begin{aligned}
\frac{2}{9}\int_{\mathbb{R}^3}\mathbf{w}\cdot\nabla w_i v_i\frac{1}{G}dx &\le \|v_i\|_{L^\infty(\mathbb{R}^3)}\frac{1}{2\varepsilon}\int_{\mathbb{R}^3}\frac{|\mathbf{w}|^2}{G}dx+2\varepsilon\|v_i\|_{L^\infty(\mathbb{R}^3)}\int_{\mathbb{R}^3}\frac{|\nabla w_i|^2}{G}dx\\
&= \|v_i\|_{L^\infty(\mathbb{R}^3)}\frac{1}{2\varepsilon}\int_{\mathbb{R}^3}\frac{|\mathbf{w}|^2}{G}dx+2\varepsilon\|v_i\|_{L^\infty(\mathbb{R}^3)}\int_{\mathbb{R}^3}\left|\nabla w_i+\frac{1}{3}xw_i\right|^2\frac{1}{G}dx\\
&\quad +2\varepsilon\|v_i\|_{L^\infty(\mathbb{R}^3)}\int_{\mathbb{R}^3}\frac{|w_i|^2}{G}dx.
\end{aligned}$$

Rescaling inequality (5.2) leads to

$$\|\mathbf{v}(t)\|_{L^\infty(\mathbb{R}^3)}\le\frac{C\|\boldsymbol{\omega}_0\|_{L^{3/2}(\mathbb{R}^3)}}{\sqrt{1-e^{-\frac{2}{3}t}}}e^{\frac{1}{3}t}\le\tilde{C}\left(1+t^{-\frac{1}{2}}\right)e^{\frac{1}{3}t}$$

and altogether we get with this bound on the velocity field that

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^3}\frac{|\mathbf{w}|^2}{G}dx &\le \sum_{i=1}^3\left(-2+6\varepsilon\tilde{C}\left(1+t^{-\frac{1}{2}}\right)e^{\frac{1}{3}t}e^{-\frac{t}{2}}\right)\int_{\mathbb{R}^3}\frac{\left|\nabla w_i+\frac{1}{3}xw_i\right|^2}{G}dx\\
&\quad+\left(2\varepsilon+\frac{3}{2\varepsilon}\right)\tilde{C}\left(1+t^{-\frac{1}{2}}\right)e^{\frac{1}{3}t}e^{-\frac{t}{2}}\int_{\mathbb{R}^3}\frac{|\mathbf{w}|^2}{G}dx\\
&= \sum_{i=1}^3\left(-2+6\varepsilon\tilde{C}\left(1+t^{-\frac{1}{2}}\right)e^{-\frac{1}{6}t}\right)\int_{\mathbb{R}^3}\frac{\left|\nabla w_i+\frac{1}{3}xw_i\right|^2}{G}dx\\
&\quad+\left(2\varepsilon+\frac{3}{2\varepsilon}\right)\tilde{C}\left(1+t^{-\frac{1}{2}}\right)e^{-\frac{1}{6}t}\int_{\mathbb{R}^3}\frac{|\mathbf{w}-\mathbf{w}_\infty|^2}{G}dx\\
&\quad+\left(2\varepsilon+\frac{3}{2\varepsilon}\right)\tilde{C}\left(1+t^{-\frac{1}{2}}\right)e^{-\frac{1}{6}t}\sum_{i=1}^3\left(\int_{\mathbb{R}^3}w_0^i dx\right)^2.
\end{aligned}$$

Using the convex Sobolev inequality for each w_i (choosing ε small enough such that we have $6\varepsilon\tilde{C}(1+t_0^{-\frac{1}{2}})e^{-\frac{1}{6}t_0}<2$) we get for all $t>t_0$

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^3}\frac{|\mathbf{w}|^2}{G}dx &\le \left(-1+\left(5\varepsilon+\frac{3}{2\varepsilon}\right)\tilde{C}\left(1+t^{-\frac{1}{2}}\right)e^{-\frac{1}{6}t}\right)\int_{\mathbb{R}^3}\frac{|\mathbf{w}-\mathbf{w}_\infty|^2}{G}dx \qquad (5.4)\\
&\quad+\left(2\varepsilon+\frac{3}{2\varepsilon}\right)\tilde{C}(1+t^{-\frac{1}{2}})e^{-\frac{1}{6}t}\sum_{i=1}^3\left(\int_{\mathbb{R}^3}w_0^i dx\right)^2.
\end{aligned}$$

Now we should mention that $(1+t^{-\frac{1}{2}})e^{-\frac{1}{6}t}$ is integrable on $[0,\infty)$, so we could use Gronwall's Lemma. But since this term is bounded on $[t_0,\infty)$ by $(1+t_0^{-\frac{1}{2}})e^{-\frac{1}{6}t_0}$, for sake of simple notation we rather use this bound, since we have to make the assumption $t>t_0$ anyways. Since $\tilde{C}=\tilde{C}\left(\|\mathbf{w}_0\|_{L^{3/2}(\mathbb{R}^3)}\right)$ is sufficiently small, we can now choose ε such that

$2\delta = \left(5\varepsilon + \frac{3}{2\varepsilon}\right) \tilde{C} \left(1 + t_0^{-\frac{1}{2}}\right) e^{-\frac{1}{6}t_0}$ and after integration from t_0 to t we get that

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{|\mathbf{w}(t) - \mathbf{w}_\infty|^2}{G} dx &\leq e^{-(1-2\delta)(t-t_0)} \int_{\mathbb{R}^3} \frac{|\mathbf{w}(t_0) - \mathbf{w}_\infty|^2}{G} dx \\ &\quad + e^{-(1-2\delta)(t-t_0)} \hat{C} \sum_{i=1}^{3} \left(\int_{\mathbb{R}^3} w_0^i dx\right)^2 \int_{t_0}^{t} e^{(1-2\delta)(s-t_0)} e^{-\frac{1}{6}s} ds \\ &\leq e^{-(1-2\delta)(t-t_0)} \int_{\mathbb{R}^3} \frac{|\mathbf{w}(t_0) - \mathbf{w}_\infty|^2}{G} dx \\ &\quad + e^{-\frac{1}{6}(t-t_0)} \hat{C} \frac{6}{5 - 12\delta} \sum_{i=1}^{3} \left(\int_{\mathbb{R}^3} w_0^i dx\right)^2,\end{aligned}$$

where $\hat{C} = \tilde{C}(1 + t_0^{-\frac{1}{2}}) \left(2\varepsilon + \frac{3}{2\varepsilon}\right)$. Now, if we take the square root, we get

$$\|\mathbf{w}(t) - \mathbf{w}_\infty\|_{L^2(G^{-1})} \leq e^{-\frac{1}{12}t} \sqrt{e^{-\frac{5}{6}t + 2\delta t} K \|\mathbf{w}(t_0) - \mathbf{w}_\infty\|^2_{L^2(G^{-1})} + \hat{K} \sum_{i=1}^{3} \left(\int_{\mathbb{R}^3} w_0^i dx\right)^2}.$$

Finally, the embedding of $L^2(G^{-1})$ in $L^p(\mathbb{R}^3)$ proves the theorem. □

In comparison to the two dimensional case, the main differences are the appearance of the second term in the root, which leads to slower convergence, and the condition $t > t_0$. The first difference appears due to the fact, that in three dimensions we have that $\sum_{i=1}^3 \int_{\mathbb{R}^3} v_i \mathbf{w}_\infty \cdot \nabla \frac{w_i}{G} dx \neq 0$ and so we had to change the proof, which leads to the appearance of this second term.
Nonetheless, if we study solutions with zero mass, this term vanishes and we have a higher order of convergence.
The second difference is caused by the estimate on the velocity field for our transformed equation. In the two dimensional case we used an estimate on the square of the L^∞-norm of the velocity field and proved that the velocity field decreased in time in this norm. For our transformed equation in three dimensions in contrary, the velocity field grows with the rate $e^{\frac{1}{3}t}$. In the proof above this is compensated by the term $e^{-\frac{t}{2}}$, with which the nonlinearity is multiplied, but this is not enough to compensate the growth of the square of the L^∞-norm. Therefore, we used a different estimate on the nonlinear terms in the quadratic entropy, which leads to a point where we have to use the convex Sobolev inequality with a time dependent factor, which becomes unbounded for $t \to 0$. So to uniformly bound this factor in time, we made the rather unnatural assumption $t > t_0$, but this does in terms of large time asymptotics not make a huge difference at all.

Again we want to undo the scaling to translate this result for the vorticity equation.

Corollary 5.3. *Under the assumptions of the previous theorem we have for solutions of the vorticity equation that*

$$\begin{aligned}\left\|\boldsymbol{\omega}(t) - \frac{1}{(1+t)^{\frac{3}{2}}} \mathbf{w}_\infty \left(\frac{3}{\sqrt{6}} \frac{\cdot}{\sqrt{1+t}}\right)\right\|_{L^p(\mathbb{R}^3)} &\leq C(p) \frac{1}{(1+t)^{\frac{13}{8} - \frac{3}{2p}}} \\ &\quad \cdot \sqrt{\frac{1}{(1+t)^{\frac{5}{4} - \frac{3}{2}\delta}} K \|\mathbf{w}(t_0) - \mathbf{w}_\infty\|^2_{L^2(G^{-1})} + \tilde{K}}\end{aligned}$$

for $t > t_0$.

This result can be extended to the case $p > 2$ with the same argument, that was made in Remark 3.5.

One should mention that a result like Corollary 3.3 does not hold in the three dimensional case. While the equations (5.3) still are invariant under rotations, this does not hold for $\mathrm{div}(\mathbf{w}) = 0$, so we can not have solutions that are radially symmetric.

Just like in the two dimensional case, if the first spectral subspaces do not appear in the eigenfunction expansion of the solution, we can get a higher order of convergence.

The spectrum of the linear operator $Lw := \Delta w + \frac{1}{3}x \cdot \nabla w + w$ can be computed with the same methods as in Lemma 4.1. So we get $\sigma(L) = \left\{-\frac{k}{3}, k \in \mathbb{N}_0\right\}$ and the eigenfunctions associated to $-\frac{k}{3}$ are given as $\phi_\alpha(x) = \partial^\alpha e^{-\frac{1}{6}|x|^2}$ for all multiindices $\alpha \in \mathbb{N}_0^3$ with $|\alpha| = k$.

Further, the improved convex Sobolev inequality from Lemma 4.3 obviously holds in the three dimensional case as well. The following lemma shows that the second part of Lemma 4.4 is also valid in the three dimensions.

Lemma 5.4. *Assume* $w_0 \in L^2(G^{-1})$ *and that* $\int_{\mathbb{R}^2} \mathbf{w}_0(x)\phi_\alpha \frac{1}{G}dx = \mathbf{0}$ *for an eigenfunction* ϕ_α *with* $|\alpha| = 1$.
Then we have $\int_{\mathbb{R}^2} \mathbf{w}(x,t)\phi_\alpha dx = \mathbf{0}$ *for all* $t > 0$.

Proof: A basis of the subspace associated to the eigenvalue $-\frac{1}{3}$ is x_1G, x_2G and x_3G and inserting this in equation (4.1) leads to

$$\frac{d}{dt}\int_{\mathbb{R}^2} w_j x_i dx = \int_{\mathbb{R}^2} Lw_j x_i dx - \frac{2}{3}e^{-\frac{t}{2}}\int_{\mathbb{R}^2} (\mathbf{v}\cdot\nabla)w_j x_i dx + \frac{2}{3}e^{-\frac{t}{2}}\int_{\mathbb{R}^2} (\mathbf{w}\cdot\nabla)v_j x_i dx.$$

Since the linear operator L is in divergence form, we get that

$$\int_{\mathbb{R}^2} Lw_j x_i dx = -\int_{\mathbb{R}^2} \partial_{x_i} w_j + \frac{1}{3}x_i w_j dx = -\frac{1}{3}\int_{\mathbb{R}^2} w_j x_i dx,$$

because w_j is decreasing fast for $|x| \to \infty$. Using integration by parts in the nonlinear terms leads to

$$-\int_{\mathbb{R}^2} (\mathbf{v}\cdot\nabla)w_j x_i dx + \int_{\mathbb{R}^2} (\mathbf{w}\cdot\nabla)v_j x_i dx = \int_{\mathbb{R}^2} v_i w_j - v_j w_i dx.$$

For $i = j$ this is obviously zero. For $i \neq j$ we will show that this vanishes too for $i = 1$, $j = 2$, the other cases follow with exactly the same arguments. Using $\mathbf{w} = \mathrm{rot}(\mathbf{v})$ and $\mathrm{div}(\mathbf{v}) = 0$ we get

$$\begin{aligned}\int_{\mathbb{R}^2} v_1 w_2 - v_2 w_1 dx &= \int_{\mathbb{R}^2} v_1(\partial_{x_3}v_1 - \partial_{x_1}v_3) - v_2(\partial_{x_2}v_3 - \partial_{x_3}v_2)dx \\ &= \int_{\mathbb{R}^2} \partial_{x_1}v_1 v_3 + \partial_{x_2}v_2 v_3 dx = -\int_{\mathbb{R}^2} \partial_{x_3}v_3 v_3 dx = 0.\end{aligned}$$

So altogether we get

$$\frac{d}{dt}\int_{\mathbb{R}^2} w_j x_i dx = -\frac{1}{3}\int_{\mathbb{R}^2} w_j x_i dx$$

for all solutions $\mathbf{w}$ and therefore, the subspace created by $\int_{\mathbb{R}^2} \mathbf{w}_0(x)\phi_\alpha \frac{1}{G} dx = \mathbf{0}$ for $|\alpha| = 1$ stays invariant by the nonlinear evolution. □

Just as in the two dimensional case for the vorticity equation we can always translate the initial condition such that the first moments vanish, since the vorticity equation in three dimensions is invariant under translations as well.

Now we are able to prove a higher order convergence result.

Corollary 5.5. *Let the assumptions of Theorem 5.2 be fulfilled and assume that* $\mathbf{w}_0 \in L^2(G^{-1})$ *with* $\int_{\mathbb{R}^3} \mathbf{w}_0 \phi_\alpha \frac{1}{G} dx = \mathbf{0}$ *for all eigenfunctions with* $0 \leq |\alpha| \leq 1$. *Then we have*

$$\|\mathbf{w}(t)\|_{L^p(\mathbb{R}^3)} \leq C_1(p, t_0) e^{-(1-2\delta)t}$$

for all $t > t_0$.

Proof: The corollary follows directly from the improved convex Sobolev inequality just as Corollary 4.6. One should mention that because of our additional assumptions, our solution has zero mass and therefore the stationary state $\mathbf{w}_\infty$ is zero as well and the second term in the square from Theorem 5.2 does not appear. If we do not study solutions with zero mass, we do not get a fast rate of convergence since the improved convex Sobolev inequality does not help with the second term of the right hand side of the differential inequality (5.4). □

Again, if more eigenspaces do not appear for all $t > 0$, we get an even faster convergence just as in the two dimensional case. As in the two dimensional case, if one chooses a basis of the second eigenspace that has a radially symmetric eigenfunction ϕ_α, the subspace generated by $\int_{\mathbb{R}^3} \mathbf{w}_0 \phi_\alpha \frac{1}{G} dx = 0$ stays invariant under the nonlinear evolution. But for all other eigenfunctions this does not hold.

In the following we want to cite and explain some interesting results about large time behavior of the vorticity equation in three dimensions, which we found in the literature, to give an overview about some further results in the three dimensional case.

An interesting connection between the two and the three dimensional case is, that the stationary solution of the nonlinear Fokker-Planck equation in two dimension appears in a family of explicit solutions of the Navier-Stokes equations in three dimensions. These particular solutions are called Burgers vortices and are given by

$$\boldsymbol{\omega}_B(x) = \begin{pmatrix} 0 \\ 0 \\ \alpha \frac{\gamma_3}{4\pi} e^{-\gamma_3 \frac{x_1^2 + x_2^2}{4}} \end{pmatrix}$$

and the velocity field

$$\mathbf{u}_B(x) = \frac{1}{2\pi} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} \frac{1}{x_1^2 + x_2^2} \left(1 - e^{-\gamma_3 \frac{x_1^2 + x_2^2}{4}}\right) + \begin{pmatrix} \gamma_1 x_1 \\ \gamma_2 x_2 \\ \gamma_3 x_3 \end{pmatrix},$$

with $\sum_{i=1}^3 \gamma_i = 1$ and $\gamma_1, \gamma_2 < 0$ and $\gamma_3 > 0$. Due to the last term, this velocity field is not satisfying the Biot-Savart law we mentioned in the introduction. Nonetheless, it is a particular solution of the equations $\boldsymbol{\omega} = \mathrm{rot}(\mathbf{u})$, $\mathrm{div}(\mathbf{u}) = 0$, since the second term is a Laplace field. One should note that this particular solution is a solution with infinite energy and therefore leads to no contradiction to our result from above.

These Burgers vortices have an interesting asymptotic behavior, since they are stable in a particular sense, which was analyzed in [GW4] and which we want to cite in the following theorem, which can be found in [GW4] Theorem 1.2.

Theorem 5.6. *Assume that* $(\gamma_1, \gamma_2, \gamma_3) = \gamma_3\left(-\frac{1}{2}, \frac{1}{2}, 1\right)$ *and that for every fixed* $\mu \in \left(0, \frac{1}{2}\right)$ *there exists* $R, \varepsilon > 0$ *such that if* $|\alpha| \leq R$ *and* $\sup_{x_3 \in \mathbb{R}} \|\boldsymbol{\omega}_0\|_{L^2((\omega_B)_3^{-1})} \leq \varepsilon$ *the solution* $\boldsymbol{\omega}$ *of* (1.4) *with initial condition* $\boldsymbol{\omega}_0 + \alpha\boldsymbol{\omega}_B$ *satisfies for* $p \in [1,2]$

$$\sup_{x_r \in I} \|\boldsymbol{\omega}(\cdot, x_3, t) - \tilde{\alpha}\boldsymbol{\omega}_B(\cdot)\|_{L^p(\mathbb{R}^2)} = \mathcal{O}(e^{-\mu\gamma_3 t}),$$

where $I \subset \mathbb{R}$ *is a compact interval and* $\tilde{\alpha} = \alpha + \delta\alpha$ *and*

$$\delta\alpha = \left(\frac{\gamma_3}{2\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}^2} e^{-\frac{\gamma_3 x_3^2}{2}} (\omega_0)_3(x_1, x_2, x_3) d(x_1, x_2) dx_3.$$

Proof: We shall sketch the idea of the proof, which is to linearize the vorticity equation at the Burgers vortex $\alpha\boldsymbol{\omega}_B$ and then to explicitly compute the integral representation of the associated semigroup. Then in the decomposition $\boldsymbol{\omega}(x,t) = \phi(x_3,t)\boldsymbol{\omega}_B(x_1,x_2) + \tilde{\boldsymbol{\omega}}(x,t)$ with $\phi(x_3,t) = \int_{\mathbb{R}^2} \omega_3(x_1,x_2,x_3,t) d(x_1,x_2)$ the function $\tilde{\boldsymbol{\omega}}$ decays exponentially to 0, since the semigroup does that acting on functions, which satisfy that the integral with respect to the first two components vanishes. Then it can be shown that $\phi(x_3,t)$ satisfies the equation

$$\phi_t + \gamma_3 x_3 \partial_{x_3}\phi = \partial_{x_3}^2 \phi,$$

which can be solved explicitly and the explicit formula shows the convergence in the desired norm to $\delta\alpha$. □

In the paper [GW4] this result has also been extended to non axisymmetric Burgers vortices, where γ_1 and γ_2 are given as $\gamma_1 = -\frac{\gamma_3}{2}(1+\lambda)$ and $\gamma_2 = -\frac{\gamma_3}{2}(1-\lambda)$, which seems to be a better fit as a model for turbulent flows.

Finally, we shall cite a result stated in [Rou], where the vorticity equation has been studied on the domain $\mathbb{R}^2 \times (0,1)$ and where $\boldsymbol{\omega} = \boldsymbol{\omega}(x,z,t)$ for $(x,z,t) \in \mathbb{R}^2 \times (0,1) \times \mathbb{R}^+$ is 1-periodic in z. The main idea in the paper [Rou] is that in this case the scaling

$$\begin{aligned}
\boldsymbol{\omega}(x,z,t) &= \frac{1}{1+t}\mathbf{w}\left(\frac{x}{\sqrt{1+t}}, z, \ln(1+t)\right) \\
u(x,z,t) &= \frac{1}{\sqrt{1+t}}\mathbf{v}\left(\frac{x}{\sqrt{1+t}}, z, \ln(1+t)\right)
\end{aligned}$$

leaves the domain invariant and leads to the equations

$$\begin{aligned}
\partial_t \mathbf{w} &= \Delta_x \mathbf{w} + \frac{1}{2} x \cdot \nabla_x \mathbf{w} + \mathbf{w} + e^t \partial_z^2 \mathbf{w} + N(\mathbf{w}) \qquad (5.5) \\
\nabla_x \cdot \mathbf{w}_x + e^{\frac{t}{2}} \partial_z w_z &= 0
\end{aligned}$$

with $N(\mathbf{w}) = (\mathbf{w}_x \cdot \nabla_x)\mathbf{v} - (\mathbf{v}_x \cdot \nabla_x)\mathbf{w} + e^{\frac{t}{2}}(w_3\partial_z\mathbf{v} - v_3\partial_z\mathbf{w})$ where $\mathbf{w}_x = (w_1, w_2)^T$ and $\nabla_x = (\partial_{x_1}, \partial_{x_2})^T$.

An important property of this equation is, that we have conservation of mass, in particular for the third component. Further, $\mathbf{G} := (0, 0, G)$ with $G = \frac{1}{4\pi}e^{-\frac{|x|^2}{4}}$, $x \in \mathbb{R}^2$ is a stationary solution of the equations above. The following result shows that this stationary solution is stable in a similar way as the Burgers vortices from above are stable.

Theorem 5.7. *There exists an $\varepsilon > 0$ such that for every $\mathbf{w}_0 \in L^2(G^{-1})$ with $div(\mathbf{w}_0) = 0$ and $\|\mathbf{w}_0\|_{L^2(G^{-1})} \leq \varepsilon$ an unique solution $\mathbf{w} \in C([0, \infty), L^p(\mathbb{R}^2 \times (0, 1)))$ with $\mathbf{w}(0) = \mathbf{w}_0$ of the equations (5.5) exists. Further for every $\mu \in (0, \frac{1}{2})$ we have*

$$\|\mathbf{w}(t) - \alpha\mathbf{G}\|_{L^p(\mathbb{R}^2\times(0,1))} \leq Ce^{-\mu t} \|\mathbf{w}_0\|_{L^2(G^{-1})}$$

for $p \in [1, 2]$ with $\alpha = \int_{\mathbb{R}^2\times(0,1)} (w_0)_3 dz dx$.

The main idea of the proof is to once again make some estimates on the integral equation as well as a decomposition of the solution similar as in Theorem 5.6.

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