TECHNISCHE
UNIVERSITÄT
WIEN

Large-time behavior in Fokker-Planck equations

Anton ARNOLD

(www.asc.tuwien.ac.at/∼arnold/)

Porto Ercole, June 2014

7 Topics – Contents:

1. Entropy method for symmetric Fokker-Planck equations
2. Non-symmetric Fokker-Planck equations, generalized Bakry-Emery condition
3. Logarithmic & convex Sobolev inequalities
4. Perturbation results & application to a semi-linear model
5. Fokker-Planck model for polymeric fluid-flow
6. Fokker-Planck equations with non-local terms
7. Hypocoercive equations

1. Entropy method for symmetric Fokker-Planck equations

Outline:

1. (non)symmetric Fokker-Planck equations
2. steady state
3. relative entropy
4. entropy method $\rightarrow$ large time convergence
5. comparison to spectral methods

linear symmetric Fokker-Planck equation

evolution of probability density $f(x,t)$, $x \in \mathbb{R}^n$, $t > 0$:

$$\begin{aligned} f_t &= \operatorname{div}\Big(\mathbf{D}(x) \cdot [\nabla A(x) f + \nabla f]\Big) =: Lf \qquad (1) \\ f(x,0) &= f_0(x); \quad f_0 \in L^1_+(\mathbb{R}^n),\ \int_{\mathbb{R}^n} f_0\, dx = 1 \quad \Rightarrow \quad f(x,t) \geq 0 \end{aligned}$$

linear symmetric Fokker-Planck equation

evolution of probability density $f(x,t)$, $x \in \mathbb{R}^n$, $t > 0$:

$$f_t = \operatorname{div}\Big(\mathbf{D}(x)\cdot[\nabla A(x) f + \nabla f]\Big) =: Lf \qquad (1)$$

$$f(x,0) = f_0(x); \quad f_0 \in L^1_+(\mathbb{R}^n),\ \int_{\mathbb{R}^n} f_0\,dx = 1 \quad \Rightarrow \quad f(x,t) \geq 0$$

$$f_\infty(x) = e^{-A(x)} \ \ldots \text{ (unique) normalized steady state}$$

$$Lf = \operatorname{div}\Big(f_\infty \mathbf{D}(x) \nabla \frac{f}{f_\infty}\Big) \ \ldots \text{ symmetric in } L^2(\mathbb{R}^n, f_\infty^{-1})$$

$\mathbf{D}(x) > 0$ ... positive definite matrix $\forall\, x \in \mathbb{R}^n$

$A(x)$ ... scalar confinement potential, i.e. $A(x) \to \infty$ as $|x| \to \infty$;
idea : $A(x) \gtrsim c|x|^2$

applications: Brownian motion, kinetic Fokker-Planck equation (for plasmas)

non-symmetric Fokker-Planck equation

$$f_t = \operatorname{div}\Big(\mathbf{D}(x)\cdot[\nabla A(x) f + \nabla f]\Big) \quad \dots \text{ symmetric FP}$$

$$f_\infty(x) = \mathrm{e}^{-A(x)} \quad \dots \text{ (unique) normalized steady state}$$

non-symmetric Fokker-Planck equation

$$f_t = \mathrm{div}\Big(\mathbf{D}(x)\cdot[\nabla A(x) f + \nabla f]\Big) \quad \ldots \text{ symmetric FP}$$

$$f_\infty(x) = e^{-A(x)} \quad \ldots \text{ (unique) normalized steady state}$$

$$f_t = \mathrm{div}\Big(\mathbf{D}(x)\cdot[\vec{G}(x) f + \nabla f]\Big) \quad \ldots \text{ non-symmetric FP, } n \geq 2$$

$\vec{G}(x)$ … not a gradient

$f_\infty(x)$ … in general not explicit

non-symmetric Fokker-Planck equation

$$f_t = \operatorname{div}\Big(\mathbf{D}(x)\cdot[\nabla A(x)f + \nabla f]\Big) \quad \ldots \text{ symmetric FP}$$

$$f_\infty(x) = e^{-A(x)} \quad \ldots \text{ (unique) normalized steady state}$$

$$f_t = \operatorname{div}\Big(\mathbf{D}(x)\cdot[\vec{G}(x)f + \nabla f]\Big) \quad \ldots \text{ non-symmetric FP}, \ n \geq 2$$

$\vec{G}(x)$ ... not a gradient

$f_\infty(x)$ ... in general not explicit

non-symmetric examples:

- kinetic Fokker-Planck equation (for plasmas): degenerate parabolic $\rightarrow$ hypocoercive
- quantum-kinetic (Wigner-) Fokker-Planck equation (for dissipative quantum systems: electron transport in crystal lattice, e.g.)

admissible relative entropies (for entropy method)

$\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ ... entropy generators

$$\psi \geq 0, \quad \psi(1) = 0, \quad \psi'' > 0, \quad (\psi''')^2 \leq \frac{1}{2}\psi''\psi^{\mathrm{IV}}$$

admissible relative entropies (for entropy method)

$\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ ... entropy generators

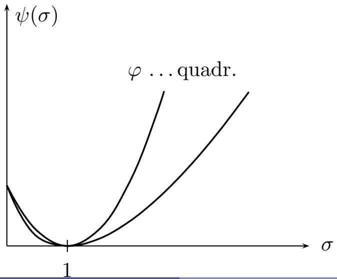
$\psi \geq 0, \quad \psi(1) = 0, \quad \psi'' > 0, \quad (\psi''')^2 \leq \frac{1}{2}\psi''\psi^{\mathrm{IV}}$

examples 1) $\psi_1(\sigma) = \sigma \ln \sigma - \sigma + 1$

2) $\psi_p(\sigma) = \sigma^p - 1 - p(\sigma - 1), \quad 1 < p \leq 2$

$$e_\psi(f_1|f_2) := \int_{\mathbb{R}^n} \psi\left(\frac{f_1}{f_2}\right) f_2 \, \mathrm{d}x \geq 0$$ ... relative entropy

for $f_{1,2}$ ... probability densities



goals of the entropy method

- prove convergence $f(t) \stackrel{t\to\infty}{\longrightarrow} f_\infty$
- (possibly) with sharp exponential rate

goals of the entropy method

- prove convergence $f(t) \stackrel{t\to\infty}{\longrightarrow} f_\infty$
- (possibly) with sharp exponential rate

- convergence in relative entropy $\Rightarrow$ L^1–convergence by (generalized) Csiszár-Kullback inequality:

$$\|f_1 - f_2\|^2_{L^1(\mathbb{R}^n)} \le \frac{2}{\psi''(1)}\, e_\psi(f_1|f_2)$$

goals of the entropy method

- prove convergence $f(t) \overset{t\to\infty}{\longrightarrow} f_\infty$
- (possibly) with sharp exponential rate

- convergence in relative entropy $\Rightarrow$ L^1–convergence by (generalized) Csiszár-Kullback inequality:

$$\|f_1 - f_2\|^2_{L^1(\mathbb{R}^n)} \le \frac{2}{\psi''(1)}\, e_\psi(f_1|f_2)$$

- as a by-product:
derivation of functional inequalities (of Poincaré / Sobolev type)

entropy dissipation in 1D; A, D ... time independent

$$f_\infty = e^{-A(x)};$$
$$f_t = \Big(D(x)\,[A_x f + f_x]\Big)_x = \left[D(x)\,f_\infty\left(\frac{f}{f_\infty}\right)_x\right]_x$$

$$\frac{\mathrm{d}}{\mathrm{d}t} e_\psi(f(t)|f_\infty) = \int_{\mathbb{R}} \psi'\left(\frac{f(x,t)}{f_\infty(x)}\right) f_t \,\mathrm{d}x$$
$$= \int \psi'\Big(\frac{f}{f_\infty}\Big)\left[D(x)\,f_\infty\Big(\frac{f}{f_\infty}\Big)_x\right]_x \mathrm{d}x$$
$$= -\int \psi''\Big(\frac{f}{f_\infty}\Big) D(x)\Big(\partial_x \frac{f}{f_\infty}\Big)^2 f_\infty \,\mathrm{d}x$$
$$=: -I_\psi(f|f_\infty) \le 0 \qquad \ldots \text{(negative) Fisher information}$$

entropy dissipation in nD; A, $\mathbf{D}$ ... time dependent

Lemma 1

Let $f(t), g(t)$ *solve Fokker-Planck equation* (1)

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t} e_\psi(f(t)|g(t)) = -\int_{\mathbb{R}^n} \psi''\left(\frac{f(t)}{g(t)}\right) \nabla^\top \frac{f}{g} \cdot \mathbf{D}(x,t) \cdot \nabla \frac{f}{g}\, g(t)\, \mathrm{d}x \leq 0$$

Rem: No *explicit* steady state needed.
Drift term does not enter the r.h.s.
2 trajectories approach each other.

Ref's: [Bolley-Gentil] J. Math. Pures Appl. 2010;
[Fontbona-Jourdain] preprint 2013

proof of Lemma 1 (even for non-symmetric FP)

$f(t)$, $g(t)$ satisfy $f_t = \mathrm{div}\Big(\mathbf{D} \cdot \Big[\nabla f + f\vec{G}\Big]\Big)$

$$\frac{\mathrm{d}}{\mathrm{d}t} e_\psi(f(t)|g(t)) = \int_{\mathbb{R}^n} \psi'\left(\frac{f}{g}\right)\left(f_t - \frac{f}{g} g_t\right) + \psi\left(\frac{f}{g}\right) g_t \,\mathrm{d}x$$

proof of Lemma 1 (even for non-symmetric FP)

$f(t),\ g(t)$ satisfy $f_t = \mathrm{div}\Big(\mathbf{D}\cdot\big[\nabla f + f\vec{G}\big]\Big)$

$$\frac{\mathrm{d}}{\mathrm{d}t} e_\psi(f(t)|g(t)) = \int_{\mathbb{R}^n} \psi'\Big(\frac{f}{g}\Big)\Big(f_t - \frac{f}{g}g_t\Big) + \psi\Big(\frac{f}{g}\Big)g_t\,\mathrm{d}x$$

$$= -\int \psi''\big(\tfrac{f}{g}\big)\nabla^\top\big(\tfrac{f}{g}\big)\cdot\mathbf{D}\cdot\Big[\nabla f + f\vec{G}\Big]\,\mathrm{d}x + \int\Big[\psi\big(\tfrac{f}{g}\big) - \tfrac{f}{g}\psi'\big(\tfrac{f}{g}\big)\Big]g_t\,\mathrm{d}x$$

proof of Lemma 1 (even for non-symmetric FP)

$f(t)$, $g(t)$ satisfy $f_t = \mathrm{div}\Big(\mathbf{D}\cdot\big[\nabla f + f\vec{G}\,\big]\Big)$

$$\frac{\mathrm{d}}{\mathrm{d}t} e_\psi(f(t)|g(t)) = \int_{\mathbb{R}^n} \psi'\Big(\frac{f}{g}\Big)\Big(f_t - \frac{f}{g}g_t\Big) + \psi\Big(\frac{f}{g}\Big)g_t\,\mathrm{d}x$$

$$= -\int \psi''\big(\tfrac{f}{g}\big)\nabla^\top\big(\tfrac{f}{g}\big)\cdot\mathbf{D}\cdot\Big[\nabla f + f\vec{G}\,\Big]\mathrm{d}x + \int\Big[\psi\big(\tfrac{f}{g}\big) - \tfrac{f}{g}\psi'\big(\tfrac{f}{g}\big)\Big]g_t\,\mathrm{d}x$$

$$= -\int \psi''\big(\tfrac{f}{g}\big)\nabla^\top\big(\tfrac{f}{g}\big)\cdot\mathbf{D}\cdot\Big[\nabla f + \cancel{f\vec{G}}\,\Big]\mathrm{d}x$$

$$+\int \tfrac{f}{g}\psi''\big(\tfrac{f}{g}\big)\nabla^\top\big(\tfrac{f}{g}\big)\cdot\mathbf{D}\cdot\Big[\nabla g + \cancel{g\vec{G}}\,\Big]\mathrm{d}x$$

proof of Lemma 1 (even for non-symmetric FP)

$f(t),\ g(t)$ satisfy $f_t = \mathrm{div}\Big(\mathbf{D}\cdot\big[\nabla f + f\vec{G}\,\big]\Big)$

$$\frac{\mathrm{d}}{\mathrm{d}t} e_\psi(f(t)|g(t)) = \int_{\mathbb{R}^n} \psi'\Big(\frac{f}{g}\Big)\Big(f_t - \frac{f}{g}g_t\Big) + \psi\Big(\frac{f}{g}\Big)g_t\,\mathrm{d}x$$

$$= -\int \psi''\big(\tfrac{f}{g}\big)\nabla^\top\big(\tfrac{f}{g}\big)\cdot\mathbf{D}\cdot\Big[\nabla f + f\vec{G}\Big]\,\mathrm{d}x + \int\Big[\psi\big(\tfrac{f}{g}\big) - \tfrac{f}{g}\psi'\big(\tfrac{f}{g}\big)\Big]g_t\,\mathrm{d}x$$

$$= -\int \psi''\big(\tfrac{f}{g}\big)\nabla^\top\big(\tfrac{f}{g}\big)\cdot\mathbf{D}\cdot\Big[\nabla f + \cancel{f\vec{G}}\Big]\,\mathrm{d}x$$

$$+\int \tfrac{f}{g}\psi''\big(\tfrac{f}{g}\big)\nabla^\top\big(\tfrac{f}{g}\big)\cdot\mathbf{D}\cdot\Big[\nabla g + \cancel{g\vec{G}}\Big]\,\mathrm{d}x$$

$$= -\int \psi''\big(\tfrac{f}{g}\big)\nabla^\top\big(\tfrac{f}{g}\big)\cdot\mathbf{D}\cdot\nabla\big(\tfrac{f}{g}\big)\,g\,\mathrm{d}x \le 0 \qquad \square$$

entropy decay to 0 (first without rate)

Lemma 2

Let $f_0 \in L^2(\mathbb{R}^n; f_\infty^{-1}) \quad \Rightarrow \quad e_\psi(f(t)|f_\infty) \xrightarrow{t\to\infty} 0$

entropy decay to 0 (first without rate)

Lemma 2

Let $f_0 \in L^2(\mathbb{R}^n; f_\infty^{-1}) \quad \Rightarrow \quad e_\psi(f(t)|f_\infty) \xrightarrow{t\to\infty} 0$

Proof: bound e_ψ by a quadratic super entropy e_φ:

$$0 \le e_\psi(f(t)|f_\infty) \le \psi''(1)\|f(t) - f_\infty\|^2_{L^2(\mathbb{R}^n; f_\infty^{-1})} \quad \searrow 0$$

with spectral representation of *symmetric* evolution of $z(t) := \frac{f(t)}{\sqrt{f_\infty}}$ in $L^2(\mathbb{R}^n)$:

$$z(t) = \sqrt{f_\infty} + \int_{(0,\infty)} \mathrm{e}^{-\lambda t} \,\mathrm{d}\Big(P_\lambda \frac{f_0}{\sqrt{f_\infty}}\Big)$$

P_λ ... projection valued spectral measure of $H = -\frac{1}{\sqrt{f_\infty}} L(\sqrt{f_\infty}\,\cdot\,)$

Idea: use spectral theory

□

exponential decay of entropy dissipation for $\mathbf{D} \equiv$ const.

$\mathbf{D} =$ const. in x, $\quad f_\infty(x) = e^{-A(x)}$

Theorem 1

Let $I_\psi(f_0|f_\infty) < \infty$. Let $\mathbf{D}, A$ satisfy a

$$\boxed{\textit{Bakry - Emery condition} \quad \frac{\partial^2 A(x)}{\partial x^2} \geq \underbrace{\lambda_1}_{>0} \mathbf{D}^{-1}} \quad \forall x \in \mathbb{R}^n \tag{2}$$

$$\Rightarrow I_\psi(f(t)|f_\infty) \leq e^{-2\lambda_1 t} I_\psi(f_0|f_\infty), \qquad t \geq 0$$

A ... uniformly convex if $\mathbf{D} = \mathbf{I}$

Ref's: [Bakry-Emery] 1984/85;
[Arnold-Markowich-Toscani-Unterreiter] Comm. PDE 2001

proof of Theorem 1 in 1D with $\mathbf{D} \equiv 1$ (BEC: $A''(x) \geq \lambda_1$)

$$I_\psi(t) = \int \psi''\Big(\frac{f(t)}{f_\infty}\Big)\Big(\underbrace{\partial_x \frac{f(t)}{f_\infty}}_{=:u}\Big)^2 f_\infty \, \mathrm{d}x \geq 0 \qquad ; \quad f_t = \Big(f_\infty \underbrace{\partial_x \frac{f}{f_\infty}}_{=:u}\Big)_x$$

proof of Theorem 1 in 1D with $\mathbf{D} \equiv 1$ (BEC: $A''(x) \geq \lambda_1$)

$$I_\psi(t) = \int \psi''\Big(\frac{f(t)}{f_\infty}\Big)\Big(\underbrace{\partial_x \frac{f(t)}{f_\infty}}_{=:u}\Big)^2 f_\infty \,\mathrm{d}x \geq 0 \qquad ; \quad f_t = \Big(f_\infty \underbrace{\partial_x \frac{f}{f_\infty}}_{=:u}\Big)_x$$

$$\frac{\mathrm{d}}{\mathrm{d}t} I = \int \psi'''\Big(\frac{f}{f_\infty}\Big) \underbrace{(f_\infty u)_x}_{=f_t} u^2 \,\mathrm{d}x \; + \; 2\int \psi''\Big(\frac{f}{f_\infty}\Big) u \underbrace{u_t f_\infty}_{= (u_x f_\infty)_x - A'' u f_\infty} \mathrm{d}x$$

proof of Theorem 1 in 1D with $\mathbf{D} \equiv 1$ (BEC: $A''(x) \geq \lambda_1$)

$$I_\psi(t) = \int \psi''\Big(\frac{f(t)}{f_\infty}\Big)\Big(\underbrace{\partial_x \frac{f(t)}{f_\infty}}_{=:u}\Big)^2 f_\infty \,\mathrm{d}x \geq 0 \qquad ; \quad f_t = \Big(f_\infty \underbrace{\partial_x \frac{f}{f_\infty}}_{=:u}\Big)_x$$

$$\frac{\mathrm{d}}{\mathrm{d}t} I = \int \psi'''\Big(\frac{f}{f_\infty}\Big) \underbrace{(f_\infty u)_x}_{=f_t} u^2 \,\mathrm{d}x \; + \; 2\int \psi''\Big(\frac{f}{f_\infty}\Big) u \underbrace{u_t f_\infty}_{= (u_x f_\infty)_x - A'' u f_\infty} \mathrm{d}x$$

$$\overset{\text{2 int. by parts}}{=} -\int \psi^{\mathrm{IV}}\Big(\frac{f}{f_\infty}\Big) u^4 f_\infty \,\mathrm{d}x - 2\int \psi'''\Big(\frac{f}{f_\infty}\Big) u_x u^2 f_\infty \,\mathrm{d}x$$

$$\underbrace{-2\int \psi''\Big(\frac{f}{f_\infty}\Big) A'' u^2 f_\infty \,\mathrm{d}x}_{\geq 0} - 2\int \psi'''\Big(\frac{f}{f_\infty}\Big) u^2 u_x f_\infty \; - \; \psi''\Big(\frac{f}{f_\infty}\Big) {u_x}^2 f_\infty \,\mathrm{d}x =$$

proof of Theorem 1 in 1D with $\mathbf{D} \equiv 1$ (BEC: $A''(x) \geq \lambda_1$)

$$\frac{\mathrm{d}}{\mathrm{d}t} I = -2 \underbrace{\int \psi''\left(\frac{f}{f_\infty}\right) A''(x) u^2 f_\infty \,\mathrm{d}x}_{\geq 0} - 2 \int \underbrace{\mathrm{Tr}\,(XY)}_{\geq 0} \underbrace{f_\infty}_{\geq 0} \,\mathrm{d}x$$

proof of Theorem 1 in 1D with $\mathbf{D} \equiv 1$ (BEC: $A''(x) \geq \lambda_1$)

$$\frac{\mathrm{d}}{\mathrm{d}t} I = -2 \underbrace{\int \psi''\left(\frac{f}{f_\infty}\right) A''(x) u^2 f_\infty \,\mathrm{d}x}_{\geq 0} - 2 \int \underbrace{\mathrm{Tr}\,(XY)}_{\geq 0} \underbrace{f_\infty}_{\geq 0} \,\mathrm{d}x$$

$$\overset{\text{BEC}}{\leq} \quad -2\lambda_1 \int \psi''\left(\tfrac{f}{f_\infty}\right) u^2 f_\infty \,\mathrm{d}x = -2\lambda_1 I(t)$$

proof of Theorem 1 in 1D with $\mathbf{D} \equiv 1$ (BEC: $A''(x) \geq \lambda_1$)

$$\frac{\mathrm{d}}{\mathrm{d}t} I = -2 \underbrace{\int \psi''\left(\frac{f}{f_\infty}\right) A''(x) u^2 f_\infty \,\mathrm{d}x}_{\geq 0} - 2 \int \underbrace{\mathrm{Tr}\,(XY)}_{\geq 0} \underbrace{f_\infty}_{\geq 0} \,\mathrm{d}x$$

$$\overset{\text{BEC}}{\leq} \quad -2\lambda_1 \int \psi''\left(\frac{f}{f_\infty}\right) u^2 f_\infty \,\mathrm{d}x = -2\lambda_1 I(t)$$

use

$$X := \begin{pmatrix} \psi'' & \psi''' \\ \psi''' & \frac{1}{2}\psi^{\mathrm{IV}} \end{pmatrix} \left(\frac{f}{f_\infty}\right) \geq 0 \quad ; Y := \begin{pmatrix} u_x^2 & u^2 u_x \\ u^2 u_x & u^4 \end{pmatrix} \geq 0$$

with $\det X = \frac{1}{2}\psi''\psi^{\mathrm{IV}} - (\psi''')^2 \geq 0$ (for admissible entropies) .

proof of Theorem 1 in 1D with $\mathbf{D} \equiv 1$ (BEC: $A''(x) \geq \lambda_1$)

$$\frac{\mathrm{d}}{\mathrm{d}t} I = \underbrace{-2 \int \psi''\Big(\frac{f}{f_\infty}\Big) A''(x) u^2 f_\infty \, \mathrm{d}x}_{\geq 0} - 2 \int \underbrace{\mathrm{Tr}\,(XY)}_{\geq 0} \underbrace{f_\infty}_{\geq 0} \, \mathrm{d}x$$

$$\overset{\text{BEC}}{\leq} \quad -2\lambda_1 \int \psi''\Big(\frac{f}{f_\infty}\Big) u^2 f_\infty \, \mathrm{d}x = -2\lambda_1 I(t)$$

use

$$X := \begin{pmatrix} \psi'' & \psi''' \\ \psi''' & \frac{1}{2}\psi^{\mathrm{IV}} \end{pmatrix} \Big(\frac{f}{f_\infty}\Big) \geq 0 \quad ; Y := \begin{pmatrix} u_x^2 & u^2 u_x \\ u^2 u_x & u^4 \end{pmatrix} \geq 0$$

with $\det X = \frac{1}{2}\psi''\psi^{\mathrm{IV}} - (\psi''')^2 \geq 0$ (for admissible entropies).

$$\Rightarrow I_\psi(f(t)|f_\infty) \leq \mathrm{e}^{-2\lambda_1 t} I_\psi(f_0|f_\infty)$$

$\square$

exponential decay of relative entropy for $\mathbf{D} \equiv$ const.

Theorem 2

Let $\mathbf{D}$, *A satisfy BEC* $\quad \frac{\partial^2 A(x)}{\partial x^2} \geq \lambda_1 \mathbf{D}^{-1} \Rightarrow$

$$e_\psi(f(t)|f_\infty) \leq e^{-2\lambda_1 t} e_\psi(f_0|f_\infty), \qquad t \geq 0$$

exponential decay of relative entropy for $\mathbf{D} \equiv$ const.

Theorem 2

Let $\mathbf{D}$, A *satisfy BEC* $\frac{\partial^2 A(x)}{\partial x^2} \geq \lambda_1 \mathbf{D}^{-1} \Rightarrow$

$$e_\psi(f(t)|f_\infty) \leq e^{-2\lambda_1 t} e_\psi(f_0|f_\infty), \qquad t \geq 0$$

Proof: from proof of Theorem 1 :

$$\frac{\mathrm{d}}{\mathrm{d}t} I(t) \leq -2\lambda_1 \underbrace{I(t)}_{=-e'(t)} \qquad \Big| \int_t^\infty \dots \mathrm{d}t$$

Since $I(t)$, $e(t) \xrightarrow{t\to\infty} 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t} e(t) \leq -2\lambda_1 e(t) \tag{3}$$

(+ density argument) □

comparison to spectral methods

example:

Hamiltonian $H := -\Delta + V(x)$ (can be transformed to FP operator $-L$)

If $V \in L^1_{loc}$, bounded below and $V(x) \overset{|x|\to\infty}{\longrightarrow} \infty$
$\Rightarrow \exists$ positive spectral gap λ_0 [Reed-Simon IV]

pros & cons:

- sharp decay if spectral gap λ_0 is known
- but explicit values / estimates for λ_0 hard to get
- method hard to generalize to non-linear problems

comparison to spectral methods

example:

Hamiltonian $H := -\Delta + V(x)$ (can be transformed to FP operator $-L$)

If $V \in L^1_{loc}$, bounded below and $V(x) \stackrel{|x|\to\infty}{\longrightarrow} \infty$

$\Rightarrow \exists$ positive spectral gap λ_0 [Reed-Simon IV]

pros & cons:

- sharp decay if spectral gap λ_0 is known
- but explicit values / estimates for λ_0 hard to get
- method hard to generalize to non-linear problems

entropy methods:

- Bakry-Emery condition $\frac{\partial^2 A}{\partial x^2} \geq \lambda_1 \mathbf{D}^{-1}$ elementary to check
- but rate λ_1 often not sharp
- robust to many non-linear generalizations (porous medium equation, Boltzmann, ...)

2. Non-symmetric Fokker-Planck equations, generalized Bakry-Emery condition

Outline:

1. non-symmetric Fokker-Planck equations
2. decomposition of generator $\rightarrow$ steady state
3. 2 entropy methods (for large time convergence)

non-symmetric Fokker-Planck equation

evolution of probability density $f(x,t)$, $x \in \mathbb{R}^n$, $t > 0$; $n \geq 2$:

$$f_t = \operatorname{div}\Big(\mathbf{D}(x) \cdot [\underbrace{\{\nabla A(x) + \vec{F}(x)\}}_{=:\vec{G}(x)} f + \nabla f]\Big) =: Lf \qquad (4)$$

$$f(x,0) = f_0(x); \quad f_0 \in L^1_+(\mathbb{R}^n), \int_{\mathbb{R}^n} f_0 \, dx = 1 \quad \Rightarrow \quad f(x,t) \geq 0$$

Let $\vec{F}$ satisfy condition $\operatorname{div}(\mathbf{D} \cdot \vec{F} f_\infty) = 0 \quad \forall x$
$\Rightarrow f_\infty := e^{-A}$ still steady state

non-symmetric Fokker-Planck equation

evolution of probability density $f(x,t)$, $x \in \mathbb{R}^n$, $t > 0$; $n \geq 2$:

$$f_t = \operatorname{div}\Big(\mathbf{D}(x) \cdot [\underbrace{\{\nabla A(x) + \vec{F}(x)\}}_{=:\vec{G}(x)} f + \nabla f]\Big) =: Lf \tag{4}$$

$$f(x,0) = f_0(x); \quad f_0 \in L^1_+(\mathbb{R}^n), \int_{\mathbb{R}^n} f_0\, dx = 1 \quad \Rightarrow \quad f(x,t) \geq 0$$

Let $\vec{F}$ satisfy condition $\operatorname{div}(\mathbf{D} \cdot \vec{F} f_\infty) = 0 \quad \forall x$
$\Rightarrow f_\infty := e^{-A}$ still steady state

decomposition of $L = L^s + L^{as}$:
$L^s f := \operatorname{div}\Big(\mathbf{D}(x) \cdot [\nabla A(x) f + \nabla f]\Big)$ ... symmetric in $L^2(\mathbb{R}^n, f_\infty^{-1})$
$L^{as} f := \operatorname{div}\Big(\mathbf{D}(x) \cdot \vec{F} f\Big)$ ... skew-symmetric in $L^2(\mathbb{R}^n, f_\infty^{-1})$

decomposition of generator

general situation:

$$f_t = \mathrm{div}\Big(\mathbf{D}(x)\cdot[\vec{G}(x)f + \nabla f]\Big)$$

$\to$ find decomposition $\vec{G} = \nabla A + \vec{F}$, with $\mathrm{div}(\mathbf{D}\vec{F}f_\infty) = 0,\ f_\infty = \mathrm{e}^{-A}$;

is equivalent to find f_∞ (similar to Helmholtz-Hodge decomposition).

orthogonality:

$$0 = \int \vec{F}\cdot\mathbf{D}\cdot\nabla A\, f_\infty\,\mathrm{d}x = -\int \vec{F}\cdot\mathbf{D}\cdot\nabla(\mathrm{e}^{-A})\,\mathrm{d}x$$

non-symmetric example: Wigner-Fokker-Planck equation 1

application:
dissipative quantum systems (electron transport in crystal lattice, e.g.)

Evolution of quantum-kinetic quasi-probability density $f(x,v,t) \in \mathbb{R}$, $x, v \in \mathbb{R}^n$; $t \geq 0$:

$$f_t + \underbrace{v \cdot \nabla_x f}_{\text{free transport}} - \underbrace{\Theta[V]f}_{\text{force term}} = \underbrace{\Delta_v f}_{\text{diffusion}} + \underbrace{2\,\mathrm{div}_v(vf)}_{\text{friction}} + \underbrace{\Delta_x f}_{\text{quantum-diffusion}}$$

non-symmetric example: Wigner-Fokker-Planck equation 1

application:
dissipative quantum systems (electron transport in crystal lattice, e.g.)

Evolution of quantum-kinetic quasi-probability density $f(x,v,t) \in \mathbb{R}$, $x, v \in \mathbb{R}^n$; $t \geq 0$:

$$f_t + \underbrace{v \cdot \nabla_x f}_{\text{free transport}} - \underbrace{\Theta[V]f}_{\text{force term}} = \underbrace{\Delta_v f}_{\text{diffusion}} + \underbrace{2\,\mathrm{div}_v(vf)}_{\text{friction}} + \underbrace{\Delta_x f}_{\text{quantum-diffusion}}$$

$\Theta[V]$ is in general a pseudo-differential operator (convolution in v for "nice" V) $\Rightarrow$
no maximum principle;
f_∞ not explicit

non-symmetric example: Wigner-Fokker-Planck equation 2

For $V = |x|^2/2$: Wigner-Fokker-Planck is a PDE (Fokker-Planck eq.):

$$f_t + v \cdot \nabla_x f - x \cdot \nabla_v f = \Delta_v f + 2\mathrm{div}_v(vf) + \Delta_x f \qquad (5)$$

non-symmetric example: Wigner-Fokker-Planck equation 2

For $V = |x|^2/2$: Wigner-Fokker-Planck is a PDE (Fokker-Planck eq.):

$$f_t + v \cdot \nabla_x f - x \cdot \nabla_v f = \Delta_v f + 2\mathrm{div}_v(vf) + \Delta_x f \qquad (5)$$

Unique normalized steady state $f_\infty(x,v) = e^{-A(x,v)}$ with:

$$A(x,v) = c + \frac{1}{4}\left(|x|^2 + 2x \cdot v + 3|v|^2\right); \quad \vec{F}(x,v) = \frac{1}{2}\begin{pmatrix} -x - 3v \\ x + v \end{pmatrix}$$

$$f_t = \mathrm{div}_{x,v}\left([\nabla_{x,v} A + \vec{F}]\, f + \nabla_{x,v} f\right)$$

non-symmetric example: Wigner-Fokker-Planck equation 2

For $V = |x|^2/2$: Wigner-Fokker-Planck is a PDE (Fokker-Planck eq.):

$$f_t + v \cdot \nabla_x f - x \cdot \nabla_v f = \Delta_v f + 2\mathrm{div}_v(vf) + \Delta_x f \qquad (5)$$

Unique normalized steady state $f_\infty(x,v) = e^{-A(x,v)}$ with:

$$A(x,v) = c + \frac{1}{4}\left(|x|^2 + 2x\cdot v + 3|v|^2\right); \quad \vec{F}(x,v) = \frac{1}{2}\begin{pmatrix} -x - 3v \\ x + v \end{pmatrix}$$

$$f_t = \mathrm{div}_{x,v}\left([\nabla_{x,v} A + \vec{F}]\, f + \nabla_{x,v} f\right)$$

Ref's: [Sparber-Carrillo-Dolbeault-Markowich] Monatshefte f. Math. 2004: $V = |x|^2$; exponential convergence via entropy method; $f_0 = f_0^+ + f_0^-$

[Arnold-Gamba-Gualdani-Mischler-Mouhot-Sparber] M3AS 2012: $V = |x|^2$ + small perturbation; expon. convergence via spectral theory

entropy dissipation

$$f_t = \mathrm{div}\Big(\mathbf{D}(x)\cdot[\underbrace{\{\nabla A(x)+\vec{F}(x)\}}_{=\vec{G}(x)} f + \nabla f]\Big)$$

$$f_\infty(x) = e^{-A(x)} \quad \text{for } \mathrm{div}(\mathbf{D}\vec{F}e^{-A}) = 0\,.$$

relative entropy: $e_\psi(f|f_\infty) := \int_{\mathbb{R}^n} \psi\left(\frac{f(x)}{f_\infty(x)}\right) f_\infty\, dx \geq 0$

entropy dissipation (from Lemma 1):

$$\frac{d}{dt} e_\psi(f(t)|f_\infty) = -I_\psi(f(t)|f_\infty)$$

$$= -\int \psi''\left(\frac{f(t)}{f_\infty}\right) \nabla^\top \frac{f(t)}{f_\infty} \cdot \mathbf{D}(x) \cdot \nabla \frac{f(t)}{f_\infty} f_\infty\, dx \leq 0$$

... independent of $\vec{F}$!

$\Rightarrow$ e, $e' = -I$ coincide for symmetric & non-symmetric FP equation

exponential entropy decay for non-symmetric FP: method 1

$$f_t = \operatorname{div}\Big(\mathbf{D}(x)\cdot[\{\nabla A(x)+\vec{F}(x)\}f+\nabla f]\Big)$$

Theorem 3

Let $\mathbf{D} \equiv$ *const and* $A(x)$ *satisfy BEC* $\quad \frac{\partial^2 A(x)}{\partial x^2} \geq \lambda_1 \mathbf{D}^{-1} \quad \forall x \in \mathbb{R}^n$

$$\Rightarrow \quad e(f(t)|f_\infty) \leq e^{-2\lambda_1 t} e(f_0|f_\infty)\,, \quad t \geq 0$$

Proof.

use convex Sobolev inequality $e' \leq -2\lambda_1 e$ from Theorem 2 for each fixed $f(t)$.

□

<u>Ref:</u> [Arnold-Markowich-Toscani-Unterreiter] Comm. PDE 2001

exponential entropy decay for non-symmetric FP: method 2

$$f_t \;=\; \mathrm{div}\Big(\mathbf{D}(x)\cdot[\vec{G}(x)f+\nabla f]\Big)$$

same strategy as for symmetric FP equation:

1. derive dissipation inequality $e'' \geq -\lambda_2 e' \quad (*)$
2. $\Rightarrow$ decay of entropy dissipation $I=-e'$: $\quad I(t) \leq e^{-2\lambda_2 t}\, I(0)$
3. integrate inequality $(*)$ over $[t,\infty)$ $\Rightarrow$ $e' \leq -2\lambda_2\, e$
   technical difficulty:
   $e(t) \stackrel{t\to\infty}{\longrightarrow} 0$ (since non-symmetric eq. $\Rightarrow$ no functional calculus)
4. $\Rightarrow \quad e(t) \leq e^{-2\lambda_2 t}\, e(0)$

Ref's: [Arnold-Carlen-Ju] Comm. Stoch. Analysis 2008;
[Bolley-Gentil] J. Math. Pures Appl. 2010;
[Fontbona-Jourdain] preprint 2013

exponential entropy decay for non-symmetric FP: method 2

$$f_t \;=\; \mathrm{div}\Big(\mathbf{D}(x)\cdot[\vec{G}(x)f+\nabla f]\Big) \qquad (6)$$

Theorem 4 (Bolley-Gentil)

Assume (6) has a unique steady state f_∞ (not explicit). Let $\mathbf{D}\equiv const$; let $\vec{G}(x)$ satisfy generalized Bakry-Emery Condition for some $\lambda_2>0$:

$$\frac{1}{2}\left(\frac{\partial\vec{G}}{\partial x}+\frac{\partial\vec{G}^\top}{\partial x}\right)\geq\lambda_2\mathbf{D}^{-1}\quad\forall x\in\mathbb{R}^n$$

$$\Rightarrow\quad e(f(t)|f_\infty)\leq e^{-2\lambda_2 t}e(f_0|f_\infty)\,,\quad t\geq 0$$

Rem: sometimes Theorem 3 better, sometimes Theorem 4 better.

entropy decay for (non)symmetric FP: combined method

$$f_t \;=\; \mathrm{div}\Big(\mathbf{D}(x)\cdot[\underbrace{\{\nabla A(x)+\kappa\vec{F}(x)\}}_{=\vec{G}_\kappa(x)} f+\nabla f]\Big)\,, \qquad \kappa\in\mathbb{R} \qquad (7)$$

with $\mathrm{div}(\mathbf{D}\vec{F}(x)f_\infty)=0$, $f_\infty=e^{-A}$

Corollary 5 (Arnold-Bolley)

Let $\mathbf{D}\equiv$ *const. For ONE fixed* κ*, let* $\vec{G}_\kappa(x)$ *satisfy generalized Bakry-Emery Condition for some* $\lambda_3>0$*:*

$$\frac{1}{2}\left(\frac{\partial\vec{G}_\kappa}{\partial x}+\frac{\partial\vec{G}_\kappa^\top}{\partial x}\right)\geq\lambda_3\mathbf{D}^{-1}\quad\forall x\in\mathbb{R}^n$$

$$\Rightarrow\quad e(f(t)|f_\infty)\leq e^{-2\lambda_3 t}e(f_0|f_\infty)\,,\qquad \textit{in (7) with ALL } \kappa\in\mathbb{R}$$

entropy decay for (non)symmetric FP: combined method

Proof.

e, e' are independent of $\vec{F}$, and hence of κ. □

Remark:

- $\kappa \vec{F}(x)$ can indeed 'help' to prove better decay rates (even for the symmetric equation);
  examples not generic [Arnold-Carlen-Ju]
  idea: lack of convexity of A in "small" regions (or points) can be fixed by rotation

entropy decay for (non)symmetric FP: combined method

Proof.

e, e' are independent of $\vec{F}$, and hence of κ. □

Remark:

- $\kappa\vec{F}(x)$ can indeed 'help' to prove better decay rates (even for the symmetric equation);
  examples not generic [Arnold-Carlen-Ju]
  idea: lack of convexity of A in "small" regions (or points) can be fixed by rotation

open questions:

- Which κ yields the best (i.e. largest) decay rate?
- For fixed f_∞, which div-free drift field $\vec{F}(x)$ yields the best decay rate?

3. Logarithmic & convex Sobolev inequalities

Outline:

1. convex Sobolev inequalities from entropy method
2. log. Sobolev & weighted Poincaré inequality
3. sharpness, extremal functions

Sobolev inequality

$$\|f\|_{L^p(\mathbb{R}^n)} \leq c_n \|\nabla f\|_{L^2(\mathbb{R}^n)} \quad \forall f \in H^1(\mathbb{R}^n);\ n \geq 3 \tag{8}$$

$2 < p = \frac{2n}{n-2} \overset{n\to\infty}{\longrightarrow} 2$

So: No additional information (from $f \in H^1$) on local singularities / integrability "survives" as $n \to \infty$.

$n \to \infty$ relevant in quantum field theory.

logarithmic Sobolev inequality 1

consider entropy inequality (3) for log. entropy at t=0 : $e(0) \le \frac{1}{2\lambda_1} \underbrace{I(0)}_{=-e'(0)}$

i.e.
$$\int_{\mathbb{R}^n} \frac{f_0}{f_\infty} \ln \frac{f_0}{f_\infty} f_\infty \, \mathrm{d}x \le \frac{4}{2\lambda_1} \int_{\mathbb{R}^n} \left| \nabla \sqrt{\frac{f_0}{f_\infty}} \right|^2 f_\infty \, \mathrm{d}x$$

$\forall$ probability densities f_0, f_∞ (evolution no longer needed !)

logarithmic Sobolev inequality 1

consider entropy inequality (3) for log. entropy at t=0 : $e(0) \le \frac{1}{2\lambda_1} \underbrace{I(0)}_{=-e'(0)}$

i.e.
$$\int_{\mathbb{R}^n} \frac{f_0}{f_\infty} \ln \frac{f_0}{f_\infty} f_\infty \, \mathrm{d}x \;\le\; \frac{4}{2\lambda_1} \int_{\mathbb{R}^n} \left| \nabla \sqrt{\frac{f_0}{f_\infty}} \right|^2 f_\infty \, \mathrm{d}x$$

$\forall$ probability densities f_0, f_∞ (evolution no longer needed !)

with $f^2 := \frac{f_0}{f_\infty}$ $\Rightarrow$

$$\int f^2 \ln f \, f_\infty \, \mathrm{d}x \le \frac{1}{\lambda_1} \int |\nabla f|^2 f_\infty \, \mathrm{d}x$$

$$\forall \int f^2 f_\infty \, \mathrm{d}x = \int f_\infty \, \mathrm{d}x$$

logarithmic Sobolev inequality

logarithmic Sobolev inequality 1

consider entropy inequality (3) for log. entropy at t=0 : $e(0) \le \frac{1}{2\lambda_1} \underbrace{I(0)}_{=-e'(0)}$

i.e.
$$\int_{\mathbb{R}^n} \frac{f_0}{f_\infty} \ln \frac{f_0}{f_\infty} f_\infty \, \mathrm{d}x \le \frac{4}{2\lambda_1} \int_{\mathbb{R}^n} \left| \nabla \sqrt{\frac{f_0}{f_\infty}} \right|^2 f_\infty \, \mathrm{d}x$$

$\forall$ probability densities f_0, f_∞ (evolution no longer needed !)

with $f^2 := \frac{f_0}{f_\infty}$ $\Rightarrow$

$$\int f^2 \ln f \, f_\infty \, \mathrm{d}x \le \frac{1}{\lambda_1} \int |\nabla f|^2 f_\infty \, \mathrm{d}x$$

$$\forall \int f^2 f_\infty \, \mathrm{d}x = \int f_\infty \, \mathrm{d}x$$

logarithmic Sobolev inequality

ex: $A(x) = c + \frac{|x|^2}{2a} \Rightarrow f_\infty(x) = (2\pi a)^{-\frac{n}{2}} \mathrm{e}^{-\frac{|x|^2}{2a}} =: M_a(x), \ \lambda_1 = \frac{1}{a}$

Ref: [Federbush] J. Math. Phys. 1969, [Gross] Amer. J. of Math. 1975

logarithmic Sobolev inequality 2

LSI: $$\int_{\mathbb{R}^n} f^2 \ln f \, M_a \, \mathrm{d}x \leq a \int_{\mathbb{R}^n} |\nabla f|^2 \, M_a \, \mathrm{d}x \quad \forall \int_{\mathbb{R}^n} f^2 \, M_a \, \mathrm{d}x = 1 \quad (9)$$

logarithmic Sobolev inequality 2

LSI: $$\int_{\mathbb{R}^n} f^2 \ln f \, M_a \, \mathrm{d}x \le a \int_{\mathbb{R}^n} |\nabla f|^2 \, M_a \, \mathrm{d}x \quad \forall \int_{\mathbb{R}^n} f^2 \, M_a \, \mathrm{d}x = 1 \quad (9)$$

comparison of Sobolev inequality (8) and LSI (9):

- integrand of l.h.s. of (9) may be negative, but it is bounded below ($f^2 \ln f \ge c$, $M_a(x)dx$ is a bounded measure)
- $f^2 \ln f$ provides less information on local singularities than f^p; $p > 2$, but it "survives" as $n \to \infty$
- The constant a in (9) is independent of n; c_n in (8) is not.
- (9) is a non-linear inequality $\Rightarrow$ makes the normalization $\int f^2 M_a dx = 1$ necessary (otherwise additional terms appear in (9))

convex Sobolev inequalities, Poincaré inequality

consider entropy inequality (3) at t=0 $\rightarrow$ convex Sobolev inequality:

$$\int \psi\left(\frac{f_0}{f_\infty}\right) f_\infty \,\mathrm{d}x \leq \frac{1}{2\lambda_1} \int \psi''\left(\frac{f_0}{f_\infty}\right) \left|\nabla \frac{f_0}{f_\infty}\right|^2 f_\infty \,\mathrm{d}x \qquad (10)$$

$\forall\, f_0,\ f_\infty \in L^1_+(\mathbb{R}^n)$ with $\int f_0 \,\mathrm{d}x = \int f_\infty \,\mathrm{d}x$ (evolution no more needed)

convex Sobolev inequalities, Poincaré inequality

consider entropy inequality (3) at t=0 $\quad \rightarrow \quad$ convex Sobolev inequality:

$$\int \psi\Big(\frac{f_0}{f_\infty}\Big)\, f_\infty \, \mathrm{d}x \le \frac{1}{2\lambda_1} \int \psi''\Big(\frac{f_0}{f_\infty}\Big) \Big|\nabla \frac{f_0}{f_\infty}\Big|^2 f_\infty \, \mathrm{d}x \qquad (10)$$

$\forall\, f_0,\ f_\infty \in L^1_+(\mathbb{R}^n)$ with $\int f_0 \, \mathrm{d}x = \int f_\infty \, \mathrm{d}x$ (evolution no more needed)

ex: Let $\psi = \psi_2(\sigma) = (\sigma - 1)^2$. Use transformation $\dfrac{f_0}{f_\infty} = \dfrac{f}{\int f\, f_\infty \, \mathrm{d}x}$;

assume $\int f_\infty \, \mathrm{d}x = 1$

$\Rightarrow$ weighted Poincaré / spectral gap inequality :

$$\int_{\mathbb{R}^n} f^2 f_\infty \, \mathrm{d}x - \Big(\int_{\mathbb{R}^n} f\, f_\infty \, \mathrm{d}x\Big)^2 \le \frac{1}{\lambda_1} \int_{\mathbb{R}^n} |\nabla f|^2 f_\infty \, \mathrm{d}x \quad \forall\, f \in L^1(\mathbb{R}^n, f_\infty)$$

convex Sobolev inequalities, Poincaré inequality

consider entropy inequality (3) at t=0 $\quad\rightarrow\quad$ convex Sobolev inequality:

$$\int \psi\Big(\frac{f_0}{f_\infty}\Big)\, f_\infty \,\mathrm{d}x \le \frac{1}{2\lambda_1}\int \psi''\Big(\frac{f_0}{f_\infty}\Big)\,\Big|\nabla \frac{f_0}{f_\infty}\Big|^2 f_\infty \,\mathrm{d}x \qquad (10)$$

$\forall\, f_0,\ f_\infty \in L^1_+(\mathbb{R}^n)$ with $\int f_0 \,\mathrm{d}x = \int f_\infty \,\mathrm{d}x$ (evolution no more needed)

<u>ex:</u> Let $\psi = \psi_2(\sigma) = (\sigma-1)^2$. Use transformation $\dfrac{f_0}{f_\infty} = \dfrac{f}{\int f\, f_\infty \,\mathrm{d}x}$;

assume $\int f_\infty \,\mathrm{d}x = 1$

$\Rightarrow$ weighted Poincaré / spectral gap inequality :

$$\int_{\mathbb{R}^n} f^2 f_\infty \,\mathrm{d}x - \Big(\int_{\mathbb{R}^n} f\, f_\infty \,\mathrm{d}x\Big)^2 \le \frac{1}{\lambda_1}\int_{\mathbb{R}^n} |\nabla f|^2 f_\infty \,\mathrm{d}x \quad \forall\, f \in L^1(\mathbb{R}^n, f_\infty)$$

meaning: Hamiltonian $H = -\frac{1}{\sqrt{f_\infty}} L(\sqrt{f_\infty}\,\cdot\,)$ has spectral gap $\ge \lambda_1$ on $L^2(\mathbb{R}^n)$

comparison: log. Sobolev inequality — weighted Poincaré inequality

intuitive idea: log. Sobolev is the strongest, Poincaré the weakest convex Sobolev inequality

Lemma 3

LSI $\Rightarrow$ *Poincaré inequality (with same* λ_1*)* $\forall\, f \in L^2(\mathbb{R}^n, f_\infty) \subset L^1(\mathbb{R}^n, f_\infty)$

comparison: log. Sobolev inequality — weighted Poincaré inequality

intuitive idea: log. Sobolev is the strongest, Poincaré the weakest convex Sobolev inequality

Lemma 3

LSI $\Rightarrow$ Poincaré inequality (with same λ_1) $\forall f \in L^2(\mathbb{R}^n, f_\infty) \subset L^1(\mathbb{R}^n, f_\infty)$

Proof:
Use $f^2(x) = 1 + \varepsilon\tilde{g}(x)$ with $\tilde{g} \in L^2(\mathbb{R}^n, f_\infty)$, $\int \tilde{g}\, f_\infty\, \mathrm{d}x = 0$ in LSI:

$$\int f^2 \ln f\, f_\infty\, \mathrm{d}x \le \frac{1}{\lambda_1} \int |\nabla f|^2\, f_\infty\, \mathrm{d}x$$

Then: $\varepsilon \to 0$; $\tilde{g} = g - \int g\, f_\infty\, \mathrm{d}x$ □

comparison: log. Sobolev inequality — weighted Poincaré inequality

intuitive idea: log. Sobolev is the strongest, Poincaré the weakest convex Sobolev inequality

Lemma 3

LSI $\Rightarrow$ *Poincaré inequality (with same* λ_1*)* $\forall f \in L^2(\mathbb{R}^n, f_\infty) \subset L^1(\mathbb{R}^n, f_\infty)$

Proof:
Use $f^2(x) = 1 + \varepsilon \tilde{g}(x)$ with $\tilde{g} \in L^2(\mathbb{R}^n, f_\infty)$, $\int \tilde{g}\, f_\infty \, \mathrm{d}x = 0$ in LSI:

$$\int f^2 \ln f \, f_\infty \, \mathrm{d}x \leq \frac{1}{\lambda_1} \int |\nabla f|^2 \, f_\infty \, \mathrm{d}x$$

Then: $\varepsilon \to 0$; $\tilde{g} = g - \int g \, f_\infty \, \mathrm{d}x$ $\square$

note: $\tilde{e}_1(f_1|f_2) \leq e_\psi(f_1|f_2) \leq \tilde{e}_2(f_1|f_2)$

connection: entropy decay — convex Sobolev inequalities

exponential decay of e_1 $\forall\, f$ $\quad\Leftrightarrow\quad$ log. Sobolev inequality holds $\forall\, f$

connection: entropy decay — convex Sobolev inequalities

exponential decay of e_1 $\forall f$ $\Leftrightarrow$ log. Sobolev inequality holds $\forall f$

exponential decay of e_2 $\forall f$ $\Leftrightarrow$ weighted Poincaré inequality holds $\forall f$

sharpness of convex Sobolev inequlities (CSI)

Q: For which $f_0 \neq f_\infty$ does the CSI become an equality?

$$\text{CSI}: \quad \int \psi\Big(\frac{f_0}{f_\infty}\Big)\, f_\infty \,\mathrm{d}x \le \frac{1}{2\lambda_1} \int \psi''\Big(\frac{f_0}{f_\infty}\Big) \Big|\nabla \frac{f_0}{f_\infty}\Big|^2 f_\infty \,\mathrm{d}x$$

Ref's: [Carlen] JFA 1991; [AMTU] CPDE 2001

sharpness of convex Sobolev inequlities (CSI)

Q: For which $f_0 \neq f_\infty$ does the CSI become an equality?

$$\text{CSI}: \quad \int \psi\left(\frac{f_0}{f_\infty}\right) f_\infty \, \mathrm{d}x \leq \frac{1}{2\lambda_1} \int \psi''\left(\frac{f_0}{f_\infty}\right) \left|\nabla \frac{f_0}{f_\infty}\right|^2 f_\infty \, \mathrm{d}x$$

Ref's: [Carlen] JFA 1991; [AMTU] CPDE 2001

- recall derivation of CSI for symmetric FP with $\mathbf{D} \equiv \mathbf{I}$:

$$\frac{\mathrm{d}}{\mathrm{d}t} I_\psi(f(t)|f_\infty) = -2\lambda_1 I_\psi(f(t)|f_\infty) - r_\psi(f(t)),$$

  with remainder $r_\psi(f(t)) \geq 0$.

- integrate w.r.t. t:

$$e_\psi(f_0|f_\infty) = \frac{1}{2\lambda_1} I_\psi(f_0|f_\infty) - \frac{1}{2\lambda_1} \int_0^\infty r_\psi(f(s)) \, \mathrm{d}s$$

sharpness conditions for CSI

- CSI $e \le \frac{1}{2\lambda_1} I$ is an equality iff $r_\psi(f(t)) = 0$ for a.e. $t \in (0,\infty)$, i.e.

$$2\int_{\mathbb{R}^n} \psi''\left(\frac{f}{f_\infty}\right) u^\top \cdot \left(\frac{\partial^2 A}{\partial x^2} - \lambda_1 \mathbf{I}\right) \cdot u f_\infty \,\mathrm{d}x + 2\int_{\mathbb{R}^n} \mathrm{Tr}(XY) f_\infty \,\mathrm{d}x = 0 \tag{11}$$

for "trajectory" $f = f(t)$ with $u := \nabla \frac{f}{f_\infty}$.

sharpness conditions for CSI

- CSI $e \le \frac{1}{2\lambda_1} I$ is an equality iff $r_\psi(f(t)) = 0$ for a.e. $t \in (0,\infty)$, i.e.

$$2\int_{\mathbb{R}^n} \psi''\left(\frac{f}{f_\infty}\right) u^\top \cdot \left(\frac{\partial^2 A}{\partial x^2} - \lambda_1 \mathbf{I}\right) \cdot u f_\infty \, \mathrm{d}x + 2\int_{\mathbb{R}^n} \mathrm{Tr}(XY) f_\infty \, \mathrm{d}x = 0 \tag{11}$$

  for "trajectory" $f = f(t)$ with $u := \nabla \frac{f}{f_\infty}$.

- conditions for (11):
$$\det X = \frac{1}{2}\psi''(\frac{f}{f_\infty})\psi^{IV}(\frac{f}{f_\infty}) - \psi'''(\frac{f}{f_\infty})^2 = 0$$
$$\det Y = |u|^4 \sum_{i,j}(\frac{\partial u_i}{\partial x_j})^2 - (u^\top \frac{\partial u}{\partial x} u)^2 = 0$$
$$\psi''(\frac{f}{f_\infty}) u^\top \frac{\partial u}{\partial x} u + \psi'''(\frac{f}{f_\infty})|u|^4 = 0$$
$$u^T \left(\frac{\partial^2 A}{\partial x^2} - \lambda_1 \mathbf{I}\right) u = 0$$

- saturation only for logarithmic or quadratic entropies (from $\det X = 0$)

extremal functions for logarithmic Sobolev inequality

Theorem 6

For $\mathbf{D} \equiv \mathbf{I}$ *and* $\psi_1(\sigma) = \sigma \ln \sigma - \sigma + 1$, *the LSI is an equality iff:*

1. $$A(x(y)) = \frac{\lambda_1}{2} y_1^2 + \beta y_1 + B(y_2, ..., y_n)$$

   for some Cartesian coordinates $y(x) = (y_1, ..., y_n)$ *and some* $\beta \in \mathbb{R}$;

2. $$f = \exp\left[-A(x(y)) + \xi y_1 - \frac{\xi^2}{2\lambda_1} + \frac{\beta\xi}{\lambda_1}\right] \tag{12}$$

   for some $\xi \in \mathbb{R}$.

extremal functions for weighted Poincaré inequality

Theorem 7

For $\mathbf{D} \equiv \mathbf{I}$ *and* $\psi_2(\sigma) = (\sigma - 1)^2$, *the Poincaré is an equality iff:*

1. $$A(x(y)) = \frac{\lambda_1}{2} y_1^2 + \beta y_1 + B(y_2, ..., y_n)\,;$$
2. $$f = (1 + \xi y_1) e^{-A(x(y))} \tag{13}$$

for some $\xi \in \mathbb{R}$.

note: For $p = 2$, $f < 0$ is allowed.

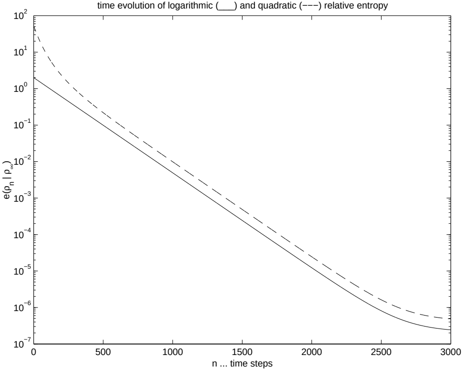
(12), (13) are "extremal functions";
(13) is first eigenfunction of L.

extremal functions for LSI: example

extremal function as $f_0 \Rightarrow$ exponential decay of $e_1(f(t)|f_\infty)$ is the *slowest* !

$A(x) = \frac{x^2}{2}, \quad n = 1$

extremal function: $f_0(x) = \exp[-\frac{x^2}{2} + 2x] > 0$



decay of relativ entropies $e_1(t), e_2(t)$

For large times numerical effects are visible.

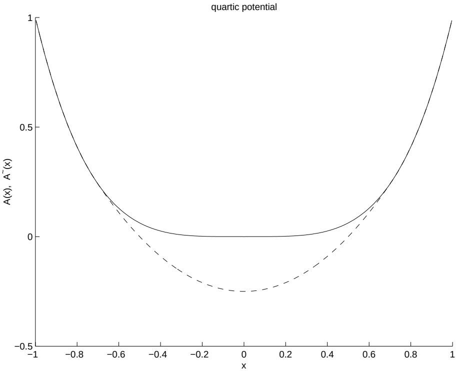
4. Perturbation results & application to a semi-linear model

Outline:

1. Holley-Stroock perturbation in CSI
2. a drift-diffusion-Poisson model

Holley-Stroock perturbation for CSI: example

$A(x) = x^4$, $x \in \mathbb{R}$; $\mathbf{D} \equiv 1$ violates BE-condition $A''(x) \geq \lambda_1 > 0$ at $x = 0$.
But L has a positive spectral gap; also: exponential decay of relative entropy



$\tilde{A}(x) = x^4$ (—) is a bounded perturbation
of the uniformly convex function $A(x)$ (- - -).

Holley-Stroock perturbation for CSI: idea

Let $\left[f_\infty = e^{-A(x)},\ \mathbf{D}(x)\right]$ satisfy the Convex Sobolev Inequality (use $\frac{f_0}{f_\infty} =: \frac{f^2}{\|f\|^2}$; $\|f\|^2 := \|f\|^2_{L^2(f_\infty)}$):

$$\int \psi\left(\frac{f^2}{\|f\|^2}\right) f_\infty \, \mathrm{d}x \le \frac{2}{\lambda_1} \int \frac{f^2}{\|f\|^4} \psi''\left(\frac{f^2}{\|f\|^2}\right) \nabla^\top f \cdot \mathbf{D} \cdot \nabla f \, f_\infty \, \mathrm{d}x \quad (14)$$

$\forall f \in L^2(f_\infty)$, for some $\psi = \psi_p(\sigma) \ge 0\,; 1 \le p \le 2$.

$\Rightarrow$ perturbed measure $\tilde{f}_\infty := e^{-A(x)-v(x)}$ with $v \in L^\infty$ also satisfies a CSI.

Holley-Stroock perturbation for CSI

Theorem 8

Let $\left[f_\infty = e^{-A(x)},\ \mathbf{D}(x)\right]$ *satisfy the CSI* (14).
Let $\tilde{f}_\infty := e^{-\tilde{A}(x)}$ *with* $\int f_\infty \, \mathrm{dx} = \int \tilde{f}_\infty \, \mathrm{dx} = 1$,

$$\tilde{A}(x) := A(x) + v(x), \quad 0 < a \le e^{-v(x)} \le b < \infty, \quad x \in \mathbb{R}^n.$$

$\Rightarrow \left[\tilde{f}_\infty,\ \mathbf{D}(x)\right]$ *also satisfy a CSI (14), with constant* $\tilde{\lambda}_1 := \frac{a}{b}\lambda_1 < \lambda_1$.

[Holley-Stroock] JSP 1987 (for logarithmic entropy e_1) ;
[AMTU] CPDE 2001 (for general admissible entropies e_ψ)

Holley-Stroock perturbation: proof 1

Proof for $\psi_p(\sigma) := \sigma^p - 1 - p(\sigma - 1);\ 1 < p \leq 2$:

Idea: use homogeneity of ψ_p'' .

$$k(t) := t^p \int_{\mathbb{R}^n} \psi_p\Big(\frac{f(x)^2}{t}\Big)\, \tilde{f}_\infty(x)\, \mathrm{d}x\,, \quad t > 0$$

$$\Rightarrow \quad \operatorname*{argmin}_{t>0} k(t) = \|f\|^2_{\tilde{f}_\infty}$$

Holley-Stroock perturbation: proof 2

$$
\begin{aligned}
\|f\|^{2p}_{\tilde f_\infty} & \int \psi\left(\frac{f^2}{\|f\|^2_{\tilde f_\infty}}\right)\tilde f_\infty \,\mathrm{d}x = k\big(\|f\|^2_{\tilde f_\infty}\big) \\
\le \quad & k\big(\|f\|^2_{f_\infty}\big) = \|f\|^{2p}_{f_\infty}\int \psi\left(\frac{f^2}{\|f\|^2_{f_\infty}}\right)\underbrace{\tilde f_\infty}_{\le b\, f_\infty}\,\mathrm{d}x \\
\overset{\text{CSI}}{\le} \quad & b\frac{2}{\lambda_1}p(p-1)\int f^{2(p-1)}\nabla^\top f\cdot \mathbf{D}\cdot\nabla f\,\underbrace{f_\infty}_{\le \tilde f_\infty/a}\,\mathrm{d}x \\
\le \quad & \|f\|^{2p}_{\tilde f_\infty}\,\frac{b}{a}\frac{2}{\lambda_1}p(p-1)\int \frac{f^2}{\|f\|^4_{\tilde f_\infty}}\psi''\left(\frac{f^2}{\|f\|^2_{\tilde f_\infty}}\right)\nabla^\top f\cdot\mathbf{D}\cdot\nabla f\,\tilde f_\infty\,\mathrm{d}x
\end{aligned}
$$

□

A drift-diffusion-Poisson model

$$\begin{cases} f_t &= \text{div}\ \left(\nabla f + \nabla\left(\frac{|x|^2}{2} + V(x,t)\right) f\right),\ x \in \mathbb{R}^3,\ t > 0 \\ f(x,t=0) &= f_0(x) \geq 0; \quad \int f_0 dx = 1 \\ V(x,t) &= \frac{1}{4\pi} \int \frac{f(y,t)}{|x-y|}\,\mathrm{d}y \ \ldots\ \text{solves}\ -\Delta V = f \end{cases} \tag{15}$$

A drift-diffusion-Poisson model

$$\begin{cases} f_t & = \operatorname{div}\ \left(\nabla f + \nabla\left(\frac{|x|^2}{2} + V(x,t)\right) f\right),\ x \in \mathbb{R}^3,\ t > 0 \\ f(x, t=0) & = f_0(x) \geq 0; \quad \int f_0 dx = 1 \\ V(x,t) & = \frac{1}{4\pi} \int \frac{f(y,t)}{|x-y|}\,\mathrm{d}y \ \ldots \text{ solves } -\Delta V = f \end{cases} \tag{15}$$

- model for electron transport in plasma, semiconductor ($f(x,t)$ ... electron density)
- (external) *confinement potential* $V_{\text{ext}} = \frac{|x|^2}{2}$ can be removed by time-dependent rescaling:

$$u(\xi,\tau) := R(\tau)^{-3}\ f\left(\frac{\xi}{R(\tau)}, \ln R(\tau)\right), \quad R(\tau) = \sqrt{2\tau + 1} \tag{16}$$

Also transforms FP equation for $f(x,t)$ into heat equation for $u(\xi,\tau)$.

drift-diffusion-Poisson: steady state

- unique normalized steady state satisfies the *mean-field equation*, a semilinear elliptic equation for V_∞:

$$\begin{cases} f_\infty(x) & := \dfrac{\exp\left[-\frac{|x|^2}{2} - V_\infty(x)\right]}{\int_{\mathbb{R}^3} \exp\left[-\frac{|y|^2}{2} - V_\infty(y)\right] \mathrm{d}y} \quad \text{and} \quad f_\infty(x) = -\Delta V_\infty \\ \text{i.e. } V_\infty(x) & = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f_\infty(y)}{|x-y|} \mathrm{d}y \end{cases}$$

drift-diffusion-Poisson: steady state

- unique normalized steady state satisfies the *mean-field equation*, a semilinear elliptic equation for V_∞:

$$\begin{cases} f_\infty(x) & := \dfrac{\exp\left[-\frac{|x|^2}{2} - V_\infty(x)\right]}{\int_{\mathbb{R}^3} \exp\left[-\frac{|y|^2}{2} - V_\infty(y)\right] \mathrm{d}y} \quad \text{and} \quad f_\infty(x) = -\Delta V_\infty \\ \text{i.e. } V_\infty(x) & = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f_\infty(y)}{|x-y|} \mathrm{d}y \end{cases}$$

- relative entropy-type functional:

$$e(f) := \int_{\mathbb{R}^3} \psi_1\left(\frac{f}{f_\infty}\right) f_\infty \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla(V[f] - V_\infty)|^2 \, \mathrm{d}x \geq 0 \quad (17)$$

drift-diffusion-Poisson: exponential convergence

Theorem 9

Let $f_0 \in L^1_+(\mathbb{R}^3) \cap L^{\frac{3}{2}+\epsilon}(\mathbb{R}^3)$
$\Rightarrow \exists \tilde{\lambda}_1 > 0$ *such that*

$$e(f(t)) \leq e^{-2\tilde{\lambda}_1 t} \cdot e(f_0), \quad t \geq 0; \qquad \|\nabla V(t) - \nabla V_\infty\|_{L^2(\mathbb{R}^3)} = c(f_0)\, e^{-\tilde{\lambda}_1 t}.$$

Proof:

$$\frac{\mathrm{d}}{\mathrm{d}t} e(t) = -\int_{\mathbb{R}^3} f(t) \left| \nabla \ln \frac{f(t)}{N(t)} \right|^2 \, \mathrm{d}x \leq 0\,,$$

with t-local equilibrium state:

$$N(t) := \exp\left[-\frac{|x|^2}{2} - V(x,t) \right] \Big/ \int_{\mathbb{R}^3} \exp\left[-\frac{|y|^2}{2} - V(y,t) \right] \, \mathrm{d}y$$

Ref: [AMTU] CPDE 2001

drift-diffusion-Poisson: convergence proof 1

a-priori estimates from drift-diffusion-Poisson (15):

$$\|f(t)\|_{L^p(\mathbb{R}^3)} \leq \text{const.} \quad \forall\, t \geq 0 \qquad (\text{for some } p > \tfrac{3}{2})\,.$$

$$\|V(t)\|_{L^\infty(\mathbb{R}^3)} \leq \text{const.} \quad \forall\, t \geq 0 \qquad (\text{from Poisson eq.})$$

$\Rightarrow$ $V(x,t)$ is a bounded perturbation (uniformly in $t \geq 0$) of the uniformly convex potential $\frac{|x|^2}{2}$.

drift-diffusion-Poisson: convergence proof 1

a-priori estimates from drift-diffusion-Poisson (15):

$$\|f(t)\|_{L^p(\mathbb{R}^3)} \le \text{const.} \quad \forall\, t \ge 0 \qquad (\text{for some } p > \tfrac{3}{2})\,.$$

$$\|V(t)\|_{L^\infty(\mathbb{R}^3)} \le \text{const.} \quad \forall\, t \ge 0 \qquad (\text{from Poisson eq.})$$

$\Rightarrow$ $V(x,t)$ is a bounded perturbation (uniformly in $t \ge 0$) of the uniformly convex potential $\frac{|x|^2}{2}$.

$\Rightarrow$ a LSI holds for the potential $\frac{|x|^2}{2} + V(x,t)$ by Holley-Stroock: $\exists \tilde\lambda_1 > 0$ (independent of t) with:

$$\underbrace{\int f \ln \frac{f}{N(t)}\, \mathrm{d}x}_{\text{rel. entropy w.r.t. } t\text{-local equilibrium}} \le \frac{1}{2\tilde\lambda_1} \underbrace{\int f \left|\nabla \ln \frac{f}{N(t)}\right|^2 \mathrm{d}x}_{\text{true entropy dissipation}}, \quad \forall \int f\, \mathrm{d}x = 1$$

drift-diffusion-Poisson: convergence proof 2

$$\frac{d}{dt}e(t) \overset{\text{LSI}}{\leq} -2\tilde{\lambda}_1 \int f \ln \frac{f}{N} \, \mathrm{d}x$$

$$\overset{\text{def. } N}{=} -2\tilde{\lambda}_1 \int f \ln \frac{f}{f_\infty} \, \mathrm{d}x - 2\tilde{\lambda}_1 \int (V(t) - V_\infty) f \, \mathrm{d}x$$

$$-2\tilde{\lambda}_1 \ln \left(\int e^{V_\infty - V(t)} f_\infty \, \mathrm{d}x \right)$$

$$\overset{\text{Jensen}}{\leq} -2\tilde{\lambda}_1 \int f \ln \frac{f}{f_\infty} \, \mathrm{d}x - 2\tilde{\lambda}_1 \int (V(t) - V_\infty)(f(t) - f_\infty) \, \mathrm{d}x$$

drift-diffusion-Poisson: convergence proof 2

$$\frac{d}{dt}e(t) \overset{\text{LSI}}{\leq} -2\tilde{\lambda}_1 \int f \ln \frac{f}{N} \, \mathrm{d}x$$

$$\overset{\text{def. } N}{=} -2\tilde{\lambda}_1 \int f \ln \frac{f}{f_\infty} \, \mathrm{d}x - 2\tilde{\lambda}_1 \int (V(t) - V_\infty) f \, \mathrm{d}x$$

$$-2\tilde{\lambda}_1 \ln \left(\int e^{V_\infty - V(t)} f_\infty \, \mathrm{d}x \right)$$

$$\overset{\text{Jensen}}{\leq} -2\tilde{\lambda}_1 \int f \ln \frac{f}{f_\infty} \, \mathrm{d}x - 2\tilde{\lambda}_1 \int (V(t) - V_\infty)(f(t) - f_\infty) \, \mathrm{d}x$$

Use Poisson equation: $f(t) - f_\infty = -\Delta(V(t) - V_\infty)$

$$\Rightarrow \frac{d}{dt}e(t) \leq -2\tilde{\lambda}_1 e(t)$$

□

drift-diffusion-Poisson: Philosophy of convergence proof

1. Holley-Stroock perturbation theorem $\Rightarrow$ t–uniform LSI
2. LSI between $e'(t)$ and relative entropy w.r.t. t–local equilibrium ($\neq e(t)$ for nonlinear equation)
3. differential inequality between $e'(t)$, $e(t)$

5. Fokker-Planck model for polymeric fluid-flow

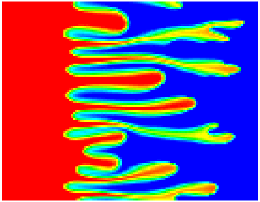
Outline:

1. multi-scale polymeric fluid-flow model
2. fixed flow $\to$ linear non-symmetric FP equation
3. convergence of nonlinear model

Dilute polymer suspension – applications

- multi-grade motor oil:
  polymer additives to improve/tune viscosity $\nu(p, T)$

- enhanced (tertiary) oil recovery:
  strong fingering at oil/water interface
  $\rightarrow$ front stabilization with polymer additives (increase viscosity of water)



- food industry:
  polymer additives to thicken sauces, ...

Macro Model: fluid flow

- dilute solution of polymers in homogeneous fluid
- coupled micro-macro model
- incompressible Navier-Stokes for macro flow $u(t,x)$:

$$\begin{aligned} u_t + (u \cdot \nabla_x) u &= \Delta_x u - \nabla_x p + \mathrm{div}_x \tau, \ \Omega \subset \mathbb{R}^n \\ \mathrm{div}_x u &= 0 \end{aligned}$$

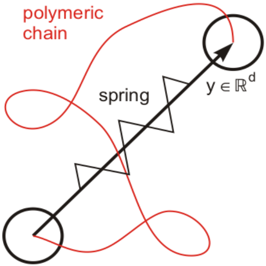
- coupling to polymer-model via extra stress tensor:

$$\tau(t,x) = \int_{\mathbb{R}^n} (y \otimes \nabla_y \Pi(y)) f(t,x,y)\, dy$$

(all parameters :=1)

Micro Model: polymer distribution

- dumbbell model for polymeric chains: $y \in \mathbb{R}^n$ ... extension, orientation



- micro-distribution (probability density in y) at each $x \in \Omega$: $f(t, x, y)$ in Fokker-Planck equ.:

$$f_t + \underbrace{u \cdot \nabla_x f}_{\substack{\text{transport} \\ \text{in flow}}} = \frac{1}{2}\text{div}_y\Big(\big[\underbrace{\nabla_y \Pi(y)}_{\substack{\text{spring force} \\ \text{in dumbbells}}} \; -2\, \underbrace{(\nabla_x \otimes u)^T \cdot y}_{\substack{\text{drag force of} \\ \text{inhom. flow field } u}}\big] f\Big) + \frac{1}{2}\Delta_y f$$

Results from [Jourdain-LeBris-Lelièvre-Otto, ARMA 2006]

linear FP: $f_t = \frac{1}{2}\mathrm{div}_y([\nabla_y \Pi(y) - 2\kappa y]f + \nabla f)$, $y \in \mathbb{R}^n$; $\underbrace{\text{const. } \kappa}_{=(\nabla_x \otimes u)^T} \in \mathbb{R}^{n\times n}$

1. Hookean, $\Pi = \frac{1}{2}|y|^2$:

Theorem 10

κ symmetric with $\lambda_j(\kappa) < \frac{1}{2}$, or κ anti-symmetric:
$\Rightarrow \exists!$ *steady state f_∞; exp. convergence of $f(t)$, $t \to \infty$; general κ open !*

Results from [Jourdain-LeBris-Lelièvre-Otto, ARMA 2006]

linear FP: $f_t = \frac{1}{2}\text{div}_y([\nabla_y \Pi(y) - 2\kappa y]f + \nabla f)$, $y \in \mathbb{R}^n$; $\underbrace{\text{const. } \kappa}_{=(\nabla_x \otimes u)^T} \in \mathbb{R}^{n\times n}$

1. Hookean, $\Pi = \frac{1}{2}|y|^2$:

Theorem 10

κ symmetric with $\lambda_j(\kappa) < \frac{1}{2}$, or κ anti-symmetric:
$\Rightarrow \exists!$ steady state f_∞; exp. convergence of $f(t)$, $t \to \infty$; general κ open !

2. FENE (=finite extensibility), $\Pi = -\frac{b}{2}\ln(1 - \frac{|y|^2}{b})$, $|y|^2 < b$:

Theorem 11

κ symmetric, or κ anti-symmetric, or $|\kappa^s| < \frac{1}{2}$:
$\Rightarrow \exists!$ steady state f_∞; exp. convergence of $f(t)$; general κ open !

Results from [Jourdain-LeBris-Lelièvre-Otto, ARMA 2006]

linear FP: $f_t = \frac{1}{2}\mathrm{div}_y([\nabla_y \Pi(y) - 2\kappa y]f + \nabla f)$, $y \in \mathbb{R}^n$; $\underbrace{\text{const. } \kappa}_{=(\nabla_x \otimes u)^T} \in \mathbb{R}^{n\times n}$

1. Hookean, $\Pi = \frac{1}{2}|y|^2$:

Theorem 10

κ symmetric with $\lambda_j(\kappa) < \frac{1}{2}$, or κ anti-symmetric:
$\Rightarrow \exists!$ *steady state f_∞; exp. convergence of $f(t)$, $t \to \infty$; general κ open !*

2. FENE (=finite extensibility), $\Pi = -\frac{b}{2}\ln(1 - \frac{|y|^2}{b})$, $|y|^2 < b$:

Theorem 11

κ symmetric, or κ anti-symmetric, or $|\kappa^s| < \frac{1}{2}$:
$\Rightarrow \exists!$ *steady state f_∞; exp. convergence of $f(t)$; general κ open !*

3. coupled nonlin. model ($\to$ log. relative entropy of $f + \|u - u_\infty\|^2_{L^2}$)

Theorem 12

FENE, if $|\kappa^s| < \frac{1}{2}$, $Tr\,\kappa = 0, [\kappa, \kappa^T]$ small (κ from BC/u_∞-steady state)
$\Rightarrow$ *exp. convergence of $(u, f) \overset{t\to\infty}{\longrightarrow} (u_\infty, f_\infty)$; Hookean open !*

linear polymer model with Hookean dumbbells: $\Pi(y)=\frac{|y|^2}{2}$

$$f_t = Lf = \frac{1}{2}\text{div}([y - 2 \underbrace{\kappa}_{:=(\nabla_x \otimes u)^T} y]f + \nabla f), \quad y \in \mathbb{R}^n, \ t \geq 0 \qquad (18)$$

Theorem 13 (steady state [AA-Carrillo-Manzini, CMS 2010])

Let $-(I - 2\kappa)$ be stable (otherwise no confinement pot.), i.e. $Re\lambda_j(\kappa) < \frac{1}{2}$:

$\Rightarrow \exists!$ normalized steady state of (18):

$$f_\infty(y) = c\, exp(-\frac{1}{2} y^T \Sigma^{-1} y),$$

$$0 < \Sigma = \Sigma^T = 2\int_0^\infty e^{-(I-2\kappa)s} e^{-(I-2\kappa^T)s}\, ds$$

if κ normal: $\Sigma = (I - 2\kappa^s)^{-1}$. $\exists$ standard algorithms to compute Σ from κ

Proof. Fourier Transform of (18) $\Rightarrow$ matrix eq. for Σ

Proof.

Fourier transform of steady state equ: $\mathrm{div}([y-2\kappa y]f+\nabla f)=0$

$$\xi^T(I-2\kappa)\nabla_\xi \hat f(\xi) = -|\xi|^2 \hat f(\xi)$$

Use ansatz $\hat f(\xi) = c\,\exp(-\frac{1}{2}\xi^T\Sigma\xi)$:

$\Rightarrow\ 0=-(I-2\kappa)\Sigma-\Sigma(I-2\kappa)^T+2I\ \ldots$ "continuous Lyapunov equ"

Since $-(I-2\kappa)$ stable; $2I$ pos.def, symm $\Rightarrow \exists!\ \Sigma$ □

$A(y)=-\ln f_\infty = \frac{1}{2}y^T\Sigma^{-1}y+c\ \ldots$ uniformly convex potential of symmetric part of L in $L^2(f_\infty^{-1})$

Theorem 14 (convergence in rel. entropy [AA-Carrillo-Manzini, 2010])

Let $-(I - 2\kappa)$ *be stable*

$$\Rightarrow \; e(f(t)|f_\infty) \le e^{-\lambda_{min}(\Sigma^{-1})t}\, e(f_0|f_\infty), \; t \ge 0,$$

with $\lambda_{min} > 0$ *computable*

Proof.

entropy method [AMTU,2001], Bakry-Emery cond. for symm. part of L:

$$\frac{\partial^2 A}{\partial y^2} \ge \lambda_{\min}(\Sigma^{-1})$$

□

Theorem 14 (convergence in rel. entropy [AA-Carrillo-Manzini, 2010])

Let $-(I-2\kappa)$ *be stable*

$$\Rightarrow \; e(f(t)|f_\infty) \le e^{-\lambda_{min}(\Sigma^{-1})t}\, e(f_0|f_\infty), \; t \ge 0,$$

with $\lambda_{min} > 0$ *computable*

Proof.

entropy method [AMTU,2001], Bakry-Emery cond. for symm. part of L:

$$\frac{\partial^2 A}{\partial y^2} \ge \lambda_{\text{min}}(\Sigma^{-1})$$

□

Rem:

decay is sharp for quadratic potentials, also for non-symmetric Fokker-Planck equ. $\forall \vec{F}$ (same entropy dissipation for "optimal functions")

Coupled micro-macro model for Hookean dumbbells: $t \to \infty$ convergence

Navier-Stokes for $u(t,x)$ on Ω :
Choose BC $u\big|_{\partial\Omega} = \kappa x$ for some $\kappa \in \mathbb{R}^{n\times n}$ ($\operatorname{div} u = \operatorname{Tr}\kappa = 0$) $\Rightarrow$

$$u_\infty = \kappa x,\ f_\infty = c\, e^{-\frac{1}{2}y^T\Sigma^{-1}y} \quad \text{is steady state.}$$

$\to$ decay of "relative entropy" (formal; if solution $\exists$):

$$E(t) := \frac{1}{2}\int_\Omega |u(t) - u_\infty|^2\, dx + \int_\Omega \int_{\mathbb{R}^n} f(t) \ln \frac{f(t)}{f_\infty}\, dy\, dx$$

Theorem 15 ([AA-Carrillo-Manzini, Comm. Math. Sc. 2010])

Let $\|\kappa^s\|_2,\ \sup_t \|\underbrace{\nabla_x \otimes u^s(t,\cdot)}_{deformation\ matrix}\|_{L^\infty(\Omega)},\ \|\int_{\mathbb{R}^n} |y|^4 f_0(x,y)\,dy\|_{L^\infty(\Omega)}$ *be small*;

let $Re\,\lambda_j(\kappa) < \frac{1}{2}$.

$\Rightarrow\ E(t) \searrow 0$ *exponentially*

Theorem 15 ([AA-Carrillo-Manzini, Comm. Math. Sc. 2010])

Let $\|\kappa^s\|_2, \ \sup_t \|\underbrace{\nabla_x \otimes u^s(t,\cdot)}_{deformation\ matrix}\|_{L^\infty(\Omega)}, \ \|\int_{\mathbb{R}^n} |y|^4 f_0(x,y)\,dy\|_{L^\infty(\Omega)}$ *be small*;

let $Re\,\lambda_j(\kappa) < \frac{1}{2}$.
$\Rightarrow \ E(t) \searrow 0$ *exponentially*

Proof.

- differential inequality between $\frac{dE}{dt}$, $E(t)$
- logarithmic Sobolev inequality for $f_\infty(y)$ ... Gaussian
- new weighted Csiszár-Kullback inequality for

$$\tau(t,x) - \tau_\infty = \int_{\mathbb{R}^n} \big(y \otimes \underbrace{\nabla_y \Pi(y)}_{=y}\big)[f(t,x,y) - f_\infty(y)]\,dy$$

(was "missing link" in [Jourdain-LeBris-Lelièvre-Otto]) □

Lemma 4 (weighted Csiszár-Kullback inequality; [AA-Carrillo-Manzini])

$f, g \in L^1_+(\mathbb{R}^n)$, $\int f = \int g = 1$, $|y|^4(f+g) \in L^1(\mathbb{R}^n)$

$$\Rightarrow \; \| |y|^2 (f-g) \|_{L^1}^2 \le 2\, e_1(f|g) \cdot \max\Big(\int |y|^4 f \, dy \, , \; \int |y|^4 g \, dy \Big)$$

Lemma 4 (weighted Csiszár-Kullback inequality; [AA-Carrillo-Manzini])

$f, g \in L^1_+(\mathbb{R}^n)$, $\int f = \int g = 1$, $|y|^4(f+g) \in L^1(\mathbb{R}^n)$

$$\Rightarrow \; \| |y|^2 (f-g) \|^2_{L^1} \leq 2\, e_1(f|g) \cdot \max\left(\int |y|^4 f\, dy \,,\; \int |y|^4 g\, dy \right)$$

Proof.

$$e_1(f|g) = \int_{\mathbb{R}^n} \frac{f}{g} \ln \frac{f}{g}\, g\, dy \;\overset{2^{nd}\ \text{Taylor}}{=}\; \frac{1}{2} \int_{\mathcal{A}} \frac{1}{\zeta(y)} (f-g)^2\, dy$$

with $\mathcal{A} := \{ f(y) \neq g(y) \}$; $\zeta =$ some intermediate value in (f, g)

$$\int_{\mathcal{A}} |y|^2 |f-g|\, dy \overset{\text{Hölder}}{\leq} \left(\int_{\mathcal{A}} \frac{1}{\zeta} (f-g)^2\, dy \right)^{\frac{1}{2}} \cdot \left(\int_{\mathcal{A}} |y|^4 \zeta\, dy \right)^{\frac{1}{2}}$$

□

6. Fokker-Planck equations with non-local terms [Stürzer-Arnold 2014]

Outline:

1. appropriate weight in L^2
2. spectrum of Fokker-Planck operator in L^2 with non-standard weight
3. perturbation by non-local operators

Fokker-Planck equation with non-local perturbation

motivation for the model:

Wigner-Fokker-Planck eq. for quantum-kinetic quasi-probability density $f(x, v, t)$ with potential $|x|^2/2 + V(x)$:

$$f_t = \text{div}_{x,v}\Big(\nabla_{x,v} f + \frac{1}{2}\begin{pmatrix} -v \\ x + 2v \end{pmatrix} f\Big) + \Theta[V]\, f$$

$\Theta[V]$ ... pseudo-differential operator / convolution in v

Q: $\exists\,!$ steady state? large-time convergence?

Fokker-Planck equation with non-local perturbation

motivation for the model:

Wigner-Fokker-Planck eq. for quantum-kinetic quasi-probability density $f(x,v,t)$ with potential $|x|^2/2 + V(x)$:

$$f_t = \operatorname{div}_{x,v}\left(\nabla_{x,v} f + \frac{1}{2}\begin{pmatrix} -v \\ x+2v \end{pmatrix} f\right) + \Theta[V]\, f$$

$\Theta[V]$ ... pseudo-differential operator / convolution in v

Q: $\exists\,!$ steady state? large-time convergence?

toy model with similar structure for $f(x,t)\,,\ x \in \mathbb{R}^n;\ t \geq 0$:

$$\begin{aligned} f_t &= \operatorname{div}(\nabla f + x\, f) + \vartheta * f \\ f(x,0) &= f_0(x) \\ \vartheta &\in \mathcal{S}'(\mathbb{R}^n)\,, \quad \int \vartheta(x)\,\mathrm{d}x = 0 \quad \rightarrow \quad \text{mass conservation} \end{aligned}$$

1D case: analytical problems

example:

$$f_t = \underbrace{(f_x + x\,f)_x}_{=:L\,f} + \underbrace{f(x+\alpha) - f(x-\alpha)}_{=:\Theta\,f}, \quad \alpha \in \mathbb{R}$$

- $\Theta\,f$ is unbounded in $L^2(\mathbb{R}, e^{x^2/2})$
- f_∞ (explicit in Fourier space) $\notin L^2(e^{x^2/2})$

So: $L^2(e^{x^2/2})$ "too small", weight grows too quickly

1D case: analytical problems

example:

$$f_t = \underbrace{(f_x + x\,f)_x}_{=: L\,f} + \underbrace{f(x+\alpha) - f(x-\alpha)}_{=:\Theta\, f}, \quad \alpha \in \mathbb{R}$$

- $\Theta\, f$ is unbounded in $L^2(\mathbb{R}, e^{x^2/2})$
- f_∞ (explicit in Fourier space) $\notin L^2(e^{x^2/2})$

So: $L^2(e^{x^2/2})$ "too small", weight grows too quickly

$\to$ find new L^2–weight $\omega(x)$

2 opposing aspects:

- slow enough growth $\Rightarrow \Theta \in \mathcal{B}(L^2(\omega))$, $f_\infty \in L^2(\omega)$
- too slow growth $\Rightarrow$ spectral gap of L is lost ($= 1$ in $L^2(e^{x^2/2})$)
  [Metafune] Ann. SNS Pisa 2001:
  in $L^2(\mathbb{R}^n)$: $\sigma(L) = \{\lambda \in \mathbb{C} \,|\, \Re\lambda \le \frac{n}{2}\}$

new L^2–weight

good weight: $\omega(x) = \cosh(\beta x)\,,\ \beta > 0$

practical for computations:

Lemma 5

$f \in L^2(\omega) \iff \hat{f}$ has an analytic continuation to the strip

$$\Omega_{\beta/2} := \{z \in \mathbb{C}\,|\,|\Im z| < \beta/2\} < \infty$$

with

$$\sup_{|b|<\beta/2, b\in\mathbb{R}} \|\hat{f}(\cdot + ib)\|_{L^2(\mathbb{R})} < \infty$$

Fokker-Planck operator in $L^2(e^{x^2/2})$

$L f := (f_x + x f)_x$ in $L^2(e^{x^2/2})$:

Theorem 16

- *L symmetric*
- $\sigma(L) = \sigma_P(L) = -\mathbb{N}_0$
- *eigenfunctions:* $\mu_k = \frac{1}{\sqrt{2\pi}} H_k \, e^{-x^2/2}$, H_k ... *Hermite polynomials*
- $\|e^{tL} f_0 - C\mu_0\| \le \|f_0\| \, e^{-t}, \; t \ge 0$

Fokker-Planck operator in $L^2(\omega)$

$L\,f := (f_x + x\,f)_x$ in $L^2(\omega)\,,\ \omega = \cosh(\beta x)$:

Theorem 17

- *L non-symmetric*
- $\sigma(L) = \sigma_P(L) = -\mathbb{N}_0$
- *eigenfunctions:* $\mu_k,\ k \in \mathbb{N}_0$ *(not orthogonal !)*
- *semigroup* $(e^{tL})_{t\geq 0}$ *is uniformly bounded*
- $\|e^{tL} f_0 - C\mu_0\|_\omega \leq C_1 \, \|f_0\|_\omega \, e^{-t},\ t \geq 0 \quad \forall\, f_0 \in L^2(\omega)$

FP operator in $L^2(\omega)$: proof-ideas

- resolvent of L is compact in $L^2(\omega)$
- solution formula:

$$\widehat{e^{tL}f}(\xi) = \exp\Big(-\frac{\xi^2}{2}(1-e^{-2t})\Big)\,\hat{f}(\xi e^{-t})$$

can be estimated on $\Omega_{\beta/2}$

- study e^{tL} on the invariant subspaces

$$\mathcal{E}_k := \mathrm{cl}_{L^2(\omega)}\,\mathrm{span}\{\mu_k,\,\mu_{k+1},\,\ldots\}$$

$\rightarrow$ decay: $c_k\, e^{-kt}$

perturbation operator Θ

$\Theta f := \vartheta * f$

conditions on ϑ :

1. $\vartheta \in \mathcal{S}'(\mathbb{R})\,, \quad \int \vartheta(x)\,\mathrm{d}x = 0$
2. $\hat{\vartheta} \in C^\omega(\Omega_{\beta/2}) \cap L^\infty(\Omega_{\beta/2}) \quad \Rightarrow \; \Theta \in \mathcal{B}(L^2(\omega))$
3. map $\quad \mathbb{C} \ni \xi \mapsto \hat{\psi}(\xi) := \Re \displaystyle\int_0^1 \hat{\vartheta}(\xi s)/s \,\mathrm{d}s \quad$ is in $\quad L^\infty(\Omega_{\beta/2})$

results on the operator Θ

- $\Theta : \mathcal{E}_k \to \mathcal{E}_{k+1} \subset \mathcal{E}_k \quad \forall\, k \in \mathbb{N}_0$
- $\sigma(L+\Theta) = \sigma_P(L+\Theta) = \sigma(L) = -\mathbb{N}_0$
  Θ is an isospectral perturbation of L !
- $\Psi : f \mapsto f * \psi$ on $L^2(\omega)$ yields the similarity transformation
  $L \to L + \Theta$

results on the evolution of $f_t = (L + \Theta)f$

Hence:

Theorem 18

1. $\exists\,!$ *(normalized) steady state* $\tilde{\mu}_0 := \Psi(e^{-x^2/2}/\sqrt{2\pi})$ *of*

$$f_t = (f_x + x\,f)_x f + \Theta\, f$$

results on the evolution of $f_t = (L + \Theta)f$

Hence:

Theorem 18

1. $\exists\,!$ *(normalized) steady state* $\tilde\mu_0 := \Psi(e^{-x^2/2}/\sqrt{2\pi})$ *of*

$$f_t = (f_x + x\,f)_x f + \Theta\, f$$

2. *decay estimate:*

$$\|e^{t(L+\Theta)} f_0 - C\tilde\mu_0\|_\omega \le C_1\, \|f_0\|_\omega\, e^{-t},\ \ t \ge 0 \quad \forall\, f_0 \in L^2(\omega)$$

Ref: [Stürzer-Arnold] Rend. Lincei Mat. Appl. 2014 (analysis in nD)

structure of the isospectral perturbation Θ

Representation of the operators in the Hermite basis $\{\mu_k\}_{k\in\mathbb{N}_0}$:

$$L = \text{ diag } \{0, -1, -2, ...\}$$

structure of the isospectral perturbation Θ

Representation of the operators in the Hermite basis $\{\mu_k\}_{k\in\mathbb{N}_0}$:

$$L = \text{ diag } \{0, -1, -2, ...\}$$

Θ is a strictly lower tridiagonal "matrix" (i.e. $\mathcal{E}_k \to \mathcal{E}_{k+1}$)

structure of the isospectral perturbation Θ

Representation of the operators in the Hermite basis $\{\mu_k\}_{k\in\mathbb{N}_0}$:

$$L = \text{ diag } \{0, -1, -2, ...\}$$

Θ is a strictly lower tridiagonal "matrix" (i.e. $\mathcal{E}_k \rightarrow \mathcal{E}_{k+1}$)

Reason: Due to $\int \vartheta(x)\, \mathrm{d}x = 0$ we have:

$$\begin{aligned} \hat{\mu}_k(0) &= 0 \quad \text{of order } k \\ \Rightarrow \widehat{\Theta\mu}_k(0) &= \hat{\vartheta}(0)\,\hat{\mu}_k(0) = 0 \quad \text{of order } k+1 \end{aligned}$$

structure of the isospectral perturbation Θ

Representation of the operators in the Hermite basis $\{\mu_k\}_{k\in\mathbb{N}_0}$:

$$L = \text{ diag } \{0, -1, -2, ...\}$$

Θ is a strictly lower tridiagonal "matrix" (i.e. $\mathcal{E}_k \rightarrow \mathcal{E}_{k+1}$)

Reason: Due to $\int \vartheta(x)\, \mathrm{d}x = 0$ we have:

$$\begin{aligned} \hat{\mu}_k(0) &= 0 \quad \text{of order } k \\ \Rightarrow \widehat{\Theta\mu}_k(0) &= \hat{\vartheta}(0)\,\hat{\mu}_k(0) = 0 \quad \text{of order } k+1 \end{aligned}$$

$\Rightarrow$ motivates $\sigma(L) = \sigma(L + \Theta)$ ... trivial for matrices

7. Hypocoercive equations [Erb-Arnold 2014]

Outline:

1. hypocoercivity, prototypic examples
2. decay of modified "entropy dissipation" functional
3. regularization of semigroup $\rightarrow$ entropy decay
4. sharp decay rates

degenerate Fokker-Planck equations with linear drift

evolution of probability density $f(x,t)$, $x \in \mathbb{R}^n$, $t > 0$:

$$\begin{aligned} f_t &= \operatorname{div}\Big(\mathbf{D}\nabla f + \mathbf{C}x\, f\Big) \\ f(x,0) &= f_0(x) \end{aligned} \tag{19}$$

$\mathbf{D} \in \mathbb{R}^{n\times n}$ ... symmetric, const in x, degenerate

w.r.o.g. (via coordinate transformation, x–scaling):
let $\mathbf{D} = \operatorname{diag}(\underbrace{1, \dots 1}_{k}, \underbrace{0, \dots 0}_{n-k})$

$\mathbf{C} \in \mathbb{R}^{n\times n}$ ... const in x

goals: existence & uniqueness of steady state $f_\infty(x)$;
convergence $f(t) \xrightarrow{t\to\infty} f_\infty$ with sharp rates;
complete theory for the equation class (19)

(hypo)coercivity 1

example 1: standard Fokker-Planck equation on $\mathbb{R}^n$:

$$f_t = \operatorname{div}\Big(\nabla f + x f\Big) =: Lf \quad \ldots \text{ symmetric on } H := L^2(f_\infty^{-1})$$

$$f_\infty(x) = c e^{-\frac{|x|^2}{2}}, \qquad \ker L = \operatorname{span}(f_\infty)$$

L is dissipative, i.e. $\langle Lf, f\rangle_H \le 0 \quad \forall f \in \mathcal{D}(L)$

$-L$ is coercive (has a spectral gap), in the sense:

$$\langle -Lf, f\rangle_H \ge \|f\|^2_{L^2(f_\infty^{-1})} \quad \forall f \in \{f_\infty\}^\perp$$

(hypo)coercivity 2

example 2:

$$f_t = \operatorname{div}\Big(\mathbf{D}\nabla f + \mathbf{C}x\, f\Big) =: Lf \tag{20}$$

with degenerate $\mathbf{D}$ is degenerate parabolic;
(symmetric part of) $-L$ is not coercive.

(hypo)coercivity 2

example 2:

$$f_t \;=\; \operatorname{div}\Big(\mathbf{D}\nabla f + \mathbf{C}x\, f\Big) =: Lf \qquad (20)$$

with degenerate $\mathbf{D}$ is degenerate parabolic;
(symmetric part of) $-L$ is not coercive.

Definition 19 (Villani 2009)

Consider L on Hilbert space H with $\mathcal{K} = \ker L$; let $\tilde{H} \hookrightarrow \mathcal{K}^\perp$ (densely)
(e.g. H ... weighted L^2, $\tilde{H}$ ... weighted H^1).
$-L$ is called hypocoercive on $\tilde{H}$ if $\exists\, \lambda > 0,\; c > 0$:

$$\|e^{Lt} f\|_{\tilde{H}} \le c\, e^{-\lambda t} \|f\|_{\tilde{H}} \qquad \forall f \in \tilde{H}$$

(hypo)coercivity 3

example 3: kinetic Fokker-Planck equation for $f(x,v,t)$, $x,v \in \mathbb{R}^n$:

$$f_t + \underbrace{v \cdot \nabla_x f}_{\text{free transport}} - \underbrace{\nabla_x V \cdot \nabla_v f}_{\text{influence of potential } V(x,t)} = \underbrace{\sigma \Delta_v f}_{\text{diffusion, } \sigma>0} + \underbrace{\nu \, \mathrm{div}_v(vf)}_{\text{friction, } \nu>0}$$

steady state: $f_\infty(x,v) = c\, e^{-\frac{\nu}{\sigma}\left[\frac{|v|^2}{2}+V(x)\right]}$

$V(x)$... given *confinement potential*

(hypo)coercivity 3

example 3: kinetic Fokker-Planck equation for $f(x,v,t)$, $x, v \in \mathbb{R}^n$:

$$f_t + \underbrace{v \cdot \nabla_x f}_{\text{free transport}} - \underbrace{\nabla_x V \cdot \nabla_v f}_{\text{influence of potential } V(x,t)} = \underbrace{\sigma \Delta_v f}_{\text{diffusion, } \sigma>0} + \nu \underbrace{\operatorname{div}_v(vf)}_{\text{friction, } \nu>0}$$

steady state: $f_\infty(x,v) = c\, e^{-\frac{\nu}{\sigma}\left[\frac{|v|^2}{2}+V(x)\right]}$

$V(x)$... given *confinement potential*

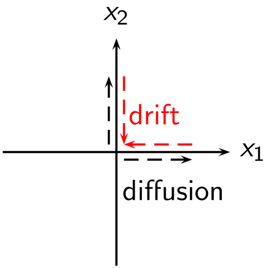
rewritten:

$$f_t = \operatorname{div}_{x,v}\left[\begin{pmatrix} 0 & 0 \\ 0 & \sigma \mathbf{I} \end{pmatrix} \nabla_{x,v} f + \begin{pmatrix} -v \\ \nabla_x V + \nu v \end{pmatrix} f\right]$$

problem 1: steady state

standard Fokker-Planck equation $f_t = \operatorname{div}(\nabla f + x\, f)$:
unique steady state $f_\infty(x) = c\, e^{-|x|^2/2}$ as a balance of drift & diffusion;
sharp decay rate $= 1$

$n = 2$:



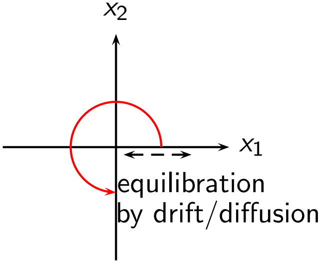
2 degenerate prototypes

prototype (a): degenerate diffusion (1D Fokker-Planck) + rotation

$$f_t = \operatorname{div}\Big[\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{=\mathbf{D}} \nabla f + \underbrace{\begin{pmatrix} x_1 - \omega\, x_2 \\ \omega\, x_1 \end{pmatrix}}_{=\mathbf{C}x} f\Big]$$

$f_\infty(x) = c\, \mathrm{e}^{-|x|^2/2}$;

sharp decay rate $= \frac{1}{2}$ $(= \min \Re \lambda_C)$ for fast enough rotation $(|\omega| > \frac{1}{2})$

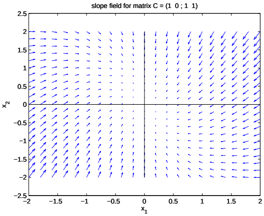


2 degenerate prototypes

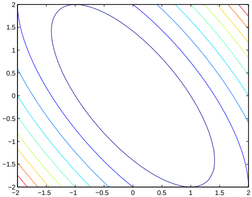
prototype (b): degenerate diffusion – not aligned with drift characteristics

$$f_t = \operatorname{div}\Big[\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{=\mathbf{D}} \nabla f + \underbrace{\begin{pmatrix} x_1 \\ x_1 + x_2 \end{pmatrix}}_{=\mathbf{C}x} f\Big]$$

$f_\infty(x) = c\, e^{-(x_1^2+2x_1x_2+2x_2^2)}$; (sharp) decay rate $= 1-\varepsilon$ $(< \min \Re \lambda_C)$



characteristics of drift: $x_t = -\mathbf{C}\,x$



contours of steady state potential $-\ln f_\infty$

coefficients **C**, **D** in Fokker-Planck equation

$$f_t = \operatorname{div}\Big(\mathbf{D}\,\nabla f + \mathbf{C}\,x\,f\Big) =: Lf$$

<u>Condition A:</u> No (nontrivial) subspace of $\ker \mathbf{D}$ is invariant under $\mathbf{C}^\top$.
(equivalent: No eigenvector v of $\mathbf{C}^\top$ satisfies $\mathbf{D}\,v = 0$. L hypoelliptic.)

Proposition 1

Let Condition A hold.

a) *Let* $f_0 \in L^1(\mathbb{R}^d) \quad \Rightarrow \quad f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+)$. *[Hörmander 1969]*

b) *Let* $f_0 \in L^1_+(\mathbb{R}^d) \quad \Rightarrow \quad f(x,t) > 0,\ \forall t > 0.$ *(Green's fct* > 0*)*

coefficients **C**, **D** in Fokker-Planck equation

$$f_t = \operatorname{div}\Big(\mathbf{D}\,\nabla f + \mathbf{C}\,x\,f\Big) =: Lf$$

Condition A: No (nontrivial) subspace of $\ker \mathbf{D}$ is invariant under $\mathbf{C}^\top$.
(equivalent: No eigenvector v of $\mathbf{C}^\top$ satisfies $\mathbf{D}\,v = 0$. L hypoelliptic.)

Proposition 1

Let Condition A hold.

a) *Let* $f_0 \in L^1(\mathbb{R}^d) \quad \Rightarrow \quad f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+)$. *[Hörmander 1969]*

b) *Let* $f_0 \in L^1_+(\mathbb{R}^d) \quad \Rightarrow \quad f(x,t) > 0,\ \forall t > 0.$ *(Green's fct* > 0*)*

Condition B: Condition A + let **C** be positively stable (i.e. $\Re\,\lambda_C > 0$)
$\rightarrow \exists$ confinement potential; drift towards $x = 0$.

steady state

$$f_t = \operatorname{div}\Big(\mathbf{D}\nabla f + \mathbf{C}x\, f\Big) \tag{21}$$

Theorem 20

(21) has a unique (normalized) steady state $f_\infty \in L^1(\mathbb{R}^n)$ iff Condition B holds.

Then: $f_\infty(x) = c_K e^{-\frac{x^\top \mathbf{K}^{-1} x}{2}}$ ... *non-isotropic Gaussian*

$0 < \mathbf{K} \in \mathbb{R}^{n\times n}$ ... *unique solution of* $2\mathbf{D} = \mathbf{C}\mathbf{K} + \mathbf{K}\mathbf{C}^\top$

(continuous Lyapunov equation)

steady state

$$f_t = \operatorname{div}\Big(\mathbf{D}\nabla f + \mathbf{C}x\, f\Big) \tag{21}$$

Theorem 20

(21) has a unique (normalized) steady state $f_\infty \in L^1(\mathbb{R}^n)$ iff Condition B holds.

Then: $f_\infty(x) = c_K e^{-\frac{x^\top \mathbf{K}^{-1} x}{2}}$ ... *non-isotropic Gaussian*

$0 < \mathbf{K} \in \mathbb{R}^{n\times n}$ ... *unique solution of* $2\mathbf{D} = \mathbf{C}\mathbf{K} + \mathbf{K}\mathbf{C}^\top$

(continuous Lyapunov equation)

proof - idea:

Fourier transformed equation for $\widehat{f}(\xi, t)$:

$$\widehat{f}_t = -(\xi^\top \mathbf{D}\xi)\widehat{f} - (\mathbf{C}^\top \xi)\cdot \nabla_\xi \widehat{f}$$

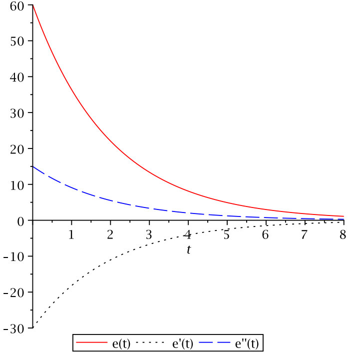
$\mathbf{D} \geq 0 \;\Rightarrow\; \exists\,!\,\mathbf{K} \geq 0$; $\quad \mathbf{K} > 0$ from Condition B.

decomposition of the generator L:

$$\begin{aligned}
\partial_t f &= Lf = \operatorname{div}\Big(\mathbf{D}\nabla f + \mathbf{C}x\, f\Big) \quad \text{in } L^2(f_\infty^{-1}), \qquad (22)\\
L &= L^s + L^{as} \quad \text{with} \quad L^s f_\infty = L^{as} f_\infty = 0,\\
L^s f &= \operatorname{div}\left(\mathbf{D}\left(\nabla\frac{f}{f_\infty}\right) f_\infty\right) \quad \ldots \text{ like in non-degenerate case,}\\
L^{as} f &= \operatorname{div}\left(\mathbf{R}\left(\nabla\frac{f}{f_\infty}\right) f_\infty\right),\\
\mathbf{R} &:= -\mathbf{R}^\top = \frac{1}{2}(\mathbf{CK} - \mathbf{KC}^\top) \neq 0 \quad \rightarrow \quad \text{(22) is non-symmetric.}
\end{aligned}$$

problem 2: entropy decay

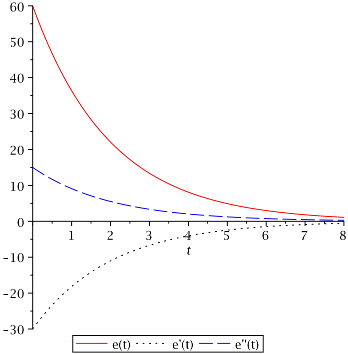
decay of quadratic entropy $e_2(t) = \|f(t) - f_\infty\|^2_{L^2}$:



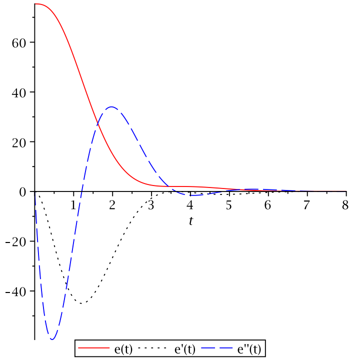
standard Fokker-Planck equation:
non-degenerate $\rightarrow$ $e(t)$ is convex;
entropy dissip. $e'(t) < 0 \; \forall \; f \neq f_\infty$;
$e' \leq -\mu e$ possible (with $\mu > 0$)

problem 2: entropy decay

decay of quadratic entropy $e_2(t) = \|f(t) - f_\infty\|^2_{L^2}$:



standard Fokker-Planck equation:
non-degenerate $\rightarrow$ $e(t)$ is convex;
entropy dissip. $e'(t) < 0 \ \forall\ f \neq f_\infty$;
$e' \leq -\mu e$ possible (with $\mu > 0$)



degenerate prototype ex. (a):
$\rightarrow$ $e(t)$ is not convex;
$e'(t) = 0$ for some $f \neq f_\infty$;
$e' \leq -\mu e$ wrong (in general)

modified entropy method for degenerate FP equation

$e'(t) = 0$ for some $f \neq f_\infty \Rightarrow$ entropy dissipation "useless":

$$\frac{\mathrm{d}}{\mathrm{d}t} e_\psi = -\int_{\mathbb{R}^n} \psi''\left(\frac{f}{f_\infty}\right) \nabla^\top \frac{f}{f_\infty} \cdot \underbrace{\mathbf{D}}_{\geq 0} \cdot \nabla \frac{f}{f_\infty} f_\infty \, \mathrm{d}x =: -I_\psi(f) \leq 0$$

modified entropy method for degenerate FP equation

$e'(t) = 0$ for some $f \neq f_\infty \Rightarrow$ entropy dissipation "useless":

$$\frac{\mathrm{d}}{\mathrm{d}t} e_\psi = -\int_{\mathbb{R}^n} \psi''\left(\frac{f}{f_\infty}\right) \nabla^\top \frac{f}{f_\infty} \cdot \underbrace{\mathbf{D}}_{\geq 0} \cdot \nabla \frac{f}{f_\infty} f_\infty \, \mathrm{d}x =: -I_\psi(f) \leq 0$$

$\Rightarrow$ define modified "entropy dissipation" as auxiliary functional:

$$S_\psi(f) := \int_{\mathbb{R}^n} \psi''\left(\frac{f}{f_\infty}\right) \nabla^\top \frac{f}{f_\infty} \cdot \underbrace{\mathbf{P}}_{>0} \cdot \nabla \frac{f}{f_\infty} f_\infty \, \mathrm{d}x \geq 0$$

goal: estimate between $S(f(t))$, $\frac{\mathrm{d}}{\mathrm{d}t} S(f(t))$ for "good" choice of $\mathbf{P} > 0$.
Then:

$$\mathbf{P} \geq c_P \, \mathbf{D} \quad \Rightarrow \quad S_\psi(f) \geq c_P \, I_\psi(f) \quad \searrow 0$$

modified "entropy dissipation" $S_\psi(f)$: choice of $\mathbf{P}$

Lemma 6

Let $\mu := \min\{\Re\,\lambda_C\}$ *(*> 0 *since* $\mathbf{C}$ *is positively stable);* $\mathbf{Q} := \mathbf{K}\mathbf{C}^\top\mathbf{K}^{-1}$ *.*

1. *If all* $\lambda_C^{\min} := \{\lambda \in \sigma(\mathbf{C}) \,|\, \Re\lambda = \mu\}$ *are non-defective*
   (i.e. geometric = algebraic multiplicity)
   $\Rightarrow \quad \exists\, \mathbf{P} \in \mathbb{R}^{n\times n},\ \mathbf{P} > 0 \ : \quad \mathbf{Q}\mathbf{P} + \mathbf{P}\mathbf{Q}^\top \geq 2\mu\mathbf{P}\,.$

modified "entropy dissipation" $S_\psi(f)$: choice of $\mathbf{P}$

Lemma 6

Let $\mu := \min\{\Re\,\lambda_C\}$ *(*> 0 *since* $\mathbf{C}$ *is positively stable);* $\mathbf{Q} := \mathbf{K}\mathbf{C}^\top\mathbf{K}^{-1}$ *.*

1. *If all* $\lambda_C^{\min} := \{\lambda \in \sigma(\mathbf{C}) \,|\, \Re\lambda = \mu\}$ *are non-defective*
   (i.e. geometric = algebraic multiplicity)
   $\Rightarrow \quad \exists\, \mathbf{P} \in \mathbb{R}^{n\times n},\ \mathbf{P} > 0 \ : \quad \mathbf{Q}\mathbf{P} + \mathbf{P}\mathbf{Q}^\top \geq 2\mu\mathbf{P}$.
2. *If (at least) one* $\lambda_C^{\min}$ *is defective* $\Rightarrow$
   $\forall\, \varepsilon > 0 \quad \exists\, \mathbf{P} = \mathbf{P}(\varepsilon) > 0 \ : \quad \mathbf{Q}\mathbf{P} + \mathbf{P}\mathbf{Q}^\top \geq 2(\mu - \varepsilon)\mathbf{P}$.

modified "entropy dissipation" $S_\psi(f)$: choice of $\mathbf{P}$

Lemma 6

Let $\mu := \min\{\Re\lambda_C\}$ *(*> 0 *since* $\mathbf{C}$ *is positively stable);* $\mathbf{Q} := \mathbf{K}\mathbf{C}^\top\mathbf{K}^{-1}$ *.*

1. *If all* $\lambda_C^{\min} := \{\lambda \in \sigma(\mathbf{C}) \,|\, \Re\lambda = \mu\}$ *are non-defective (i.e. geometric = algebraic multiplicity)*
   $\Rightarrow \quad \exists\, \mathbf{P} \in \mathbb{R}^{n\times n},\ \mathbf{P} > 0 \ : \quad \mathbf{Q}\mathbf{P} + \mathbf{P}\mathbf{Q}^\top \geq 2\mu\mathbf{P}$.
2. *If (at least) one* $\lambda_C^{\min}$ *is defective* $\Rightarrow$
   $\forall\, \varepsilon > 0 \quad \exists\, \mathbf{P} = \mathbf{P}(\varepsilon) > 0 \ : \quad \mathbf{Q}\mathbf{P} + \mathbf{P}\mathbf{Q}^\top \geq 2(\mu - \varepsilon)\mathbf{P}$.

Proof: $\mathbf{P}$ can be constructed explicitly; e.g. for $\mathbf{C}$ non-defective / diagonalizable:

$$\mathbf{P} := \sum_{j=1}^{n} z_j \otimes \bar{z}_j^\top \ ; \qquad z_j \ \dots \text{ eigenvectors of } \mathbf{Q}$$

- $\mathbf{P}$ not unique; but decay are rates independent of $\mathbf{P}$

exponential decay of auxiliary functional $S_\psi(f)$

Proposition 2

$\mu := \min\{\Re \lambda_C\}$. *Let* f_0 *satisfy:*

$$\int \psi''\left(\frac{f_0}{f_\infty}\right)\left|\nabla \frac{f_0}{f_\infty}\right|^2 f_\infty \, \mathrm{d}x < \infty \quad (\sim \text{ weighted } H^1\text{–seminorm})$$

1. *If all* $\lambda_C^{\min}$ *are non-defective* $\Rightarrow \quad S(f(t)) \le e^{-2\mu t} S(f_0)\,, \quad t \ge 0;$

exponential decay of auxiliary functional $S_\psi(f)$

Proposition 2

$\mu := \min\{\Re\,\lambda_C\}$. *Let* f_0 *satisfy:*

$$\int \psi''\left(\frac{f_0}{f_\infty}\right)\left|\nabla \frac{f_0}{f_\infty}\right|^2 f_\infty \, \mathrm{d}x < \infty \quad (\sim \text{ weighted } H^1\text{–seminorm})$$

1. *If all* $\lambda_C^{\min}$ *are non-defective* $\Rightarrow \quad S(f(t)) \leq e^{-2\mu t} S(f_0)\,, \quad t \geq 0;$
2. *If one* $\lambda_C^{\min}$ *is defective* $\Rightarrow \quad S(f(t), \varepsilon) \leq e^{-2(\mu-\varepsilon)t} S(f_0, \varepsilon)\,, \quad t \geq 0.$

exponential decay of auxiliary functional $S_\psi(f)$

Proposition 2

$\mu := \min\{\Re\,\lambda_C\}$. *Let* f_0 *satisfy:*

$$\int \psi''\left(\frac{f_0}{f_\infty}\right)\left|\nabla \frac{f_0}{f_\infty}\right|^2 f_\infty \,\mathrm{d}x < \infty \quad (\sim \textit{ weighted } H^1\textit{–seminorm})$$

1. *If all* $\lambda_C^{\min}$ *are non-defective* $\Rightarrow \quad S(f(t)) \le e^{-2\mu t} S(f_0)\,, \quad t \ge 0;$
2. *If one* $\lambda_C^{\min}$ *is defective* $\Rightarrow \quad S(f(t),\varepsilon) \le e^{-2(\mu-\varepsilon)t} S(f_0,\varepsilon)\,, \quad t \ge 0.$

notation:

$$u(x,t) := \nabla \frac{f(x,t)}{f_\infty(x)}\ ;$$

$$f_\infty(x) = c_K e^{-\frac{x^\top \mathbf{K}^{-1} x}{2}} = c_K e^{-V(x)}$$

Proof of Proposition 2 – modified entropy method

$$\frac{\mathrm{d}}{\mathrm{d}t}S(f(t)) = -\int \psi''\Big(\frac{f}{f_\infty}\Big) u^\top \underbrace{\left[(\mathbf{D}-\mathbf{R})\frac{\partial^2 V}{\partial x^2}\mathbf{P} + \mathbf{P}\frac{\partial^2 V}{\partial x^2}(\mathbf{D}+\mathbf{R})\right]}_{=\mathbf{QP}+\mathbf{PQ}^\top \geq 2\mu\mathbf{P} \text{ ... replaces BEC}} u f_\infty \, \mathrm{d}x$$

$$-2\int \underbrace{\mathrm{Tr}\,(XY)}_{\geq 0} f_\infty \, \mathrm{d}x \leq -2\mu\, S(f(t))$$

Proof of Proposition 2 – modified entropy method

$$\frac{\mathrm{d}}{\mathrm{d}t} S(f(t)) = -\int \psi''\Big(\frac{f}{f_\infty}\Big) u^\top \underbrace{\Big[(\mathbf{D}-\mathbf{R})\frac{\partial^2 V}{\partial x^2}\mathbf{P} + \mathbf{P}\frac{\partial^2 V}{\partial x^2}(\mathbf{D}+\mathbf{R})\Big]}_{=\mathbf{QP}+\mathbf{PQ}^\top \geq 2\mu \mathbf{P}\ \ldots \text{ replaces BEC}} u f_\infty \,\mathrm{d}x$$

$$-2\int \underbrace{\mathrm{Tr}\,(XY)}_{\geq 0} f_\infty \,\mathrm{d}x \leq -2\mu\, S(f(t))$$

use (similar to non-degenerate case)

$$X := \begin{pmatrix} \psi'' & \psi''' \\ \psi''' & \frac{1}{2}\psi^{\mathrm{IV}} \end{pmatrix}\Big(\frac{f}{f_\infty}\Big) \geq 0 \quad ;$$

$$Y := \begin{pmatrix} \mathrm{Tr}\,(\mathbf{D}\frac{\partial u}{\partial x}\mathbf{P}\frac{\partial u}{\partial x}) & u^\top \mathbf{D}\frac{\partial u}{\partial x}\mathbf{P}u \\ u^\top \mathbf{D}\frac{\partial u}{\partial x}\mathbf{P}u & (u^\top \mathbf{P}u)\,(u^\top \mathbf{D}u) \end{pmatrix} \geq 0\,, \quad \text{with}$$

Cauchy-Schwarz for $(u^\top \mathbf{D}\frac{\partial u}{\partial x}\mathbf{P}u)^2 = \mathrm{Tr}\,(\sqrt{\mathbf{P}}uu^\top\sqrt{\mathbf{D}}\cdot\sqrt{\mathbf{D}}\frac{\partial u}{\partial x}\sqrt{\mathbf{P}})^2$

exponential decay of relative entropy

Theorem 21

Let f_0 satisfy:

$$\int \psi''\left(\frac{f_0}{f_\infty}\right) |u|^2 \, f_\infty \, \mathrm{d}x < \infty \, .$$

$$\Rightarrow \quad \boxed{e(f(t)|f_\infty) \leq c\, S(f(t)) \leq c\, e^{-2\mu t} S(f_0)\,, \quad t \geq 0}$$

(reduced rate for a defective $\lambda_C^{\min}$: $2(\mu - \varepsilon)$)

exponential decay of relative entropy

Theorem 21

Let f_0 satisfy:
$$\int \psi''\left(\frac{f_0}{f_\infty}\right) |u|^2 \; f_\infty \, \mathrm{d}x < \infty \, .$$

$$\Rightarrow \quad \boxed{e(f(t)|f_\infty) \leq c\, S(f(t)) \leq c\, e^{-2\mu t} S(f_0)\,, \quad t \geq 0}$$

(reduced rate for a defective $\lambda_C^{\min}$: $2(\mu - \varepsilon)$)

Proof: Consider non-degenerate (auxiliary) symmetric FP equation:

$$g_t = \operatorname{div}\left(\mathbf{P}(\nabla \frac{g}{f_\infty}) f_\infty\right) \; ; \quad g_\infty = f_\infty = c_K \, e^{-V(x)} \qquad (23)$$

It satisfies the Bakry-Emery condition $\frac{\partial^2 V}{\partial x^2} = \mathbf{K}^{-1} \geq \lambda_P \mathbf{P}^{-1}$.

$$\Rightarrow \quad \text{convex Sobolev inequality:} \quad e_\psi(g|f_\infty) \leq \frac{1}{2\lambda_P} S_\psi(g) \quad \forall \, g$$

Remark: $S_\psi(g)$ is the true entropy dissipation for (23) !

(parabolic) regularization of semigroup e^{Lt}

Proposition 3

Let $m \le n - k$ *(*$k =$*rank* $\mathbf{D}$*) be the minimum such that*

$$\sum_{j=0}^{m} \mathbf{C}^j \mathbf{D} (\mathbf{C}^\top)^j \ge \kappa \, \mathbf{I} \quad \textit{for some } \kappa > 0 \, .$$

(Existence of m *is equivalent to Condition A, i.e. hypoellipticity of* L*.)*

$$\Rightarrow \quad S_\psi(f(t)) \le c \, t^{-(2m+1)} \, e_\psi(f_0 | f_\infty) \, , \quad 0 < t \le 1 \, .$$

(parabolic) regularization of semigroup e^{Lt}

Proposition 3

Let $m \le n - k$ ($k =$rank $\mathbf{D}$) be the minimum such that

$$\sum_{j=0}^{m} \mathbf{C}^j \mathbf{D} (\mathbf{C}^\top)^j \ge \kappa \, \mathbf{I} \quad \textit{for some } \kappa > 0\,.$$

(Existence of m is equivalent to Condition A, i.e. hypoellipticity of L.)

$$\Rightarrow \quad S_\psi(f(t)) \le c\, t^{-(2m+1)}\, e_\psi(f_0|f_\infty)\,, \quad 0 < t \le 1\,.$$

<u>Ref's:</u>
Prop. 3 is generalization to all admissible relative entropies of:
[Hérau] JFA 2007;
[Villani] book 2009 (only for quadratic & logarithmic entropies)

Proof of Proposition 3

Prove decay of the auxiliary functional $\mathcal{F}(t)$:

$$\mathcal{F}(t) := \underbrace{c_1}_{>0} e_\psi(f(t)|f_\infty) + \underbrace{\int \psi''\left(\frac{f}{f_\infty}\right) u^\top \tilde{\mathbf{P}}(t) u \, f_\infty \, \mathrm{d}x}_{\text{"similar" to } S_\psi(f(t))} \geq 0 \,,$$

$\tilde{\mathbf{P}}(t)$ ... matrix polynomial in t of order $2m+1$ (coeff's depend on $\mathbf{D}$, $\mathbf{Q}$) with

$$\tilde{\mathbf{P}}(t) \geq c_2 t^{2m+1} \mathbf{I} \geq c_3 t^{2m+1} \mathbf{P} > 0 \,; \quad \tilde{\mathbf{P}}(0) = 0 \,.$$

Proof of Proposition 3

Prove decay of the auxiliary functional $\mathcal{F}(t)$:

$$\mathcal{F}(t) := \underbrace{c_1}_{>0} e_\psi(f(t)|f_\infty) + \underbrace{\int \psi''\left(\frac{f}{f_\infty}\right) u^\top \tilde{\mathbf{P}}(t) u\, f_\infty \,\mathrm{d}x}_{\text{“similar” to } S_\psi(f(t))} \geq 0\,,$$

$\tilde{\mathbf{P}}(t)$ ... matrix polynomial in t of order $2m+1$ (coeff's depend on $\mathbf{D}$, $\mathbf{Q}$) with

$$\tilde{\mathbf{P}}(t) \geq c_2 t^{2m+1}\mathbf{I} \geq c_3 t^{2m+1}\mathbf{P} > 0\,; \quad \tilde{\mathbf{P}}(0) = 0\,.$$

- computations like in modified entropy method (for $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}$) $\Rightarrow$

$$c_1\, e_\psi(f_0|f_\infty) = \mathcal{F}(0) \geq \mathcal{F}(t) \geq c_3 t^{2m+1} S_\psi(f(t))$$

□

exp. decay of rel. entropy for $f_t = \operatorname{div}(\mathbf{D}\nabla f + \mathbf{C}x\, f) =: Lf$

combination of regularization for initial time with Th.21 (entropy decay) $\Rightarrow$

Theorem 22 (Erb-Arnold 2014)

Let L satisfy Condition B; $\mu := \min\{\Re\, \lambda_C\}$. $\Rightarrow \exists\, c > 0$:

$$\boxed{e_\psi(f(t)|f_\infty) \le c\, e^{-2\mu t} e_\psi(f_0|f_\infty)\,, \quad t \ge 0}$$

(reduced rate for a defective $\lambda_C^{\min}$: $2(\mu - \varepsilon)$)

exp. decay of rel. entropy for $f_t = \mathrm{div}(\mathbf{D}\nabla f + \mathbf{C}x\, f) =: Lf$

combination of regularization for initial time with Th.21 (entropy decay) $\Rightarrow$

Theorem 22 (Erb-Arnold 2014)

Let L satisfy Condition B; $\mu := \min\{\Re\,\lambda_C\}$. $\Rightarrow \exists\, c > 0$:

$$\boxed{e_\psi(f(t)|f_\infty) \le c\, e^{-2\mu t} e_\psi(f_0|f_\infty)\,, \quad t \ge 0}$$

(reduced rate for a defective $\lambda_C^{\min}$: $2(\mu - \varepsilon)$)

Proof:

$$e(t) \overset{\text{CSI}}{\le} \frac{1}{2\lambda_P} S(f(t)) \overset{\text{decay}}{\le} \frac{1}{2\lambda_P}\, e^{-2\mu(t-\delta)}\, S(f(\delta)) \overset{\text{regulariz.}}{\le} c(\delta)\, e^{-2\mu t} e(0)$$

□

exp. decay of rel. entropy for $f_t = \mathrm{div}(\mathbf{D}\nabla f + \mathbf{C}x\, f) =: Lf$

combination of regularization for initial time with Th.21 (entropy decay) $\Rightarrow$

Theorem 22 (Erb-Arnold 2014)

Let L satisfy Condition B; $\mu := \min\{\Re\,\lambda_C\}$. $\Rightarrow \exists\, c > 0$:

$$\boxed{e_\psi(f(t)|f_\infty) \le c\, e^{-2\mu t} e_\psi(f_0|f_\infty)\,, \quad t \ge 0}$$

(reduced rate for a defective $\lambda_C^{\min}$: $2(\mu - \varepsilon)$ *)*

Proof:

$$e(t) \overset{\text{CSI}}{\le} \frac{1}{2\lambda_P} S(f(t)) \overset{\text{decay}}{\le} \frac{1}{2\lambda_P} \mathrm{e}^{-2\mu(t-\delta)} S(f(\delta)) \overset{\text{regulariz.}}{\le} c(\delta)\, e^{-2\mu t} \mathrm{e}\,(0)$$

$\square$

Remark: Rate μ is sharp, but constant c is not.

Ref: [Villani] 2009: exponential decay in weighted H^1, but *no sharp rates*

sharpness of the exponential decay 1

Theorem 23

Let $\mu := \min\{\Re\,\lambda_C\}$. $\quad\Rightarrow$

1. *If* $\lambda_C^{\min} \in \mathbb{R}$ *with eigenvector* $v_0 \in \mathbb{R}^n$*:*

$$f_0(x) := f_\infty(x)\, e^{\,v_0\cdot x - \frac{v_0^\top \mathbf{K} v_0}{2}} \quad \textit{yields}$$

$$e_1(f(t)|f_\infty) = e^{-2\mu t}\, e_1(f_0|f_\infty)\,. \tag{24}$$

sharpness of the exponential decay 1

Theorem 23

Let $\mu := \min\{\Re\,\lambda_C\}$. $\quad \Rightarrow$

1. *If* $\lambda_C^{\min} \in \mathbb{R}$ *with eigenvector* $v_0 \in \mathbb{R}^n$*:*

$$f_0(x) := f_\infty(x)\, e^{\,v_0\cdot x - \frac{v_0^\top \mathbf{K} v_0}{2}} \quad \textit{yields}$$

$$e_1(f(t)|f_\infty) = e^{\,-2\mu t}\, e_1(f_0|f_\infty)\,. \tag{24}$$

$$g_0(x) := f_\infty(x)\,(1 + v_0\cdot x) \quad \textit{yields}$$

$$e_2(g(t)|f_\infty) = e^{\,-2\mu t}\, e_2(g_0|f_\infty)\,. \tag{25}$$

sharpness of the exponential decay 1

Theorem 23

Let $\mu := \min\{\Re\,\lambda_C\}$. $\quad \Rightarrow$

1. *If* $\lambda_C^{\min} \in \mathbb{R}$ *with eigenvector* $v_0 \in \mathbb{R}^n$*:*

$$f_0(x) := f_\infty(x)\, e^{\,v_0\cdot x - \frac{v_0^\top \mathbf{K} v_0}{2}} \quad \textit{yields}$$

$$e_1(f(t)|f_\infty) = e^{-2\mu t}\, e_1(f_0|f_\infty)\,. \tag{24}$$

$$g_0(x) := f_\infty(x)\,(1 + v_0\cdot x) \quad \textit{yields}$$

$$e_2(g(t)|f_\infty) = e^{-2\mu t}\, e_2(g_0|f_\infty)\,. \tag{25}$$

2. *If* $\lambda_C^{\min} \notin \mathbb{R}$ *with eigenvector* $v_0 \in \mathbb{C}^n$*:*
   $\exists\, f_0,\, g_0$ *such that* "... $\le$ *const.* ..." *in* (24), (25);
   and equality for $t = t_0 + n\tau$ *(with some* $t_0 \ge 0$, $\tau > 0$, $n \in \mathbb{N}_0$*).*

sharpness of the exponential decay 2

Remark:

- General entropies: bounded below by $c_1\, e_1$, bounded above by $c_2\, e_2$.

  $\Rightarrow$ Decay rate from Theorem 22 is sharp $\forall$ admissible entropies e_ψ.
- for a defective $\lambda_C^{\min}$: $\exists\ f_0$ with

$$e_{1,2}(t) = \mathrm{e}^{\,-2\mu t}\,(c_0 + c_1\, t + c_2\, t^2)\,.$$

spectrum of L in $L^2(\mathbb{R}^n, f_\infty^{-1})$

L ... non-symmetric

Theorem 24

1. $$\sigma(L) = \sigma_p(L) = \left\{ -\sum_{j=1}^{n} \alpha_j \lambda_j \middle| \alpha \in \mathbb{N}_0^n \right\} \subset \{0\} \cup (\mathbb{R}^- \times i\mathbb{R})$$

   with $\{\lambda_j\}$ ... *eigenvalues of* $\mathbf{C}$ *.*

spectrum of L in $L^2(\mathbb{R}^n, f_\infty^{-1})$

L ... non-symmetric

Theorem 24

1. $$\sigma(L) = \sigma_p(L) = \left\{ -\sum_{j=1}^{n} \alpha_j \lambda_j \middle| \alpha \in \mathbb{N}_0^n \right\} \subset \{0\} \cup (\mathbb{R}^- \times i\mathbb{R})$$

   with $\{\lambda_j\}$ *... eigenvalues of* $\mathbf{C}$.

2. $L^2(\mathbb{R}^n, f_\infty^{-1})$ *has an orthogonal decomposition in* e^{Lt}*–invariant subspaces, defined by* $\{|\alpha| = const.\}$.
   They are spanned by (generalized) eigenfunctions of L *which lie in* $\mathcal{P}(\mathbb{R}^n) \cdot f_\infty$.