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Classical and Quantum Mechanical Models of Many-Particle Systems

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Abstract. The topic of this meeting were non-linear partial differential and integro-differential equations (in particular kinetic equations and their macroscopic/fluid-dynamical limits) modeling the dynamics of many-particle systems with applications in physics, engineering, and mathematical biology. Typical questions of interest were the derivation of macro-models from micro-models, the mathematical analysis (well-posedness, stability, asymptotic behavior of solutions), and "to a lesser extent" numerical aspects of such equations. A highlight of this meeting was a mini-course on the recent mathematical theory of Landau damping.

Mathematics Subject Classification (2000): 35Qxx, 82Cxx, 82B40, 81S30.

Introduction by the Organisers

The Oberwolfach meeting described here aimed at presenting the latest mathematical results in the field of kinetic theory (both related to classical mechanics and quantum mechanics). There were 50 participants, among which 14 young participants (PhD students, post-docs or young assistant professors). Three of them (G. Raoul, E. Dolera, G. Aki) were partially sponsored by the program "Oberwolfach Leibnitz Graduate Student", and two others (C. Sparber and R. Strain) were invited within the program "US Junior Oberwolfach Fellows": they are promising young researchers working in the US.

About half of the participants gave a presentation, where the length of talks were either 30 or 45 minutes. The longer talks included an expository introduction to the subject and the shorter talks were rather on specialized results. One of the

highlights of the meeting was the presentation by C. Mouhot (in three one-hour lectures) of the remarkable theorem obtained by C. Villani and himself about the Landau damping (one of two works which were considered for awarding to C. Villani the Fields medal): the main steps of the proof were presented at this occasion, together with many important technical aspects.

Two talks (S. Mischler on one side, and F. Salvarani on the other side) are directly related to hypocoercivity, the other concept which was celebrated when the Fields medal was awarded to C. Villani.

New results on long-standing problems were reported in the talks on the derivation of kinetic equations (two on the Lorentz gas model, by E. Caglioti and B. Wennberg, still another one by M. Pulvirenti on nonlinear kinetic equations), and in the talk by R. Strain on perturbative solutions of the singular Boltzmann equation. Dolera presented his work with Regazzini in which it is proved for Maxwellian molecules under only a fourth moment hypothesis and a very mild smoothness condition that solutions of the spatially homogenous Boltzmann equation converge to equilibrium exponentially fast at a rate given by the spectral gap, even for large initial data. This had been proved earlier for interactions harder than Maxwellian by Mouhot, but his analytic proof did not extend to the Maxwellian case. Dolera and Regazini use probabilistic methods.

Non-traditional applications of kinetic theory were visited, in particular in the talks by M. Bisi (application to chemistry), J. A. Carrillo (application to the collective motion of animals), M. Herty (application to supply chain models), D. Matthes (application to socio-economics), G. Raoul (application to biomechanics and chemotaxis models). E. Sonnendrücker presented a recent numerical approach (semi-Lagrangian) to the gyrokinetic model for plasmas. It shall be used to simulate the turbulent evolution in a fusion reactor, as in the ITER project. Finally, C. Schmeiser presented a proof of existence and uniqueness for traveling waves in a chemical reaction model. Those various fields show that kinetic theory (considered in a broad sense) remains, 150 years after its discovery, a powerful tool for exploring various aspects of the reality.

The quantum mechanical talks presented here covered a broad spectrum of topics: the semi-classical limit (in particular the relation of Bohmian and Wigner measures for the linear Schröedinger equation discussed by C. Sparber, and the asymptotic dynamics for nonlinear Schrödinger equations as presented by R. Carles), macroscopic quantum models (in particular global-in-time existence results for higher order, nonlinear quantum-diffusion equations, which exploit new entropy estimates obtained by A. Jüngel and D. Matthes). Relativistic and nonrelativistic gravitational Hartree models for boson stars were analyzed in the talks of A. Michelangeli and G. Aki. M. Escobedo und X. Lu discussed the quantum Boltzmann equation for bosons. The latter author found a dichotomy betweeen an oscillatory solution and (Bose-Einstein) condensation, i.e. a singular solution. C. Negulescu presented an efficient, WKB-based numerical scheme for the stationary Schrödinger equation in the highly oscillatory classical limit.

Another emerging development stems from the interplay of entropy methods for (classical) kinetic equations on the one hand and the dissipative behavior of open quantum systems on the other hand, as presented by F. Fagnola. This seems to pave the way for a "quantum entropy method" and a better understanding of quantum hypercontractivity.

A highlight involving entropy in particle systems was the talk of F. Otto, on joint work with G. Menz that solves a long-standing problem in hydrodynamic limits. The problem is to prove a logarithmic Sobolev inequality for a Gibbs measure for N non-interacting spins, coupled only by a constraint on the total spin, and to obtain a constant that is independent of N and the value of the total spin. This had been done earlier only for the case in which the single particle energy is exactly quadratic outside a compact set. Otto and Menz introduce an interesting asymmetric variant of the Brascamp-Lieb inequality, and use it to treat a much wider range of energy functions, including the standard quartic double well.

Y. Brenier described recent work on a novel variational approach to the dynamics of a system of self-gravitating particles with sticky collisions, and J. Dolbeault also discussed variational problems relating to gravitational systems, presenting a recent result obtained with Campos and Del Pino on the existence of distinct equilibrium solutions in a stellar dynamics model. Another interesting development in particle models was presented by Y. Guo who described his work with B. Pausander giving the construction of global smooth irrotational solutions for ion dynamics in a two–fluid plasma model. Finally, J.-A. Canizo presented his results (in collaboration with L. Desvillettes and K. Fellner) about non-gelation in diffusive coagulation breakup systems.

Workshop: Classical and Quantum Mechanical Models of Many-Particle Systems

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Abstracts

Thermal effects in gravitational Hartree systems

Gonca L. Aki
(joint work with Jean Dolbeault, Christof Sparber)

In this work we investigate the *non-relativistic gravitational Hartree system*. This model can be seen as a mean-field description of a system of self-gravitating quantum particles. It is used in astrophysics to describe so-called *Boson stars*. In the present work, we are particularly interested in *thermal effects*, i.e. (qualitative) differences to the zero temperature case.

A physical state of the system will be represented by a density matrix operator $\rho \in \mathfrak{S}_1(L^2(\mathbb{R}^3))$, i.e. a positive self-adjoint trace class operator acting on $L^2(\mathbb{R}^3;\mathbb{C})$. Such an operator ρ can be decomposed as

$$\rho = \sum_{j\in\mathbb{N}} \lambda_j \, |\psi_j\rangle\langle\psi_j| \tag{1}$$

with an associated sequence of eigenvalues $(\lambda_j)_{j\in\mathbb{N}} \in \ell^1$, $\lambda_j \geq 0$, usually called *occupation numbers*, and a corresponding sequence of eigenfunction $(\psi_j)_{j\in\mathbb{N}}$, forming a complete orthonormal basis of $L^2(\mathbb{R}^3)$, cf. [15]. By evaluating the kernel $\rho(x,y)$ on its diagonal, we obtain the corresponding particle density

$$n_\rho(x) = \sum_{j\in\mathbb{N}} \lambda_j \, |\psi_j(x)|^2 \in L^1_+(\mathbb{R}^3) \; .$$

In the following we shall assume that

$$\int_{\mathbb{R}^3} n_\rho(x) \, \mathrm{d}x = M \; , \tag{2}$$

for a given total mass $M > 0$. We assume that the particles interact solely via gravitational forces. The corresponding *Hartree energy* of the system is then given by

$$\mathcal{E}_H[\rho] := \mathcal{E}_{\mathrm{kin}}[\rho] - \mathcal{E}_{\mathrm{pot}}[\rho] = \mathrm{tr}(-\Delta\,\rho) - \frac{1}{2}\mathrm{tr}(V_\rho\,\rho) \; ,$$

where V_ρ denotes the *self-consistent potential*

$$V_\rho = n_\rho * \frac{1}{|\,\cdot\,|}$$

and '$*$' is the usual convolution w.r.t. $x \in \mathbb{R}^3$. Using the decomposition (1) for ρ, the Hartree energy can be rewritten as

$$\mathcal{E}_H[\rho] = \sum_{j\in\mathbb{N}} \lambda_j \int_{\mathbb{R}^3} |\nabla\psi_j(x)|^2 \, \mathrm{d}x - \frac{1}{2} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{n_\rho(x)\, n_\rho(y)}{|x-y|} \, \mathrm{d}x \, \mathrm{d}y \; .$$

To take into account thermal effects, we consider the associated *free energy functional*

$$\mathcal{F}_T[\rho] := \mathcal{E}_H[\rho] - T\,\mathcal{S}[\rho] \tag{3}$$

where $T \geq 0$ denotes the temperature and $\mathcal{S}[\rho]$ is the *entropy functional*

$$\mathcal{S}[\rho] := -\mathrm{tr}\beta(\rho) \ .$$

The *entropy generating function* β is assumed to be convex, of class C^1 and will satisfy some additional properties to be prescribed later on. The purpose of this paper is to investigate the existence of *minimizers* for $\mathcal{F}_T$ with fixed mass $M > 0$ and temperature $T \geq 0$ and study their qualitative properties. These minimizers, often called *ground states*, can be interpreted as stationary states for the time-dependent system

$$i\,\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = [H_{\rho(t)}, \rho(t)] \ , \quad \rho(0) = \rho_{\mathrm{in}} \ . \tag{4}$$

Here $[A, B] = A\,B - B\,A$ denotes the usual commutator and H_ρ is the mean-field *Hamiltonian operator*

$$H_\rho := -\Delta - n_\rho * \frac{1}{|\,\cdot\,|} \ . \tag{5}$$

Using again the decomposition (1), this can equivalently be rewritten as a system of (at most) countably many Schrödinger equations coupled through the mean field potential V_ρ:

$$\begin{cases} i\,\partial_t\psi_j + \Delta\,\psi_j + V(t,x)\,\psi_j = 0 \ , \quad j \in \mathbb{N} \ , \\ -\Delta V_\rho \ = \ 4\pi \sum_{j\in\mathbb{N}} \lambda_j\,|\psi_j(t,x)|^2 \ . \end{cases} \tag{6}$$

This system is a generalization of the gravitational Hartree equation (also known as the *Schrödinger-Newton model*, see [1]) to the case of mixed states. Notice that it reduces to a finite system as soon as only a finite number of λ_j are non-zero. In such a case, ρ is a finite rank operator.

Establishing the existence of stationary solutions to nonlinear Schrödinger models by means of variational methods is a classical idea, cf. for instance [6]. A particular advantage of such an approach is that in most cases one can directly deduce *orbital stability* of the stationary solution w.r.t. the dynamics of (4) or, equivalently, (6). In the case of *repulsive* self-consistent interactions, describing e.g. electrons, this has been successfully carried out in [2, 3, 4, 11]. In addition, existence of stationary solutions in the repulsive case has been obtained in [10, 12, 13, 14] using convexity properties of the corresponding energy functional.

In sharp contrast to the repulsive case, the gravitational Hartree system of stellar dynamics, does *not* admit a convex energy and thus a more detailed study of minimizing sequences is required. To this end, we first note that at zero temperature, i.e. $T = 0$, the free energy $\mathcal{F}_T[\rho]$ reduces to the gravitational Hartree energy $\mathcal{E}_H[\rho]$. For this model, existence of the corresponding zero temperature ground states has been studied in [5, 7, 9] and, more recently, in [1]. Most of these works rely on the so-called *concentration-compactness method* introduced by Lions in [8]. According to [5], it is known that for $T = 0$ the minimum of the Hartree energy is uniquely achieved by an appropriately normalized *pure state*, i.e. a rank one density matrix $\rho_0 = M\,|\psi_0\rangle\langle\psi_0|$.

The results of this work can be summarized as follows: First, we shall prove the existence of minimizers for $\mathcal{F}_T$, extending the results of [5, 7, 9, 1] to the case of non-zero temperature. As we shall see, a *threshold in temperature* arises due to the competition between the Hartree energy and the entropy term and we find that minimizers of $\mathcal{F}_T$ exist only *below a certain maximal temperature* $T^* > 0$, which depends on the specific form of the entropy generating function β. Moreover, depending on the choice of β, it could happen that $T^* = +\infty$, in which case minimizers of $\mathcal{F}_T$ would exist even if the temperature is taken arbitrarily large. In a second step, we shall also study the qualitative properties of the ground states with respect to the temperature $T \in [0, T^*)$. In particular, we will prove that there exists a certain *critical temperature* $T_c > 0$, above which minimizers correspond to *mixed quantum states*, i.e. density matrix operators with rank higher than one. If $T < T_c$, minimizers are pure states, as in the zero temperature model.

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BGK models and hydrodynamic limits for chemically reacting mixtures

Marzia Bisi

Chemically reacting mixtures have been extensively dealt with from a kinetic point of view in recent scientific literature. The most usual physical frame is given by a mixture of four gases A^s, $s = 1, \dots, 4$, whose particles, besides elastic collisions, are subject to the reversible bimolecular chemical reaction

$$A^1 + A^2 \rightleftharpoons A^3 + A^4. \tag{1}$$

The evolution of the four distribution functions f^s, $s = 1, \dots, 4$, may be described by nonlinear integro–differential Boltzmann–like equations, but relaxation-time-approximations of the cumbersome Boltzmann collision operators (the so called BGK models) may be adopted for practical purposes. In particular, the consistent BGK strategy proposed in [1] has been extended to the much more complicated chemical frame [7], taking into account exchange of mass and of energy of chemical link in the reactive encounters. More recently we have built up a different relaxation model which, unlike previous approximations of the reactive Boltzmann equations, is not affected by the constraint that the chemical characteristic time should be large compared to the mechanical one, and therefore it can be applied also to situations with fast chemical reactions [3]. Specifically, model kinetic equations read as

$$\frac{\partial f^s}{\partial t} + \mathbf{v} \cdot \frac{\partial f^s}{\partial \mathbf{x}} = \nu_s \big(\mathcal{M}_s - f^s\big) \qquad s = 1, \dots, 4, \tag{2}$$

where $\mathcal{M}_s$ are the family of local Maxwellians

$$\mathcal{M}_s(\mathbf{v}) = \tilde{n}^s \left(\frac{m^s}{2\pi K \tilde{T}}\right)^{\frac{3}{2}} \exp\left[-\frac{m^s}{2K\tilde{T}}(\mathbf{v} - \tilde{\mathbf{u}})^2\right] \qquad s = 1, \dots, 4 \tag{3}$$

with seven disposable scalar parameters, provided by $\tilde{n}^s, \tilde{\mathbf{u}}, \tilde{T}$, bound together by the mass action law

$$\frac{\tilde{n}^1 \tilde{n}^2}{\tilde{n}^3 \tilde{n}^4} = \left(\frac{m^1 m^2}{m^3 m^4}\right)^{3/2} \exp\left(\frac{\Delta E}{K\tilde{T}}\right). \tag{4}$$

Here m^s denote particle masses of species s, while $\Delta E = E^3 + E^4 - E^1 - E^2 > 0$, with E^s standing for the energy of chemical link of gas A^s. The auxiliary fields $\tilde{n}^s, \tilde{\mathbf{u}}, \tilde{T}$ in (3)–(4) are determined in terms of the actual species number densities n^s, mass velocities $\mathbf{u}^s$, and temperatures T^s by requiring that the present approximation and the actual Boltzmann description share the same collision invariants. In (2) the factor ν_s is the inverse of the s–th relaxation time, possibly depending on macroscopic fields, but independent of $\mathbf{v}$.

This model preserves the main features of the correct kinetic approach, including the mass action law of chemical equilibrium, and it allows an analytical proof

of the H–theorem in terms of the actual reactive Boltzmann H–functional

$$H[\underline{f}] = \sum_{s=1}^{4} \int f^s(\mathbf{v}) \log \left[\frac{f^s(\mathbf{v})}{(m^s)^3} \right] d_3\mathbf{v} \,. \tag{5}$$

Owing to a suitable Chapman–Enskog asymptotic procedure, hydrodynamic equations, at Euler or Navier–Stokes accuracy, have been consistently derived from both the "slow" and the "fast" reacting BGK models [5, 4]. The performance of such kinetic relaxation models has been numerically tested, to verify their adherence to physical expectations, both in space–homogeneous problems (relaxation to equilibrium) [2], and also in the space–dependent setting (profiles of macroscopic quantities) [6]. The application of our models to other meaningful physical problems would be a challenging future work. Another interesting research line, still unexploited to our knowledge, could be the derivation of a consistent BGK model for a reacting gas mixture diffusing in a host medium.

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A modified least action principle for systems of self-gravitating particles with sticky collisions

Yann Brenier

1. A geometric description of systems of self-gravitating particles

Let H be an euclidean space with norm denoted $||\cdot||$ and inner product by $((\cdot,\cdot))$. Given a closed bounded subset S of H, we consider the dynamical system

$$\frac{d^2X}{dt^2} = (\nabla_H \Phi)[X], \tag{1}$$

$$\Phi[X] = \inf\{\frac{||X-s||^2}{2} \,;\, s \in S\} = \frac{||X||^2}{2} - K[X],$$
$$K[X] = \sup\{((X,s)) - \frac{||s||^2}{2} \,;\, s \in S\}. \tag{2}$$

Our main application is the description of the motion of N self-gravitating particles moving on the real line with a neutralizing background (which makes sense in the framework of cosmology)

$$\frac{d^2X_i}{dt^2} = X_i - \frac{1}{N}\sum_j (1\{X_i > X_j\} - 1/2) \tag{3}$$

Indeed, this exactly corresponds to (1), with $H = \mathbf{R^N}$ and $S = \{(I_{\sigma_i}, \cdots, I_{\sigma_N}), \sigma \in \Sigma_N\}$, where Σ_N is the group of all permutations of the N first integers and $I_j = j/N - 1/2$, as can be easily checked. (This can be extended to higher dimensions in the framework of "Monge-Ampère gravitation".) Let us now return to the general framework. Notice that K is a Lipschitz convex function and is differentiable on a large set D (whose complement is both Lebesgue negligible and contained in a countable union of nowhere dense closed sets). On D, $\nabla_H \Phi[X]$ is well defined and takes value $X - \pi[X]$, where $\pi[X]$ is just the unique closest point to X in the set S. In particular Φ is solution to the Hamilton-Jacobi equation

$$\Phi[X] = \frac{||\nabla_H \Phi[X]||^2}{2}, \quad \forall X \in D. \tag{4}$$

The Cauchy problem for (1) admits unique global C^1 solutions for almost every initial condition $(X_0, \frac{dX_0}{dt})$, according to the theory of Bouchut (and Ambrosio) [1]. As expected, for large systems of particles, in the continuous limit $N \to +\infty$, the Bouchut-Ambrosio solutions will correspond to solutions $f(t, x, \xi)$ of the Vlasov model

$$\begin{aligned} \partial_t f + \partial_x(\xi f) - \partial_\xi(\partial_x \phi f) = 0, \\ \rho(t,x) = \int f(t,x,\xi) d\xi = 1 + \partial_{xx}\phi(t,x). \end{aligned} \tag{5}$$

2. A modified least action principle

Alternately, the dynamical equation (1) can be seen as the optimality equation obtained by minimizing the action, defined for all pairs of times $t_1 > t_0$ by

$$A_{[t_0,t_1]}[t \to X(t)] = \int_{t_0}^{t_1} \frac{1}{2}||\frac{dX}{dt}||^2 + \Phi[X(t)] \quad dt, \tag{6}$$

as the end points $(t_0, X(t_0))$ and $(t_1, X(t_1))$ are fixed. We observe, using (4) in a crucial way, that for all curves $t \to X(t)$ valued in D for a.e. t, the action can be as well written as

$$\int_{t_0}^{t_1} \frac{1}{2}||\frac{dX}{dt} - \nabla_H \Phi[X]||^2 \ dt + \Phi[X(t_1)] - \Phi[X(t_0)]. \tag{7}$$

This immediately implies that all solutions of the $first$ order equation

$$\frac{dX}{dt} = (\nabla_H \Phi)[X] = x - (\nabla_H K)[X], \tag{8}$$

automatically are action minimizers, provided they take values in D for a.e. t. (This phenomenon occurs in a similar way in Yang-Mills theory, and in Ghoussoub's theory of self-dual lagrangians [6].) However, according to maximal monotone theory [5], the first order equation (8), for every initial condition X_0, admits a unique global Lipschitz generalized solution, which is everywhere right-differentiable with

$$\frac{dX(t+0)}{dt} = X(t) - d^0K[X(t)], \quad \forall t \tag{9}$$

where $d^0K[X]$ denotes the element in the subdifferential $\partial K[X]$ with minimal $H-$norm. Usually, such a solution takes values outside of D for a non-negligible amount of time and is not C^1. In particular, it *cannot* be solution of (1) in the Bouchut-Ambrosio sense. We get a different type of solutions. For systems of particles on the real line, this exactly corresponds to sticky collisions (which do not occur for Bouchut-Ambrosio solutions, in which case particles just cross each other without dissipation of kinetic energy). This suggests the introduction of the modified action

$$\int_{t_0}^{t_1} \frac{1}{2}||\frac{dX}{dt} - d^0\Phi[X]||^2 \; dt + \Phi[X(t_1)] - \Phi[X(t_0)]. \tag{10}$$

where $d^0\Phi[X]$ stands for $X - d^0K[X]$, to get solutions of (1) including sticky collisions. A possible application of this idea is the resolution of the Early Universe reconstruction problem, introduced by Peebles and studied by Frisch and co [4, 2], which precisely amounts to minimizing a similar action (integrated from the big bang up to now!). A numerical scheme is proposed in [2] for this purpose.

3. The Cauchy problem for sticky collisions

In the one dimensional setting, a very simple approach can be used to include sticky collisions, following the ideas of [3] (and recently revisited by and collaborators, as in [7]). Instead of the positions X_i, we look at the uniquely defined sequence in increasing order Y_i so that $X_i = Y_{\sigma_i}$, for some permutation σ. We denote by K the closed convex cone of all sequences of real numbers in increasing order and by 1_K the corresponding indicator function, with value 0 in K and $+\infty$ outside. Then, the 1d gravitational model (3), with sticky collisions, reduces to a simple sub-differential system in $H = \mathbf{R}^N$:

$$\begin{aligned} -\frac{dY}{dt} + V \in \partial 1_K[Y], \\ \frac{dV}{dt} = Y - I. \end{aligned} \tag{11}$$

It turns out that the macroscopical limit is well described by the Navier-Stokes-Poisson system, in the limit of both vanishing pressure and viscosity

$$\begin{aligned} \partial_t\rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho v) + \partial_x(\rho v^2 + \epsilon\rho) = \epsilon\partial_x(\rho\partial_x v) - \rho\partial_x\phi, \\ \rho = 1 + \partial_{xx}\phi, \end{aligned} \tag{12}$$

as $\epsilon \to 0$. In spite of its simplicity, formulation (11) seems limited to the 1d setting and we think that the modified least action principle is more suitable for multidimensional extensions.

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The Boltzmann Grad Limit for the 2D Periodic Lorentz Gas

Emanuele Caglioti

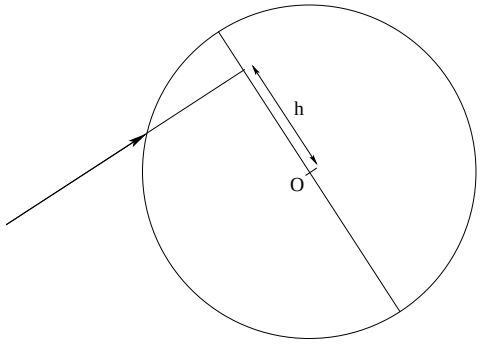
(joint work with Francois Golse)

The Lorentz gas is the dynamical system corresponding to the free motion of a single point particle in a periodic system of fixed spherical obstacles, assuming that collisions between the particle and any of the obstacles are elastic.

In the case of random (Poisson) obstacles G. Gallavotti (see [1]) proved that in the Boltzmann-Gad limit the particle distribution $f(t, x, v)$ evolves according to the linear Boltzmann Equation.

We consider here the 2D periodic Lorentz gas: the obstacles are disks of radius r centered in the points of $\mathbf{Z}^2$. Moreover the particles are assumed to have velocity 1.

In the Boltzmann Grad limit: that is $r \to 0$ and position and time scaled by $\frac{1}{r}$, we show that the particles distribution can be described by a kinetic equation in an **extended phase space** x, v, h, τ, where $h \in [-1, 1]$ is the (suitably scaled) impact parameter of the next collision and $\tau \in \mathbf{R}^+$ is the time to the next collision,



Denoting $v = (\cos\theta, \sin\theta)$, the governing equation for f is

$$(\partial_t + v \cdot \nabla_x) f(t, x, \theta, h, \tau) = \partial_\tau f(t, x, \theta, h, \tau) + \int_{-1}^{1} p(h, \tau | h') f(t, x, \theta'[h], h', 0) dh'$$

where $p(h, \tau|h')$ is the probability of a reflection with impact parameter h' creating a particle with next impact parameter h and life expectancy $\tau > 0$, and where $\theta'[h]$ is the outgoing angle which is a function of h and θ.

The scattering kernel $p(h, \tau|h')$ can be computed by means of Farey fractions.

This equation has been derived in [2]] as a conjecture; see also [4] for similar results.

The same equation has been also derived, by means of completely different techniques in [5], and the existence of the limit in any space dimension has been proved in [6]).

For the solutions of the above equation it is possibile to prove an H theorem and the convergence to the unique equilibrium (see [3]).

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Regularity and mass conservation for discrete coagulation-fragmentation equations with diffusion

José A. Cañizo
(joint work with Laurent Desvillettes, Klemens Fellner)

We present a new *a priori* estimates for discrete coagulation-fragmentation systems with size-dependent diffusion within an open, bounded, regular domain Ω confined by homogeneous Neumann boundary conditions [5, 6]. The initial-boundary problem for the concentrations $c_i = c_i(t,x) \geq 0$ of clusters with integer size $i \geq 1$ at position $x \in \Omega$ and time $t \geq 0$ is given by the following system:

$$\partial_t c_i - d_i \Delta_x c_i = Q_i + F_i \qquad \text{for } x \in \Omega, t \geq 0, i \in \mathbb{N}^*, \tag{1a}$$

$$\nabla_x c_i \cdot n = 0 \qquad \text{for } x \in \partial\Omega, t \geq 0, i \in \mathbb{N}^*, \tag{1b}$$

$$c_i(0,x) = c_i^0(x) \qquad \text{for } x \in \Omega, i \in \mathbb{N}^*, \tag{1c}$$

where $n = n(x)$ represents a unit normal vector at a point $x \in \partial\Omega$, d_i is the diffusion constant for clusters of size i, and

$$\begin{aligned} Q_i \equiv Q_i[c] &:= Q_i^+ - Q_i^- := \frac{1}{2}\sum_{j=1}^{i-1} a_{i-j,j}\, c_{i-j}\, c_j - \sum_{j=1}^{\infty} a_{i,j}\, c_i\, c_j, \\ F_i \equiv F_i[c] &:= F_i^+ - F_i^- := \sum_{j=1}^{\infty} B_{i+j}\, \beta_{i+j,i}\, c_{i+j} - B_i\, c_i. \end{aligned} \tag{2}$$

The parameters B_i, $\beta_{i,j}$ and $a_{i,j}$, for integers $i, j \geq 0$, represent the total rate B_i of fragmentation of clusters of size i, the average number $\beta_{i,j}$ of clusters of size j produced due to fragmentation of a cluster of size i, and the coagulation rate $a_{i,j}$ of clusters of size i with clusters of size j. We refer to these parameters as *the coefficients* of the system of equations. We assume the following conditions on them:

$$a_{i,j} = a_{j,i} \geq 0, \qquad \beta_{i,j} \geq 0, \qquad (i, j \in \mathbb{N}^*), \tag{3a}$$

$$B_1 = 0, \qquad B_i \geq 0, \qquad (i \in \mathbb{N}^*), \tag{3b}$$

$$i = \sum_{j=1}^{i-1} j\, \beta_{i,j}, \qquad (i \in \mathbb{N}, i \geq 2). \tag{3c}$$

We always work with the global weak solutions constructed in [17] under the assumption

$$\lim_{j\to+\infty} \frac{a_{i,j}}{j} = \lim_{j\to+\infty} \frac{B_{i+j}\, \beta_{i+j,i}}{i+j} = 0, \qquad \text{(for fixed } i \geq 1). \tag{4}$$

Under the extra assumptions on the diffusion constants and the initial data

$$0 < \inf_i\{d_i\} =: d, \qquad D := \sup_i\{d_i\} < +\infty, \tag{5}$$

$$\sum_{i=1}^{\infty} i c_i^0 \in L^2(\Omega), \tag{6}$$

we are in fact able to prove the following L^2 bound on the mass density $\rho(t,x) := \sum_{i=1}^{\infty} i\, c_i(t,x)$: Denoting by Ω_T the cylinder $[0,T]\times\Omega$, we have the

Proposition 1. *Assume that* (3), (4), (5) *and* (6) *hold. Then, for all* $T > 0$ *the mass* ρ *of a weak solution to system* (1) – (2) *lies in* $L^2(\Omega_T)$ *and the following estimate holds:*

$$\|\rho\|_{L^2(\Omega_T)} \le \left(1 + \frac{\sup_i\{d_i\}}{\inf_i\{d_i\}}\right) T\, \|\rho(0,\cdot)\|_{L^2(\Omega)}. \tag{7}$$

This kind of estimates have been previously used, for instance, in the context of reaction-diffusion equations [11, 18, 19].

We give two lines of applications for such an estimate: on the one hand, it enables to simplify parts of the known existence theory and allows to show existence of solutions for generalised models involving collision-induced, quadratic fragmentation terms for which the previous existence theory seems difficult to apply. On the other hand and most prominently, it proves mass conservation (and thus the absence of gelation) for almost all the coagulation coefficients for which mass conservation is known to hold true in the space homogeneous case:

Theorem 1. *Assume that* (3), (4) *and* (6) *hold. Assume also that the following extra relationship between the coefficients of coagulation and diffusion holds:*

$$d_i \ge Cst\, i^{-\gamma}, \qquad a_{i,j} \le Cst\Big(i^{\alpha}\, j^{\beta} + i^{\beta}\, j^{\alpha}\Big), \tag{8}$$

with $\alpha+\beta+\gamma \le 1$, $\alpha,\beta \in [0,1)$, $\gamma \in [0,1]$. *Then, all weak solutions to the system* (1) *conserve mass:*

$$\int_\Omega \rho_0(x)\, dx = \int_\Omega \rho(t,x)\, dx \quad \text{ for all } t \ge 0. \tag{9}$$

If we assume also that (5) holds (the case $\gamma = 0$ in the theorem), then the proof of this can be obtained as a consequence of Proposition 1. In fact, in [6] we give slightly more relaxed conditions on $a_{i,j}$ for which the theorem still holds. When $\gamma > 0$ (and hence d_i may decay as $i \to +\infty$), the proof involves also a weaker estimate instead of (7) (see [5]).

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Concentration in space and velocity: Swarming models

José Antonio Carrillo

The talk presented in this Oberwolfach meeting was devoted to discuss several concentration phenomena in nonlocal PDEs of macroscopic and kinetic type arising in some applications such as swarming modelling. More precisely, we discussed the methods to deal with measure solutions to the nonlocal aggregation equation with

non-smooth interaction potential as the typical Morse potential. A variational scheme based on optimal transport tools allows to show global well-posedness of the problem for measure initial data. Moreover, the stability with respect to the initial data allows to show a total collapse result for this problem, i.e., all the mass of the solution arrives to the center of mass of the solution in finite time for all initial data. This part corresponds to the work in [2].

In the second part of this talk, a similar strategy was used to deal with some kinetic models of swarming incorporating effects such as local repulsion, long-range attraction, and alignment. These models can be treated by using optimal transport tools to pass from particle models to kinetic models. For the example of the Cucker-Smale model a concentration in velocity result was shown by taking advantage again of the well-posedness for measures initial data together with particle approximations. Some simulation were shown to point out the complexity of the patterns obtained from these seemingly simple models. These results are a summary of the works [1, 3, 5, 4].

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Nonlinear coherent states and Ehrenfest time for Schrödinger equation

Rémi Carles

(joint work with Clotilde Fermanian-Kammerer)

We consider the semi-classical limit $\varepsilon \to 0$ for the nonlinear Schrödinger equation

$$(1)\quad i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon + \lambda|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon, \quad (t,x)\in \mathbf{R}_+\times\mathbf{R}^d \quad ; \quad \psi^\varepsilon_{|t=0} = \psi^\varepsilon_0,$$

with $\lambda\in\mathbf{R}$, $d\geq 1$. The external potential V is smooth, real-valued, and at most quadratic:

$$V\in C^\infty(\mathbf{R}^d;\mathbf{R}) \quad \text{and} \quad \partial_x^\gamma V\in L^\infty\left(\mathbf{R}^d\right), \quad \forall|\gamma|\geq 2.$$

We assume that the initial data ψ_0^ε is a localized wave packet of the form

$$(2)\qquad \psi_0^\varepsilon(x) = \varepsilon^\beta\times\varepsilon^{-d/4}a\left(\frac{x-x_0}{\sqrt{\varepsilon}}\right)e^{i(x-x_0)\cdot\xi_0/\varepsilon}, \quad a\in\mathcal{S}(\mathbf{R}^d), \quad x_0,\xi_0\in\mathbf{R}^d.$$

Such data, which are called *semi-classical wave packets* (or *coherent states*), have been extensively studied in the linear case (see e.g. [2, 4, 5, 10, 11]). In particular, Gaussian wave packets are used in numerical simulation of quantum chemistry like Initial Value Representations methods (see [12, 13, 14] and references therein). These methods rely on the fact that if the data is a wave packet, then the solution of the linear equation ($\lambda = 0$) associated with (1) still is a wave packet at leading order up to times of order $C \log\left(\frac{1}{\varepsilon}\right)$: such a large (as $\varepsilon \to 0$) time is called *Ehrenfest time*, see e.g. [1, 7, 8]. Our aim here is to investigate what remains of these facts in the nonlinear case ($\lambda \neq 0$), since typically (1) appears as a model for Bose–Einstein Condensation, where, for instance, V may be exactly a harmonic potential, or a truncated harmonic potential (hence not exactly quadratic); see e.g. [6, 9].

In the present nonlinear setting, a new parameter has to be considered: the size of the initial data, hence the factor ε^β in (2). There exists a notion of criticality for β: for $\beta > \beta_c := 1/(2\sigma) + d/4$, the initial data are too small to ignite the nonlinearity at leading order, and the leading order behavior of ψ^ε as $\varepsilon \to 0$ is the same as in the linear case $\lambda = 0$, up to Ehrenfest time. On the other hand, if $\beta = \beta_c$, the function ψ^ε is given at leading order by a wave packet whose envelope satisfies a *nonlinear* equation, up to a nonlinear analogue of the Ehrenfest time. We show moreover a nonlinear superposition principle: when the initial data is the sum of two wave packets of the form (2), then ψ^ε is approximated at leading order by the sum of the approximations obtained in the case of a single initial coherent state.

Up to changing ψ^ε to $\varepsilon^{-\beta}\psi^\varepsilon$, we may assume that the initial data are of order $\mathcal{O}(1)$ in $L^2(\mathbf{R}^d)$, and we consider

$$\text{(3)} \qquad \begin{cases} i\varepsilon\partial_t\psi^\varepsilon + \dfrac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon + \lambda\varepsilon^\alpha|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon, \quad (t,x) \in \mathbf{R}_+ \times \mathbf{R}^d, \\ \psi^\varepsilon(0,x) = \varepsilon^{-d/4} a\left(\dfrac{x - x_0}{\sqrt{\varepsilon}}\right) e^{i(x-x_0)\cdot\xi_0/\varepsilon}, \end{cases}$$

where $\alpha = 2\beta\sigma$.

Consider the classical trajectories associated with the Hamiltonian $\frac{|\xi|^2}{2} + V(x)$:

$$\text{(4)} \qquad \dot{x}(t) = \xi(t), \;\; \dot{\xi}(t) = -\nabla V(x(t)); \quad x(0) = x_0, \; \xi(0) = \xi_0.$$

We associate with these trajectories the *classical action*

$$\text{(5)} \qquad S(t) = \int_0^t \left(\frac{1}{2}|\xi(s)|^2 - V(x(s))\right) ds.$$

We observe that if we change the unknown function ψ^ε to u^ε by

$$\psi^\varepsilon(t,x) = \varepsilon^{-d/4} u^\varepsilon\left(t, \frac{x - x(t)}{\sqrt{\varepsilon}}\right) e^{i(S(t) + \xi(t)\cdot(x - x(t)))/\varepsilon},$$

then, in terms of $u^\varepsilon = u^\varepsilon(t,y)$, (3) is equivalent

$$i\partial_t u^\varepsilon + \frac{1}{2}\Delta u^\varepsilon = V^\varepsilon(t,y)u^\varepsilon + \lambda\varepsilon^{\alpha-\alpha_c}|u^\varepsilon|^{2\sigma}u^\varepsilon \quad ; \quad u^\varepsilon(0,y) = a(y),$$

where the external time-dependent potential V^ε is given by

$$V^\varepsilon(t,y) = \frac{1}{\varepsilon}\left(V(x(t)+\sqrt{\varepsilon}y) - V(x(t)) - \sqrt{\varepsilon}\,\langle \nabla V(x(t)), y\rangle\right), \tag{6}$$

and $\alpha_c = 1 + \dfrac{d\sigma}{2}$. The real number α_c appears as a critical exponent. The expression (6) reveals the first terms of the Taylor expansion of V about the point $x(t)$. Passing formally to the limit, V^ε converges to the Hessian of V at $x(t)$ evaluated at (y,y). One does not even need to pass to the limit if V is a polynomial of degree at most two: in that case, we see that the solution ψ^ε remains exactly a coherent state for all time. Let us denote by $Q(t)$ the symmetric matrix

$$Q(t) = \operatorname{Hess} V(x(t)).$$

If $\lambda = 0$ or $\alpha > \alpha_c$, then ψ^ε is approximated by $\varphi^\varepsilon_{\rm lin}$, up to time of order $C\log\frac{1}{\varepsilon}$, where

$$\varphi^\varepsilon_{\rm lin}(t,x) = \varepsilon^{-d/4} v\left(t, \frac{x-x(t)}{\sqrt{\varepsilon}}\right) e^{i(S(t)+\xi(t)\cdot(x-x(t)))/\varepsilon},$$

and v is given by

$$i\partial_t v + \frac{1}{2}\Delta v = \frac{1}{2}\langle Q(t)y, y\rangle\, v \quad ; \quad v(0,y) = a(y).$$

In the critical nonlinear case $\lambda \neq 0$ and $\alpha = \alpha_c$, we have typically the following result. Consider the solution to

$$i\partial_t u + \frac{1}{2}\Delta u = \frac{1}{2}\langle Q(t)y, y\rangle u + \lambda|u|^{2\sigma}u \quad ; \quad u(0,y) = a(y), \tag{7}$$

and let

$$\varphi^\varepsilon(t,x) = \varepsilon^{-d/4} u\left(t, \frac{x-x(t)}{\sqrt{\varepsilon}}\right) e^{i(S(t)+\xi(t)\cdot(x-x(t)))/\varepsilon}. \tag{8}$$

Theorem 2. *Assume $d = \sigma = 1$, and let $a \in \mathcal{S}(\mathbf{R})$. There exist $C, C_0 > 0$ independent of ε, and $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0]$,*

$$\|\psi^\varepsilon(t) - \varphi^\varepsilon(t)\|_{L^2(\mathbf{R})} \lesssim \sqrt{\varepsilon}\exp(C_0 t), \quad 0 \le t \le C\log\frac{1}{\varepsilon}.$$

Consider now initial data corresponding to the superposition of two wave packets:

$$\psi^\varepsilon(0,x) = \varepsilon^{-d/4} a_1\left(\frac{x-x_1}{\sqrt{\varepsilon}}\right) e^{i(x-x_1)\cdot\xi_1/\varepsilon} + \varepsilon^{-d/4} a_2\left(\frac{x-x_2}{\sqrt{\varepsilon}}\right) e^{i(x-x_2)\cdot\xi_2/\varepsilon},$$

with $a_1, a_2 \in \mathcal{S}(\mathbf{R})$, $(x_1,\xi_1), (x_2,\xi_2) \in \mathbf{R}^2$, and $(x_1,\xi_1) \neq (x_2,\xi_2)$. For $j \in \{1,2\}$, $(x_j(t), \xi_j(t))$ are the classical trajectories solutions to (4) with initial data (x_j, ξ_j). We denote by S_j the action associated with $(x_j(t), \xi_j(t))$ by (5) and by u_j the solution of (7) for the curve $x_j(t)$ and with initial data a_j. We consider φ^ε_j associated by (8) with u_j, x_j, ξ_j, S_j, and ψ^ε solution to (3) with $\alpha = \alpha_c$ and the above data.

Theorem 3. *Assume $d = \sigma = 1$, and let $a_1, a_2 \in \mathcal{S}(\mathbf{R})$. Suppose $E_1 \neq E_2$, where*

$$E_j = \frac{\xi_j^2}{2} + V(x_j).$$

There exist $C, C_1 > 0$ independent of ε, and $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0]$,

$$\|\psi^\varepsilon(t) - \varphi_1(t)^\varepsilon - \varphi_2^\varepsilon(t)\|_{L^2(\mathbf{R})} \lesssim \varepsilon^\gamma e^{C_1 t}, \quad 0 \leq t \leq C \log \frac{1}{\varepsilon}, \quad \text{with } \gamma = \frac{k-2}{2k-2}.$$

Even though the profiles are nonlinear, the superposition principle, which is a property of linear equations, still holds. The assumption $E_1 \neq E_2$ is probably only technical, but we cannot conclude without it, unless we consider time intervals which do not depend upon ε. Detailed proofs can be found in [3].

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Relative equilibria in continuous stellar dynamics

Jean Dolbeault
(joint work with Juan Campos and Manuel del Pino)

Consider the gravitational Vlasov-Poisson system

$$(1)\qquad \begin{cases} \partial_t f + v\cdot\nabla_x f - \nabla_x\phi\cdot\nabla_v f = 0\,,\\ \phi = -\frac{1}{4\pi\,|\cdot|} * \rho\,, \quad \rho = \displaystyle\int_{\mathbb{R}^3} f\,dv\,. \end{cases}$$

We look for time-periodic solutions which are in rotation at constant angular velocity ω. Replacing $x=(x',x^3)$ and $v=(v',v^3)$ respectively by $(e^{i\,\omega\,t}\,x',x^3)$ and $(i\,\omega\,x'+e^{i\,\omega\,t}\,v',v^3)$ and using complex notations so that x', $v'\in\mathbb{R}^2\approx\mathbb{C}$, Problem (1) becomes

$$(2)\qquad \begin{cases} \partial_t f + v\cdot\nabla_x f - \nabla_x\phi\cdot\nabla_v f - \omega^2\,x'\cdot\nabla_{v'} f + 2\,\omega\,i\,v'\cdot\nabla_{v'} f = 0\,,\\ \phi = -\frac{1}{4\pi\,|\cdot|} * \rho\,, \quad \rho = \displaystyle\int_{\mathbb{R}^3} f\,dv\,, \end{cases}$$

where we have abusively used the same notations for the potential ϕ and the distribution function f, for sake of simplicity. A *relative equilibrium* of (1) is a stationary solution of (2) and can be obtained by considering critical points of the *free energy* functional

$$\mathcal{F}[f]=\iint_{\mathbb{R}^3\times\mathbb{R}^3}\beta(f)\,dx\,dv+\frac12\iint_{\mathbb{R}^3\times\mathbb{R}^3}\big(|v|^2-\omega^2\,|x'|^2\big)f\,dx\,dv-\frac12\int_{\mathbb{R}^3}|\nabla\phi|^2\,dx$$

for some arbitrary convex function β, under a mass constraint $\iint_{\mathbb{R}^3\times\mathbb{R}^3} f\,dx\,dv=M$. Notice that as soon as $\omega\neq0$, $\mathcal{F}$ is not bounded from below anymore. A typical example of a function β, corresponding to the *polytropic gas model*, is $\beta(f)=\frac1q\,\kappa_q^{q-1}\,f^q$ for some $q\in(9/7,\infty)$ and some positive constant κ_q. Any relative equilibrium takes the form $f(x,v)=\gamma\big(\lambda+\frac12\,|v|^2+\phi(x)-\frac12\,\omega^2\,|x'|^2\big)$ where $\gamma(s)=\kappa_q^{-1}\,(-s)_+^{1/(q-1)}$ and λ is constant on each component of the support of f. The problem is now reduced to solve a nonlinear Poisson equation, namely

$$\Delta\phi=g\big(\lambda+\phi(x)-\tfrac12\,\omega^2\,|x'|^2\big)\quad\text{if } x\in\mathrm{supp}(\rho)$$

and $\Delta\phi=0$ otherwise, with $g(\mu)=(-\mu)_+^p$ and $p=\frac{1}{q-1}+\frac32$, if κ_q is appropriately chosen. Assuming that the solution has N disjoint connected components K_i, denoting by λ_i the value of λ on K_i and by χ_i the characteristic function of K_i, we end up looking for a positive solution $u=-\phi$ of

$$-\Delta u=\sum_{i=1}^N\rho_i^\omega\quad\text{in }\mathbb{R}^3\,,\quad \rho_i^\omega=\big(u-\lambda_i+\tfrac12\,\omega^2\,|x'|^2\big)_+^p\,\chi_i$$

under the asymptotic boundary condition $\lim_{|x|\to\infty}u(x)=0$. We define the mass and the center of mass associated to each component by $m_i=\int_{\mathbb{R}^3}\rho_i^\omega\,dx$ and $\xi_i^\omega=\frac{1}{m_i}\int_{\mathbb{R}^3}x\,\rho_i^\omega\,dx$ respectively. The main result in [1] goes as follows.

Theorem 4. *Let $N \geq 2$ and $p \in (3/2, 3) \cup (3, 5)$. For almost any masses m_i, $i = 1, \dots N$, and for any sufficiently small $\omega > 0$, there exist at least $[2^{N-1}(N-2)+1]\,(N-2)\,!$ distinct stationary solutions f_ω of (2) which are such that*

$$\int_{\mathbb{R}^3} f_\omega \, dv \, = \sum_{i=1}^N \rho_i^\omega \, + \, o(1)$$

where $o(1)$ means that the remainder term uniformly converges to 0 as $\omega \to 0_+$ and identically vanishes away from $\cup_{i=1}^N B_R(\xi_i^\omega)$, for some $R > 0$, independent of ω.

With the above notations, for all $i = 1, \dots N$, we have that

$$\rho_i^\omega(x - \xi_i^\omega) = \lambda_i^p \, \rho_*\big(\lambda_i^{(p-1)/2} \, x\big) \, + \, o(1)$$

where ρ_* is non-negative, radially symmetric, non-increasing, compactly supported function, depending only on p, and λ_i is such that $m_i = \lambda_i^{(3-p)/2} \int_{\mathbb{R}^3} \rho_* \, dx + o(1)$.

The points ξ_i^ω are such that $\xi_i^\omega = \omega^{-2/3}\,(\zeta_i^\omega, 0)$ where, for any $i = 1, \dots N$, $\zeta_i^\omega \in \mathbb{R}^2$ converges as $\omega \to 0$ to a critical point of

$$\mathcal{V}(\zeta_1, \dots \zeta_N) \, = \, \frac{1}{8\pi} \sum_{i \neq j = 1}^N \frac{m_i \, m_j}{|\zeta_i - \zeta_j|} + \frac{1}{2} \sum_{i=1}^N m_i \, |\zeta_i|^2 \, .$$

This theorem relies on a classification of relative equilibria for the N-body problems which has been established mostly by J.I. Palmore. See [1, Theorem 4] for a summary of these results. Here *distinct* solutions means that one solution cannot be deduced from another one by a simple scaling or by a rotation. The strategy is to find critical points of

$$\mathsf{J}\,[u] = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx - \frac{1}{p+1} \sum_{i=1}^N \int_{\mathbb{R}^3} \big(u - \lambda_i + \tfrac{1}{2}\,\omega^2 \, |x'|^2\big)_+^{p+1} \, \chi_i \, dx \, ,$$

by using the solution of

$$-\Delta w_* = (w_* - 1)_+^p =: \rho_* \quad \text{in } \mathbb{R}^3$$

as "building brick" on each of the connected components. With $W_\xi := \sum_{i=1}^N w_i$, $w_i(x) = \lambda_i \, w_*\big(\lambda_i^{(p-1)/2}\,(x - \xi_i)\big)$ and $\xi = (\xi_1, \dots \xi_N)$, we want to solve the problem

$$\Delta\phi + \sum_{i=1}^N p\big(W_\xi - \lambda_i + \tfrac{1}{2}\,\omega^2 \, |x'|^2\big)_+^{p-1} \chi_i \, \phi = -\mathsf{E} - \mathsf{N}[\phi]$$

with $\lim_{|x| \to \infty} \phi(x) = 0$, where $\mathsf{E} = \Delta W_\xi + \sum_{i=1}^N \big(W_\xi - \lambda_i + \frac{1}{2}\,\omega^2 \, |x'|^2\big)_+^p \, \chi_i$ and $\mathsf{N}[\phi]$ is a nonlinear correction. A lengthy computation shows that

$$\mathsf{J}\,[W_\xi] \, = \sum_{i=1}^N \lambda_i^{(5-p)/2} \, \mathsf{e}_* \, - \, \omega^{2/3} \, \mathcal{V}(\zeta_1, \dots \zeta_N) \, + \, O(\omega^{4/3})$$

where $\mathsf{e}_* = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla w|^2 \, - \frac{1}{p+1} \int_{\mathbb{R}^3} (w-1)_+^{p+1} \, dx$ and $\zeta_i = \omega^{2/3} \, \xi_i'$ if the points ξ_i are such that, for a large, fixed $\mu > 0$, and all small $\omega > 0$, we have $|\xi_i| < \mu \, \omega^{-2/3}$

and $|\xi_i - \xi_j| > \mu^{-1}\,\omega^{-2/3}$. To localize each K_i in a neighborhood of ξ_i, we impose the orthogonality conditions

$$\int_{\mathbb{R}^3} \phi\,\partial_{x_j} w_i\,\chi_i\,dx = 0 \quad \forall\, i = 1,\,2\ldots N,\ j = 1,\,2,\,3\,, \tag{3}$$

to the price of Lagrange multipliers. Fixed point methods allow to find a constrained solution ϕ. Since $\xi \mapsto \mathsf{J}\,[W_\xi]$ is a finite dimensional function, if $\xi_i = (\zeta_i, 0)$ is such that $(\zeta_1, \ldots \zeta_N)$ is in a neighborhood of a non-degenerate critical point of $\mathcal{V}$, we can find a critical point ϕ for which the Lagrange multipliers associated to (3) are all equal to zero. This completes the scheme of the proof, up to a last technicality. All above computations have been done in terms of fixed Lagrange multipliers (corresponding to the mass constraints associated to each K_i). These constraints still need to be inverted (in order to fix the masses), thus introducing an additional restriction, namely $p \neq 3$.

In this approach, relative equilibria have been obtained in an asymptotic regime in which each component of the distribution function behaves like a minimizer of the free energy when $\omega = 0$, slightly perturbed by the other components, and can be seen at large scale like *pseudo-particles*. These pseudo-particles are located close to the relative equilibria of the N-body problem which are obtained when the centrifugal force in the rotating frame equilibrates the force of gravitation. In the rotating frame, the centrifugal force gives rise to an harmonic potential in the variable x', with negative sign, which competes with the nonlinearity. The nonlinearity indeed tends to aggregate the mass into spherically symmetric functions.

Such *symmetry breaking* phenomena due to rotation effects have been investigated in [2] in the so-called *flat* case, which is slightly simpler (no x_3 variable) to the price of a nonlocal interaction. In such a case, a different branch of solutions has been considered, which originates from the radial solution corresponding to $\omega = 0$ and gets deformed as $|\omega|$ increases. These solutions can be defined as minimizers, provided their support is restricted to a well chosen ball. It is probably not very difficult to find similar solutions in the full three-dimensional setting, although they will certainly be harder to compute numerically. It would then be of interest to understand if such solutions can co-exists with the ones found in Theorem 4 and to extend them as ω increases as a branch of solutions depending on ω. If solutions co-exist, and after restricting the support of the solutions to a large but finite ball, comparing their energy would definitely provide a new insight into the physics of gravitating systems. This is also a very nice problem of symmetry breaking, for which almost nothing is known in case of a nonlocal nonlinearity such as the one corresponding to the Newtonian potential found by solving the attractive Poisson equation.

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New probabilistic methods for spatially homogeneous Maxwellian molecules

Emmanuele Dolera

The talk deals with the *spatially homogeneous Boltzmann equation for Maxwellian molecules*, namely

$$\begin{aligned} \frac{\partial}{\partial t} f(\mathbf{v},t) &= \int_{\mathbb{R}^3}\int_{S^2} [f(\mathbf{v}_*,t)f(\mathbf{w}_*,t) - f(\mathbf{v},t)f(\mathbf{w},t)] \times \\ &\times \; b\left(\frac{\mathbf{w}-\mathbf{v}}{|\mathbf{w}-\mathbf{v}|}\cdot\boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\mathbf{w} \end{aligned} \tag{1}$$

with $(\mathbf{v},t)$ in $\mathbb{R}^3\times(0,+\infty)$. The post-collisional velocities are given by $\mathbf{v}_* = \mathbf{v} + [(\mathbf{w}-\mathbf{v})\cdot\boldsymbol{\omega}]\,\boldsymbol{\omega}$ and $\mathbf{w}_* = \mathbf{w} - [(\mathbf{w}-\mathbf{v})\cdot\boldsymbol{\omega}]\,\boldsymbol{\omega}$. The *angular collision kernel* $b:(-1,1)\to[0,+\infty)$ meets the symmetry condition

$$b(x) = b(\sqrt{1-x^2})\frac{|x|}{\sqrt{1-x^2}} = b(-x) \tag{2}$$

and, at least in the first part of the talk, the *Grad cutoff assumption*, here written as

$$\int_0^1 b(x)\mathrm{d}x = 1\ . \tag{3}$$

The first new result concerns a new probabilistic representation for measure solutions $\mu(\cdot,t)$ of (1). In fact, recalling that $\mu(\cdot,t)$ is a probability measure (p.m., in short) for every $t\geq 0$, one can construct a measurable space $(\Omega,\mathcal{F})$ and a family of p.m.s $(\mathsf{P}_t)_{t\geq 0}$ on $(\Omega,\mathcal{F})$ in such a way that

$$\mu(B,t) = \mathsf{E}_t[\mathcal{M}(B)] \tag{4}$$

for every Borel set B of $\mathbb{R}^3$ and $t\geq 0$, where E_t denotes the expectation w.r.t. P_t and $\mathcal{M}:\Omega\to\mathcal{P}(\mathbb{R}^3)$ is a *random probability measure* depending only on the initial datum μ_0 and not on b. This construction, based on some seminal ideas by McKean [4], is carefully described in [2]. Then, one can establish a link between $\mathcal{M}$ and certain sums of independent random variables, in such a way that the techniques related to the *central limit theorem* can be exploited to obtain some new results. First, in the Grad cutoff setting, one can use the *Rosenthal inequalities* to improve some estimations, due to Elmroth, on the global boundedness of the moments of $\mu(\cdot,t)$ (assuming that they are initially bounded), by showing that the upper bounds can be chosen independently of b. But the most important conclusion drawn from (4) is the validation of a conjecture by McKean on the rapidity of convergence of $\mu(\cdot,t)$ – assuming the normalizations $\int_{\mathbb{R}^3}\mathbf{v}\mathrm{d}\mu_0 = \mathbf{0}$ and $\int_{\mathbb{R}^3}|\mathbf{v}|^2\mathrm{d}\mu_0 = 3$ – to the *standard Maxwellian equilibrium* $\gamma(\mathrm{d}\mathbf{v}) = \left(\frac{1}{2\pi}\right)^{3/2}\exp\{-\frac{1}{2}|\mathbf{v}|^2\}\mathrm{d}\mathbf{v}$ in the *total variation distance* d_{TV}. More precisely, *let b satisfy* (2)-(3) *and suppose that*

$$\begin{aligned} \mathfrak{m}_4 &:= \int_{\mathbb{R}^3}|\mathbf{x}|^4\mu_0(\mathrm{d}\mathbf{x}) < +\infty \\ |\hat{\mu}_0(\boldsymbol{\xi})| &= o(|\boldsymbol{\xi}|^{-p}) \qquad (|\boldsymbol{\xi}|\to+\infty) \end{aligned}$$

for some strictly positive p, $\hat{}$ *denoting the Fourier transform. Then,*

$$\mathrm{d_{TV}}(\mu(\cdot,t);\gamma) \leq C(\mu_0)e^{\Lambda_b t} \tag{5}$$

for every $t \geq 0$, *where* Λ_b *is the* spectral gap of the linearized Boltzmann collision operator (*given by* $\Lambda_b = -2\int_0^1 x^2(1-x^2)b(x)\mathrm{d}x$ *and* $C(\mu_0)$ *is a constant depending only on* μ_0. The proof of (5) is contained in [2]. Moreover, *the bound* (5) *is* optimal *in the sense that it cannot be improved* in general: *Indeed, for an entire class of initial data the lower bound*

$$\mathrm{d_{TV}}(\mu(\cdot,t);\gamma) \geq C_*(\mu_0;b)e^{\Lambda_b t}$$

also holds for every $t \geq 0$, *with some constant* $C_*(\mu_0;b)$. This statement is proved in [3].

Inequality (5) was firstly proved Grad *only for initial data very close to the equilibrium*, and the problem of its extension to the general case remained open. The best attempt in this direction was achieved by Carlen, Gabetta and Toscani [1], who proved, always in the Grad cutoff setting, that

$$\mathrm{d_{TV}}(\mu(\cdot,t);\gamma) \leq C_\varepsilon(\mu_0,b)e^{(\Lambda_b+\varepsilon)t} \tag{6}$$

holds true, provided that μ_0 has a density f_0 such that: i) all its moments are finite; ii) f_0 belongs to all the Sobolev spaces $H^k(\mathbb{R}^3)$; the Linnik functional $I[f_0]$ is finite. Lastly, one can note that $\lim_{\varepsilon\downarrow 0} C_\varepsilon(\mu_0,b) = +\infty$. Hence, the probabilistic approach shows that ε in (6) can be removed, together with many of the regularity hypotheses on the initial datum. Finally, inequality (5) proves to be *stable when passing from the Grad cutoff setting to the weak cutoff*, according to which b satisfies $\int_0^1 x^2 b(x)\mathrm{d}x < +\infty$. In fact, the validity of (5) is preserved, unlike (6), when b is replaced by $\frac{b\wedge l}{\int_0^1 b\wedge l(x)\mathrm{d}x}$ and t by $t\cdot\int_0^1 b\wedge l(x)\mathrm{d}x$, with $l \in \mathbb{N}$ and hence in the limit $l \to \infty$, according to a strategy described in [5].

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Linearised quantum Boltzmann equations for bosons

Miguel Escobedo
(joint work with F. Pezzotti, M. A. Valle)

We consider the linearisation of the quantum Boltzmann equation for bosons around a thermodynamic equilibrium. The first stage of the evolution of a homogeneous gas of Bosons is described by the equation:

$$\begin{aligned}
\frac{\partial f}{\partial t}(t,p) &= Q(f)(t,p), \qquad t>0,\ p\in\mathbb{R}^3.\\
Q(f)(t,p) &= \int\int\int_{\mathbb{R}^9} W\left(p,p_2,p_3,p_4\right) q(f) dp_2 dp_3 dp_4\\
q\left(f\right) &= f_3 f_4(1+f)(1+f_2) - f\, f_2(1+f_3)(1+f_4)\\
W\left(p,p_2,p_3,p_4\right) &= \delta(p+p_2-p_3-p_4)\ \delta\left(\omega(p)+\omega(p_2)-\omega(p_3)-\omega(p_4)\right)
\end{aligned}$$

where $f_i \equiv f(p_i,t)$ is the density function of particles with momentum p_i at time t and $\omega(p)$ is the energy of the particle with momentum p.

The equlibria with zero momentum ($P=0$) are as follows. For any N anf E such that:

$$N \le \left(\frac{3\zeta(5/2)}{2\,\pi\zeta(3/2)}E\right)^{3/5}$$

there exists an equilibria:

$$\mathcal{F}_{N,E}(p) = \frac{1}{e^{\beta\,|p|^2-\mu}-1}, \quad \beta>0,\ \mu\le 0$$

such that:

$$\int_{\mathbb{R}^3}\mathcal{F}_{N,E}(p)dp = N, \quad \int_{\mathbb{R}^3}\mathcal{F}_{N,E}(p)\,p\,dp = 0, \quad \int_{\mathbb{R}^3}\mathcal{F}_{N,E}(p)\,|p|^2\,dp = E$$

If on the other hand N and E are such that

$$N > \left(\frac{3\zeta(5/2)}{2\,\pi\zeta(3/2)}E\right)^{3/5}$$

the distribution:

$$\mathcal{F}_{N,E}(p) = \frac{1}{e^{\beta\omega(p)}-1} + \rho\,\delta_0, \quad \beta>0,\ \rho>0$$

is an equilibria with zero momentum such that:

$$\int_{\mathbb{R}^3} d\,\mathcal{F}_{N,E}(p) = N, \quad \int_{\mathbb{R}^3}|p|^2\,d\,\mathcal{F}_{N,E}(p) = E$$

For non zero momentum: $F(p-p_0)$ for some $p_0\in\mathbb{R}^3$

1. The small p limit of the linearised equation above the critical temperature with zero chemical potential

In the limit $p \to 0$ the equilibrium with zero chemical potential is $\frac{1}{e^{\beta|p|^2}-1} \sim \frac{1}{\beta|p|^2} \equiv F_\beta$ we obtain the linear equation.

$$N = \int_{\mathbb{R}^9} dp_2\, dp_3\, dp_4\, W(\cdots)\, (F_\beta(p_2)(F_\beta(p_3)+F_\beta(p_4)) - F_\beta(p_3)F_\beta(p_4)) \equiv cst.$$

$$L[g] = \int_{\mathbb{R}^9} dp_2\, dp_3\, dp_4\, W(\cdots)\, (-F_\beta(p_3)F_\beta(p_4)g(p_2,t) + 2\, F_\beta(p_2)F_\beta(p_4)g(p_3,t))$$

$$W(\cdots) = \delta(p+p_2-p_3-p_4)\ \delta\left(|p|^2+|p_2|^2-|p_3|^2-|p_4|^2\right)$$

If we restrict to spherically symmetric solutions

$$g(t,p) = g(t,|p|), \quad |p| = x$$

we may perform the angular integrations and the equation reduces to:

$$\begin{aligned} \frac{\partial g}{\partial t}(t,x) &= -\nu_N\, g(t,x) + \int_0^\infty V\left(\frac{x}{y}\right)\, g(y,t)\frac{d\,y}{y} \\ \nu_N &= \int_0^\infty Q(x)\, dx \approx 4,71 \\ V(x) &= R(x)+S(x);\ \ Q(x) = x\, R(x) \end{aligned}$$

with R and S explicitly know functions.

Theorem. (With M. Valle P. R. E '09) Consider $\beta = 1$, denote $F_\beta = F \equiv |p|^{-2}$.

For any radially symmetric initial data $g_{in} \in C_0^\infty(\mathbb{R}^3)$:

There exists a unique global, bounded, radial, classical solution to the linearised equation

$$\frac{\partial g}{\partial t} = -N\, g + L[g]$$

For all $t > 0$: $: \frac{d}{dt}\int_{\mathbb{R}^3} F(p)\, g(t,p)\, dp = 0$ (Conservation of number of particles.)

Moreover

$$\lim_{t\to+\infty} F(p)\, g(t,p) = M\, \delta_0, \quad \text{with}: \ M = \int_{\mathbb{R}^3} F(p)\, g_{in}(p)\, dp.$$

(Concentration of particles).

2. The small p limit of the linearised equation below but close to the critical temperature

We perturb now a thermodynamic equilibrium $\mathcal{F}_{N,E} = \frac{1}{e^{\beta|p|^2}-1} + \rho\, \delta_0$ that lies below the curve $E = \frac{2\,\pi\zeta(3/2)}{3\zeta(5/2)} N^{5/3}$ The system describing the homogeneous set of

particles after the condensation is:

$$\begin{aligned} \partial_t f(t) &= Q(f) + Q_2(f;n), \\ \frac{d}{dt}\, n(t) &= -\int_{\mathbb{R}^3} Q_2(f;n)(t,p)\, dp, \end{aligned}$$

where $n \equiv n(t)$ is the condensate density and

$$\begin{aligned} Q_2(f;n)(p,t) &= \int_{\mathbb{R}^9} dp_1 dp_2 dp_3\, |A(p_1,p_2,p_3;n)|^2 \delta(p_1-p_2-p_3)\delta(\omega_1-\omega_2-\omega_3) \\ &\quad \times[\delta(p-p_1)-\delta(p-p_2)-\delta(p-p_3)]\,[f_2 f_3 - f_1 f_2 - f_1 f_3 - f_1]\,. \end{aligned}$$

Q_2 preserves the energy but not the number of particles.

M. Imamovic-Tomasovic, A. Griffin (PRA 1999), H. T. C. Stoof (J. Low. Temp. Phys. 1999), R. Baier, T. Stockkamp (hep-ph/0412310).

At temperature below but of the same order than the critical temperature one has $\omega(p) \approx |p|^2$ and $|A(p_1,p_2,p_3;n)|^2 \approx C\, n$. While, at very low temperature, $\omega(p) \approx c\,|p|$ and $|A(p_1,p_2,p_3;n)|^2 \approx C\, \dfrac{p_1 p_2 p_3}{n}$. It is expected that in both cases $Q(f) << Q_2(f;n)$. We consider the simplified system at "low temperature":

$$\partial_t f(t) = Q_2(f),\ \ \partial_t n_0(t) = -\int_{\mathbb{R}^3} Q_2(f)(t,p)\, dp$$

We linearise as follows

$$\begin{aligned} &f(t,p) = F_0(p) + F_0(p)(1+F_0(p))|p|^2\, g(t,p) \\ &F_0(p) = \frac{1}{e^{\beta|p|^2}-1} \\ &n(t) = n_0 + m(t) \end{aligned}$$

and consider the small $|p|$ limit and radially symmetric case.

$$\frac{\partial g}{\partial t}(t,x) = C\, n_0\, x^{-1} \int_0^\infty \mathcal{H}\left(\frac{x}{y}\right)\, g(t,y)\, \frac{dy}{y}$$

for some explicit kernel $\mathcal{H}(x)$ singular at $x=1$. Notice the presence of the singular factor x^{-1} and the absence of any function $\nu(p)$ nor constant ν_N.

Classical asymptotic methods give then:

$$\begin{aligned} g(t,x) &= \frac{2\, G(0,3)}{W'(0)\, W(1)\, W'(2)}\, t^{-3} + o(t^{-3}) \quad \text{as } x \to 0,\ t \to +\infty \\ m(t) &= C\, n_0\, G(0,3)\, t^{-3} + o(t^{-3}) \quad \text{as } t \to +\infty. \end{aligned}$$

(Algebraic convergence to zero).

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Quantum Fokker-Planck models: an Open System Approach

Franco Fagnola
(joint work with A. Arnold, L. Neumann)

Quantum Fokker–Planck (QFP) models are special types of Open Quantum Systems arising in the study of quantum mechanical charge-transport including diffusive effects, in the description of quantum Brownian motions, quantum optics and semiconductor device simulations [1].

The state of an open quantum system, interacting with an external environment, is given by a density matrix, namely a positive operator on a Hilbert space h with unit trace.

In the quantum kinetic picture of QFP models, the state at time t is described by its Wigner function $w(x,v,t)$, where $(x,v)\in\mathbb{R}^2$ (we choose the one-dimensional case for simplicity), and its evolution, in a harmonic confinement potential $V_0(x)=\omega^2\frac{x^2}{2}$ with $\omega>0$ is given by (see [1] and the references therein)

$$\begin{aligned}(1)\qquad \partial_t w &= \omega^2 x\partial_v w - v\partial_x w + Qw\,,\\ Qw &= 2\gamma\partial_v(vw) + D_{pp}\Delta_v w + D_{qq}\Delta_x w + 2D_{pq}\partial_v\partial_x w + \Theta(V)\,.\end{aligned}$$

The real valued diffusion constants D_{pp}, D_{pq}, D_{qq} and the friction $\gamma>0$ satisfy

$$(2)\qquad \Delta := D_{pp}D_{qq} - D_{pq}^2 - \gamma^2/4 \geq 0\,,$$

$D_{pp}, D_{qq}\geq 0$, $V$is a perturbation of the external harmonic oscillator potential and

$$\Theta(V)w(x,v) = \frac{i}{(2\pi)^{1/2}}\int_{\mathbb{R}^2} d\tilde{v}\,d\eta\left(V(x+\frac{\eta}{2}) - V(x-\frac{\eta}{2})\right)w(x,\tilde{v})\mathrm{e}^{i(v-\tilde{v})\eta}.$$

In the theory of Markovian Open Quantum Systems, the evolution of a state is described by a strongly continuous semigroup of completely positive normal maps on the Banach space of trace-class operators on h. The dual semigroup on the von Neumann algebra $\mathcal{B}(\mathsf{h})$ of all bounded operators on h is called a quantum Markov semigroup, or quantum dynamical semigroup following the physical terminology, and its generator $\mathcal{L}$ has the typcal structure

$$(3)\qquad \mathcal{L}(x) = i[H,x] + \frac{1}{2}\sum_\ell\left(-L_\ell^*L_\ell x + 2L_\ell^* x L_\ell - xL_\ell^*L_\ell\right),$$

for x in the domain of $\mathcal{L}$, where H, L_ℓ are operators on h with H self-adjoint and $i[H,x]=i(Hx-xH)$. All bounded generators of norm-continuous quantum Markov semigroups have this structure, called GKSL (Gorini-Kossakowski-Sudarshan-Lindblad) ([10]), with bounded operators H and L_ℓ. Moreover, several generators of weakly*-continuous quantum Markov semigroups have the same algebraic structure.

This is the case for quantum Markov semigroups describing the evolution in QFP models (1). Here $\mathsf{h}=L^2(\mathbb{R},\mathbb{C})$ and, letting p and q denote the momentum and position operator on h, it is not difficult to show that considering

$$H = \frac{1}{2}\left(p^2+\omega^2q^2+\gamma(pq+qp)\right) + V(q)\,,$$

and the operators L_1 and L_2 given by

$$L_1 = \frac{-2D_{pq} + i\gamma}{\sqrt{2D_{pp}}} p + \sqrt{2D_{pp}}\, q\,, \qquad L_2 = \frac{2\sqrt{\Delta}}{\sqrt{2D_{pp}}}\, p\,,$$

the Wigner function $w(\cdot,\cdot,t)$ of the state ρ_t satisfying the quantum master equation

$$\frac{d}{dt}\rho_t = \mathcal{L}_*(\rho_t) \tag{4}$$

where $\mathcal{L}_*$ is the pre-dual generator of $\mathcal{L}$ (3) with the above H, L_1, L_2 solves (1). In this model $w(\cdot,\cdot,t)$ is explicitly given by

$$w(x,v,t) = \left(\frac{1}{2\pi}\right)^2 \int_{\mathbb{R}^2} \mathrm{e}^{i(\xi x+\eta v)} \mathrm{trace}(\rho_t \mathrm{e}^{-i(\xi q+\eta p)} d\xi d\eta\,.$$

Applying the methods developed by the author and R. Rebolledo ([8, 9, 6] and the references therein) for the study of the asymptotic behaviour of quantum Markov semigroups, we show the existence and uniqueness of a solution to (1) as Wigner transforms of density matrices $(\rho_t)_{t\geq 0}$ solving the master equation (4) with the above H, L_1, L_2, when V is a twice differentiable potential with strictly sub-linearly growing first derivative. Moreover, we show that ρ_t converges to a steady state ρ_∞ as time goes to infinity via a compactness argument in the Banach space of trace class operators.

The existence of a unique stationary solution to (1) in a weighted Sobolev space, for "small" (bounded, in particular) and smooth potentials V, was also proved in [3] where, in addition, it was shown that the steady state corresponds to a positive density matrix operator with unit trace and solutions converge towards the steady state with an exponential rate.

The Open System approach, working in the natural space of trace-class operators, has the advantage of leading directly to solutions that are already density matrices of quantum states. Moreover, it allows us to prove existence of a unique steady state and convergence towards this state in trace-norm for a bigger class of potentials V. However, only few results are available on convergence rates and hypercontractivity ([5, 4, 11] and the references therein) in a fully non-commutative framework, namely, in spaces of operators instead of space of functions.

Quantum mechanical effects are now becoming more and more relevant in the mathematical modeling of nano-scale devices; therefore further investigation in this direction would be interesting and useful. We would like to explore deeper properties of Quantum Fokker-Planck models with a genuinely non-commutative approach.

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Global Smooth Ion Dynamics in hte Euler-Poisson System

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(joint work with Benoit Pausader)

A fundamental two-fluid model for describing dynamics of a plasma in the Euler-Poisson system, in which compressible ion and electron fluids interact with their self-consistent electrostatic force. Global smooth electron dynamics were constructed in 1998 due to dispersive effect of the electric field. We are able to construct global smooth irrotational solutions with small amplitude for the ion dynamics equations, derived as the electron mass tend to zero from the Euler-Poisson system.

Kinetic Equations on Networks with Applications in Traffic Flow and Supply Chains

Michael Herty

We are interested in transport problems on networks as arising for example in the theory of gas transportation, water flow in open canals or traffic and production system dynamics. Such problems are typically described by conservation laws or

kinetic equations. In recent years there has been a progress in the theoretical investigations of such problems and among the many references we refer to [8] for a study of scalar equations on a network and to [5] for general 2×2-systems. Using classical solutions additional results are available for example in [11, 6, 9]. Here, we review recent results obtained in collaboration with R. M. Colombo, P. Degond, S. Göttlich, G. Guerra, A. Klar, M. Rascle and V. Schleper.

First, we introduce some notation. A network is considered as finite directed graph consisting of arcs $i = 1, \dots, K$ and vertices. Each arc is parameterized by an interval $[a_i, b_i]$ where a_i or b_i might be $-\infty$ and ∞, respectively. In the following we consider for simplicity a single vertex. We denote by x_0 either a_i or b_i depending on whether the arc is incoming $i \in I \subset K$ or outgoing $i \in J \subset K$ to the vertex with $J \cap I = \emptyset$. Note that the direction of flow is independent of the parameterization. In the following paragraphs we discuss results where the dynamics on each arc is governed by either a general 2×2 hyperbolic balance law [5], the Aw–Rascle [2] traffic flow model or a kinetic supply chain model [1, 7]. In order to present and discuss the results we give in all cases not the full statement but highlight a few important assumptions and properties.

We consider the general problem consisting of the 2×2 systems of balance laws where $y_i \in: [a_i, b_i] \times [0, \infty] \to \mathbb{R}^2$.

$$\partial_t y_i (t, x) + \partial_x f_l (y_i (t, x)) = g (t, x, y_i (t, x)) \tag{1}$$

and where the dynamics is coupled at $x = x_0$ by

$$\Psi (y_1 (t, x_0), \dots, y_n (t, x_0)) = U (t) . \tag{2}$$

The system has to be accompanied by initial conditions and some given function $U : \mathbb{R}^+ \to \mathbb{R}^n$. Applications fitting into the previous context are gas flow in pipelines, where typically the isothermal Euler equations are considered, or open canals where the dynamics is given by the shallow–water equations. For the homogenous Cauchy problem we consider weak solutions $y_i^{1,2} \in C^{0,1}(0, T; L^1[a_i, b_i]) \cap C(0, T; BV(a_i, b_i))$ to (1). Those solutions are constructed for example by wave–front tracking [4]. The main step in the construction are suitable solutions at the vertex satisfying $\Psi = 0$. This is achieved by extending the standard Riemann solver: Consider the case of constant data on each connected arc. The local invertibility of Ψ along the Lax–wave curves yields existence of new constant states at the vertex satifying the condition and leading to simple wave solutions of (1). Estimates on the total variation of the arising waves and Helly's theorem then yields also existence for a problem with non–constant data [5]. Extensions to source terms and functions U of small total variation are possible. We discuss some crucial assumptions in the general result [5, Theorem 3.8]. First, there is an assumption on the determinant of derivatives of Ψ which is satisfied by conditions taking from the engineering literature. Second, the obtained result is a local result in the following sense: Existence of solutions is proven for initial data $(y_i, U) \in (\bar{y}_i, \bar{U}) + L^1([a_i, b_i]) \times L^1(0, T)$ and $TV(y_i) + TV(U) \leq \delta$ for some (possibly small) value of δ. Here, we need to assume that $\Psi(\bar{y}) = \bar{U}$ and additionally $Df(\bar{y})$

has a striclty negative and a strictly positive eigenvalue. The latter condition implies no transition between sub- and supercritical situations at the vertex.

A different approach towards coupled dynamics on networks has been discussed in [10] for traffic flow on road networks described by the Aw–Rascle–Zhang model [2]

$$\partial_t(\rho_i u_i) + \partial_x(\rho_i u_i) = 0,\ \partial_t(\rho_i w_i) + \partial_x(\rho_i u_i w_i) = 0,\ w_i = u_i + p(\rho_i). \tag{3}$$

Here, ρ_i and u_i are density and velocity of the cars and p measures the deviation of the cars from the average velocity. Instead of prescribing a fixed function Ψ a microscopic approach is used. It can be shown that the microscopic description corresponds to a semi–discretization of the equations (3) in Lagragian coordinates. The key point is the observation that w_i is a Lagrangian property: On a microscopic level we therefore observe on any outgoing arc $j \in J$ a mixture of cars (w_i) arriving from different incoming roads $i \in I$. In order to obtain a macroscopic description we use homogenization [3] in the $w-$variable on each arc $j \in J$. For constant initial data the homogenziation procedure yields a new homogenized pressure p_i^* on $(x,t) : x \leq u_{j,0}t$ for an outgoing road j. The precise formulation of p^* depends on a flow distribution rate. The rate is assumed to be known. Then, it has been shown [10] that the total flow $\sum_{i\in I}(\rho_i u_i)(t,x_0) = \sum_{j\in J}(\rho_j u_j)(t,x_0)$ and similarly, the homogenized 'momentum' $\rho_i w_i$ is conserved for weak solutions to (3).

Jet a different procedure to couple dynamics on arcs has been discussed in the context of a kinetic supply chain model [7]. Here, the dynamics of the particle density f_i with priority y is given by

$$\partial_t f_i + \partial_x \Phi_i[f] + \partial_y E\ f_i = 0,\ \Phi_i[f] = v\ H(\mu_i - \int_{-\infty}^{v} f_i dv')f, \tag{4}$$

where $f_i : \mathbb{R}^+ \times [a_i, b_i] \times \mathbb{R} \to \mathbb{R}$, H is the heavi-side function, E the decay rate of the priority, μ_i the capacity of arc i and $v \geq 0$ the transport velocity. The exceeding parts entering in an outgoing road $j \in J$ will be stored in a buffer Q_j if the total incoming flow exceeds the capacitiy, i.e., $\sum_{i\in I} \Phi_i > \mu_j$,

$$\partial_t Q_j(y,t) = \sum_{i\in I} \Phi_i - \Phi_j. \tag{5}$$

Additionally, boundary conditions $f_j(y, x_0, t),\ j \in J$ are prescribed. Using moments $m_i^l = \int y^l f_i dy$ and an equilbrium closure relation
$f^e = \sum_{k=1}^{K} \rho_k(x,t)\delta\left(y - Y_k(x,t)\right)$ a system of conservation laws

$$\partial_t m_i^l + \partial_x \int y^l \Phi_i[f] dy,\ f = f^e$$

is derived. The corresponding conditions Ψ at the vertex $x = x_0$ for this system are obtained by applying the same closure procedure to the queue and the (kinetic) boundary conditions [7].

Summarizing, different approaches to coupling conditions depending on model and type of equation have been presented. Results on different conditions are

available but except for general 2×2-systems a complete theory is still absent. Open questions include the transition from sub- to supersonic in the gas of general $2 \times 2-$systems, global existence in time for the Aw–Rascle model under the given conditions and long–term behavior for the supply chain model at the vertex.

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Diffusion in macroscopic quantum models: drift-diffusion and Navier-Stokes approximations

Ansgar Jüngel

Diffusive quantum fluid models are used in superfluidity modeling and quantum semiconductor theory, for instance, to investigate the interplay between quantum dispersion and dissipative phenomena. In this work, we review some analytical results for three types of dissipative quantum fluid models.

Fourth-order quantum diffusion model. Nonlocal quantum drift-diffusion equations have been derived from a Wigner-BGK equation, in the diffusive scaling, by Degond et al. [3]. In the $O(\hbar^4)$ approximation, where $\hbar$ is the reduced Planck constant, local quantum drift-diffusion equations have been derived. In a simplified

setting, we can write the nonlinear parabolic equation as

$$n_t + \operatorname{div}\left(n\nabla\Big(\frac{\Delta\sqrt{n}}{\sqrt{n}}\Big)\right) = 0, \quad n(\cdot,0) = n_0 \geq 0 \quad \text{in } \mathbb{T}^d,\ t>0, \tag{1}$$

where $\mathbb{T}^d$ is the d-dimensional torus and $n(x,t)$ is the particle density.

Global existence results for the multidimensional equations have been achieved only recently [4, 7]. Whereas the framework of the first approach in [4] is that of mass transportation theory, the second approach in [7] is based on two tools. The first tool is the entropy estimate

$$\frac{d}{dt}\int_{\mathbb{T}^d} n(\log n - 1)dx + c\int_{\mathbb{T}^d} n|\nabla^2 \log n|^2 dx \leq 0,$$

where $c>0$ is some constant and $|\nabla^2 \log n|$ is the eulidean norm of the Hessian of $\log n$. Lacking a lower bound on n, the above inequality does not yield an H^2 estimate for $\log n$. However, it is possible to prove that

$$\int_{\mathbb{T}^d} n|\nabla^2 \log n|^2 dx \geq \kappa \int_{\mathbb{T}^d} |\nabla^2\sqrt{n}|^2 dx, \tag{2}$$

where $\kappa = 4(4d-1)/(d(d+2))$ [7]. The proof of this inequality is based on the method of systematic integration by parts developed in [6]. The second tool is the exponential variable transformation $n = \exp y$, motivated by the equivalent formulation of (1), $n_t + \frac{1}{2}\nabla^2 : (n\nabla^2 \log n) = 0$, where the double points signify summation over both indices of the Hessian matrix ∇^2. Indeed, if y is bounded, the particle density $n = \exp y$ becomes positive, circumventing the maximum principle.

Open problem: Is the solution strictly positive for all $t>0$, even if the nonnegative initial datum vanishes at some point?

Sixth-order quantum diffusion model. When expanding the nonlocal quantum diffusion model of [3] up to order $O(\hbar^6)$, we obtain the sixth-order parabolic equation

$$n_t = \operatorname{div}\left(n\nabla\Big(\frac{1}{2}|\nabla^2 \log n|^2 + \frac{1}{n}\nabla^2 : (n\nabla^2 \log n)\Big)\right), \quad x\in\mathbb{T}^d,\ t>0, \tag{3}$$

$$n(\cdot,0) = n_0 \geq 0, \quad x\in\mathbb{T}^d. \tag{4}$$

As for the fourth-order equation (1), the mathematical challenges are the nonnegativity of the particle density n and the proof of suitable a priori estimates. Interestingly, these difficulties can be overcome by similar tools employed for (1). Indeed, motivated by the equivalent formulation of (3),

$$n_t = \nabla^3 : (n\nabla^3 \log n) + 2\nabla^2 : (n(\nabla^2 \log)^2),$$

where $\nabla^3 \log n$ is the tensor of all third-order derivatives of $\log n$, we introduce the exponential variable $n = \exp y$. The second tool are the entropy estimates

$$\frac{d}{dt}\int_{\mathbb{T}^d} n(\log n - 1)dx + c\int_{\mathbb{T}^d} \big(|\nabla^3\sqrt{n}|^2 + |\nabla^2\sqrt[4]{n}\nabla\sqrt[4]{n}|^2 + |\nabla\sqrt[6]{n}|^6\big)dx \leq 0,$$

which are proved by using systematic integration by parts. With these techniques, the global-in-time existence of weak solutions to (3)-(4) is proved in [8] for the one-dimensional equation and in [2] for the multidimensional case.

Open problems: Do there exist (special) weak solutions to (3) which vanish at some points? Is there uniqueness of weak solutions?

Quantum Navier-Stokes equations. Brull and Méhats [1] derived from the Wigner-BGK equation in the hydrodynamic scaling a Chapman-Enskog correction to the moment equations, leading to the quantum Navier-Stokes equations for the particle density $n(x,t)$ and the mean velocity $u(x,t)$,

$$n_t + \operatorname{div}(nu) = 0, \quad x \in \mathbb{T}^d,\ t > 0, \tag{5}$$

$$(nu)_t + \operatorname{div}(nu \otimes u) + \nabla p(n) - \frac{\varepsilon^2}{6} n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right) - nf = 2\alpha \operatorname{div}(nD(u)), \tag{6}$$

$$n(\cdot,0) = n_0, \quad (nu)(\cdot,0) = n_0 u_0 \quad \text{in } \mathbb{T}^d, \tag{7}$$

where $\varepsilon > 0$ is the scaled Planck constant, f is a force field, $D(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$, and $\alpha > 0$ is the mean-free path coming from the Enskog-Chapman expansion. We have allowed for a barotropic pressure function $p(n) = n^\beta$ with $\beta > 1$. Besides the lack of a maximum principle due to the third-order differential term, another problem is the density-dependent viscosity $\nu(n) = \alpha n$ which degenerates at vacuum and does not allow for H^1 estimates on u. A third problem is the lack of suitable a priori estimates. Indeed, let us define the energy by

$$E_\varepsilon(n,u) = \int_{\mathbb{T}^d} \Big(\frac{n}{2}|u|^2 + H(n) + \frac{\varepsilon^2}{6}|\nabla\sqrt{n}|^2\Big)dx,$$

where $H(n) = n^\beta/(\beta-1)$ if $\beta > 1$ and $H(n) = n(\log n - 1)$ if $\beta = 1$. A formal computation shows that, without force field $f = 0$,

$$\frac{dE_\varepsilon}{dt}(n,u) + \alpha\int_{\mathbb{T}^d} n|D(u)|^2 dx = 0,$$

which provides an H^1 estimate for $\sqrt{n}$ only.

Our main idea to solve these problems is to transform the quantum Navier-Stokes system by means of the *"osmotic velocity"* $w = u + \alpha\nabla\log n$. Then (5)-(7) can be equivalently written as the viscous quantum hydrodynamic equations [5]

$$n_t + \operatorname{div}(nw) = \alpha\Delta n, \tag{8}$$

$$(nw)_t + \operatorname{div}(nw \otimes w) + \nabla p(n) - \frac{\varepsilon_0^2}{6} n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right) - n\nabla V = \alpha\Delta(nw), \tag{9}$$

with initial conditions $n(\cdot,0) = n_0$ and $(nw)(\cdot,0) = n_0u_0 + \alpha\nabla n_0$, where $\varepsilon_0^2 = \varepsilon^2 - 12\alpha^2$. This formulation has two advantages. The first advantage is that it allows for the additional energy estimate (if $\varepsilon^2 > 12\alpha^2$ and $f = 0$)

$$\frac{dE_{\varepsilon_0}}{dt}(n,w) + \alpha\int_{\mathbb{T}^d}\Big(n|\nabla w|^2 + H'(n)|\nabla n|^2 + \frac{\varepsilon_0^2}{12}n|\nabla^2\log n|^2\Big)dx = 0,$$

and hence, (2) provides an H^2 bound for $\sqrt{n}$. The second advantage is that we can apply the maximum principle to (8) to deduce strict positivity of the density n if n_0 is strictly positive and the velocity w is smooth. Employing a Faedo-Galerkin method and assuming that $\beta > 3$, the global-in-time existence of weak solutions to (8)-(9) and hence to (5)-(7) was shown in [5].

Open problems: Prove global existence of solutions to (5)-(6) for $\beta \leq 3$. Develop an existence theory for the quantum Navier-Stokes system including the energy equation (see [9] for the model).

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The Boltzmann equation for Bose-Einstein particles: Concentration or oscillation in finite time

Xuguang Lu

The Boltzmann equation for Bose-Einstein particles (also called Uehling-Uhlenbeck equation) can be used to study the formation of Bose-Einstein condensation of Bose gases. For spatially homogeneous and isotropic solutions with the hard sphere model, the equation can be written as

$$\frac{\partial}{\partial t} f(x,t) = \frac{1}{\sqrt{x}} \int_{\mathbf{R}_+^2} q(f(\cdot,t))(x,y,z) \min\{\sqrt{x}, \sqrt{x_*}, \sqrt{y}, \sqrt{z}\} \mathrm{d}y\mathrm{d}z, \qquad (1)$$

$$t \in (0,T),\ x > 0, \quad \mathbf{R}_+ = [0,\infty), \quad 0 < T \leq \infty, \quad x_* = (y+z-x)_+,$$
$$q(f)(x,y,z) = f(y)f(z)[1+f(x)+f(x_*)] - f(x)f(x_*)[1+f(y)+f(z)].$$

A measure solution of Eq.(1) on $\mathbf{R}_+ \times [0,T)$ is a family $\{F_t\}_{t\in[0,T)}$ of positive Borel measures on $\mathbf{R}_+$ with $\sup_{t\in[0,T)} \int_{\mathbf{R}_+}(1+x)\mathrm{d}F_t(x) < \infty$, such that the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{R}_+}\varphi \mathrm{d}F_t = \int_{\mathbf{R}_+^2} J[\varphi]\mathrm{d}(F_t\otimes F_t) + \int_{\mathbf{R}_+^3} K[\varphi]\mathrm{d}(F_t\otimes F_t\otimes F_t)$$

holds for all $t\in[0,T)$ and all $\varphi \in C_b^2(\mathbf{R}_+)$, where $J[\varphi], K[\varphi]$ are explicit linear operators of φ. The existence of mass-energy conserved measure solutions is known [2]. Let $f(x,t)\geq 0$ and $\mu_t \geq 0$ be the $L^1(\sqrt{x}\,\mathrm{d}x)$ part and the singular part of F_t in terms of the Lebesgue decomposition, i.e. $\mathrm{d}F_t(x) = f(x,t)\sqrt{x}\,\mathrm{d}x + \mathrm{d}\mu_t(x)$. We are interested in the case where the initial datum $F_0 = F_t|_{t=0}$ has no singular part, i.e. $\mathrm{d}F_0(x) = f_0(x)\sqrt{x}\,\mathrm{d}x$, and f_0 is large near the origin $x=0$ (the zero energy of particles) so as to include some cases of low temperatures. Although physical experiments show that the Bose-Einstein condensation occurs in finite time, this has not been clear for the present quantum Boltzmann model:

Does there exist a measure solution F_t and a time $t_0 \in (0,T)$, such that $F_{t_0}(\{0\}) = \mu_{t_0}(\{0\}) > 0$? or in a weaker version, $\lim_{\varepsilon\to 0^+}\sup_{0\leq t\leq t_0} F_t([0,\varepsilon]) > 0$?

To this problem we obtain an alternative result:

Theorem 1. *Suppose the initial datum F_0 of a measure solution F_t of Eq.(1) on $\mathbf{R}_+\times[0,T)$ satisfies $\liminf_{\varepsilon\to 0^+}\frac{F_0([0,\varepsilon])}{\varepsilon^{1/3}} > 0$. Then for almost every $t\in(0,T)$, one of the following* (a), (b) *holds:*

(a) $\forall\, 0<\eta<1$

$$\liminf_{\varepsilon\to 0^+}\frac{1}{\varepsilon^{1/3}}\int_{\eta\varepsilon}^{\varepsilon} f(x,t)\sqrt{x}\,\mathrm{d}x = 0, \quad \limsup_{\varepsilon\to 0^+}\frac{1}{\varepsilon^{1/3}}\int_0^{\varepsilon} f(x,t)\sqrt{x}\,\mathrm{d}x = +\infty.$$

(b)

$$\limsup_{\varepsilon\to 0^+}\frac{\mu_t([0,\,\varepsilon])}{\varepsilon^{1/3}} = +\infty\,.$$

Theorem 1 implies in particular that if the singular part μ_t has no contribution near the origin for all $t\in(0,T)$, then $x^{7/6}f(x,t)$ will be always oscillating near the origin: $\liminf_{x\to 0^+} x^{7/6}f(x,t) = 0$ and $\limsup_{x\to 0^+} x^{7/6}f(x,t) = +\infty$ a.e. $t\in(0,T)$.

However if the restriction of conservation of mass is removed, the situation will be very different:

Theorem 2. *Let $f(x,t)\geq 0$ be a weak solution of Eq.(1) on $\mathbf{R}_+\times[0,T)$ with respect to the subspace $C_b^2(\mathbf{R}_+)_0 = \{\varphi\in C_b^2(\mathbf{R}_+)\,|\,\varphi(0)=0\}$ of test functions. Suppose*

$$\liminf_{x\to 0+} x^{7/6}f_0(x) > 0, \qquad \sup_{(x,t)\in(0,\infty)\times[0,T)} x^{7/6}f(x,t) < \infty.$$

Then $f(x,t)$ cannot be the $L^1(\sqrt{x}\,\mathrm{d}x)$ part of any measure solution of Eq.(1) on $\mathbf{R}_+ \times [0,T)$. Moreover the mass $t \mapsto N(t) = \int_{\mathbf{R}_+} f(x,t)\sqrt{x}\mathrm{d}x$ is not conserved:

$$N(t) = N(0) - B\int_0^t [a(s)]^3 \mathrm{d}s, \qquad \int_0^t [a(s)]^3 \mathrm{d}s > 0 \qquad \forall\, t \in (0,T) \tag{2}$$

where $B > 0$ is a universal constant, $a_*(t) \le a(t) \le a^*(t)$, and

$$a_*(t) = \liminf_{x\to 0+} x^{7/6} f(x,t), \quad a^*(t) = \limsup_{x\to 0+} x^{7/6} f(x,t).$$

The authors of [1] constructed a classical solution $f(x,t)$ (we call it the EMV solution) of Eq.(1) that satisfies all conditions in Theorem 2 with $a_*(t) = a^*(t)$, and $f(x,t)$ conserves the energy. Spohn [3] then proved the mass equality in (2). These together with the strict positivity in (2) and the first conclusion of Theorem 2 show that the EMV solution $f(x,t)$ gives a process of increasing temperature, and for any function $n(t) \ge 0$ on $[0,T)$, the measure F_t defined by

$$\mathrm{d}F_t(x) = f(x,t)\sqrt{x}\mathrm{d}x + n(t)\delta(x)\mathrm{d}x \tag{3}$$

is not a mass-conserved weak solution of Eq.(1), where $\delta(x)$ is the delta function. Recall that to study the concentration in finite time physicists use splitting method: Hope to solve the couple $(f(x,t), n(t))$ in (3) as a solution of an equation system of the following form :

$$\begin{aligned} &\frac{\partial}{\partial t} f(x,t) = Q(f)(x,t) + n(t)\tilde{Q}(f)(x,t)\,, \quad f(x,0) = f_0(x), \\ &\frac{\mathrm{d}}{\mathrm{d}t} n(t) = -n(t)\int_{0+}^{\infty} \tilde{Q}(f)(x,t)\sqrt{x}\mathrm{d}x\,, \qquad n(0) = 0 \end{aligned}$$

where $Q(f)(x,t)$ is defined by the right hand side of Eq.(1). From Theorems 1 & 2 and the properties of the EMV solution above we feel that either this splitting method is oversimplified, or the problem of concentration in finite time is more difficult than what we thought (maybe need very sharp estimates for proving $n(t) > 0$ for $t > t_c > 0$?). See [3] for relevant results and discussions.

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Kinetic models with non-strict conservations

Daniel Matthes

(joint work with Federico Bassetti, Lucia Ladelli, and Giuseppe Toscani)

In his seminal works from the 60s, McKean introduced the idea of applying probabilistic methods to study the long time behavior of solutions to the Kac equation. Surprisingly, it was much later [2] that the powerful machinery of the central limit theorem has been employed in this spirit to derive estimates on the rate of equilibration, which eventually [7] lead to the confirmation of McKean's original conjecture. This talk is concerned with yet another application of the probabilistic ideas to study the long time behavior of Maxwell-type equations.

Specifically, we consider the spatially homogeneous Boltzmann equation $\partial_t f + f = Q_+[f,f]$ for the velocity distribution $f = f(t;v)$ in a d-dimensional gas, whose collision mechanism turns the pre-collisional velocities v, v_* of the two interacting molecules into the post collisional velocities

$$v' = Av + Bv_*, \quad v'_* = A_*v_* + B_*v, \tag{1}$$

with *random* matrices A, B, A_*, B_*. We generalize the classical theory for Maxwell molecules (which fits in by choosing $A = \mathbf{1} - \omega\omega^T$ and $B = \omega\omega^T$, with a random vector ω uniformly distributed on $\mathbf{S}^{d-1}$) by *not* requiring strict energy conservation but only conservation in the ensemble average. I.e., although it will no longer be true that $|v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2$ holds almost surely, one still has that the statistical expectation of the left-hand side equals the right-hand side, so the total kinetic energy of the system remains constant. The original motivation to study such models arose in applications of kinetic theory to economics [5].

Under the relaxed hypothesis of conservation in the mean, the possibilities for the long time behavior of solutions is much richer than for Maxwell molecules. Breaking the strict conservation of energy also breaks the H-Theorem, and neither Boltzmann's entropy functional nor the Fisher information are time-monotone anymore. Typically, there is still a (unique up to rescaling) stationary state f_∞, but f_∞ is not a Gaussian in general; in fact, f_∞ is a Gaussian if and only if the collisions (1) are strictly energy conservative. In the general situation, we prove [1] that f_∞ is a scale *mixture* of Gaussians instead,

$$f_\infty(v) = \int_{\Sigma\geq 0} \frac{1}{\det\sqrt{2\pi\Sigma}} \exp\Big(-\frac{1}{2}v^T\Sigma^{-1}v\Big)\,\mathrm{d}\nu(\Sigma), \tag{2}$$

where ν is a probability measure on the set of non-negative symmetric matrices Σ. We find that, depending on the particular choice of the rules (1), f_∞ might have a concentration in $v = 0$ and/or might possess a fat (Pareto) tail, $f_\infty(v) \propto |v|^{-(1+\beta)}$. Moreover, f_∞ is weakly attractive for all transient solutions of finite kinetic energy.

These results are derived in application of the central limit theorem. The key ingredient is the probabilistic representation [3] of the transient solution,

$$f(t) = e^{-t}\sum_{n=1}^{\infty}(1-e^{-t})^{n-1}\,\mathrm{Law}\left(\sum_{j=1}^{n} C_{nj}X_j\right),$$

with i.i.d. random variables $X_1, X_2, \ldots$ representing the initial distribution, and random matrix weights C_{nj} that are obtained by "taking a random walk in the McKean forrest". More precisely, first pick a binary tree of order n at random according to McKean's construction. Then assign i.i.d. copies (A_k, B_k) of the random matrix pair (A, B) to each interior node, with A_k being associated to the node's left branching, and B_k to its right branching. Now the matrix C_{nj} is the product of all A_ℓ's and B_ℓ's along the route from the jth leaf to the tree's root. The applicability of the central limit theorem then rests on suitable estimates on the convergence of the summed covariance matrices

$$\Theta_n := \sum_{j=1}^{n} C_{nj} \operatorname{Cov}[X_j] C_{nj}^T$$

towards a limiting random matrix Θ as $n \to \infty$. It follows that ν in (2) is the law of Θ. All qualitative information of f_∞ are thus encoded in Θ, which we characterize as solution of the fixed point equation $\Theta \stackrel{D}{=} A\Theta A^T + B\Theta B^T$.

In $d = 1$ spatial dimension, and under mild conditions on the random matrices in (1), the weak convergence of transient solutions to the stationary state can be improved to strong convergence [8], provided that the initial condition f_0 is a genuine density function with a bit of Sobolev regularity, namely $f_0 \in H^\epsilon(\mathbf{R})$ and $\sqrt{f_0} \in H^\epsilon(\mathbf{R})$ for an arbitrarily small $\epsilon > 0$. The proof combines the classical strategy from [4] with direct estimates a la [6] to compensate for the lack of entropy control. Unless f_0 has just barely finite energy (i.e., all of its moments above the second diverge), the convergence $f(t) \to f_\infty$ is even exponentially fast in the L^1-norm, with a rate that depends sensitively on growth of f_0's moments.

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Dynamical collapse of boson stars

Alessandro Michelangeli
(joint work with B. Schlein)

We consider systems of gravitating bosons known as *boson stars*. Assuming the particles to have a relativistic dispersion, but the interaction to be treated classically (Newtonian gravity), we arrive at the N-particle Hamiltonian

$$H_N = \sum_{j=1}^{N} \sqrt{1-\Delta_{x_j}} - \frac{G}{N} \sum_{i<j}^{N} \frac{1}{|x_i - x_j|} \tag{1}$$

acting on the Hilbert space $L^2_s(\mathbb{R}^{3N})$, the subspace of $L^2(\mathbb{R}^{3N})$ containing all functions symmetric with respect to arbitrary permutations (here we use units with $\hbar = 1$, $c = 1$, and $m = 1$, where m denotes the mass of the bosons).

The mean-field type scaling in front of the interaction is dictated by the emergence of an effective theory in the limit of infinitely many particles [5, 6]. More precisely, as long as G is smaller that a certain critical value G_{crit} ($G_{\mathrm{crit}} > 0$), then $\frac{1}{N} \inf \sigma(H_N)$ stays finite as $N \to \infty$ and its asymptotics is captured by the Hartree energy. In this case the Schrödinger dynamics governed by H_N is well-posed and it makes sense to study the "propagation of chaos"-like phenomenon described by the following diagram

$$\begin{array}{ccccc} \Psi_N & \xrightarrow{\text{partial trace}} & \gamma_N^{(k)} & \xrightarrow{N\to\infty} & |\varphi^{\otimes k}\rangle\langle\varphi^{\otimes k}| \\ \Big\downarrow{\scriptstyle \substack{\text{many-body} \\ \textbf{linear} \text{ dynamics}}} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \substack{\textbf{non linear} \\ \text{Hartree}}} \\ \Psi_{N,t} & \longrightarrow & \gamma_{N,t}^{(k)} & \xrightarrow{N\to\infty} & |\varphi_t^{\otimes k}\rangle\langle\varphi_t^{\otimes k}| \end{array} \tag{2}$$

Here the underlying effective non-linear Schrödinger equation for the one-body orbital φ_t is the semi-relativistic Hartree-equation

$$i\partial_t \varphi_t = \sqrt{1-\Delta}\,\varphi_t - G\left(\frac{1}{|\cdot|} * |\varphi_t|^2\right)\varphi_t\,. \tag{3}$$

Eqn. (3) is known to be globally well-posed for $G < G_{\mathrm{crit}}$ [4], and to undergo blow-up in finite time as $G > G_{\mathrm{crit}}$ [3]. In this *sub-critical* regime, diagram (2) was closed first in [1] and later in [7], the general result reading

$$\mathrm{Tr}\Big|\gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t|\Big| \;\leqslant\; \frac{C}{\sqrt{N}} \tag{4}$$

uniformly in $t \in [0,T]$ (T being fixed, arbitrary) and $\forall N \geqslant 1$. Analogous results were proved in the non-relativistic version of our model (1), [2, 8].

The generally believed scenario in the *super-critical* regime $G > G_{\mathrm{crit}}$ is that (3) still describes the effective dynamics of the boson star and that the one-body blow-up describes the occurrence in time of the many-body collapse.

Our work substantiates this picture as follows. We monitor the loss of regularity of the many-body wave function along the time evolution, as the time approaches

the one-body blow-up time. To this aim, we control the evolution of the fluctuations around the limiting Hartree dynamics. Our control being for the first time in the energy sense, and with a quantitative rate of convergence, we can deduce from the blow-up of the Hartree dynamics the loss of regularity along the many-body quantum dynamics.

Theorem. Fix $G \in \mathbb{R}$, $\varphi \in H^2(\mathbb{R}^3)$ with $\|\varphi\|_2 = 1$ and set $\Psi_N = \varphi^{\otimes N}$. Let $\alpha_N = N^{-k}$ for an arbitrary positive integer k. Let $\Psi_{N,t} = e^{-iH_N^{(\alpha)}t}\Psi_N$ be the evolution of the the initial wave function Ψ_N with respect to the regularised Hamiltonian

$$H_N^{(\alpha)} = \sum_{j=1}^{N} \sqrt{1-\Delta_{x_j}} - \frac{G}{N}\sum_{i<j}^{N} \frac{1}{|x_i - x_j| + \alpha_N} \tag{5}$$

and let $\gamma_{N,t}^{(1)}$ be the one-particle reduced density matrix associated with $\Psi_{N,t}$. (Note that the regularisation is removed as $N \to \infty$.) Denote by φ_t the solution to the non-linear Hartree equation (3) with initial data $\varphi_{t=0} = \varphi$. Suppose that $T_{\text{blow-up}}$ is the first time of blow-up for φ_t. In other words, assume that $\kappa(t) := \sup_{s\in(0,t)} \|\varphi_s\|_{H^{1/2}} < \infty$ for all $t \in [0, T_{\text{blow-up}})$, and $\|\varphi_t\|_{H^{1/2}} \to \infty$ as $t \to T_{\text{blow-up}}^-$. Then, for $t \in [0, T_{\text{blow-up}})$, there exists $N(t) \in \mathbb{N}$, such that $N(t) \to \infty$, as $t \to T_{\text{blow-up}}^-$, and

$$\|(1-\Delta_{x_1})^{1/4}\,\Psi_{N(t),t}\|^2 = \operatorname{Tr}(1-\Delta)^{1/2}\gamma_{N(t),t}^{(1)} \to \infty \qquad \text{as} \quad t \to T_{\text{blow-up}}^- . \tag{6}$$

In other words, the kinetic energy per particle diverges in the limit $t \to T_{\text{blow-up}}^-$, if at the same time, the number of particles tends to infinity appropriately.

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Hypocoercivity of non-symetric operators on large functional space

Stéphane Mischler
(joint work with Maria Pia Gualdani, Clément Mouhot)

Consider a Hilbert space H, an (unbounded integro-differential) operator L on H and the associated semigroup $e^{t\,L}$ or equivalently the linear evolution PDE

$$\partial_t f = L\,f \quad \text{on} \quad H.$$

We assume that L is hypocoercive in the sense that the following "spectral gap like" information holds:

(i) $\Sigma(L) \cap \Delta_a = \{0\}$, $a < 0$, and $H_1 := \mathrm{null}(L)$ is finite dimensional,
(ii) $\exists\, C_a\ \forall\, f_0 \in H\ \|e^{t\,L}\,f_0 - \Pi_1 f_0\|_H \le C_a\, e^{a\,t}\,\|f_0\|_H$,

where $\Sigma(L)$ is the spectrum set of L, $\Delta_a := \{z \in \mathbb{C},\ \Re e\, z > a\}$ is the complex half plane and Π_1 is the projection operator on the eigenspace H_1 associated to the first eigenvalue 0. It is clear that (i) $\Leftarrow$ (ii) and that (ii) is a kind of partial spectral mapping theorem in the sense that $\Sigma(e^{tL}) \cap \Delta_{e^{a\,t}} = e^{t\,\Sigma(L)\cap\Delta_a}$.

Considering now a larger Banach space $\mathcal{E} \supset H$ and an extension operator $\mathcal{L}$ on $\mathcal{E}$, in the sense that $\mathcal{L}_{|H} = L$, we ask whether the same spectral analysis holds for $\mathcal{L}$ on $\mathcal{E}$. We give a positive answer by exhibiting some conditions on $\mathcal{L}$, namely and roughly speaking that $\mathcal{L}$ splits as

$$\mathcal{L} = \mathcal{A} + \mathcal{B}, \qquad \mathcal{A} : \mathcal{E} \to E, \quad \Sigma(\mathcal{B}) \cap \Delta_a = \emptyset,$$

under which $\mathcal{L}$ is still an hypocoercive operator on $\mathcal{E}$. Our proof is based on a factorization trick from which we may estimate the resolvent of $\mathcal{L}$ thanks to $\mathcal{A}$ and the resolvents of $\mathcal{B}$ and $\mathcal{L}$ as well as a representation formula of $e^{t\mathcal{L}}$ thanks to a complex path integral of its resolvent (inverse Laplace formula).

The above abstract result is motivated and applied to the (possibly torus space inhomogeneous) Fokker-Planck operator

$$Lf = \Delta_v\, f + \mathrm{div}_v(v\,f) + v \cdot \nabla_x f, \quad v \in \mathbb{R}^d,\ x \in \mathbb{T}^d,$$

and the (possibly torus space inhomogeneous) linear Boltzmann

$$Lf = \int_{\mathbb{R}^d} k(v,v')\,f(x,v')\,dv' - K(v)\,f + v \cdot \nabla_x f \quad v \in \mathbb{R}^d,\ x \in \mathbb{T}^d.$$

For such operators it is known that the centered and normalized Gaussian M is an equilibrium and that they are coercive in $L^2(\mathbb{R}^d; M^{-1}(v)\,dv)$ (space homogeneous case) and hypocoercive in $L^2(\mathbb{T}^d \times \mathbb{R}^d; M^{-1}(v)\,dvdx)$ (space inhomogeneous case).

Thanks to the above mentioned abstract result, we establish that the associated extended operators $\mathcal{L}$ are also hypocoercive in $L^p(\mathbb{T}^d \times \mathbb{R}^d; m(v)\,dvdx)$ for $p = 1, 2$ and for some weight polynomial function $m(v) = (1 + |v|)^\alpha$, $\alpha > d$, or stretch exponential function $m(v) = e^{a\,|v|^s}$, $a > 0$, $s \in (0, 2]$. Some applications to nonlinear related problems are stated.

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Landau damping

Clément Mouhot
(joint work with Cédric Villani)

We report on the mini-course the first author of this joint work has given in the Oberwolfach workshop "Classical and Quantum Mechanical Models of Many-Particle Systems" 5-11 december 2010. This work is concerned with the phenomenon of "collisionless relaxation" in plasma physics, the so-called "Landau damping effect" and provides the first mathematical proof of this damping effect in the framework of the non-linear Vlasov-Poisson equation.

The "standard model" of classical plasma physics (in the non-relativistic and non-magnetic case) is the Vlasov–Poisson equation [9, 5], here written with periodic boundary conditions and in adimensional units:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F[f] \cdot \nabla_v f = 0, \tag{1}$$

where $f = f(t, x, v)$ is the electron distribution function ($t \geq 0$, $v \in \mathbb{R}^3$, $x \in \mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$),

$$F[f](t, x) = - \iint \nabla W(x - y)\, f(t, y, w)\, dw\, dy \tag{2}$$

is the self-induced force, $W(x) = 1/|x|$ is the Coulomb interaction potential.

On very large time scales, collisional dissipative phenomena play a non-negligible role, the equation has to be complemented with a collision operator, and the entropy increase is supposed to force the (slow) convergence to a maxwellian (gaussian) velocity distribution.

Ten years after devising this collisional scenario, Landau [5] formulated a much more subtle prediction: the stability of homogeneous equilibria satisfying certain conditions — for instance any function of $|v|$, not necessarily Gaussian — on much shorter time scales, by means of purely conservative mechanisms. This phenomenon, called **Landau damping**, is a property of the (collisionless) Vlasov equation. This is a theoretical cornerstone of the classical plasma physics (among a large number of references let us mention [1]). Similar damping phenomena also occur in other domains of physics.

The Landau damping has been since long understood at the linearized level [3, 6, 8], but the study of the full (nonlinear) equation poses important conceptual and technical problems. As a consequence, up to now the only existing results were proving existence of *some* damped solutions with prescribed behavior as $t \to \pm\infty$ [2, 4]. We fill this gap in the the recent work [7], whose main result we shall now describe.

If f is a function defined on $\mathbb{T}^d \times \mathbb{R}^d$, we note, for any $k \in \mathbb{Z}^d$ and $\eta \in \mathbb{R}^d$,

$$\hat{f}(k,v) = \int_{\mathbb{T}^d} f(x,v)\, e^{-2i\pi k\cdot x}\, dx,\ \tilde{f}(k,\eta) = \iint_{\mathbb{T}^d\times\mathbb{R}^d} f(x,v)\, e^{-2i\pi k\cdot x}\, e^{-2i\pi \eta\cdot v}\, dv\, dx.$$

We also set, for $\lambda, \mu, \beta > 0$,

$$\|f\|_{\lambda,\mu,\beta} = \sup_{k,\eta}\Big(|\tilde{f}(k,\eta)|\, e^{2\pi\lambda|\eta|}\, e^{2\pi\mu|k|}\Big) + \iint_{\mathbb{T}^d\times\mathbb{R}^d} |f(x,v)|\, e^{2\pi\beta|v|}\, dv\, dx. \tag{3}$$

Theorem 5. *Let $d \geq 1$, and $f^0 : \mathbb{R}^d \to \mathbb{R}_+$ an analytic velocity profile. Let $W : \mathbb{T}^d \to \mathbb{R}$ be an interaction potential. For any $k \in \mathbb{Z}^d$, $\xi \in \mathbb{C}$, we set*

$$\mathcal{L}(k,\xi) = -4\pi^2\, \hat{W}(k) \int_0^\infty e^{2\pi|k|\xi^* t}\, |\tilde{f}^0(kt)|\, |k|^2\, t\, dt.$$

We assume that there is $\lambda > 0$ such that, for ϵ small enough,

$$\sup_{\eta\in\mathbb{R}^d} |\tilde{f}^0(\eta)|\, e^{2\pi\lambda|\eta|} \leq C_0, \qquad \sum_{n\in\mathbb{N}^d} \frac{\lambda^n}{n!} \|\nabla_v^n f^0\|_{L^1(dv)} \leq C_0,$$

$$\inf_{k\in\mathbb{Z}^d} \inf_{0\leq \Re\,\xi <\lambda} \big|\mathcal{L}(k,\xi) - 1\big| \geq \kappa > 0$$

$$\exists\, \gamma \geq 1;\ \forall\, k \in \mathbb{Z}^d; \qquad |\hat{W}(k)| \leq \frac{C_W}{|k|^{1+\gamma}}.$$

Then as soon as $0 < \lambda' < \lambda$, $0 < \mu' < \mu$, $\beta > 0$, $r \in \mathbb{N}$, there are $\varepsilon > 0$ and $C > 0$, depending on $d, \gamma, \lambda, \lambda', \mu, \mu', C_0, \kappa, C_W, \beta, r$, such that if $f_i \geq 0$ satisfies

$$\delta := \|f_i - f^0\|_{\lambda,\mu,\beta} \leq \varepsilon,$$

then the unique solution of the nonlinear Vlasov equation (1-2). Futhermore, there are analytic profiles $f_{+\infty}(v)$, $f_{-\infty}(v)$ such that

$$f(t,\cdot) \xrightarrow{t\to\pm\infty} f_{\pm\infty} \qquad \textit{weakly}$$

$$\int f(t,x,\cdot)\, dx \xrightarrow{t\to\pm\infty} f_{\pm\infty} \qquad \textit{strongly (in } C^r(\mathbb{R}^d_v)),$$

these convergences being also $O(\delta\, e^{-2\pi\lambda'|t|})$.

The conditions on f^0 are satisfied for instance by any analytic radially symmetric velocity profile in dimension 3. This theorem is entirely constructive and almost optimal in many aspects. The mini-course given at MFO first described the linearized stability theory, second sketched the general scheme of the proof in the nonlinear case, and third outlined the main difficulties and new ideas from the proof. More details as well as more comments, both from the mathematical and the physical sides, can be found in [7].

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WKB-based scheme for the highly oscillatory Schrödinger equation

Claudia Negulescu

(joint work with Anton Arnold, Naoufel Ben Abdallah)

Multi-scale phenomena and high frequency problems are common in nature and their study is rather challenging from a mathematical and numerical point of view. Substantial effort is done currently to solve these problems in a precise manner and with affordable numerical costs. The used techniques are based either on mathematical or numerical approaches.

The present paper deals with an asymptotic scheme for the numerical solution of highly oscillating differential equations of the type

$$\varepsilon^2 \varphi''(x) + a(x)\varphi(x) = 0\,, \tag{1}$$

where $0 < \varepsilon \ll 1$ is a very small parameter and $a(x) \geq a_0 > 0$ a sufficiently smooth function. For very small $\varepsilon > 0$, the wave length $\lambda = \frac{2\pi\varepsilon}{\sqrt{a(x)}}$ is very small, such that the solution φ becomes highly oscillating. In a classical ODE–scheme such a situation requires a very fine mesh in order to accurately resolve the oscillations, typically at least 10 grid points per oscillation. Hence, standard numerical methods would be very costly and inefficient here. The goal of this paper is to present a new method, based on a mathematical reformulation and which uses a coarse spatial grid with step size $h > \lambda$ (see Figure 1). The detailed study of this scheme is presented in [1].

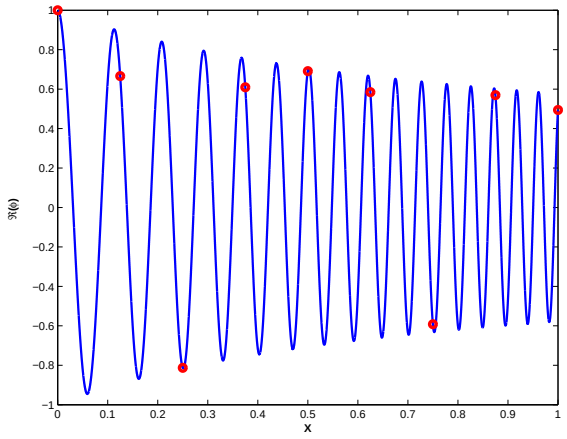

FIGURE 1. In standard numerical methods highly oscillating solutions require a very fine mesh to capture the oscillations. However, with the analytic pre-processing of our method an accurate solution can be obtained on a coarse grid (dots). Plotted is the solution $\Re\varphi(x)$ of (1) with $\varepsilon = 0.01$, $h = 0.125$, and $a = (x + \frac{1}{2})^2$.

Our method is closely related to the well-known WKB-approximation (cf. [2]) for the singularly perturbed ODE (1). The WKB-ansatz

$$\varphi(x) = \exp\left(\frac{1}{\varepsilon}\sum_{p=0}^{\infty}\varepsilon^p\phi_p(x)\right), \tag{2}$$

inserted in (1), leads after comparison of the ε^p-terms to

$$\phi_0(x) = \pm 1\int_0^x \sqrt{a(\tau)}\,d\tau + const., \tag{3}$$

$$\phi_1(x) = \ln a(x)^{-1/4} + const., \tag{4}$$

$$\phi_2(x) = \mp 1\int_0^x \beta(\tau)\,d\tau + const., \quad \beta := \frac{a''}{8a^{3/2}} - \frac{5(a')^2}{32a^{5/2}}. \tag{5}$$

Based on these ideas, we shall sketch now briefly the strategy of the construction of our "asymptotically correct" scheme, devided into three steps. It is closely related to the procedure in [3], but yields a refinement to higher ε–order:

(1) *Analytic pre-processing* of (1) by a second order WKB-transformation of the form (3)-(5). The equation (1) is transformed into a smoother problem that can be solved accurately and efficiently on a coarse grid (see Fig. 1), i.e.

$$\begin{cases} \dfrac{dZ}{dx} = \varepsilon N^{\varepsilon} Z\,, \quad 0 < x < 1\,, \\ Z(0) = Z_I\,, \end{cases} \tag{6}$$

where N^ε is bounded independently on ε. The transformation from the "smooth" unknown Z to the oscillatory unknown U is given by

$$U(x) = P^{-1} e^{\frac{i}{\varepsilon}\Phi^\varepsilon(x)} Z(x)\,, \quad U(x) := \begin{pmatrix} a^{1/4}\varphi(x) \\ \dfrac{\varepsilon(a^{1/4}\varphi)'(x)}{\sqrt{a(x)}} \end{pmatrix}, \tag{7}$$

and

$$P := \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}, \quad \Phi^\varepsilon(x) = \int_0^x \left(\sqrt{a(\tau)} - \varepsilon^2\beta(\tau)\right) d\tau \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(2) ε–uniform *discretization of the oscillatory integral* $\int \beta(y) \exp\left(\frac{2i}{\varepsilon}\phi(y)\right) dy$ (and multiple iterates of it) appearing in the numerical scheme for the transformed, smoother problem (6).

(3) *Numerical integration of the phase* $\Phi(x)^\varepsilon$ in (7). Here, numerical errors of order $\mathcal{O}(h^\gamma)$ in the phase-computation will typically induce $\mathcal{O}(h^\gamma/\varepsilon)$ errors in the oscillatory integral. However, this phase integral can be computed explicitly in several relevant examples (e.g. RTDs). But even then, machine precision round-off errors will also introduce "small" $\mathcal{O}(1/\varepsilon)$ errors.

Following this procedure, first and second order schemes were constructed and a detailed numerical analysis permits to show that the global errors of the first order scheme satisfy

$$||Z(x_n) - Z_n|| \le C\varepsilon^2 \min(\varepsilon, h)\,, \quad 1 \le n \le N\,, \tag{8}$$

$$||U(x_n) - U_n|| \le C\frac{h^\gamma}{\varepsilon} + C\varepsilon^2 \min(\varepsilon, h)\,, \quad 1 \le n \le N\,. \tag{9}$$

whereas for the second order scheme one has the estimates

$$||Z(x_n) - Z_n|| \le C\varepsilon^3 h^2\,, \quad ||U(x_n) - U_n|| \le C\frac{h^\gamma}{\varepsilon} + C\varepsilon^3 h^2\,, \quad 1 \le n \le N\,, \tag{10}$$

with C independent of n, h, and ε. Here, $\gamma > 0$ is the order of the chosen numerical integration method for computing of the phase integral Φ^ε.

The errors obtained via the here introduced "asymptotically correct" numerical scheme are plotted in Figure 2. This *asymptotic correctness* w.r.t. ε is an additional novel feature of our scheme: RTD-models typically have piecewise linear potentials. Hence, the phase function Φ^ε can be integrated exactly. As a consequence, the numerical error decreases to zero as $\varepsilon \to 0$, even when using a fixed step size $h > 0$. Hence, the scheme with e.g. just 2 grid points on $[0, 1]$ becomes asymptotically correct in the highly oscillatory limit, which was a-priori the most difficult scenario.

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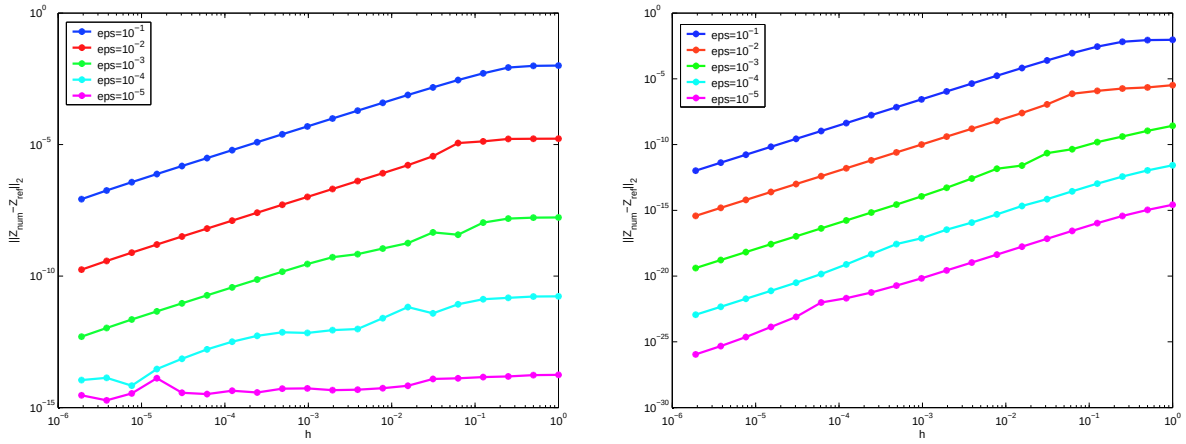

FIGURE 2. Absolute error (in the $L^2(0,1)$–norm and log – log scale) between the computed solution Z_{num} and a reference solution Z_{ref} as a function of h and for several ε-values. Left: first order scheme. Right: second order scheme.

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Logarithmic Sobolev Inequality for a conservative spin system with single-site potentials of arbitrary super-quadratic growth

FELIX OTTO
(joint work with Georg Menz)

We consider an Ising model on a lattice of $N \gg 1$ sites, with unbounded spin space, say $\mathbb{R}$, and conservation of the mean spin m. Hence the configuration space is given by $X_{N,m} := \{(x_1, \cdots, x_N) \in \mathbb{R}^N | \frac{1}{N}\sum_{n=1}^N x_n = m\}$. We consider a Hamiltonian with no interaction and thus determined by the single-site potential $\psi(x)$. Such an ensemble

$$d\mu_{N,m} = \frac{1}{Z}\exp(-\sum_{n=1}^N \psi(x_n))\, dx_1 \cdots dx_{N|_{X_{N,m}}} \tag{1}$$

is natural for the continuous version of spin-exchange dynamics (Kawasaki dynamics), that conserve the mean spin.

We are interested in the Logarithmic Sobolev Inequality (LSI). A measure μ on a Euclidean space $X_{N,m}$ is said to satisfy LSI with constant $\rho > 0$, provided

$$\forall\, f(x) \geq 0 \text{ s. t. } \int f\, d\mu = 1: \quad \int f \ln f\, d\mu \leq \frac{2}{\rho}\int \frac{1}{f}|\nabla f|^2\, d\mu. \tag{2}$$

LSI is a nonlinear version of the Spectral Gap (SG) estimate

$$\forall\, f(x) \text{ s. t. } \int f\, d\mu = 0: \quad \int f^2\, d\mu \leq \frac{1}{\rho}\int |\nabla f|^2\, d\mu,$$

that bounds the spectral gap of the generator of the corresponding reversible dynamics on X. One of the advantages of LSI over SG becomes apparent when dealing with hydrodynamic limits: The entropy $\int f_0 \ln f_0 \, d\mu$ of the initial data f_0 is an *extensive* quantity in the number N of sites, so that it makes sense to assume that the initial specific entropy $\frac{1}{N} \int f_0 \ln f_0 \, d\mu$ is bounded for $N \uparrow \infty$, whereas such a statement would be meaningless for $\int f_0^2 \, d\mu$. For the system $d\mu_{N,m}$ at hand, the fact that LSI holds uniformly in N and m has been used in [2] to reprove a hydrodynamic limit by Guo & Papanicolaou & Varadhan.

Landim & Panzio & Yau [1] proved that $\mu_{N,m}$ satisfies LSI uniformly in the number of sites N and the prescribed mean m, provided that the single site-potential $\psi(x)$ is *perturbed quadratic*, i. e.

$$\psi(x) \;=\; \frac{1}{2}x^2 + \delta\psi(x) \quad \text{with} \quad |\delta\psi| + |\delta\psi'| + |\delta\psi''| \;\lesssim\; 1. \tag{3}$$

Note that the standard criteria for LSI do not apply to $\mu_{N,m}$: Because the mean spin is constrained to m, the tensorization principle of Gross does not apply; because the single-site potential is non convex, the Bakry-Emery criterion does not apply; the Holley-Stroock criterion would not give uniformity in N.

Caputo [3] proved that $\mu_{N,m}$ satisfies SG uniformly in N and m provided that ψ is *perturbed strictly convex*, i. e.

$$\psi \;=\; \psi_0 + \delta\psi \quad \text{with} \quad \psi_0'' \;\gtrsim\; 1 \quad \text{and} \quad |\delta\psi| + |\delta\psi'| \;\lesssim\; 1. \tag{4}$$

We establish

Theorem 6. *The measure $\mu_{N,m}$, c. f. (1), satisfies LSI, c. f. (2), uniformly in N and m provided that ψ satisfies (4).*

Note that (4) (as opposed to (3)) includes the standard potential $\psi(x) = \frac{1}{4}(x^2 - 1)^2$; the statement was unknown even for the convex $\psi(x) = x^4$.

For the proof of Theorem 1, we adapt the coarse-graining argument used in the perturbed quadratic case [2]. More precisely, we coarse grain via block spins, and carry out a finite number of coarse-graining steps, where at each step, a block is formed by two sites. The first new ingredient is an asymmetric Brascamp-Lieb inequality. It is needed to transfer LSI from a coarse-grained level to the next finer level.

Proposition 2. *Let ψ be perturbed strictly convex, cf. (4). The covariance $\mathrm{cov}_\mu[f;g]$ of two functions f and g w. r. t. the measure $d\mu = \frac{1}{Z}\exp(-\psi(x))dx$ satisfies*

$$|\mathrm{cov}_\mu[f;g]| \;\le\; \exp(6\,\mathrm{osc}|\delta\psi|) \int |f'| \, d\mu \; \sup \frac{|g'|}{\psi_0''}. \tag{5}$$

Recall that the traditional Brascamp-Lieb inequality states that in case of $\delta\psi \equiv 0$, one has $|\mathrm{cov}_\mu[f;g]|^2 \;\le\; \int \frac{1}{\psi_0''}|f'|^2 \, d\mu \int \frac{1}{\psi_0''}|g'| \, d\mu$. The merit of (5) is the *asymmetric* distribution of the weight $\frac{1}{\psi_0''}$ over the two factors.

The second new ingredient is the following quantitative version of the Central Limit Theorem. Via Cramèr's transform, it is used to derive strict convexity of the coarse-grained Hamiltonian after $N = 2^K$ coarse-graining steps.

Proposition 3. *Let ψ be perturbed strictly convex, cf. (4). For any σ consider the measure $d\mu_\sigma = \frac{1}{Z}\exp(\sigma x - \psi(x))dx$. Consider N independent random variables $X_1, \cdots, X_N$ identically distributed according to μ_σ. Let g_N denote the Lebesgue density of the distribution of the normalized sum $\frac{1}{\sqrt{N}}\sum_{n=1}^{N}\frac{X_n - m}{s}$, where m and s^2 denote the mean and variance of μ_σ. Then $g_N(0)$ converges for $N \uparrow \infty$ to the corresponding value for the normalized Gaussian:*

$$|g_N(0) - \frac{1}{\sqrt{2\pi}}| + |\frac{1}{s}\frac{d}{d\sigma}g_N(0)| + |(\frac{1}{s}\frac{d}{d\sigma})^2 g_N(0)| \lesssim \frac{1}{\sqrt{N}}. \tag{6}$$

The merit of (6) is the uniformity in the "field strength" σ up to second derivatives.

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Two scaling limits: from particles to kinetic equations

Mario Pulvirenti

The physical system of interest is a thin layer (the vertical dimension is much smaller than the orizontal one) of a rarefied gas described by an hard sphere system. We want to derive (in term of the underlying particle system) for such a system, a two-dimensional Boltzmann equation

$$(\partial_t + v \cdot \nabla_y)g(y, v; t) = Q(g, g)(y, v; t)$$

where $Q(g, g)(y, v; t)$ is the usual collision operator and $(y, v) \in \mathbf{R}^2 \times \mathbf{R}^3$. Note that the space variable is two-dimensional while the velocity variable lives in the three-dimensional space.

The particle system consists of N hard spheres of diameter $\varepsilon > 0$. The one-particle phase space is

$$\{x.v | x = (y, q) \in \mathbf{R}^2 \times T_\varepsilon, v \in \mathbf{R^3}\}$$

$$T_\varepsilon = (-\frac{\varepsilon^\gamma}{2}, \frac{\varepsilon^\gamma}{2}]$$

with periodic boundary conditions.

The low density regime means that the density (in macroscopic unities) is $O(\varepsilon^{-2})$. In the usual Boltzmann-Grad limit N = density $\times$ vol=$O(\varepsilon^{-2})$. Here N = density $\times$ vol=$O(\varepsilon^{-2+\gamma})$.

This ensures that each particle has a finite number of collissions in a unitary time. Hence the Boltzmann-Grad limit in this context is

$$\varepsilon \to 0, \quad N \to \infty, \quad N\varepsilon^{2-\gamma} = 1.$$

We denote by $f_j^\varepsilon(t)$ the $j-$ particle marginals of the particle system and by $g_j(t) = g(t)^{\otimes}$ products of solution to the Boltzmann eqaution.

Under suitable assumptions on the initial state for the particle system, we can prove:

Theorem *There exist $t_0 > 0$ such that, for $t < t_0$,*

$$\lim_{\varepsilon \to 0} \int_{T_\varepsilon} dq[\, g_j^\varepsilon(t) - f_j^\varepsilon(t)]$$

a.e.. In particular we have propagation of chaos.

The main difference with the classical Lanford's result is that here we need an explicit estimate of the recollisions. By geometrical and mechanical considerations one can prove that they can be estimated by

$$C_\mu \varepsilon^\mu \varepsilon^{-\gamma}$$

for any $\mu < 1$. Hence it must be $\gamma < 1$. It is actually necessary as it is ahown by an explicit counterexample.

This result has been obtained in collaboration with R. Esposito (see [1]).

The second example of particle approximation I am going to discuss, is a set of interacting Brownian motions. There is only the velocity variable $v \in \mathbf{R^3}$. The time evolution for a probability distributions $W^N = W^N(v_1 \dots v_N)$ is given by

$$\partial_t W^N = \tilde{L}^N W^N$$

where

$$\tilde{L}^N = div_{V_N} B \cdot \nabla_{V_N} W^N,$$
$$B : \mathbf{R}^{3N} \to \mathbf{R}^{3N \times 3N}$$

is the matrix

$$B_{i,j}(V_N) = -\frac{a(v_i - v_j)}{N}, \quad i \neq j,$$
$$B_{i,i}(V_N) = \frac{1}{N} \sum_j a(v_i - v_j),$$

where the 3×3 matrix a is given by

$$a(w) = \frac{1}{|w|}(1 - \hat{w} \otimes \hat{w}) = \frac{1}{|w|} P(w), \quad w \in \mathbf{R}^3,$$

and $\hat{w} = \frac{w}{|w|}$, with $P(w)$ the orthogonal projection on the plane orthogonal to w.

This model is obtained as the grazing collision limit of the Kac mean-field model for the Boltzmann equation.

Unfortunately t $\tilde{L}^N$ is not uniformly elliptic and not smooth, due to the divergence for $|v_i - v_j| \approx 0$. Then we slightly modify $\tilde{L}^N$ to obtain

$$L^N = div_{V_N}(B^N \nabla_{V_N})$$

where B^N is obtained by making the matrix B smooth and bounded from below. This regularization vanishes when $N \to \infty$.

We (this is a work in collaboration with E. Miot and C. Saffirio) prove that, in the limit $N \to \infty$, the j-particle marginals of the measure W^N converge weakly to a family of marginals which satisfy a weak form of the Landau hierarchy.

The technique of the proof (see ref. [2]), is based on a previous result due to C. Villani [3], for the study of the Landau equation with Coulomb kernel.

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Aggregation equations: Stationary states and stability analysis

Gael Raoul

(joint work with Klemens Fellner)

We present a qualitative study (from works [3, 4, 7]) of the so-called aggregation equation:

$$\partial_t \rho(t,x) = \nabla \cdot (\nabla W *_x \rho(t,x) + V(x)) \,, \tag{1}$$

where $V \in C^2$, $W \in W^{1,\infty}$, $W|_{\{0\}^c} \in C^2$ and W is radially symmetric. This type of equation appears in many biological or physical models, with various singularities of W at the origin :

- W has an attractive singularity in Chemiotaxis models (see [1]),
- W is regular in cell bio-mechanical models (see [6]),
- W has a repulsive singularity in swarming models (see [5]).

We have tried to understand the dynamics of the solution ρ of (1), and how this dynamics depends on the singularity of W at the origin. Our study is done in dimension 1 only.

(1) is a gradient flow equation associated to the energy $E(\rho) = \int\int W(x-y)\, d\rho(x)\, d\rho(y) + \int V(x)\, d\rho(x)$. This provides an existence theory for (1) (see [2]), when W is not too singular. Moreover, E is a strict entropy, we can thus only expect the convergence of the solution to a steady-state (or a set of steady-states).

If W is regular or attractive, we show that stable steady-states are generically sums of Dirac masses: If a steady-state $\bar{\rho}$ had an L^1 part, we could find an arbitrarily small perturbation $\tilde{\rho}$ of $\bar{\rho}$ with a smaller energy.

If W has a repulsive singularity, then, we can show that for any bounded initial data, the solution of (1) remains uniformly bounded for all times. The stable steady-states can thus only be L^∞ functions in this case.

We have then investigated the connection between those two cases. We have considered an interaction potential W having a repulsive singularity, and smoothed versions W^ε of W. Under some assumptions, we have shown that the steady solutions of the smoothed problem converge weakly to the steady-state of the initial singular problem, when the smoothing parameter ε goes to 0. In particular, this shows that a simple double well potential can lead to steady-states consisting of an arbitrarily large number of Dirac masses.

Finally, we have characterised stables steady solutions of (1) consisting of a finite number of Dirac masses, for regular or attractive interaction potentials. We have recognise two simple stability conditions, which are necessary for the linear stability of the steady-state, and imply its local nonlinear stability.

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Asymptotic behaviour of degenerate linear transport equations

Francesco Salvarani

(joint work with Laurent Desvillettes)

In this presentation we have shown the results of our paper [1].

We consider non-homogeneous (in space) transport equations of the type

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \sigma(x)\,(\bar{f} - f), \tag{1}$$

where $f := f(t, x, v)$ is the density of particles which at time t and point x move with velocity v. Here $\bar{f}(t, x) = \int_V f(t, x, v)\, dv$, where V is a bounded set (of $\mathbb{R}^d$) of velocities of measure 1. The right-hand side of Equation (1) describes a process of isotropization of the velocities of the particles. This process has an intensity

$\sigma(x) \geq 0$ which is not necessarily bounded below by a strictly positive constant (in the vocabulary of radiative transfer, the points belonging to the set $\{x \,:\, \sigma(x) = 0\}$ would correspond to points of transparency).

For the sake of simplicity, we shall systematically consider that the solutions are periodic (of period 1) in all components of x, that is $x \in \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, and defined for all nonnegative times.

We finally introduce initial data

$$f(0,x,v) = f_0(x,v). \tag{2}$$

We shall also consider a simplified one-dimensional model of (1), in which the velocities are $v = \pm 1$. This is a variant of the well-known Goldstein-Taylor model, which describes the behavior of a gas composed of two kinds of particles moving parallel to the x-axis with constant speeds, of equal modulus $c = 1$, one in the positive x-direction with density u, the other in the negative x-direction with density v. The corresponding system of equations is:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = \sigma(x)(v-u) \\ \\ \dfrac{\partial v}{\partial t} - \dfrac{\partial v}{\partial x} = \sigma(x)(u-v), \end{cases} \tag{3}$$

where $u := u(t,x)$, $v := v(t,x)$, $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, $t \geq 0$.

Such set of equations will satisfy the initial conditions

$$u(0,x) = u_0(x), \qquad v(0,x) = v_0(x). \tag{4}$$

If σ were bounded from below by a strictly positive constant, then a variant of the strategy proposed by Mouhot and Neumann in [4] would lead to prove the exponential decay (with explicit rates) in time of the solutions of Equation (1) or System (3) towards the unique equilibrium state of the system.

However, this result has no obvious extension in the case of a vanishing cross section (even if such a degeneracy happens at only one point). A reasonable conjecture is that when the equilibrium is still unique, then some explicit (non necessarily exponential) rate should still exist.

Our goal is to prove this property under reasonable assumptions on the cross section. More precisely, we shall suppose that it satisfies the properties given in the following assumption:

Assumption 1: Let $\sigma : \mathbb{T}^d \to \mathbb{R}_+$ be a function satisfying the following property: there exist $x_i \in \mathbb{T}^d$, $i = 1, \ldots, N$, $C_\sigma > 0$ and $\lambda_\sigma > 0$ such that

$$\text{for a.e. } x \in \mathbb{T}^d, \qquad \sigma(x) \geq C_\sigma \inf_{i=1,\ldots,N} |x - x_i|^{\lambda_\sigma}.$$

Our results are summarized in the following theorems:

Theorem 1: *Consider the linear transport model* (1)-(2) *in the domain* $\mathbb{T}^d$ *(*$d \in \mathbb{N}$*) with a cross section* $\sigma \in L^\infty \cap H^1(\mathbb{T}^d)$ *satisfying Assumption* 1 *and* $f_0 \geq 0$ *a.e. such that* $f_0 \in L^\infty(\mathbb{T}^d \times V)$, $\nabla_x \bar{f}_0 \in L^2(\mathbb{T}^d)$, *and* $v \otimes v : \nabla_x \nabla_x f_0 \in L^2(\mathbb{T}^d \times V)$.

Then there exists a unique nonnegative solution $f := f(t,x,v)$ *to this system in* $C(\mathbb{R}_+; L^2(\mathbb{T}\times V))$. *The solution* f *converges when* $t \to +\infty$ *to its asymptotic profile*

$$f_\infty(x,v) := \int_{\mathbb{T}^d}\int_V f_0(y,w)\,dwdy.$$

Moreover, the following estimate holds:

$$||f(t,\cdot,\cdot) - f_\infty||^2_{L^2(\mathbb{T}\times V)} \leq C_1\, t^{-\frac{1}{1+2\,\lambda_\sigma}} \tag{5}$$

where C_1 *is a constant depending on* C_σ, λ_σ, $||\sigma||_{H^1(\mathbb{T})\cap L^\infty(\mathbb{T})}$, *and* f_0, *which can be explicitly estimated in terms of those quantities.*

Theorem 2: *Consider the generalized Goldstein-Taylor model* (3)-(4) *in the domain* $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ *with a cross section* $\sigma \in H^1(\mathbb{T})$ *satisfying Assumption* 1 *and with initial conditions* (u_0, v_0) *in* $H^2(\mathbb{T}) \times H^2(\mathbb{T})$ *such that* u_0, $v_0 \geq 0$ *a.e..*

Then there exists a unique nonnegative solution $(u,v) := (u(t,x), v(t,x))$ *to this system in* $C(\mathbb{R}_+; L^2(\mathbb{T}))^2$. *This solution converges when* $t \to +\infty$ *to its asymptotic profile*

$$(u_\infty, v_\infty) := \left(\frac{1}{2}\int (u_0+v_0)\,dx, \frac{1}{2}\int (u_0+v_0)\,dx\right).$$

Moreover, the following estimate holds:

$$||u(t,\cdot) - u_\infty||^2_{L^2} + ||v(t,\cdot) - v_\infty||^2_{L^2} \leq C_2\, t^{-\frac{1}{1+\lambda_\sigma}}, \tag{6}$$

where C_2 *is a constant depending on* C_σ, λ_σ, $||\sigma||_{H^1(\mathbb{T})}$ *and* u_0, v_0, *which can be explicitly estimated in terms of those quantities.*

Finally, if the initial data (u_0, v_0) *belong to* $C^\infty(\mathbb{T}) \times C^\infty(\mathbb{T})$, *and if the cross section* σ *also lies in* $C^\infty(\mathbb{T})$, *then estimate* (6) *can be replaced by*

$$||u(t,\cdot) - u_\infty||^2_{L^2} + ||v(t,\cdot) - v_\infty||^2_{L^2} \leq C_3\, t^{-\frac{3}{\lambda_\sigma}+\delta}, \tag{7}$$

for any $\delta > 0$. *Here* C_3 *is a constant which now depends on* C_σ, λ_σ, δ, $||\sigma||_{W^{k,\infty}}$ *(for all* $k \leq k_0(\delta)$*) and* u_0, v_0, *which can be explicitly estimated in terms of those quantities.*

Note that it is not known if exponential (or even "almost exponential") convergence holds for these models. It is also not known if the method of hypocoercivity such as described (for example) in [4], [5], [6] can be used (though this seems likely). The proof presented here relies on the older method introduced in [2] and [3], based on the following proposition, that replaces Gronwall's lemma in the context of hypocoercive equations, whose proof is a direct consequence of Lemma 12 in [3]:

Proposition 1: *Let* z *and* y *be two nonnegative* C^2 *functions defined on* $\mathbb{R}_+$ *and satisfying (for all* $t > 0$*)*

$$\begin{cases} -z'(t) \geq \alpha_1\, y^{1+\delta}(t), \\ y''(t) \geq \alpha_3\, z(t) - \alpha_2 y^{1-\varepsilon}(t), \end{cases} \tag{8}$$

for some constants $\delta \geq 0$, $\varepsilon \in]0,1[$ *and* α_1, α_2, $\alpha_3 > 0$.

Then there exists a constant $\alpha_4 > 0$ depending only on $x(0)$, α_1, α_2, α_3, δ and ε such that (for all $t > 0$),

$$z(t) \leq \alpha_4 \, t^{-\frac{1-\varepsilon}{\delta+\varepsilon}}.$$

The different rates of convergence obtained in Theorems 1 and 2 reflect the possibility to use interpolations which have a different power, depending on the a priori smoothness of the solution of the equations.

The difference in the exponents appearing in (5) and (6) is due to the possibility to use, in the case of the Goldstein-Taylor model, some a-priori estimates on u_x and v_x, while no a-priori estimate is available for $\nabla_x f$ in the case of the non-homogeneous transport equation (1)-(2).

The ideas developed in this work are presented on very simple models on purpose. We think that they can be used for many variants of Equation (1), changing for example the boundary conditions, or the cross section.

There is no a-priori reason why it should not also work in nonlinear situations, provided that uniform in time smoothness estimates are known for the solution of the problem under study (such estimates are often difficult to obtain for general data, but they can sometimes be proven in special regimes).

Note that the challenging problems of cross sections σ such that $\sigma = 0$ on a set of strictly positive measure has not been treated here.

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Traveling waves of a kinetic transport model for the KPP-Fisher equation

Christian Schmeiser

(joint work with Carlota Cuesta, Sabine Hittmeir)

This note shortly reports on the results of [1]. The KPP-Fisher equation

$$\partial_t u - D\partial_x^2 u = \overline{\rho} u - u^2$$

with diffusivity $D > 0$ and equilibrium density $\overline{\rho} > 0$ models the chemical reaction $A + B \leftrightarrow 2A$ ($u = \rho_A \ll \rho_B$) and diffusion. It possesses traveling front wave

solutions $u_{TW}(\xi) \geq 0$, $\xi = x - st$, with $u_{TW}(-\infty) = \overline{\rho}$, $u_{TW}(\infty) = 0$, $u'_{TW} < 0$, for $s \geq s_0 := 2\sqrt{D\overline{\rho}}$.

These waves are stable under perturbations decaying faster than the wave as $\xi \to \infty$ (see, e.g., Sattinger (1976) for a L^∞-based result). For an L^2-setting the following can be proven:

Theorem: Let $s > s_0$, $u(x,0) \geq \gamma u_{TW}(x)$, $\gamma > 0$, and

$$\int (u(x,0) - u_{TW}(x))^2 (1 + e^{xs/D}) dx < \infty \,.$$

Then there exist $c, \lambda > 0$, such that

$$\int (u(x,t) - u_{TW}(x - st))^2 (1 + e^{(x-st)s/D}) dx \leq c e^{-\lambda t} \,.$$

Replacing diffusion by collisions with a (nonmoving) background medium leads to a kinetic model

$$\varepsilon^2 \partial_t f + \varepsilon v \, \partial_x f = Q_c(f) + \varepsilon^2 Q_r(f) \,,$$

with the *collision operator* $Q_c(f)(v) = \rho_f M(v) - f(v)$, $\rho_f = \int f \, dv$, where we assume

$$M \text{ even} \,, \quad \int_V M \, dv = 1 \,, \quad \int_V v^2 M \, dv = D \,,$$

and the *reaction operator*

$$Q_r(f) = \int \int [\bar{\rho} M M^* f' - M' f^* f] \, dv' dv^* = \rho_f (\bar{\rho} M - f)$$

Global existence and a comparison principle can be shown, where contraction in $C([0,T];\, L^\infty(dx\, dv/M))$ leads to local existence:

Theorem: Let $0 \leq f(x,v,0) \leq \hat{\rho} M(v)$. Then there is a unique mild solution $f \in C([0,\infty);\, L^\infty(R \times V))$, satisfying

$$0 \leq f(x,v,t) \leq \max\{\hat{\rho}, \bar{\rho}\} M(v) \,.$$

Lemma: Let $f_1(x,v,0) \geq \gamma f_2(x,v,0) \geq 0$, $0 \leq \gamma \leq 1$, and $f_2(x,v,0) \leq (\bar{\rho} + \varepsilon^{-2}) M(v)$. Then

$$f_1(x,v,t) \geq \gamma f_2(x,v,t) \quad \text{for all } x, v, t.$$

For $\varepsilon \ll 1$ solutions of the traveling wave problem

$$\varepsilon(v - \varepsilon s) \partial_\xi f = Q_c(f) + \varepsilon^2 Q_r(f)$$

$$f(-\infty, v) = \bar{\rho} M(v), \quad f(\infty, v) = 0$$

can be approximated by Chapman-Enskog expansion:

$$f_{as} = u_{TW} M - \varepsilon u'_{TW} v M + \varepsilon^2 u''_{TW} (v^2 - D) M \,,$$

where u_{TW} is a traveling wave of the KPP-Fisher equation. The approximation f_{as} satisfies the far-field conditions and solves the traveling wave equation up to an $O(\varepsilon^3)$ residual, whose v-integral vanishes.

The existence proof of traveling waves is based on three main ideas.

Step 1: The Caflisch-Nicolaenko-micro-macro decomposition of the error:

$$f(\xi, v) - f_{as}(\xi, v) = \varepsilon^2(z(\xi)\Phi(v) + \varepsilon w(\xi, v))$$

with

$$\Phi(v) = \left(1 + \frac{\varepsilon s}{D + \varepsilon^2 s^2}(v - \varepsilon s)\right) M(v)\,, \quad \int (v - \varepsilon s)^2 w\, dv = 0\,,$$

implying $\int (v - \varepsilon s)\Phi\, dv = 0$. Appropriate projections lead to equations for the macroscopic and microscopic solution components:

$$Dz'' + sz' + (\bar\rho - 2u_{TW})z = \varepsilon B^z[z, w] + \varepsilon^2 R^z[z, w] + h^z$$

$$\varepsilon(v - \varepsilon s)\partial_\xi w - Q_c(w) = A[z] + \varepsilon B^w[z, w] + \varepsilon^3 R^w[z, w] + h^w$$

Step 2: removal of the null space (again inspired by Caflisch-Nicolaenko (1982)): replace Q_c by

$$\overline{Q}(w) := Q_c(w) - (v - \varepsilon s)^2 M \int (v - \varepsilon s)^2 w\, dv\,.$$

Lemma: $-\overline{Q}$ is symmetric and coercive in $L^2(dv/M)$.

Lemma: Replacing Q_c by $\overline{Q}$ leads to an equivalent problem.

Step 3: solution of the linearized problem:

Lemma The operators

$$z \mapsto \left(Dz'' + sz' + (\bar\rho - 2u_{TW})z,\ z(0)\right)$$

$$w \mapsto \varepsilon(v - \varepsilon s)\partial_\xi w - \overline{Q}(w)$$

are boundedly (uniformly in ε) invertible.

Step 4: contraction for solving the nonlinear problem

Theorem: For $s \geq s_0$ and ε small enough, there exists a locally unique (up to translations) traveling wave solution f_{TW}, satisfying $f_{TW} = f_{as} + O(\varepsilon^2)$ and $0 \leq f_{TW}(\xi, v) \leq \bar\rho M(v)$.

Stability of traveling waves is proven by employing the Lyapunov functional

$$H(t) = \int (f(x, v, t) - f_{TW}(x - st, v))^2 (1 + \alpha e^{(x-st)s/D}) dx\, dv$$

Theorem: For $s > s_0$, ε small enough, α big enough, $H(0) < \infty$, and $f(x, v, 0) \geq \gamma f_{TW}(x, v)$ (with $0 < \gamma \leq 1$), there exist $c, \lambda > 0$ such that

$$H(t) \leq ce^{-\lambda t}$$

The proof uses the standard micro-macro decomposition of the deviation (see also Liu-Yu (2004)), the H-theorem for the collision operator and the macroscopic stability estimate.

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Gyrokinetic Modelling and Simulation

Eric Sonnendrücker

Energy can be obtained by fusion of a Deuterium and a Tritium atom, which are two isotopes of hydrogen. For this reaction to yield a positive energy balance, the particles need to be confined at a very high temperature of the order of a hundred million degrees at a high enough density for a long enough time. At this range of temperature the particles are fully ionized and in the plasma state. Hence, one way of confining the particles is to use a very large magnetic field in a toroidal vessel. Such a device is called a Tokamak. The international project ITER, that started in Cadarache, in the south of France, consist of building and operating a large Tokamak that should demonstrate the feasibility of energy production though fusion reactions.

Magnetic plasmas are very prone to instabilities and the success of ITER depends in particular on their control. One important issue is to understand the development of turbulence in the core of the plasma and its influence on energy confinement. Recent works in physics have emphasized that fluid models cannot account for the energy confinement time observed in experiments and that although more complex, kinetic models are necessary.

As the particle mean free path is very large in tokamaks, a collisionless Vlasov equation self-consistently coupled with Maxwell's equations is the right model to use on the turbulence time scale. However, the very large external magnetic field puts strong constraints on the particle dynamics, introducing in particular a very large cyclotron frequency, which is the frequency of rotation of the particles around the magnetic field lines. The resolution of cyclotron period would put unacceptable constraint on the time step in a numerical simulation. Therefore a reduced model, the so-called gyrokinetic model is used. It consists in averaging out the fast motion of particles around the magnetic field lines by getting an equation for the guiding centers of the trajectory an older straightforward derivation is given by Frieman and Chen [6]. For a survey on the physics literature on the subject see [3].

There has also been a rich mathematical literature in the last ten years on the subject, proposing different tools for the rigorous derivation of the gyrokinetic model or related models generally in some specific cases [1, 2, 5, 7, 8, 10, 11, 12, 13].

The gyrokinetic model takes its simplest form in slab geometry, in which the torus is approximated by a periodic cylinder, removing all curvature effects. In this case the confinement field can be a uniform magnetic field in the direction of the axis of the cylinder. We obtain a 5D model describing the evolution of the guiding center distribution $f(r, \theta, \phi, v_\parallel, \mu)$

$$\frac{\partial f}{\partial t} + v_D \cdot \nabla_x f + v_\parallel \cdot \nabla_\parallel f + \frac{q}{m} \mathbf{E}_\parallel \cdot \nabla_v f = 0,$$

with $v_D = -\frac{\nabla J(\phi) \times \mathbf{B}}{B^2}$. Generally in regimes where the gyrokinetic model applies, the scaled Debye length is very small and the plasma is quasi-neutral. Hence the self consistent electric field is obtained via its potential using the quasi-neutrality equation

$$-\nabla_\perp \cdot (\frac{n_0(r)}{B\omega_c} \nabla_\perp \phi) + \frac{e\, n_0(r)}{T_e(r)} (\phi - \lambda \langle \phi \rangle) = \int J(f)\, dv_\parallel d\mu - n_0.$$

The gyroaverage operator J transforms the guiding-center distribution onto the actual particle distribution enabling to take into account the finite Larmor radius, which is the radius of gyration of the particles around the magnetic field lines.

The gyroaverage operator is applied to a function g depending on the guiding center distribution. For a particle of velocity v, the Larmor radius is $\vec{\rho} = \frac{\vec{v}_\perp}{\omega_c} = (\rho \cos\alpha, \rho \sin\alpha)$ where $\vec{v}_\perp = (-v_y, v_x)$

$$J(g)(\vec{x}, \vec{v}) = \frac{1}{2\pi} \int_0^{2\pi} g(x + \rho\cos\alpha, y + \rho \sin\alpha)\, d\alpha.$$

A Fourier transform in $\vec{x}$ yields

$$\widehat{J(g)}(\vec{k}, \vec{v}) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\vec{k}\cdot\vec{\rho}}\, d\alpha\ \hat{g}(\vec{k}),$$

moreover denoting by $\vec{k}_\perp = (k_\perp \cos\beta, k_\perp \sin\beta)$,

$$\frac{1}{2\pi} \int_0^{2\pi} e^{i\vec{k}\cdot\vec{\rho}}\, d\alpha = \frac{1}{2\pi} \int_0^{2\pi} e^{i\rho k_\perp \cos(\alpha-\beta)}\, d\alpha = J_0(\rho k_\perp).$$

In a real torus, the gyrokinetic model takes the following form [3] that is used in most simulation codes

$$\frac{\partial f}{\partial t} + \frac{d\mathbf{X}}{dt} \cdot \nabla_x f + \frac{dV_\parallel}{dt} \frac{\partial f}{\partial v_\parallel} = 0,$$

with

$$\begin{aligned} B^* \frac{d\mathbf{X}}{dt} &= \mathbf{b} \times \nabla J(\phi) + \frac{1}{q}(mV_\parallel^2 \nabla \times \mathbf{b} + \mu \mathbf{b} \times \nabla B) + V_\parallel \mathbf{B} \\ B^* \frac{dV_\parallel}{dt} &= -(\mathbf{B} + \frac{m}{q} V_\parallel \nabla \times \mathbf{b}) \cdot (\frac{\mu}{m} \nabla B + \frac{q}{m} \nabla J(\phi)) \end{aligned}$$

and $B^* = B + \frac{m}{q} V_\parallel \nabla \times \mathbf{b} \cdot \mathbf{b}$.

Note that in this model is still hamiltonian. In particular, we have the relations We have the relations

$$\begin{aligned} \nabla \cdot (B^* \frac{d\mathbf{X}}{dt}) &= \nabla \cdot (\mathbf{b} \times \nabla J(\phi)) + \frac{1}{q} \nabla \cdot (\mathbf{b} \times \mu \nabla B) \\ &= \nabla J(\phi) \cdot \nabla \times \mathbf{b} + \frac{\mu}{q} \nabla B \cdot \nabla \times \mathbf{b} \end{aligned}$$

On the other hand

$$\frac{\partial}{\partial v_\parallel} (B^* \frac{dV_\parallel}{dt}) = -\nabla \times \mathbf{b} \cdot (\frac{\mu}{q} \nabla B + \nabla J(\phi)).$$

Hence the phase-space divergence vanishes, which leads to the conservatity. So that the model is conservative, which is an essential point to take into account when designing a numerical method.

The Gysela code we are developing is based on a semi-Lagrangian method [14, 9]. We have recently introduced a conservative formalism for this method and linked it to the classical method [4]. Specific limiters for the gyrokinetic model in the conservative form have also been developed in this work.

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Bohmian measures and their classical limit

Christof Sparber
(joint work with Thierry Paul, Peter Markowich)

We consider the time-evolution of quantum mechanical wave functions $\psi^\varepsilon(t,\cdot) \in L^2(\mathbb{R}^d;\mathbb{C})$ governed by the Schrödinger equation:

$$i\varepsilon\partial_t\psi^\varepsilon = -\frac{\varepsilon^2}{2}\Delta\psi^\varepsilon + V(x)\psi^\varepsilon, \quad \psi^\varepsilon(t=0,x) = \psi_0^\varepsilon \in L^2(\mathbb{R}^d), \tag{1}$$

where $x \in \mathbb{R}^d$, $t \in \mathbb{R}$, and $V \in L^\infty(\mathbb{R}^d;\mathbb{R})$ a given bounded potential (satisfying some additional regularity assumptions). In addition, we have rescaled all physical parameters such that only one *semi-classical parameter* $0 < \varepsilon \leq 1$ remains. In quantum mechanics one defines out of $\psi^\varepsilon(t,x) \in \mathbb{C}$ real-valued observable densities from. Possibly, the two most important such densities are the *position* and the *current-density*, given by

$$\rho^\varepsilon(t,x) = |\psi^\varepsilon(t,x)|^2, \quad J^\varepsilon(t,x) = \varepsilon\mathrm{Im}\big(\overline{\psi^\varepsilon}(t,x)\nabla\psi^\varepsilon(t,x)\big). \tag{2}$$

In *Bohmian mechanics* [2, 3], one defines an ε-dependent flow-map $X_t^\varepsilon : x \mapsto X^\varepsilon(t,x)$ via the following differential equation

$$\dot{X}^\varepsilon(t,x) = u^\varepsilon(t, X^\varepsilon(t,x)), \quad X^\varepsilon(0,x) = x \in \mathbb{R}^d,$$

where the velocity field u^ε is (formally) given by

$$u^\varepsilon(t,x) := \frac{J^\varepsilon(t,x)}{\rho^\varepsilon(t,x)} = \varepsilon\mathrm{Im}\left(\frac{\nabla\psi^\varepsilon(t,x)}{\psi^\varepsilon(t,x)}\right)$$

and the initial data is assumed to be distributed according to $\rho_0^\varepsilon(x) \equiv |\psi_0^\varepsilon(x)|^2$. It has been rigorously proved in [1] that $X^\varepsilon(t,\cdot)$ is for all $t \in \mathbb{R}$ well-defined $\rho_0^\varepsilon - a.e.$ and that $\rho^\varepsilon(t,x) = X_t^\varepsilon \# \rho_0^\varepsilon(x)$, i.e. $\rho^\varepsilon(t,x)$ is the *push-forward* of the initial density $\rho_0^\varepsilon(x)$ under the mapping $X_t^\varepsilon : x \mapsto X^\varepsilon(t,x)$. This can be seen as the Eulerian viewpoint of Bohmian mechanics.

Bohmian mechanics can be reformulated in its Lagrangian form, by using the concept of *Bohmian measures*, recently introduced by the authors in [8]:

Definition 1. *For $\psi^\varepsilon \in H^1(\mathbb{R}^d)$, with associated densities $\rho^\varepsilon, J^\varepsilon$ as in (2), and a given $\varepsilon > 0$, we define the corresponding* Bohmian measure $\beta^\varepsilon \equiv \beta^\varepsilon[\psi^\varepsilon] \in \mathcal{M}^+(\mathbb{R}^d_x \times \mathbb{R}^d_p)$ *via*

$$\langle\beta^\varepsilon, \varphi\rangle := \int_{\mathbb{R}^d} \rho^\varepsilon(x)\varphi\left(x, \frac{J^\varepsilon(x)}{\rho^\varepsilon(x)}\right)dx, \quad \forall\,\varphi \in C_0(\mathbb{R}^d_x \times \mathbb{R}^d_p),$$

where $C_0(\mathbb{R}^d_x \times \mathbb{R}^d_p)$ denotes the space of continuous function vanishing at infinity and $M^+(\mathbb{R}^d_x \times \mathbb{R}^d_p)$ the set of non-negative Radon measures on phase space.

It has been shown shown in [8] that if $\psi^\varepsilon(t,x)$ solves (1), then the corresponding Bohmian $\beta^\varepsilon(t,x,p)$ measure is the push-forward of

$$\beta^\varepsilon[\psi_0^\varepsilon] \equiv \beta_0^\varepsilon(x,p) = \rho_0^\varepsilon(x)\delta(p-u_0(x)), \tag{3}$$

under the ε-dependent phase space flow $\Phi_t^\varepsilon : (x,p) \mapsto (X^\varepsilon(t,x,p), P^\varepsilon(t,x,p))$ induced by

$$\begin{cases} \dot{X}^\varepsilon = P^\varepsilon, \\ \dot{P}^\varepsilon = -\nabla V(X^\varepsilon) - \nabla V_B^\varepsilon(t,X^\varepsilon), \end{cases} \tag{4}$$

where $V_B^\varepsilon(t,x)$, denotes the so-called *Bohm potential*

$$V_B^\varepsilon(t,x) := -\frac{\varepsilon^2}{2}\frac{\Delta\sqrt{\rho^\varepsilon(t,x)}}{\sqrt{\rho^\varepsilon(t,x)}}.$$

More precisely, under mild regularity assumptions on V, the flow Φ_t^ε is shown to exists globally in time for almost all $(x,p) \in \mathbb{R}^{2d}$, *relative to the measure* β_0^ε and is continuous in time on its maximal open domain, cf. [8, Lemma 2.5].

The fact that $\beta^\varepsilon(t) = \Phi_t^\varepsilon \# \beta_0^\varepsilon$ makes it a natural starting point for investigations of the *classical limit* $\varepsilon \to 0_+$ (of Bohmian mechanics). In [8] we were able to establish the existence of a limiting non-negative phase space measure $\beta(t) \in \mathcal{M}^+(\mathbb{R}_x^d \times \mathbb{R}_p^d)$, such that, after extracting an appropriate sub-sequence (denoted by the same symbol):

$$\beta^\varepsilon \overset{\varepsilon\to 0_+}{\longrightarrow} \beta \quad \text{in } C_{\mathrm{b}}(\mathbb{R}_t; \mathcal{M}^+(\mathbb{R}_x^d \times \mathbb{R}_p^d))\,\mathrm{w}-*.$$

If, in addition, $\sup_{0<\varepsilon\leq 1}(\|\psi^\varepsilon(t)\|_{L^2} + \|\varepsilon\nabla\psi^\varepsilon(t)\|_{L^2}) < +\infty$, one can prove that the limiting phase space measure $\beta(t)$ incorporates the classical limit of the particle and current density in the sense that

$$\rho^\varepsilon(t,x) \overset{\varepsilon\to 0_+}{\longrightarrow} \int_{\mathbb{R}^d} \beta(t,x,dp), \quad J^\varepsilon(t,x) \overset{\varepsilon\to 0_+}{\longrightarrow} \int_{\mathbb{R}^d} p\beta(t,x,dp). \tag{5}$$

Hereby the limits have to be understood in $\mathcal{M}^+(\mathbb{R}_x^d)\,\mathrm{w}-*$, uniformly on compact time-intervals $I \subset \mathbb{R}_t$. In other words, the limiting Bohmian measure $\beta(t)$ therefore yields the classical limit of the quantum mechanical position and current densities, by taking the zeroth and first moment with respect to $p \in \mathbb{R}^d$. This is analogous to the, by now classical, theory of Wigner measures $w(t) \in \mathcal{M}^+(\mathbb{R}_x^d \times \mathbb{R}_p^d)$ developed in [7] and [5].

Several sufficient conditions for having $\beta = w$ have been established in [8]. In addition, we show that $\beta \neq w$ in general by means of several examples. Finally, in the follow-up work [9], we were able to prove that in the case where $\psi^\varepsilon(t)$ is a so-called *semi-classical wave packet* [4, 6], the re-scaled Bohmian trajectories

$$Y^\varepsilon(t,y) = X^\varepsilon(t, x_0 + \sqrt{\varepsilon}y), \quad Z^\varepsilon(t,y) = P^\varepsilon(t, x_0 + \sqrt{\varepsilon}y),$$

satisfy:

$$Y^\varepsilon \overset{\varepsilon\to 0_+}{\longrightarrow} X, \quad Z^\varepsilon \overset{\varepsilon\to 0_+}{\longrightarrow} P,$$

locally in measure on $\mathbb{R}_t \times \mathbb{R}^d_x$, where (X, P) are the classical particle trajectories induced by the Hamiltonian system

$$\begin{cases} \dot{X} = P, \quad X(0) = x_0, \\ \dot{P} = -\nabla V(X), \quad P(0) = p_0. \end{cases} \tag{6}$$

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Around the Boltzmann equation without angular cut-off

Robert M. Strain

(joint work with Philip T. Gressman)

In this report, we will describe briefly several recent developments [2, 3, 4, 5] for the Boltzmann equation without the Grad angular cut-off assumption [1]:

$$\frac{\partial F}{\partial t} + v \cdot \nabla_x F = \mathcal{Q}(F, F), \quad F(0, x, v) = F_0(x, v). \tag{1}$$

Here the unknown is $F = F(t, x, v) \geq 0$ with $t \geq 0$. The spatial coordinates we consider are $x \in \Omega$ ($\Omega \in \{\mathbb{T}^n, \mathbb{R}^n\}$), and the velocities are $v \in \mathbb{R}^{\dim}$ with $n \geq 2$. The Boltzmann collision operator, $\mathcal{Q}$, acts only on the velocity variables, v, as

$$\mathcal{Q}(G, F)(v) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} dv_* \int_{\mathbb{S}^{n-1}} d\sigma \; B(v - v_*, \sigma) \left[G(v'_*) F(v') - G(v_*) F(v) \right]. \tag{2}$$

Above the velocities of a pair of particles before and after collision are connected by $v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma$ and $v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma$, where $\sigma \in \mathbb{S}^{n-1}$. The collision kernel, $B(v-v_*, \sigma)$, depends upon the deviation angle θ through $\cos\theta = (v-v_*)\cdot\sigma/|v-v_*|$. Furthermore $B(v - v_*, \sigma)$ can be taken to be zero for $\theta > \frac{\pi}{2}$.

We suppose that $B(v-v_*,\sigma) = \Phi(|v-v_*|)\, b(\cos\theta)$ where b and Φ are non-negative. The angular function is not locally integrable; for $c_b > 0$ it satisfies

$$\frac{c_b}{\theta^{1+2s}} \le \sin^{n-2}\theta\; b(\cos\theta) \le \frac{1}{c_b\theta^{1+2s}}, \quad s\in(0,1), \quad \forall\,\theta\in\left(0,\frac{\pi}{2}\right]. \tag{3}$$

Additionally the kinetic factor satisfies for some $C_\Phi > 0$ that

$$\Phi(|v-v_*|) = C_\Phi |v-v_*|^\gamma, \quad \gamma > -n. \tag{4}$$

We distinguish between the cases $\gamma \ge -2s$, which are called "hard potentials", and the cases $-2s > \gamma > -n$, furthermore called "soft potentials" herein.

These collision kernels are physically motivated since they can be derived from an intermolecular repulsive potential such as $\phi(r) = r^{-(p-1)}$ with $p\in(2,\infty)$ as was shown by Maxwell in 1866. In the physical dimension ($n=3$), B satisfies the conditions above with $\gamma = (p-5)/(p-1)$ and $s = 1/(p-1)$. The vast majority of previous work requires the Grad angular cut-off assumption [1] from 1963 which usually means either $b(\cos\theta)\in L^\infty(\mathbb{S}^{n-1})$ or $b(\cos\theta)\in L^1(\mathbb{S}^{n-1})$. Neither of these assumptions are satisfied for singular angular factors such as (3).

The Boltzmann H-theorem is a hallmark of statistical physics. Define the H-functional by $H(t) \stackrel{\text{def}}{=} -\int_\Omega dx \int_{\mathbb{R}^{\dim}} dv\; f\log f$. Then the H-theorem predicts that, for solutions of the Boltzmann equation, the entropy is increasing over time:

$$\frac{dH(t)}{dt} = \int_\Omega dx\; D(f,f) \ge 0.$$

This is a demonstration of the second law of thermodynamics. The entropy production functional is $D(g,f) \stackrel{\text{def}}{=} -\int_{\mathbb{R}^{\dim}} dv\; \mathcal{Q}(g,f)\log f$. This functional is zero if and only if it is operating on a Maxwellian equilibrium. The prediction is thus that the Boltzmann equation exhibits irreversible dynamics and should experience convergence to Maxwellian in large time.

We will study the linearization of (1) around the Maxwellian equilibrium

$$F(t,x,v) = \mu(v) + \sqrt{\mu(v)}f(t,x,v), \tag{5}$$

where the Maxwellian is given by $\mu(v) \stackrel{\text{def}}{=} (2\pi)^{-n/2}e^{-|v|^2/2}$. We linearize the Boltzmann equation (1) around (5). This grants an equation for the perturbation:

$$\partial_t f + v\cdot\nabla_x f + L(f) = \Gamma(f,f), \quad f(0,x,v) = f_0(x,v), \tag{6}$$

where the bilinear operator, Γ, is given by $\Gamma(g,h) \stackrel{\text{def}}{=} \mu^{-1/2}\mathcal{Q}(\sqrt{\mu}g,\sqrt{\mu}h)$. Then the linearized collision operator, L, is defined as $L(g) \stackrel{\text{def}}{=} -\Gamma(g,\sqrt{\mu}) - \Gamma(\sqrt{\mu},g)$. The null space of L is: $N(L) \stackrel{\text{def}}{=} \operatorname{span}\left\{\sqrt{\mu},\; v\sqrt{\mu},\; (|v|^2-n)\sqrt{\mu}\right\}$. Now, for fixed (t,x), we denote the orthogonal projection from $L^2(\mathbb{R}^{\dim}_v)$ into $N(L)$ by $\mathbf{P}$.

In recent works [2, 3, 4], we introduced the norm: $|f|^2_{N^{s,\gamma}} \stackrel{\text{def}}{=} |f|^2_{L^2_{\gamma+2s}} + |f|^2_{\dot{N}^{s,\gamma}}$. Here, for $\ell\in\mathbb{R}$, we use the norm $|f|^p_{L^p_\ell} \stackrel{\text{def}}{=} \int_{\mathbb{R}^{\dim}} dv\; \langle v\rangle^\ell\; |f(v)|^p$ for $p = 1, 2$. The

weight is $\langle v\rangle \stackrel{\text{def}}{=} \sqrt{1+|v|^2}$. We also use the "dotted" semi-norm

$$|f|^2_{\dot{N}^{s,\gamma}} \stackrel{\text{def}}{=} \int_{\mathbb{R}^{\dim}} dv \int_{\mathbb{R}^{\dim}} dv' \ (\langle v\rangle \langle v'\rangle)^{\frac{\gamma+2s+1}{2}} \ \frac{(f(v')-f(v))^2}{d(v,v')^{n+2s}} \mathbf{1}_{d(v,v')\le 1}.$$

The fractional differentiation effects are measured using the anisotropic metric:

$$d(v,v') \stackrel{\text{def}}{=} \sqrt{|v-v'|^2 + \frac{1}{4}\left(|v|^2-|v'|^2\right)^2}.$$

Here the quadratic difference $|v|^2-|v'|^2$ is an essential component of the anisotropic fractional differentiation effects induced by the Boltzmann collision operator, which occur on a "lifted" paraboloid. This metric encodes the anisotropic changes in the power of the weight, which are entangled with the fractional differentiation effects.

It is known that $L \ge 0$ and $Lg = 0$ if and only if $g = \mathbf{P}g$. Our anisotropic space then sharply characterizes the Dirichlet form of the linearized collision operator as

$$\frac{1}{C}|\{\mathbf{I}-\mathbf{P}\}g|^2_{N^{s,\gamma}} \le \langle Lg, g\rangle \le C|\{\mathbf{I}-\mathbf{P}\}g|^2_{N^{s,\gamma}},$$

with a constructive constant $C > 0$. Above $\langle\cdot,\cdot\rangle$ is the standard $L^2(\mathbb{R}^{\dim}_v)$ inner product. It follows that a spectral gap exists if and only if $\gamma+2s \ge 0$. Furthermore

Theorem 7 ([4])**.** *The diffusive behavior of the operator* (2) *in* $L^2(\mathbb{R}^{\dim}_v)$ *is*

$$-\langle \mathcal{Q}(g,f), f\rangle \approx |f|^2_{\dot{N}^{s,\gamma}} + |f|^2_{L^2_\gamma} - \textit{l.o.t.}$$

Furthermore, the entropy production satisfies the estimate

$$D(g,f) \gtrsim |\sqrt{f}|^2_{\dot{N}^{s,\gamma}} + |f|^2_{L^1_\gamma} - \textit{l.o.t.}$$

In each statement above $g \gtrsim\neq 0$ *is a parameter function. The precise assumptions needed are in* [4]*. Also the lower order terms in "l.o.t." are non-differentiating.*

Since this is a three page report, we state our results without complete precision although we illustrate the main conclusions in detail. Otherwise we refer to the precise statements in [2, 3, 4, 5]. In the following we use H^K_ℓ to denote a weighted L^2 Sobolev space with K space-velocity derivatives and ℓ velocity weights.

Theorem 8 ([3], [5])**.** *Fix* $K \ge 2\lfloor \frac{n}{2}+1\rfloor$ *and* $\ell \ge 0$*. Suppose* (3) *and* (4)*. Choose initially* $f_0(x,v) \in H^K_\ell(\Omega\times\mathbb{R}^n)$ *in* (5)*. If* $\|f_0\|_{H^K_\ell}$ *is sufficiently small, then there exists a unique global classical solution to the Boltzmann equation* (1)*, in the form* (5)*. If* $\gamma+2s \ge 0$ *then, for some fixed* $\lambda > 0$*, we have exponential decay as*

$$\|f(t)\|_{H^K_\ell(\mathbb{T}^n\times\mathbb{R}^{\dim})} \lesssim e^{-\lambda t}\|f_0\|_{H^K_\ell(\mathbb{T}^n\times\mathbb{R}^{\dim})}.$$

When $\gamma < -2s$*, if* $\|f_0\|_{H^K_{\ell+m}}$ *is sufficiently small for* $\ell, m \ge 0$*, then we have*

$$\|f(t)\|_{H^K_\ell(\mathbb{T}^n\times\mathbb{R}^n)} \le C_m(1+t)^{-m}\|f_0\|_{H^K_{\ell+m}(\mathbb{T}^n\times\mathbb{R}^n)}.$$

We also have positivity, i.e. $F = \mu + \sqrt{\mu}f \ge 0$ *if* $F_0 = \mu + \sqrt{\mu}f_0 \ge 0$*.*

Theorem 8 provides a global existence theorem on Ω, and also shows rapid time decay on the torus $\mathbb{T}^n$ with $n \geq 2$. In the whole space, $\mathbb{R}^n$ with $n \geq 3$, the presence of dispersion slows down the decay rates. For $r \geq 1$, we define the mixed norm $\|g\|_{Z_r} \stackrel{\text{def}}{=} \left(\int_{\mathbb{R}^{\dim}} \left(\int_{\mathbb{R}^{\dim}} |g(x,v)|^r dx \right)^{2/r} dv \right)^{1/2}$. For $\ell \geq 0$, further set $\epsilon_{K,\ell} \stackrel{\text{def}}{=} \|f_0\|_{H^K_\ell} + \|f_0\|^2_{Z_1}$. We then have time decay rates in the Z_r norm:

Theorem 9 ([5]). *Let $f(t,x,v)$ be the solution to the Cauchy problem, $\Omega = \mathbb{R}^n$, of the Boltzmann equation from Theorem 8. Suppose $\epsilon_{K,\ell'(n)}$ is sufficiently small, where $\ell'(n) > 0$ is fixed. Then for any $2 \leq r \leq \infty$, we have the uniform estimate:*

$$\|f(t)\|_{Z_r} \lesssim (1+t)^{-\frac{n}{2}+\frac{n}{2r}}. \tag{7}$$

Furthermore $\|\{\mathbf{I}-\mathbf{P}\}f(t)\|_{Z_r} \lesssim (1+t)^{-\frac{n}{2}-\frac{1}{r}+\frac{n}{2r}}$. These hold for any $t \geq 0$.

These time decay rates for the Z_r-norms in (7) are optimal in the sense that they are the same as those for the linearized system, which is studied using Fourier analysis. These rates also coincide with those of the Boltzmann equation for hard-sphere particles, and they are the same in $L^r(\mathbb{R}^{\dim}_x)$ as those for the Heat equation.

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Free path length distributions of a Lorentz gas in a quasi crystal

Bernt Wennberg

The Lorentz gas is a mathematical model for the motion of (point) particles in e.g. a crystal, consisting of spherical, elastic scatterers of radius a with centers at a set of points $\Gamma_\varepsilon \subset \mathbb{R}^n$. A point particle moves in straight lines between the obstacles, on which it is specularly reflected. At least two very different point distributions, Γ_ε have been studied thoroughly: the standard lattice $\mathcal{L} \subset \mathbb{R}^n$, with interstitial distance ε, or a random distribution, where Γ_ε is Poisson distributed with intensity ε^{-n} (almost; one must assume a hard core condition, preventing the obstacles from overlapping).

Here we are interested in the so-called Boltzmann-Grad limit, in which one lets $a \to 0$ and $\varepsilon \to 0$ in such a way that $a\varepsilon^{-n}$ is constant, and the question is:

Suppose that we are given an initial distribution $f_0(x,v)$[1], $x \in \mathbb{R}^n$, $v \in S^{n-1}$, and we let $f^\varepsilon(x,v,t)$ denote the point distribution at time t under the dynamics described above. Does the sequence f^ε converge to some $f = f(x,v,t)$ when $\varepsilon \to 0$? If so, can one identify an equation that is satisfied by f?

Gallavotti [3] proved that when $n = 2$, and Γ_ε is Poisson distributed, the limit exists, and Ψ satisfies a linear Boltzmann equation,

$$(1)\quad \partial_t f(x,v,t) + v \cdot \nabla_x f(x,v,t) \;=\; \frac{1}{2}\int_{S^-} (f(x,v',t) - f(x,v,t))\,|\omega \cdot v|\,dv'$$

and similar results hold *e.g.* when is obtained from a periodic distribution by removing some of them, randomly, independently, provided that the correct obstacle density is maintained [7, 5]. In the purely periodic case, the result is false, as has been rigorously established e.g. in Caglioti,Golse [2] and in Marklof, Strmbergsson [4]. By different means the authors prove that the limit is again a linear transport equation, but in a larger phase space.

An important concept in these studies is the *mean free path*, and the distribution of free path lengths. The latter could be defined as

$$(2)\qquad \tau(x,v) = \inf\{t > 0 \,:\, x + vt \in \partial\Omega_\varepsilon\},$$

where $\Omega_\varepsilon = \mathbb{R}^n \setminus (\Gamma_\varepsilon + B_\varepsilon(0))$, and $B_R(0)$ is the closed ball of radius ε and center at the origin. Hence $\tau_\varepsilon(x,v)$ is the time until a particle starting at (x,v) hits an obstacle. When Γ_ε satisfies some kind of translation invariance, it makes sense to define the free-path length distribution as

$$(3)\ \phi_\varepsilon(]a,b[) = \lim_{R\to\infty} \frac{m\left(\{(x,v) \in (\Omega_\varepsilon \cap B_R(0)) \times S^{n-1} \,|\, \tau_\varepsilon(x,v) \in]a,b[\right)\}}{m\left((\Omega_\varepsilon \cap B_R(0)) \times S^{n-1}\right)},$$

where m is the Lebesgue measure in $\mathbb{R}^n$. ϕ_ε is a positive measure on $\mathbb{R}^+$, and if the Lorentz model converges to a linear Boltzmann equation, then the limiting path length distribution is exponentially decreasing. This is true for the random distribution of scatters, but not for the periodic Lorentz gas, where the limiting distribution satisfies $\phi(]T,\infty[) \sim T^{-1}$ (independently of the dimension n, see [1]).

In this talk I show a simulation result which indicates that the behaviour of a *quasi crystal* is similar to that of a periodic lattice.

A quasi crystal is, by definition, *a solid with an essentially discrete diffraction pattern exhibiting a symmetry forbidden by the crystallographic restriction* (See e.g. the book by M. Senechal [6]). For example, periodic crystals in $\mathbb{R}^3$ do not admit 5-fold symmetries. Experiments indicate that solids of this kind exist, and a matematical construction which yields an aperiodic structures with an essentially discrete diffraction pattern is the following: Let $\mathcal{L}$ be a lattice in $\mathbb{R}^k$, and let $\mathcal{E}$ be a n-dimensional subspace of $\mathbb{R}^k$, such that $\mathcal{E} \cap \mathcal{L} = \{0\}$ and $\mathcal{E}^\perp$ its orthogonal complement. Let Π and $\Pi^\perp$ be the orthogonal projections on $\mathcal{E}$ and $\mathcal{E}^\perp$ respectively.

[1]Of course, one must handle the fact that when ε is positive, there should be no particles inside the obstacles

If K is the Voronoi cell of the origin, let

$$X = \mathcal{L} \cap \left(\Pi^{\perp}(K) \oplus \mathcal{E}\right) , \tag{4}$$

and finally consider the set $\Pi(X) \subset \mathcal{E}$. This will be a non-periodic, discrete set, which is a candidate for a quasi crystal. To obtain a crystal with a five fold symmetry in $\mathbb{R}^2$, one may chose a two-dimensional subspace $\mathcal{E}$ of $\mathbb{R}^5$. With a construction similar to this, one may construct the famous Penrose tiling of the plane (see [6]).

It is now natural to study a Lorentz gas in an aperiodic crystal, and to determine whether the Boltzmann-Grad limit would yield a linear Boltzmann equation, as in the random case, or if the limiting procedure would give a result similar to the periodic case. The Boltzmann-Grad limit here consist of taking $\Gamma_\varepsilon = \sqrt{\varepsilon}\Pi(X)$.

A simple one-dimensional model, which actually corresponds exactly to the two-dimensional periodic Lorentz gas is as follows: On the line, take $\Omega_\varepsilon = \mathbb{R} \setminus (]-\varepsilon, \varepsilon[+\mathbb{Z})$. Chose $x_0, q \in]0, 1-\varepsilon[$ randomly (*e.g.* uniformly), and let $x_N = x_0 + Nq$. The discrete free path length is then

$$\tau_\varepsilon(x_0, q) = \min\{N > 0 \mid x_N \notin \Omega_\varepsilon\} \tag{5}$$

and

$$\phi_\varepsilon(]a, b[) = m(\{(x_0, q) \in]0, 1-\varepsilon[^2 \mid \tau_\varepsilon(x_0, q) \in]a/\varepsilon, b/\varepsilon[\}) \tag{6}$$

In the corresponding one-dimensional quasi-crystal, $\mathbb{Z}$ is replaced by a so-called Fibonacci sequence, in which the intervals between to points is either α or $\alpha + \beta$, where $\alpha/\beta = \gamma$, the golden ratio. The ordering of long and short intervals can be determined in several ways. One can, for example, use the projection method described above. Let then $\mathcal{L}$ be the regular lattice in $\mathbb{R}^3$, and let $\mathcal{E} \subset \mathbb{R}^2$ be the subset consisting of a line through the origin with slope $1/\gamma$, and then $\mathcal{E}'$ is the line with slope $-\gamma$.

Figure 1 shows one example of a Monte Carlo simulation, where $\phi_\varepsilon(]T, \infty[)$ with $\varepsilon = 10^{-5}$ has been estimated based on more than 1.5 10^7 trajectories. The graphs show $\log \phi_\varepsilon(]T, \infty[)$ as a function of log, and the the three curves represent different obstacle distributions. The green curve shows the result for the Fibonacci sequence, the blue curve for a periodic distribution of scatterers, and the magenta colored curve the result for a periodic (although non-Poissonian) distribution.

Although the blue and green curves, corresponding to a periodic crystal and the quasi crystal, do not coincide, they are both very close to a straight lines, which means that the decay rate of the mean free path distribution decays polynomially; $\phi_\varepsilon(]T, \infty[) \sim T^{-1}$. The magenta colored curve decays exponentially as a plot of $\log \phi_\varepsilon(]T, \infty[)$shows. But the scatterers are not Poisson distributed; instead a a point (the obstacle center) is chosen at random, uniformly, in each integer interval.

The next step in this project will be to make a true two-dimensional simulation of the Lorentz gas in a quasi-crystal with fivefold symmetry, as well as some other aperiodic crystals.

The, at least in the simple case studied here, it should be possible to rigorously establish the decay rate of the free path length distribution.

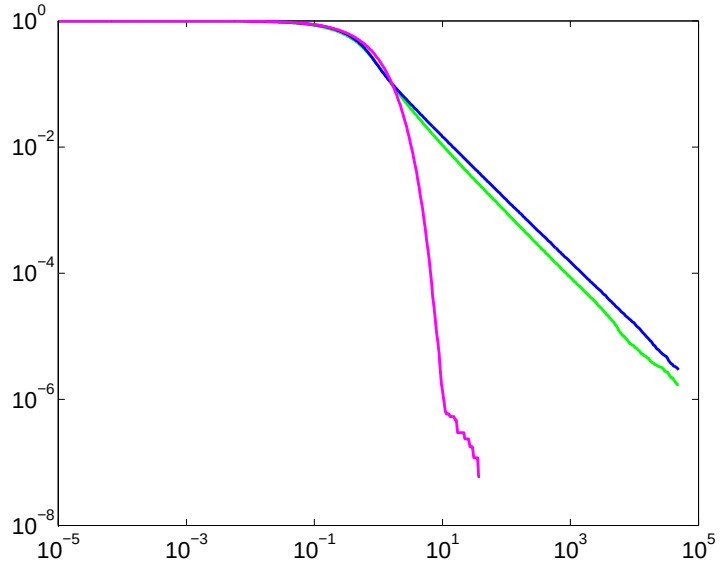

FIGURE 1. A log log plot of the free path length distribution, as described in the text

An interesting question to address is to determine all possible decay rates.

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