Entropy methods for diffusion equations

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Multi-species systems

Examples:

- Animal populations: observing, predicting, harvesting
- Fluid mixtures: heliox (diving, asthma), biofilm reactors, air pollution
- Cell dynamics: tumor growth, ion transport through membranes
- Electrolysis: lithium-ion batteries, production of hydrogen from water

Nature is generally composed of multi-species systems!

Modeling: diffusion equations

Literature

Needed prerequisites:

- Partial differential equations (PDEs)
- Sobolev spaces, basics of functional analysis
- Optional: basics of probability theory, nonlinear PDEs

Main reference

A. Jüngel. Entropy methods for diffusive partial differential equations. Chap. 4, Springer Briefs, 2016.

- A. Jüngel. The boundedness-by-entropy method for cross-diffusion systems. *Nonlinearity*, 2015.
- A. Jüngel. Cross-diffusion systems with entropy structure. *Proceedings of Equadiff 2017*.
- X. Chen, E. Daus. A. Jüngel. Global existence analysis of cross-diffusion population systems for multiple species. *Arch. Ration. Mech. Anal.*, 2018.

Diffusion equations

Heat equation:

$$\partial_t u - \Delta u = 0 \quad \text{in } \Omega,\ t > 0,\ \text{initial \& boundary conditions}$$

- Strongly regularizing: $u(0) \in L^2(\Omega) \Rightarrow u(t) \in C^\infty(\Omega)$
- Preserves nonnegativity: $u(0) \geq 0 \Rightarrow u(t) \geq 0$

Reaction-diffusion equations:

$$\partial_t u_i - \operatorname{div}(D_i \nabla u_i) = f_i(u) \quad \text{in } \Omega,\ t > 0,\ D_i > 0$$

- Still regularizing and nonnegativity preserving (if $f_i \leq 0$ at $u_i = 0$)
- Global existence of weak solutions if f_i at most quadratic growth
- Global existence of classical solutions not always guaranteed!

Problem:

- Flux $D_i \nabla u_i$ only depends on ∇u_i (Fick's law)
- In multicomponent systems, flux may depend on all ∇u_j
  $\Rightarrow$ cross-diffusion systems

What are cross-diffusion systems?

$$\partial_t u_i - \operatorname{div}\left(\sum_{j=1}^{n} A_{ij}(u)\nabla u_j\right) = f_i(u) \quad \text{in } \Omega,\ t>0,\ i=1,\ldots,n$$

- Systems of quasilinear parabolic equations
- Initial and (no-flux) boundary conditions

What makes these systems special?

- Adding (cross-) diffusion, constant equilibria may become unstable even if equilibria of associated ODE system linearly stable
- May lead to physically desired pattern formation (Turing 1952)
- Uphill diffusion: diffusion flux in higher concentration area
- Segregation: $\operatorname{supp}(u_i(t)) \cap \operatorname{supp}(u_j(t)) = \emptyset\ \forall\, t$ (Bertsch et al. 1985)
- Blow-up in L^∞ norm in finite time possible (Stará-John 1995)

Aim: Develop mathematical theory only for systems from applications

❶ Multicomponent gas mixtures

$$\partial_t u - \mathrm{div}(A(u)\nabla u) = f(u) \text{ in } \Omega,\ t > 0, \quad u(0) = u^0, \quad \text{no-flux b.c.}$$

- Volume fractions of gas components u_1, u_2, $u_3 = 1 - u_1 - u_2$
- Diffusion matrix: $\delta(u) = d_1 d_2(1 - u_1 - u_2) + d_0(d_1 u_1 + d_2 u_2)$

$$A(u) = \frac{1}{\delta(u)} \begin{pmatrix} d_2 + (d_0 - d_2)u_1 & (d_0 - d_1)u_1 \\ (d_0 - d_2)u_2 & d_1 + (d_0 - d_1)u_2 \end{pmatrix}$$

- Application: Patients with airway obstruction inhale Heliox to speed up diffusion
- Proposed by Maxwell 1866/Stefan 1871
- Duncan-Toor 1962: Fick's law ($J_i \sim \nabla u_i$) not sufficient, include cross-diffusion terms
- Derivation: Boudin-Grec-Salvarani 2015
- $A(u)$ not symm., generally not pos. definite

❷ Segregating populations

$$\partial_t u - \operatorname{div}(A(u)\nabla u) = f(u) \text{ in } \Omega,\ t > 0, \quad u(0) = u^0, \quad \text{no-flux b.c.}$$

- $u = (u_1, u_2)$ and u_i models population density of ith species
- Diffusion matrix: $(a_{ij} \geq 0)$

$$A(u) = \begin{pmatrix} a_{10} + a_{11}u_1 + a_{12}u_2 & a_{12}u_1 \\ a_{21}u_2 & a_{20} + a_{21}u_1 + a_{22}u_2 \end{pmatrix}$$

- Suggested by Shigesada-Kawasaki-Teramoto 1979 to model segregation
- Lotka-Volterra functions: $f_i(u) = (b_{i0} - b_{i1}u_1 - b_{i2}u_2)u_i$
- Diffusion matrix is not symmetric, generally not positive definite

Figure: Black residential segregation in Milwaukee (blue dots) US Census Bureau 2002

Difficulties and objectives

$$\partial_t u - \mathrm{div}(A(u)\nabla u) = f(u) \text{ in } \Omega,\ t > 0, \quad u(0) = u^0$$

Main features:

- Diffusion matrix $A(u)$ nondiagonal (cross-diffusion)
- Matrix $A(u)$ may be neither symmetric nor positive definite
- Variables u_i expected to be bounded from below and/or above

Objectives:

- Derivation of equations (formal or rigorous)
- Global-in-time existence of weak solutions
- Positivity and boundedness of solution (if physically expected)
- Large-time behavior, uniqueness & regularity of solutions

Mathematical difficulties:

- No general theory for diffusion systems
- Generally no maximum principle, no regularity theory
- Lack of positive definiteness ⇒ global existence nontrivial

Overview

- Introduction
- Derivation
- Existence analysis
- Further topics

Derivation of cross-diffusion systems

- From random-walk lattice models: Taylor expansion of transition rates and cell size $h \to 0$

  $\to$ Shigesada-Kawasaki-Teramoto (SKT) model

- From stochastic differential equations: many particle & small interaction limit

  $\to$ Shigesada-Kawasaki-Teramoto (SKT) model

- From fluid models: high-friction limit and forces proportional to velocity differences

  $\to$ Maxwell-Stefan model

- From kinetic transport equations for distribution function $f(x,v,t)$: mean-free path limit in moments $\int f(x,v,t)\phi(v)dv$ close to equilib.

  $\to$ Maxwell-Stefan model

❶ From lattice random walk to cross diffusion

Single species: one space dimension to simplify

- Master equation: time variation = incoming − outgoing

$$\partial_t u(x_i) = p(u(x_{i-1}) + u(x_{i+1})) - 2pu(x_i)$$

- Taylor expansion: (h = grid size, $x_i = ih$)

$$u(x_{i\pm1}) - u(x_i) = \pm h\partial_x u(x_i) + \frac{1}{2}h^2\partial_x^2 u(x_i) + O(h^3)$$

- Diffusion scaling: $t \mapsto t/h^2 \Rightarrow \partial_t \rightsquigarrow h^2\partial_t$

$$\begin{aligned} h^2\partial_t u(x_i) &= p(u(x_{i-1}) - u(x_i)) + p(u(x_{i+1}) - u(x_i)) \\ &= ph^2\partial_x^2 u(x_i) + O(h^3) \end{aligned}$$

- Limit $h \to 0$ gives $\partial_t u(x) = p\partial_x^2 u(x)$ (heat equation)
- Rigorous limit: De Masi, Lebowitz, Sinai, Spohn etc. (from 1980s on)

❶ From lattice random walk to cross diffusion

Multiple species:

- Master equation for particle number $u_j(x_i)$ at ith cell:

$$\partial_t u_j(x_i) = p^+_{j,i} u_j(x_{i-1}) + p^-_{j,i+1} u_j(x_{i+1}) - (p^+_{j,i} + p^-_{j,i}) u_j(x_i)$$

- Transition rates: $p^\pm_{j,i} = p_i(u(x_j)) q_i(u_n(x_{j\pm 1}))$
- Taylor expansion, diffusion scaling and limit $h \to 0$ leads to system

$$\partial_t u_j = \partial_x \left(\sum_{k=1}^n A_{jk}(u) \partial_x u_k \right), \quad j = 1, \ldots, n$$

- Multi-dimensional case analogous

Example: $q_i = 1$, $p_i(u) = a_{i0} + \sum_{k=1}^n a_{ik} u_k$

$$A_{ij}(u) = p_i(u) + u_i \frac{\partial p_i}{\partial u_j}(u) = \delta_{ij} a_{i0} + \delta_{ij} \sum_{k=1}^n a_{ik} u_k + a_{ij} u_i$$

$\to$ n-species generalization of SKT population model

❷ From SDEs to cross diffusion

Aim: Many-particle limits in single-species particle system in $\mathbb{R}^d$

$$dX^k = -\frac{1}{N}\sum_{j=1,j\neq k}^{N} \nabla B(X^k - X^j)dt + \sqrt{2\sigma}dW^k(t), \quad k = 1,\dots,N$$

- $X^k(t)$ stochastic processes ("random position"), $W^k(t)$ independent Wiener processes, B interaction potential, $\sigma > 0$
- Expectation for many-particle limit $N \to \infty$:

$$\frac{1}{N}\sum_j \nabla B(x - X^j) \sim \mathrm{E}(\nabla B) = \int_{\mathbb{R}^d} \nabla B(x-y)u(y)dy = \nabla B * u$$

- Limit eq. for probability density $u(x,t)$: $\partial_t u = \sigma\Delta u + \mathrm{div}(u\nabla B * u)$ (Oelschläger 1989, Sznitman 1991, Méléard 1996)
- Localization limit $B \to \delta_0$: $\partial_t u = \sigma\Delta u + \mathrm{div}(u\nabla u)$
- Goal: extend to multi-species case, expect cross-diffusion $\mathrm{div}(u_j\nabla u_i)$

❷ From SDEs to cross diffusion

First attempt: Stochastic processes $X_i^k(t)$ solve SDE

$$dX_i^k = -\sum_{j=1}^n \frac{1}{N} \sum_{\ell=1,\,\ell\neq k}^N \nabla B_{ij}^\eta(X_i^k - X_j^\ell)dt + \sqrt{2\sigma_i}dW_i^k(t) \quad \text{in } \mathbb{R}^d$$

- Species index $i = 1, \ldots, n$, particle index $k = 1, \ldots, N$
- W_i^k independent Wiener processes, $\sigma_i > 0$
- Interaction potential: $B_{ij}^\eta(x) = \eta^{-d}B_{ij}(|x|/\eta)$, $\int_{\mathbb{R}^d} B_{ij}dx = a_{ij} \Rightarrow$

$$\|B_{ij}^\eta\|_{L^1(\mathbb{R}^d)} = \|B_{ij}\|_{L^1(\mathbb{R}^d)}, \quad B_{ij}^\eta \to a_{ij}\delta_0 \quad \text{in } \mathcal{D}' \text{ as } \eta \to 0$$

- Limit: $N \to \infty$ leads to “nonlocal” SDE, $\eta \to 0$ leads to local SDE with probability density u_i satisfying PDE (by Itô’s lemma)
- Rigorous limit (L. Chen-Daus-A.J. 2019):

$$\partial_t u_i = \operatorname{div}\big(\sigma_i \nabla u_i + u_i \nabla p_i(u)\big), \quad p_i(u) = \sum_{k=1}^n a_{ik}u_k$$

- This is not the SKT population system $\partial_t u_i = \Delta(\sigma_i u_i + u_i p_i(u))$

❷ From SDEs to cross diffusion

Second attempt: $X_i^k(t)$ stochastic processes, $i=1,\ldots,n$, $k=1,\ldots,N$

$$dX_i^k = \sqrt{2\sigma_i + 2F_N^\eta(X)}\,dW_i^k(t) \quad \text{in } \mathbb{R}^d$$

$$F_N^\eta(X) = \sum_{j=1}^n \frac{1}{N}\sum_{\ell=1}^N B_{ij}^\eta(X_i^k - X_j^\ell)$$

- W_i^k independent Wiener processes, $\sigma_i > 0$ constants
- Interaction potential: as before, with $B_{ij}^\eta \to a_{ij}\delta_0$ as $\eta \to 0$
- Limit $N\to\infty$, $\eta\to 0$ with $\eta^{-2d-3} \le \delta \log N$ for "small" $\delta > 0$ (L. Chen-Daus-Holzinger-A.J. 2020)
- Density function u_i solves SKT population model

$$\partial_t u_i = \Delta\big(\sigma_i u_i + u_i p_i(u)\big), \quad p_i(u) = \sum_{k=1}^n a_{ik}\nabla u_k$$

❸ From fluid models to cross diffusion

Mass and momentum balance equations: (Huo-A.J. Tzavaras 2019)

$$\partial_t u_i + \operatorname{div}(u_i v_i) = 0, \quad i = 1, \ldots, n$$
$$\partial_t(u_i v_i) + \operatorname{div}(u_i v_i \otimes v_i) + \nabla u_i = \frac{1}{\varepsilon} \sum_{j=1}^{n} b_{ij} u_i u_j (v_j - v_i)$$

- Rigorous high-friction limit $\varepsilon \to 0$: expand $u_i = u_i^0 + \varepsilon u_i^1 + O(\varepsilon^2)$, etc.
- Simplify (for presentation only): $\sum_{i=1}^n u_i = 1$, barycentr. velocity zero
- Equations up to order $O(\varepsilon^2)$, i.e. for $u_i^0 + \varepsilon u_i^1$, etc.

Maxwell-Stefan equations: set $J_i := u_i v_i$, b_{ij} symmetric

$$\partial_t u_i + \operatorname{div} J_i = 0, \quad \nabla u_i = -\sum_{j=1}^{n} b_{ij}(u_j J_i - u_i J_j), \quad \sum_{i=1}^{n} u_i v_i = 0$$

- Invert relation $J_i \leftrightarrow \nabla u_i$: problem $\sum_i J_i = 0 \Rightarrow$ invert on $\operatorname{span}\{1\}^\perp$
- Result: $J_i = -\sum_{j=1}^{n-1} A_{ij}(u) \nabla u_j$

❹ From kinetic models to cross diffusion

Boltzmann equation: $f_i = f_i(x,v,t)$ (Boudin-Grec-Salvarani 2015)

$$\varepsilon\partial_t f_i + v\cdot\nabla_x f_i = \frac{1}{\varepsilon}Q_i(f_i,f_i) + \frac{1}{\varepsilon}\sum_{j\neq i} Q_{ij}(f_i,f_j), \quad i=1,\ldots,n$$

- Q_i mono-species, Q_{ij} bi-species collision operators
- Collisions are elastic & conserve mass: $\int_{\mathbb{R}^3}(Q_i + \sum_{j\neq i} Q_{ij})dv = 0$
- Particle density: $u_i = \int_{\mathbb{R}^3} f_i dv$, flux: $\varepsilon u_i v_i = \int_{\mathbb{R}^3} f_i v dv$
- Ansatz: f_i close to equilibrium:

$$f_i(x,v,t) = \frac{u_i(x,t)}{(2\pi)^{3/2}}\exp\left(-\frac{1}{2}|v-\varepsilon v_i(x,t)|^2\right)$$

- Insert into Boltzmann eq., integrate, limit $\varepsilon\to 0 \to$ Maxwell-Stefan

$$\partial_t u_i + \operatorname{div}(u_i v_i) = 0, \quad \nabla u_i = -\sum_{j=1}^n b_{ij}u_i u_j(v_i - v_j)$$

- Rigorous derivation for small initial data: Briant-Grec 2020

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State of the art

$$\partial_t u - \mathrm{div}(A(u)\nabla u) = f(u) \text{ in } \Omega,\ t > 0, \quad u(0) = u^0, \quad \text{no-flux b.c.}$$

Aim: Develop existence theory (uniqueness, regularity)

- Ladyženskaya et al. 1968: growth conditions on nonlinearities needed
- Many results for small cross diffusion (Kim 1984, Deuring 1987,...)
- Alt-Luckhaus 1983: global solutions if Onsager matrix unif. pos. def.
- Kawashima-Shizuta 1988: hyperbolic-parabolic systems, entropies
- Amann 1990: parabolic in the sense of Petrovskii $\Rightarrow \exists!$ local classical solution; bounds in $W^{1,p}(\Omega)$, $p > d \Rightarrow \exists$ global classical solution
- D. Le 2016: BMO bound & condition on $A(u) \Rightarrow$ classical solution
- Burger et al. 2010: global bounded weak solutions for special model

Novelty of approach: degeneracies allowed, global L^∞ solutions

Key idea: exploit formal gradient-flow / entropy structure

Gradient-flow or entropy structure

Main assumption

$\partial_t u - \operatorname{div}(A(u)\nabla u) = f(u)$ possesses formal gradient-flow structure

$$\partial_t u - \operatorname{div}\left(B\nabla \frac{\delta H(u)}{\delta u}\right) = f(u),$$

where Onsager matrix B is pos. semi-definite, $H(u) = \int_\Omega h(u)dx$ entropy

Equivalent formulation: $\delta H(u)/\delta u \simeq h'(u) =: w$ (entropy variable)

$$\partial_t u(w) - \operatorname{div}(B(w)\nabla w) = f(u(w)), \quad B(w) = A(u(w))h''(u(w))^{-1}$$

Consequences:

1. H is Lyapunov functional if $f = 0$:

$$\frac{dH}{dt} = \int_\Omega \partial_t u \cdot \underbrace{h'(u)}_{=w} dx = -\int_\Omega \nabla w : B\nabla w dx \le 0$$

2. L^∞ bounds for u: Let $h' : D \to \mathbb{R}^n$ ($D \subset \mathbb{R}^n$) be invertible $\Rightarrow$ $u(x,t) = (h')^{-1}(w(x,t)) \in D$ (no maximum principle needed!)

Gradient-flow and thermodynamic structure

$$\partial_t u_i(w) - \operatorname{div}\Big(\sum_{j=1}^{n-1} B_{ij}(w)\nabla w_j\Big) = f_i(u(w)), \quad i = 1, \ldots, n-1$$

Gradient-flow structure: write equations as

$$\partial_t u_i - \operatorname{div}\Big(\sum_{j=1}^{n-1} B_{ij}\nabla \frac{\delta H}{\delta u_j}\Big) = 0, \quad w_i = \frac{\delta H}{\delta u_i}$$

- Entropy $H = \int_\Omega h(u)dx$
- Gradient flow: $\partial_t u = -K[u^*]\mathrm{grad} H|_u$ on differential manifold, where $K[u^*]w = -\operatorname{div}(B\nabla w)$ Onsager operator

Thermodynamic structure:

- Mathematical entropy density $h = -s$ physical entropy density
- Entropy variable = chemical potential $w_i = \partial h/\partial u_i$
- Onsager reciprocal relations: B is symmetric
- Entropy production: $-\frac{dH}{dt} = \int_\Omega \nabla w : B\nabla w dx \geq 0$
  $\to$ expresses second law of thermodynamics

❶ Maxwell-Stefan models

$$\partial_t u - \operatorname{div}(A(u)\nabla u) = 0 \quad \text{in } \Omega,\ t > 0$$

$$A(u) = \frac{1}{\delta(u)} \begin{pmatrix} d_2 + (d_0 - d_2)u_1 & (d_0 - d_1)u_1 \\ (d_0 - d_2)u_2 & d_1 + (d_0 - d_1)u_2 \end{pmatrix}$$

- Entropy: $H(u) = \int_\Omega h(u)dx,\ u \in D = \{u : u_i > 0,\ u_1 + u_2 < 1\} \subset \mathbb{R}^2$

  $h(u) = u_1(\log u_1 - 1) + u_2(\log u_2 - 1) + \underbrace{(1 - u_1 - u_2)}_{=u_3}(\log(1 - u_1 - u_2) - 1)$

- Entropy production:

$$\frac{dH}{dt} + c\int_\Omega \sum_{i=1}^{3} |\nabla\sqrt{u_i}|^2 dx \le 0$$

- Entropy variables: $w = h'(u) \in \mathbb{R}^2$ or $u = (h')^{-1}(w)$

$$w_i = \frac{\partial h}{\partial u_i} = \log\frac{u_i}{u_3}, \quad u_i = \frac{e^{w_i}}{1 + e^{w_1} + e^{w_2}} \in D$$

- Consequences: gradient estimate for $\sqrt{u_i}$, $u_i(x,t)$ is bounded

❷ Population model

$$\partial_t u - \operatorname{div}(A(u)\nabla u) = f(u) \text{ in } \Omega,\ t > 0, \quad u(0) = u_0, \quad \text{no-flux b.c.}$$

$$A(u) = \begin{pmatrix} a_{10} + a_{11}u_1 + a_{12}u_2 & a_{12}u_1 \\ a_{21}u_2 & a_{20} + a_{21}u_1 + a_{22}u_2 \end{pmatrix}, \ a_{ij} \geq 0$$

- Lotka-Volterra terms: $f_i(u) = (b_{i0} - b_{i1}u_1 - b_{i2}u_2)u_i$, $i = 1, 2$
- Entropy: $h(u) = a_{21}u_1(\log u_1 - 1) + a_{12}u_2(\log u_2 - 1)$, $D = (0, \infty)^2$
- Entropy production:

$$\frac{dH}{dt} + C\sum_{i=1}^{2}\int_\Omega \big(a_{i0}|\nabla\sqrt{u_i}|^2 + a_{ii}|\nabla u_i|^2\big)dx \leq C_f$$

- Entropy variables: $w_i = \partial h/\partial u_i = \log u_i \Rightarrow u_i = \exp(w_i) > 0$
- Consequences: gradient estimates for $\sqrt{u_i}$ if $a_{i0} > 0$ and u_i if $a_{ii} > 0$, nonnegativity for u_i

Intermediate summary

- Multicomponent systems omnipresent in applications, leads to cross-diffusion systems
- Diffusion matrix generally neither symmetric nor positive semidefinite
- Generally, no full regularity, no maximum principle
- Derivation from
  - diffusion limit for random walks on lattices
  - many-particle limit in interacting particle systems
  - high-friction limit in Euler equations
  - diffusion limit in system of Boltzmann equations
- Mathematical theory based on formal gradient-flow / entropy structure
- Structure inspired from thermodynamics

Next lecture: global existence analysis, uniqueness of solutions, large-time asymptotics, regularity of solutions

Existence analysis

$$\partial_t u_i - \operatorname{div}\left(\sum_{j=1}^{n} A_{ij}(u)\nabla u_j\right) = f_i(u), \quad i = 1, \ldots, n$$

where $A(u) = (A_{ij}(u))$ generally neither symm. nor positive semidefinite

Main assumption

$\partial_t u - \operatorname{div}(A(u)\nabla u) = f(u)$ possesses formal gradient-flow structure

$$\partial_t u - \operatorname{div}(B\nabla w) = f(u), \quad w = \nabla h(u)$$

where Onsager matrix B is pos. semi-definite, $H(u) = \int_\Omega h(u)dx$ entropy

Consequences:

1. H is Lyapunov functional if $f = 0$: $dH/dt + \int_\Omega \nabla w : B\nabla w dx = 0$
2. L^∞ bounds for u: Let $h' : D \to \mathbb{R}^n$ ($D \subset \mathbb{R}^n$) be invertible $\Rightarrow$ $u(x,t) = (h')^{-1}(w(x,t)) \in D$ (no maximum principle needed!)

Boundedness-by-entropy method

$$\partial_t u(w) - \operatorname{div}(B(w)\nabla w) = f(u(w)) \text{ in } \Omega, \qquad u(0) = u^0, \text{ no-flux b.c.}$$

$$\frac{d}{dt}\int_\Omega h(u)dx + \int_\Omega \nabla u : h''(u)A(u)\nabla u dx = \int_\Omega f(u)\cdot h'(u)dx$$

Assumptions:

1. $\exists$ convex entropy $h \in C^2(D;[0,\infty))$, h' invertible on $D \subset \mathbb{R}^n$
2. "Degenerate" positive definiteness: for all $u \in D$,

$$z^\top h''(u)A(u)z \geq c\sum_{i=1}^n u_i^{2m_i-2}z_i^2, \ m_i \geq \frac{1}{2} \Rightarrow \text{ estimate for } |\nabla u_i^{m_i}|^2$$

3. A continuous on D, $\exists C > 0 : \forall u \in D$: $f(u)\cdot h'(u) \leq C(1 + h(u))$

Theorem (A.J. 2015)

Let the above assumptions hold, let $D \subset \mathbb{R}^n$ be bounded, $\int_\Omega h(u^0) < \infty$, $u_i^0(x) \in \overline{D}$. Then $\exists$ global weak solution such that $u(x,t) \in \overline{D}$ and

$$u \in L^2_{\rm loc}(0,\infty;H^1(\Omega)), \quad \partial_t u \in L^2_{\rm loc}(0,\infty;H^1(\Omega)')$$

Boundedness-by-entropy method

Theorem (A.J. 2015)

Let the above assumptions hold, let $D \subset \mathbb{R}^n$ be bounded, $\int_\Omega h(u^0) < \infty$, $u_i^0(x) \in \overline{D}$. Then $\exists$ global weak solution such that $u(x,t) \in \overline{D}$ and

$$u \in L^2_{\rm loc}(0,\infty; H^1(\Omega)), \quad \partial_t u \in L^2_{\rm loc}(0,\infty; H^1(\Omega)')$$

Remarks:

- Result valid for rather general model class
- Yields L^∞ bounds without using a maximum principle
- Boundedness assumption on D is strong but can be weakened in some cases; see SKT model below
- Main assumptions: existence of entropy h, pos. def. of $h''(u)A(u)$
- How to find entropy functions h? Physical intuition, trial and error
- Yields immediately global existence for Maxwell-Stefan ($m_i = \frac{1}{2}$)

Proof of existence theorem

$$\partial_t u - \mathrm{div}(A(u)\nabla u) = f(u) \quad \text{or} \quad \partial_t u(w) - \mathrm{div}(B(w)\nabla w) = f(u(w))$$

Key ideas:

- Discretize in time: replace $\partial_t u(w)$ by $(u(w^k) - u(w^{k-1}))/\tau$, $\tau > 0$
  Benefit: avoid issues with time regularity
- Regularize in space by adding "$\varepsilon(-\Delta)^m w^k$", $\varepsilon > 0$
  Benefit: yields solutions $w^k \in H^m(\Omega) \subset L^\infty(\Omega)$ if $m > d/2$
  (note that $\mathrm{div}(B(w)\nabla w)$ not uniformly elliptic)
- Solve problem in w^k by fixed-point argument
  Benefit: elliptic problem in w-formulation (not true for u-formulation)
- Perform limit $(\varepsilon, \tau) \to 0$, obtain solution $u(t) = \lim u(w^k)$
  Benefit: compactness comes from entropy estimate; L^∞ bounds coming from $u(w^k) \in D \Rightarrow u \in \overline{D}$

Strategy: problem in u $\to$ solve in w $\to$ limit solves problem in u

Existence proof: more details

- Approximate problem: Given $w^{k-1} \in L^\infty(\Omega)$, solve for $\phi \in H^m(\Omega)$,

$$\frac{1}{\tau}\int_\Omega (u(w^k) - u(w^{k-1})) \cdot \phi dx + \int_\Omega \nabla\phi : B(w^k)\nabla w^k dx + \varepsilon \int_\Omega \Big(\sum_{|\alpha|=m} D^\alpha w^k \cdot D^\alpha \phi + w^k \cdot \phi \Big) dx = \int_\Omega f(u(w^k)) \cdot \phi dx$$

- Linearized system: $S : L^\infty(\Omega) \times [0,1] \to L^\infty(\Omega)$, $S(y,\delta) = w^k$ and w^k solves linear problem (by Lax-Milgram)
- Fixed-point argument: show that S compact, entropy estimate for all fixed points $\Rightarrow \exists\, w^k \in H^m(\Omega)$: $S(w^k, 1) = w^k$ (by Leray-Schauder)

$$\delta \int_\Omega h(u^k)dx + \tau \int_\Omega \nabla w^k : B\nabla w^k dx + \varepsilon\tau \|w^k\|^2_{H^m(\Omega)} \le \delta \int_\Omega h(u^{k-1})dx + \underbrace{C\tau}_{<1}\delta \int_\Omega (1 + h(u^k))dx, \quad u^k := u(w^k)$$

- Limit $(\varepsilon, \tau) \to 0$: Aubin-Lions compactness lemma

Aubin-Lions lemma

- Estimates uniform in (τ,ε): set $u^{(\tau)}(\cdot,t)=u(w^k)$, $t\in((k-1)\tau,k\tau]$

$$\|(u_i^{(\tau)})^{m_i}\|_{L^2(0,T;H^1)}+\sqrt{\varepsilon}\|w^{(\tau)}\|_{L^2(0,T;H^m)}\le C$$
$$\tau^{-1}\|u^{(\tau)}(t)-u^{(\tau)}(t-\tau)\|_{L^2(\tau,T;(H^m)')}\le C$$

Lemma (Aubin-Lions 1963/69)

Let $\|u^{(\tau)}\|_{L^2(0,T;H^1)}+\|\partial_t u_i^{(\tau)}\|_{L^2(0,T;H^m(\Omega)')}\le C$.
Then exists subsequence $u^{(\tau)}\to u$ strongly in $L^2(0,T;L^2)$.

Problem: discrete time derivative and nonlinear estimate

Lemma (Discrete Aubin-Lions; Simon 1987)

Let $X\hookrightarrow B$ compact and $B\hookrightarrow Y$ continuous, $1\le p<\infty$, and

$$\|u^{(\tau)}\|_{L^p(0,T;X)}\le C,\quad \sup_{\tau>0}\lim_{h\to 0}\|u^{(\tau)}(t)-u^{(\tau)}(t-h)\|_{L^1(\tau,T;Y)}=0$$

Then $(u^{(\tau)})$ is relatively compact in $L^p(0,T;B)$.

Aubin-Lions lemma

Lemma (Discrete Aubin-Lions; Dreher-A.J., 2012)

If additionally, $(u^{(\tau)})$ piecewise constant in time, and

$$\|u^{(\tau)}\|_{L^p(0,T;X)} + \tau^{-1}\|u^{(\tau)}(t) - u^{(\tau)}(t-\tau)\|_{L^1(\tau,T;Y)} \le C$$

Then $(u^{(\tau)})$ is relatively compact in $L^p(0,T;B)$.

Benefit: study $u^{(\tau)}(t) - u^{(\tau)}(t-\tau)$, not all $u^{(\tau)}(t) - u^{(\tau)}(t-h)$

Theorem (Nonlinear Aubin-Lions lemma, Chen-A.J.-Liu 2014)

Let $(u^{(\tau)})$ be piecewise constant in time, $k \in \mathbb{N}$, $s \ge \frac{1}{2}$, and

$$\|(u^{(\tau)})^s\|_{L^2(0,T;H^1)} + \tau^{-1}\|u^{(\tau)}(t) - u^{(\tau)}(t-\tau)\|_{L^1(\tau,T;(H^k)')} \le C$$

Then exists subsequence $u^{(\tau)} \to u$ strongly in $L^{2s}(0,T;L^{2s})$

Remark: Alt-Luckhaus 1983: $s = 1$, Maître 2003: nonlinear version of Simon 1987, Moussa 2016: monotone nonlinearities

SKT population model

- Diffusion matrix: $(a_{ij} \geq 0)$
$$A(u) = \begin{pmatrix} a_{10} + a_{11}u_1 + a_{12}u_2 & a_{12}u_1 \\ a_{21}u_2 & a_{20} + a_{21}u_1 + a_{22}u_2 \end{pmatrix}$$
- Entropy $H(u) = \int_\Omega h(u)dx$, $h(u) = \sum_{i=1}^2 u_i(\log u_i - 1)$ for $u \in D = (0,\infty)^2$ but no L^∞ bound
- Positivity: $u_i = \exp(w_i) > 0$ and entropy inequality:
$$\frac{dH}{dt} + C_1 \sum_{i=1}^2 \int_\Omega (a_{i0}|\nabla\sqrt{u_1}|^2 + a_{ii}|\nabla u_i|^2)dx \leq C_2$$
- $a_{ii} > 0$: Gagliardo-Nirenberg $u_i \in L^{2+2/d}_{x,t} \to$ enough to treat $u_i \nabla u_j$
- $a_{i0} > 0$: more sophisticated estimates since $u_i \in L^{1+1/d}_{x,t}$ only

Theorem (L. Chen-A.J. 2004/2006)

Let $H(u^0) < \infty$. Then $\exists$ solution (u_1, u_2) with u_1, $u_2 \geq 0$ in Ω and
$$a_{i0} > 0: \ \sqrt{u_i} \in L^2_{\rm loc}(0,\infty; H^1(\Omega)), \quad a_{ii} > 0: \ u_i \in L^2_{\rm loc}(0,\infty; H^1(\Omega))$$

Generalization 1: nonlinear coefficients

Macroscopic limit of random walk on lattice:

$$A(u) = \begin{pmatrix} p_1(u) + u_1\frac{\partial p_1}{\partial u_1}(u) & u_1\frac{\partial p_1}{\partial u_2}(u) \\ u_2\frac{\partial p_2}{\partial u_1}(u) & p_2(u) + u_2\frac{\partial p_2}{\partial u_2}(u) \end{pmatrix}$$

- p_i linear: Chen-A.J. 2004
- p_i sublinear: Desvillettes-Lepoutre-Moussa 2014
- p_i superlinear: $p_i(u) = a_{i0} + a_{i1}u_1^s + a_{i2}u_2^s$ $(i = 1, 2)$, entropy density: $h_s(u) = a_{21}u_1^s + a_{12}u_2^s$, $s > 1$

Theorem (A.J. 2015)

Let $1 < s < 4$ and $(1 - \frac{1}{s})a_{12}a_{21} \le a_{11}a_{22}$, $H(u^0) < \infty$.
Then $\exists$ nonnegative weak solution $u_i^{s/2} \in L^2_{\rm loc}(0, \infty; H^1(\Omega))$

Idea of proof: use entropy $h_s(u) + \varepsilon\sum_i u_i(\log u_i - 1)$

- p_i superlinear, $s > 1$: Desvillettes-Lepoutre-Moussa-Trescases 2015

Generalization 2: more than two species

$$A_{ij}(u) = (a_{i0} + a_{i1}u_1 + \cdots + a_{in}u_n)\delta_{ij} + a_{ij}u_i$$

- Entropy: $H(u) = \int_\Omega h(u)dx = \int_\Omega \sum_{i=1}^n \pi_i u_i(\log u_i - 1)$
- Key assumption: $\pi_i a_{ij} = \pi_j a_{ji}$ (detailed balance), $\pi_i > 0$

Why detailed balance?

- Detailed balance $\Leftrightarrow$ (π_i) reversible measure $\Leftrightarrow$ $h''(u)A(u)$ symmetric $\Rightarrow$ entropy $H(u(t))$ decreases $\forall t$
- Detailed balance not satisfied: a_{ii} "large" $\Rightarrow$ $H(u(t))$ decreases, otherwise $\exists$ $u(0)$ such that $H(u(t))$ increases

Theorem (X. Chen-Daus-A.J. 2018)

Let $a_{ij} > 0$ and detailed balance hold. Then $\exists$ nonnegative weak solution $u_i \in L^2_{\mathrm{loc}}(0,\infty; H^1(\Omega))$, $i = 1,\ldots,n$

Nonlinear coefficients: Chen-Daus-A.J. 2018, Lepoutre-Moussa 2017

Overview

- Introduction
- Derivation
- Existence analysis
- Further topics

❶ Entropy structure and normal ellipticity

$$\partial_t u - \mathrm{div}(A(u)\nabla u) = f(u) \qquad (*)$$

Definition: $A(u)$ normally elliptic = $A(u)$ positively stable = eigenvalues of $A(u)$ have positive real parts = $(*)$ parabolic in sense of Petrovskii

Theorem (X. Chen-A.J. 2019)

- *If $(*)$ has entropy structure then $A(u)$ normally elliptic*
  $\Rightarrow$ local existence of smooth solutions by Amann 1990
- *If $A(u)$ normally elliptic & $h''(u)A(u)$ symmetric then $(*)$ has an entropy structure and $A(u)$ diagonalizable with positive eigenvalues*
  symmetry of $h''(u)A(u)$ corresponds to Onsager relations
- *If $A = A_0$ constant: A normally elliptic $\Leftrightarrow$ $(*)$ has entropy structure*
  $A(u) = A_0$ + nonlinear perturbation $\Rightarrow \exists$ entropy structure

Proof: Use Lyapunov theorem and matrix factorization

Entropy structure

Application: Keller-Segel model with additional cross-diffusion

$$\partial_t u_i = \operatorname{div}(\nabla u_i - u_i \nabla c), \quad i = 1, \ldots, n$$

$$\partial_t c = \Delta c + \delta \sum_{j=1}^n \Delta u_j + \sum_{j=1}^n b_{ij} u_j - c, \quad \text{no-flux b.c.}$$

- u_i: cell density of ith species, c: concentration of chemical signal
- $\delta > 0$: strength of additional cross-diffusion, avoids blow-up
- Diffusion matrix $A(u)$ is normally elliptic
- Factorization: $A(u) = A_1 A_2$, A_1 symm. pos. def., A_2 pos. def.

$$A_1 = \begin{pmatrix} u_1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & u_n & 0 \\ 0 & \cdots & 0 & \delta \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1/u_1 & & 0 & -1 \\ & \ddots & & \vdots \\ 0 & & 1/u_n & -1 \\ 1 & \cdots & 1 & 1/\delta \end{pmatrix}$$

- Set $h''(u) = A_1^{-1}$, then $A_2 = A_1^{-1} A(u) = h''(u) A(u)$ pos. def.
- Compute entropy: $h(u) = \sum_{i=1}^n u_i (\log u_i - 1) + u_2^2/(2\delta)$

❷ Uniqueness of weak solutions

- Alt-Luckhaus 1983: linear elliptic operator, $\partial_t u_i \in L^1$
- Gajewski 1994: elliptic Onsager operator monotone in special sense
- Berendsen et al. 2020: weak-strong uniqueness for special system

Result based on entropy method:

$$\partial_t u_i = \operatorname{div}\bigg(\sum_{j=1}^n A_{ij}(u)\nabla u_j\bigg), \quad A_{ij}(u) = p(u_0)\delta_{ij} + a_j u_i q(u_0)$$

$$u_0 = \sum_{i=1}^n a_i u_i, \quad \text{initial \& no-flux boundary conditions}$$

- u_i: species' concentrations, u_0: solvent concentration
- Example: ion transport in membrane and nanopore

Theorem (X. Chen-A.J. 2018)

Let $p(s) \geq 0$, $p(s) + sq(s) \geq 0$. Then uniqueness in class of functions $p(u_0)^{1/2}\nabla u_i$, $|q(u_0)|^{1/2}\nabla u_i \in L^2$, $\partial_t u_i \in L^2(0, T; H^1(\Omega)')$.

Idea of proof: combine H^{-1} method and entropy technique of Gajewski

Uniqueness of weak solutions

$$\partial_t u_i = \operatorname{div}\Big(\sum_{j=1}^n A_{ij}(u)\nabla u_j\Big), \quad A_{ij}(u) = p(u_0)\delta_{ij} + a_j u_i q(u_0)$$

Step ❶: $H^{-1}(\Omega)$ method

- Sum equations for $i = 1, \ldots, n$, use $u_0 = \sum_{j=1}^n a_j u_j$

$$\partial_t u_0 = \operatorname{div}\big((p(u_0) + u_0 q(u_0))\nabla u_0\big) = \Delta Q(u_0), \text{ no-flux b.c.}$$

where $Q'(z) = p(z) + zq(z) \geq 0$ (assumption) $\Rightarrow$ Q monotone

- Let u_0, v_0 be two weak solutions, let ξ solve $-\Delta\xi = u_0 - v_0$ & b.c.

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\xi|^2 dx &= -\int_\Omega \partial_t(\Delta\xi)\xi dx = \int_\Omega \Delta(Q(u_0) - Q(v_0))\xi dx \\ &= -\int_\Omega (Q(u_0) - Q(v_0))(u_0 - v_0)dx \leq 0, \quad \xi(0) = 0\end{aligned}$$

- Implies that $\xi(t) = 0$ and hence $(u_0 - v_0)(t) = 0 \Rightarrow$ uniqueness for u_0

Uniqueness of weak solutions

$$\partial_t u_0 = \Delta Q(u_0), \quad \partial_t u_i = \operatorname{div}(p(u_0)\nabla u_i + u_i q(u_0)\nabla u_0)$$

Step ❷: Define Gajewski's semimetric

$$G(u,v) = \sum_{i=1}^n \int_\Omega \left(h(u_i) + h(v_i) - 2h\left(\frac{u_i+v_i}{2}\right)\right) dx, \ \ h(s) = s(\log s - 1)$$

- Compute time derivative

$$\frac{dG}{dt}(u,v) = -4\sum_{i=1}^n \int_\Omega p(u_0)\big(|\nabla\sqrt{u_i}|^2 + |\nabla\sqrt{v_i}|^2 - |\nabla\sqrt{u_i+v_i}|^2\big)dx \le 0$$

- Test function $\partial h/\partial u_i = \log u_i$ requires to regularize $h(u)$
- $G(u(0),v(0)) = 0$ implies that $G(u(t),v(t)) = 0$ and $u(t) = v(t)$

Theorem (X. Chen-A.J. 2018)

Let $p(s) \ge 0$, $p(s) + sq(s) \ge 0$. Then uniqueness of weak solutions satisfying $\sqrt{p(u_0)}\nabla u_i$, $\sqrt{|q(u_0)|}\nabla u_i \in L^2(\Omega\times(0,T))$.

Weak-strong uniqueness of renormalized solutions

$$\partial_t u_i = \operatorname{div} \sum_{j=1}^n A_{ij}(u)\nabla u_j + f_i(u), \quad i = 1, \ldots, n$$
$$A_{ij}(u) = (a_{i0} + a_{i1}u_1 + \cdots + a_{in}u_n)\delta_{ij} + a_{ij}u_i$$

Theorem (X. Chen-A.J. 2019)

u: renormalized solution, v: strong solution to SKT model. Then $u = v$.

Renormalized solution: Use test function $(\partial\xi/\partial u_i)\phi_i$, where $\xi \in C^\infty$ with $\xi' \in C_0^\infty$; needed since no growth condition for $f_i(u)$ supposed

Idea of proof: use relative entropy $H(u|v) = \int_\Omega h(u|v)dx$ with

$$h(u|v) = h(u) - h(v) - h'(v)\cdot(u - v), \quad h(u) = \sum_{i=1}^n u_i(\log u_i - 1)$$

- Aim: Show that $dH/dt \le CH \Rightarrow H(u(t)|v(t)) = 0 \Rightarrow u(t) = v(t)$
- Several cutoffs required (J. Fischer 2017), very technical
- Relative entropy related to Gajewski's semimetric

❸ Large-time asymptotics

$$\partial_t u + \mathcal{A}(u) = f(u), \quad t > 0, \quad u(0) = u^0$$

- Entropy production:

$$\frac{dH}{dt} + \langle \mathcal{A}(u), H'(u) \rangle = \langle f(u), H'(u) \rangle$$

- Assume: $\langle f(u), H'(u) \rangle \le 0$ and $\langle \mathcal{A}(u), H'(u) \rangle \ge \lambda H$. Then

$$\frac{dH}{dt} + \lambda H \le 0 \;\Rightarrow\; H(u(t)) \le H(u^0)e^{-\lambda t} \;\Rightarrow\; u(t) \to 0$$

- Convex Sobolev inequality: $\langle \mathcal{A}(u), H'(u) \rangle \ge \lambda H$

Example: SKT population model

$$\frac{dH}{dt} + C_1 \sum_{i=1}^n \int_\Omega |\nabla \sqrt{u_i}|^2 dx \le 0, \quad H(u) = \sum_{i=1}^n \int_\Omega u_i(\log u_i - 1)$$

Use logarithmic Sobolev inequality:

$$\int_\Omega u_i(\log u_i - 1)dx \le C_S \int_\Omega |\nabla \sqrt{u_i}|^2 dx \;\Rightarrow\; \frac{dH}{dt} + \frac{C_1}{C_S} H \le 0$$

Large-time asymptotics for reactive mixtures

$$\partial_t u + \mathcal{A}(u) = f(u), \quad t > 0, \quad u(0) = u^0$$

Question: What happens if we do not have $\langle f(u), H'(u)\rangle \le 0$?

Example: Maxwell-Stefan systems and mass action kinetics

$$f_i(u) = \sum_{a=1}^{N} (\beta_i^a - \alpha_i^a)\big(k_f^a u^{\alpha^a} - k_b^a u^{\beta^a}\big), \quad i = 1, \ldots, n$$

- k_f^a: forward reaction rate, k_b^a: backward reaction rate
- α_i^a, β_i^a: stoichiometric coefficients, $u^{\alpha^a} := \prod_{j=1}^n u_j^{\alpha_j^a}$
- Conservation of total mass: $\sum_{i=1}^n f_i(u) = 0$

Aim: Show that $u(t) \to u_\infty$ as $t \to \infty$, use relative entropy $H[u|u_\infty]$

Entropy inequality: $\frac{dH}{dt} + P[u] \le 0$, we need $P[u] \ge \lambda H[u|u_\infty]$

$$P[u] = \int_\Omega \nabla w : B\nabla w dx + \sum_{a=1}^{N} \int_\Omega \big(k_f^a u^{\alpha^a} - k_b^a u^{\beta^a}\big) \log \frac{k_f^a u^{\alpha^a}}{k_b^a u^{\beta^a}} \ge 0$$

Large-time asymptotics for reactive mixtures

$$P[u] = \int_\Omega \nabla w : B\nabla w dx + \sum_{a=1}^{a} \int_\Omega \left(k_f^a u^{\alpha^a} - k_b^a u^{\beta^a}\right) \log \frac{k_f^a u^{\alpha^a}}{k_b^a u^{\beta^a}} \geq 0$$

- Homogeneous equilibrium: $\nabla w = 0 \Rightarrow u_\infty = u(w)$ constant
- Detailed-balance equilibrium u: $k_f^a u_\infty^{\alpha^a} = k_b^a u_\infty^{\beta^a}$ (there are many!)
- Wegscheider matrix: $W = (\beta_i^a - \alpha_i^a)_{ia}$, $q_1, \ldots, q_m$ basis of $\ker(W^\top)$, $Q = (q_1, \ldots, q_m)^\top$
- Conservation laws: $\partial_t Q \int_\Omega u(t)dx = \int_\Omega Qf(u)dx = 0$, $t > 0$

Theorem (Daus-A.J.-Tang 2020)

- *$\exists$ unique detailed-balance equilibrium u_∞^* satisfying conserv. laws*
- *$\exists$ $\lambda > 0$: $P[u] \geq \lambda H[u|u_\infty^*]$*
- *Exponential convergence to equilibrium for $1 \leq p < \infty$:*

$$\|u(t) - u_\infty^*\|_{L^p(\Omega)} \leq C(u^0)e^{-\lambda t/(2p)}, \quad t > 0$$

❹ Regularity of solutions

$$\partial_t u - \mathrm{div}(A(u)\nabla u) = f(u) \quad \text{in } \Omega_T = \Omega \times (0,T), \quad u(0) = u^0$$

Negative result:

- Stará-John 1995: $\exists\, A \in L^\infty$: $u(t)$ Hölder blows up at $t = 1$ in L^∞

Full regularity:

- Amann 1990: $u(t)$ bounded in $W^{1,p}(\Omega)$, $p > d \Rightarrow u$ classical solution
- D. Le 2017: $A(u)$ has polynomial growth of order ≤ 5, $u(t) \in \mathrm{BMO}$ $\Rightarrow u$ classical solution ("Bounded Mean Oscillation", $L^\infty \subset \mathrm{BMO} \subset L^p_{\mathrm{loc}}$)

Partial regularity:

- Giaquinta-Struwe 1982 ($A(u)$ pos. def.): u is Hölder continuous in $\Omega_T \setminus S$, where $\mathcal{H}_{d-\varepsilon}(S) = 0$ for some $\varepsilon > 0$
- Braukhoff-Raithel-Zamponi 2020 ($h''(u)A(u)$ pos. def.): u bounded $\Rightarrow u$ is Hölder continuous in $\Omega_T \setminus S$, $\mathcal{H}_{d-\varepsilon}(S) = 0$

  Idea: Use relative entropy $h(u|v) = h(u) - h(v) - h'(v) \cdot (u - v)$ and $h(u|v) \sim |u - v|^2$ for u_i far from zero, $A_{ij}(u)$ diagonal for $u_i \to 0$

Summary

$$\partial_t u - \operatorname{div}(A(u)\nabla u) = f(u), \quad t > 0, \quad u \in \mathbb{R}^n$$

Boundedness-by-entropy method:

- Gives global existence of bounded weak solutions
  - Compared to Alt-Luckhaus: degeneracies allowed, bounded solutions
  - Compared to Amann: "easy-to-verify" conditions for global results
- Main ingredient: $\exists$ entropy $h(u)$ such that $h''(u)A(u)$ pos. semidef.
- Relation to thermodynamics: $w = h'(u)$ are chemical potentials

Entropy methods are used to prove:

- Global existence of bounded weak solutions: for volume-filling models
- Uniqueness of weak solutions, weak-strong uniqueness
- Large-time asymptotics: exponential decay to equilibrium
- Regularity of solutions: only partial results, problem mainly open

Perspectives

$$\partial_t u - \mathrm{div}(A(u)\nabla u) = f(u), \quad t > 0, \quad u \in \mathbb{R}^n$$

Further topics:

- Numerical schemes preserving entropy structure: finite volumes, finite elements, finite differences (A.J.-Zurek 2020)
- Cross-diffusion systems with stochastic noise (Dhariwal-Huber-A.J.-Kuehn-Neamtu 2020)

Open problems:

- Existence of global weak solutions to n-species SKT population model without detailed balance, for all $a_{ij} > 0$
- Size of class of diffusion systems having an entropy structure
- Generalization of relative entropy for weak-strong uniqueness
- Analysis of models for nonisothermal, compressible fluid mixtures
- Derivation of noise terms for cross-diffusion systems