MINI-COURSE ON CROSS-DIFFUSION SYSTEMS WITH ENTROPY STRUCTURE

ANSGAR JÜNGEL

Abstract. Cross-diffusion equations describe the diffusive interaction in multicomponent systems arising in, for instance, population dynamics, biological cells, and gas mixtures. The models consist of quasilinear parabolic equations with a full diffusion matrix. The major challenge is that the diffusion matrix is generally neither symmetric nor positive definite in applications. A common feature of the models is that they possess an entropy structure. This structure provides gradient estimates and is exploited to analyze the cross-diffusion systems mathematically. In this mini-course, the global existence analysis of such systems is presented (boundedness-by-entropy method) and applied to some examples from biology and physics. An outlook to further topics like regularity of solutions, large-time asymptotics, and weak-strong uniqueness is also given.

1. Introduction

Diffusion plays a pivotal role in many applications, ranging from physics and chemistry to biology, finance, and even social sciences. Diffusion generally means the movement of particles from a region of high concentration to a region of lower concentration. Interestingly, diffusion may lead to some counter-intuitive phenomena. For instance, constant equilibria that are stable in ordinary differential equations may become unstable if diffusion is added (Turing instability). This is not necessarily a negative effect: It gives rise to, for instance, **pattern formation** in reaction-diffusion equations, which models chemical kinetics, wind pattern formed in sand, or pattern on animal skins.

The situation is even more complex in diffusion systems. From a mathematical viewpoint, there is generally no general maximum principle (like for the heat equation) and no general regularity theory (like for scalar equations). From an application viewpoint, diffusion systems may show novel phenomena like

- **uphill diffusion**: A chemical flows in the direction of lower concentrations due to the influence of the other chemicals;
- **segregation**: A population species segregates completely from another population species, possibly in such a way that the population densities become discontinuous.

Both phenomena can be explained by **cross diffusion**, i.e. the diffusion effect of one species to another one. In these notes, we will discuss some mathematical results for such systems.

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2. General cross-diffusion systems

Cross-diffusion equations are **quasilinear parabolic equations** for the unknown $u(x,t) = (u_1(x,t),\ldots,u_n(x,t))$ of the form

$$\partial_t u_i - \sum_{j=1}^n \sum_{k,\ell=1}^d \frac{\partial}{\partial x_k}\left(A_{ij}^{k\ell}(x,u)\frac{\partial u}{\partial x_\ell}\right) = f_i(u) \quad \text{in } \Omega,\ t>0,\ i=1,\ldots,n, \tag{1}$$

where $\Omega\subset\mathbb{R}^d$ $(d\ge 1)$ is a bounded domain, $A_{ij}^{k\ell}(u)$ are the diffusion coefficients, and $f_i(u)$ the reaction terms. We impose the initial conditions

$$u_i(x,0) = u_i^0(x) \quad \text{for } x\in\Omega,\ i=1,\ldots,n, \tag{2}$$

and some boundary conditions. Usually, to avoid boundary integrals when formulating the weak formulation, we suppose no-flux boundary conditions,

$$\sum_{j=1}^n \sum_{k,\ell=1}^d \nu_k A_{ij}^{k\ell}(u)\frac{\partial u_j}{\partial x_\ell} = 0 \quad \text{on } \partial\Omega,\ t>0,\ i=1,\ldots,n, \tag{3}$$

where $\nu = (\nu_1,\ldots,\nu_d)$ is the exterior unit normal vector to $\partial\Omega$ (which is assumed to exist). Other boundary conditions (Dirichlet, Robin, nonhomogeneous) can be used as well, making however the analysis more technical. Amann [3] has shown that if $\partial\Omega\in C^3$, $A_{ij}^{k\ell}\in C^2$, the initial data and the source term $f(u)$ are smooth, and if

$$\sigma\left(\sum_{k,\ell=1}^d A_{ij}^{k\ell}(x,u)\xi_i^k\xi_j^\ell\right)_{i,j=1}^n \subset \{z\in\mathbb{C}: \mathrm{Re}(z)>0\} \tag{4}$$

for all $(x,u)\in\overline{\Omega}\times\mathbb{R}^n$, $(\xi_i^k)\in\mathbb{R}^{n\times d}$ with $\xi\neq 0$, then there exists a **unique local-in-time classical solution** u to (1)-(3). (Here, $\sigma(\cdot)$ denotes the spectrum of a matrix.)

In many applications, the diffusion matrix $A_{ij}^{k\ell}(x,u)$ simplifies to $A_{ij}^{k\ell}(x,u) = A_{ij}(u)\delta_{k\ell}$. Then the initial-boundary value problem (1)–(3) becomes

$$\partial_t u_i - \operatorname{div}\left(\sum_{j=1}^n A_{ij}(u)\nabla u_j\right) = f_i(u) \quad \text{in } \Omega,\ t>0,\ i=1,\ldots,n, \tag{5}$$

$$u(\cdot,0) = u_i^0 \quad \text{in } \Omega, \quad \sum_{j=1}^n A_{ij}(u)\nabla u_j\cdot\nu = 0 \quad \text{on } \partial\Omega,\ t>0. \tag{6}$$

It is convenient to write equations (5) more compactly in the vector form

$$\partial_t u - \operatorname{div}(A(u)\nabla u) = f(u).$$

In this situation, the positivity condition (4) becomes $\sigma(A_{ij}(u))_{i,j=1}^n\subset\{\mathrm{Re}(z)>0\}$ for all $u\in\overline{\Omega}$. This means that the real parts of all eigenvalues of $A(u)$ are positive. According to Amann [3], we call the diffusion matrix $A(u)$ (together with the corresponding boundary conditions) **normally elliptic**.

Amann's results show that the **local solvability** of such equations can be shown under rather general conditions. Note that normal ellipticity is much weaker than positive definiteness of $A(u)$ and weaker than the uniform Legendre–Hadamard condition. However, we obtain solutions only in a possibly small time interval. Under which conditions the solutions become global?

Amann [2] has shown **global solvability** if the local solution with maximal existence time T satisfies the estimate

$$\sup_{0\le t<T} \|u(t)\|_{W^{1,p}(\Omega)} < \infty \quad \text{for } p > d.$$

Often, we obtain weak solutions with bounds in $H^1(\Omega)$ only, corresponding to $p = 2$, which is not sufficient to conclude global solvability in in two- or higher-dimensional problems. Unfortunately, higher-order estimates with $p > d$ are usually very difficult to prove.

Even worse: Stará and John [25] have shown the following **negative results** for systems (1) with $f(u) = 0$:

- There exists a uniformly strongly elliptic real analytic diffusion matrix $(A_{ij}^{k\ell}(u))$ and a real analytic solution u to (1) in $\mathbb{R}^d$ with $d \ge 3$ and $n = d$ such that u develops a singularity in finite time. (Uniformly strongly elliptic means that $A_{ij}^{k\ell}(u)\xi_i^k\xi_j^\ell \ge c|\xi|^2$ for all $\xi \in \mathbb{R}^{d\times d}$ and some $c > 0$.)
- There exists a uniformly elliptic bounded *linear* diffusion matrix $(A_{ij}^{k\ell}(x,t))$ and a solution u to (1) in $\mathbb{R}^d$ with $d \ge 3$ such that u is Hölder continous up to $t = 1$ and blows up in the $L^\infty(\mathbb{R}^d)$ norm if $t \to 1$.

Even for reaction-diffusion systems with **diagonal diffusion matrix**, Pierre and Schmitt [23] have found solutions that blow up in finite time in the $L^\infty(\Omega)$ norm, while still existing as a weak solutions beyond the blow-up time. This shows that parabolic systems generally do not have regularizing properties like the heat equation. Moreover, we cannot expect global-in-time solutions for general cross-diffusion systems and some conditions are needed to guarantee global solvability.

Alt and Luckhaus [1] considered a variant of (5), namely,

$$\partial_t u_i - \operatorname{div} a_i(u, \nabla w) = f_i(u) \quad \text{in } \Omega,\ t > 0, \quad \text{where } u = b(w), \tag{7}$$

and $b : \mathbb{R}^n \to \mathbb{R}^n$ is monotone, i.e. $(b(u) - b(w)) \cdot (v - w) \ge 0$ for all $v, w \in \mathbb{R}^n$, and a gradient, i.e., there exists a convex $\Phi : \mathbb{R}^n \to \mathbb{R}$ such that $b = \Phi'$. Then, under some regularity and growth conditions and the uniform **strongly ellipticity** requirement

$$(a(u,p) - a(u,q)) \cdot (p - q) \ge c|p - q|^r \quad \text{for all } p, q \in \mathbb{R}^{n\times d},\ u \in \mathbb{R}^n,$$

where $c > 0$ and $r > 1$ are some constants and $a = (a_1, \ldots, a_n)$, they could show that there exists a **global weak solution** u to (7) with initial conditions (2) and Dirichlet boundary conditions. Let $h := \Phi^*$, where Φ^* is the Legendre transform of Φ. Then $h'(u) = (\Phi^*)'(u) = (\Phi')^{-1}(u) = b^{-1}(u) = w$, and we recover (5) by choosing $a(u, \nabla w) = A(u)h''(u)^{-1}\nabla w$, since $\nabla w = \nabla h'(u) = h''(u)\nabla u$. The existence proof is based on the observation that (for the sake of simplicity, we assume that no source terms are present, the Dirichlet conditions are homogeneous, and $a(u,0)=0$), the test function w leads to the inequality

$$\frac{d}{dt}\int_\Omega h(u)dx = \int_\Omega \underbrace{h'(u)}_{=w}\cdot\partial_t u dx = -\int_\Omega a(u,\nabla w)\cdot\nabla w dx \le -c\int_\Omega |\nabla w|^r dx,$$

where the last step follows from the ellipticity of $a(u,\cdot)$. This shows that $\int_\Omega h(u)dt$ is a Lyapunov functional along the solutions, and it provides a bound for ∇w in $L^r(\Omega)$.

Our observation is that many cross-diffusion systems from applications have a structure similar to (7). However, the ellipticity condition of Alt and Luckhaus may be not satisfied. Therefore, we wish to **extend** their approach and prove the global existence of weak solutions such that

- the diffusion matrix is not necessarily uniformly strongly elliptic;
- the solutions are bounded in $L^\infty(\Omega)$ for all time.

We identify the functional $h:\mathbb{R}^n\to\mathbb{R}$ as the **entropy** or **free energy** of the diffusion system. Thus, our approach is fundamentally **physics-oriented**, i.e., we aim to prove exactly those mathematical properties (global solvability, nonnegativity, boundedness, etc.) that are expected from the physical modeling. It turns out that the thermodynamic principles underlying the model play a role also in other applications like in biology or chemistry. Thus, we wish to consider **cross-diffusion systems with an entropy structure**.

3. Examples

We present two concrete examples from applications, one from thermodynamics and another one from biology.

3.1. **Two-species Maxwell–Stefan equations.** The Maxwell–Stefan equations describe the evolution of the volume fractions of a multicomponent gas. Examples are air, perfumes, and heliox (mixture of helium and oxygene). For notational simplicity, we consider a three-component mixture with volume fractions u_1, u_2, and u_3. These variables are dimensionless since they are percentages, and they add up to one, $u_1+u_2+u_3=1$. Therefore, we only need two evolution equations for u_1 and u_2, and u_3 is determined from $u_3=1-u_1-u_2$. The equations are given by $\partial_t u-\operatorname{div}(A(u)\nabla u)=f(u)$ in Ω, where $u=(u_1,u_2)$ and

$$A(u)=\frac{1}{\delta(u)}\begin{pmatrix} d_2+(d_0-d_2)u_1 & (d_0-d_1)u_1 \\ (d_0-d_2)u_2 & d_1+(d_0-d_1)u_2\end{pmatrix}$$

is the diffusion matrix with constants $d_0,d_1,d_2>0$, where $\delta(u)=d_1d_2(1-u_1-u_2)+d_0(d_1u_1+d_2u_2)$. The source term $f(u)$ models, for instance, chemical reactions. These equations have been suggested by J. Maxwell in 1866 [22] and J. Stefan in 1871 [26]. There are several mathematical issues associated with this problem:

- The matrix $A(u)$ is neither symmetric nor positive definite, and it is not easy to determine whether the real parts of its eigenvalues are positive for all u.
- Since there is generally no maximum principle, how to determine the bounds $u_1,u_2\ge 0$ and $u_1+u_2\le 1$?

The key of the analysis is the observation that the Maxwell–Stefan system possesses an **entropy structure**. Let

$$h(u) = u_1(\log u_1 - 1) + u_2(\log u_2 - 1) + u_3(\log u_3 - 1) \quad \text{for } u \in [0,\infty)^2,$$

where $u_3 = 1 - u_1 - u_2$ is considered as a function of u_1 and u_2. Then $w_i = \partial h/\partial u_i = \log(u_i/u_3)$ is called an **entropy variable**. (In thermodynamics, this variable is known as the chemical potential of the ith species.) We find after a computation that

$$\frac{d}{dt}\int_\Omega h(u)dx = \int_\Omega w\cdot\partial_t u dx = -\int_\Omega \big(\alpha_{11}|\nabla u_1|^2 + 2\alpha_{12}\nabla u_1\cdot\nabla u_2 + \alpha_{22}|\nabla u_2|^2\big)dx,$$

where

$$\alpha_{11} = \frac{u_2(d_0u_1 + d_2u_3)}{u_1u_2u_3}, \quad \alpha_{12} = \frac{d_0u_1u_2}{u_1u_2u_3}, \quad \alpha_{22} = \frac{u_1(d_0u_2 + d_1u_3)}{u_1u_2u_3}.$$

This derivative is nonpositive since

$$\alpha_{11}\alpha_{22} - \alpha_{12}^2 = \frac{d_0(d_1u_1 + d_2u_2) + d_1d_2u_3}{u_1u_2u_3} \ge 0.$$

This shows that $t \mapsto \int_\Omega h(u(t))dx$ is a Lyapunov functional. It can be shown [5, 21] that there exists a constant $c > 0$, only depending on the coefficients d_i, such that

$$\frac{d}{dt}\int_\Omega h(u)dx + c\int_\Omega \big(|\nabla\sqrt{u_1}|^2 + |\nabla\sqrt{u_2}|^2 + |\nabla\sqrt{1-u_1-u_2}|^2\big)dx \le 0,$$

where $c > 0$ only depends on $\min\{d_0, d_1, d_2\}$, which provides gradient bounds for $\sqrt{u_1}$ and $\sqrt{u_2}$.

3.2. **Shigesada–Kawasaki–Teramoto (SKT) population model.** The SKT population system models the evolution of two segregating populations. It can be generalized to n population species [9] but for the sake of simiplicity, we consider two species only. The equations are given by $\partial_t u - \operatorname{div}(A(u)\nabla u) = f(u)$ in Ω, where $u = (u_1, u_2)$ is the vector of population densities of the two species and

$$A(u) = \begin{pmatrix} a_{10} + 2a_{11}u_1 + a_{12}u_2 & a_{12}u_1 \\ a_{21}u_2 & a_{20} + a_{21}u_1 + 2a_{22}u_2 \end{pmatrix},$$

where $a_{ij} \ge 0$ for $i = 0, 1, 2$ and $j = 1, 2$. The function $f(u)$ on the right-hand side is given by Lotka–Volterra terms,

$$f_i(u) = (b_{i0} - b_{i1}u_1 - b_{i2}u_2)u_i, \quad i = 1, 2,$$

where $b_{ij} \ge 0$. These source terms are not problematic in the analysis since $f_i(u)$ is the sum of a linear term and a nonpositive term. This model has been suggested by Shigesada, Kawasaki, and Teramoto to describe the segregation of two population species [24]. Mathematically, the problems are similar as in the previous subsection:

- The matrix $A(u)$ is neither symmetric nor positive definite, and it is not easy to determine whether the real parts of its eigenvalues are positive. (In fact, this property has been recently shown in [12] if $a_{i0} + a_{ii} > 0$ for $i = 1, 2$.)

- Since there is generally no maximum principle, how to determine the bounds $u_1, u_2 \geq 0$? (In fact, using a Stampacchia truncation technique, this question can be easily answered.)
- How to find suitable gradient estimates? (This is the key problem.)

Also this example has an entropy structure. Let

$$h(u) = a_{21}u_1(\log u_1 - 1) + a_{12}u_2(\log u_2 - 1), \quad u_1, u_2 \geq 0.$$

Then the entropy variables become $w_1 = \partial h/\partial u_1 = a_{21}\log u_1$ and $w_2 = \partial h/\partial u_2 = a_{12}\log u_2$, and the time derivative (in case $f(u) = 0$) is computed according to

$$\frac{d}{dt}\int_\Omega h(u)dx = \int_\Omega w\cdot\partial_t u dx = -\int_\Omega \bigg(a_{21}(a_{10} + a_{11}u_1)\frac{|\nabla u_1|^2}{u_1} + a_{12}(a_{20} + a_{22}u_2)\frac{|\nabla u_2|^2}{u_2} + a_{21}a_{12}|\nabla(u_1+u_2)|^2\bigg)dx \leq 0.$$

This shows that $t \mapsto \int_\Omega h(u(t))dx$ is a Lyapunov functional, and if the coefficients a_{ij} are positive, we obtain $L^2(\Omega)$ bounds for $\sqrt{u_i}$ and u_i, $i = 1, 2$.

4. Boundedness-by-entropy method

We consider cross-diffusion systems of the type

$$\partial_t u_i - \operatorname{div}\left(\sum_{j=1}^n A_{ij}(u)\nabla u_j\right) = f_i(u) \quad \text{in } \Omega,\ t > 0,\ i = 1, \ldots, n, \tag{8}$$

$$u(\cdot, 0) = u_i^0 \quad \text{in } \Omega, \quad \sum_{j=1}^n A_{ij}(u)\nabla u_j\cdot\nu = 0 \quad \text{on } \partial\Omega,\ t > 0. \tag{9}$$

We wish to determine conditions that ensure the global solvability of this system. First, let $f(u) = 0$; we discuss the general case below.

Motivated by the previous examples, we want to find a functional $h : D \to \mathbb{R}$ such that $(d/dt)\int_\Omega h(u)dx \leq 0$. The domain $D \subset \mathbb{R}^n$ equals the simplex $D = \{(u_1, u_2) : u_1, u_2, 1 - u_1 - u_2 < 0\}$ for the Maxwell–Stefan system and $D = (0, \infty)^2$ for the SKT system. We set $w = h'(u)$. Then $\nabla w = h''(u)\nabla u$ and

$$0 \geq \frac{d}{dt}\int_\Omega h(u)dx = \int_\Omega w\cdot\partial_t u dx = -\int_\Omega \nabla w^T A(u)\nabla u dx = -\int_\Omega (h''(u)\nabla u)^T A(u)\nabla u dx = -\int_\Omega \nabla u^T h''(u)A(u)\nabla u dx.$$

We need the positive semidefiniteness of $h''(u)A(u)$ to fulfill this inequality. In fact, we also need gradient estimates. This is possible if $h''(u)A(u)$ is positive definite (in the sense $z^T h''(u)A(u)z \geq c|z|^2$ for all $z \in \mathbb{R}^n$ and some $c > 0$), and this condition yields gradient bounds for u_i. However, we have seen in the examples that we obtain gradient estimates for $\sqrt{u_i}$ but not necessarily for u_i. Thus, we need a condition that is weaker than positive definiteness but stronger than positive semidefiniteness. We suggest that the matrix is positive definite but with a constant depending on $u \in D$, which we call "nonlinearly" positive definite. Thus, we impose the following **first condition**: There exist $\alpha \geq 0$ and $c > 0$ such that

$$z^T h''(u)A(u)z \geq c\sum_{i=1}^n u_i^{2\alpha-1}z_i^2 \quad \text{for all } z \in \mathbb{R}^n,\ u \in D. \tag{10}$$

Note that we have $\alpha = 0$ in the Maxwell–Stefan system and $\alpha = 1/2$ in the SKT system if $a_{11}, a_{22} > 0$. Condition (10) can be generalized by replacing the exponent $2\alpha - 1$ by $2\alpha_i - 1$ for $\alpha_i \geq 0$.

Remark 1 (Comparison with Alt and Luckhaus [1])**.** Because of $\nabla w = \nabla h'(u) = h''(u)\nabla u$, we have $A(u)\nabla u = A(u)h''(u)^{-1}\nabla w$ and hence $a(u, \nabla w) = A(u)h''(u)^{-1}\nabla w$. Alt and Luckhaus assume strong ellipticity for $a(u, p)$, which here reads as

$$(a(u,p) - a(u,q)) \cdot (p-q) = (p-q)^T A(u)h''(u)^{-1}(p-q) \geq c|p-q|^2.$$

This means that $A(u)h''(u)^{-1}$ is required to be positive definite. We are able to consider more general conditions. □

Next, let us consider source terms $f(u)$ in (8). Then

$$\frac{d}{dt}\int_\Omega h(u)dx + c\sum_{i=1}^n \int_\Omega u_i^{2\alpha-1}|\nabla u_i|^2 dx \leq \int_\Omega f(u) \cdot h'(u)dx.$$

If the right-hand side is nonpositive, $t \mapsto \int_\Omega h(u(t))dx$ is again a Lyapunov functional. On finite time interval, we may assume the weaker **second condition**: There exists $C > 0$ such that

$$f(u) \cdot h'(u) \leq C(1 + h(u)) \quad \text{for } u \in D. \tag{11}$$

If $h(u)$ is of the type $u_i \log u_i$ like in our examples, this condition means that $f(u)$ should be of at most linear growth. This is rather restrictive, but it is satisfied for the Lotka–Volterra terms, for instance. With these conditions, the **entropy inequality** becomes

$$\frac{d}{dt}\int_\Omega h(u)dx + c\sum_{i=1}^n \int_\Omega |\nabla u_i^{\alpha+1/2}|^2 dx \leq C + C\int_\Omega h(u)dx, \tag{12}$$

and Gronwall's lemma gives some estimates for $h(u)$ and $\nabla u_i^{\alpha+1/2}$.

The idea now is to transform system (8) into a system in the entropy variable w,

$$\partial_t u(w) - \operatorname{div}(B(w)\nabla w) = f(u(w)), \quad \text{where } B(w) = A(u(w))h''(u(w))^{-1},$$

where u is interpreted as a function of w, and we have used the fact that $\nabla w = \nabla h'(u) = h''(u)\nabla u$ is equivalent to $\nabla u = h''(u)^{-1}\nabla w$. Observe that $B(w)$ is positive semidefinite, by the first condition. Then we transform back to the variable $u = (h')^{-1}(w)$, for which we need to assume that h' is invertible. This leads to our **third condition**:

$$h \in D^2(D) \text{ is convex and } h' : D \to \mathbb{R}^n \text{ is invertible.} \tag{13}$$

The convexity of h is needed to obtain the positive definiteness of the Hessian $h''(u)$. Note that if $h : D \to \mathbb{R}$ is a differentiable on a convex domain $D \subset \mathbb{R}^n$ and if h is strictly convex, then its gradient has a well-defined inverse. In this situation, our third condition is satisfied.

Example 1 (Maxwell–Stefan system)**.** We recall that $w_i = \log(u_i/u_3)$ for $i = 1, 2$. Summing $u_3 \exp(w_1) = u_1$ and $u_3 \exp(w_2)$ gives $u_3(1+\exp(w_1)+\exp(w_2)) = 1$ and consequently,

$$u_i = u_3 \exp(w_i) = \frac{\exp(w_i)}{1 + \exp(w_1) + \exp(w_2)} \quad \text{for } i = 1, 2.$$

Interestingly, the transform $u(x,t) = (h')^{-1}(w(x,t))$ shows, since $(h')^{-1} : \mathbb{R}^n \to D$, that the solution $u(x,t)$ lies in D, i.e. $0 < u_1(x,t) + u_2(x,t) < 1$ for $x \in D$, $t > 0$.

The example shows that if D is **bounded**, we conclude automatically the **boundedness** of u. In the SKT model, we have $D = (0,\infty)^2$ and then $u_1 = \exp(w_1/a_{21}) > 0$ and $u_2 = \exp(w_2/a_{12}) > 0$. This proves the positivity of the populations densities. (Since we need to perform the limit of vanishing approximation parameters, we obtain eventually only nonnegativity.) *We stress the fact that these pointwise bounds are obtained without the use of a maximum principle, which usually cannot be applied to cross-diffusion systems.*

We can prove the following existence theorem.

Theorem 2 (Global existence)**.** *Let conditions* (10)*,* (11)*, and* (13) *hold and assume that* $T > 0$, $A \in C^0(D; \mathbb{R}^{n\times n})$, $f \in C^0(D; \mathbb{R}^n)$, $u^0 \in L^1(\Omega; \mathbb{R}^n)$ *such that* $u^0(x) \in \overline{D}$ *for* $x \in \Omega$. *Then there exists a bounded weak solution to* (8)*–*(9) *satisfying*

$$u_i(x,t) \in \overline{D} \quad \text{for } x \in \Omega,\ t > 0, \quad u_i \in L^\infty(0,T; L^\infty(\Omega)) \cap L^2(0,T; H^1(\Omega)),$$
$$\partial_t u_i \in L^2(0,T; H^1(\Omega)'), \quad i = 1, \ldots, n.$$

We say that u is a weak solution to (8)–(9) if $u(\cdot, 0) = u^0$ in the sense of $H^1(\Omega)'$ and if for all test functions $\phi \in L^2(0,T; H^1(\Omega; \mathbb{R}^n))$,

$$\int_0^T \langle \partial_t u, \phi\rangle dt + \int_0^T \int_\Omega \nabla\phi^T A(u) \nabla u dx dt = \int_0^T \int_\Omega f(u) \cdot \phi dx dt,$$

where $\langle \cdot, \cdot \rangle$ is the dual product between $H^1(\Omega; \mathbb{R}^n)'$ and $H^1(\Omega; \mathbb{R}^n)$.

Strictly speaking, this theorem holds for $0 \le \alpha \le 1/2$; if $\alpha > 1/2$, we need a growth condition for $A_{ij}(u)$. The proof of this theorem, which is called the **boundedness-by-entropy method** is rather technical; see [19, 20] for details. We only sketch the main steps:

(1) Definition of the approximate scheme: We discretize in time by using an implicit Euler method and add a higher-order regularization:

$$\frac{1}{\Delta t}(u^k - u^{k-1}) - \operatorname{div}(B(w^k)\nabla w^k) + \varepsilon \widetilde{a}(w^k, \cdot) = f(u^k),$$

where $\Delta t > 0$, $u^k := u(w^k)$ approximates $u(\cdot, k\Delta t)$, $\widetilde{a}(\cdot,\cdot)$ is the bilinear form

$$\widetilde{a}(w,\phi) = \int_\Omega \bigg(\sum_{|\alpha|=m} D^\alpha w \cdot D^\alpha \phi + w \cdot \phi \bigg) dx, \quad w,\phi \in H^m(\Omega;\mathbb{R}^n),$$

$\varepsilon > 0$, and $m \in \mathbb{N}$ is chosen such that $H^m(\Omega) \hookrightarrow L^\infty(\Omega)$, i.e. $m > d/2$. By the generalized Poincaré inequality, we have $\widetilde{a}(w,w) \geq C\|w\|^2_{H^m(\Omega)}$, so we expect solutions $w \in H^m(\Omega;\mathbb{R}^n)$. This ensures that $w \in L^\infty(\Omega;\mathbb{R}^n)$ and the function $w \mapsto u$ is well defined.

(2) Solution of the approximate scheme: We linearize the approximated system by replacing $A(u(w))$ and $f(u(w))$ by $A(u(\widehat{w}))$ and $f(u(\widehat{w}))$ for some $\widehat{w} \in L^\infty(\Omega;\mathbb{R}^n)$. By the Lax–Milgram lemma, there exists a unique solution $w \in H^m(\Omega;\mathbb{R}^n)$ to the linear system. This defines the fixed-point operator $F : L^\infty(\Omega;\mathbb{R}^n) \to L^\infty(\Omega;\mathbb{R}^n)$, $\widehat{w} \mapsto w$. Then we verify the conditions of the Leray–Schauder fixed-point theorem. The most delicate one is the uniform bound which is obtained from the discrete regularized entropy estimate similar to (12).

(3) Estimates uniform in Δt and ε: By summing the discrete entropy inequality over the time steps k, we conclude some uniform bounds for $(u_i^k)^{\alpha+1/2}$ in $H^1(\Omega)$. Since $u_i^k(x) \in D$ for $x \in \Omega$ and consequently $u_i^k \in L^\infty(\Omega)$, we infer that $\nabla u_i^k = (u_i^k)^{1/2-\alpha}\nabla(u_i^k)^{\alpha+1/2}$, which gives an $H^1(\Omega)$ bound for u_i^k if $\alpha \leq 1/2$. If $\alpha > 1/2$, the exponent in $\nabla(u_i^k)^{\alpha+1/2}$ is larger than one, and we are in the situation of the porous-medium equation. This requires some conditions on $A(u)$ and we refer to [20, Sec. 4.4] for details.

(4) Limit $(\Delta t, \varepsilon) \to 0$: Thanks to the uniform estimates, we are able to perform the simultaneous limit $\Delta t \to 0$ and $\varepsilon \to 0$. This is done by applying the discrete Aubin–Lions lemma in the version of [13], which yields the a.e. convergence of u_i^k. This allows us to identify the limits in the nonlinearities.

One may ask how the entropy structure of a cross-diffusion system can be determined if there is any? This question cannot be answered in full generality but we can give some hints:

- Often, there are thermodynamic considerations behind a cross-diffusion system at hand (sometimes even if it describes biological phenomena). Then the free energy of the physical system is a good candidate for an entropy functional in the sense presented here.
- It is shown in [12] that any cross-diffusion system with an entropy structure has a normally elliptic diffusion matrix $A(u)$ (meaning that the real parts of all eigenvalues of the matrix are positive). Thus, if $A(u)$ is *not* normally elliptic, the cross-diffusion system cannot have an entropy structure.
- If $A(u)$ is normally elliptic for all $u \in D$ and if $h''(u)A(u)$ is symmetric for some convex function $h(u)$, then the cross-diffusion system has an entropy structure [12].

5. Further topics

5.1. **Regularity of solutions.** Generally, the fixed-point theorems like the Leray–Schauder or Schauder theorems yield only weak solutions, i.e., functions whose gradients are integrable. One may ask whether solutions to cross-diffusion systems are more regular. As shown by Stará and John [25], this is generally not true. However, numerical simulations of cross-diffusion systems from applications, in particular those with an entropy structure, often appear to be smooth. Therefore: Can we prove more regularity for such solutions? The answer is open up to now, but there are some partial results, which are presented in the following:

- Solutions to cross-diffusion systems may become **discontinuous**. An example is the population system [4]

$$\partial_t u_1 = \operatorname{div}(u_1\nabla(u_1+u_2)), \quad \partial_t u_2 = \operatorname{div}(u_2\nabla(u_1+u_2)) \quad \text{in } \Omega,\ t>0,$$

  with initial and no-flux boundary conditions. This system models two populations that have the tendency to segregate. It can be reformulated as

$$\begin{aligned} \partial_t w &= \operatorname{div}(w\nabla w), \quad \text{where } w := u_1+u_2, \\ \partial_t u_2 &= \operatorname{div}(u_2\nabla w). \end{aligned}$$

  The first equation is a porous-medium equation, while the second one is a transport equation. Hence, it can happen that $u_2(t)$ becomes discontinuous in finite time, which is supported by numerical experiments [8]. This property is not surprising since the determinant of the diffusion matrix vanishes, so there is a vanishing eigenvalue, and Amann's theory cannot be applied, even not locally. Still, the existence of solutions can be shown; see [4, 14].
- The boundedness-by-entropy method yields **bounded** weak solutions. This is better than just $H^1(\Omega)$ weak solutions but still not sufficient to conclude strong or classical solutions.
- **Local regularity:** Higher integrability holds for cross-diffusion systems if the diffusion matrix is bounded and uniformly positive definite [18]. More precisely, if u solves (1) with the right-hand side $f_i + \operatorname{div} g_i$, where $f_i, g_i \in L^p(0,T;L^p(\Omega))$ for $p\in[2,p_0)$ and some $p_0>2$. Then $u\in L^\infty(0,T;L^p_{\rm loc}(\Omega))$. Moreover, with vanishing source terms and in two space dimensions, the solution to (1) is locally Hölder continuous if the diffusion matrix is differentiable in u [18].
- For cross-diffusion systems in **Laplacian form**, the duality method of Pierre and Schmitt [23] provides solutions in $L^q(Q_T)$ for some $q>2$ (we have set $Q_T = \Omega\times(0,T)$), which improves the usual $L^2(Q_T)$ bound. The result is as follows (we omit some technical details on $\partial\Omega$ and the initial datum) [7, 23]:

  Theorem 3 (Duality method)**.** *Let $M\ge c>0$ be integrable in Q_T for some $c>0$. Any nonnegative smooth solution to*

$$\begin{aligned} &\partial_t v - \Delta(M(x,t)v) = 0 \quad \text{in } Q_T, \\ &v(\cdot,0) = v^0 \quad \text{in } \Omega, \quad \nabla(Mv)\cdot\nu = 0 \quad \text{on } \partial\Omega,\ t>0, \end{aligned}$$

satisfies the a priori estimate

$$\int_0^T\int_\Omega Mv^2dxdt \le C\bigg(T, v^0, \int_0^T\int_\Omega Mdxdt\bigg).$$

If $0 < m_* \le M \le m^*$ *in* Q_T *for some* $m_*, m^* > 0$ *and* $v^0 \in L^p(\Omega)$ *for some* $p > 2$ *then* $\|v\|_{L^p(Q_T)} \le C(T, v^0, m_*, m^*)$ *provided that* $m^* - m_*$ *is sufficiently small.*

Example 2 (SKT model)**.** The SKT model without Lotka–Volterra terms can be written as

$$\partial_t u_i = \Delta(u_i p_i(u)), \quad \text{where } p_i(u) = a_{i0} + a_{ii}u_i + a_{ij}u_j.$$

for $i, j = 1, 2$, $i \ne j$. Then the duality estimates gives

$$\int_0^T\int_\Omega (a_{i0} + a_{ii}u_i + a_{ij}u_j)u_i^2dxdt \le C, \quad i = 1, 2,$$

and consequently an $L^3(Q_T)$ bound for u_i if $a_{ii} > 0$.

- Braukhoff, Raithel, and Zamponi [6] have shown the **partial Hölder regularity** of bounded weak solutions to cross-diffusion systems which possess a so-called glued entropy. *Partial Hölder regularity* means that the solution u_i is Hölder continuous on $B \subset Q_T$ such that $u_i \in C^{0,\alpha}(B)$ for some $\alpha \in (0, 1)$. The singular set $Q_T \setminus B$ is of parabolic Hausdorff dimension smaller than $d - \gamma$ for some $\gamma > 0$. A *glued entropy* is a regularized entropy which distinguishes the cases u_i being close to zero (assuming that $A(u)$ becomes close to a constant diagonal matrix) and u_i being far from zero (and the Hessian of the entropy is bounded). For instance, this theorem applies to the Maxwell–Stefan equations and to the SKT system (provided that the weak solution is bounded).

5.2. **Large-time asymptotics.** In some situations, the entropy inequality (12) yields exponential decay rates for the solution $u(t)$ to (8)–(9) towards a steady state. We assume that the cross-diffusion system has a unique *constant* steady state u_∞ (to simplify). We introduce the relative entropy

$$H(u|u_\infty) = \int_\Omega \big(h(u) - h(u_\infty) - h'(u_\infty)\cdot(u - u_\infty)\big)dx.$$

Indeed, let $f(u)\cdot(h'(u) - h'(u_\infty)) \le 0$ for $u \in D$. Then (12) becomes

$$\begin{aligned}\frac{dH}{dt}(u|u_\infty) &= \int_\Omega (h'(u) - h'(u_\infty))\cdot\partial_t u dx \\ &= -\int_\Omega \nabla u^T h''(u)A(u)\nabla u dx + \int_\Omega f(u)\cdot(h'(u) - h'(u_\infty))dx \\ &\le -c\sum_{i=1}^n\int_\Omega |\nabla u_i^{\alpha+1/2}|^2dx.\end{aligned}$$

Suppose that the following inequality holds:

$$H(u|u_\infty) \le C_1 \sum_{i=1}^n \int_\Omega |\nabla u_i^{\alpha+1/2}|^2 dx. \tag{14}$$

Then

$$\frac{dH}{dt}(u|u_\infty) + \frac{c}{C_1} H(u|u_\infty) \le 0,$$

and Gronwall's lemma implies that

$$H(u(t)|u_\infty) \le e^{-ct/C_1} H(u^0|u_\infty), \quad t > 0,$$

proving exponential decay in the "measure of $H(u|u_\infty)$". It remains the question in which situations the functional inequality (14) holds? We consider two simple examples.

- Let $\alpha = 1/2$ and $h(u) = \frac{1}{2}\sum_{i=1}^n u_i^2$. Then $H(u|u_\infty) = \frac{1}{2}\int_\Omega |u-u_\infty|^2 dx$. If $f(u) = 0$, the steady state u_∞ is given by that constant that has the same total mass as u_i. By mass conservation, the total mass of $u_i(t)$ is the same as of u_i^0. Thus,

  $$u_{\infty,i} = \bar u_i := \frac{1}{|\Omega|}\int_\Omega u_i^0 dx, \quad i = 1, \ldots, n.$$

  The Poincaré–Wirtinger inequality

  $$\int_\Omega (u_i - \bar u_i)^2 dx \le C_P \int_\Omega |\nabla u_i|^2 dx$$

  provides the desired inequality, since the previous inequality can be written as

  $$H(u|u_\infty) \le \frac{C_P}{2}\int_\Omega |\nabla u_i|^2 dx.$$

- Let $\alpha = 0$ and $h(u) = \sum_{i=1}^n u_i(\log u_i - 1)$. (This is the situation of the Maxwell–Stefan system if $f(u) = 0$ as well as of the SKT model if $f(u) = 0$ and $a_{10}, a_{20} > 0$.) Then we use the so-called logarithmic Sobolev inequality:

  $$\sum_{i=1}^n \int_\Omega u_i \log\frac{u_i}{\bar u_i} dx \le C_L \sum_{i=1}^n \int_\Omega |\nabla\sqrt{u_i}|^2 dx,$$

  which is the desired inequality since

  $$H(u|u_\infty) = \sum_{i=1}^n \int_\Omega \left(u_i \log\frac{u_i}{u_{\infty,i}} - (u_i - u_{\infty,i})\right) dx = \sum_{i=1}^n \int_\Omega u_i \log\frac{u_i}{u_{\infty,i}} dx,$$

  if the total mass of $u_{\infty,i}$ is the same as the total mass of u_i, as in the first example.

5.3. **Weak-strong uniqueness.** The question of the uniqueness of weak solutions to system (8)–(9) is mainly open. A uniqueness result for a very special class of cross-diffusion systems was proved in [10]. It seems to be easier to prove the weak-strong uniqueness of solutions. This means that a weak solution coincides with the strong solution, emanating from the same initial data, as long as the latter exists. In particular, strong solutions are unique within the class of weak solutions. The idea to prove such a result is based on the **relative entropy method**. Given two solutions u and v, the relative entropy is defined by

$$H(u|v) = \int_\Omega (h(u) - h(v) - h'(v)(u-v))dx.$$

We consider only the case $h(u) = \sum_{i=1}^n u_i(\log u_i - 1)$. Then

$$H(u|v) = \sum_{i=1}^n \int_\Omega \bigg(u_i \log \frac{u_i}{v_i} - (u_i - v_i) \bigg) dx, \tag{15}$$

where u and v are two solutions to (8)–(9). If u_i and v_i are bounded functions, we can estimate $H(u|v)$ from below by [17, Appendix B]

$$H(u|v) \ge \frac{1}{2} \sum_{i=1}^n \int_\Omega (u_i - v_i)^2 dx, \tag{16}$$

and the relative entropy "measures" the distance $u_i - v_i$ with respect to the $L^2(\Omega)$ norm.

Let u be a weak solution and v be a strong solution to the cross-diffusion problem with $f(u) = 0$. We differentiate (15) formally with respect to time and insert the evolution equation (8):

$$\begin{aligned}
\frac{dH}{dt}(u|v) &= \sum_{i=1}^n \int_\Omega \bigg(\partial_t u_i \log \frac{u_i}{v_i} - \partial_t v_i \frac{u_i}{v_i} - \partial_t (u_i - v_i) \bigg) dx \\
&= -\sum_{i,j=1}^n \int_\Omega \bigg\{ A_{ij}(u) u_j \nabla \log u_j \cdot \nabla \log \frac{u_i}{v_i} - A_{ij}(v) \nabla v_j \cdot \nabla \bigg(\frac{u_i}{v_i} \bigg) \bigg\} dx \\
&= -\sum_{i,j=1}^n \int_\Omega \bigg\{ A_{ij}(u) u_j \nabla \log \frac{u_j}{v_j} \cdot \nabla \log \frac{u_i}{v_i} + A_{ij}(u) u_j \nabla \log v_j \cdot \nabla \log \frac{u_i}{v_i} \\
&\quad - A_{ij}(v) \frac{u_i}{v_i} \nabla v_j \cdot \nabla \log \frac{u_i}{v_i} \bigg\} dx \\
&= -\sum_{i,j=1}^n \int_\Omega A_{ij}(u) u_j \nabla \log \frac{u_i}{v_i} \cdot \nabla \log \frac{u_j}{v_j} dx \\
&\quad - \sum_{i,j=1}^n \int_\Omega \bigg(A_{ij}(u) \frac{u_j}{v_j} - A_{ij}(v) \frac{u_i}{v_i} \bigg) \nabla v_j \cdot \nabla \log \frac{u_i}{v_i} \bigg\} dx =: I_1 + I_2.
\end{aligned}$$

Now, the calculations depend strongly on the diffusion coefficients A_{ij}. The aim is to prove that

$$I_1 \le -c_0 \sum_{i=1}^n \int_\Omega u_i^2 \left|\nabla \log \frac{u_i}{v_i}\right|^2 dx,$$
$$I_2 \le c_0 \sum_{i=1}^n \int_\Omega u_i^2 \left|\nabla \log \frac{u_i}{v_i}\right|^2 dx + C(v)H(u|v)$$

for some $c_0 > 0$, since this yields

$$\frac{dH}{dt}(u|v) \le C(v) \int_\Omega H(u|v)dx,$$

where $C(v) > 0$ depends on some norm for v. Here we need v to be a strong solution, since usually, $L^\infty(\Omega)$ bounds for v or ∇v are needed. Assuming that both solutions have the same initial data, we have $H(u(0)|v(0)) = 0$, and we infer from Gronwall's lemma that $H(u(t)|v(t)) = 0$ and hence $u(t) = v(t)$, proving the weak-strong uniqueness.

The Maxwell–Stefan system possesses the weak-strong uniqueness property [17], but in this case we cannot work in the formulation of (2×2)-matrices (since we cannot estimate I_1). Instead, we analyze the equations for (u_1, u_2, u_3) which leads to (3×3)-matrices. Therefore, we illustrate the relative entropy method only for the SKT model.

Example 3 (SKT model)**.** We consider the SKT system with diffusion coefficients $A_{ij}(u) = \delta_{ij}(a_{i0} + \sum_{k=1}^2 a_{ik}u_k) + a_{ij}u_j$. If $a_{11}, a_{22} > 0$, the matrix $(A_{ij}(u)u_j)$ is positive definite in the sense $\sum_{i,j=1}^2 z_iA_{ij}(u)u_jz_j \ge c_0 \sum_{i=1}^2 u_i^2z_i^2$. Then

$$I_1 \le -c_0 \sum_{i=1}^n \int_\Omega u_i^2 \left|\nabla \log \frac{u_i}{v_i}\right|^2 dx,$$
$$\begin{aligned} I_2 &= -\sum_{i,j=1}^n \int_\Omega \bigg\{ \sum_{k=1}^2 a_{ik} \underbrace{\delta_{ij}\left(u_k\frac{u_j}{v_j} - v_k\frac{u_i}{v_i}\right)}_{=u_i(u_k-v_k)/v_i} + a_{ij} \bigg(\underbrace{u_i\frac{u_j}{v_j} - v_i\frac{u_i}{v_i}}_{=u_i(u_j-v_j)/v_j} \bigg) \bigg\} \nabla v_j \cdot \nabla \log \frac{u_i}{v_i} dx \\ &= -\sum_{i,j=1}^n \int_\Omega a_{ij}u_i(u_j - v_j)\nabla \log(v_iv_j) \cdot \nabla \log \frac{u_i}{v_i} dx \\ &\le c_0 \sum_{i=1}^n \int_\Omega u_i^2 \left|\nabla \log \frac{u_i}{v_i}\right|^2 + C(v, c_0) \sum_{i=1}^n \int_\Omega |u_i - v_i|^2 dx, \end{aligned}$$

($C(v, c_0)$ depends on the L^∞ norm of $\nabla \log v_i$ for $i = 1, \ldots, n$) and we conclude that

$$\frac{dH}{dt}(u|v) \le I_1 + I_2 \le C(v, c_0) \sum_{i=1}^n \int_\Omega |u_i - v_i|^2 dx \le CH(u|v),$$

proving, after an application of Gronwall's lemma and using $H(u(0)|v(0)) = 0$, the weak-strong uniqueness.

Unfortunately, the proof is not corrected as stated. Indeed, we have used the lower bound (16) to estimate the $L^2(\Omega)$ norm of $u_i - v_i$ from above by the relative entropy. However, this is only possible if u_i (and v_i) are bounded, which is generally not clear for the weak solutions to the SKT system. This problem can be overcome by truncating u_i as in [11] (the idea goes back to [15]). More precisely, we replace the relative entropy by

$$H_L(u|v) = \sum_{i=1}^n \int_\Omega \big(u_i \log u_i - \phi_L(u) u_i \log v_i - (u_i - v_i)\big) dx,$$

where ϕ_L is a smooth cutoff function satisfying $\phi_L(u) = 1$ if $\sum_{k=1}^n u_k \le L$ and $\phi_L(u) = 0$ if $\sum_{k=1}^n u_k > L + 1$. It holds that [16, Lemma 9]

$$\int_\Omega 1_{\{\sum_{k=1}^n u_k \le L\}} \sum_{i=1}^n |u_i - v_i|^2 dx \le C(L) H_L(u|v). \tag{17}$$

Then, after some computations,

$$\frac{dH_L}{dt}(u|v) \le C(L) H_L(u|v),$$

which, because of $H_L(u(0)|v(0)) = 0$, implies that $H_L(u(t)|v(t)) = 0$ for $t > 0$ and hence, because of (17), $u(t) = v(t)$ in $\{\sum_{k=1}^n u_k \le L\}$. The limit $L \to \infty$ then gives $u(t) = v(t)$ in Ω for $t > 0$.

References

[1] H.-W. Alt and S. Luckhaus. Quasilinear elliptic-parabolic differential equations. *Math. Z.* 183 (1983), 311–341.

[2] H. Amann. Dynamic theory of quasilinear parabolic systems III. Global existence. *Math. Z.* 202 (1989), 219–250.

[3] H. Amann. Dynamic theory of quasilinear parabolic equations II. Reaction-diffusion equations. *Differ. Int. Eqs.* 3 (1990), 13–75.

[4] M. Bertsch, M. Gurtin, D. Hilhorst, and L. Peletier. On interacting populations that disperse to avoid crowding: preservation of segregation. *J. Math. Biol.* 23 (1985), 1–13.

[5] D. Bothe. On the Maxwell–Stefan equations to multicomponent diffusion. *Progr. Nonlin. Differ. Eqs. Appl.*, Springer, Basel, 2011, pp. 81–93.

[6] M. Braukhoff, C. Raithel, and N. Zamponi. Partial Hölder regularity for solutions of a class of cross-diffusion systems with entropy structure. *J. Math. Pures Appl.* 166 (2022), 30–69.

[7] J. Cañizo, L. Desvillettes, and K. Fellner. Improved duality estimates and applications to reaction-diffusion equations. *Commun. Partial Differ. Eqs.* 39 (2014), 1185–1204.

[8] J. A. Carrillo, F. Filbet, and M. Schmidtchen. Convergence of a finite volume scheme for a system of interacting species with cross-diffusion. *Numer. Math.* 145 (2020), 473–511.

[9] X. Chen, E. Daus, and A. Jüngel. Global existence analysis of cross-diffusion population systems for multiple species. *Arch. Rational Mech. Anal.* 227 (2018), 715–747.

[10] X. Chen and A. Jüngel. A note on the uniqueness of weak solutions to a class of cross-diffusion systems. *J. Evol. Eqs.* 18 (2018), 805–820.

[11] X. Chen and A. Jüngel. Weak-strong uniqueness of renormalized solutions to reaction-cross-diffusion systems. *Math. Models Meth. Appl. Sci.* 29 (2019), 237–270.

[12] X. Chen and A. Jüngel. When do cross-diffusion systems have an entropy structure? *J. Differ. Eqs.* 278 (2021), 60–72.

[13] M. Dreher and A. Jüngel. Compact families of piecewise constant functions in $L^p(0,T;B)$. *Nonlin. Anal.* 75 (2012), 3072–3077.
[14] P.-E. Druet and A. Jüngel. Analysis of cross-diffusion systems for fluid mixtures driven by a pressure gradient. *SIAM J. Math. Anal.* 52 (2020), 2179–2197.
[15] J. Fischer. Global existence of renormalized solutions to entropy-dissipating reaction-diffusion systems. *Arch. Ration. Mech. Anal.* 218 (2015) 553–587.
[16] J. Fischer, Weak-strong uniqueness of solutions to entropy-dissipating reaction-diffusion equations,. *Nonlin. Anal.* 159 (2017), 181–207.
[17] X. Huo, A. Jüngel, and A. Tzavaras. Weak-strong uniqueness for Maxwell–Stefan systems. *SIAM J. Math. Anal.* 54 (2022), 3215–3252.
[18] O. John and J. Stará. On the regularity of weak solutions to parabolic systems in two spatial dimensions. *Commun. Partial. Differ. Eqs.* 23 (1998), 1159–1170.
[19] A. Jüngel. The boundedness-by-entropy method for cross-diffusion systems. *Nonlinearity* 28 (2015), 1963–2001.
[20] A. Jüngel. *Entropy Methods for Diffusive Partial Differential Equations.* Springer Briefs Math., Springer, 2016.
[21] A. Jüngel and I. Stelzer. Existence analysis of Maxwell–Stefan systems for multicomponent mixtures. *SIAM J. Math. Anal.* 45 (2013), 2421–2440.
[22] C. Maxwell. On the dynamical theory of gases. *Phil. Trans. R. Soc. London* 157 (1866), 49–88.
[23] M. Pierre and D. Schmitt. Blow up in reaction-diffusion systems with dissipation of mass. *SIAM J. Math. Anal.* 28 (1997), 259–269.
[24] N. Shigesada, K. Kawasaki, and E. Teramoto. Spatial segregation of interacting species. *J. Theor. Biol.* 79 (1979), 83–99.
[25] J. Stará and O. John. Some (new) counterexamples of parabolic systems. *Comment. Math. Univ. Carolinae* 36 (1995), 503–510.
[26] J. Stefan. Über das Gleichgewicht und Bewegung, insbesondere die Diffusion von Gemischen. *Sitzungsberichte Kaiserl. Akad. Wissensch. Wien* 63 (1871), 63–124.

Institute of Analysis and Scientific Computing, Technische Universität Wien, Wiedner Hauptstrasse 8–10, 1040 Wien, Austria
Email address: juengel@tuwien.ac.at