A Quantum Regularization of the One-dimensional Hydrodynamic Model for Semiconductors

Myo Thein Gyi
Fachbereich Mathematik, Technische Universität Berlin,
Straße des 17. Juni 136, D – 10623 Berlin, Germany, and
Department of Mathematics, University of Yangon, University Post Office,
Yangon 11041, Myanmar, e-mail: gyi@math.tu-berlin.de

Ansgar Jüngel
Fachbereich Mathematik, Technische Universität Berlin,
Straße des 17. Juni 136, D – 10623 Berlin, Germany, e-mail: jungel@math.tu-berlin.de

Abstract

The steady-state hydrodynamic equations for isothermal states including the quantum Bohm potential are analyzed. The one-dimensional equations for the electron current density and the particle density are coupled self-consistently to the Poisson equation for the electric potential. The quantum correction can be interpreted as a dispersive regularization of the classical hydrodynamic equations. Physically motivated Dirichlet and Neumann boundary conditions for the electron density are prescribed.

The existence and uniqueness of strong solutions for sufficiently small current densities are proven. Furthermore, the classical limit (vanishing scaled Planck constant) and the zero-space-charge limit (vanishing scaled Debye length) are performed. The proofs are based on a transformation of variable for the electron density, yielding an forth-order, elliptic equation for the new variable. As a by-product of the classical limit, the existence of subsonic solutions to the hydrodynamic system is obtained. Finally, numerical examples are presented showing that for "large" current densities, fast oscillations in the particle density occur as the scaled Planck constant tends to zero.

Keywords. Quantum hydrodynamic model, nonlinear forth-order elliptic equation, existence and uniqueness of solutions, singular perturbation limits, subsonic flow.

1991 Mathematics Subject Classification. 35J60, 35J40, 76Y05, 35B25.

Acknowledgments. The authors acknowledges support from the DAAD-PROCOPE Program, from the Deutsche Forschungsgemeinschaft, grant numbers MA 1662/ -1 and -2, and from the TMR Project "Asymptotic Methods in Kinetic Theory", grant number ERB FMBX CT97 0157. The first author is partly supported by the DAAD Exchange Program.

1 Introduction

For the simulation of submicron semiconductor devices, usually the energy-transport equations or the hydrodynamic equations are used [5, 11, 19]. However, for ultra-small devices in which quantum effects are present, semiconductor models have to incorporate the quantum mechanical phenomena. Recently, various so-called quantum hydrodynamic models were used in semiconductor simulations of tunneling diodes [12, 18]. These models are macroscopic models describing the electron flow in semiconductor crystals, in terms of macroscopic variables like the electron density and the electron current density. The quantum hydrodynamic models are extensions of the classical hydrodynamic equations including quantum corrections. In one space dimension, the scaled equations for the electron current density J, the electron density n and the electric potential V read

$$\begin{aligned} n_t + J_x &= 0, & (1.1)\\ J_t - \delta^2 \left(n(\ln n)_{xx}\right)_x + \left(\frac{J^2}{n} + Tn\right)_x - nV_x &= -\frac{J}{\tau}, & (1.2)\\ \lambda^2 V_{xx} &= n - C & (1.3) \end{aligned}$$

in the bounded semiconductor domain $\Omega = (0,1)$. The physical constants are the scaled Planck constant δ, the scaled electron temperature T, the scaled momentum relaxation time τ, and the scaled Debye length λ. For the scaling, we refer to [20]. The doping profile $C = C(x)$ models fixed charged background ions in the semiconductor crystal. The equation (1.1) represents the conservation of the mass, whereas the second equation (1.2) expresses the conservation of the momentum.

The quantum hydrodynamic equations (1.1)-(1.3) without relaxation term can be derived from a system of countable Schrödinger equations coupled selfconsistently to the Poisson equation (1.3), assuming isothermal states for the electron temperature [16, 17]. Furthermore, quantum hydrodynamic models can be obtained by taking the first velocity moments of the quantum Boltzmann equation [12] or of the Wigner equation and introducing the mixed-state Wigner function [17]. Models with smooth quantum potentials are derived in [13].

In this paper we are concerned with the steady-state equations of (1.1)-(1.3). Then, J is a constant and the stationary equations read:

$$\begin{aligned} -\delta^2 \left(n(\ln n)_{xx}\right)_x + \left(\frac{J^2}{n} + Tn\right)_x - nV_x &= -\frac{J}{\tau}, & (1.4)\\ \lambda^2 V_{xx} &= n - C & (1.5) \end{aligned}$$

in $\Omega = (0,1)$, subject to the boundary conditions

$$n(0) = n_0, \quad n(1) = n_1, \quad n_x(0) = n_x(1) = 0, \quad V(0) = V_0. \tag{1.6}$$

In this formulation, the electron current density J is a given (positive) constant. From the equations, the applied voltage U can be computed by $U = V(1) - V(0)$.

We mention two points of view to the quantum hydrodynamic equations. First, the model (without relaxation term) can be seen as a reformulation of a mixed-state Schrödinger-Poisson system via a transformation of variables, for isothermal states. For $\delta = 0$ we obtain the classical hydrodynamic model. For $\delta > 0$, the quantum correction term

$$-\delta^2 \left(n(\ln n)_{xx}\right)_x = -2\delta^2 n\Big(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\Big)_x \tag{1.7}$$

can be interpreted either as internal self-potential, the so-called Bohm potential $Q = -2\delta^2(\sqrt{n})_{xx}/\sqrt{n}$, or as a pressure term $P = -\delta^2 n(\ln n)_{xx}$.

Another point of view is to consider the quantum correction as a regularization of the hydrodynamic equations. Indeed, existence of weak solutions to the steady-state hydrodynamic equations

$$\Big(\frac{J^2}{n} + Tn\Big)_x - nV_x = -\frac{J}{\tau}, \tag{1.8}$$

$$\lambda^2 V_{xx} = n - C \tag{1.9}$$

in Ω with boundary conditions

$$n(0) = n_0, \quad n(1) = n_1, \quad V(0) = U_0 \tag{1.10}$$

can be established by using the vanishing viscosity method. In this method, a term of the form $-\delta^2(J/n)_{xx}$, for instance, is added to the left-hand side of (1.8). Then, existence of approximate solutions (n_δ, V_δ) to the resulting regularized elliptic system and appropriate *a priori* bounds for the solutions independent of the viscosity parameter are shown and the limit $\delta \to 0$ is performed. There are several choices for the viscosity term, see, e.g., [7, 8, 9, 10, 22]. Here we present a "regularizing" term which can be interpreted as a quantum correction term. Moreover, we are able to perform the vanishing "regularizing" limit $\delta \to 0$ (classical limit) for subsonic flow.

In the following we present the main results of this paper.

Existence and uniqueness results. We show the existence and uniqueness of solutions for small current values. More precisely, if the electron mean velocity J/n is smaller than $\sqrt{T + 2\delta^2}$, then there exists a solution $n \in H^4(\Omega)$, $V \in H^2(\Omega)$ to (1.4)-(1.6) such that n is strictly positive. For sufficiently small $\delta > 0$, this solution is unique. We notice that a similar condition like the smallness assumption on the electron current was needed in [21], where the multi-dimensional quantum hydrodynamic equations in the potential flow formulation were considered. Moreover, for a so-called reduced quantum model, it could be proven that the electron density cavitates for large enough current densities.

The existence and uniqueness of solutions to the hydrodynamic equations for $J = 0$ (thermal equilibrium) is studied in one [3] or several space dimensions [14, 24]. The equations for $J > 0$ are considered in [21, 25]. However, in these papers different boundary conditions than (1.6) are employed. In [25] *ad hoc* nonlinear boundary conditions involving $n(0)$, $n_x(0)$, and $n_{xx}(0)$ and Dirichlet boundary conditions for V are used. Dirichlet boundary conditions for the electron density, the electric potential and the velocity potential are prescribed in [21]. The boundary conditions (1.6) are used in numerical simulations [12, 23]. They seem very reasonable from a physical point of view for the modelling of tunneling devices.

The classical limit $\delta \to 0$. If $J/n < \sqrt{T}$ we can perform the classical limit $\delta \to 0$. This limit is important in semiconductor applications, since the scaled Planck constant is usually very small compared to one ($\delta^2 \sim 10^{-3}$, 10^{-4}; see [20]). The solutions (n_δ, V_δ) to (1.4)-(1.6) converge in some Sobolev spaces to a solution (n, V) to (1.8)-(1.10) (up to a subsequence). Thus, we can prove the existence of *subsonic solutions* of the hydrodynamic model. In contrast to the vanishing viscosity method, we cannot expect that for "large" current densities, the solutions to the quantum hydrodynamic model converge to a *transonic* solution to the hydrodynamic equations. Due to the dispersive quantum term, the solutions of the quantum hydrodynamic system may develop fast oscillations which are not damped as δ tends to zero, and in this case, the limiting solution is not expected to be a solution of the hydrodynamic model. This behavior is similar to the small-dispersion limit of the Korteveg-de-Vries equation and is first observed in [23]. We illustrate the oscillating behavior by numerical experiments.

The classical limit for $J = 0$ is studied in [14, 24]. For quantum hydrodynamic problems without relaxation term, the dispersive limit $\delta \to 0$ is performed in [15], in the context of Schrödinger equations.

The zero-space-charge limit $\lambda \to 0$. We show that the solutions (n_λ, V_λ) to (1.4)-(1.6) converge in some spaces to (n, V), where $n = \ln C$ and V is given by

$$V(x) = -2\delta^2 \frac{(\sqrt{C})_{xx}}{\sqrt{C}} + \frac{J^2}{2C^2} + T \ln C + \frac{J}{\tau} \int_0^x \frac{ds}{C(s)}, \qquad x \in \Omega.$$

The limit $\lambda \to 0$ in the thermal equilibrium state $J = 0$ is performed in [24]. We observe that in semiconductor modeling, the scaled Debye length is usually "small" ($\lambda^2 \sim 10^{-2}$, 10^{-3}).

The idea of the existence proof as well as of the uniform *a priori* estimates is to transform the equation (1.4) via $u = \ln n$ as in [3]. Let $(n, V) \in H^4(\Omega) \times H^2(\Omega)$ be a strong solution to (1.4)-(1.6) such that $n(x) \geq \underline{n} > 0$ in Ω. By dividing (1.4) by n, taking the derivative with respect to x, and using the Poisson equation (1.5), we get the following forth-order equation for u:

$$\delta^2 \left(u_{xx} + \frac{u_x^2}{2} \right)_{xx} + J^2 (e^{-2u} u_x)_x - T u_{xx} + \frac{e^u - C}{\lambda^2} = \frac{J}{\tau} (e^{-u})_x \tag{1.11}$$

with the boundary conditions

$$u(0) = u_0, \quad u(1) = u_1, \quad u_x(0) = u_x(1) = 0, \tag{1.12}$$

where $u_0 = \ln n_0$ and $u_1 = \ln n_1$. To obtain the electric potential, divide (1.4) by n, use the relation (1.7) and integrate over $x \in (0, 1)$:

$$V(x) = -2\delta^2 e^{-u/2} \left(e^{-u/2} \right)_{xx} + \frac{J^2}{2} e^{-2u} + Tu + \frac{J}{\tau} \int_0^x e^{-u(s)} ds. \tag{1.13}$$

The integration constant can be assumed to be zero by fixing the reference point for the electric potential. This implies that (see (1.6))

$$V_0 = -2\delta^2 e^{-u_0/2} \left(e^{-u/2}\right)_{xx}(0) + \frac{J^2}{2} e^{-2u_0} + T u_0. \tag{1.14}$$

Every strong solution (n, V) to (1.4)-(1.6) satisfying $\underline{n} \le n \in H^4(\Omega)$, $V \in H^2(\Omega)$ defines a strong solution $(u, V) \in H^4(\Omega) \times H^2(\Omega)$ to (1.11)-(1.13). Inversely, we show in Section 2 that every solution $(u, V) \in H^4(\Omega) \times H^2(\Omega)$ to (1.11)-(1.13) defines a solution (n, V) to (1.4)-(1.6) satisfying $\underline{n} \le n \in H^4(\Omega)$ and $V \in H^2(\Omega)$.

The transformation of variables has two advantages. First, we can avoid the problem of not having boundary conditions for the electric potential. This point is also noticed in the paper [3]. Secondly, we obtain a lower bound for the electron density by the Sobolev embedding $u \in H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ and the relation $n = \exp(u)$. Notice that maximum principles are not available for the third-order equation (1.4).

This paper is organized as follows. In Sections 2 and 3 the existence and uniqueness of solutions to (1.11)-(1.13) and hence, to (1.4)-(1.6) are proved. The Sections 4 and 5 are concerned with the classical limit and the zero-space-charge limit. In Section 6 numerical simulations of a semiconductor n^+n diode are presented to illustrate the asymptotic limits.

2 Existence of solutions

In this section we show the existence of weak solutions (u, V) to the problem (1.11)-(1.13). First we prove the existence of solutions to the truncated problem

$$\delta^2 \left(u_{xx} + \frac{u_x^2}{2}\right)_{xx} + J^2 \left(e^{-2u_K} u_x\right)_x - T u_{xx} + \frac{e^u - C}{\lambda^2} = \frac{J}{\tau} \left(e^{-u_K}\right)_x \quad \text{in } \Omega \tag{2.1}$$

subject to the boundary conditions (1.12), where we have set $u_K = \min(K, \max(-K, u))$ for $K > 0$. We assume that

$$\delta,\ J,\ T,\ \lambda,\ \tau > 0; \quad u_0, u_1 \in \mathbb{R}; \quad C \in L^2(\Omega). \tag{2.2}$$

For simplicity, let $\delta < 1$. Furthermore, for given $\gamma \in (0,1)$, we suppose that J satisfies

$$J \le e^{-K(\gamma)} \gamma \sqrt{T + 2\delta^2}, \tag{2.3}$$

where $K(\gamma) > 0$ only depends on γ, T, τ, u_0, u_1, λ, and $\|C\|_{0,2,\Omega}$. This constant is defined below (see (2.14)-(2.18)).

The following lemma provides *a priori* estimates for the solution to the truncated problem.

Lemma 2.1 *Assume (2.2) and (2.3). Let $u \in H^2(\Omega)$ be a solution to (2.1), (1.12) and let $K = K(\gamma)$. Then*

$$\delta \|u_{xx}\|_{0,2,\Omega} + \sqrt{T} \|u_x\|_{0,2,\Omega} \le K_0, \tag{2.4}$$

where $K_0 > 0$ is defined in (2.17) and only depends on γ, T, λ, τ, u_0, u_1, and $\|C\|_{0,2,\Omega}$. Furthermore, $K(\gamma) = |u_0| + K_0/\sqrt{T}$ and

$$\|u\|_{0,\infty,\Omega} \leq K(\gamma). \tag{2.5}$$

Remark 2.2 In the case of the hydrodynamic model, i.e. $\delta = 0$ in (1.11), (1.13), the condition (2.3) means that the flow is subsonic. Indeed, subsonic flow is characterized by the fact that the mean velocity J/n is smaller than $\sqrt{T}$. Now, the estimate (2.5) implies

$$\frac{J}{n} = Je^{-u} \leq J\, e^{K(\gamma)} \leq \gamma\sqrt{T} < \sqrt{T} \quad \text{for } \delta = 0,$$

that means, the flow is subsonic.

Another interpretation of the condition (2.3) is the observation that the second-order term $(J^2e^{-2u} - T)u_{xx}$ is non-positive if (2.3) holds. Thus, in general, the second-order term is of sign-changing type (cf. [4]).

Remark 2.3 From the proof of Lemma 2.1 it can be seen that the condition on τ can be weakened. In fact, it suffices to assume that $\tau : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying $\tau(x, u) \geq \tau_0 > 0$ for all $(x, u) \in \Omega \times \mathbb{R}$. In this case, the right-hand side of (1.11) has to be replaced by $J(e^{-u}/\tau(u))_x$ and the integral in equation (1.13) by

$$J \int_0^x \frac{e^{-u(s)}}{\tau(s, u(s))} ds.$$

Proof. We define, following [3], a function $u_D \in C^2\left(\overline{\Omega}\right)$ satisfying the boundary conditions

$$u_D(0) = u_0, \quad u_D(1) = u_1, \quad u_{D,x}(0) = u_{D,x}(1) = 0$$

with piecewise linear second derivative

$$u_{D,xx}(x) = \begin{cases} \frac{4\,\alpha}{\varepsilon^2(1-\varepsilon)}\, x & \text{for} \quad x \in [0, \frac{\varepsilon}{2}) \\ \frac{4\,\alpha}{\varepsilon^2(1-\varepsilon)}\, (\varepsilon - x) & \text{for} \quad x \in [\frac{\varepsilon}{2}, \varepsilon) \\ 0 & \text{for} \quad x \in [\varepsilon, \frac{1}{2}] \end{cases}$$

and $u_{D,xx}(x) = -u_{D,xx}(1-x)$ for $x \in (1/2, 1]$, where $\alpha = u_1 - u_0$ and $\varepsilon \in (0, 1/2)$. Elementary computations show that

$$\int_0^{1/2} x|u_{D,xx}(x)|dx + \int_{1/2}^1 (1-x)|u_{D,xx}(x)|dx = \frac{\varepsilon|\alpha|}{1-\varepsilon}, \tag{2.6}$$

$$\int_0^1 |u_{D,xx}(x)|^2 dx = \frac{8\alpha^2}{3\varepsilon(1-\varepsilon)}, \tag{2.7}$$

$$\int_0^1 |u_{D,x}(x)|dx = |\alpha|, \tag{2.8}$$

$$\int_0^1 |u_{D,x}(x)|^2 dx = \frac{\alpha^2(23\varepsilon + 1)}{30(1-\varepsilon)^2} < \frac{\alpha^2}{2}. \tag{2.9}$$

Use $u - u_D \in H_0^2(\Omega)$ as test function in the weak formulation of (2.1):

$$\delta^2 \int_0^1 \left(u_{xx} + \frac{u_x^2}{2} \right) (u - u_D)_{xx} - J^2 \int_0^1 e^{-2u_K} u_x (u - u_D)_x + T \int_0^1 u_x (u - u_D)_x$$

$$+ \frac{1}{\lambda^2} \int_0^1 (e^u - e^{u_D})(u - u_D) + \frac{1}{\lambda^2} \int_0^1 (e^{u_D} - C)(u - u_D) = -\frac{J}{\tau} \int_0^1 e^{-u_K} (u - u_D)_x.$$

Employing Young's inequality in the first three integrals and the monotonicity of $u \mapsto e^u$ yields, for any $\eta > 0$,

$$\begin{aligned}
&\left(1 - \frac{\eta}{2}\right) \delta^2 \int_0^1 u_{xx}^2 + \frac{\delta^2}{2} \int_0^1 u_x^2 u_{xx} - (1+\eta) J^2 \int_0^1 e^{-2u_K} u_x^2 + \left(1 - \frac{\eta}{2}\right) T \int_0^1 u_x^2 \\
&\quad \leq \frac{\delta^2}{2\eta} \int_0^1 |u_{D,xx}|^2 + \frac{\delta^2}{2} \int_0^1 u_x^2 u_{D,xx} + \frac{J^2}{4\eta} \int_0^1 e^{-2u_K} |u_{D,x}|^2 \qquad (2.10) \\
&\qquad + \frac{T}{2\eta} \int_0^1 |u_{D,x}|^2 + \frac{1}{\lambda^2} \int_0^1 (e^{u_D} - C)(u - u_D) - \frac{J}{\tau} \int_0^1 e^{-u_K} (u - u_D)_x.
\end{aligned}$$

Observe that, in view of the boundary conditions for u_x,

$$\int_0^1 u_x^2 u_{xx} = \frac{1}{3} \int_0^1 \left(u_x^3\right)_x = \frac{1}{3} \left(u_x(1)^3 - u_x(0)^3\right) = 0. \qquad (2.11)$$

We estimate the right-hand side of (2.10) term by term. From (2.7) we obtain

$$\frac{\delta^2}{2\eta} \int_0^1 |u_{D,xx}|^2 = \frac{4\delta^2 \alpha^2}{3\varepsilon(1-\varepsilon)\eta} < \frac{4\alpha^2}{3\varepsilon\eta},$$

since $\delta < 1$. Using the Cauchy-Schwarz inequality and the boundary conditions for u_x, we get for $x \in \Omega$

$$|u_x(x)| \leq \sqrt{x} \|u_{xx}\|_{0,2,\Omega}, \quad |u_x(x)| \leq \sqrt{1-x} \|u_{xx}\|_{0,2,\Omega}, \qquad (2.12)$$

which gives, together with (2.6),

$$\begin{aligned}
\frac{\delta^2}{2} \int_0^1 u_x^2 u_{D,xx} &\leq \frac{\delta^2}{2} \|u_{xx}\|_{0,2,\Omega}^2 \left(\int_0^{1/2} x |u_{D,xx}(x)| dx + \int_{1/2}^1 (1-x) |u_{D,xx}(x)| dx \right) \\
&\leq \frac{\delta^2}{2} \|u_{xx}\|_{0,2,\Omega}^2 \varepsilon |\alpha| \\
&\leq \frac{\delta^2}{4} \eta \|u_{xx}\|_{0,2,\Omega}^2,
\end{aligned}$$

choosing $\varepsilon = \min(1/2, \eta/2|\alpha|)$. The assumption (2.3) and the inequality (2.9) yield

$$\frac{J^2}{4\eta} \int_0^1 e^{-2u_K} |u_{D,x}|^2 \leq \frac{\alpha^2}{8\eta} J^2 e^{2K(\gamma)} \leq \frac{\alpha^2 \gamma}{8\eta} (T + 2\delta^2)$$

and

$$\frac{T}{2\eta}\int_0^1 |u_{D,x}|^2 \leq \frac{\alpha^2 T}{4\eta}.$$

By the Poincaré inequality

$$\|u-u_D\|_{0,2,\Omega} \leq \frac{1}{2}\|(u-u_D)_x\|_{0,2,\Omega}$$

and the Young inequality, we conclude

$$\begin{aligned}
\frac{1}{\lambda^2}\int_0^1 \left(e^{u_D}-C\right)(u-u_D) &\leq \frac{\eta T}{8}\|(u-u_D)_x\|^2_{0,2,\Omega} + \frac{1}{2\eta\lambda^4 T}\|e^{u_D}-C\|^2_{0,2,\Omega} \\
&\leq \frac{\eta T}{4}\|u_x\|^2_{0,2,\Omega} + \frac{\eta T}{4}\|u_{D,x}\|^2_{0,2,\Omega} \\
&\quad + \frac{1}{2\eta\lambda^4 T}\|e^{u_D}-C\|^2_{0,2,\Omega} \\
&\leq \frac{\eta T}{4}\int_0^1 u_x^2 + \frac{\alpha^2\eta T}{8} + \frac{1}{2\eta\lambda^4 T}\|e^{u_D}-C\|^2_{0,2,\Omega}.
\end{aligned}$$

Finally, again using the hypothesis (2.3), we get

$$\begin{aligned}
-\frac{J}{\tau}\int_0^1 e^{-u_K}(u-u_D)_x &\leq \frac{J}{\tau}e^{K(\gamma)}\int_0^1 \left(|u_x|+|u_{D,x}|\right) \\
&\leq \frac{\eta T}{4}\int_0^1 u_x^2 + \frac{J^2}{\eta\tau^2 T}e^{2K(\gamma)} + \frac{J}{\tau}e^{K(\gamma)}\int_0^1 |u_{D,x}| \\
&\leq \frac{\eta T}{4}\int_0^1 u_x^2 + \frac{\gamma(T+2)}{\eta\tau^2 T} + \frac{|\alpha|}{\tau}\sqrt{\gamma(T+2)},
\end{aligned}$$

employing (2.8) and $\delta < 1$.

With the above estimates, we conclude from (2.10)

$$\left(1-\frac{3\eta}{4}\right)\delta^2\int_0^1 u_{xx}^2 + \left[(1-\eta)T-(1+\eta)J^2e^{2K(\gamma)}\right]\int_0^1 u_x^2 \leq K_1, \tag{2.13}$$

where K_1 is defined by

$$\begin{aligned}
K_1 &\stackrel{\text{def}}{=} \frac{8\alpha^2}{3\eta^2}\max(\eta,|\alpha|) + \gamma(T+2)\left(\frac{\alpha^2}{8\eta}+\frac{1}{\tau^2\eta T}\right) + \frac{\alpha^2 T}{4}\left(\frac{1}{\eta}+\frac{\eta}{2}\right) \\
&\quad + \frac{1}{2\eta\lambda^4 T}\|e^{u_D}-C\|^2_{0,2,\Omega} + \frac{|\alpha|}{\tau}\sqrt{\gamma(T+2)}.
\end{aligned} \tag{2.14}$$

From (2.12) follows that

$$\|u_x\|^2_{0,2,\Omega} \leq \frac{1}{2}\|u_{xx}\|^2_{0,2,\Omega}$$

and therefore

$$\left(1-\frac{3\eta}{4}\right)\delta^2\int_0^1 u_{xx}^2 \geq \frac{\eta}{4}\delta^2\int_0^1 u_{xx}^2 + 2(1-\eta)\delta^2\int_0^1 u_x^2.$$

With this inequality and assumption (2.3) we obtain from (2.13)

$$\frac{\eta}{4}\delta^2\int_0^1 u_{xx}^2 + \left[(1-\eta)\left(T+2\delta^2\right) - (1+\eta)\gamma\left(T+2\delta^2\right)\right]\int_0^1 u_x^2 \le K_1.$$

Choosing

$$\eta = \frac{1-\gamma}{2(1+\gamma)} \in (0, 1/2), \tag{2.15}$$

we get

$$\frac{1-\gamma}{8(1+\gamma)}\delta^2\int_0^1 u_{xx}^2 + \frac{1}{2}(1-\gamma)\left(T+2\delta^2\right)\int_0^1 u_x^2 \le K_1.$$

Hence, setting $K_2 = 8(1+\gamma)K_1/(1-\gamma)$,

$$\delta^2\int_0^1 u_{xx}^2 + T\int_0^1 u_x^2 \le K_2, \tag{2.16}$$

and the estimate (2.4) follows after defining $K_0 = \sqrt{2K_2}$. The constant K_0 equals to

$$K_0 = 4\sqrt{\frac{1+\gamma}{1-\gamma}K_1}. \tag{2.17}$$

Finally, the inequality (2.5) is obtained from

$$|u(x)| \le |u_0| + \int_0^x |u_x(s)|ds \le |u_0| + \|u_x\|_{0,2,\Omega} \le |u_0| + \frac{K_0}{\sqrt{T}} = K(\gamma),$$

where we have set

$$K(\gamma) = |u_0| + \frac{K_0}{\sqrt{T}}. \tag{2.18}$$

The lemma is proved. □

Remark 2.4 We have used in the above proof several times the assumption of one space dimension. In particular, the integral (2.11) does not vanish in the case of several space dimensions.

The existence result for the problem (1.11)-(1.12) is as follows:

Theorem 2.5 *Under the assumptions (2.2) and (2.3), there exists a weak solution $u \in H^2(\Omega)$ to (1.11)-(1.12).*

Proof. We show that there exists a weak solution $u \in H^2(\Omega)$ to (2.1), (1.12) satisfying $\|u\|_{0,\infty,\Omega} \le K(\gamma)$. Then u solves (1.11)-(1.12) after setting $K = K(\gamma)$. The existence of a solution to the truncated system is shown by using the Leray-Schauder fixed point theorem. For this, let $v \in X = C^{0,1}([0,1])$ and consider the linear problem

$$\begin{aligned} &\delta^2\left(u_{xx} + \frac{\sigma}{2}v_x^2\right)_{xx} + \sigma J^2\left(e^{-2v_K}v_x\right)_x - Tu_{xx} \\ &+\frac{\sigma}{\lambda^2}\left(\frac{e^v-1}{v}u + 1 - C\right) = \sigma\frac{J}{\tau}\left(e^{-v_K}\right)_x \quad \text{in } \Omega, \end{aligned} \tag{2.19}$$

$$u(0) = \sigma u_0, \quad u(1) = \sigma u_1, \quad u_x(0) = u_x(1) = 0, \tag{2.20}$$

where $\sigma \in [0, 1]$. Define the bilinear form

$$a(u, \phi) = \int_0^1 \left(\delta^2 u_{xx} \phi_{xx} + T u_x \phi_x + \frac{\sigma}{\lambda^2} \frac{e^v - 1}{v} u \phi \right)$$

for $u, \phi \in H^2(\Omega)$ and the functional

$$F(\phi) = \int_0^1 \left(-\frac{\delta^2 \sigma}{2} v_x^2 \phi_{xx} + \sigma J^2 e^{-2v_K} v_x \phi_x + \frac{\sigma}{\lambda^2}(C - 1)\phi - \sigma \frac{J}{\tau} e^{-v_K} \phi_x \right)$$

for $\phi \in H^2(\Omega)$. Since $v \in X \hookrightarrow W^{1,\infty}(\Omega)$, we see that $a(.,.)$ is continuous and coercive in $H^2(\Omega)$ and that F is linear and continuous in $H^2(\Omega)$. By Lax–Milgram's Lemma, there exists a unique solution $u \in H^2(\Omega)$ to (2.19)-(2.20). This defines the fixed point operator $S : X \times [0, 1] \to X, (v, \sigma) \mapsto u$. It can be easily seen that S is continuous. Thanks to the compact embedding $H^2(\Omega) \hookrightarrow X$, the mapping S is compact. Furthermore, $S(v, 0) = 0$ for all $v \in X$, and there exists a constant $c > 0$ such that for all $(u, \sigma) \in X \times [0, 1]$ satisfying $S(u, \sigma) = u$ it holds

$$\|u\|_X \leq c.$$

Indeed, Lemma 2.1 settles the case $\sigma = 1$. For $\sigma < 1$, the same steps as in the proof of Lemma 2.1 lead to (cf. (2.13))

$$\left(1 - \frac{\eta}{2} - \frac{\sigma\varepsilon|\alpha|}{2}\right)\delta^2 \int_0^1 u_{xx}^2 + \left[(1-\eta)T - (1+\eta)\gamma\left(T + 2\delta^2\right)\right] \int_0^1 u_x^2 \leq K_1.$$

By choosing $\varepsilon = \min(1/2, \eta/2|\alpha|)$ as in the proof of Lemma 2.1 and observing that $\sigma \leq 1$, we get

$$\delta^2 \|u_{xx}\|_{0,2,\Omega}^2 + T\|u_x\|_{0,2,\Omega}^2 \leq K_2,$$

with $K_2^2 = K_0^2/2$ (see (2.16)). Using Poincaré's inequality, we obtain

$$\|u\|_X \leq c_1 \|u\|_{2,2,\Omega} \leq c_2,$$

where $c_2 > 0$ is independent of u and σ. Now, the existence of a fixed point u with $S(u, 1) = u$ follows from the Leray-Schauder fixed point theorem. $\square$

Corollary 2.6 *Under the assumptions (2.2)-(2.3) there exists a solution $(u, V) \in H^4(\Omega) \times H^2(\Omega)$ to (1.11)-(1.13).*

Proof. Let $(u, V) \in H^2(\Omega) \times L^2(\Omega)$ be a solution to (1.11)-(1.13) (see Theorem 2.5). Since $u \in H^2(\Omega)$, it holds $(u_x)^2 \in H^1(\Omega)$ and thus $(u_x^2)_{xx} \in H^{-1}(\Omega)$. Observing that, by (1.11),

$$\delta^2 u_{xxxx} = -\delta^2 (u_x^2)_{xx} - J^2 (e^{-2u} u_x)_x + T u_{xx} - \frac{1}{\lambda^2}(e^u - C) + \frac{J}{\tau}(e^{-u})_x, \tag{2.21}$$

we get $u_{xxxx} \in H^{-1}(\Omega)$, i.e. there exists $w \in L^2(\Omega)$ such that $w_x = u_{xxxx}$ or $(u_{xxx} - w)_x = 0$ in Ω. Therefore, $u_{xxx} = w + \text{const.} \in L^2(\Omega)$. This implies $(u_x^2)_{xx} \in L^2(\Omega)$ and, by (2.21), $u_{xxxx} \in L^2(\Omega)$. We conclude that $u \in H^4(\Omega)$. The regularity on u and the definition (1.13) for V immediately give $V \in H^2(\Omega)$. □

The main result of this section is as follows:

Theorem 2.7 *Assume (2.2) and (2.3). Then there exists a solution $(n, V) \in H^4(\Omega) \times H^2(\Omega)$ to the quantum hydrodynamic problem (1.4)-(1.6). Moreover, it holds*

$$n(x) \geq \underline{n} > 0 \qquad \text{in } \Omega,$$

where $\underline{n} = \exp(-K(\gamma))$ and $K(\gamma) > 0$ is as in Lemma 2.1.

Proof. Define $n = \exp(u)$ where u is as in Corollary 2.6. Then $n \in H^4(\Omega)$ and $n \geq \underline{n} > 0$ in Ω, where $\underline{n} = \exp(-K(\gamma))$, by Corollary 2.6 and Lemma 2.1. We can rewrite (1.11) as follows:

$$\delta^2 \left[\frac{1}{n}(n(\ln n)_{xx})_x\right]_x - \left(\frac{J^2}{2n^2}\right)_{xx} - T(\ln n)_{xx} + \frac{n - C}{\lambda^2} = \left(\frac{J}{\tau n}\right)_x . \tag{2.22}$$

We differentiate equation (1.13) for V twice with respect to x:

$$V_{xx} = -2\delta^2 \left[\frac{1}{\sqrt{n}}\left(\sqrt{n}\right)_{xx}\right]_{xx} + \left(\frac{J^2}{2n^2}\right)_{xx} + T(\ln n)_{xx} + \left(\frac{J}{\tau n}\right)_x . \tag{2.23}$$

Since

$$2\left[\frac{1}{\sqrt{n}}\left(\sqrt{n}\right)_{xx}\right]_{xx} = \left[\frac{1}{n}(n(\ln n)_{xx})_x\right]_x ,$$

we get from (2.22) and (2.23) the Poisson equation

$$\lambda^2 V_{xx} = n - C.$$

Taking the derivative with respect to x in (1.13) and multiplying by n, we obtain

$$\begin{aligned} nV_x &= -2\delta^2 n \left(\frac{1}{\sqrt{n}}\left(\sqrt{n}\right)_{xx}\right)_x + n\left(\frac{J^2}{2n^2}\right)_x + Tn_x + \frac{J}{\tau} \\ &= -\delta^2(n(\ln n)_{xx})_x + \left(\frac{J^2}{n}\right)_x + Tn_x + \frac{J}{\tau}, \end{aligned}$$

which is equal to (1.4). Furthermore,

$$V(0) = -2\delta^2 \frac{1}{\sqrt{n_0}}\left(\sqrt{n}\right)_{xx}(0) + \frac{J^2}{2n_0^2} + T\ln n_0 = V_0,$$

taking into account (1.14). □

3 Uniqueness of solutions

We show that there exists a unique solution to (1.11)-(1.13) if the current density $J > 0$ and the scaled Planck constant $\delta > 0$ are small enough. Notice that in applications, the scaled Planck constant is usually very small compared to one (see [20]). We cannot expect that uniqueness of solutions holds for arbitrary large current densities. Indeed, in tunneling diodes, so-called negative differential resistance effects can occur, that means, $dJ/dU < 0$ holds in some region, where $U = V(1) - V(0)$ is the applied voltage. Therefore, there exist current densities $J \geq J_0 > 0$ for which several states exist and consequently, uniqueness of solutions does not hold in general.

Theorem 3.1 *Assume (2.2). Let $\delta_0 = T^{3/2}/K_0^2$, where $K_0 > 0$ is defined in Lemma 2.1, and let $0 < \delta \leq \delta_0$. Then there exists $J_0 = J_0(\delta) > 0$ such that if $0 < J \leq J_0$, there exists a unique weak solution to (1.11)-(1.13).*

Proof. We proceed as in [3]. Let $u, v \in H^2(\Omega)$ be two solutions to (1.11)-(1.12). Observe that, in view of the boundary conditions for u_x,

$$|u_x(x)|^2 = \int_0^x u_{xx}(s)u_x(s)ds \leq \|u_{xx}\|_{0,2,\Omega}\|u_x\|_{0,2,\Omega}$$

and hence, for any $\eta > 0$,

$$\|u_x\|_{0,\infty,\Omega} \leq \frac{\eta}{2}\|u_{xx}\|_{0,2,\Omega} + \frac{1}{2\eta}\|u_x\|_{0,2,\Omega}.$$

By Lemma 2.1, we obtain

$$\begin{aligned}
\|u_x\|_{0,\infty,\Omega} &\leq \frac{\eta}{2\delta}\delta\|u_{xx}\|_{0,2,\Omega} + \frac{\sqrt{T}}{2\eta\sqrt{T}}\|u_x\|_{0,2,\Omega} \\
&\leq \max\left(\frac{\eta}{2\delta}, \frac{1}{2\eta\sqrt{T}}\right) K_0.
\end{aligned}$$

Choose

$$\eta = \sqrt{\frac{T}{2K_0^2}} \quad \text{and} \quad 0 < \delta \leq \delta_0 \stackrel{\text{def}}{=} \frac{T^{3/2}}{2K_0^2}$$

to get

$$\frac{\eta}{2\delta} \geq \frac{1}{2\eta\sqrt{T}}$$

and therefore

$$\|u_x\|_{0,\infty,\Omega} \leq \frac{\eta K_0}{2\delta} = \sqrt{\frac{T}{2}}\frac{1}{2\delta}.$$

An analogous inequality holds for v_x. Hence

$$\|u_x + v_x\|_{0,\infty,\Omega} \leq \frac{1}{\delta}\sqrt{\frac{T}{2}}. \tag{3.1}$$

Now use $u-v \in H_0^2(\Omega)$ as test function in the weak formulation of the difference of the equations satisfied by u and v:

$$
\begin{aligned}
&\delta^2 \int_0^1 (u-v)_{xx}^2 + \frac{\delta^2}{2}\int_0^1 (u+v)_x(u-v)_x(u-v)_{xx} + T\int_0^1 (u-v)_x^2 \\
&= \frac{J^2}{2}\int_0^1 \left(e^{-2u}-e^{-2v}\right)(u-v)_{xx} - \frac{1}{\lambda^2}\int_0^1 (e^u-e^v)(u-v) \\
&\quad -\frac{J}{\tau}\int_0^1 (e^{-u}-e^{-v})(u-v)_x \\
&\le \frac{\delta^2}{2}\int_0^1 (u-v)_{xx}^2 + \frac{J^4}{8\delta^2}\int_0^1 \left(e^{-2u}-e^{-2v}\right)^2 - \frac{1}{\lambda^2}e^{-K(\gamma)}\int_0^1 (u-v)^2 \\
&\quad +\frac{T}{2}\int_0^1 (u-v)_x^2 + \frac{J^2}{2T\tau^2}\int_0^1 (e^{-u}-e^{-v})^2,
\end{aligned}
$$

employing Young's inequality and the mean value theorem. Taking into account the L^∞ bound (2.5) for u and v, we get further

$$
\begin{aligned}
&\frac{\delta^2}{2}\int_0^1 (u-v)_{xx}^2 + \frac{\delta^2}{2}\int_0^1 (u+v)_x(u-v)_x(u-v)_{xx} + \frac{T}{2}\int_0^1 (u-v)_x^2 \\
&\qquad \le \left[\frac{J^4}{8\delta^2}e^{4K(\gamma)} + \frac{J^2}{2T\tau^2}e^{2K(\gamma)} - \frac{1}{\lambda^2}e^{-K(\gamma)}\right]\int_0^1 (u-v)^2.
\end{aligned} \tag{3.2}
$$

The left-hand side of this inequality is estimated as follows:

$$
\begin{aligned}
&\frac{\delta^2}{4}\int_0^1 (u_{xx}-v_{xx})^2 + \frac{\delta^2}{2}\int_0^1 (u_x+v_x)(u_x-v_x)(u_{xx}-v_{xx}) + \frac{T}{2}\int_0^1 (u_x-v_x)^2 \\
&\ge \int_0^1 \left[\frac{\delta^2}{4}(u_{xx}-v_{xx})^2 - \frac{\delta^2}{2}|u_x+v_x||u_x-v_x||u_{xx}-v_{xx}| + \frac{T}{2}(u_x-v_x)^2\right] \\
&= \int_0^1 \Bigg[\left(\frac{\delta}{2}|u_{xx}-v_{xx}| - \sqrt{\frac{T}{2}}|u_x-v_x|\right)^2 + \delta\sqrt{\frac{T}{2}}|u_x-v_x||u_{xx}-v_{xx}| \\
&\qquad \times\left(1-\frac{\delta}{\sqrt{2T}}|u_x+v_x|\right)\Bigg] \\
&\ge \int_0^1 \left[\left(\frac{\delta}{2}|u_{xx}-v_{xx}| - \sqrt{\frac{T}{2}}|u_x-v_x|\right)^2 + \frac{\delta}{2}\sqrt{\frac{T}{2}}|u_x-v_x||u_{xx}-v_{xx}|\right] \\
&= \int_0^1 \left[\frac{\delta^2}{4}(u_{xx}-v_{xx})^2 + \frac{T}{2}(u_x-v_x)^2 - \frac{\delta}{2}\sqrt{\frac{T}{2}}|u_x-v_x||u_{xx}-v_{xx}|\right] \\
&\ge \frac{\delta^2}{8}\int_0^1 (u-v)_{xx}^2 + \frac{T}{4}\int_0^1 (u-v)_x^2,
\end{aligned}
$$

where we have used the bound (3.1) and Young's inequality. Therefore, the inequality (3.2) becomes

$$\frac{\delta^2}{8}\int_0^1 (u-v)_{xx}^2 + \frac{T}{4}\int_0^1 (u-v)_x^2$$
$$\leq \left[\frac{J^4 e^{4K(\gamma)}}{8\delta^2} + \frac{J^2 e^{2K(\gamma)}}{2T\tau^2} - \frac{1}{\lambda^2} e^{-K(\gamma)}\right]\int_0^1 (u-v)^2. \tag{3.3}$$

We choose $J > 0$ such that

$$J^2 \leq J_0^2 \stackrel{\text{def}}{=} \min\left\{e^{-2K(\gamma)}\gamma(T + 2\delta_0^2), e^{-3K(\gamma)}\lambda^{-2}T\tau^2, 2e^{-5K(\gamma)/2}\lambda^{-1}\delta\right\}.$$

Then the right-hand side of (3.3) is nonpositive and we get $u = v$ in Ω. $\square$

4 The classical limit

We show in this section that the solution (u_δ, V_δ) to the quantum hydrodynamic model converges, as $\delta \to 0$, to a solution (u, V) to the hydrodynamic model

$$J^2(e^{-2u}u_x)_x - Tu_{xx} + \frac{1}{\lambda^2}(e^u - C) = \frac{J}{\tau}(e^{-u})_x, \tag{4.1}$$

$$u(0) = u_0, \quad u(1) = u_1, \tag{4.2}$$

$$V(x) = \frac{J^2}{2}e^{-2u} + Tu + \frac{J}{\tau}\int_0^x e^{-u(s)}ds, \quad x \in \Omega. \tag{4.3}$$

Theorem 4.1 *Let (2.2) and (2.3) hold and let (u_δ, V_δ) be a solution to (1.11)-(1.13) for $\delta > 0$. Then there exists a subsequence $(u_{\delta'}, V_{\delta'})$ such that*

$$u_{\delta'} \rightharpoonup u \qquad \text{in } H^1(\Omega) \text{ weakly}, \tag{4.4}$$
$$u_{\delta'} \to u \qquad \text{in } C^0\left(\overline{\Omega}\right), \tag{4.5}$$
$$V_{\delta'} \rightharpoonup V \qquad \text{in } L^2(\Omega) \text{ weakly as } \delta' \to 0, \tag{4.6}$$

where $(u, V) \in H^1(\Omega) \times L^2(\Omega)$ is a solution of (4.1)-(4.3).

Proof. We conclude from Lemma 2.1 and Poincaré's inequality that (u_δ) is uniformly bounded in $H^1(\Omega)$, i.e. there exists a subsequence $(u_{\delta'})$ such that (4.4) and (4.5) hold. The weak formulation of (1.11) reads

$$(\delta')^2\int_0^1 u_{\delta'}\phi_{xxxx} + \frac{(\delta')^2}{2}\int_0^1 (u_{\delta'})_x^2\phi_{xx} - J^2\int_0^1 e^{-2u_{\delta'}}(u_{\delta'})_x\phi_x + T\int_0^1 (u_{\delta'})_x\phi_x$$
$$+\frac{1}{\lambda^2}\int_0^1 (e^{u_{\delta'}} - C)\phi = -\frac{J}{\tau}\int_0^1 e^{-u_{\delta'}}\phi_x \tag{4.7}$$

for $\phi \in C_0^\infty(\Omega)$. Thanks to (4.5) it holds

$$e^{\beta u_{\delta'}} \to e^{\beta u} \quad \text{in } L^2(\Omega) \text{ as } \delta' \to 0, \tag{4.8}$$

for any $\beta \in \mathbb{R}$. Furthermore, the integral $\int (u_{\delta'})_x^2 \phi_{xx}$ remains bounded and, in view of (4.4) and (4.8),

$$\int_0^1 e^{-2u_{\delta'}}(u_{\delta'})_x \phi_x \to \int_0^1 e^{-2u} u_x \phi_x \quad \text{as } \delta' \to 0.$$

Hence we can pass to the limit $\delta' \to 0$ in (4.7) to get

$$-J^2 \int_0^1 e^{-2u} u_x \phi_x + T \int_0^1 u_x \phi_x + \frac{1}{\lambda^2} \int_0^1 (e^u - C)\phi = -\frac{J}{\tau} \int_0^1 e^{-u} \phi_x$$

for all $\phi \in C_0^\infty(\Omega)$, which gives (4.1). The boundary conditions (4.2) follow from (4.5) and (1.12).

The equation (1.13) can be rewritten as

$$V_\delta(x) \;=\; \delta^2 e^{-u_\delta}(u_\delta)_{xx} - \frac{\delta^2}{2} e^{-u_\delta}(u_\delta)_x^2 + \frac{J^2}{2} e^{-2u_\delta} + T u_\delta + \frac{J}{\tau} \int_0^x e^{-u_\delta(s)} ds.$$

Taking into account the uniform L^∞ bound for u_δ and the uniform L^2 bound for $\delta(u_\delta)_{xx}$ (see (2.4)), we get a uniform L^2 bound for V_δ and hence, there exists a subsequence $(V_{\delta'})$ such that (4.6) holds.

Multiply (1.13) by a test function $\phi \in C_0^\infty(\Omega)$, integrate over $(0,1)$ and use integration by parts in

$$\begin{aligned}
\delta^2 \int_0^1 e^{-u_\delta/2} \left(e^{-u_\delta/2}\right)_{xx} \phi \;&=\; -\delta^2 \int_0^1 \left[\left(e^{-u_\delta/2}\right)_x\right]^2 \phi - \delta^2 \int_0^1 e^{-u_\delta/2} (e^{-u_\delta/2})_x \phi_x \\
&=\; -\frac{\delta^2}{4} \int_0^1 e^{-u_\delta}(u_\delta)_x^2 \phi + \frac{\delta^2}{2} \int_0^1 e^{-u_\delta}(u_\delta)_x \phi_x
\end{aligned}$$

to obtain

$$\begin{aligned}
\int_0^1 V_{\delta'}\phi \;&=\; \frac{(\delta')^2}{2} \int_0^1 e^{-u_{\delta'}}(u_{\delta'})_x^2 \phi - (\delta')^2 \int_0^1 e^{-u_{\delta'}}(u_{\delta'})_x \phi_x \\
&\quad + \frac{J^2}{2} \int_0^1 e^{-2u_{\delta'}} \phi + T \int_0^1 u_\delta \phi + \frac{J}{\tau} \int_0^1 \int_0^x e^{-u_{\delta'}(s)} ds \phi(x) dx.
\end{aligned} \tag{4.9}$$

Since $e^{-u_\delta}(u_\delta)_x^2$ is uniformly bounded in $L^2(\Omega)$ and

$$\int_0^x e^{-u_{\delta'}(s)} ds \to \int_0^x e^{-u(s)} ds \quad \text{in } L^2(\Omega) \text{ as } \delta' \to 0,$$

we can let $\delta' \to 0$ in (4.9) to get

$$\int_0^1 V\phi = \int_0^1 \left(\frac{J^2}{2} e^{-2u} + Tu + \frac{J}{\tau} \int_0^x e^{-u(s)} ds \right) \phi$$

for all $\phi \in C_0^\infty(\Omega)$. This proves the theorem. $\square$

Remark 4.2 For sufficiently small current densities $J > 0$, the hydrodynamic equations (4.1)-(4.3) are uniquely solvable. This can be seen as in the proof of Theorem 3.1 (also see [6]). Then the whole sequence (u_δ, V_δ) converges to (u, V) in the sense of (4.4)-(4.6).

Now, we can prove the main result of this section.

Theorem 4.3 *Assume (2.2), (2.3) for $\delta = 0$ and let $(n_\delta, V_\delta) \in H^4(\Omega) \times H^2(\Omega)$ be a solution to (1.4)-(1.6). Then there exists a subsequence $(n_{\delta'}, V_{\delta'})$ such that*

$$n_{\delta'} \rightharpoonup n \quad \text{in } H^1(\Omega) \text{ weakly,} \tag{4.10}$$

$$n_{\delta'} \to n \quad \text{in } C^0\left(\overline{\Omega}\right), \tag{4.11}$$

$$V_{\delta'} \rightharpoonup V \quad \text{in } L^2(\Omega) \text{ weakly as } \delta' \to 0, \tag{4.12}$$

where $(n, V) \in H^1(\Omega) \times H^1(\Omega)$ is a solution to the hydrodynamic problem (1.8)-(1.10). Moreover, $n(x) \geq \underline{n} > 0$ in Ω, where $\underline{n} = \exp(-K(\gamma))$ and $K(\gamma) > 0$ is defined in Lemma 2.1. The value U_0 in (1.10) is defined by $U_0 = J^2/2n_0^2 + T \ln n_0$.

Notice that we only get weak L^2 convergence of V_δ with the limit function V being in $H^1(\Omega)$. We cannot conclude strong convergence or convergence in H^1 since we have no convergence results for the boundary value V_0, defined in (1.14), which depends on δ.

Proof. By Theorem 4.1, there exists a solution (u, V) to (4.1)-(4.3), obtained as the limit of solutions $(u_{\delta'}, V_{\delta'})$ to (1.11)-(1.13) as $\delta' \to 0$, where $(u_{\delta'}, V_{\delta'})$ is a subsequence of (u_δ, V_δ). Setting $n_{\delta'} = \exp(u_{\delta'})$ and $n = \exp(u)$, the convergence results (4.10)-(4.12) follow from (4.4)-(4.6), since the embedding $H^1(\Omega) \hookrightarrow C^0(\Omega)$ is compact in one dimension.

We show that (n, V) solves (1.8)-(1.10). We rewrite the equation (4.1) in the variable n:

$$-\left(\frac{J^2}{2n^2}\right)_{xx} - T(\ln n)_{xx} + \frac{1}{\lambda^2}(n - C) = \left(\frac{J}{\tau n}\right)_x. \tag{4.13}$$

Notice that $n \geq \underline{n} = \exp(-K(\gamma))$ since $K(\gamma)$ does not depend on δ. Differentiating (4.3) twice with respect to x gives

$$V_{xx} = \left(\frac{J^2}{2n^2}\right)_{xx} + T(\ln n)_{xx} + \left(\frac{J}{\tau n}\right)_x \in H^{-1}(\Omega). \tag{4.14}$$

Thus, the relations (4.13) and (4.14) give the Poisson equation (1.9). Taking the derivative with respect to x in (4.3) gives

$$V_x = \left(\frac{J^2}{2n^2}\right)_x + T(\ln n)_x + \frac{J}{\tau n} \in L^2(\Omega), \tag{4.15}$$

and therefore, $V \in H^1(\Omega)$. Multiplying (4.15) by n, the resulting equation equals (1.8). Finally, from (4.3) we get

$$V(0) = U_0 \stackrel{\text{def}}{=} \frac{J^2}{2n_0^2} + T \ln n_0.$$

□

5 The zero-space-charge limit

This section is devoted to the limit $\lambda \to 0$. We cannot use directly the estimates of Lemma 2.1 since the constant K_0 depends on λ such that $K_0 \to \infty$ as $\lambda \to 0$ (see (2.14)-(2.17)), and no uniform bounds are available. From the definition of K_1 (2.14) we see that if we choose the boundary function u_D such that $\exp(u_D) = C$ in Ω, then the constant K_0 (and hence K_1) is independent of λ. This provides uniform estimates in $H^2(\Omega)$. However, we have constructed u_D in the proof of Lemma 2.1 in such a way that the term

$$\delta^2 \int_0^1 u_x^2 u_{D,xx}$$

is estimated by $\delta^2 \eta \|u_{xx}\|_{0,2,\Omega}$, for any $\eta > 0$. Choosing $u_D = \ln C$, we have to use different arguments to treat the above integral. We can prove the following estimates:

Lemma 5.1 *Assume (2.2), (2.3) and*

$$\begin{aligned} &C \in H^2(\Omega) \ \textit{satisfies}\ C(x) > 0 \ \textit{in}\ \Omega,\ C_x(0) = C_x(1) = 0, \\ &C(0) = e^{u_0},\ C(1) = e^{u_1}. \end{aligned} \tag{5.1}$$

Then there exists a constant $\delta_0 > 0$ only depending on C, T, and γ such that for all $0 < \delta \le \delta_0$ there exists a solution (u, V) to (1.11)-(1.13) satisfying the uniform bounds

$$\delta^2 \int_0^1 u_{xx}^2 + T \int_0^1 u_x^2 + \frac{1}{\lambda^2} \int_0^1 (e^u - C)(u - \ln C) \le K(\gamma), \tag{5.2}$$

where $K(\gamma) > 0$ is defined in (5.6), (5.5), (5.3) and only depends on C, T, τ, and γ.

Remark 5.2 In the modelling of tunneling diodes, the doping profile satisfies the assumption (5.1). Indeed, usually the doping is positive and constant near the contacts (see, e.g., [12]). Furthermore, the boundary data is usually chosen such that the space charge vanishes at the contact boundary, i.e. $e^u - C = 0$ on $\partial\Omega$. This implies $e^{u_0} = C(0)$ and $e^{u_1} = C(1)$ (see Section 6).

Proof. We define $u_D = \ln C$. Then, in view of (5.1), $u_D \in H^2(\Omega), u_{D,x}(0) = u_{D,x}(1) = 0$, and

$$\|u_D\|_{2,2,\Omega} \le M_1, \tag{5.3}$$

where $M_1 > 0$ only depends on $\min_\Omega C$ and $\|C\|_{2,2,\Omega}$. Let (u, V) be a weak solution to (2.1), (1.12). Using $u - u_D \in H_0^2(\Omega)$ as test function in the weak formulation of (2.1), we can proceed as in the proof of Lemma 2.1, but we have to estimate the integrals

$$\frac{\delta^2}{2} \int_0^1 u_x^2 u_{D,xx} \quad \text{and} \quad \frac{1}{\lambda^2} \int_0^1 (e^{u_D} - C)(u - u_D)$$

in a different way. Thanks to the choice of u_D, the latter integral vanishes, whereas the former integral can be estimated by using (5.3):

$$\frac{\delta^2}{2}\int_0^1 u_x^2 u_{D,xx} \le \frac{\delta^2}{2}M_1\int_0^1 u_x^2.$$

Similarly to (2.13) we get

$$\begin{aligned}\delta^2\left(1-\frac{3\eta}{4}\right)\int_0^1 u_{xx}^2 + \left[(1-\eta)T-(1+\eta)J^2e^{2K(\gamma)}-\frac{\delta^2}{2}M_1\right]\int_0^1 u_x^2 \\ +\frac{1}{\lambda^2}\int_0^1 \left(e^u-e^{u_D}\right)(u-u_D)\le M_2,\end{aligned}\tag{5.4}$$

where

$$M_2 \stackrel{\text{def}}{=} \frac{1+T}{2\eta}M_1^2+\frac{\gamma(T+2)}{4\eta\tau^2 T}(4+\tau^2 T M_1^2)+\frac{1}{\tau}\sqrt{\gamma(T+2)}M_1. \tag{5.5}$$

Set

$$\delta_0^2=\frac{2T\eta}{M_1}\quad\text{and}\quad \eta=\frac{1-\gamma}{2(2+\gamma)},$$

and let $0<\delta\le\delta_0$. Then we get from (5.4), as in the proof of Lemma 2.1,

$$\begin{aligned}\delta^2\frac{\eta}{4}\int_0^1 u_{xx}^2 + \left[(1-\eta)\left(T+2\delta^2\right)-(1+\eta)\gamma\left(T+2\delta^2\right)-T\eta\right]\int_0^1 u_x^2 \\ +\frac{1}{\lambda^2}\int_0^1 (e^u-e^{u_D})(u-u_D)\le M_2.\end{aligned}$$

Since

$$\left(T+2\delta^2\right)(1-\eta-(1+\eta)\gamma)-T\eta \ge T(-\eta(1+\gamma)+1-\gamma)-T\eta=\frac{1-\gamma}{2}T,$$

we conclude

$$\delta^2\frac{1-\gamma}{8(2+\gamma)}\int_0^1 u_{xx}^2+\frac{T}{2}(1-\gamma)\int_0^1 u_x^2+\frac{1}{\lambda^2}\int_0^1 (e^u-C)(u-\ln C)\le M_2.$$

The estimate (5.2) follows after setting

$$K(\gamma)=\frac{8M_2(2+\gamma)}{1-\gamma}. \tag{5.6}$$

Now the existence of a solution to (1.11)-(1.13) satisfying (5.2) can be shown as in the proof of Theorem 2.5. □

The main result of this section is the following theorem.

Theorem 5.3 *Assume (2.2), (2.3), and (5.1). Let* $0 < \delta \leq \delta_0$*, where* $\delta_0 > 0$ *is as in Lemma 5.1 and let* (u_λ, V_λ) *be a solution to (1.11)-(1.13). Then, as* $\lambda \to 0$,

$$\begin{aligned} u_\lambda &\rightharpoonup \ln C \quad \text{in } H^2(\Omega) \text{ weakly,} \\ u_\lambda &\to \ln C \quad \text{in } C^1\left(\overline{\Omega}\right), \\ V_\lambda &\rightharpoonup V \quad \text{in } L^2(\Omega) \text{ weakly,} \end{aligned}$$

where V *is defined by*

$$V(x) = -\frac{2\delta^2}{\sqrt{C}}(\sqrt{C})_{xx} + \frac{J^2}{2C^2} + T\ln C + \frac{J}{\tau}\int_0^x \frac{ds}{C(s)}.$$

Proof. From Lemma 5.1 and Poincaré's inequality follow that

$$\|u_\lambda\|_{2,2,\Omega} \leq c, \quad \text{and} \quad \int_0^1 (e^{u_\lambda} - e^{u_D})(u_\lambda - u_D) \leq M_1\lambda^2,$$

where $c > 0$ and $M_1 > 0$ are independent of λ. Therefore, for a subsequence $(u_{\lambda'})$,

$$u_{\lambda'} \rightharpoonup u_D \quad \text{in } H^2(\Omega) \text{ weakly as } \lambda' \to 0, \tag{5.7}$$

where $u_D = \ln C$. Thus, V_λ is uniformly bounded in $L^2(\Omega)$ (see (1.13)) and, for a subsequence $(V_{\lambda'})$,

$$V_{\lambda'} \rightharpoonup V \quad \text{in } L^2(\Omega) \text{ weakly.}$$

To identify the limit, we observe that (5.7) implies $u_{\lambda'} \to u_D$ in $C^1\left(\overline{\Omega}\right)$ and therefore,

$$e^{-u_{\lambda'}}(u_{\lambda'})_x^2 \to e^{-u_D}u_{D,x}^2 \quad \text{in } L^2(\Omega).$$

By (4.9), it holds for all $\phi \in H_0^1(\Omega)$

$$\begin{aligned} \int_0^1 V_{\lambda'}\phi &= \frac{\delta^2}{2}\int_0^1 e^{-u_{\lambda'}}(u_{\lambda'})_x^2\phi - \delta^2\int_0^1 e^{-u_{\lambda'}}(u_{\lambda'})_x\phi_x \\ &\quad + \frac{J^2}{2}\int_0^1 e^{-2u_{\lambda'}}\phi + T\int_0^1 u_{\lambda'}\phi + \frac{J}{\tau}\int_0^1\int_0^x e^{-u_{\lambda'}(s)}ds\phi(x)dx. \end{aligned}$$

Performing the limit $\lambda' \to 0$ gives

$$\begin{aligned} \int_0^1 V\phi &= \frac{\delta^2}{2}\int_0^1 e^{-u_D}u_{D,x}^2\phi - \delta^2\int_0^1 e^{-u_D}u_{D,x}\phi_x \\ &\quad + \frac{J^2}{2}\int_0^1 e^{-2u_D}\phi + T\int_0^1 u_D\phi + \frac{J}{\tau}\int_0^1\int_0^x e^{-u_D(s)}ds\phi(x)dx \\ &= \int_0^1\left[-\frac{2\delta^2}{\sqrt{C}}\left(\sqrt{C}\right)_{xx} + \frac{J^2}{2C^2} + T\ln C + \frac{J}{\tau}\int_0^x\frac{ds}{C(s)}\right]\phi dx. \end{aligned}$$

The uniqueness of the limit implies convergence of the whole sequence. This proves the theorem. □

6 Numerical examples

In this section, the limits $\delta \to 0$ and $\lambda \to 0$ are illustrated by numerical examples. The experiments were performed by employing the general purpose boundary-value-problem solver COLSYS, which uses piecewise polynomial collocation at Gaussian points [1]. We model an n^+n diode defined by the doping profile

$$C(x) = 0.75 - 0.25 \tanh(100(x - 0.5)), \qquad x \in (0, 1).$$

This doping concentration is a smooth approximation of the piecewise constant function $\tilde{C}(x) = 1$ for $x \in (0, 1/2)$ and $\tilde{C}(x) = 0.5$ for $x \in (1/2, 1)$. Note that $C(0) = 1$, $C(1) = 0.5$ and $C_x(0) = C_x(1) \simeq 10^{-42}$. We choose the scaled parameters $T = 1$, $\alpha \stackrel{\text{def}}{=} 1/\tau = 10$ and the boundary densities $n_0 = 1$, $n_1 = 0.5$.

In Figure 1 the electron densities for $J = 0.4$, $\lambda = 0.01$ and different values for δ are shown (logarithmic scaling). This situation corresponds in the classical hydrodynamic model to fully subsonic flow since $J/n < 1 = \sqrt{T}$ holds in $(0, 1)$. According to Theorem 4.1, the classical limit $\delta \to 0$ can be performed, and the electron density converges strongly to the hydrodynamic particle density. A similar statement holds for the limit $\lambda \to 0$, see Figure 2. Here, the values $J = 0.4$ and $\delta = 0.01$ are choosen.

For larger current densities, the situation becomes more involved. In Figure 3 the current value $J = 1$ is taken. This corresponds to supersonic flow in the n region since $J/n > 1 = \sqrt{T}$. We observe large-frequency oscillations in this region. For smaller λ, small-frequency oscillations appear (Figure 4). They are also present in the corresponding hydrodynamic problem (see [2]), whereas the large-frequency oscillations are coming from the dispersive quantum term. Figure 5 shows a blow-up of Figure 4 in the region $x \in (0.85, 1)$ for two different values of δ. The frequency is of order $1/\delta$ and the amplitude is of order 1 which shows a typical dispersive behavior. For even smaller λ, the large-frequency oscillations are damped and the wavelength of the small-frequency oscillations decreases (Figure 6). Therefore, for the classical limit and the zero-space-charge limit, we cannot expect to get strong convergence of the particle density in the case $J/n > \sqrt{T}$.

References

[1] U. Ascher, J. Christiansen, and R. D. Russel. Collocation software for boundary value ODEs. *Trans. Math. Soft.*, 7:209–222, 1981.

[2] U. Ascher, P. Markowich, P. Pietra, and C. Schmeiser. A phase plane analysis of transonic solutions for the hydrodynamic semiconductor model. *Math. Mod. Meth. Appl. Sci.*, 1:347–376, 1991.

[3] F. Brezzi, I. Gasser, P. Markowich, and C. Schmeiser. Thermal equilibrium states of the quantum hydrodynamic model for semiconductors in one dimension. *Appl. Math. Lett.*, 8:47–52, 1995.

[4] L. Caffarelli and N. Muler. An L^∞ bound for solutions of the Cahn-Hilliard equations. *Arch. Rat. Mech. Anal.*, 133:129-144, 1995.

[5] D. Chen, E. Kan, U. Ravaioli, C. Shu, and R. Dutton. An improved energy transport model including nonparabolicity and non-maxwellian distribution effects. *IEEE Electr. Dev. Letters*, 13:26–28, 1992.

[6] P. Degond and P. Markowich. On a one-dimensional steady-state hydrodynamic model for semiconductors. *Appl. Math. Lett.*, 3:25–29, 1990.

[7] W. Fang and K. Ito. Steady-state solutions of a one-dimensional hydrodynamic model for semiconductors. *J. Diff. Eqs.*, 133:224–244, 1997.

[8] I. Gamba. Stationary transonic solutions of a one-dimensional hydrodynamic model for semiconductors. *Comm. P. D. E.*, 17:553–577, 1992.

[9] I. Gamba. Sharp uniform bounds for steady potential fluid-Poisson systems. *Proc. Roy. Soc. Edinb., Sect. A*, 127:479-516, 1997.

[10] I. Gamba and C. Morawetz. A viscous approximation for a 2D steady semiconductor or transonic gas dynamic flow: Existence theorem for potential flow. *Comm. Pure Appl. Math.*, 49:999–1049, 1996.

[11] C. Gardner. Numerical simulation of a steady-state electron shock wave in a submicron semiconductor device. *IEEE Trans. El. Dev.*, 38:392–398, 1991.

[12] C. Gardner. The quantum hydrodynamic model for semiconductor devices. *SIAM J. Appl. Math.*, 54:409–427, 1994.

[13] C. Gardner and C. Ringhofer. The smooth quantum potential for the hydrodynamic model. *Phys. Rew. E*, 53:157–167, 1996.

[14] I. Gasser and A. Jüngel. The quantum hydrodynamic model for semiconductors in thermal equilibrium. *ZAMP*, 48:45–59, 1997.

[15] I. Gasser and P. A. Markowich. Quantum hydrodynamics, Wigner transforms and the classical limit. *Asympt. Anal.*, 14:97–116, 1997.

[16] I. Gasser, P. A. Markowich, D. Schmidt, and A. Unterreiter. *Macroscopic theory of charged quantum fluids.* Proceedings of the Workshop on "Mathematical Problems in Semiconductor Physics", Rome, 1995.

[17] I. Gasser, P. A. Markowich, and A. Unterreiter. *Quantum Hydrodynamics.* Proceedings of the SPARCH GdR Conference, St. Malo, 1995.

[18] H. Grubin and J. Kreskovsky. Quantum moment balance equations and resonant tunneling structures. *Solid-State Electr.*, 32:1071–1075, 1989.

[19] X. Jiang. A streamline-upwinding/Petrov-Galerkin method for the hydrodynamic semiconductor device model. *Math. Models Meth. Appl. Sci.*, 5:659–681, 1995.

[20] A. Jüngel. A note on current-voltage characteristics from the quantum hydrodynamic equations for semiconductors. *Appl. Math. Lett.*, 10:29–34, 1997.

[21] A. Jüngel. A steady-state quantum Euler-Poisson system for potential flows. To appear in *Comm. Math. Phys.*, 1998.

[22] Y.-G. Lu. An explicit lower bound of viscosity solutions to isentropic gas dynamics and Euler equation. Preprint 95-18, Universität Heidelberg, 1995.

[23] P. Pietra and C. Pohl. Weak limits of the quantum hydrodynamic model. To appear in *VLSI Design, Proceedings of the "Workshop on Quantum Kinetic Theory"*, 1998.

[24] A. Unterreiter. On the thermal equilibrium solution of a bipolar quantum hydrodynamic model. Preprint, TU Berlin, Fachbereich Mathematik, Germany, 1995.

[25] B. Zhang and J. Jerome. On a steady-state quantum hydrodynamic model for semiconductors. *Nonlin. Anal.*, 26:845–856, 1996.

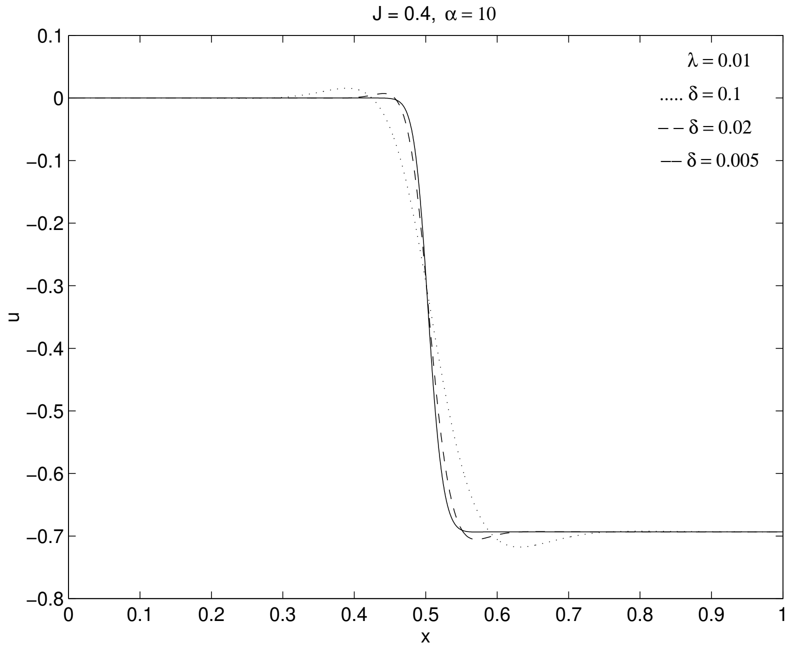

Figure 1: Logarithm of the electron density for $J = 0.4$.

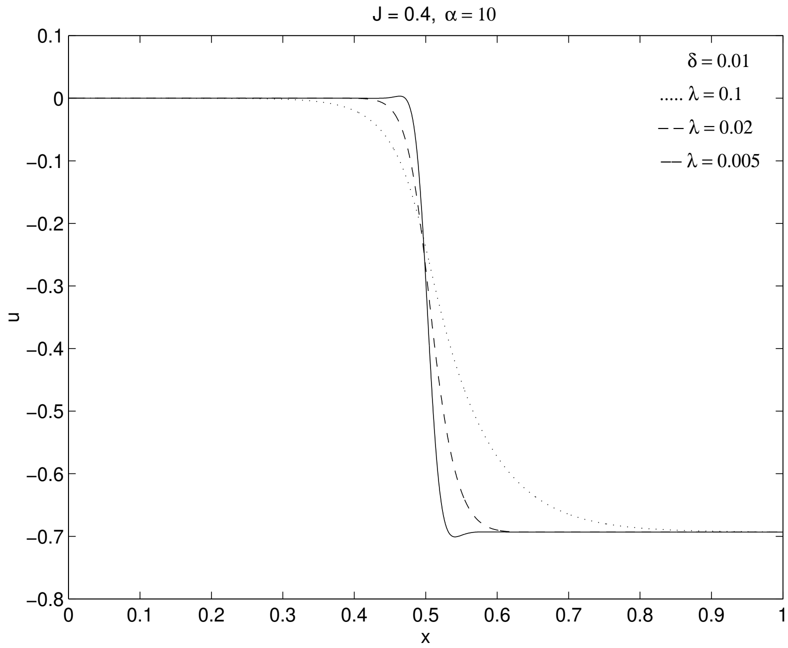

Figure 2: Logarithm of the electron density for $J = 0.4$.

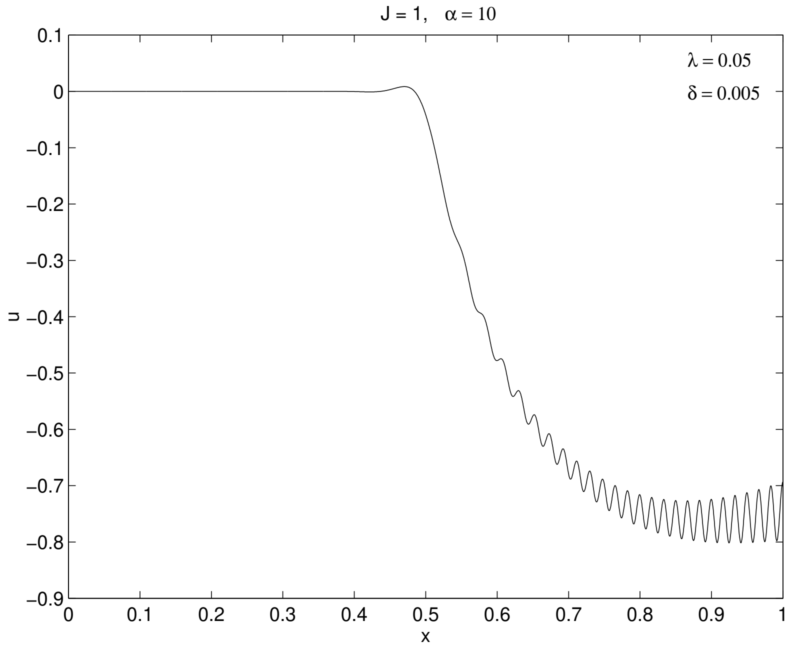

Figure 3: Logarithm of the electron density for $J = 1$.

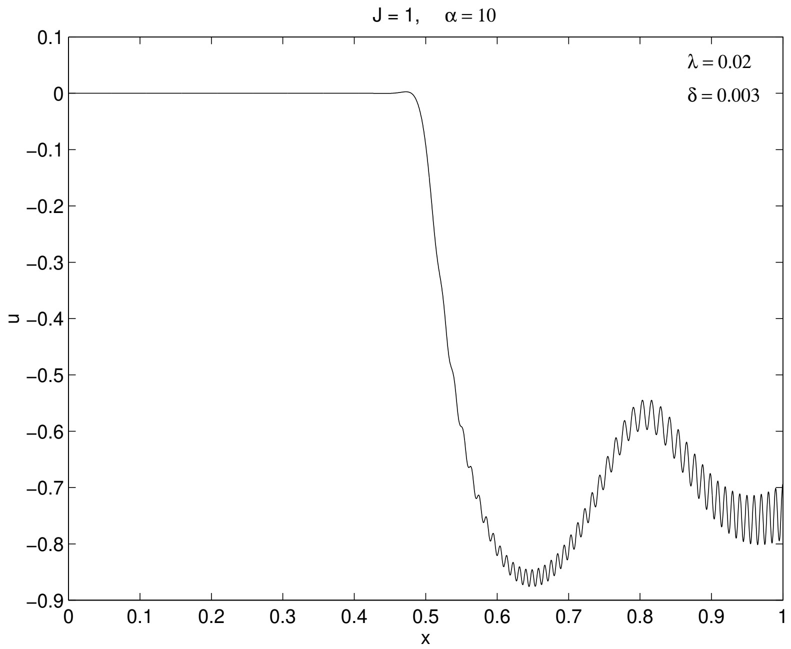

Figure 4: Logarithm of the electron density for $J = 1$.

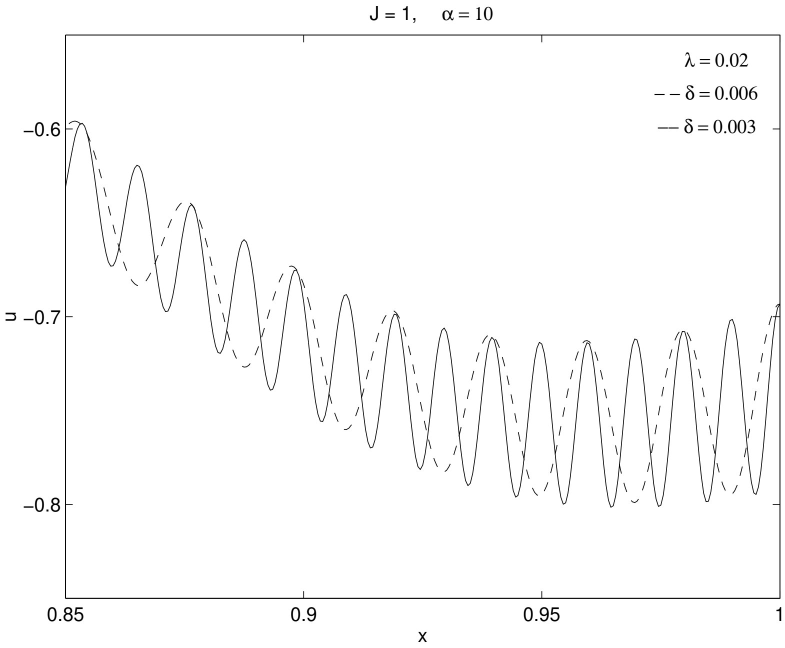

Figure 5: Blow-up of Figure 4.

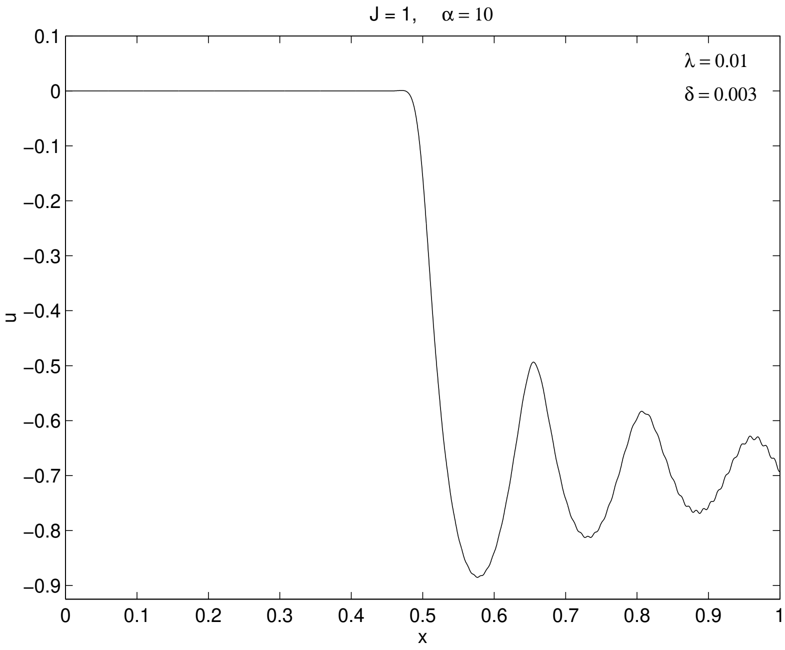

Figure 6: Logarithm of the electron density for $J = 1$.