Analysis and numerical solution of a nonlinear cross-diffusion system arising in population dynamics

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Abstract

A nonlinear population model with cross-diffusion terms for two competing species is studied analytically and numerically. Due to the cross diffusion terms, the problem is strongly nonlinear and so, no maximum principle generally applies. We show first the existence of weak solutions to the parabolic system in any space dimension. Then the one-dimensional stationary problem is investigated analytically and the notion of segregation is discussed. Finally, we present numerical results for the one-dimensional stationary problem underlining the effects of segregation of the species.

1 Introduction

For the evolution of two competing species with homogeneous population density, usually the classical Lotka-Volterra differential equations are used as an appropriate mathematical model. In the case of non-homogeneous densities, diffusion effects have to be taken into account leading to reaction-diffusion equations. Shigesada *et al.* proposed in their pioneering work [18] to introduce further so-called cross-diffusion terms modeling inter-specific influences of the species. Denoting by n_i the population density of the i-th

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species ($i = 1, 2$) and by J_i the corresponding current densities, the time-dependent equations can be written as

$$\partial_t n_1 + \operatorname{div} J_1 = f_1(n_1, n_2), \tag{1}$$
$$\partial_t n_2 + \operatorname{div} J_2 = f_2(n_1, n_2), \tag{2}$$
$$J_1 := -\nabla\big((c_1 + \beta_{11} n_1 + \beta_{12} n_2) n_1\big) + d_1 n_1 \nabla U, \tag{3}$$
$$J_2 := -\nabla\big((c_2 + \beta_{21} n_1 + \beta_{22} n_2) n_2\big) + d_2 n_2 \nabla U \tag{4}$$

in $Q_T := \Omega \times (0, T)$, with $\Omega \subset \mathbb{R}^n$ bounded and $n \geq 1$. The function U is the (given) environmental potential, modeling areas where the environmental conditions are more or less favorable [18, 14]. The diffusion coefficients c_i and β_{ij} are non-negative, and $d_i \in \mathbb{R}$ ($i, j = 1, 2$). The source terms are usually given by a Lotka-Volterra form

$$f_1(n_1, n_2) \;:=\; (R_1 - \gamma_{11} n_1 - \gamma_{12} n_2) n_1, \tag{5}$$
$$f_2(n_1, n_2) \;:=\; (R_2 - \gamma_{21} n_1 - \gamma_{22} n_2) n_2, \tag{6}$$

where R_i are the intrinsic growth rates of the i-th species ($i = 1, 2$), γ_{11} and γ_{22} are the coefficients of intra-specific competition, and γ_{12} and γ_{21} are those of inter-specific competitions.

The above equations are completed by no-flux boundary conditions and inital conditions:

$$J_1 \cdot \nu = J_2 \cdot \nu = 0 \qquad \text{on } \Gamma_T := \partial\Omega \times (0, T), \tag{7}$$
$$n_1(\cdot, 0) = n_{10}, \quad n_2(\cdot, 0) = n_{20} \qquad \text{in } \Omega, \tag{8}$$

with ν denoting the exterior unit normal to $\partial\Omega$.

The above problem contains several types of reaction-convection-diffusion equations. Indeed, in the case $\beta_{ij} = 0$ ($i, j = 1, 2$), Eqs. (1)-(4) reduce to the semilinear drift-diffusion equations which are studied in many fields of applications, e.g. population dynamics [15], electro-chemistry or semiconductors [13]. When $c_1 = c_2 = 0$ and $\beta_{12} = \beta_{21} = 0$, the above problem is of degenerate type. These kind of problems arise, e.g., in multi-phase filtration problems [1], plasma physics and semiconductor theory [7, 9].

For $\beta_{ij} > 0$ ($i, j = 1, 2$), the problem becomes strongly coupled with full diffusion matrix. Such problems arise, for instance, in nonequilibrium thermodynamics [3]. It is well known that in this case, maximum principle arguments generally do not apply so different techniques have to be used or special situations have to be studied.

A possible technique is described in [3], based on entropy dissipation methods. There, the diffusion matrix is symmetric, positive definite and uniformly bounded. However, in our case, the matrix is non-symmetric and, in fact, it is uniformly bounded only when certain L^∞ bounds (which we do not obtain) are available. Nevertheless, we are able to give an existence result for weak solutions under the condition

$$8\beta_{11} > \beta_{12}, \quad 8\beta_{22} > \beta_{21}. \tag{9}$$

Under this hypothesis it turns out that the diffusion matrix is positive definite. The lack of uniform upper bounds for the diffusion terms is compensated by a weaker definition of solution, see Definition 1. However, it seems that (9) is merely a technical condition, and numerical experiments indicate a continuous dependence of the solutions with respect to all the parameters.

The time-dependent problem (1)-(8) has been studied in the literature in several special situations, see [21] for a review. Global existence of solutions and their qualitative behavior for $\beta_{11} = \beta_{22} = \beta_{21} = 0$ has been proved in, e.g., [16, 17, 20]. In this case Eq. (2) is only weakly coupled. For sufficiently small cross-diffusion parameters $\beta_{12} > 0$ and $\beta_{21} > 0$ (or equivalently, "small" initial data) and vanishing self-diffusion coefficients $\beta_{11} = \beta_{22} = 0$, Deuring could show the global existence of solutions [4]. When $c_1 = c_2$, a global existence result in one space dimension has been obtained by Kim in [10]. Finally, under the condition (9) and in two space dimensions, Yagi has shown the global existence of solutions [22]. He supposes $H^{1+\varepsilon}$ inital data and obtains strict solutions (in the sense of semigroups). However, no existence result for positive self-diffusion and cross-diffusion coefficients and L^2 initial data in any space dimension seems to be available. In this paper we prove such a result assuming (9).

The stationary problem corresponding to (1)-(8), but without the transport term $\mathrm{div}(n_i \nabla U)$ was studied by Lou and Ni [11]. They focused on the existence and non-existence of non-constant solutions of the problem depending on the relationship among the parameters of the equations. The question they adress is how self- and cross-diffusion affect to the equilibria points of the system of ODE's of Lotka-Volterra type. Dropping time and transport terms in (1)-(4) allows him to use maximum principle arguments, obtaining in this way L^∞ estimates which are crucial. Unfortunately, this approach seems to fail for obtaining L^∞ estimates both in the time dependent problem and in the stationary problem with transport. We present in this paper an example of existence of solutions for the one-dimensional stationary problem with transport but without reaction term. In this way we illustrate that condition (9) may be not a necessary but a technical condition. Numerical experiments also point out in this direction, as we already mentioned.

The paper is organized as follows. In Section 2 we present the notion of solution of problem (1)-(8), as well as the assumptions which will hold throughout the paper and the main results. Section 3 is devoted to the proof of the existence of solutions of the time-dependent problem. In Section 4 we study one case of the stationary problem for which condition (9) is no longer necessary. We prove existence of solutions for this problem and give an integral representation formula for the solution. We also define and discuss the notion of segregation. Finally, in Section 5 we present numerical examples for the one-dimensional stationary model.

2 Assumptions and main results

First we reformulate the problem. If either β_{12} or β_{21} are zero then at least one of the equations is only weakly coupled, and the results of [16, 17, 20] can be used. Therefore we assume that both β_{12} and β_{21} are positive and perform the following change of unknowns:

$$u := \beta_{21} n_1, \quad v := \beta_{12} n_2 \quad \text{ and } \mathbf{q} := -\nabla U.$$

Then the problem can be reformulated as:

$$\begin{aligned} \partial_t u - \operatorname{div}(\nabla(c_1 u + a_1 u^2 + vu) + d_1 \mathbf{q} u) &= F_1(u,v), && (10) \\ \partial_t v - \operatorname{div}(\nabla(c_2 v + a_2 v^2 + uv) + d_2 \mathbf{q} v) &= F_2(u,v), && (11) \end{aligned}$$

in Q_T, where $a_1 = \beta_{11}/\beta_{21}$, $a_2 = \beta_{22}/\beta_{12}$ and F_1 and F_2 are general source terms including the Lotka-Volterra type terms (5), (6) (see below for the precise assumptions). We impose the no-flux boundary conditions

$$\begin{aligned} \nabla(c_1 u + a_1 u^2 + vu)\cdot\nu + d_1 u \mathbf{q}\cdot\nu &= 0, && (12) \\ \nabla(c_2 v + a_2 v^2 + uv)\cdot\nu + d_2 v \mathbf{q}\cdot\nu &= 0 \quad \text{on } \Gamma_T, && (13) \end{aligned}$$

and the initial conditions

$$u(\cdot,0) = u_0, \quad v(\cdot,0) = v_0 \qquad \text{in } \Omega \qquad (14)$$

with $u_0 := \beta_{21} n_{10}$ and $v_0 := \beta_{12} n_{20}$. From now on we shall refer to problem (10)-(14) as **Problem P**.

We specify our notion of weak solution.

Definition 1. We say that (u, v) is a *weak solution* of Problem P if:

(i) $u, v \geq 0$ satisfy the regularity properties

$$u, v \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)) \cap H^1(0,T; (W^{1,\infty}(\Omega))^*). \tag{15}$$

(ii) Equations (10)-(13) are satisfied in the following sense:

$$\int_0^T \langle u_t, \varphi \rangle + \int_{Q_T} \big((c_1 + 2a_1 u + v)\nabla u + u \nabla v + d_1 u \mathbf{q} \big) \cdot \nabla \varphi$$
$$= \int_{Q_T} F_1(u,v)\varphi, \tag{16}$$
$$\int_0^T \langle v_t, \psi \rangle + \int_{Q_T} \big((c_2 + 2a_2 v + u)\nabla v + v \nabla u + d_2 v \mathbf{q} \big) \cdot \nabla \psi$$
$$= \int_{Q_T} F_2(u,v)\psi, \tag{17}$$

for all $\varphi, \psi \in L^2(0,T; W^{1,\infty}(\Omega))$, where $\langle \cdot, \cdot \rangle$ denotes the duality product of $(W^{1,\infty}(\Omega))^* \times W^{1,\infty}(\Omega)$.

(iii) The initial conditions (14) are satisfied in the sense

$$\begin{matrix} \lim_{t\to 0} \|u(\cdot,t) - u_0\|_{(W^{1,\infty}(\Omega))^*} = 0 \\ \lim_{t\to 0} \|v(\cdot,t) - v_0\|_{(W^{1,\infty}(\Omega))^*} = 0 \end{matrix} \quad \text{as} \quad t \to 0. \tag{18}$$

Observe that Definition 1 is not the usual L^2 based notion of weak solution. However, if the weak solution in the sense of Definition 1 satisfies additionally $u, v \in L^\infty(Q_T)$, then it is straightforward to show that u, v is a weak solution in the usual sense.

We consider the following assumptions on the data:

$$\begin{cases} \text{(a) } \Omega \subset \mathbb{R}^n \text{ is a bounded domain with Lipschitz continuous} \\ \qquad \text{boundary } \partial\Omega, \text{ and } T > 0, \\ \text{(b) } \min\{c_1, c_2\} > 0, \min\{a_1, a_2\} \geq 0, d_1, d_2 \in \mathbb{R}, \\ \text{(c) } \mathbf{q} \in L^\infty(Q_T), \\ \text{(d) } u_0, v_0 \in L^2(\Omega), \text{ with } u_0, v_0 \geq 0, \text{ and} \\ \text{(e) } F_1, F_2 : \mathbb{R}^2 \to \mathbb{R} \text{ are continuous and} \\ \qquad F_1(s,\sigma)s \leq c_F s^2, \ F_2(s,\sigma)\sigma \leq c_F \sigma^2 \text{ for any } s, \sigma \in \mathbb{R}. \end{cases} \tag{19}$$

In particular, assumption (e) includes the Lotka-Volterra source terms. The main result for Problem P is the following.

Theorem 1. *Let the assumptions (19) hold and assume*

$$\min\{a_1, a_2\} \geq 1/8. \tag{20}$$

Then there exists a weak solution of Problem P.

3 Existence of solutions of Problem P

The proof of Theorem 1 is divided into three steps.
Step 1. Introduce the function

$$f_\varepsilon(s) = \frac{s^+}{\varepsilon s^+ + 1}, \quad \text{with } s^+ = \max\{0, s\}. \tag{21}$$

Then

$$0 \leq f_\varepsilon(s) \leq \min\{s^+, 1/\varepsilon\} \quad \text{for any } s \in \mathbb{R} \tag{22}$$

and

$$f_\varepsilon(s) \to s \quad \text{pointwise in } \mathbb{R} \quad \text{as } \varepsilon \to 0. \tag{23}$$

Let $U, V \in L^2(Q_T)$ be given and consider the linear problem

$$\begin{aligned}
&\partial_t u - \operatorname{div}\big(a_{11}\nabla u + a_{12}\nabla v + d_1\mathbf{q}u\big) = f_1, && && (24)\\
&\partial_t v - \operatorname{div}\big(a_{21}\nabla u + a_{22}\nabla v + d_2\mathbf{q}v\big) = f_2 && \text{in } Q_T, && (25)\\
&\big(a_{11}\nabla u + a_{12}\nabla v + d_1\mathbf{q}u\big)\cdot\nu = 0, && && (26)\\
&\big(a_{21}\nabla u + a_{22}\nabla v + d_2\mathbf{q}v\big)\cdot\nu = 0 && \text{on } \Gamma_T, && (27)\\
&u(\cdot,0) = u_0, \quad v(\cdot,0) = v_0 && \text{on } \Omega, && (28)
\end{aligned}$$

where the diffusion matrix $\mathbf{A} = (a_{ij})$ is given by

$$\mathbf{A}(U,V) = \begin{pmatrix} c_1 + 2a_1 f_\varepsilon(U) + f_\varepsilon(V) & f_\varepsilon(V) \\ f_\varepsilon(U) & c_2 + 2a_2 f_\varepsilon(V) + f_\varepsilon(U) \end{pmatrix}, \tag{29}$$

and $f_i := F_i(f_\varepsilon(U), f_\varepsilon(V))$, $i = 1, 2$.

To obtain a solution of (24)-(28) we apply a general result for linear systems of equations associated to uniformly parabolic operators, see [12]. For this, we observe that $a_{ij}, \mathbf{q} \in L^\infty(Q_T)$ and $f_i \in L^2(Q_T)$, $1 \leq i, j \leq 2$.

Moreover, the matrix $\mathbf{A}$ is uniformly positive definite, since for any $\mathbf{x} = (x, y) \in \mathbb{R}^2$ we have

$$\begin{aligned}
\mathbf{x}^T\mathbf{A}\mathbf{x} &= c_1x^2 + c_2y^2 + (2a_1f_\varepsilon(U) + f_\varepsilon(V))x^2 + (2a_2f_\varepsilon(V) + f_\varepsilon(U))y^2 \\
&\quad + (f_\varepsilon(U) + f_\varepsilon(V))xy \\
&\geq c_1x^2 + c_2y^2 + (2a_1 - 1/4)f_\varepsilon(U)x^2 + (2a_2 - 1/4)f_\varepsilon(V)y^2 \\
&\geq c_1x^2 + c_2y^2 \\
&\geq \min\{c_1, c_2\}|\mathbf{x}|^2, \qquad (30)
\end{aligned}$$

where we used condition (9). Then, the results of [12] assert the existence of a unique solution, (u, v), of problem (24)-(28) such that

$$u, v \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \cap H^1(0, T; (H^1(\Omega))^*). \qquad (31)$$

Observe that, in particular, this solution is a weak solution in the sense of Definition 1.

Step 2. We consider the Banach space

$$\mathcal{X}_T := L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \cap H^1(0, T; (H^1(\Omega))^*),$$

and define the set

$$K := \{g \in L^2(Q_T) : \|g\|_{\mathcal{X}_T} \leq \rho\},$$

with some positive ρ to be fixed. Aubin's lemma [19] implies that the imbedding $K \subset L^2(Q_T)$ is compact. It is straightforward to check that K is convex. For fixed $\varepsilon > 0$ and for any $(U, V) \in K^2$, we define the fixed-point operator $S : K^2 \to L^2(Q_T)^2$ by $S(U, V) = (u, v)$, where (u, v) is the solution of the linear problem obtained in Step 1 of this proof. Note that a fixed point of this operator is a solution of problem (24)-(28) with the matrix $\mathbf{A} := (a_{ij})$ given by

$$\mathbf{A}(u, v) := \begin{pmatrix} c_1 + 2a_1f_\varepsilon(u) + f_\varepsilon(v) & f_\varepsilon(v) \\ f_\varepsilon(u) & c_2 + 2a_2f_\varepsilon(v) + f_\varepsilon(u) \end{pmatrix}, \qquad (32)$$

and $f_i := F_i(f_\varepsilon(u), f_\varepsilon(v))$, $i = 1, 2$. We shall refer to this problem as **Problem $\mathbf{P}_\varepsilon$**.

We verify now the assumptions of the Schauder fixed-point theorem. First we determine $\rho > 0$ such that $S(K^2) \subset K^2$. Due to the regularity

(31) we can use u, v as test functions for Eqs. (24)-(25) to obtain, for a.e. $t \in (0,T)$,

$$\frac{d}{dt}\int_\Omega (u^2+v^2) + c\int_\Omega (|\nabla u|^2 + |\nabla v|^2) \leq k\int_\Omega (u^2+v^2) + c_F^2 \int_\Omega (f_\varepsilon^2(U) + f_\varepsilon^2(V)), \tag{33}$$

with $c = \min\{c_1, c_2\}$ and $k = 1/4 + (|d_1|^2 + |d_2|^2)\|\mathbf{q}\|_\infty / c$. Here we used assumption (e) of (19) and estimate (30). Then Gronwall's lemma implies

$$\int_\Omega (u^2(t) + v^2(t)) \leq e^{kt}\Big(\int_\Omega (u_0^2 + v_0^2) + c_F^2 \int_{Q_T} (f_\varepsilon^2(U) + f_\varepsilon^2(V))\Big). \tag{34}$$

From (33), (34) and (22) we obtain

$$\int_\Omega (u^2(t) + v^2(t)) + c\int_{Q_T} (|\nabla u|^2 + |\nabla v|^2) \leq (1 + kTe^{kT})\int_\Omega (u_0^2 + v_0^2) + 2Tc_F^2(1+kT)\mathrm{meas}(\Omega)/\varepsilon^2. \tag{35}$$

In view of definition (29) for the coefficients a_{ij} and of definition (21) for the function f_ε we obtain, for φ such that $\|\varphi\|_{L^2(0,T;H^1(\Omega))} = 1$,

$$\begin{aligned}\|u_t\|_{L^2(0,T;(H^1(\Omega))^*)} &= \sup_\varphi \int_{Q_T} \big(\big[(a_{11}\nabla u + a_{12}\nabla v + d_1\mathbf{q}u\big]\cdot\nabla\varphi \\ &\quad + F_1(f_\varepsilon(U), f_\varepsilon(V))\varphi\big) \\ &\leq c_\varepsilon(\|\nabla u\|_{L^2(Q_T)} + \|\nabla v\|_{L^2(Q_T)})\|\nabla\varphi\|_{L^2(Q_T)} \\ &\quad + |d_1|\|\mathbf{q}\|_{L^\infty}\|u\|_{L^2(Q_T)}\|\nabla\varphi\|_{L^2(Q_T)} + 2c_F\varepsilon^{-1}\|\varphi\|_{L^1(Q_T)},\end{aligned}$$

for $c_\varepsilon = c_1 + (2a_1+1)/\varepsilon$. Hence

$$\|u_t\|_{L^2(0,T;(H^1(\Omega))^*)} \leq c_\varepsilon(\|\nabla u\|_{L^2(Q_T)} + \|\nabla v\|_{L^2(Q_T)}) + |d_1|T^{1/2}\|\mathbf{q}\|_{L^\infty}\|u\|_{L^\infty(0,T;L^2(\Omega))} + c(\varepsilon). \tag{36}$$

A similar estimate holds for v_t in the space $L^2(0,T;(H^1(\Omega))^*)$. We deduce from (35) and (36) that for any $(U,V) \in K^2$, $(u,v) := S(U,V)$ satisfies

$$\|u\|_{\mathcal{X}_T} + \|v\|_{\mathcal{X}_T} \leq \sigma_1,$$

with $\sigma_1 > 0$ independent of ρ. Finally, choosing $\rho \geq \sigma_1$ we obtain $S(K^2) \subset K^2$.

The continuity of the fixed-point operator follows from standard arguments. Furthermore, $S(K^2)$ is relatively compact since the embedding $K \hookrightarrow L^2(Q_T)$ is compact. Thus Schauder's fixed-point theorem provides the existence of a weak solution of Problem P_ε.

We finish this step by proving that the solution of Problem P_ε, which we denote now by $(u_\varepsilon, v_\varepsilon)$, satisfies $u_\varepsilon, v_\varepsilon \geq 0$ in Q_T. Since $u_\varepsilon, v_\varepsilon \in \mathcal{X}_T$, we can use $u_\varepsilon^- := \min\{0, u_\varepsilon\}$ and $v_\varepsilon^- := \min\{0, v_\varepsilon\}$ as admissible test functions to obtain

$$\frac{d}{dt}\int_\Omega ((u_\varepsilon^-)^2 + (v_\varepsilon^-)^2) + c\int_\Omega (|\nabla u_\varepsilon^-|^2 + |\nabla v_\varepsilon^-|^2) \leq k\int_\Omega ((u_\varepsilon^-)^2 + (v_\varepsilon^-)^2),$$

since $F_1(f_\varepsilon(u_\varepsilon), f_\varepsilon(v_\varepsilon)) = F_1(0, f_\varepsilon(v_\varepsilon)) = 0$ on $\{u_\varepsilon < 0\}$ and similarly for F_2. Gronwall's lemma implies

$$\int_\Omega (u_\varepsilon^-(t)^2 + v_\varepsilon^-(t)^2) \leq e^{kt}\int_\Omega ((u_0^-)^2 + (v_0^-)^2) = 0,$$

for a.e. $t \in (0, T)$. The result follows.

Step 3. For each $\varepsilon > 0$ consider the sequence of solutions $(u_\varepsilon, v_\varepsilon)$ of Problem P_ε, obtained in Step 2. We show that $(u, v) := \lim_{\varepsilon\to 0}(u_\varepsilon, v_\varepsilon)$ exists and is a weak solution of Problem P. Using $u_\varepsilon, v_\varepsilon \in \mathcal{X}_T$ as test functions in Eqs. (24), (25), respectively, we get

$$\begin{aligned}\frac{d}{dt}\int_\Omega (u_\varepsilon^2 + v_\varepsilon^2) + c\int_\Omega (|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) \leq{}& k\int_\Omega (u_\varepsilon^2 + v_\varepsilon^2) \\ &+ c_F^2\int_\Omega (f_\varepsilon^2(u_\varepsilon) + f_\varepsilon^2(u_\varepsilon)). \end{aligned} \tag{37}$$

We use again property (22) and Gronwall's lemma to obtain

$$\int_\Omega (u_\varepsilon(t)^2 + v_\varepsilon(t)^2) \leq e^{(k+c_F^2)t}\int_\Omega (u_0^2 + v_0^2). \tag{38}$$

From (37) and (38) we deduce

$$\begin{aligned} \|u_\varepsilon\|_{L^2(0,T;H^1(\Omega))} + \|v_\varepsilon\|_{L^2(0,T;H^1(\Omega))} &\leq C, \\ \|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|v_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} &\leq C, \end{aligned}$$

with $C > 0$ independent of ε. Moreover, from Eq. (24) we obtain, for a_{ij} given by (32) with (u, v) replaced by $(u_\varepsilon, v_\varepsilon)$,

$$\begin{aligned}\|u_{\varepsilon t}\|_{L^2(0,T;(W^{1,\infty}(\Omega))^*)} \leq \sup_\varphi \int_{Q_T} &\big[(a_{11}\nabla u_\varepsilon + a_{12}\nabla v_\varepsilon + d_1\mathbf{q}u_\varepsilon\big]\nabla\varphi \\ &+ \int_{Q_T} F_1(f_\varepsilon(u_\varepsilon), f_\varepsilon(v_\varepsilon))\varphi, \end{aligned}$$

with φ such that $\|\varphi\|_{L^2(0,T;W^{1,\infty}(\Omega))} = 1$. We have

$$
\begin{aligned}
\|u_{\varepsilon t}\|_{L^2(0,T;(W^{1,\infty}(\Omega))^*)} &\le \|a_{11}\|_{L^\infty(0,T;L^2(\Omega))}\|\nabla u_\varepsilon\|_{L^2(Q_T)} \\
&\quad + \|a_{12}\|_{L^\infty(0,T;L^2(\Omega))}\|\nabla v_\varepsilon\|_{L^2(Q_T)} \\
&\quad + |d_1|\|\mathbf{q}\|_{L^\infty(Q_T)}\|u_\varepsilon\|_{L^2(Q_T)} \\
&\le C,
\end{aligned}
$$

and similarly, $\|v_{\varepsilon t}\|_{L^2(0,T;(W^{1,\infty}(\Omega))^*)} \le C$. We deduce the existence of a subsequence of $(u_\varepsilon, v_\varepsilon)$ (again denoted with the subindex ε) and a pair (u,v) such that

$$
\begin{aligned}
&(u_{\varepsilon t}, v_{\varepsilon t}) \overset{*}{\rightharpoonup} (u_t, v_t) && \text{weakly* in } L^2(0,T;(W^{1,\infty}(\Omega))^*), \\
&(\nabla u_\varepsilon, \nabla v_\varepsilon) \rightharpoonup (\nabla u, \nabla v) && \text{weakly in } L^2(Q_T).
\end{aligned}
\tag{39}
$$

Furthermore, Aubin's lemma implies that, up to a new subsequence,

$$
(u_\varepsilon, v_\varepsilon) \to (u,v) \quad \text{a.e. in } Q_T \text{ and strongly in } L^2(Q_T). \tag{40}
$$

We then have, using Hölder's inequality, for $p = 2d/(d-2)$ ($p<\infty$ if $d \le 2$),

$$
\begin{aligned}
\int_{Q_T} (f_\varepsilon(u_\varepsilon) - u)^2 &= \int_{Q_T} \frac{(u_\varepsilon - u - \varepsilon u u_\varepsilon)^2}{(\varepsilon u_\varepsilon + 1)^2} \\
&\le 2\int_{Q_T} (u_\varepsilon - u)^2 + 2\int_{Q_T} \frac{(\varepsilon u_\varepsilon u)^2}{(\varepsilon u_\varepsilon)^{2-4/d}(\varepsilon u_\varepsilon + 1)^{4/d}} \\
&\le 2\int_{Q_T} (u_\varepsilon - u)^2 + 2\varepsilon^{4/d}\int_{Q_T} u_\varepsilon^{4/d} u^2 \\
&\le 2\|u_\varepsilon - u\|_{L^2(Q_T)} + 2\varepsilon^{4/d}\|u_\varepsilon\|^{4/d}_{L^\infty(L^2)}\|u\|^2_{L^2(L^p)} \\
&\le 2\|u_\varepsilon - u\|_{L^2(Q_T)} + 2\varepsilon^{4/d} c\|u_\varepsilon\|^{4/d}_{L^\infty(L^2)}\|u\|^2_{L^2(H^1)} \\
&\to 0,
\end{aligned}
\tag{41}
$$

as $\varepsilon \to 0$. Here we have used the continuous embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ and the uniform bounds on u_ε.

The continuity of F_1 implies

$$
F_1(f_\varepsilon(u_\varepsilon), f_\varepsilon(v_\varepsilon)) \to F_1(u,v) \quad \text{a.e. in } Q_T \text{ and strongly in } L^2(Q_T). \tag{42}
$$

Since $(u_\varepsilon, v_\varepsilon)$ is a weak solution in the sense of Definition 1 we have, for any $\varphi \in L^2(0,T;(W^{1,\infty}(\Omega)))$,

$$
\begin{aligned}
\int_0^T \langle u_{\varepsilon t}, \varphi\rangle + \int_{Q_T} \big((c_1 + 2a_1 f_\varepsilon(u_\varepsilon) + f_\varepsilon(v_\varepsilon))\nabla u_\varepsilon + f_\varepsilon(u_\varepsilon)\nabla v_\varepsilon + d_1 u_\varepsilon \mathbf{q}\big)\cdot\nabla\varphi \\
= \int_{Q_T} F_1(f_\varepsilon(u_\varepsilon), f_\varepsilon(v_\varepsilon))\varphi,
\end{aligned}
$$

Letting $\varepsilon \to 0$, using (39)-(42), we obtain

$$\int_0^T \langle u_t, \varphi \rangle + \int_{Q_T} \big((c_1 + 2a_1 u) + v)) \nabla u + u \nabla v + d_1 u \mathbf{q}\big) \cdot \nabla \varphi$$
$$= \int_{Q_T} F_1(u,v)\varphi.$$

Finally, since $u \in H^1(0,T; W^{1,\infty}(\Omega)^*) \subset C^0([0,T]; W^{1,\infty}(\Omega)^*)$, the initial condition is satisfied in the sense of $W^{1,\infty}(\Omega)^*$, as prescribed in (18). Similarly, we obtain Eq. (11) in the limit $\varepsilon \to 0$. This proves the theorem.

4 The stationary problem

We consider in this section the one-dimensional version of the stationary problem corresponding to Problem P, with $F_1(u,v) = F_2(u,v) = 0$ (see [18]). Integrating both equations in $(0,x)$ with $x \in \Omega$ and using the boundary conditions leads to the following problem. Find $(u,v) : \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}^2_+$ such that

$$\begin{cases} \big(u(c_1 + a_1 u_+ + v_+)\big)' + d_1 q u = 0 \\ \big(v(c_2 + a_2 v_+ + u_+)\big)' + d_2 q v = 0 \end{cases} \quad \text{in } \Omega, \tag{43}$$

with

$$\int_\Omega u = \tilde{u} \quad \text{and} \quad \int_\Omega v = \tilde{v}. \tag{44}$$

Here $\tilde{u}$ and $\tilde{v}$ stand for the *mass* of u and v, conserved along time in the evolution problem and determined by the initial distributions:

$$\tilde{u} := \int_\Omega u_0 \quad \text{and} \quad \tilde{v} := \int_\Omega v_0.$$

In [18] the authors followed a dynamical system approach to equations (43), for which the origin $(u,v) = (0,0)$ is the unique stable stationary point. They also deduced a necessary condition for segregation and analyzed and gave an explicit solution of the simplified case $\beta_{11} = \beta_{12} = \beta_{22} = 0$. However, it is clear from the property of conservation of mass that $(u,v) = (0,0)$ is not an admissible solution of Problem (43)-(44). In this section we shall give a proof of existence of solutions of this problem. We shall see that condition (2) is no longer necessary and only positiveness of c_i will be assumed. The end of this section is devoted to discuss the notion of segregation.

Let us start by transforming the system (43). Splitting the equations (43) we find

$$u'(c_1 + 2a_1u_+ + v_+) + u(v'_+ + d_1q) = 0, \tag{45}$$
$$v'(c_2 + 2a_2v_+ + u_+) + v(u'_+ + d_2q) = 0. \tag{46}$$

We have from (46)

$$v' = -\frac{d_2q + u'_+}{c_2 + 2a_2v_+ + u_+}v,$$

and subsituting in (45) we obtain

$$u' = -\frac{qh_1(u,v)}{g_1(u,v)}u, \tag{47}$$

with

$$\begin{aligned} g_1(u,v) &:= (c_1 + 2a_1u_+ + v_+)(c_2 + 2a_2v_+ + u_+) - u_+v_+ > 0, \\ h_1(u,v) &:= d_1(c_2 + 2a_2v_+ + u_+) - d_2v_+. \end{aligned} \tag{48}$$

In a similar way we get

$$v' = -\frac{qh_2(u,v)}{g_2(u,v)}v, \tag{49}$$

with

$$\begin{aligned} g_2(u,v) &:= (c_1 + 2a_1u_+ + v_+)(c_2 + 2a_2v_+ + u_+) - u_+v_+ > 0, \\ h_2(u,v) &:= d_2(c_1 + 2a_1u_+ + v_+) - d_1u_+. \end{aligned} \tag{50}$$

We now state the result on existence of solutions.

Theorem 2. *Let c_1 and c_2 be positive and assume $q \in L^\infty(\Omega)$. There exists a solution (u,v) of Problem (43)-(44) such that $u,v \geq 0$ in Ω and*

$$u,v \in W^{1,\infty}(\Omega). \tag{51}$$

In addition, if $q \in W^{m,\infty}(\Omega)$ then the solution (u,v) satisfies

$$u,v \in W^{m+1,\infty}(\Omega). \tag{52}$$

Proof. A general solution of (47), (49) is given by

$$u(x) = u(0)\exp\{\int_0^x \frac{-qh_1(u,v)}{g_1(u,v)}\} \quad v(x) = v(0)\exp\{\int_0^x \frac{-qh_2(u,v)}{g_2(u,v)}\}, \tag{53}$$

and by (44) we determine the value of $u(0)$ and $v(0)$ as

$$u(0) = \tilde{u}\Big(\int_\Omega \exp\{\int_0^x \frac{-qh_1(u,v)}{g_1(u,v)}\}\Big)^{-1}, \tag{54}$$

$$v(0) = \tilde{u}\Big(\int_\Omega \exp\{\int_0^x \frac{-qh_2(u,v)}{g_2(u,v)}\}\Big)^{-1}. \tag{55}$$

Thus, if $u, v \in L^\infty(\Omega)$ satisfy (53)-(55) then (u, v) is a solution of (43)-(44) which in fact belongs to $W^{1,\infty}(\Omega)$.

To find a solution of (53)-(55) we start obtaining a priori estimates. We have

$$\begin{aligned}|\frac{h_1(u,v)}{g_1(u,v)}| &\le \frac{|2a_2d_1 - d_2|v_+}{g_1(u,v)} + \frac{d_1u_+}{g_1(u,v)} + \frac{c_2d_1}{g_1(u,v)} \le \\ &\le \frac{|2a_2d_1 - d_2|}{c_2 + 2a_2c_1} + \frac{d_1}{c_1 + 2a_1c_2} + \frac{d_1}{c_1} =: k_1,\end{aligned} \tag{56}$$

and therefore, using (53)

$$\frac{\tilde{u}}{|\Omega|}\exp\{-2k_1\|q\|_{L^1}\} \le u(x) \le \frac{\tilde{u}}{|\Omega|}\exp\{2k_1\|q\|_{L^1}\}, \tag{57}$$

for $x \in \Omega$. ¿From (47), (56) and (57) we obtain

$$|u'(x)| \le |q(x)||u(x)||\frac{h_1(u,v)}{g(u,v)}| \le \frac{k_1\tilde{u}}{|\Omega|}\|q\|_{L^\infty}\exp\{2k_1\|q\|_{L^1}\}, \tag{58}$$

for $x \in \Omega$. Estimates similar to (57)-(58) hold for v. We introduce the following iterative scheme based on (53)-(55)

$$u_0 := \frac{\tilde{u}}{|\Omega|}, \quad u_{n+1}(x) := u_{n+1}(0)\exp\{\int_0^x \frac{-qh_1(u_n,v_n)}{g_1(u_n,v_n)}\}, \tag{59}$$

with

$$u_{n+1}(0) := \tilde{u}\Big(\int_\Omega \exp\{\int_0^x \frac{-qh_1(u_n,v_n)}{g_1(u_n,v_n)}\}\Big)^{-1},$$

and similar expressions for v. Since $\tilde{u}, \tilde{v} > 0$ and qh_i/g_i, $i = 1, 2$, are bounded we have $u_n, v_n > 0$ uniformly in $n \in \mathbb{N}$. The a priori estimates (57)-(58) imply that the sequences u_n and v_n are bounded in $W^{1,\infty}(\Omega)$. We can therefore extract subsequences (still denoted by n) such that

$$\begin{array}{llll} u_n \stackrel{*}{\rightharpoonup} u & \text{and} & v_n \stackrel{*}{\rightharpoonup} v & \text{in } W^{1,\infty}(\Omega) \\ u_n \to u & \text{and} & v_n \to v & \text{in } L^p(\Omega) \text{ for all } p < \infty \text{ and a.e. in } \Omega. \end{array} \tag{60}$$

Moreover, $u, v \in C^{0,\alpha}(\Omega)$ for any $\alpha < 1$ and are non-negative in Ω . To check that u, v is actually a solution of (43)-(44) is straightforward. The mass condition is a consequence of the strong convergence in $L^p(\Omega)$ (in fact, only weak convergence would be sufficient). Using expression (59) and a similar one for v we find

$$\left\{ \begin{array}{l} u'_{n+1} g_1(u_n, v_n) + qh_1(u_n, v_n) u_{n+1} = 0 \\ v'_{n+1} g_2(u_n, v_n) + qh_2(u_n, v_n) v_{n+1} = 0 \end{array} \right. \quad \text{in } \Omega.$$

Property (60) and the continuity of g_i, h_i allow us to pass to the limit and to obtain a solution of (45)-(46), which is equivalent to have a solution of (43)-(44). Finally, a standard bootstrap argument allows us to deduce (52). $\square$

¿From now on we shall write u and v instead of u_+ and v_+ in the diffusion coefficients, and define $g(u, v) := g_1(u, v) = g_2(u, v)$.

In the following we shall analyze the concept of segregation in a simplified framework. We shall assume, following [18], that the environmental potential $U(x)$ is a smooth function such that the corresponding enviromental flux satisfies $q(x_0) = 0$ for a single point $x_0 \in \Omega$.

Definition 2. We say that the stationary problem (43)-(44) has the property of *segregation* if there exists a point $x_0 \in \Omega$ such that u and v have a local maximum and minimum at x_0, respectively (or vice versa).

Observe that since $q(x_0) = 0$, from (47) and (49) we find

$$u'(x_0) = v'(x_0) = 0,$$

so we deduce the existence of a critical point, x_0, both for u and v. Taking derivatives in (47) and (49) and evaluating at x_0 we have

$$g(u(x_0), v(x_0)) u''(x_0) = -q'(x_0) h_1(u(x_0), v(x_0)),$$
$$g(u(x_0), v(x_0)) v''(x_0) = -q'(x_0) h_2(u(x_0), v(x_0)).$$

Therefore, segregation will take place if

$$\mathrm{sign}(h_1(u(x_0), v(x_0))) \neq \mathrm{sign}(h_2(u(x_0), v(x_0))).$$

We may rewrite h_1, h_2 as follows:

$$\begin{pmatrix} h_1(u,v) \\ h_2(u,v) \end{pmatrix} = \mathbf{H} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} := \begin{pmatrix} c_2 + 2a_2 v + u & -v \\ -u & c_1 + 2a_1 u + v \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix},$$

and since $\mathbf{H}$ is positive definite we also have

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \frac{1}{\det \mathbf{H}} \begin{pmatrix} c_1 + 2a_1 u + v & v \\ u & c_2 + 2a_2 v + u \end{pmatrix} \begin{pmatrix} h_1(u,v) \\ h_2(u,v) \end{pmatrix}.$$

¿From these expressions we see

$$h_1(u,v)h_2(u,v) > 0 \Longrightarrow d_1 d_2 > 0,$$
$$d_1 d_2 < 0 \Longrightarrow h_1(u,v)h_2(u,v) < 0.$$

Therefore, segregation does always occur if $\mathrm{sign}(d_1) \neq \mathrm{sign}(d_2)$. Note that in this case x_0 is an attractive point (from the environmental point of view) for one of the populations and a repulsive point for the other. On the other hand, the only possibility not to have segregation (so x_0 is a point of local maximum or local minimum for both u and v) is that $\mathrm{sign}(d_1) = \mathrm{sign}(d_2)$. However, this is merely a necessary condition and it is still possible to have $\mathrm{sign}(d_1) = \mathrm{sign}(d_2)$ and segregation taking place.

Since segregation occurs for $d_1 d_2 < 0$ let us assume, to fix ideas, that d_1 and d_2 are positive (the case $d_1, d_2 < 0$ can be analyzed in a similar way). A first non-trivial situation is the following:

$$\text{if} \quad 2a_2 d_1 > d_2 \quad \text{and} \quad 2a_1 d_2 > d_1 \tag{61}$$

then $h_1 > 0$ and $h_2 > 0$ and therefore, x_0 is a point of maximum both for u and v. As a consequences, segregation does not occur. Observe also that if (61) holds then $a_1 a_2 > 1/4$, which is a condition in the same spirit as (20). We could say that this is the case of strong diffusion. Solutions corresponding to this case do spread along the whole domain without noticing the effects of cross diffussion and enviromental potential.

Non-trivial cases in which segregation occurs are not so clearly determined by the size of the parameters. We have $h_1 > 0$ and $h_2 < 0$ if

$$c_2 + 2a_2 v(x_0) + u(x_0) > \frac{d_2}{d_1} v(x_0) \quad \text{and} \quad \frac{d_2}{d_1}(c_1 + 2a_1 u(x_0) + v(x_0)) < u(x_0). \tag{62}$$

Observe that the L^∞ norms of u and v are bounded when $d_2 \to 0$, see (57). Therefore, fixing the other parameters arbitrarily and taking d_2 small enough we have condition (62) fulfilled and segregation occurs. In fact, the important parameter here is the ratio d_2/d_1, which leads to segregation either if it is large or small, see Figure 3.

Finally, the numerical experiments indicate that this sufficient condition on the ratio d_2/d_1 is not necessary, as Figure 1 shows.

5 Numerical results

For the discretization of Problem P, a semi-implicit finite difference method is used. A rectangular mesh over the domain $D = \{(x,t) : a \le x \le b, 0 \le t \le T\}$ defines a set of points $D_\Delta = \{(x_i, t_n) : x_i = ih,\ t_n = n\tau, i = 0, \ldots, N,\ n = 0, \ldots, M\}$, with $N+1$, $M+1$ the number of mesh points in the space and time dimensions, respectively, and $h = \frac{b-a}{N}$, $\tau = \frac{T}{M}$ the correspondent mesh sizes. The following notation is used: let u_i^n, v_i^n be the numerical aproximations of the population densities $u(x_i, t_n), v(x_i, t_n)$ respectively and $\Delta u_i = u_{i+1} - u_{i-1}$, $\delta^2 u_i = u_{i-1} - 2u_i + u_{i+1}$ the usual difference operators. An obvious discretization of equations (10)-(11) which leads to a linear system of equations to be solved at each time step, is the following:

$$\delta_t u_i - \frac{1}{h^2}(\alpha_u)_i^n \delta^2 u_i^{n+1} - \frac{1}{2h^2}(\beta_u)_i^n \Delta u_i^{n+1} - \frac{1}{h^2} u_i^{n+1} \delta^2 v_i^n \\ - d_1\left(u_i^{n+1} q_x + \frac{1}{2h} q \Delta u_i^{n+1}\right) = 0,$$

$$\delta_t v_i - \frac{1}{h^2}(\alpha_v)_i^n \delta^2 v_i^{n+1} - \frac{1}{2h^2}(\beta_v)_i^n \Delta v_i^{n+1} - \frac{1}{h^2} v_i^{n+1} \delta^2 u_i^n \\ - d_2\left(v_i^{n+1} q_x + \frac{1}{2h} q \Delta v_i^{n+1}\right) = 0,$$

for $i = 1, \ldots, N-1$, where: $\delta_t u_i = (u_i^{n+1} - u_i^n)/\tau$,

$$\begin{aligned} (\alpha_u)_i^n &= (c_1 + 2a_1 u_i^n + v_i^n), & (\alpha_v)_i^n &= (c_2 + 2a_2 v_i^n + u_i^n), \\ (\beta_u)_i^n &= a_1 \Delta u_i^n + \Delta v_i^n, & (\beta_v)_i^n &= a_2 \Delta v_i^n + \Delta u_i^n. \end{aligned} \tag{63}$$

The discrete boundary conditions are

$$\begin{aligned} (c_1 + 2a_1 u_0^n + v_0^n)\delta_x^+ u_0^{n+1} + u_0^{n+1}\delta_x^+ v_0^n + d_1 u_0^{n+1} q = 0, \\ (c_2 + 2a_2 v_0^n + u_0^n)\delta_x^+ v_0^{n+1} + v_0^{n+1}\delta_x^+ u_0^n + d_1 v_0^{n+1} q = 0, \end{aligned}$$

for $i = 0$, and

$$(c_1 + 2a_1 u_N^n + v_N^n)\delta_x^- u_N^{n+1} + u_N^{n+1}\delta_x^- v_N^n + d_1 u_N^{n+1} q = 0,$$
$$(c_2 + 2a_2 v_N^n + u_N^n)\delta_x^- v_N^{n+1} + v_N^{n+1}\delta_x^- u_N^n + d_1 v_N^{n+1} q = 0,$$

for $i = N$. Here

$$\delta_x^+ u_0^n = \frac{-3u_0^n + 4u_1^n - u_2^n}{2h}, \quad \delta_x^- u_N^n = \frac{u_{N-2}^n - 4u_{N-1}^n + 3u_N^n}{2h}$$

are a forward three point formula to approximate $u_x(x_0, t_n)$ and a backward three point formula to approximate $u_x(x_N, t_n)$, respectively. Therefore, discretization errors are of orders $O(h^2)$ and $O(\tau)$.

The following data has been taken: $\Omega = [0, 3]$; the environmental potential function $U(x) = 1.5(x - 0.5)^2$. The mesh sizes are $h = 10^{-2}$, $\tau = 10^{-3}$ and we considered the stationary solution to be attained when

$$\max_i \frac{|u_i^{n+1} - u_i^n|}{|u_i^n|} < 10^{-3}.$$

The mass conservation condition

$$\tilde{u} = \int_\Omega u_0, \quad \tilde{v} = \int_\Omega v_0$$

is fullfilled by the numerical solution at each time step. Let $\tilde{u}_a, \tilde{v}_a$ be the approximated values of $\tilde{u}$, $\tilde{v}$ respectively, obtained by numerical integration. The mass errors are

$$\max_{0<n\leq M+1} (|\tilde{u} - \tilde{u}_a|, |\tilde{v} - \tilde{v}_a|) < 10^{-2}.$$

The performed numerical experiments show that, for a wide range of model parameters and initial conditions, the numerical scheme is stable, which allows us to show some different behaviours of two interacting species included in the model. In particular it is interesting to observe when segregation of the two species, due to habitat hetereogenity, does appear.

We run the numerical experiments to study the behaviour of the model in the following cases:

(a) Large and small cross-diffusion terms. $c_i = d_i = 1$ and initial conditions $u(x, 0) = 10, v(x, 0) = 20$ are fixed, while small a_i corresponds to large cross diffusion and vice versa. Figure 1 shows the numerical solution corresponding to a_i values of $0, 0.1, 10$.

(b) Large diffusion coefficients c_i compared to a_i, i.e. $c_i \gg a_i$. The transport coefficient $d_i = 1$ and the same initial conditions as in case (a) are fixed. Figure 2 shows the numerical solutions corresponding to $a_i = 0.01$ and $c_i = 1, 10, 100$.

(c) Segregation effects due to a large ratio of the transport coefficients. In this case $d_1 \gg d_2$. $c_i = d_2 = 1$ and $u(x,0) = 10$, $v(x,0) = 10$ are fixed. Figure 3 shows the numerical solutions corresponding to $a_i = 1$ and $d_1 = 4, 8, 20, 40$. As case (a) shows, large cross-diffusion enhance segregation effects, therefore we repeat the last simulation with $a_i = 0.1$. Figure 4 shows same results for this case.

(d) Discontinuous initial data. The model parameters are $c_i = a_i = 1$, $d_1 = 8$, $d_2 = 1$. The initial conditions are $u(x,0) = 12$, $0 \leq x \leq 1.5$ and $u(x,0) = 8$, $1.5 < x \leq 3$, $v(x,0) = 10$. Figure 5 shows that the stationary solution $u(x), v(x)$ is independent of the initial density distributions, provided they have same mass (compare to Figure 3).

(e) $c_i = 1$, $a_1 = 1$, $a_2 = 0.01$, $d_1 = 40$, $d_2 = 1$, $u(x,0) = v(x,0) = 10$ (Figure 6). Here, a neat segregation of the two species can be observed.

For large density gradients arising, for instance, when $d_i \gg 1, a_i \ll 1$, the numerical scheme fails to converge to a mass conserving solution. In these situations, numerical schemes for convection-dominated problems have to be employed.

In view of the numerical results the following conclusions can be drawn: Either large diffusion coefficients or small cross-diffusion terms, with equal transport coefficients, have similar effects on the final stationary density distributions and no segregation occurs. Segregation effects are outstanding with ratios of $d_1/d_2 > 10$ and $a_1/a_2 > 10$.

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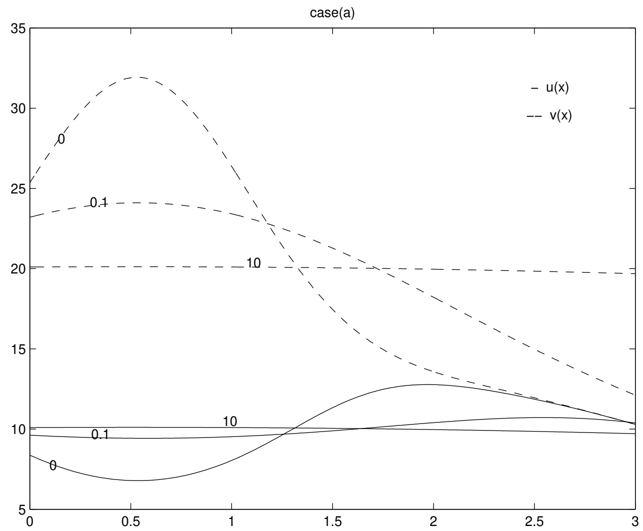

Figure 1: Stationary density distributions $u(x), v(x)$ corresponding to case (a). Different curves are labeled with the corresponding a_i values.

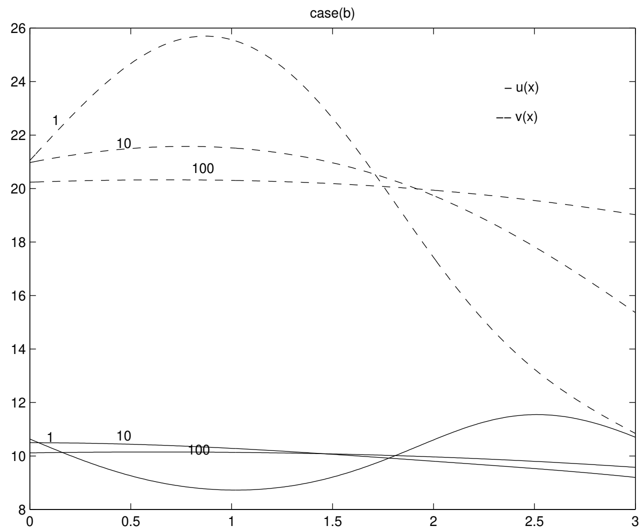

Figure 2: Stationary density distributions $u(x), v(x)$ corresponding to case (b). Different curves are labeled with the corresponding c_i values.

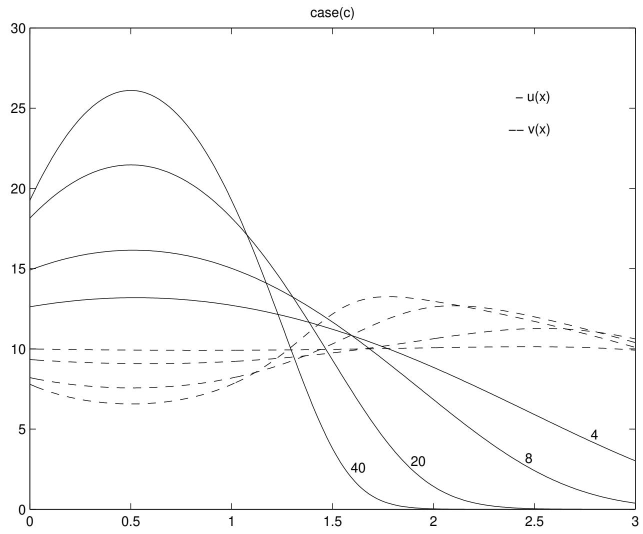

Figure 3: Stationary density distributions $u(x), v(x)$ corresponding to case (c), $a_i = 1$. Different curves are labeled with the corresponding d_1 values.

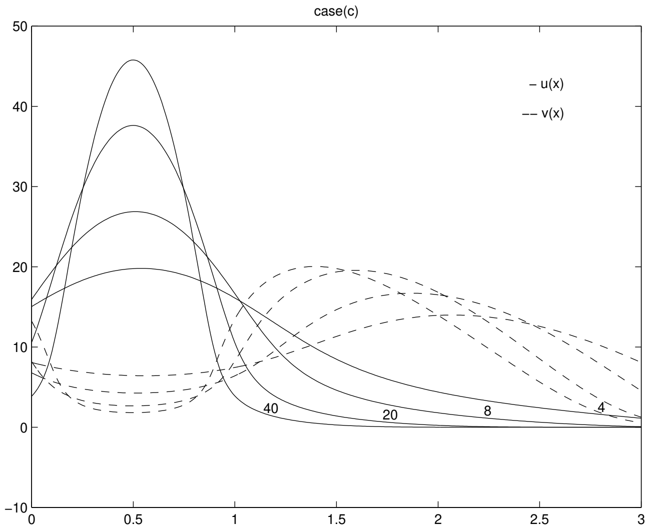

Figure 4: Stationary density distributions $u(x), v(x)$ corresponding to case (c), $a_i = 0.1$. Different curves are labeled with the corresponding d_1 values.

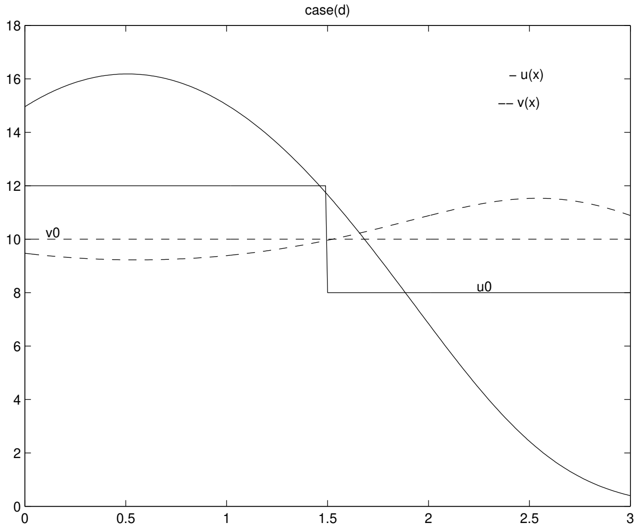

Figure 5: Stationary density distributions $u(x), v(x)$ corresponding to case (d).

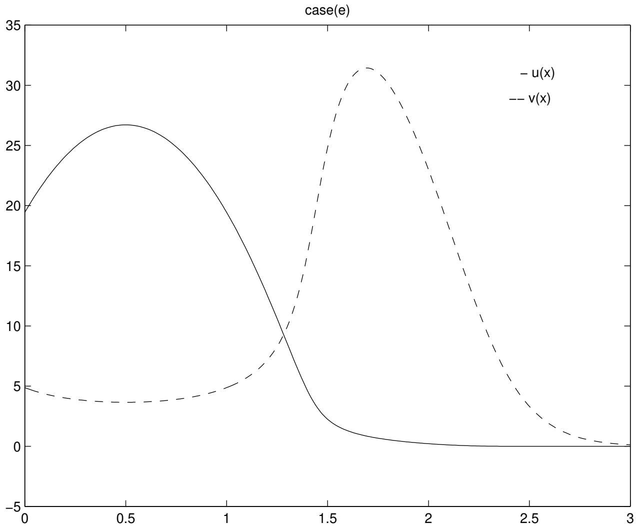

Figure 6: Stationary density distributions $u(x), v(x)$ corresponding to case (e).