Quantum Euler-Poisson Systems: Existence of Stationary States

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Abstract

A one-dimensional quantum Euler-Poisson system for semiconductors for the electron density and the electrostatic potential in bounded intervals is considered. The existence and uniqueness of strong solutions with positive electron density is shown for quite general (possibly non-convex or non-monotone) pressure-density functions under a "subsonic" condition, i.e. assuming sufficiently small current densities. The proof is based on a reformulation of the dispersive third-order equation for the electron density as a nonlinear elliptic fourth-order equation using an exponential transformation of variables.

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1 Introduction

In 1927, Madelung gave a fluiddynamical description of quantum systems governed by the Schrödinger equation for the wave function ψ :

$$
\begin{aligned}
i\varepsilon\partial_t\psi &= -\frac{\varepsilon^2}{2}\Delta\psi - \phi\psi \quad \text{in } \mathbb{R}^d \times (0,T), \\
\psi(\cdot,0) &= \psi_0 \quad \text{in } \mathbb{R}^d,
\end{aligned}
$$

where $T > 0$, $d \geq 1$, $\varepsilon > 0$ is the scaled Planck constant, and $\phi = \phi(x,t)$ is some (given) potential. Separating the amplitude and phase of $\psi = |\psi|\exp(iS/\varepsilon)$, the particle

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density $\rho = |\psi|^2$ and the particle current density $j = \rho\nabla S$ for irrotational flows satisfy the so-called *Madelung equations* [23]

$$\partial_t \rho + \mathrm{div} j = 0, \tag{1.1}$$

$$\partial_t j + \mathrm{div}\left(\frac{j \otimes j}{\rho}\right) - \rho\nabla\phi - \frac{\varepsilon^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = 0 \quad \text{in } \mathbb{R}^d \times (0,T), \tag{1.2}$$

where the i-th component of the convective term $\mathrm{div}(j \otimes j/\rho)$ equals

$$\sum_{k=1}^{d} \frac{\partial}{\partial x_k}\left(\frac{j_i j_k}{\rho}\right).$$

The equations (1.1)-(1.2) can be interpreted as the pressureless Euler equations including the quantum Bohm potential

$$\frac{\varepsilon^2}{2}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}.$$

They have been used for the modeling of superfluids like Helium II [20, 22].

Recently, Madelung-type equations have been derived for the modeling of quantum semiconductor devices, like resonant tunneling diodes, starting from the Wigner-Boltzmann equation [8] or from a mixed-state Schrödinger-Poisson system [11, 12]. There are several advantages of the fluiddynamical description of quantum semiconductors. First, kinetic equations, like the Wigner equation, or Schrödinger systems are computationally very expensive, whereas for Euler-type equations efficient numerical algorithms are available [7, 27]. Second, the macroscopic description allows for a coupling of classical and quantum models. Indeed, setting the Planck constant ε in (1.2) equal to zero, we obtain the classical pressureless equations, so in both pictures, the same (macroscopic) variables can be used. Finally, as semiconductor devices are modeled in bounded domains, it is easier to find physically relevant boundary conditions for the macroscopic variables than for the Wigner function or for the wave function.

The Madelung-type equations derived by Gardner [8] and Gasser et al. [11] also include a pressure term and a momentum relaxation term taking into account interactions of the electrons in the semiconductor crystal, and are self-consistently coupled to the Poisson equation for the electrostatic potential ϕ:

$$\partial_t \rho + \mathrm{div} j = 0, \tag{1.3}$$

$$\partial_t j + \mathrm{div}\left(\frac{j \otimes j}{\rho}\right) + \nabla p(\rho) - \rho\nabla\phi - \frac{\varepsilon^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = -\frac{j}{\tau}, \tag{1.4}$$

$$\lambda^2 \Delta\phi = \rho - \mathcal{C}(x) \qquad \text{in } \Omega \times (0,T), \tag{1.5}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain, $\tau > 0$ is the (scaled) momentum relaxation time constant, $\lambda > 0$ the (scaled) Debye length, and $\mathcal{C}(x)$ is the doping concentration modeling the semiconductor device under consideration [15, 17, 25]. The pressure is assumed to depend only on the particle density and, like in classical fluid dynamics, often the expression

$$p(\rho) = \frac{T}{\gamma}\rho^\gamma, \quad \rho \geq 0, \tag{1.6}$$

with the temperature constant $T > 0$ is employed [8, 16]. *Isothermal* fluids correspond to $\gamma = 1$, *isentropic* fluids to $\gamma > 1$. Notice that the particle temperature is $T(\rho) = T\rho^{\gamma-1}$.

The equations (1.3)-(1.5) are referred to as the *quantum Euler-Poisson system* or as the quantum hydrodynamic model.

In this paper we study the stationary system (1.3)-(1.5) in one space dimension with $\lambda = 1$:

$$
\begin{aligned}
j_0 &= \text{const.}, && (1.7)\\
\left(\frac{j_0^2}{\rho} + p(\rho)\right)_x - \rho\phi_x - \frac{\varepsilon^2}{2}\rho\left(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\right)_x &= -\frac{j_0}{\tau}, && (1.8)\\
\phi_{xx} &= \rho - \mathcal{C}(x) \quad \text{in } \Omega = (0,1) && (1.9)
\end{aligned}
$$

subject to the boundary conditions

$$\rho(0) = \rho_1, \quad \rho(1) = \rho_2, \quad \rho_x(0) = \rho_x(1) = 0, \tag{1.10}$$

$$\phi(0) = 0, \quad \phi(1) = \Phi_0, \tag{1.11}$$

where ρ_1, $\rho_2 > 0$ and $\Phi_0 \in \mathbb{R}$. In this formulation, the electron current density is a given constant. From the equations, the applied voltage U can be computed by $U = \phi(1) - \phi(0)$.

The existence and uniqueness of (weak) solutions to the quantum Euler-Poisson system for $j_0 = 0$ (thermal equilibrium) has been studied in [1, 9]. The equations for $j_0 > 0$ have been considered in [6, 16, 28] for general monotone pressure functions, however, with different boundary conditions, assuming Dirichlet data for the velocity potential S [16] or employing nonlinear boundary conditions [6, 28]. Existence of solutions to (1.7)–(1.9) subject to the boundary conditions (1.10)–(1.11) is proven in [13], however, only for the linear pressure function $p(\rho) = \rho$. In this paper, we generalize the results of [13] for *general* pressure functions $p(\rho)$, assuming some conditions which are specified below. We stress the fact that we also allow for *non-convex* or *non-monotone* pressure-density relations.

We mention some related results on the stationary quantum Euler-Poisson system. The semi-classical limit $\varepsilon \to 0$ in the case of thermal equilibrium $j_0 = 0$ and in the case

$j_0 > 0$ has been studied in [9, 26] and [13], respectively (also see [10]). For results on the limit problem $\varepsilon = 0$ (Euler-Poisson system) we refer to the review paper [21]. The local existence of strong solutions to the *transient* quantum Euler-Poisson model has been shown in [18]. The global existence of "small" solutions to the transient model and its asymptotic behavior for large times will be studied in our forthcoming work [19] based on the results of this paper for the steady state.

In all the cited papers, the existence of (strong) steady-state solutions to the quantum hydrodynamic equations is shown for sufficiently small current densities $j_0 > 0$. In fact, in the case of the nonlinear boundary conditions assumed in [6], the *non-existence* of weak solutions to the quantum Euler-Poisson system for sufficiently large $j_0 > 0$ has been proved. We also need the smallness condition on $|j_0|$ to prove the existence of solutions to (1.7)–(1.11).

In order to explain our main results in detail, we rewrite the equation for the electron density (1.8) as a nonlinear elliptic *fourth-order* equation. Writing

$$\frac{\varepsilon^2}{2}\rho\left(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\right)_x = \frac{\varepsilon^2}{4}(\rho(\ln\rho)_{xx})_x,$$

dividing (1.8) by $\rho > 0$, differentiating with respect to x and using (1.7) and (1.9) to remove the electrostatic potential from the equation, we obtain

$$\left(\frac{p'(\rho)\rho_x}{\rho} - \frac{j_0^2}{\rho^3}\rho_x\right)_x - (\rho - \mathcal{C}(x)) - \frac{\varepsilon^2}{4}(\rho^{-1}(\rho(\ln\rho)_{xx})_x)_x = -\left(\frac{j_0}{\tau\rho}\right)_x.$$

It is convenient to introduce the new variable $u = \ln\rho$. Then the above equation can be written as

$$\frac{\varepsilon^2}{4}\left(u_{xx} + \frac{1}{2}u_x^2\right)_{xx} - ((p'(e^u) - j_0^2 e^{-2u})u_x)_x + e^u - \mathcal{C}(x) = \frac{j_0}{\tau}(e^{-u})_x. \tag{1.12}$$

The boundary conditions (1.10) transform to

$$u(0) = u_1, \quad u(1) = u_2, \quad u_x(0) = u_x(1) = 0, \tag{1.13}$$

where $u_1 = \ln\rho_1$, $u_2 = \ln\rho_2$.

The electrostatic potential can be computed from the formulae

$$\phi(x) = \Phi_0 + \int_0^1 G(x,y)(e^{u(y)} - \mathcal{C}(y))dy, \tag{1.14}$$

where the Green's function $G(x,y)$ is defined by

$$G(x,y) = \begin{cases} x(1-y), & x < y, \\ y(1-x), & x > y. \end{cases}$$

The advantage of the above formulation is that bounded solutions $u \in L^\infty(0,1)$ define *positive* densities $\rho = e^u$ and in this case, both formulations (1.8)–(1.9) and (1.12)–(1.14) are equivalent. Notice that for third-order or fourth-order equations, no maximum principle is available such that other methods for proving the positivity of the variables have to be devised. Here we use the exponential transformation of variables combined with Sobolev embeddings as in [6, 13].

Assume that

$$\varepsilon, \tau, \rho_1, \rho_2 > 0, \ \Phi_0, j_0 \in \mathbb{R}, \ \mathcal{C} \in L^2(0,1). \tag{1.15}$$

Then our main results are as follows:

1. Suppose that the pressure function is given by (1.6) for $\gamma > 0$. Then there exist constants J_0, $\gamma_1 > 0$ such that if $|j_0| \leq J_0$ and $|\gamma - 1| \leq \gamma_1$, there exists a unique strong solution $u, \phi \in H^4(0,1)$ to (1.12)-(1.14). Since $u \in L^\infty(0,1)$, we have $\rho = e^u > 0$ in $(0,1)$, and $\rho, \phi \in H^4(0,1)$ is a solution of (1.7)-(1.11). The constant j_1 can be given explicitly (see section 2).

2. Suppose that $p \in C^3(0,\infty)$, that there exists a function $\mathcal{A} \in H^2(0,1)$ such that
$$\mathcal{A} > 0 \text{ in } (0,1), \quad \mathcal{A}(0) = \rho_1, \quad \mathcal{A}(1) = \rho_2, \quad \mathcal{A}_x(0) = \mathcal{A}_x(1) = 0$$
and that there is a set $E \subset [0,1]$ such that
$$p'(\mathcal{A}) - \frac{j_0^2}{\mathcal{A}^2} \begin{cases} \leq 0, & x \in E, \\ > 0, & x \in [0,1] \backslash E. \end{cases} \tag{1.16}$$
Then if $|j_0|$ is small enough, there exists a unique strong solution $u, \phi \in H^4(0,1)$ to (1.12)-(1.14).

Notice that we allow for *non-convex* pressure functions (1.6) with $\gamma < 1$ and for *non-monotone* pressures satisfying (1.16). This means that the left part of (1.2) may be not hyperbolic. The assumption (1.16) implies that the interval under consideration may consist of subsonic, transonic and supersonic regions in the classical sense [3]. To guarantee the well-posedness of strong solutions, we assume a "subsonic" condition.

Finally, we notice that our estimates allow to perform the semi-classical limit $\varepsilon \to 0$ in (1.12)-(1.14) by employing the same techniques as in [13] (also see [5] and Remark 3.4).

The paper is organized as follows. In section 2 our first main result is formulated and proved. The second main result is shown in section 3.

Notation. The Lebesgue space of square integrable functions with the norm $\|\cdot\|$ is denoted by $L^2(0,1)$, and $H^k(0,1)$ or simply H^k denotes the usual Sobolev space of functions f satisfying $\partial_x^i f \in L^2(0,1)$, $0 \leq i \leq k$, with the norm $\|\cdot\|_k$. In particular, $\|\cdot\|_0 = \|\cdot\|$.

2 Pressure functions satisfying (1.6)

In this section, we consider the steady-state solutions to the BVP (1.7)–(1.11) when the pressure-density relation satisfies the γ-law (1.6).

Theorem 2.1 *Assume that* (1.6) *and* (1.15) *hold. Let* $\kappa \in (0,1)$. *Then there exist two constants* $\gamma_0 > 0$ *and* $K(\kappa) > 0$ *such that if*

$$|j_0| \le e^{-K(\kappa)}\kappa\sqrt{Te^{-|\gamma-1|K(\kappa)} + \frac{1}{2}\varepsilon^2} \quad \text{and} \quad |\gamma - 1| \le \gamma_0, \tag{2.1}$$

then there exists a solution $u \in H^4$ *to the BVP* (1.12)–(1.13) *satisfying*

$$\frac{1}{2}\varepsilon\|u_{xx}\| + \sqrt{Te^{-|\gamma-1|K(\kappa)} + \frac{1}{2}\varepsilon^2}\,\|u_x\| \le K_0, \tag{2.2}$$

$$|u(x)| \le K(\kappa), \tag{2.3}$$

where K_0 *is defined by* (2.21) *and* $(\gamma, K(\kappa))$ *is the unique solution to* (2.22).

Furthermore, there are J_0, ε_0, $\gamma_1 > 0$ *such that if* $|j_0| \le J_0$ *and* $|\gamma - 1| \le \gamma_1$, *the solution* u *is unique for any* $\varepsilon \in (0, \varepsilon_0]$.

Proof: *Step 1. A-priori estimates.*

Assume that $u \in H^2$ is a weak solution of the boundary-value problem (BVP) (1.12)–(1.13) satisfying a-priorily that

$$-K(\kappa) \le u \le K(\kappa). \tag{2.4}$$

Following [1] we introduce a function $u_D \in C^2([0,1])$ satisfying

$$u_D(0) = u_1,\ u_D(1) = u_2,\ u_{D,x}(0) = u_{D,x}(1) = 0, \tag{2.5}$$

with piecewise linear second order derivative

$$u_{D,xx}(x) = \begin{cases} \frac{4\zeta}{\mu^2(1-\mu)}x, & x \in [0, \frac{\mu}{2}), \\ \frac{4\zeta}{\mu^2(1-\mu)}(\mu - x), & x \in [\frac{\mu}{2}, \mu], \\ 0, & x \in (\mu, \frac{1}{2}], \end{cases} \tag{2.6}$$

and $u_{D,xx}(x) = -u_{D,xx}(1-x)$ for $x \in (\frac{1}{2}, 1]$, where $\zeta = |u_1 - u_2|$ and $\mu \in (0, \frac{1}{2})$. Elementary computations show that

$$\int_0^{1/2} x|u_{D,xx}(x)|dx + \int_{1/2}^1 (1-x)|u_{D,xx}(x)|dx = \frac{\mu\zeta}{1-\mu}, \tag{2.7}$$

$$\int_0^1 |u_{D,xx}(x)|^2 dx = \frac{8\zeta^2}{3\mu(1-\mu)}, \tag{2.8}$$

$$\int_0^1 |u_{D,x}(x)| dx = \zeta, \tag{2.9}$$

$$\int_0^1 |u_{D,x}(x)|^2 dx = \frac{\zeta^2(23\mu+1)}{30(1-\mu)^2} \le \frac{1}{2}\zeta^2. \tag{2.10}$$

Use $u - u_D \in H_0^2$ as an admissible test function in the weak formulation of (1.12) to obtain

$$\begin{aligned}
&\frac{1}{4}\varepsilon^2 \int_0^1 (u_{xx} + \frac{1}{2}u_x^2)(u-u_D)_{xx} dx \\
&+ \int_0^1 (Te^{(\gamma-1)u} - j_0^2 e^{-2u}) u_x (u-u_D)_x dx \\
&+ \int_0^1 (e^u - e^{u_D})(u-u_D) dx - \int_0^1 (e^{u_D} - \mathcal{C})(u-u_D) dx \\
= &-\frac{j}{\tau}\int_0^1 e^{-u}(u-u_D)_x dx,
\end{aligned} \tag{2.11}$$

By Cauchy's inequality, (2.4) and (2.11), it follows for $\eta \in (0,1)$

$$\begin{aligned}
&\frac{1}{4}\varepsilon^2 \left(1 - \frac{\eta}{2}\right) \int_0^1 u_{xx}^2 dx + \frac{1}{8}\varepsilon^2 \int_0^1 u_x^2 u_{xx} dx \\
&+ \int_0^1 \left[\left(1-\frac{\eta}{2}\right) Te^{-|\gamma-1|K(\kappa)} - (1+\eta) j_0^2 e^{2K(\kappa)}\right] u_x^2 dx \\
&+ \int_0^1 (e^u - e^{u_D})(u-u_D) dx \\
\le &\frac{1}{8}\frac{\varepsilon^2}{\eta}\int_0^1 |u_{D,xx}|^2 dx + \frac{1}{8}\varepsilon^2 \int_0^1 u_x^2 u_{D,xx} dx + \frac{j_0^2}{4\eta}\int_0^1 e^{2K(\kappa)}|u_{D,x}|^2 dx \\
&+ \frac{1}{2\eta} Te^{2|\gamma-1|K(\kappa)} \int_0^1 |u_{D,x}|^2 dx + \int_0^1 (e^{u_D} - \mathcal{C})(u-u_D) dx \\
&- \frac{j_0}{\tau}\int_0^1 e^{-u}(u-u_D)_x dx,
\end{aligned} \tag{2.12}$$

Using the boundary condition (1.13), and applying (2.5) and (2.7)–(2.8), we have as the proof of Lemma 2.1 in [13]:

$$\begin{gathered}
\int_0^1 u_x^2 u_{xx} dx = 0, \quad \frac{1}{8}\frac{\varepsilon^2}{\eta}\int_0^1 |u_{D,xx}|^2 dx \le \frac{\zeta^2\varepsilon^2}{3\mu\eta}, \\
\frac{1}{8}\varepsilon^2 \int_0^1 u_x^2 u_{D,xx} dx \le \frac{\varepsilon^2\eta}{16}\|u_{xx}\|^2,
\end{gathered} \tag{2.13}$$

where we have used the Poincaré inequality and chosen $\mu = \min(1/2, \eta/2\zeta)$. From (2.5) and (2.10) follows

$$\frac{j_0^2}{4\eta}\int_0^1 e^{2K(\kappa)}|u_{D,x}|^2dx \le \frac{\zeta^2}{8\eta}j_0^2e^{2K(\kappa)} \le \frac{\zeta^2}{8\eta}\left(Te^{-|\gamma-1|K(\kappa)}+\frac{1}{2}\varepsilon^2\right) \tag{2.14}$$

and

$$\frac{1}{2\eta}Te^{2|\gamma-1|K(\kappa)}\int_0^1 |u_{D,x}|^2dx \le \frac{\zeta^2}{4\eta}Te^{2|\gamma-1|K(\kappa)}. \tag{2.15}$$

Since

$$\|u-u_D\| \le \frac{1}{2}||(u-u_D)_x||,$$

we have by Cauchy's inequality

$$\begin{aligned}
&\int_0^1 (e^{u_D}-\mathcal{C})(u-u_D)dx \\
\le&\frac{\eta}{8}\left(Te^{-|\gamma-1|K(\kappa)}+\frac{1}{2}\varepsilon^2\right)||(u-u_D)_x||^2 \\
&+\frac{1}{2\eta}\left(Te^{-|\gamma-1|K(\kappa)}+\frac{1}{2}\varepsilon^2\right)^{-1}||e^{u_D}-\mathcal{C}||^2 \\
\le&\frac{\eta}{8}\left(Te^{-|\gamma-1|K(\kappa)}+\frac{1}{2}\varepsilon^2\right)||u_x||^2+\frac{\eta\zeta^2}{16}\left(Te^{-|\gamma-1|K(\kappa)}+\frac{1}{2}\varepsilon^2\right) \\
&+\frac{1}{2\eta}\left(Te^{-|\gamma-1|K(\kappa)}+\frac{1}{2}\varepsilon^2\right)^{-1}||e^{u_D}-\mathcal{C}||^2.
\end{aligned} \tag{2.16}$$

Using (2.1) and (2.9), the last term in (2.12) can be estimated as

$$\begin{aligned}
&-\frac{j_0}{\tau}\int_0^1 e^{-u}(u-u_D)_xdx \\
\le&\frac{|j_0|}{\tau}e^{K(\kappa)}\int_0^1(|u_x|+|u_{D,x}|)dx \\
\le&\frac{\eta}{4}\left(Te^{-|\gamma-1|K(\kappa)}+\frac{1}{2}\varepsilon^2\right)||u_x||^2+\frac{1}{\eta\tau^2}+\frac{\zeta\kappa}{\tau}\sqrt{Te^{-|\gamma-1|K(\kappa)}+\frac{1}{2}\varepsilon^2}.
\end{aligned} \tag{2.17}$$

Setting $\eta = (1-\kappa)/[2(1+\kappa)]$ in (2.13)–(2.17) and substituting η into (2.12), we have, estimating as in [13],

$$\frac{1}{4}\varepsilon^2||u_{xx}||^2+\left(Te^{-|\gamma-1|K(\kappa)}+\frac{1}{2}\varepsilon^2\right)||u_x||^2 \le a_c^2\frac{(1+\kappa)^2}{(1-\kappa)^2}K_1, \tag{2.18}$$

where K_1 is given by

$$K_1 = 1+Te^{-|\gamma-1|K(\kappa)}+\frac{1}{2}\varepsilon^2+Te^{2|\gamma-1|K(\kappa)}+T^{-1}e^{|\gamma-1|K(\kappa)}, \tag{2.19}$$

and a_c a generic positive constant which only depends on ζ, T, and τ. From (2.18) follows

$$\frac{1}{2}\varepsilon||u_{xx}|| + \sqrt{Te^{-|\gamma-1|K(\kappa)} + \frac{1}{2}\varepsilon^2}||u_x|| \le K_0, \tag{2.20}$$

where

$$K_0 = a_c\frac{1+\kappa}{1-\kappa}\sqrt{K_1}. \tag{2.21}$$

Now, consider the equation for $(\gamma, K(\kappa))$

$$K(\kappa) - a_c\frac{1+\kappa}{1-\kappa}\sqrt{K_2(K(\kappa),\gamma)} - |u_1| = 0, \tag{2.22}$$

where

$$K_2(K(\kappa),\gamma) = 1 + T^{-1}e^{|\gamma-1|K(\kappa)} + e^{3|\gamma-1|K(\kappa)} + T^{-2}e^{2|\gamma-1|K(\kappa)}.$$

It has a solution

$$(\gamma, K(\kappa)) = (1, |u_1| + a_c\frac{1+\kappa}{1-\kappa}\sqrt{2+T^{-1}+T^{-2}}\ ).$$

By the implicit function theorem, there exists a $\gamma_0 > 0$ such that for $|\gamma - 1| < \gamma_0$, the equation (2.22) has a solution $(\gamma, K(\kappa))$.

Therefore, in view of (2.19)–(2.22), we obtain

$$\begin{aligned}
|u(x)| &\le |u_1| + ||u_x|| \\
&= |u_1| + \frac{K_0}{\sqrt{Te^{-|\gamma-1|K(\kappa)} + \frac{1}{2}\varepsilon^2}} \\
&= |u_1| + a_c\frac{1+\kappa}{1-\kappa}\sqrt{\frac{1 + Te^{-|\gamma-1|K} + \frac{1}{2}\varepsilon^2 + Te^{2|\gamma-1|K(\kappa)} + T^{-1}e^{|\gamma-1|K(\kappa)}}{Te^{-|\gamma-1|K(\kappa)} + \frac{1}{2}\varepsilon^2}} \\
&\le |u_1| + a_c\frac{1+\kappa}{1-\kappa}\sqrt{K_2(K(\kappa),\gamma)} \\
&= K(\kappa).
\end{aligned} \tag{2.23}$$

Step 2. Existence.

We apply the Leray-Schauder fixed point theorem to prove the existence of strong solutions. Let $v \in X := C^{0,1}([0,1])$. Consider the linear BVP

$$\begin{aligned}
\frac{1}{4}\varepsilon^2\left(u_{xx} + \frac{1}{2}\lambda v_x^2\right)_{xx} &- \left(Te^{(\gamma-1)v_K}u_x - \lambda j_0^2 e^{2v_K}u_x\right)_x \\
+ \lambda\left(\frac{e^v-1}{v}u + 1 - \mathcal{C}\right) &= \lambda\frac{j_0}{\tau}(e^{-v_K})_x, \\
u(0) = \lambda u_1,\ u(1) = \lambda u_2,\ &u_x(0) = u_x(1) = 0,
\end{aligned}$$

where $v_K = \text{mim}\{K(\kappa), \max\{-K(\kappa), v\}\}$ and $\lambda \in [0,1]$. Define the bilinear form

$$a(u,\psi) = \int_0^1 \Big(\frac{1}{4}\varepsilon^2 u_{xx}\psi_{xx} + Te^{(\gamma-1)v_K}u_x\psi_x + \lambda\frac{e^v - 1}{v}u\psi\Big)dx,$$

for $u, \psi \in H^2$ and the functional

$$F(\psi) = \int_0^1 \left(-\frac{1}{8}\varepsilon^2\lambda v_x^2\psi_{xx} + \lambda j_0^2 e^{-2v_K}v_x\psi_x + \lambda(\mathcal{C}-1)\psi - \lambda\frac{j_0}{\tau}e^{-v_K}\psi_x\right)dx.$$

Since $X \hookrightarrow W^{1,\infty}$, $a(.,.)$ is continuous and coercive in H^2, and F is linear and continuous in H^2, the Lax-Milgram theorem yields the existence of a solution $u \in H^2$. This means that the map $\mathcal{S} : X \times [0,1] \to X$, $(v,\lambda) \mapsto u$ is well defined. Moreover, it is not difficult to see that $\mathcal{S}$ is continuous and compact. Since $\mathcal{S}(v,0) = 0$ for all $v \in H^2$, and by similar estimates as in Step 1 with e^u replaced by e^{v_K}, we can verify that it holds for all $\lambda \in [0,1]$,

$$||u||_X \le c_1, \tag{2.24}$$

where $c_1 > 0$ is a constant independent of u and λ. Then the existence of $u \in H^2$ follows from the Leray-Schauder fixed point theorem. It is not difficult to prove that indeed $u \in H^4$ (see [13] for details).

Step 3. Uniqueness.

Let $u, v \in H_0^2$ be two weak solutions of the BVP (1.12)–(1.13), which satisfy (2.20). Using $u - v \in H_0^2$ as an admissible test function in the weak formulation derived for $u-v$, we obtain

$$\begin{aligned}
&\frac{1}{4}\varepsilon^2\int_0^1 (u-v)_{xx}^2 dx + \frac{1}{8}\varepsilon^2\int_0^1 (u_x+v_x)(u-v)_x(u-v)_{xx}dx \\
&\quad + Te^{(\gamma-1)u}\int_0^1 (u-v)_x^2 dx \\
&= -\int_0^1 \left(Te^{(\gamma-1)u} - Te^{(\gamma-1)v}\right)v_x(u-v)_x dx \\
&\quad + \frac{1}{2}\int_0^1 (j_0^2e^{-2u} - j_0^2e^{-2v})(u-v)_{xx}dx \\
&\quad - \int_0^1 (e^u - e^v)(u-v)dx - \frac{j_0}{\tau}\int_0^1 (e^{-u} - e^{-v})(u-v)_x dx \\
&\le \int_0^1 Te^{|\gamma-1|K(\kappa)}\left(e^{(\gamma-1)u} - e^{(\gamma-1)v}\right)^2 v_x^2 dx \\
&\quad + \frac{1}{8}\varepsilon^2\int_0^1 (u-v)_{xx}^2 dx + \frac{1}{2\varepsilon^2}j_0^4\int_0^1 (e^{-2u} - e^{-2v})^2 dx
\end{aligned}$$

$$- e^{-K(\kappa)} \int_0^1 (u-v)^2 dx + \frac{1}{2} T e^{-|\gamma-1|K(\kappa)} \int_0^1 (u-v)_x^2 dx$$
$$+ \frac{j_0^2}{\tau^2 T} e^{|\gamma-1|K(\kappa)} \int_0^1 (e^{-u} - e^{-v})^2 dx,$$

which implies

$$\begin{aligned}
&\frac{1}{8}\varepsilon^2 \int_0^1 (u-v)_{xx}^2 dx + \frac{1}{8}\varepsilon^2 \int_0^1 (u_x+v_x)(u-v)_x(u-v)_{xx} dx \\
&+ \frac{1}{2} T e^{-|\gamma-1|K} \int_0^1 (u-v)_x^2 dx \\
\le& \left(\frac{1}{2\varepsilon^2} j_0^4 e^{4K(\kappa)} + \frac{j_0^2}{\tau^2 T} e^{(|\gamma-1|+2)K} + T(\gamma-1)^2 e^{3|\gamma-1|K(\kappa)} |v_x|_{L^\infty}^2 \right) \int_0^1 (u-v)^2 dx \\
&- e^{-K(\kappa)} \int_0^1 (u-v)^2 dx.
\end{aligned} \tag{2.25}$$

From (1.13) and (2.20), we have by Hölder's inequality

$$\begin{aligned}
|u_x(x)| \le& \sqrt{2||u_{xx}|| \cdot ||u_x||} \\
\le& \frac{1}{2}\theta ||u_{xx}|| + \theta^{-1} ||u_x|| \\
\le& \max\left\{ \frac{\theta}{\varepsilon}, \frac{1}{\theta T^{\frac{1}{2}} e^{-\frac{1}{2}|\gamma-1|K(\kappa)}} \right\} \tilde{K}_0,
\end{aligned}$$

where

$$\tilde{K}_0 = a_c \frac{1+\kappa}{1-\kappa} \sqrt{K_1}.$$

Choose

$$\theta = \frac{T^{\frac{1}{2}} e^{-\frac{1}{2}|\gamma-1|K(\kappa)}}{\tilde{K}_0} \quad \text{and} \quad 0 < \varepsilon \le \varepsilon_0 =: \frac{T^{\frac{3}{2}} e^{-\frac{3}{2}|\gamma-1|K(\kappa)}}{\tilde{K}_0^2}$$

to obtain

$$\frac{\theta}{\varepsilon} \ge \frac{1}{\theta T^{\frac{1}{2}} e^{-\frac{1}{2}|\gamma-1|K(\kappa)}}$$

and

$$|u_x(x)| \le \frac{\theta}{\varepsilon} \tilde{K}_0 \le \frac{1}{\varepsilon} T^{\frac{1}{2}} e^{-\frac{1}{2}|\gamma-1|K(\kappa)}. \tag{2.26}$$

From this estimate and Cauchy's inequality follows that the left-hand side of (2.25) is bounded from below by

$$\frac{1}{8}\varepsilon^2 \int_0^1 (u-v)_{xx}^2 dx + \frac{1}{8}\varepsilon^2 \int_0^1 (u_x+v_x)(u-v)_x(u-v)_{xx} dx$$

$$
\begin{aligned}
&+\frac{1}{2}\int_0^1 Te^{-|\gamma-1|K(\kappa)}(u-v)_x^2dx \\
&\geq \frac{1}{16}\varepsilon^2\int_0^1 (u-v)_{xx}^2dx+\frac{1}{4}Te^{-|\gamma-1|K(\kappa)}\int_0^1 (u-v)_x^2dx.
\end{aligned}
$$

On the other hand, by the implicit function theorem, there is a $\gamma_1 \leq \gamma_0$ such that for $|\gamma-1|<\gamma_1$ it holds

$$
|\gamma-1|^2e^{(2|\gamma-1|+1)K(\kappa)} \leq \frac{1}{8}T^{-2}\varepsilon^2, \tag{2.27}
$$

which implies

$$
\frac{2}{\varepsilon^2}T^2(\gamma-1)^2e^{2|\gamma-1|K(\kappa)} \leq \frac{1}{4}e^{-K(\kappa)}. \tag{2.28}
$$

Thus, there exists J_0 such that if

$$
j_0^2 \leq J_0^2 =: \min\left\{e^{-2K(\kappa)}\kappa\left(Te^{-|\gamma-1|K(\kappa)}+\frac{1}{2}\varepsilon^2\right),\ \frac{\sqrt{2}}{2}\varepsilon e^{-\frac{5}{2}K(\kappa)},\ \frac{1}{4}T\tau^2e^{-(|\gamma-1|+3)K(\kappa)}\right\}, \tag{2.29}
$$

then the first integral on the right-hand side of (2.25) is bounded, in view of (2.26) and (2.28), by

$$
\begin{aligned}
&\left(\frac{1}{2\varepsilon^2}j_0^4e^{4K(\kappa)}+\frac{j_0^2}{\tau^2T}e^{(|\gamma-1|+2)K(\kappa)}+T(\gamma-1)^2e^{3|\gamma-1|K(\kappa)}|v_x|_{L^\infty}^2-e^{-K(\kappa)}\right)\int_0^1 (u-v)^2dx \\
\leq &\left(\frac{1}{2\varepsilon^2}j_0^4e^{4K(\kappa)}+\frac{j_0^2}{\tau^2T}e^{(|\gamma-1|+2)K(\kappa)}+\frac{2}{\varepsilon^2}T^2(\gamma-1)^2e^{2|\gamma-1|K(\kappa)}-e^{-K(\kappa)}\right)\int_0^1 (u-v)^2dx \\
\leq &-\frac{1}{4}e^{-K(\kappa)}\int_0^1 (u-v)^2dx.
\end{aligned} \tag{2.30}
$$

Therefore, the weak solution is unique if both (2.27) and (2.29) holds. The proof of Theorem 2.1 is complete. $\square$

As in [13], we can conclude from Theorem 2.1 the following result.

Theorem 2.2 *Assume that* (1.6) *and* (1.15) *hold. Then there exist two constants* $\gamma_0 > 0$ *and* $K(\kappa) > 0$ ($\kappa \in (0,1)$) *such that if*

$$
|j_0| \leq e^{-K(\kappa)}\kappa\sqrt{Te^{-|\gamma-1|K(\kappa)}+\frac{1}{2}\varepsilon^2} \quad \text{and} \quad |\gamma-1| \leq \gamma_0, \tag{2.31}
$$

then there is a solution $(\rho,\phi) \in H^4 \times H^2$ *to the BVP* (1.7)–(1.11) *such that*

$$
\rho \geq \bar{\rho} =: e^{-K(\kappa)} > 0, \tag{2.32}
$$

where $(\gamma, K(\kappa))$ *solves the equation* (2.22). *Moreover, if* $|j_0|$, ε, *and* $|\gamma-1|$ *are small enough, the solution is unique.*

3 Pressure functions satisfying (1.16)

In this section, we consider the BVP (1.12)-(1.13) (and (1.7)–(1.11)) with pressure functions satisfying the condition (1.16).

Set $u_D := \ln \mathcal{A}$. Then $e^{u_D} = \mathcal{A}$. We have the following theorem.

Theorem 3.1 *Assume that* (1.15) *and* (1.16) *holds and that* $p \in C^3(0,\infty)$*. For* $\kappa \in (0,1)$ *assume that it holds*

$$\min_{x\in[0,1]} \mathcal{A}(x)^2 \left(\frac{1}{4}\kappa\varepsilon^2 + p'(\mathcal{A}(x))\right) > j_0^2. \tag{3.1}$$

Then the BVP (1.12)–(1.13) *has a unique solution* $u \in H^4$ *provided that* $||\mathcal{A}'||_1 + ||\mathcal{A} - \mathcal{C}||$ *is sufficiently small. Moreover, it holds*

$$\begin{aligned} \mathcal{A}_*||u-u_D||^2 + A_0||u_{xx}||^2 + \varepsilon^4||(u_{xxx}, u_{xxxx})||^2 \\ + \int_{I\setminus E}\left(p'(\mathcal{A}^2) - \frac{j_0^2}{\mathcal{A}^4}\right)(u-u_D)_x^2 dx \le K_c\delta_0, \end{aligned} \tag{3.2}$$

where

$$\mathcal{A}_* = \min_{x\in[0,1]} \mathcal{A}(x), \quad \delta_0 = ||\mathcal{A}'||_1 + ||\mathcal{A} - \mathcal{C}||, \tag{3.3}$$

$$A_0 = \min_{x\in[0,1]} \left(\frac{1}{4}\kappa\varepsilon^2 + p'(\mathcal{A}) - j_0^2\mathcal{A}^{-2}\right) > 0, \tag{3.4}$$

and $K_c > 0$ *is a constant depending on* $\mathcal{A}$*,* τ *and* j_0*.*

Remark 3.2 (1) We call the main assumption (3.1) a "subsonic" condition for the quantum Euler-Poisson system. When $\varepsilon = 0$, the assumption (3.1) is exactly the subsonic condition for the classical hydrodynamic model [4].

(2) One can verify that the assumption (3.1) can be replaced by

$$\frac{1}{4}\kappa\varepsilon^2 + \text{meas}(E)\min_{x\in E}(p'(\mathcal{A}) - j_0^2\mathcal{A}^{-2}) > 0, \quad \kappa \in (0,1), \tag{3.5}$$

in order to obtain the existence and uniqueness of strong solutions. Here, we recall that the region $E \subset [0,1]$ is defined such that it holds $p'(\mathcal{A})\mathcal{A}^2 - j_0^2 < 0$ in E.

Proof: We prove Theorem 3.1 by the same steps as Theorem 2.1.

Step 1. The a-priori estimates.

Let $u \in H^4$ be a solution of the BVP (1.12)–(1.13) satisfying

$$u_D - \delta_1 \le u \le u_D + \delta_1, \tag{3.6}$$

where $\delta_1 > 0$ is chosen such that

$$\frac{4}{5}\mathcal{A}_* \le e^{-\delta_1} e^{u_D} \le e^u \le e^{\delta_1} e^{u_D} \le \frac{5}{4}\mathcal{A}^*, \tag{3.7}$$

$$\max_{x\in[0,1]} (|p''(\mathcal{A})|\mathcal{A} + 2j_0^2\mathcal{A}^{-2})\delta_1 \le \frac{2}{9}(1+\theta)A_0, \tag{3.8}$$

$$\max_{\ln\frac{16}{25}\mathcal{A}_* \le y \le \ln\frac{25}{16}\mathcal{A}^*} (|p'''(e^y)|e^{2y} + |p''(e^y)|e^y + 4j_0^2 e^{-2y})\delta_1^2 \le \frac{4}{9}(1+\theta)A_0. \tag{3.9}$$

where $\mathcal{A}^* = \max_{x\in[0,1]}\mathcal{A}(x)$, $\mathcal{A}_* = \min_{x\in[0,1]}\mathcal{A}(x)$, and $\theta = \frac{1-\kappa}{1+\kappa}$ (then $\kappa = \frac{1-\theta}{1+\theta}$).

Assume that $\delta_0 = ||\mathcal{A}-\mathcal{C}|| + ||(\mathcal{A}_x, \mathcal{A}_{xx})||$ is so small that it holds

$$|u_{D,x}|_\infty + |u_{D,xx}|_\infty \le \theta, \tag{3.10}$$

where $|\cdot|_\infty$ denotes the L^∞ norm.

Taking $(u-u_D) \in H_0^2$ as an admissible test function in the weak formulation of (1.12), we have, by Cauchy's inequality and (3.7), that

$$\begin{aligned}
&\frac{1}{4}\left(1-\frac{1}{2}\theta-\frac{1}{2}|u_{D,xx}|_\infty\right)\varepsilon^2\int_0^1 u_{xx}^2 dx \\
&+\int_0^1 (p'(e^u)-j_0^2e^{-2u})u_x(u-u_D)_x dx + \frac{1}{2}\mathcal{A}_*\int_0^1 (u-u_D)^2 dx \\
\le&\frac{1}{\mathcal{A}_*}\int_0^1(\mathcal{A}-\mathcal{C})^2dx + \frac{1}{8\theta}\varepsilon^2\int_0^1 |u_{D,xx}|^2 dx - \frac{j_0}{\tau}\int_0^1 e^{-u}(u-u_D)_x dx,
\end{aligned} \tag{3.11}$$

where we have used the facts that

$$\int_0^1 u_x^2 u_{xx} dx = 0, \quad \int_0^1 u_x^2 dx \le \int_0^1 u_{xx}^2 dx. \tag{3.12}$$

The last term in (3.11) can be estimated as

$$\begin{aligned}
-\frac{j_0}{\tau}\int_0^1 e^{-u}(u-u_D)_x dx &= \frac{j_0}{\tau}e^{-u_D}e^{-(u-u_D)}\Big|_{x=0}^1 + \frac{j_0}{\tau}\int_0^1 e^{-u}u_{D,x}dx \\
&= -\frac{j_0}{\tau}\int_0^1 (e^{-u_D}-e^{-u})u_{D,x}dx \\
&\le K_c\delta_0,
\end{aligned} \tag{3.13}$$

where here and in the following $K_c > 0$ is a generic constant depending on $\mathcal{A}$, τ and j_0.

By Taylor's expansion and Cauchy's inequality, the second term on the left-hand side of (3.11) can be estimated as

$$\int_0^1 (p'(e^u)-j_0^2e^{-2u})u_x(u-u_D)_x dx$$

$$
\begin{aligned}
&= \int_0^1 \mathcal{A}^{-2}(p'(\mathcal{A})\mathcal{A}^2 - j_0^2)(u_x^2 - u_x u_{D,x})dx \\
&\quad + \int_0^1 \left(p''(\mathcal{A})\mathcal{A} + 2j_0^2\mathcal{A}^{-2}\right)(u-u_D)u_x(u-u_D)_x dx \\
&\quad + \frac{1}{2}\int_0^1 \left(p'''(e^y)e^{2y} + p''(e^y)e^y - 4j_0^2 e^{-2y}\right) u_x(u-u_D)_x(u-u_D)^2 dx \\
&\geq \int_0^1 \mathcal{A}^{-2}(p'(\mathcal{A})\mathcal{A}^2 - j_0^2)(u_x^2 - u_x u_{D,x})dx \\
&\quad - \frac{9}{8}\max_{x\in[0,1]}\left(|(p''(\mathcal{A})|\mathcal{A} + 2j_0^2\mathcal{A}^{-2}\right)\delta_1 \int_0^1 u_x^2 dx \\
&\quad - \frac{9}{16}\max_{\ln\frac{16}{25}\mathcal{A}_*\leq y\leq \ln\frac{25}{16}\mathcal{A}^*}\left(|p'''(e^y)|e^{2y} + |p''(e^y)e^y| + 4j_0^2e^{-2y}\right)\delta_1^2\int_0^1 u_x^2 dx \\
&\quad - K_c\delta_0,
\end{aligned}
$$

where $y = u + \theta_1(u-u_D)$ for some $\theta_1 \in (0,1)$. This implies, using (3.7)–(3.9) and (3.12),

$$
\begin{aligned}
&\int_0^1 (p'(e^u) - j_0^2e^{-2u})u_x(u-u_D)_x dx \\
&\geq \int_0^1 \mathcal{A}^{-2}(p'(\mathcal{A})\mathcal{A}^2 - j_0^2)(u_x^2 - u_x u_{D,x})dx - \frac{1}{2}(1+\theta)A_0\int_0^1 u_{xx}^2 dx \\
&\quad - K_c\delta_0. \qquad (3.14)
\end{aligned}
$$

Furthermore, it follows from (1.16)

$$
\begin{aligned}
&\int_0^1 (p'(\mathcal{A}) - j_0^2\mathcal{A}^{-2})(u_x^2 - u_x u_{D,x})dx \\
&\geq (1+\theta)\min_{x\in E}(p'(\mathcal{A})\mathcal{A}^2 - j_0^2)\mathcal{A}^{-2}\int_0^1 u_x^2 dx + (1-\theta)\int_{I\setminus E}(p'(\mathcal{A})\mathcal{A}^2 - j_0^2)\mathcal{A}^{-2}u_x^2 dx \\
&\quad - \frac{1}{\theta}\max_{x\in[0,1]}|(p'(\mathcal{A}) - j_0^2\mathcal{A}^{-2})|\int_0^1 |u_{D,x}|^2 dx \\
&\geq (1+\theta)\min_{x\in[0,1]}(p'(\mathcal{A})\mathcal{A}^2 - j_0^2)\mathcal{A}^{-2}\int_0^1 u_{xx}^2 dx + (1-\theta)\int_{I\setminus E}(p'(\mathcal{A})\mathcal{A}^2 - j_0^2)\mathcal{A}^{-2}u_x^2 dx \\
&\quad - \frac{1+\kappa}{1-\kappa}K_c\delta_0, \qquad (3.15)
\end{aligned}
$$

where we have used

$$
\int_E u_x^2 dx \leq \int_0^1 u_{xx}^2 dx.
$$

The estimates (3.15) and (3.14) yield

$$
\int_0^1 (p'(e^u) - j_0^2e^{-2u})u_x(u-u_D)_x dx
$$

$$
\begin{aligned}
\geq&(1+\theta)\min_{x\in[0,1]}(p'(\mathcal{A})\mathcal{A}^2-j_0^2)\mathcal{A}^{-2}\int_0^1 u_{xx}^2dx-\frac{1}{2}(1+\theta)A_0\int_0^1 u_{xx}^2dx\\
&+(1-\theta)\int_{I\setminus E}\left(p'(\mathcal{A})-\frac{j_0^2}{\mathcal{A}^2}\right)u_x^2dx-\frac{1+\kappa}{1-\kappa}K_c\delta_0. \qquad (3.16)
\end{aligned}
$$

Substituting (3.13) and (3.16) into (3.11) and using (3.10), we have

$$
A_0||u_{xx}||^2+\mathcal{A}_*||u-u_D||^2+\int_{I\setminus E}(p'(\mathcal{A})\mathcal{A}^2-j_0^2)\mathcal{A}^{-2}(u-u_D)_x^2dx\leq\frac{1+\kappa}{1-\kappa}K_c\delta_0, \quad (3.17)
$$

where we recall that $\mathcal{A}_*$ and A_0 are given by (3.3) and (3.4), respectively.

Now, we turn to higher order estimates. Let $u\in H^4$ be a solution to the BVP (1.12). Multiply this equation with $\varepsilon^2 u_{xxxx}$ and integrate over $(0,1)$ to obtain

$$
\begin{aligned}
&\frac{1}{4}\varepsilon^4\int_0^1[u_{xxxx}^2-(u_xu_{xxx}+u_{xx}^2)u_{xxxx}]dx\\
=&\varepsilon^2\int_0^1((p'(e^u)-j_0^2e^{2u})u_x)_xu_{xxxx}dx-\varepsilon^2\int_0^1(e^u-e^{u_D})u_{xxxx}dx\\
&-\varepsilon^2\int_0^1(\mathcal{A}-\mathcal{C})u_{xxxx}dx-\varepsilon^2\frac{j_0}{\tau}\int_0^1 e^{-u}u_xu_{xxxx}dx. \qquad (3.18)
\end{aligned}
$$

Due to (1.13), there are y_1, y_2, y_3, $y_4\in(0,1)$ such that

$$
u_x(y_1)=u_{xx}(y_2)=u_{xx}(y_3)=u_{xxx}(y_4)=0,
$$

and

$$
u_{xx}^2(x)+\int_0^1 u_{xx}^2dx\leq\int_0^1 u_{xxx}^2dx,\quad \int_0^1 u_{xxx}^2dx\leq\int_0^1 u_{xxxx}^2dx. \qquad (3.19)
$$

Thus, it follows from (3.17), (3.19) and Hölder's inequality

$$
\begin{aligned}
&\int_0^1(u_xu_{xxx}+u_{xx}^2)u_{xxxx}dx\\
\leq&|u_x|_\infty||u_{xxx}||\cdot||u_{xxxx}||+|u_{xx}|_\infty||u_{xx}||\cdot||u_{xxxx}||\\
\leq&\frac{1+\kappa}{1-\kappa}K_c\delta_0\int_0^1 u_{xxxx}^2dx. \qquad (3.20)
\end{aligned}
$$

Then, we obtain from (3.18), in view of (3.7), (3.17), (3.20), (3.19), and Cauchy's inequality, that

$$
\begin{aligned}
&\varepsilon^4\int_0^1 u_{xxxx}^2dx\\
\leq&K_c\int_0^1[(p'(e^u)-j_0^2e^{-2u})^2u_{xx}^2+(p''(e^u)e^u+2j_0^2e^{-2u})^2u_x^4]dx
\end{aligned}
$$

$$
\begin{aligned}
&+ K_c \int_0^1 [(u-u_D)^2 + (\mathcal{A}-\mathcal{C})^2 + u_x^2]dx \\
\leq &\frac{1+\kappa}{1-\kappa} K_c \delta_0,
\end{aligned}
\tag{3.21}
$$

provided that δ_0 is small enough. By (3.19) and (3.21), we have

$$
\varepsilon^4 \int_0^1 [u_{xxx}^2 + u_{xxxx}^2]dx \leq \frac{1+\kappa}{1-\kappa} K_c \delta_0.
\tag{3.22}
$$

The combination of (3.17) and (3.22) finally leads to

$$
\begin{aligned}
&\mathcal{A}_*||u-u_D||^2 + A_0||u_{xx}||^2 + \varepsilon^4||(u_{xxx}, u_{xxxx})||^2 \\
&\qquad + \int_{I\backslash E} (p'(\mathcal{A}^2)\mathcal{A}^2 - j_0^2)\mathcal{A}^{-2}(u-u_D)_x^2 dx \leq \frac{1+\kappa}{1-\kappa} K_c \delta_0.
\end{aligned}
\tag{3.23}
$$

<u>*Step 2. Existence*</u>

It is not difficult to prove that there exists a solution $u \in H^4$ to the BVP (1.12)–(1.13). The argument is similar to that used in section 2 based on the Leray-Schauder fixed point theorem. The function space is $X := C^{0,1}([0,1])$. The corresponding linear BVP is

$$
\begin{aligned}
\frac{1}{4}\varepsilon^2 (u_{xx} + \frac{1}{2}\lambda v_x^2)_{xx} - ((p'(e^{v_K})u_x - \lambda j_0^2 e^{-2v_K} v_x)_x & \\
+\lambda(\frac{e^v - 1}{v} u + 1 - \mathcal{C}) =& \lambda \frac{j_0}{\tau}(e^{-v_K})_x, \\
u(0) = \lambda u_1, \ u(1) = \lambda u_2, \ u_x(0) = u_x(1) =& 0,
\end{aligned}
$$

where $\lambda \in [0,1]$, $v \in X$ and $v_K = \text{mim}\{\delta_1 \ln \mathcal{A}^*, \max\{-\delta_1 \ln \mathcal{A}_*, v\}\}$ with δ_1 chosen such that (3.7)–(3.9) hold. The bilinear form and functional are defined respectively by

$$
a(u,\psi) = \int_0^1 \left(\frac{1}{4}\varepsilon^2 u_{xx} v_{xx} + p'(e^{v_K}) u_x \psi_x + \lambda \frac{e^v - 1}{v} u\psi \right) dx
$$

and

$$
F(\psi) = \int_0^1 \left(-\frac{1}{8}\varepsilon^2 \lambda v_x^2 \psi_{xx} - \lambda j_0^2 e^{-2v_K} v_x \psi_x + \lambda(\mathcal{C}-1)\psi - \lambda \frac{j_0}{\tau} e^{-v_K}\psi_x \right) dx.
$$

where v_K. We omit the details.

<u>*Step 3. Uniqueness.*</u>

Let $u,\ v \in H^4$ be two solutions to the BVP (1.12)–(1.13) satisfying (3.23). Using $u - v \in H_0^2$ as an admissible test function in the weak formulation derived for $u - v$, we have

$$
\frac{1}{4}\varepsilon^2 \int_0^1 (u-v)_{xx}^2 dx + \frac{1}{8}\varepsilon^2 \int_0^1 (u_x + v_x)(u-v)_x(u-v)_{xx} dx
$$

$$
\begin{aligned}
&+\int_0^1 \left((p'(e^u)-j_0^2e^{-2u})u_x-(p'(e^v)-j_0^2e^{-2v})v_x\right)(u-v)_x dx\\
=&-\int_0^1 (e^u-e^v)(u-v)dx+\frac{j_0}{\tau}\int_0^1 (e^{-u}-e^{-v})_x(u-v)dx.
\end{aligned}
\tag{3.24}
$$

The last term in (3.24) can be estimated as

$$
\begin{aligned}
&\frac{j_0}{\tau}\int_0^1 (e^{-u}-e^{-v})_x(u-v)dx\\
=&-\frac{j_0}{\tau}\int_0^1 [(e^{-u}-e^{-v})(u-v)u_x+\frac{1}{2}e^{-v}[(u-v)^2]_x]dx\\
\le&\sqrt{\frac{1+\kappa}{1-\kappa}\frac{K_c}{A_0}\delta_0}\ \int_0^1 (u-v)^2dx,
\end{aligned}
\tag{3.25}
$$

where we have used

$$
|(u_x,v_x)|_\infty \le \sqrt{\frac{1+\kappa}{1-\kappa}\frac{K_c}{A_0}\delta_0},
\tag{3.26}
$$

derived from (3.23), and $K_c > 0$ is a generic constant depending on $\mathcal{A}$, τ, and j_0. From (3.7), (3.24), and (3.25) follows

$$
\begin{aligned}
&\frac{1}{4}\varepsilon^2\int_0^1 (u-v)_{xx}^2dx+\frac{1}{8}\varepsilon^2\int_0^1 (u_x+v_x)(u-v)_x(u-v)_{xx}dx\\
&+\int_0^1 \left((p'(e^u)-j_0^2e^{-2u})u_x-(p'(e^v)-j_0^2e^{-2v})v_x\right)(u-v)_x dx\\
\le&-\left(\frac{3}{4}\mathcal{A}_*-\sqrt{\frac{1+\kappa}{1-\kappa}\frac{K_c}{A_0}\delta_0}\right)\int_0^1 (u-v)^2dx\\
\le&-\frac{1}{2}\mathcal{A}_*\int_0^1 (u-v)^2dx,
\end{aligned}
\tag{3.27}
$$

provided that δ_0 is small enough.

By (3.7), (3.26), Hölder's inequality and

$$
\int_0^1 (u-v)_x^2dx \le \int_0^1 (u-v)_{xx}^2dx,
\tag{3.28}
$$

we have

$$
\begin{aligned}
&\frac{1}{4}\varepsilon^2\int_0^1 (u-v)_{xx}^2dx+\frac{1}{8}\varepsilon^2\int_0^1 (u_x+v_x)(u-v)_x(u-v)_{xx}dx\\
\ge&\frac{1}{4}\varepsilon^2\left(1-\sqrt{\frac{1+\kappa}{1-\kappa}\frac{K_c}{A_0}\delta_0}\right)\int_0^1 (u-v)_{xx}^2dx.
\end{aligned}
\tag{3.29}
$$

By (1.16), (3.7) and (3.28), the third term on the left-hand side of (3.27) can be estimated as follows, using an approach similar to (3.14):

$$
\begin{aligned}
&\int_0^1 \left((p'(e^u) - j_0^2 e^{-2u})u_x - (p'(e^v) - j_0^2 e^{-2v})v_x\right)(u-v)_x dx \\
=&\int_0^1 (p'(e^u) - j_0^2 e^{-2u})(u-v)_x^2 dx \\
&+\int_0^1 (p'(e^u) - p'(e^v) - j_0^2 e^{-2u} + j_0^2 e^{-2v})v_x (u-v)_x dx \\
\geq& \min_{x\in E}(p'(\mathcal{A})\mathcal{A}^2 - j_0^2)\mathcal{A}^{-2}\int_0^1 (u-v)_x^2 dx + \int_{I\backslash E}(p'(\mathcal{A})\mathcal{A}^2 - j_0^2)\mathcal{A}^{-2}(u-v)_x^2 dx \\
&+\int_0^1 \left(p'(e^u) - j_0^2 e^{-2u} - p'(\mathcal{A}) + j_0^2\mathcal{A}^{-2}\right)(u-v)_x^2 dx \\
&-|v_x|_\infty \int_0^1 |p'(e^u) - p'(e^v) - j_0^2 e^{-2u} + j_0^2 e^{-2v}||(u-v)_x| dx \\
\geq& \min_{x\in E}(p'(\mathcal{A})\mathcal{A}^2 - j_0^2)\mathcal{A}^{-2}\int_0^1 (u-v)_{xx}^2 dx + \int_{I\backslash E}(p'(\mathcal{A})\mathcal{A}^2 - j_0^2)\mathcal{A}^{-2}(u-v)_x^2 dx \\
&-\sqrt{\frac{1+\kappa}{1-\kappa}\frac{K_c}{A_0}\delta_0}\int_0^1 [(u-v)^2 + (u-v)_{xx}^2] dx \\
\geq& \min_{x\in E}(p'(\mathcal{A})\mathcal{A}^2 - j_0^2)\mathcal{A}^{-2}\int_0^1 (u-v)_{xx}^2 + \int_{I\backslash E}(p'(\mathcal{A})\mathcal{A}^2 - j_0^2)\mathcal{A}^{-2}(u-v)_x^2 dx \\
&-\frac{1}{3}\mathcal{A}_*\int_0^1 (u-v)^2 dx - \sqrt{\frac{1+\kappa}{1-\kappa}\frac{K_c}{A_0}\delta_0}\int_0^1 (u-v)_{xx}^2 dx,
\end{aligned}
\tag{3.30}
$$

provided that δ_0 is small enough.

Substituting (3.29) and (3.30) into (3.27) leads to

$$
\begin{aligned}
&\min_{x\in[0,1]}\left(\frac{1}{4}\kappa\varepsilon^2 + p'(\mathcal{A}) - j_0^2\mathcal{A}^{-2}\right)\int_0^1 (u-v)_{xx}^2 dx \\
&+\left(\frac{1}{4}\varepsilon^2(1-\kappa) - \sqrt{\frac{1+\kappa}{1-\kappa}\frac{K_c}{A_0}\delta_0}\right)\int_0^1 (u-v)_{xx}^2 dx + \frac{1}{6}\mathcal{A}_*\int_0^1 (u-v)^2 dx \\
\leq& 0,
\end{aligned}
\tag{3.31}
$$

which implies that $u = v$ in $(0,1)$ if δ_0 is so small that

$$\frac{1}{4}\varepsilon^2(1-\kappa) - \sqrt{\frac{1+\kappa}{1-\kappa}\frac{K_c}{A_0}\delta_0} > 0$$

and if the condition

$$\min_{x\in[0,1]}\left\{\frac{1}{4}\kappa\varepsilon^2 + p'(\mathcal{A}) - j_0^2\mathcal{A}^{-2}\right\} > 0 \tag{3.32}$$

holds. The proof of Theorem 3.1 is completed. $\square$

The existence and uniqueness of stationary solutions of (1.7)–(1.11) follows immediately from Theorem 3.1:

Theorem 3.3 *Assume that* (1.15) *and* (1.16) *hold and that* $p \in C^3(0,\infty)$. *For* $\kappa \in (0,1)$, *assume that it holds*

$$\min_{x\in[0,1]} \mathcal{A}(x)^2 \left(\frac{1}{4}\kappa\varepsilon^2 + p'(\mathcal{A}(x))\right) > j_0^2. \tag{3.33}$$

Then, the BVP (1.7)–(1.11) *has a unique solution* $(\rho_0, \phi_0) \in H^4 \times H^2$ *such that*

$$\mathcal{A}_*\|\rho_0 - \mathcal{A}\|^2 + A_0\|\rho_{0xx}\|^2 + \varepsilon^4\|(\rho_{0xxx}, \rho_{0xxxx})\|^2 + \|\phi_{0x}\|_1^2 \leq \tilde{K}_c\delta_0, \tag{3.34}$$

provided that $\|\mathcal{A}'\|_1 + \|\mathcal{A} - \mathcal{C}\|$ *is sufficiently small. The constant* A_0 *is defined in* (3.4) *and* $\tilde{K}_c > 0$ *is a constant depending on* j_0, τ *and* $\mathcal{A}$.

Remark 3.4 The estimates of Theorem 2.1 and Theorem 3.1 show that one can pass to the limit $\varepsilon \to 0$ in the quantum Euler-Poisson system (similarly as in [13]) to obtain a solution to the classical Euler-Poisson system:

$$\partial_t\rho + j_x = 0, \tag{3.35}$$

$$\partial_t j + \left(\frac{j^2}{\rho} + p(\rho)\right)_x - \rho\phi_x - \frac{\varepsilon^2}{2}\rho\left(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\right)_x = -\frac{j}{\tau}, \tag{3.36}$$

$$\lambda^2\phi_{xx} = \rho - \mathcal{C}(x) \qquad \text{in } [0,1]\times(0,T), \tag{3.37}$$

The solution of this system (together with appropriate boundary conditions) is classical because the condition (2.1) or (3.1) reduces to the classical subsonic condition for (3.35)–(3.37). For the mathematical analysis on the Euler-Poisson system, we refer to [2, 14, 21] and references therein.

The following theorem is important for the stability analysis of stationary solutions obtained by Theorem 3.3 (see [19] for details).

Theorem 3.5 *Let* (ρ_0, j_0, ϕ_0) *the unique strong solution given by Theorem 3.3. Let* $\omega_0 = \sqrt{\rho_0}$. *Then* (ω_0, j_0, ϕ_0) *is the unique solution of the following BVP*

$$j_x = 0, \tag{3.38}$$

$$\left(\frac{j^2}{w^2} + p(w^2)\right)_x = w^2\phi_x + \frac{1}{2}\varepsilon^2 w^2\left(\frac{w_{xx}}{w}\right)_x - \frac{j}{\tau}, \tag{3.39}$$

$$\phi_{xx} = w^2 - \mathcal{C}(x), \tag{3.40}$$

with boundary conditions

$$w(0) = \sqrt{\rho_1}, \quad w(1) = \sqrt{\rho_2}, \quad w_x(0) = w_x(1) = 0, \tag{3.41}$$

$$\phi(0) = 0, \quad \phi(1) = \Phi_0. \tag{3.42}$$

Moreover, it holds

$$\|\omega_0 - \sqrt{\mathcal{A}}\|^2 + \|\omega_{0x}\|_3^2 + \|\phi_{0x}\|_1^2 \leq \tilde{C}_0\delta_0, \tag{3.43}$$

where $\tilde{C}_2 > 0$ is a constant depending on $\mathcal{A}$, j_0, τ and ε.

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