Quantum Euler-Poisson Systems: Global Existence and Exponential Decay

Ansgar Jüngel* Hailiang Li†

*Fachbereich Mathematik und Statistik, Universität Konstanz, Fach D193, 78457 Konstanz, Germany; e-mail: ansgar.juengel@uni-konstanz.de

†Institut für Mathematik, Universität Wien, Boltzmanngasse 9, 1090 Vienna, Austria; e-mail: hailiang.li@univie.ac.at

Abstract

A one-dimensional transient quantum Euler-Poisson system for the electron density, the current density and the electrostatic potential in bounded intervals is considered. The equations include the Bohm potential accounting for quantum mechanical effects and are of dispersive type. They are used, for instance, for the modeling of quantum semiconductor devices.

The existence of local-in-time solutions with small initial velocity is proven for general (possibly non-convex or non-monotone) pressure-density functions. If a stability condition related to the subsonic condition for the classical Euler equations is imposed, the local solutions are proven to exist globally in time and tend to the corresponding steady-state solution exponentially fast as the time tends to infinity.

Keywords. Quantum Euler-Poisson system, existence of global-in-time classical solutions, nonlinear fourth-order wave equation, exponential decay rate, long-time behavior of the solutions.

1 Introduction

1.1 The Model Equations

In 1927, Madelung gave a fluid-dynamical description of quantum systems governed by the Schrödinger equation for the wave function ψ:

$$\begin{aligned} i\varepsilon\partial_t\psi &= -\frac{\varepsilon^2}{2}\Delta\psi - V\psi \quad \text{in } \mathbb{R}^d\times(0,\infty), \\ \psi(\cdot,0) &= \psi_0 \quad \text{in } \mathbb{R}^d, \end{aligned}$$

where $d \geq 1$ is the space dimension, $\varepsilon > 0$ denotes the scaled Planck constant, and $V = V(x,t)$ is some (given) potential. Separating the amplitude and phase of $\psi = |\psi| \exp(iS/\varepsilon)$, the particle density $\rho = |\psi|^2$ and the particle current density $j = \rho\nabla S$ for irrotational flow satisfy the so-called *Madelung equations* [20]

$$\partial_t \rho + \mathrm{div} j = 0, \tag{1.1}$$

$$\partial_t j + \mathrm{div}\left(\frac{j \otimes j}{\rho}\right) - \rho\nabla\phi - \frac{\varepsilon^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = 0 \quad \text{in } \mathbb{R}^d \times (0,\infty), \tag{1.2}$$

where the i-th component of the convective term $\mathrm{div}(j \otimes j/\rho)$ equals

$$\sum_{k=1}^{d} \frac{\partial}{\partial x_k}\left(\frac{j_i j_k}{\rho}\right).$$

The equations (1.1)-(1.2) can be interpreted as the pressureless Euler equations including the quantum Bohm potential

$$\frac{\varepsilon^2}{2}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}. \tag{1.3}$$

They have been used for the modeling of superfluids like Helium II [16, 19].

Recently, Madelung-type equations have been derived to model quantum phenomena in semiconductor devices, like resonant tunneling diodes, starting from the Wigner-Boltzmann equation [6] or from a mixed-state Schrödinger-Poisson system [8, 9]. There are several advantages of the fluid-dynamical description of quantum semiconductors. First, kinetic equations, like the Wigner equation, or Schrödinger systems are computationally very expensive, whereas for Euler-type equations efficient numerical algorithms are available [5, 24]. Second, the macroscopic description allows for a coupling of classical and quantum models. Indeed, setting the Planck constant ε in (1.2) equal to zero, we obtain the classical pressureless equations, so in both pictures, the same (macroscopic) variables can be used. Finally, as semiconductor devices are modeled in bounded domains, it is easier to find physically relevant boundary conditions for the macroscopic variables than for the Wigner function or for the wave function.

The Madelung-type equations derived by Gardner [6] and Gasser et al. [8] also include a pressure term and a momentum relaxation term taking into account interactions of the electrons with the semiconductor crystal, and are self-consistently coupled to the Poisson equation for the electrostatic potential ϕ:

$$\partial_t \rho + \mathrm{div} j = 0, \tag{1.4}$$

$$\partial_t j + \mathrm{div}\left(\frac{j \otimes j}{\rho}\right) + \nabla P(\rho) - \rho\nabla\phi - \frac{\varepsilon^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = -\frac{j}{\tau}, \tag{1.5}$$

$$\lambda^2 \Delta\phi = \rho - \mathcal{C}(x) \qquad \text{in } \Omega \times (0,\infty), \tag{1.6}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain, $\tau > 0$ is the (scaled) momentum relaxation time constant, $\lambda > 0$ the (scaled) Debye length, and $\mathcal{C}(x)$ is the doping concentration modeling the semiconductor device under consideration [12, 23]. The pressure is assumed to depend only on the particle density and, like in classical fluid dynamics, often the expression

$$P(\rho) = \frac{T_0}{\gamma}\rho^\gamma, \quad \rho \ge 0, \quad \gamma \ge 1, \tag{1.7}$$

with the temperature constant $T_0 > 0$ is employed [6, 11]. *Isothermal* fluids correspond to $\gamma = 1$, *isentropic* fluids to $\gamma > 1$. Notice that the particle temperature is $T(\rho) = T_0\rho^{\gamma-1}$. In this paper we consider *general* (smooth) pressure functions.

The equations (1.4)-(1.6) are referred to as the *quantum Euler-Poisson system* or as the *quantum hydrodynamic model.*

In this paper, we investigate the (local and global) existence and long-time behavior of solutions of the following one-dimensional quantum Euler-Poisson system:

$$\rho_t + j_x = 0, \tag{1.8}$$

$$j_t + \left(\frac{j^2}{\rho} + P(\rho)\right)_x = \rho\phi_x + \frac{1}{2}\varepsilon^2\rho\left(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\right)_x - \frac{j}{\tau}, \tag{1.9}$$

$$\phi_{xx} = \rho - \mathcal{C}(x), \tag{1.10}$$

with the following initial and boundary conditions

$$\rho(x,0) = \varrho_1(x) > 0, \quad j(x,0) = j_1(x) =: \varrho_1(x)v_1(x), \tag{1.11}$$

$$\rho(0,t) = \rho_1, \quad \rho(1,t) = \rho_2, \quad \rho_x(0,t) = \rho_x(1,t) = 0, \tag{1.12}$$

$$\phi(0,t) = 0, \quad \phi(1,t) = \Phi_0, \tag{1.13}$$

for $(x,t) \in (0,1) \times (0,\infty)$, where ρ_1, ρ_2, $\Phi_0 > 0$, and v_1 is the initial velocity.

The existence and uniqueness of *steady-state* (classical) solutions to the quantum Euler-Poisson system for current density $j_0 = 0$ (thermal equilibrium) has been studied in [1, 7]. The stationary equations for $j_0 > 0$ have been considered in [4, 11, 26] for general monotoneous pressure functions, however, with different boundary conditions, assuming Dirichlet data for the velocity potential S [11] or employing nonlinear boundary conditions [4, 26]. Existence of steady-state solutions to (1.8)–(1.10) subject to the boundary conditions (1.12)–(1.13) is proven in [10] for the linear pressure function $P(\rho) = \rho$ and in [14] for *general* pressure functions $P(\rho)$ also allowing for non-convex or non-monotone pressure-density relations. So far, to our knowledge, the only known results on the existence of the *time-dependent* system (1.4)-(1.6) have been obtained in [13] for smooth local-in-time solutions on bounded domains and in [17] for "small" irrotational global-in-time solutions in the whole space assuming strictly convex pressure functions and a constant doping profile.

In the present paper, we consider the initial-boundary-value problem (IBVP) (1.8)–(1.13) for *general* pressure and *non-constant* doping profile, and we focus on the *local* and *global* existence of classical solutions (ρ, j, ϕ) of the IBVP (1.8)–(1.13) and their time-asymptotic convergence to the stationary solutions (ρ_0, j_0, ϕ_0) obtained in [14].

First, we show that there exists a classical local-in-time solution for regular initial data. Second, we prove that if a certain "subsonic" condition (see (1.25)) holds and if the initial data is a perturbation of the stationary solution (ρ_0, j_0, ϕ_0), a classical solution (ρ, j, ϕ) exists globally in time and tends to (ρ_0, j_0, ϕ_0) exponentially fast as time tends to infinity.

In dealing with the IBVP (1.8)–(1.13) we have to overcome the following difficulties. First, since the general pressure $P(\rho)$ can be non-convex (even zero or "negative", see Remark 1.5), the left part of equations (1.8)–(1.10) may not be hyperbolic any more. Unlike [17], we cannot apply the local existence theory of quasilinear symmetric hyperbolic systems [3, 15, 21, 22]. We have to establish a new local existence theory. Second, the appearance of the nonlinear quantum Bohm potential in (1.9) requires that the density is strictly positive for regular solutions. This together with the structure of the quantum term causes problems in the local and global existence proofs.

1.2 Main results

Before stating our main results we introduce some notation. We denote by $L^2 = L^2(0,1)$ and $H^k = H^k(0,1)$ the Lebesgue space of square integrable functions and the Sobolev space of functions with square integrable weak derivatives of order k, respectively. The norm of L^2 is denoted by $\|\cdot\|_0 = \|\cdot\|$, and the norm of H^k is $\|\cdot\|_k$. The space $H_0^k = H_0^k(0,1)$ is the closure of $C_0^\infty(0,1)$ in the norm of H^k. Let $T > 0$ and let $\mathcal{B}$ be a Banach space. Then $C^k(0,T;\mathcal{B})$ ($C^k([0,T];\mathcal{B})$, respectively) denotes the space of $\mathcal{B}$-valued k-times continuously differentiable functions on $(0,T)$ ($[0,T]$, respectively), $L^2(0,T;\mathcal{B})$ is the space of $\mathcal{B}$-valued L^2-functions on $(0,T)$, and $W^{k,p}(0,T;\mathcal{B})$ the space of $\mathcal{B}$-valued $W^{k,p}$-functions on $(0,T)$. Finally, C always denotes a generic positive constant.

It is convenient to make use of the variable transformation $\rho = w^2$ in (1.8)–(1.10) which yields the following IBVP for (w, j, ϕ):

$$2ww_t + j_x = 0, \tag{1.14}$$

$$j_t + \left(\frac{j^2}{w^2} + P(w^2)\right)_x = w^2\phi_x + \frac{1}{2}\varepsilon^2 w^2\left(\frac{w_{xx}}{w}\right)_x - \frac{j}{\tau}, \tag{1.15}$$

$$\phi_{xx} = w^2 - \mathcal{C}(x), \tag{1.16}$$

with the initial and boundary conditions

$$(w, j)(x, 0) = (w_1, j_1)(x) = (\sqrt{\varrho_1}, \varrho_1 v_1)(x), \tag{1.17}$$

$$w(0,t) = \sqrt{\rho_1}, \quad w(1,t) = \sqrt{\rho_2}, \quad w_x(0,t) = w_x(1,t) = 0, \tag{1.18}$$

$$\phi(0,t) = 0, \quad \phi(1,t) = \Phi_0, \tag{1.19}$$

for $x \in (0,1)$, $t \geq 0$. This problem is equivalent to (1.8)–(1.13) for classical solutions with positive particle density.

We will assume throughout this paper compatibility conditions for the IBVP (1.14)–(1.19) in the sense that the time derivatives of the boundary values and the spatial derivatives of the initial data are compatible at $(x,t)=(0,0)$ and $(x,t)=(1,0)$ in (1.14)–(1.16).

We will prove the following local existence result for the IBVP (1.14)–(1.19):

Theorem 1.1 *Assume that*

$$P(\rho) \in C^4(0,+\infty), \quad \mathcal{C} \in H^2, \tag{1.20}$$

$(w_1, j_1) \in H^6 \times H^5$ *such that* $w_1(x) > 0$ *for* $x \in [0,1]$, *and for some* $\alpha \in [(1+2\sqrt{2}\varepsilon)^{-1}, 1)$

$$\|v_1\|_{C^1([0,1])} < \frac{(1-\alpha)w_*}{8\sqrt{2}\|w_1\|_1}, \tag{1.21}$$

where

$$w_* = \min_{x\in[0,1]} w_1(x) > 0.$$

Then, there is a number $T_* > 0$ *(determined by* (3.62)*), such that there exists a classical solution* (w, j, ϕ) *of the IBVP* (1.14)–(1.19) *in the time interval* $[0, T_*]$ *satisfying* $w \geq (1-\alpha)w_* > 0$ *in* $[0,1]\times[0,T_*]$ *and*

$$\|w(t)\|_6^2 + \|j(t)\|_5^2 + \|\phi(t)\|_4^2 < \infty \quad \text{for } t \leq T_*.$$

Remark 1.2 (1) Condition (1.21) is needed to prove the positivity of the particle density. A similar condition has been also used to prove the existence of stationary solutions [11]. This condition allows for arbitrarily large current densities $j_1 = w_1^2 v_1$, for instance if w_1 is a sufficiently large constant.

(2) We are able to show the statements of Theorem 1.1 under the slightly more general condition

$$\|v_1\|_{C^1([0,1])} < \min\left\{\alpha\varepsilon, \frac{(1-\alpha)}{2\sqrt{2}}\right\} \frac{w_*}{4\|w_1\|_1}, \quad \alpha \in (0,1). \tag{1.22}$$

Then (1.21) is a special case for $\alpha > (1+2\sqrt{2}\varepsilon)^{-1}$ which is equivalent to $\alpha\varepsilon \geq (1-\alpha)/2\sqrt{2}$.

(3) The local existence of the Cauchy problem in $\mathbb{R}^d$ or $\mathbb{T}^d$ can be proven in the same framework and will be done in [18]. $\square$

Theorem 1.1 is proven by an iteration method and compactness arguments. More precisely, we construct a sequence of approximate solutions which is uniformly bounded in a certain Sobolev space in a fixed (maybe small) time interval. Compactness arguments then imply that there is a limiting solution which proves to be a local-in-time solution of (1.14)–(1.19). Unlike [17] we cannot apply the theory of quasilinear symmetric hyperbolic systems [3, 15, 21, 22] to construct (local) approximate solutions and obtain uniform bounds in Sobolev spaces because the pressure can be non-convex causing the loss of entropy and hyperbolicity of (1.14)–(1.15).

The idea of the local existence result is first to linearize the system (1.14)–(1.16) around its initial state (w_1, j_1, ϕ_1), where ϕ_1 solves the Dirichlet problem (1.16) and (1.19) with w replaced by w_1, and to consider the equations for the perturbation $(\psi, \eta, e) = (w - w_1, j - j_1, \phi - \phi_1)$. The main idea is to write the evolution equation for the perturbed particle density as a semilinear fourth-order wave equation. Then, we construct approximate solutions (ψ_i, η_i, e_i) $(i \geq 1)$ from a fixed-point procedure, which are expected to converge to a solution (ψ, η, e) of the perturbed problem as $i \to \infty$. For this, we derive uniform bounds in Sobolev spaces on a uniform time interval and apply standard compactness arguments (see Section 3). A further analysis shows that $(w, j, \phi) = (w_1 + \psi, j_1 + \eta, \phi_1 + e)$ with $w > 0$ is the expected local (in time) solution of the original problem (1.14)–(1.19).

To extend the local classical solution globally in time, we need to establish uniform estimates. We consider the situation when the initial data is close to the stationary solution (w_0, j_0, ϕ_0) of (1.14)–(1.16) with boundary conditions (1.18)–(1.19). The existence of stationary solutions (w_0, j_0, ϕ_0) of the boundary-value problem (1.14)–(1.16) and (1.18)–(1.19) for *general* pressure functions $P(\rho)$ was obtained in [14] (see Theorem 1.3 below).

Assume that there is a function $\mathcal{A} \in H^2(0,1)$ satisfying

$$\mathcal{A}(x) > 0 \text{ for } x \in (0,1), \quad \mathcal{A}(0) = \rho_1, \quad \mathcal{A}(1) = \rho_2, \quad \mathcal{A}_x(0) = \mathcal{A}_x(1) = 0 \tag{1.23}$$

such that for a set $E \subseteq [0,1]$, it holds

$$P'(\mathcal{A}) - \frac{j_0^2}{\mathcal{A}^2} \begin{cases} \leq 0, & x \in E, \\ > 0, & x \in [0,1] \backslash E. \end{cases} \tag{1.24}$$

Then we conclude the existence of stationary solutions (w_0, j_0, ϕ_0) of (1.14)–(1.16) satisfying the boundary conditions (1.18)–(1.19):

Theorem 1.3 ([**14**]) *Let* (1.20), (1.23)–(1.24) *hold. For given* $\kappa \in (0,1)$, *assume that*

$$\min_{x \in [0,1]} \mathcal{A}^2 \left(\frac{1}{4} \kappa \varepsilon^2 + P'(\mathcal{A}) \right) > j_0^2. \tag{1.25}$$

Then there is a unique solution (w_0, j_0, ϕ_0) *of the stationary version of the boundary-value problem* (1.14)–(1.16) *and* (1.18)–(1.19) *such that*

$$\mathcal{A}_* \| w_0 - \sqrt{\mathcal{A}} \|^2 + A_0 \| w_{0x} \|_3^2 + \| \phi_{0x} \|_1^2 \leq C \delta_0,$$

provided $\delta_0 := \|\mathcal{A}'\|_1 + \|\mathcal{A} - \mathcal{C}\|$ *is small enough. Here,* $\mathcal{A}_* = \min_{x \in [0,1]} \mathcal{A}(x)$,

$$A_0 = \min_{x \in [0,1]} \left(\frac{1}{4} \kappa \varepsilon^2 + P'(\mathcal{A}) - j_0^2 \mathcal{A}^{-2} \right) > 0, \tag{1.26}$$

and $C > 0$ *is a constant depending on* j_0, τ *and* $\mathcal{A}$.

Let $\rho_0 = w_0^2$. *Then* (ρ_0, j_0, ϕ_0) *is a solution of the stationary version of the boundary-value problem* (1.8)–(1.10) *and* (1.12)–(1.13) *satisfying*

$$\mathcal{A}_* \| \rho_0 - \mathcal{A} \|^2 + A_0 \| \rho_{0x} \|_3^2 + \| \phi_{0x} \|_1^2 \leq C' \delta_0, \qquad \square$$

and $C' > 0$ *is a constant depending on* j_0, τ *and* $\mathcal{A}$.

In the following, we use the abreviation

$$\psi_0 = w_1 - w_0, \quad \eta_0 = j_1 - j_0. \tag{1.27}$$

In view of the uniform a-priori estimates of Section 2, we are able to extend the local classical solution globally in time and prove its exponential convergence to the stationary solution (w_0, j_0, ϕ_0):

Theorem 1.4 *Assume that* (1.20), (1.23)–(1.25) *hold. Let* (w_0, j_0, ϕ_0) *be the stationary solution of the boundary-value problem* (1.14)–(1.16) *and* (1.18)–(1.19) *given by Theorem 1.3 for sufficiently small* δ_0. *Assume that the initial datum* $(w_1, j_1) \in H^6 \times H^5$ *satisfying* (1.21) *and* $w_1 > 0$ *in* $[0,1]$. *Then, there is a number* $m_1 > 0$ *such that if*

$$\|\psi_0\|_6 + \|\eta_0\|_5 = \|w_1 - w_0\|_6 + \|j_1 - j_0\|_5 \le m_1,$$

the (classical) solution (w, j, ϕ) *of the IBVP* (1.14)–(1.19) *exists globally in time and satisfies*

$$\|(w-w_0)(t)\|_6^2 + \|(j-j_0)(t)\|_5^2 + \|(\phi-\phi_0)(t)\|_4^2 \le C(\|\psi_0\|_6^2 + \|\eta_0\|_5^2)e^{-\Lambda_0 t} \tag{1.28}$$

for all $t \ge 0$. *Here,* $C > 0$ *and* $\Lambda_0 > 0$ *are constants independent of time* t.

Remark 1.5 (1) Theorems 1.1–1.4 also apply for non-monotone or even "negative" pressure functions. These functions are related to quantum mechanical phenomena in which the motion of the particles is affected by their attractive interaction [16]. A typical example is the focusing nonlinear Schrödinger equation. In fact, this equation is formally equivalent to the quantum Euler-Poisson system with infinite relaxation time and with "negative" pressure.

(2) When $E = \emptyset$ the assumption (1.24) can be interpreted as a 'subsonic' condition in the sense of classical fluid dynamics [2]. However, notice that we can allow for *non-empty* sets E. □

Using Theorems 1.1–1.4 and the variable transformation $\rho = w^2$, we also obtain the local and global existence of classical solutions of the original IBVP (1.8)–(1.13) and can establish their large-time behavior:

Theorem 1.6 *Let* (1.20) *hold. Assume that* $(\sqrt{\varrho_1}, j_1) \in H^6 \times H^5$ *such that* $\varrho_1 > 0$ *in* $[0,1]$ *and*

$$\|v_1\|_{C^1([0,1])} < \min\left\{\alpha\varepsilon, \frac{(1-\alpha)}{2\sqrt{2}}\right\} \frac{\varrho_*}{4\|\sqrt{\varrho}_1\|_1}, \quad \alpha \in (0,1),$$

where

$$\varrho_* = \min_{x\in[0,1]} \sqrt{\varrho}_1(x).$$

Then there is a number $T_*' > 0$ such that there exists a classical solution (ρ, j, ϕ) of the IBVP (1.8)–(1.13) *in $t \in [0, T_*']$ satisfying $\rho > 0$ in $[0,1] \times [0, T_*']$ and*

$$\|\rho(t)\|_6^2 + \|j(t)\|_5^2 + \|\phi(t)\|_4^2 < \infty, \quad t \le T_*'. \tag{1.29}$$

Furthermore, assume that (1.23)–(1.25) *hold and let (ρ_0, j_0, ϕ_0) be the stationary solution of the boundary-value problem* (1.8)–(1.10) *and* (1.12)–(1.13) *given by Theorem 1.3 with sufficiently small δ_0. Then, there is a number $m_2 > 0$ such that if $\|\sqrt{\varrho_1} - \sqrt{\rho_0}\|_6 + \|\eta_0\|_5 \le m_2$, the solution (ρ, j, ϕ) of the IBVP* (1.14)–(1.19) *exists globally in time and satisfies*

$$\|(\rho - \rho_0)(t)\|_6^2 + \|(j - j_0)(t)\|_5^2 + \|(\phi - \phi_0)(t)\|_4^2 \le C(\|\psi_0\|_6^2 + \|\eta_0\|_5^2) e^{-\Lambda_1 t},$$

for all $t \ge 0$, where $C > 0$ and $\Lambda_1 > 0$ are constants independent of t and the pair (ψ_0, η_0) is defined in (1.27).

This paper is arranged as follows. Section 2 is concerned with uniform a-priori estimates of local (in time) solutions. We reformulate the original problem in Section 2.1 as a nonlinear fourth-order wave equation and establish the a-priori estimates for local solutions in Section 2.2. The a-priori estimates and the local existence result of Section 3 imply the global existence. In order to prove the local existence result, we first give a result on the existence of solutions of an abstract fourth-order wave equation (Section 3.1). This wave equation allows us to construct a sequence of approximate solutions converging to a local solution of the problem under consideration (Section 3.2).

2 Proof of Theorem 1.4

In this section, we establish uniform a-priori estimates for local classical solutions of (1.14)–(1.16). This yields, together with the usual continuity argument, the existence of global-in-time solutions and proves Theorem 1.4. For notational simplicity, we set $\tau = 1$.

2.1 Reformulation of the original problem

Let (w_0, j_0, ϕ_0) be the steady-state solution of the boundary-value problem (1.14)–(1.16) and (1.18)–(1.19). For any $T > 0$, assume that (w, j, ϕ) is a solution to the IBVP (1.14)–(1.19) in $[0, T]$.

Differentiating (1.14) with respect to t and (1.15) with respect to x and adding the resulting equations leads to a nonlinear fourth-order wave equation for w:

$$w_{tt} + w_t + \frac{1}{w} w_t^2 + \frac{1}{2w}(w^2 \phi_x)_x - \frac{1}{2w}\left[P(w^2) + \frac{j^2}{w^2}\right]_{xx} + \frac{1}{4}\varepsilon^2 w_{xxxx} - \frac{1}{4}\varepsilon^2 \frac{w_{xx}^2}{w} = 0, \tag{2.1}$$

where we have used the identity

$$\left[w^2 \left(\frac{w_{xx}}{w}\right)_x\right]_x = w\left[w_{xxxx} - \frac{w_{xx}^2}{w}\right]. \tag{2.2}$$

Similarly, the steady-state solution of (1.14)–(1.15) satisfies

$$\frac{1}{2w_0}(w_0^2\phi_{0x})_x - \frac{1}{2w_0}\left[P(w_0^2) + \frac{j_0^2}{w_0^2}\right]_{xx} + \frac{1}{4}\varepsilon^2 w_{0xxxx} - \frac{1}{4}\varepsilon^2\frac{w_{0xx}^2}{w_0} = 0. \tag{2.3}$$

Introduce the perturbations of the steady-state

$$\psi = w - w_0, \quad \eta = j - j_0, \quad e = \phi - \phi_0. \tag{2.4}$$

Then, using (1.14), (2.1)–(2.3), and (1.16), the evolution equations for (ψ, η, e) read as follows:

$$\eta_t + \eta = g_0(x,t), \tag{2.5}$$

$$\begin{aligned} &\psi_{tt} + \psi_t + \frac{1}{4}\varepsilon^2\psi_{xxxx} + \frac{1}{2}(2w_0^2 + 3w_0\psi + \phi_{0xx} + \psi^2)\psi \\ &\quad - \frac{1}{w_0}\left[\frac{j_0}{w_0^2}\eta\right]_{xx} - \left[\left(P'(w_0^2) - \frac{j_0^2}{w_0^4}\right)\psi_x\right]_x = g_1(x,t) + g_2(x,t), \end{aligned} \tag{2.6}$$

$$e_{xx} = (2w_0 + \psi)\psi, \tag{2.7}$$

with the following initial-boundary values

$$\eta(x,0) = \eta_0(x), \quad x \in (0,1), \tag{2.8}$$

$$\psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \theta_0(x) =: -\frac{\eta_{0x}(x)}{2(w_0+\psi_0)(x)}, \quad x \in (0,1), \tag{2.9}$$

$$\psi(0,t) = \psi(1,t) = \psi_x(0,t) = \psi_x(1,t) = 0, \quad t \ge 0, \tag{2.10}$$

$$e(0,t) = e(1,t) = 0, \quad t \ge 0, \tag{2.11}$$

and the definitions

$$\begin{aligned} g_0(x,t) = &-\left[\frac{(j_0+\eta)^2}{(w_0+\psi)^2} - \frac{j_0^2}{w_0^2} + P((w_0+\psi)^2) - P(w_0^2)\right]_x \\ &+ \frac{1}{2}\varepsilon^2(w_0+\psi)^2\left(\frac{(w_0+\psi)_{xx}}{w_0+\psi}\right)_x - \frac{1}{2}\varepsilon^2 w_0^2\left(\frac{w_{0xx}}{w_0}\right)_x \\ &+ (2w_0+\psi)\psi\phi_{0x} + (w_0+\psi)^2 e_x, \end{aligned} \tag{2.12}$$

$$\begin{aligned} g_1(x,t) = &\frac{\varepsilon^2(2w_{0xx}+\psi_{xx})}{4w_0}\psi_{xx} - \frac{\varepsilon^2(w_0+\psi)_{xx}^2}{4(w_0+\psi)w_0}\psi - \frac{\psi_t^2}{(w_0+\psi)} \\ &- (\phi_{0x} + e_x)\psi_x - w_{0x}e_x, \end{aligned} \tag{2.13}$$

$$\begin{aligned} g_2(x,t) = &\frac{1}{2(w_0+\psi)}\left[P((w_0+\psi)^2) + \frac{(j_0+\eta)^2}{(w_0+\psi)^2}\right]_{xx} - \frac{1}{2w_0}\left[P(w_0^2) + \frac{j_0^2}{w_0^2}\right]_{xx} \\ &- \left[\left(P'(w_0^2) - \frac{j_0^2}{w_0^4}\right)\psi_x\right]_x - \frac{1}{w_0}\left[\frac{j_0}{w_0^2}\eta\right]_{xx}. \end{aligned} \tag{2.14}$$

Notice that we can write (1.14) equivalently as

$$2(w_0+\psi)\psi_t + \eta_x = 0, \tag{2.15}$$

which allows us to estimate the derivatives of η in terms of ψ_t.

2.2 The a-priori estimates

We assume that for given $T > 0$, there is a classical solution (ψ, η, e) of the IBVP (2.5)–(2.11) satisfying the regularity condition

$$(\psi, \eta, e) \in X(T) := C^0([0,T]; H^6) \times C^0([0,T]; H^5) \times C^0([0,T]; H^4).$$

We also use the definition

$$\delta_T := \max_{0 \le t \le T} (\|\psi(t)\|_6 + \|\eta(t)\|_5). \tag{2.16}$$

It is easy to verify that if δ_T is sufficiently small, there are constants w_-, w_+, j_-, and j_+ such that

$$0 < w_- \le w_0 + \psi \le w_+, \quad j_- \le j_0 + \eta \le j_+.$$

In the following we assume that δ_T is sufficiently small such that the above estimates hold.

Lemma 2.1 *It holds for $(\psi, \eta, e) \in X(T)$ and $(x,t) \in (0,1) \times (0,T)$,*

$$e_x(x,t)^2 + \|e(t)\|_4^2 \le C\|\psi(t)\|_2^2, \quad e_{xt}(x,t)^2 + e_t(x,t)^2 + \|e_t(t)\|_4^2 \le C\|\psi_t(t)\|_2^2, \tag{2.17}$$

$$\|\eta(t)\|^2 \le C\|\eta_0\|^2 \exp\{-c_0 t\} + C\|(\psi_t, \psi, \psi_{xxx})\|^2, \tag{2.18}$$

$$\eta(x,t)^2 \le C\|\eta_0\|^2 \exp\{-c_0 t\} + C\|(\psi_t, \psi, \psi_{xxx})\|^2, \tag{2.19}$$

$$\|\eta_t(t)\|^2 \le C\|\eta_0\|^2 \exp\{-c_0 t\} + C\|(\psi_t, \psi, \psi_{xxx})\|^2, \tag{2.20}$$

$$\|(\psi_{xxxx}, \psi_{xxx})\|^2 \le C\|(\psi_{tt}, \psi_t, \psi, \psi_{xx}, \psi_x, \psi_{xt})\|^2, \tag{2.21}$$

provided that $\delta_T + \delta_0$ is small enough (see Theorem 1.3 for the definition of δ_0). Here, $c_0, C > 0$ are constants independent of time t.

The notation $\|(f, g, \ldots)\|^2$ means $\|f\|^2 + \|g\|^2 + \cdots$.

Proof: The estimates (2.17) follow directly from the formula

$$e = \int_0^1 G(x,y)(2w_0(y) + \psi(y,t))\psi(y,t)dy,$$

and Hölder's inequality. Here, $G(x,y)$ denotes the Green's function

$$G(x,y) = \begin{cases} x(1-y), & x < y, \\ y(1-x), & x > y. \end{cases} \tag{2.22}$$

To prove (2.18)–(2.20), it is sufficient to prove (2.18). In fact, from (2.15) follows

$$\eta^2 \le \int_0^1 \eta^2 dx + 2\int_0^1 |\eta_x \eta| dx \le C\int_0^1 \eta^2 dx + C\int_0^1 \psi_t^2 dx,$$

which gives (2.19) if (2.18) is proved. In order to see that also (2.20) follows from (2.18), we proceed as follows.

We conclude from the boundary condition (2.10) that there exists $0 \le x_1(t) \le 1$ such that

$$\psi_x(x_1(t), t) = 0,$$

and that there are $x_2(t)$, $x_3(t)$ and $x_4(t)$ satisfying $0 \le x_2(t) \le x_1(t) \le x_3(t) \le 1$ and $0 \le x_2(t) \le x_4(t) \le x_3(t) \le 1$ such that

$$\psi_{xx}(x_2(t), t) = \psi_{xx}(x_3(t), t) = \psi_{xxx}(x_4(t), t) = 0.$$

Thus, by Poincaré's and Hölder's inequality, we obtain

$$\int_0^1 \psi_x^2 dx \le C \int_0^1 \psi_{xx}^2 dx, \tag{2.23}$$

$$\int_0^1 \psi_{xx}^2 dx = \int_0^1 \left(\int_{x_3(t)}^x \psi_{xxx}(y,t) dy \right)^2 dx \le \int_0^1 \psi_{xxx}^2 dx, \tag{2.24}$$

$$\int_0^1 \psi_{xxx}^2 dx = \int_0^1 \left(\int_{x_4(t)}^x \psi_{xxxx}(y,t) dy \right)^2 dx \le \int_0^1 \psi_{xxxx}^2 dx. \tag{2.25}$$

Then, using (2.5), (2.17), (2.15), and (2.23)–(2.24), we can estimate

$$\begin{aligned}
\int_0^1 \eta_t^2 dx \le & C \int_0^1 \left\{ \left(\frac{(j_0+\eta)^2}{(w_0+\psi)^2} - \frac{j_0^2}{w_0^2} + P((w_0+\psi)^2) - P(w_0^2) \right)_x \right\}^2 dx \\
& + C \int_0^1 [\eta^2 + \psi^2 \phi_{0x}^2 + e_x^2] dx + C \int_0^1 [((2w_0+\psi)\psi)_{xxx}^2 + ((2w_{0x}+\psi_x)\psi_x)_x^2] dx \\
\le & C \int_0^1 [\eta^2 + \psi^2 + \psi_t^2 + \psi_x^2 + \psi_{xx}^2 + \psi_{xxx}^2] dx \\
\le & C \int_0^1 [\eta^2 + \psi^2 + \psi_t^2 + \psi_{xxx}^2] dx.
\end{aligned}$$

Hence, the estimate (2.20) follows as soon as (2.18) is shown.

We now prove (2.18). Multiplying (2.5) by η, integrating over $x \in (0,1)$ and integrating by parts gives, in view of the boundary conditions (2.10),

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\int_0^1 \eta^2 dx \right) + \int_0^1 \eta^2 dx \\
\le & - \left[\eta \frac{(j_0+\eta)^2 - j_0^2}{w_0^2} \right] \Bigg|_0^1 + \int_0^1 \left| \eta((2w_0+\psi)\psi\phi_{0x} + (w_0+\psi)^2 e_x) \right| dx \\
& + \int_0^1 \left| \eta_x \left(\frac{(j_0+\eta)^2}{(w_0+\psi)^2} - \frac{j_0^2}{w_0^2} + P((w_0+\psi)^2) - P(w_0^2) \right) \right| dx \\
& + C \int_0^1 \left(|\eta((2w_0+\psi)\psi)_{xxx}| + |\eta_x \psi_x (2w_{0x}+\psi_x)| \right) dx
\end{aligned}$$

$$\stackrel{\Delta}{=} I_0 + I_1 + I_2 + I_3. \tag{2.26}$$

The integrals I_0, I_1, I_2, and I_3 are estimated as follows.

$$\begin{aligned} I_0 \leq & \int_0^1 \left| \eta_x \frac{(j_0+\eta)^2 - j_0^2}{w_0^2} + 2\eta\eta_x \frac{j_0+\eta}{w_0^2} - 2\eta w_{0x} \frac{(j_0+\eta)^2 - j_0^2}{w_0^3} \right| dx \\ \leq & \Big(C\delta_0 + \frac{1}{12} \Big) \int_0^1 \eta^2 dx + C \int_0^1 \psi_t^2 dx, \end{aligned} \tag{2.27}$$

$$I_1 \leq \frac{1}{12} \int_0^1 \eta^2 dx + C \int_0^1 (\psi^2 + e_x^2) dx \leq \frac{1}{12} \int_0^1 \eta^2 dx + C \int_0^1 \psi^2 dx, \tag{2.28}$$

$$I_2 \leq C \int_0^1 |\psi_t \eta| + |\psi\psi_t| dx \leq \frac{1}{12} \int_0^1 \eta^2 dx + C \int_0^1 [\psi_t^2 + \psi^2] dx, \tag{2.29}$$

$$I_3 \leq \frac{1}{12} \int_0^1 \eta^2 dx + C \int_0^1 [\psi_t^2 + \psi^2 + \psi_{xxx}^2] dx, \tag{2.30}$$

provided that $\delta_T + \delta_0$ is small enough. In the above estimates we have used (2.15), (2.17), (2.23) and (2.24). Substituting (2.27)–(2.30) into (2.26) yields

$$\frac{d}{dt} \left(\int_0^1 \eta^2 dx \right) + c_0 \int_0^1 \eta^2 dx \leq C \int_0^1 [\psi_t^2 + \psi^2 + \psi_{xxx}^2] dx, \tag{2.31}$$

where $c_0 \in (0, \frac{4}{3} - C\delta_0]$ is a constant and δ_0 is chosen so small that $C\delta_0 < \frac{4}{3}$. Applying Gronwall's inequality to (2.31) gives (2.18).

Finally, we prove (2.21). By (2.6) and (2.18), it holds

$$\begin{aligned} \int_0^1 \psi_{xxxx}^2 dx \leq & C \int_0^1 (\psi_{tt}^2 + \psi_t^2 + \psi^2 + \psi_{xx}^2 + \psi_x^2 + \psi_{xt}^2) dx \\ & + C(\delta_T + \delta_0) \int_0^1 \psi_{xxx}^2 dx. \end{aligned} \tag{2.32}$$

The combination of (2.32) and (2.25) leads to (2.21) provided that $\delta_T + \delta_0$ is small enough such that $C(\delta_T + \delta_0) < 1$. □

We prove now uniform estimates in Sobolev spaces for ψ, ψ_t and ψ_{tt}.

Lemma 2.2 *It holds for $(\psi, \eta, e) \in X(T)$ and $0 < t < T$,*

$$\begin{aligned} & \|\psi(t)\|_4^2 + \|\psi_t(t)\|_2^2 + \|\psi_{tt}(t)\|^2 + \|e(t)\|_2^2 + \int_{(0,1)\backslash E} \left(P'(\mathcal{A}) - \frac{j_0^2}{\mathcal{A}^2} \right) (\psi_x^2 + \psi_{xt}^2) dx \\ & \leq C(\|(\psi_0\|_4^2 + \|\eta_0\|_3^2) \exp\{-\beta_3 t\}, \end{aligned} \tag{2.33}$$

provided that $\delta_T + \delta_0$ is small enough. Here, $C, \beta_3 > 0$ are constants independent of t.

Proof: Step 1: estimates for ψ and ψ_t in L^2. We multiply (2.6) by ψ, integrate the resulting equation over $(0,1)$ and integrate by parts, taking into account the boundary conditions (2.10):

$$\begin{aligned}
&\frac{d}{dt}\left(\int_0^1\left[\frac{1}{2}\psi^2+\psi\psi_t\right]dx\right)-\int_0^1\psi_t^2dx+\frac{1}{2}\int_0^1(2w_0^2+3w_0\psi+\phi_{0xx}+\psi^2)\psi^2dx\\
&=-\int_0^1\left[\frac{1}{4}\varepsilon^2\psi_{xx}^2+\left(P'(w_0^2)-\frac{j_0^2}{w_0^4}\right)\psi_x^2\right]dx+\int_0^1\frac{\psi}{w_0}\left(\frac{j_0}{w_0^2}\eta\right)_{xx}dx\\
&\quad+\int_0^1 g_1\psi dx+\int_0^1 g_2\psi dx\\
&\triangleq I_4+I_5+I_6+I_7.
\end{aligned}\tag{2.34}$$

We estimate the integrals $I_4,\ldots,I_7$ term by term. ¿From (2.23) follows

$$\begin{aligned}
I_4=&-\int_0^1\left[\frac{1}{4}\varepsilon^2\psi_{xx}^2+\left(P'(\mathcal{A})-\frac{j_0^2}{\mathcal{A}^2}\right)\psi_x^2\right]dx-\int_0^1\left(P'(w_0)-P'(\mathcal{A})-\frac{j_0^2}{w_0^4}+\frac{j_0^2}{\mathcal{A}^2}\right)\psi_x^2dx\\
\le&-\frac{1}{4}\varepsilon^2\int_0^1\psi_{xx}^2dx-\min_{x\in[0,1]}\left(P'(\mathcal{A})-\frac{j_0^2}{\mathcal{A}^2}\right)\int_E\psi_x^2dx-\int_{(0,1)\setminus E}\left(P'(\mathcal{A})-\frac{j_0^2}{\mathcal{A}^2}\right)\psi_x^2dx\\
&-\int_0^1\left(P'(w_0^2)-P'(\mathcal{A})-\frac{j_0^2}{w_0^4}+\frac{j_0^2}{\mathcal{A}^2}\right)\psi_x^2dx\\
\le&-(b_0+A_0)\int_0^1\psi_{xx}^2dx-\int_{(0,1)\setminus E}\left(P'(\mathcal{A})-\frac{j_0^2}{\mathcal{A}^2}\right)\psi_x^2dx+C\delta_0\int_0^1\psi_{xx}^2dx,
\end{aligned}\tag{2.35}$$

where A_0 is given by (1.26) and $b_0=\frac{1}{4}(1-\kappa)\varepsilon^2$. Notice that $A_0>0$ by assumption (1.25).

Elementary computations, employing (2.15) and (2.16), lead to

$$\begin{aligned}
\left(\frac{j_0}{w_0^2}\eta\right)_{xx}=&-2\frac{j_0}{w_0^2}[(w_0+\psi)\psi_{xt}+(w_0+\psi)_x\psi_t]\\
&-4\left(\frac{j_0}{w_0^2}\right)_x(w_0+\psi)\psi_t+\eta\left(\frac{j_0}{w_0^2}\right)_{xx}.
\end{aligned}\tag{2.36}$$

With this identity, Cauchy's inequality, integration by parts, (2.18) and (2.23), we have

$$\begin{aligned}
|I_5|\le&C(\delta_T+\delta_0)\int_0^1(\psi^2+\psi_t^2+\eta^2)dx+\int_0^1\left|\frac{2j_0}{w_0^2}\psi_x\psi_t\right|dx\\
\le&C(\delta_T+\delta_0)\int_0^1(\psi_t^2+\psi^2+\psi_{xxx}^2)dx+a_0\int_0^1\psi_t^2dx+\frac{1}{4}b_0\int_0^1\psi_{xx}^2dx\\
&+C\exp\{-c_0t\}\int_0^1\eta_0^2dx,
\end{aligned}$$

where $a_0=4j_0^2/\min_{[0,1]}w_0^4b_0$. In view of

$$|g_1(x,t)|\le C(|\psi_{xx}|+|\psi|+|\psi_t|+|\psi_x|+|e_x|),\tag{2.37}$$

Cauchy's inequality, (2.17) and (2.23), we infer

$$|I_6| \le C(\delta_T+\delta_0)\int_0^1 [\psi_{xx}^2+\psi^2+\psi_t^2]dx.$$

By (2.16), (2.15), (2.18) and (2.23), we obtain, after a tedious calculation, that

$$|g_2(x,t)| \le C(\delta_T+\delta_0)(|\psi_{xx}|+|\psi_x|+|\psi|+|\psi_t|+|\eta|). \tag{2.38}$$

¿From the above estimate, (2.19) and Cauchy's inequality follows

$$|I_7| \le C(\delta_T+\delta_0)\int_0^1 [\psi^2+\psi_{xx}^2+\psi_{xxx}^2+\psi_t^2]dx + C\exp\{-c_0t\}\int_0^1 \eta_0^2dx.$$

Substituting the estimates for $I_4,\ldots,I_7$ into (2.34), we conclude

$$\begin{aligned}
&\frac{d}{dt}\left(\int_0^1\left[\frac12\psi^2+\psi_t\psi\right]dx\right)-(1+a_0)\int_0^1\psi_t^2dx\\
&+\frac12\int_0^1(2w_0^2+3w_0\psi+\phi_{0xx}+\psi^2)\psi^2dxdx\\
&+(A_0+\frac34 b_0)\int_0^1\psi_{xx}^2dx+\int_{(0,1)\setminus E}\left(P'(\mathcal{A})-\frac{j_0^2}{\mathcal{A}^2}\right)\psi_x^2dx\\
\le& C(\delta_T+\delta_0)\int_0^1(\psi^2+\psi_{xx}^2+\psi_{xxx}^2+\psi_t^2)dx+C\exp\{-c_0t\}\int_0^1\eta_0^2dx.
\end{aligned}\tag{2.39}$$

Multiply now (2.6) by ψ_t, integrate the resulting equation over $(0,1)$ and integrate by parts, noticing $\psi_t(0,t)=\psi_t(1,t)=0$:

$$\begin{aligned}
&\frac12\frac{d}{dt}\left(\int_0^1\left[\psi_t^2+\left(w_0^2+\frac32 w_0\psi+\frac12\phi_{0xx}+\frac14\psi^2\right)\psi^2\right]dx\right)\\
&+\frac12\frac{d}{dt}\left(\int_0^1\left[\frac14\varepsilon^2\psi_{xx}^2+\left(P'(w_0^2)-\frac{j_0^2}{w_0^4}\right)\psi_x^2\right]dx\right)+\int_0^1\psi_t^2dx\\
=&\int_0^1 w_0^{-1}\psi_t\left(\frac{j_0}{w_0^2}\eta\right)_{xx}dx+\int_0^1 g_1\psi_tdx+\int_0^1 g_2\psi_tdx\\
\triangleq& I_8+I_9+I_{10}.
\end{aligned}\tag{2.40}$$

Employing (2.36), integration by parts and (2.18), we estimate

$$\begin{aligned}
I_8 \le& -\int_0^1\frac{2j_0}{w_0^2}\psi_t\psi_{xt}dx+C(\delta_T+\delta_0)\int_0^1(\psi^2+\psi_t^2+\eta^2)dx\\
\le& C(\delta_T+\delta_0)\int_0^1(\psi_t^2+\psi^2+\psi_{xxx}^2)dx+C\exp\{-c_0t\}\int_0^1\eta_0^2dx.
\end{aligned}$$

In view of (2.37), (2.38), (2.17), (2.18), and (2.20), the integrals I_9 and I_{10} can be bounded as follows:

$$|I_9| \le C(\delta_T+\delta_0)\int_0^1(\psi_{xx}^2+\psi^2+\psi_t^2)dx,$$

$$|I_{10}| \leq C(\delta_T+\delta_0)\int_0^1 (\psi^2+\psi_{xx}^2+\psi_{xxx}^2+\psi_t^2)dx + C\exp\{-c_0 t\}\int_0^1 \eta_0^2 dx.$$

Substitution of the above three estimates into (2.40) yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\int_0^1 \left[\psi_t^2+\left(w_0^2+\frac{3}{2}w_0\psi+\frac{1}{2}\phi_{0xx}+\frac{1}{4}\psi^2\right)\psi^2\right]dx\right)\\
&+\frac{1}{2}\frac{d}{dt}\left(\int_0^1 \left[\frac{1}{4}\varepsilon^2\psi_{xx}^2+\left(P'(w_0^2)-\frac{j_0^2}{w_0^4}\right)\psi_x^2\right]dx\right)+\int_0^1 \psi_t^2 dx\\
\leq& C(\delta_T+\delta_0)\int_0^1 (\psi^2+\psi_{xx}^2+\psi_{xxx}^2+\psi_t^2)dx + C\exp\{-c_0 t\}\int_0^1 \eta_0^2 dx. \qquad (2.41)
\end{aligned}$$

We add (2.39) and (2.41), multiplied by $2(1+a_0)$, to obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\int_0^1 \left[\frac{1}{2}\psi^2+\psi_t\psi+(1+a_0)\psi_t^2\right]dx\right)\\
&+\frac{d}{dt}\left(\int_0^1 (1+a_0)\left[w_0^2+\frac{3}{2}w_0\psi+\frac{1}{2}\phi_{0xx}+\frac{1}{4}\psi^2\right]\psi^2 dx\right)\\
&+\frac{d}{dt}\left(\int_0^1 (1+a_0)\left[\frac{1}{4}\varepsilon^2\psi_{xx}^2+\left(P'(w_0^2)-\frac{j_0^2}{w_0^4}\right)\psi_x^2\right]dx\right)\\
&+\frac{1}{2}\int_0^1 \left[(2w_0^2+3w_0\psi+\phi_{0xx}+\psi^2)\psi^2+2(1+a_0)\psi_t^2\right]dx\\
&+\left(A_0+\frac{3}{4}b_0\right)\int_0^1 \psi_{xx}^2 dx+\int_{(0,1)\setminus E}\left(P'(\mathcal{A})-\frac{j_0^2}{\mathcal{A}^2}\right)\psi_x^2 dx\\
\leq& C(\delta_T+\delta_0)\int_0^1 (\psi^2+\psi_{xx}^2+\psi_{xxx}^2+\psi_t^2)dx + C\exp\{-c_0 t\}\int_0^1 \eta_0^2 dx. \qquad (2.42)
\end{aligned}$$

Step 2: estimates for ψ_{tt} in L^2. The starting point of the following estimates is (2.6), differentiated with respect to t:

$$\begin{aligned}
&\psi_{ttt}+\psi_{tt}+\frac{1}{4}\varepsilon^2\psi_{xxxxt}+\left(w_0^2+3w_0\psi+\frac{1}{2}\phi_{0xx}+\frac{3}{2}\psi^2\right)\psi_t\\
&\quad-\frac{1}{w_0}\left(\frac{j_0}{w_0^2}\eta_t\right)_{xx}-\left[\left(P'(w_0^2)-\frac{j_0^2}{w_0^4}\right)\psi_{xt}\right]_x = g_{1t}(x,t)+g_{2t}(x,t). \qquad (2.43)
\end{aligned}$$

This equation holds pointwise almost everywhere in $(0,1)\times(0,T)$ since $\psi \in C^0([0,T];H^6)\cap H^3(0,T;L^2)$ (see the proof of Theorem 1.1). We multiply (2.43) first by ψ_t, integrate the resulting equation over $(0,1)$ and integrate by parts, using the boundary conditions $\psi_t(0,t)=\psi_t(1,t)=\psi_{xt}(0,t)=\psi_{xt}(1,t)=0$ and (2.7):

$$\begin{aligned}
&\frac{d}{dt}\left(\int_0^1 \left[\frac{1}{2}\psi_t^2+\psi_t\psi_{tt}\right]dx\right)-\int_0^1 \psi_{tt}^2 dx+\int_0^1 \left[w_0^2+3w_0\psi+\frac{1}{2}\phi_{0xx}+\frac{3}{2}\psi^2\right]\psi_t^2 dx\\
=&-\int_0^1 \left[\frac{1}{4}\varepsilon^2\psi_{xxt}^2+\left(P'(w_0^2)-\frac{j_0^2}{w_0^4}\right)\psi_{xt}^2\right]dx+\int_0^1 \frac{1}{w_0}\psi_t\left(\frac{j_0}{w_0^2}\eta_t\right)_{xx}dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 g_{1t}\psi_t dx+\int_0^1 g_{2t}\psi_t dx\\
&\triangleq I_{12}+I_{13}+I_{14}+I_{15}.
\end{aligned}
\tag{2.44}
$$

Applying an argument similar to (2.35), it follows

$$
I_{12}\le -(A_0+b_0)\int_0^1 \psi_{xxt}^2dx-\int_{(0,1)\setminus E}\left(P'(\mathcal{A})-\frac{j_0^2}{\mathcal{A}^2}\right)\psi_{xt}^2dx+C\delta_0\int_0^1\psi_{xxt}^2dx,
$$

where we have used

$$
\int_E \psi_{xt}^2dx\le\int_0^1\psi_{xt}^2dx\le\int_0^1\psi_{xxt}^2dx,
\tag{2.45}
$$

based on the facts $\psi_{xt}(0,t)=\psi_{xt}(1,t)=0$. By (2.15), (2.20), and (2.45), we have, after integration by parts,

$$
\begin{aligned}
I_{13}=&-2\int_0^1\frac{j_0}{w_0^3}\psi_t((\psi+w_0)\psi_{xtt}+2\psi_t\psi_{xt}+(w_0+\psi)_x\psi_{tt})dx\\
&-4\int_0^1 w_0^{-1}\left(\frac{j_0}{w_0^2}\right)_x\psi_t(\psi_t^2+(w_0+\psi)\psi_{tt})dx+\int_0^1 w_0^{-1}\left(\frac{j_0}{w_0^2}\right)_{xx}\psi_t\eta_t dx\\
\le& C(\delta_T+\delta_0)\int_0^1(\psi_{tt}^2+\psi_{xt}^2+\psi_t^2+\eta_t^2)dx\\
&+2\int_0^1\left|\left(\frac{j_0(w_0+\psi)}{w_0^3}\right)_x\psi_t\psi_{tt}\right|dx+2\int_0^1\left|\frac{j_0}{w_0^3}(w_0+\psi)\psi_{xt}\psi_{tt}\right|dx\\
\le& C(\delta_T+\delta_0)\int_0^1(\psi_{tt}^2+\psi_t^2+\psi_{xxx}^2+\psi_{xxt}^2)dx\\
&+C\exp\{-c_0t\}\int_0^1\eta_0^2dx+a_0\int_0^1\psi_{tt}^2dx+\frac{1}{4}b_0\int_0^1\psi_{xxt}^2dx.
\end{aligned}
$$

Elementary computations yield the estimate

$$
|g_{1t}(x,t)|\le C(\delta_T+\delta_0)(|\psi_{xxt}|+|\psi_{tt}|+|\psi_t|+|\psi_{xt}|+|e_{xt}|),
\tag{2.46}
$$

which implies, in view of Cauchy's inequality, (2.21) and (2.45), that

$$
|I_{14}|\le C(\delta_T+\delta_0)\int_0^1(\psi_{xxt}^2+\psi_t^2+\psi_{tt}^2)dx.
$$

After a tedious computation, it follows from (2.16) that

$$
\begin{aligned}
|g_{2t}(x,t)|\le& C(\delta_T+\delta_0)(|\psi_{xxt}|+|\psi_{tt}|+|\psi_{xt}|+|\psi_t|+|\eta_t|)\\
&+\left(\frac{j_0+\eta}{(w_0+\psi)^3}-\frac{j_0}{w_0^3}\right)\eta_{xxt}\\
\le& C(\delta_T+\delta_0)(|\psi_{xxt}|+|\psi_{xx}|+|\psi_{tt}|+|\psi_{xt}|+|\psi_t|+|\eta_t|)\\
&-2(w_0+\psi)\left(\frac{j_0+\eta}{(w_0+\psi)^3}-\frac{j_0}{w_0^3}\right)\psi_{xtt},
\end{aligned}
\tag{2.47}
$$

where we have used the equation

$$4\psi_{xx}\psi_t + 2(w_0+\psi)\psi_{xtt} + 2(w_0+\psi)_x\psi_{tt} + \eta_{xxt} = 0. \tag{2.48}$$

Using (2.47), (2.18), (2.45), (2.24), and the fact $\psi_t(0,t) = \psi_t(1,t) = 0$, we can estimate I_{15}, after integration by parts, as follows:

$$\begin{aligned}
I_{15} \leq & (\delta_T+\delta_0)\int_0^1 (\psi_{xxt}^2 + \psi_{tt}^2 + \psi_t^2 + \psi^2 + \psi_{xx}^2 + \psi_{xxx}^2)dx \\
& + 2\int_0^1 (w_0+\psi)\left(\frac{j_0+\eta}{(w_0+\psi)^3} - \frac{j_0}{w_0^3}\right)\psi_{xt}\psi_{tt}dx \\
& + 2\int_0^1 \left((w_0+\psi)\left[\frac{j_0+\eta}{(w_0+\psi)^3} - \frac{j_0}{w_0^3}\right]\right)_x \psi_t\psi_{tt}dx \\
\leq & C(\delta_T+\delta_0)\int_0^1 (\psi_{xxt}^2 + \psi_{tt}^2 + \psi_t^2 + \psi^2 + \psi_{xx}^2 + \psi_{xxx}^2)dx.
\end{aligned}$$

Substituting the above estimates for $I_{12}, \ldots, I_{15}$ into (2.44), we conclude

$$\begin{aligned}
& \frac{d}{dt}\left(\int_0^1 \left[\frac{1}{2}\psi_t^2 + \psi_t\psi_{tt}\right]dx\right) - (1+a_0)\int_0^1 \psi_{tt}^2 dx \\
& + \int_0^1 \left[w_0^2 + 3w_0\psi + \frac{1}{2}\phi_{0xx} + \frac{3}{2}\psi^2\right]\psi_t^2 dx \\
& + \left(A_0 + \frac{3}{4}b_0\right)\int_0^1 \psi_{xxt}^2 dx + \int_{(0,1)\setminus E}\left(P'(\mathcal{A}) - \frac{j_0^2}{\mathcal{A}^2}\right)\psi_{xt}^2 dx \\
\leq & C(\delta_T+\delta_0)\int_0^1 (\psi_t^2 + \psi^2 + \psi_{tt}^2 + \psi_{xx}^2 + \psi_{xxt}^2 + \psi_{xxx}^2)dx \\
& + C\exp\{-c_0 t\}\int_0^1 \eta_0^2 dx.
\end{aligned} \tag{2.49}$$

The next step is to multiply (2.43) by ψ_{tt}, to integrate the resulting equation over $(0,1)$ and to integrate by parts, using $\psi_{tt}(0,t) = \psi_{tt}(1,t) = 0$, which yields

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\left(\int_0^1 \left[\psi_{tt}^2 + \left(w_0^2 + 3w_0\psi + \frac{1}{2}\phi_{0xx} + \frac{3}{2}\psi^2\right)\psi_t^2\right]dx\right) \\
& + \frac{1}{2}\frac{d}{dt}\left(\int_0^1 \left[\frac{1}{4}\varepsilon^2\psi_{xxt}^2 + \left(P'(w_0^2) - \frac{j_0^2}{w_0^4}\right)\psi_{xt}^2\right]dx\right) \\
& - \frac{3}{2}\int_0^1 (w_0+\psi)\psi_t^3 dx + \int_0^1 \psi_{tt}^2 dx \\
= & \int_0^1 w_0^{-1}\psi_{tt}\left(\frac{j_0}{w_0^2}\eta_t\right)_{xx} dx + \int_0^1 g_{1t}\psi_{tt}dx + \int_0^1 g_{2t}\psi_{tt}dx \\
\triangleq & I_{16} + I_{17} + I_{18}.
\end{aligned} \tag{2.50}$$

By (2.15), (2.48), (2.20), (2.23), (2.45), and integration by parts, we have

$$
\begin{aligned}
I_{16} =& -2\int_0^1 \frac{j_0}{w_0^3}(w_0+\psi)\psi_{tt}\psi_{xtt}dx - 2\int_0^1 \frac{j_0}{w_0^3}(2\psi_t\psi_{xt} + (w_0+\psi)_x\psi_{tt})dx \\
&- 4\int_0^1 \frac{1}{w_0}\left(\frac{j_0}{w_0^2}\right)_x \psi_{tt}(\psi_t^2 + (w_0+\psi)\psi_{tt})dx + \int_0^1 \frac{1}{w_0}\left(\frac{j_0}{w_0^2}\right)_{xx}\psi_{tt}\eta_t dx \\
\le& C(\delta_T+\delta_0)\int_0^1 (\psi_{tt}^2 + \psi_t^2 + \psi^2 + \psi_{xxt}^2 + \psi_{xxx}^2)dx + C\exp\{-c_0 t\}\int_0^1 \eta_0^2 dx,
\end{aligned}
$$

¿From (2.46), (2.17) and (2.45) follows

$$
|I_{17}| \le C(\delta_T+\delta_0)\int_0^1 [\psi_{xxt}^2 + \psi_t^2 + \psi_{tt}^2]dx.
$$

Finally, in view of (2.47), (2.18), (2.20) and integration by parts, it holds

$$
\begin{aligned}
I_{18} \le& (\delta_T+\delta_0)\int_0^1 (\psi_{xxt}^2 + \psi_{tt}^2 + \psi_t^2 + \psi^2 + \psi_{xxx}^2)dx \\
&- 2\int_0^1 (w_0+\psi)\left(\frac{j_0+\eta}{(w_0+\psi)^3} - \frac{j_0}{w_0^3}\right)\psi_{tt}\psi_{xtt}dx \\
\le& (\delta_T+\delta_0)\int_0^1 [\psi_{xxt}^2 + \psi_{tt}^2 + \psi_t^2 + \psi^2 + \psi_{xxx}^2]dx.
\end{aligned}
$$

Substituting the estimates for the integrals I_{16}, I_{17} and I_{18} into (2.50) gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\int_0^1 \left[\psi_{tt}^2 + \left(w_0^2 + 3w_0\psi + \frac{1}{2}\phi_{0xx} + \frac{3}{2}\psi^2\right)\psi_t^2\right]dx\right) \\
&+ \frac{1}{2}\frac{d}{dt}\left(\int_0^1 \left[\frac{1}{4}\varepsilon^2\psi_{xxt}^2 + \left(P'(w_0^2) - \frac{j_0^2}{w_0^4}\right)\psi_{xt}^2\right]dx\right) \\
&- \frac{3}{2}\int_0^1 (w_0+\psi)\psi_t^3 dx + \int_0^1 \psi_{tt}^2 dx \\
\le& C(\delta_T+\delta_0)\int_0^1 (\psi_{xxt}^2 + \psi_{tt}^2 + \psi_t^2 + \psi^2 + \psi_{xx}^2 + \psi_{xxx}^2)dx \\
&+ C\exp\{-c_0 t\}\int_0^1 \eta_0^2 dx. \qquad (2.51)
\end{aligned}
$$

Now we add the inequalities (2.49) and (2.51), multiplied by $2(1+a_0)$, to infer

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_0^1 \left[\frac{1}{2}\psi_t^2 + \psi_t\psi_{tt} + (1+a_0)\psi_{tt}^2\right]dx\right) \\
&+ (1+a_0)\frac{d}{dt}\left(\int_0^1 \left[w_0^2 + 3w_0\psi + \frac{1}{2}\phi_{0xx} + \frac{3}{2}\psi^2\right]\psi_t^2 dx\right) \\
&+ (1+a_0)\frac{d}{dt}\left(\int_0^1 \left[\frac{1}{4}\varepsilon^2\psi_{xxt}^2 + \left(P'(w_0^2) - \frac{j_0^2}{w_0^4}\right)\psi_{xt}^2\right]dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&+(1+a_0)\int_0^1 \psi_{tt}^2 dx+\int_0^1\left[w_0^2+3w_0\psi+\frac{1}{2}\phi_{0xx}+\frac{3}{2}\psi^2\right]\psi_t^2 dx\\
&+\left(A_0+\frac{3}{4}b_0\right)\int_0^1 \psi_{xxt}^2 dx+\int_{(0,1)\setminus E}\left(P'(\mathcal{A})-\frac{j_0^2}{\mathcal{A}^2}\right)\psi_{xt}^2 dx\\
\le& C(\delta_T+\delta_0)\int_0^1(\psi_t^2+\psi_{tt}^2+\psi_{xx}^2+\psi_{xxt}^2+\psi_{xxx}^2+\psi^2)dx\\
&+\exp\{-c_0 t\}\int_0^1 \eta_0^2 dx. \qquad (2.52)
\end{aligned}
$$

Step 3: combination of the estimates for ψ, ψ_t and ψ_{tt}. We combine the estimates (2.42) and (2.52) and obtain for some constant $\beta_1>0$, using (2.21),

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_0^1\left[\frac{1}{2}(\psi^2+\psi_t^2)+\psi_t(\psi+\psi_{tt})+(1+a_0)(\psi_t^2+\psi_{tt}^2)\right]dx\right)\\
&+(1+a_0)\frac{d}{dt}\left(\int_0^1\left[w_0^2+\frac{3}{2}w_0\psi+\frac{1}{2}\phi_{0xx}+\frac{1}{4}\psi^2\right]\psi^2 dx\right)\\
&+(1+a_0)\frac{d}{dt}\left(\int_0^1\left[w_0^2+3w_0\psi+\frac{1}{2}\phi_{0xx}+\frac{3}{2}\psi^2\right]\psi_t^2 dx\right)\\
&+(1+a_0)\frac{d}{dt}\left(\int_0^1\left[\frac{1}{4}\varepsilon^2(\psi_{xx}^2+\psi_{xxt}^2)+\left(P'(w_0^2)-\frac{j_0^2}{w_0^4}\right)(\psi_x^2+\psi_{xt}^2)\right]dx\right)\\
&+\beta_1\int_0^1[\psi^2+\psi_t^2+\psi_t^2+\psi_{tt}^2+\psi_{xxt}^2+\psi_{xx}^2]dx\\
&+\beta_1\int_{(0,1)\setminus E}\left(P'(\mathcal{A})-\frac{j_0^2}{\mathcal{A}^2}\right)(\psi_x^2+\psi_{xt}^2)dx\\
\le& C\exp\{-c_0 t\}\int_0^1 \eta_0^2 dx, \qquad (2.53)
\end{aligned}
$$

provided that $\delta_T+\delta_0$ is small enough.

There exist constants $\beta_2,\beta_3>0$ such that

$$
\begin{aligned}
&\beta_2\int_0^1\left[\psi^2+\psi_t^2+\psi_{xt}^2+\psi_{tt}^2+\psi_{xxt}^2+\psi_{xx}^2\right]dx\\
&+\beta_2\int_{(0,1)\setminus E}\left(P'(\mathcal{A})-\frac{j_0^2}{\mathcal{A}^2}\right)(\psi_x^2+\psi_{xt}^2)dx\\
\le&\int_0^1\left[\frac{1}{2}(\psi^2+\psi_t^2)+\psi_t(\psi+\psi_{tt})+(1+a_0)(\psi_t^2+\psi_{tt}^2)\right]dx\\
&+(1+a_0)\int_0^1\left[w_0^2+\frac{3}{2}w_0\psi+\frac{1}{2}\phi_{0xx}+\frac{1}{4}\psi^2\right]\psi^2 dx\\
&+(1+a_0)\int_0^1\left[w_0^2+3w_0\psi+\frac{1}{2}\phi_{0xx}+\frac{3}{2}\psi^2\right]\psi_t^2 dx\\
&+(1+a_0)\int_0^1\left[\frac{1}{4}\varepsilon^2(\psi_{xx}^2+\psi_{xxt}^2)+\left(P'(w_0^2)-\frac{j_0^2}{w_0^4}\right)(\psi_x^2+\psi_{xt}^2)\right]dx
\end{aligned}
$$

$$
\begin{aligned}
&\leq \beta_3^{-1}\beta_1 \int_0^1 \left[\psi^2 + \psi_t^2 + \psi_{xt}^2 + \psi_{tt}^2 + \psi_{xxt}^2 + \psi_{xx}^2\right] dx \\
&\quad + \beta_3^{-1}\beta_1 \int_{(0,1)\setminus E} \left(P'(\mathcal{A}) - \frac{j_0^2}{\mathcal{A}^2}\right)(\psi_x^2 + \psi_{xt}^2)dx,
\end{aligned}
$$

Thus, applying Gronwall's inequality to (2.53), we obtain finally

$$
\begin{aligned}
&\int_0^1 \left[\psi^2 + \psi_t^2 + \psi_{xt}^2 + \psi_{tt}^2 + \psi_{xxt}^2 + \psi_{xx}^2\right] dx + \int_{(0,1)\setminus E} \left(P'(\mathcal{A}) - \frac{j_0^2}{\mathcal{A}^2}\right)(\psi_x^2 + \psi_{xt}^2)dx \\
\leq & C(\|\psi_0\|_4^2 + \|\eta_0\|_3^2)\exp\{-\beta_3 t\},
\end{aligned}
\tag{2.54}
$$

provided that $\delta_T + \delta_0$ is small enough.

The combination of (2.54), (2.17), (2.21) and (2.7) gives the assertion (2.33). Thus, the lemma is proved. □

We also obtain bounds for higher-order estimates for ψ.

Lemma 2.3 *It holds for $(\psi, \eta, e) \in X(T)$ and $0 < t < T$*

$$
\begin{aligned}
&\|\partial_x^4\psi(t)\|_2^2 + \|\psi_{xtt}(t)\|_1^2 + \|\psi_{ttt}(t)\|^2 + \|e(t)\|_4^2 + \int_{(0,1)\setminus E} \left(P'(\mathcal{A}) - \frac{j_0^2}{\mathcal{A}^2}\right)\psi_{xtt}^2 dx \\
\leq & C(\|(\psi_0\|_6^2 + \|\eta_0\|_5^2)\exp\{-\beta_4 t\},
\end{aligned}
\tag{2.55}
$$

provided that $\delta_T + \delta_0$ is small enough. Here, $C, \beta_4 > 0$ are constants independent of t.

Proof: For the proof of the lemma take the time derivative of (2.43) and estimate similarly as in Lemmas 2.1 and 2.2. As the estimates are analogous to those of the proofs of Lemmas 2.1–2.2, we omit the details. □

Proof of Theorem 1.4. By Theorem 1.1, there exists a solution (w, j, ϕ) of the IBVP (1.14)–(1.19) for $t \in [0, T_*]$. With the help of Lemmas 2.1–2.3, we infer that the local solution (w, j, ϕ) of the IBVP (1.14)–(1.19) satisfies, for $t \in [0, T_*]$,

$$
\|(w - w_0, j - j_0, \phi - \phi_0)(t)\|_{H^6\times H^5\times H^4}^2 \leq C(\|\psi_0\|_6^2 + \|\eta_0\|_5^2)\exp\{-\Lambda_0 t\},
\tag{2.56}
$$

where $C, \Lambda_0 > 0$ are constants independent of t. Choosing the initial data $\|\psi_0\|_6 + \|\eta_0\|_5$ so small that

$$
C(\|\psi_0\|_6^2 + \|\eta_0\|_5^2) < \delta_{T_*}
$$

we conclude first, by the Sobolv embedding theorem and (2.56), that $w > 0$ in $[0,1]\times[0,T_*]$, and second, by the usual continuity argument, that (w, j, ϕ) exists globally in time and satisfies (1.28). □

3 Proof of Theorem 1.1

The idea of the proof of Theorem 1.1 is to linearize the equations (1.14)–(1.16) around the initial state and to construct a sequence of approximate solutions of the linearized problem converging to a solution of the original problem. First we need to study the regularity properties of a certain semilinear fourth-order wave equation.

3.1 A semilinear fourth-order wave equation

Consider the two Hilbert spaces H_0^2 and L^2 on $(0,1)$, endowed with the scalar products $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)$ and corresponding norms $|\cdot|_{H_0^2} = |\cdot|_2$ and $\|\cdot\|$, respectively. Furthermore, we consider the following initial-value problem on L^2:

$$u'' + \frac{1}{\tau}u' + \nu Au + u + \mathcal{L}u' = F(t), \quad t > 0, \tag{3.1}$$

$$u(0) = u_0, \quad u'(0) = u_1, \tag{3.2}$$

where the primes denote derivatives with respect to time, $\tau, \nu > 0$ are constants, $A = \partial_x^4$ is an operator defined on

$$D(A) = H_0^2 \cap H^4 = \{u \in H^4; u|_{x=0,1} = u_x|_{x=0,1} = 0\}, \tag{3.3}$$

and the operators $\mathcal{L}$ and F are given by

$$\begin{aligned} \langle \mathcal{L}u, v\rangle &= \int_0^1 b(x,t)u_x v dx, \quad u, v \in H_0^2, \\ (F(t), v) &= \int_0^1 f(x,t)v dx, \quad v \in L^2, \end{aligned}$$

where $b, f : [0,1] \times [0,T] \to \mathbb{R}$ are measurable functions.

Related to the operator A, we introduce the coercive, continuous, symmetric bilinear form $a(u,v)$

$$a(u,v) = \nu \int_0^1 u_{xx}v_{xx}dx \quad \forall\, u, v \in H_0^2.$$

There exist a complete orthonormal family of eigenvectors $\{r_i\}_{i\in\mathbb{N}}$ of L^2 and a family of eigenvalues $\{\mu_i\}_{i\in\mathbb{N}}$ such that $0 < \mu_1 \le \mu_2 \le \cdots$ and $\mu_i \to \infty$ as $i \to \infty$. The family $\{r_i\}_{i\in N}$ is also orthogonal for $a(u,v)$ on H_0^2, i.e.

$$(r_i, r_j) = \delta_{ij}, \quad a(r_i, r_j) = \nu\langle Ar_i, r_j\rangle = \nu\delta_{ij} \quad \forall\, i, j.$$

Using the Faedo-Galerkin method [25, 27], it is possible to prove the existence of solutions of (3.1)-(3.2). The result is summarized in the following theorem.

Theorem 3.1 *Let $T_0 > 0$ and assume that*

$$F \in H^1(0,T_0; L^2), \quad b \in C^1([0,T_0]; H^2) \cap W^{2,\infty}(0,T_0; H^1). \tag{3.4}$$

Then, if $u_0 \in H^4 \cap H_0^2$ *and* $u_1 \in H_0^2$*, there exists a solution of* (3.1)–(3.2) *satisfying*

$$u \in C([0,T_0]; H^4 \cap H_0^2) \cap C^1([0,T_0]; H_0^2) \cap C^2([0,T_0]; L^2). \tag{3.5}$$

Moreover, assume additionally that

$$F \in H^2(0,T_0; L^2) \cap C([0,T_0]; H^2).$$

Then, if $u_0 \in H^6 \cap H_0^2$ *and* $u_1 \in H^4 \cap H_0^2$ *satisfy* $\nu A u_0 + \mathcal{L}(u_1) - F(0) \in H_0^2$*, it holds*

$$u \in C^i([0,T_0]; H^{6-2i} \cap H_0^2) \cap C^3([0,T_0]; L^2), \quad i = 0,1,2. \tag{3.6}$$

Proof: The existence of solutions of (3.1)–(3.2) and the regularity property (3.5) can be shown by applying the Faedo-Galerkin method as in [18]. The regularity property (3.6) follows from (3.5) by considering the problem for the new variable $v = u'$. As the proof is standard, we omit the details. □

3.2 Local existence

In this section we prove Theorem 1.1. We linearize the equations (1.14)–(1.16) around the initial state (w_1, j_1, ϕ_1) where ϕ_1 solves the Poisson equation (1.16) and (1.19) with w replaced by w_1, and prove the local-in-time existence for the perturbation $(\psi, \eta, e) = (w - w_1, j - j_1, \phi - \phi_1)$. For this, we reformulate the original initial-boundary value problem (1.14)–(1.19). It is sufficient to carry out the reformulation for the equations (1.14), (1.16) and (2.1) because of (2.15). For given $U^p = (\psi_p, \eta_p, e_p)$ we obtain the following linearized problems for $U^{p+1} = (\psi_{p+1}, \eta_{p+1}, e_{p+1})$, $p \in \mathbb{N}$, writing "∂_x" for the spatial derivative and "$'$" for the time derivative:

$$\begin{cases} \eta'_{p+1} + \eta_{p+1} = g_3(x, U^p), \\ \eta_{p+1}(x,0) = 0, \end{cases} \tag{3.7}$$

$$\begin{cases} \psi''_{p+1} + \dfrac{1}{\tau}\psi'_{p+1} + \nu \partial_x^4 \psi_{p+1} + \psi_{p+1} + k(x, U^p)\partial_x \psi'_{p+1} = g_4(x, U^p), \\ \psi_{p+1}(x,0) = 0, \quad \psi'_{p+1}(x,0) = \theta_1(x) := -\dfrac{\partial_x j_1}{2w_1}, \\ \psi_{p+1}(0,t) = \psi_{p+1}(1,t) = \partial_x \psi_{p+1}(0,t) = \partial_x \psi_{p+1}(1,t) = 0, \end{cases} \tag{3.8}$$

$$\begin{cases} \partial_x^2 e_{p+1} = (2w_1 + \psi_p)\psi_p, \\ e_{p+1}(0,t) = e_{p+1}(1,t) = 0, \end{cases} \tag{3.9}$$

where $\nu = \frac{1}{4}\varepsilon^2$ and

$$g_3(x, U^p) = \frac{4(j_1 + \eta_p)}{w_1 + \psi_p}\psi'_p - (j_1 + \eta_p)^2 \left[\frac{1}{(w_1 + \psi_p)^2}\right]_x - P((w_1 + \psi_p)^2)_x - j_1$$

$$
\begin{aligned}
&+(w_1+\psi_p)^2(\phi_1+e)_x+\frac{1}{2}\varepsilon^2(w_1+\psi_p)^2\left[\frac{(w_1+\psi_p)_{xx}}{w_1+\psi_p}\right]_x,\\
k(x,U^p)=&\frac{2(j_1+\eta_p)}{(w_1+\psi_p)^2},\\
g_4(x,U^p)=&-\frac{1}{2(w_1+\psi_p)}((w_1+\psi_p)^2(\phi_1+e_p)_x)_x+\frac{1}{2(w_1+\psi_p)}P((w_1+\psi_p)^2)_{xx}\\
&-\frac{1}{4}\varepsilon^2 w_{1xxxx}+\frac{1}{4}\varepsilon^2\frac{(w_1+\psi_p)_{xx}^2}{w_1+\psi_p}-\psi_p+\frac{3(\psi_p')^2}{w_1+\psi_p}\\
&+\frac{(j_1+\eta_p)^2}{2(w_1+\psi_p)}\left[\frac{1}{(w_1+\psi_p)^2}\right]_{xx}-3\left[\frac{1}{(w_1+\psi_p)^2}\right]_x(j_1+\eta_p)\psi_p'. \qquad (3.10)
\end{aligned}
$$

We apply an induction argument to prove the existence of solutions of (3.7)–(3.9).

Lemma 3.2 *Under the assumptions of Theorem 1.1, i.e.,* $P\in C^4(0,\infty)$, $\mathcal{C}\in H^2$, $(w_1,j_1)\in H^6\times H^5$ *with* $w_1>0$ *in* $(0,1)$ *and, for some* $\alpha\in[(1+2\sqrt{2}\varepsilon)^{-1},1)$,

$$
\|v_1\|_{C^1([0,1])}<\frac{(1-\alpha)w_*}{8\sqrt{2}\|w_1\|_1} \qquad (3.11)
$$

with

$$
w_*=\min_{x\in[0,1]}w_1(x), \qquad (3.12)
$$

there exists a sequence $\{U^i\}_{i=1}^\infty$ *of solutions of* (3.7)–(3.9) *in the time interval* $t\in[0,T_*]$ *for some* $T_*>0$ *which is independent of* i, *satisfying the regularity properties*

$$
\begin{cases}
\eta_i\in C^1([0,T_*];H^3)\cap C^2([0,T_*];H^1), \quad e_i\in C^1([0,T_*];H^4\cap H_0^1),\\
\psi_i\in C^l([0,T_*];H^{6-2l}\cap H_0^2)\cap C^3([0,T_*];L^2), \quad l=0,1,2,\ i\in\mathbb{N},
\end{cases} \qquad (3.13)
$$

and the uniform bounds

$$
\begin{cases}
\|\eta_i'(t)\|_3^2+\|\eta_i''(t)\|_1^2+\|(e_i,e_i')(t)\|_4^2\le M_0,\\
\|(\psi_i,\psi_i',\psi_i'',\psi_i''')(t)\|^2_{H^6\times H^4\times H^2\times L^2}\le M_0, \quad i\ge 1,\ t\in[0,T_*],\\
\|\eta_i(t)\|_3^2\le 1, \quad \|\partial_x^2\psi_i(t)\|^2\le\alpha^2 w_*^2,
\end{cases} \qquad (3.14)
$$

where $M_0>0$ *is a constant independent of* U^i $(i\ge 1)$ *and* T_*.

Proof: Step 1: solution of (3.7)–(3.9) for $p\ge 1$. Obviously, $U^1=(0,0,0)$ satisfies (3.13)–(3.14). Starting with $U^1=(0,0,0)$, we prove the existence of a solution $U^2=(\psi_2,\eta_2,e_2)$ of (3.7)–(3.9) satisfying (3.13)–(3.14). The functions $g_3(x,U^1)$, $g_4(x,U^1)$ and $k(x,U^1)$ only depend on the initial state (w_1,j_1,ϕ_1) and satisfy

$$
\begin{aligned}
g_3(x,U^1)=:\tilde g_3(x)\in H^3,\quad g_4(x,U^1)=:\tilde g_4(x)\in H^2,\quad k(x,U^1)=:\tilde k(x)\in H^3,\\
\partial_t g_3=\partial_t g_4=\partial_t k=0,\quad \|\tilde g_3\|_3^2+\|\tilde g_4\|_2^2+\|\tilde k\|_3^2\le a_0(I_0+1), \qquad (3.15)
\end{aligned}
$$

where $a_0 > 0$ is some constant and

$$I_0 = \|(w_1 - \sqrt{\mathcal{C}})\|^2 + \|w_{1x}\|_5^2 + \|j_1\|_5^2. \tag{3.16}$$

The existence of a solution $U^2 = (\psi_2, \eta_2, e_2)$ of the linear system (3.7)–(3.9) follows from the theory of ordinary differential equations, applied to (3.7), Theorem 3.1 with $f(x,t) = \tilde{g}_4(x)$ and $b(x,t) = \tilde{k}(x)$, applied to (3.8), and elliptic theory, applied to (3.9). The solution U^2 exists on any time interval $[0,T]$, $T > 0$, and satisfies (3.13) with $T_* = T$ and the first two inequalities of (3.14) with $i = 2$.

We show in the following that U^2 satisfies the last two inequalities of (3.14) for $t \in [0, T_1]$, where $T_1 > 0$ is given by

$$T_1 = \min\left\{ \frac{\ln 2}{2 + a_0(I_0+1)}, \ \frac{\nu\alpha^2 w_*^2 - 4\|v_1\|^2_{C^1([0,1])}\|w_1\|_1^2}{2a_0(I_0+1)}, \ \frac{1}{a_0(I_0+1)} \right\}. \tag{3.17}$$

We recall that $a_0 > 0$ is a constant and I_0 and w_* are given by (3.16) and (3.12), respectively. It holds

$$\nu\alpha^2 w_*^2 - 4\|v_1\|^2_{C^1([0,1])}\|w_1\|_1^2 > 0, \tag{3.18}$$

since (3.11) implies

$$4\|v_1\|^2_{C^1([0,1])}\|w_1\|_1^2 < \frac{(1-\alpha)^2 w_*^2}{32} \le \nu\alpha^2 w_*^2.$$

¿From (3.7) we obtain by integrating

$$\eta_2(t) = \tilde{g}_3(x) \int_0^t \exp\{-(t-s)\} ds, \qquad t \in [0, T_1],$$

and hence, in view of (3.17),

$$\|\eta_2(t)\|_3^2 \le T_1^2 \|\tilde{g}_3\|_3^2 \le 1, \quad t \in [0, T_1]. \tag{3.19}$$

Multiplying the differential equation in (3.8) by ψ_2', integrating the resulting equation over $(0,1) \times (0,t)$ for $t \in [0, T_1]$ and integrating by parts gives

$$\begin{aligned} \|\psi_{2xx}(t)\|^2 \le& \frac{1}{\nu} \left(\|\theta_1\|^2 + a_0 T_1 (I_0+1)\right) e^{T_1(2+a_0(I_0+1))} \\ \le& \frac{2}{\nu} \left(2\|v_1\|^2_{C^1([0,1])}\|w_1\|_1^2 + a_0 T_1(I_0+1)\right) \\ \le& \alpha^2 w_*^2, \end{aligned} \tag{3.20}$$

where we have used

$$\|\theta_1\|^2 = \left\| w_{1x} v_1 + \frac{1}{2} w_1 v_{1x} \right\|^2 \le 2\|v_1\|^2_{C^1([0,1])}\|w_1\|_1^2. \tag{3.21}$$

This proves the last two bounds in (3.14). Moreover, by the Sobolev embedding theorem, it follows from (3.20) that

$$\|\psi_2(t)\|_1^2 \le 2\alpha^2 w_*^2, \quad t \in [0, T_1]. \tag{3.22}$$

Now, assume that there exist solutions $\{U^i\}_{i=1}^p$ $(p \ge 2)$ of (3.7)–(3.9) on the time interval $[0, T_1]$ where T_1 is given by (3.17), satisfying (3.13)–(3.14). As above we obtain, for given U^p, the existence of a solution $U^{p+1} = (\psi_{p+1}, \eta_{p+1}, e_{p+1})$ of (3.7)–(3.9) in the interval $[0, T_1]$, satisfying

$$\eta_{p+1} \in C^1([0,T_1]; H^3) \cap C^2([0,T_1]; H^1), \quad e_{p+1} \in C^1([0,T_1]; H^4 \cap H_0^1),$$
$$\psi_{p+1} \in C^l([0,T_1]; H^{6-2l} \cap H_0^2) \cap C^3([0,T_1]; L^2), \quad l = 0,1,2.$$

We prove that there exist constants $T_* \in (0, T_1]$ and $K_i > a_0$ $(i = 1,2,3,5,6,7)$ independent of $\{U^i\}_{i=1}^p$, such that if U^p satisfies on $[0, T_*]$

$$\|\partial_x^2 \psi_p(t)\|^2 \le \alpha^2 w_*^2, \tag{3.23}$$
$$\|(\psi_p'', \psi_p')(t)\|_2^2 + \|\psi_p'''(t)\|^2 \le K_0, \tag{3.24}$$
$$\|\partial_x^3 \psi_p'(t)\|_1^2 \le K_1, \quad \|\partial_x^3 \psi_p(t)\|_1^2 \le K_2, \quad \|\partial_x^5 \psi_p(t)\|_1^2 \le K_3, \tag{3.25}$$
$$\|\eta_p(t)\|_3^2 \le 1, \quad \|\eta_p'(t)\|^2 \le K_5, \quad \|\partial_x \eta_p'(t)\|_2^2 \le K_6, \quad \|\eta_p''(t)\|_1^2 \le K_7, \tag{3.26}$$

then U^{p+1} also satisfies on $[0, T_*]$

$$\|\partial_x^2 \psi_{p+1}(t)\|_2^2 \le \alpha^2 w_*^2, \tag{3.27}$$
$$\|(\psi_{p+1}'', \psi_{p+1}')(t)\|_2^2 + \|\psi_{p+1}'''(t)\|^2 \le K_0, \tag{3.28}$$
$$\|\partial_x^3 \psi_{p+1}'(t)\|_1^2 \le K_1, \quad \|\partial_x^3 \psi_{p+1}(t)\|_1^2 \le K_2, \quad \|\partial_x^5 \psi_{p+1}(t)\|_1^2 \le K_3, \tag{3.29}$$
$$\|\eta_{p+1}(t)\|_3^2 \le 1, \quad \|\eta_{p+1}'(t)\|^2 \le K_5, \quad \|\partial_x \eta_{p+1}'(t)\|_2^2 \le K_6, \quad \|\eta_{p+1}''(t)\|_1^2 \le K_7. \tag{3.30}$$

Notice that it follows from (3.23) and (3.27), employing the boundary conditions in (3.8) and Poincaré's inequality,

$$\|\psi_p(t)\|_1^2 \le 2\alpha^2 w_*^2, \quad \|\psi_{p+1}(t)\|_1^2 \le 2\alpha^2 w_*^2, \quad t \in [0, T_*]. \tag{3.31}$$

Step 2: estimates for g_3, g_4, and k. Let U^p satisfy (3.23)–(3.26). Then a direct computation shows the following estimates for $g_3(x, U^p)$ and $g_4(x, U^p)$, for $t \in [0, T_*]$,

$$\begin{aligned}
\|g_3(\cdot, U^p)(t)\|_1^2 \le& N(I_0 + 1 + \|\psi_p(t)\|_4^2 + \|\psi_p'(t)\|_1^2)^5 \\
\le& N(I_0 + 1 + K_0 + K_2)^5, &\quad (3.32)\\
\|g_{3x}(\cdot, U^p)(t)\|_2^2 \le& N(I_0 + 1 + \|\psi_p(t)\|_6^2 + \|\psi_p'(t)\|_3^2)^7 \\
\le& N(I_0 + 1 + K_0 + K_1 + K_2 + K_3)^7, &\quad (3.33)\\
\|g_3'(\cdot, U^p)(t)\|_1^2 \le& N(I_0 + 1 + \|\eta_p'(t)\|_1^2 + \|\psi_p'(t)\|_4 + \|\psi_p(t)\|_4^2 + \|\psi_p''(t)\|_1^2)^6
\end{aligned}$$

$$
\begin{aligned}
&\leq N(I_0+1+K_0+K_1+K_2+K_3+K_5+K_6)^6, && (3.34)\\
\|g_4(\cdot,U^p)(t)\|^2 &\leq N(I_0+1+\|\psi_p(t)\|_4^2+\|\psi_p'(t)\|_1^2)^3\\
&\quad+16\nu^2 a_1\|\partial_x^2\psi_p(t)\|^2\|\partial_x^4\psi_p(t)\|^2 && (3.35)\\
&\leq N(I_0+1+K_0+K_2)^3, && (3.36)\\
\|g_{4x}(\cdot,U^p)(t)\|_1^2 &\leq N(I_0+1+\|\psi_p(t)\|_4^2+\|\psi_p'(t)\|_2^2)^5\\
&\leq N(I_0+1+K_0+K_2)^5, && (3.37)\\
\|g_4'(\cdot,U^p)(t)\|^2 &\leq N(I_0+1+\|\eta_p'(t)\|^2+\|\psi_p(t)\|_3^2+\|\psi_p'(t)\|_2^2+\|\psi_p''(t)\|^2)^4\\
&\leq N(I_0+1+K_0+K_2+K_5)^4, && (3.38)\\
\|g_4''(\cdot,U^p)(t)\|^2 &\leq N(I_0+1+\|(\eta_p',\eta_p'',\psi_p''')(t)\|^2+\|(\psi_p,\psi_p')(t)\|_3^2+\|\psi_p''(t)\|_2^2)^5\\
&\leq N(I_0+1+K_0+K_1+K_2+K_5+K_7)^5, && (3.39)
\end{aligned}
$$

where

$$a_1=\max_{x\in[0,1]}(w_1+\psi_p)^{-2}=(1-\alpha)^{-2}w_*^{-2}, \tag{3.40}$$

and estimates for $k(x,U^p)$ and $e_p(x,t)$, for $t\in[0,T_*]$,

$$
\begin{aligned}
\|k(\cdot,U^p)(t)\|_2^2 &\leq N(I_0+1+\|\psi_p(t)\|_2^2)^3,\\
&\leq N(I_0+1)^3, && (3.41)\\
\|k'(\cdot,U^p)(t)\|^2 &\leq N(I_0+1)(\|\eta_p'(t)\|^2+\|\psi_p'(t)\|^2)\\
&\leq N(I_0+1+K_0+K_5)^2, && (3.42)\\
\|k''(\cdot,U^p)(t)\|^2 &\leq N(I_0+1)(\|(\eta_p'',\eta_p',\psi_p'')(t)\|^2+\|\psi_p'(t)\|_1^2)^2\\
&\leq N(I_0+1+K_0+K_5+K_7)^3, && (3.43)\\
\|e_p\|_4^2+\|(e_p',e_p'')\|_2^2 &\leq N(\|\psi_p(t)\|_4^2+\|(\psi_p',\psi_p'')(t)\|_2^2)\\
&\leq N(I_0+1+K_0+K_2), && (3.44)
\end{aligned}
$$

where $N>1$ is a constant independent of K_i $(i=1,2,3,5,6,7)$.

Step 3: estimates for η_{p+1}. Integration of (3.7) yields

$$\eta_{p+1}(x,t)=\int_0^t \exp\{-(t-s)\}g_3(x,U^1)(s)ds,\quad 0\leq t\leq T_*\leq T_1,\quad x\in[0,1], \tag{3.45}$$

and

$$\eta_{p+1}\in C^1([0,T_1];H^3)\cap C^2([0,T_*];H^1). \tag{3.46}$$

¿From (3.32)–(3.34) we obtain the estimates

$$
\begin{aligned}
\|\eta_{p+1}(t)\|^2 &\leq T_*^2\|g_3(\cdot,U^p)\|^2\leq T_*^2N(I_0+1+K_0+K_2)^5,\\
\|\partial_x\eta_{p+1}(t)\|_2^2 &\leq T_*^2\|g_{3x}(\cdot,U^p)\|_2^2\leq T_*^2N(I_0+1+K_0+K_1+K_2+K_3)^7,\\
\|\eta_{p+1}'(t)\|^2 &\leq 2(T_1^2+1)\|g_3(\cdot,U^p)\|^2\leq 2N(T_1^2+1)(I_0+1+K_0+K_2)^5,\\
\|\partial_x\eta_{p+1}'(t)\|_2^2 &\leq 2(T_1^2+1)\|g_{3x}(\cdot,U^p)\|_2^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq 2N(T_1^2+1)(I_0+1+K_0+K_1+K_2+K_3)^7,\\
\|\eta''_{p+1}(t)\|_1 &\leq 4(T_1^2+1)\|g_3(\cdot,U^p)\|_1^2+2\|g_3'(\cdot,U^p)\|_1^2\\
&\leq 4N(T_1^2+1)(I_0+1+K_0+K_1+K_2+K_3)^7\\
&\quad +2N(I_0+1+K_0+K_1+K_2+K_3+K_5+K_6)^6.
\end{aligned}
$$

Thus, η_{p+1} satisfies (3.30) if

$$
\begin{aligned}
K_5 &= 2N(T_1^2+1)(I_0+1+K_0+K_2)^5, &(3.47)\\
K_6 &= 2N(T_1^2+1)(I_0+1+K_0+K_1+K_2+K_3)^7, &(3.48)\\
K_7 &= 2N(T_1^2+1)(I_0+1+K_0+K_1+K_2+K_3)^7,\\
&\quad +2N(I_0+1+K_0+K_1+K_2+K_3+K_5+K_6)^6 &(3.49)
\end{aligned}
$$

and if T_* satisfies

$$
T_* \leq \frac{1}{\sqrt{L_1}}, \tag{3.50}
$$

where

$$
L_1 = \min\left\{2N(I_0+1+K_0+K_2)^5,\ 2N(I_0+1+K_0+K_1+K_2+K_3)^7\right\}. \tag{3.51}
$$

Step 4: estimates for ψ_{p+1}. We multiply the differential equation in (3.8) by ψ'_{p+1}, ψ''_{p+1} and ψ'''_{p+1}, respectively, integrate the sum of the resulting equations over $(0,1)\times(0,T_*)$ and integrate by parts. In view of (3.35), (3.36), (3.38), (3.39), (3.41), (3.42), and (3.43), we obtain after tedious computations

$$
\begin{aligned}
\|\partial_x^2\psi_{p+1}(t)\|^2 &\leq \frac{1}{\nu}\left(\|\theta_1\|_2^2+T_*\|g_4(\cdot,U^p)\|^2\right)e^{T_*(2+N(I_0+1)^2)}\\
&\leq \frac{1}{\nu}\left(2\|v_1\|^2_{C^1([0,1])}\|w_1\|_1^2+T_*L_3\right)e^{T_*(2+N(I_0+1)^2)} &(3.52)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\psi'''_{p+1}(t)\|^2+\|\psi''_2(t)\|^2+\nu\|\partial_x^2\psi''_{p+1}(t)\|^2+\nu\|\partial_x^2\psi'_{p+1}(t)\|^2+\|\psi'_{p+1}\|^2\\
&\leq \left(\|\psi'''_{p+1}(0)\|^2+2\|\psi''_2(0)\|+\nu\|\partial_x^2\psi''_{p+1}(0)\|^2+\|\theta_1\|^2+\nu\|\partial_x^2\theta_1\|^2\right)e^{T_*L_4}\\
&\quad +T_*(\|g_4''(\cdot,U^p)\|^2+\|g_4'(\cdot,U^p)\|^2)e^{T_*L_4}\\
&\leq N\left(I_0+T_*L_5\right)e^{T_*L_4}, &(3.53)
\end{aligned}
$$

where

$$
\begin{aligned}
L_3 &= N(I_0+1+K_0+K_2)^3, &(3.54)\\
L_4 &= 8+6N(I_0+1+K_0+K_5+K_7)^3 > 2+N(I_0+1)^2, &(3.55)\\
L_5 &= 2N(I_0+1+K_0+K_1+K_2+K_5+K_7)^5. &(3.56)
\end{aligned}
$$

Define

$$K_0 = 20NI_0 \cdot \frac{1}{\min\{1,\nu\}} = 20NI_0 \cdot \max\left\{1,\nu^{-1}\right\}. \tag{3.57}$$

Using (3.18) (which is a consequence of (3.11)), we see that ψ_{p+1} satisfies

$$\begin{aligned} \|\partial_x^2\psi_{p+1}(t)\|^2 &\le \alpha^2 w_*^2, \\ \|(\psi'_{p+1},\psi''_{p+1})(t)\|_2^2 + \|\psi'''_{p+1}(t)\|^2 &\le 20NI_0 \cdot \max\left\{1,\nu^{-1}\right\} = K_0 \end{aligned} \tag{3.58}$$

if

$$T_* \le \min\left\{\frac{1}{\sqrt{L_1}},\ \frac{B_1}{4L_3},\ \frac{\ln 2}{2+N(I_0+1)^2},\ \frac{\ln 2}{2+N(I_0+1)},\ \frac{\ln 2}{L_4},\ \frac{I_0}{L_5}\right\}, \tag{3.59}$$

where

$$B_1 = \nu\alpha^2 w_*^2 - 4\|v_1\|_{C^1([0,1])}^2\|w_1\|_1^2.$$

This proves (3.27) and (3.28).

To verify (3.29) we employ the differential equation in (3.8) again. We use (3.58), (3.35), (3.21), and (3.59) to estimate

$$\begin{aligned} \|\partial_x^4\psi_{p+1}(t)\|^2 \le& \frac{N}{\nu^2}(\|\psi''_{p+1}(t)\|^2 + \|\psi'_{p+1}(t)\|^2 + \|\psi_{p+1}(t)\|^2 + \|k\partial_x\psi'_{p+1}(t)\|^2) \\ &+ \frac{2}{\nu^2}\|g_4(\cdot,U^p)(t)\|^2 \\ \le& \frac{2N}{\nu^2}(I_0+1)^3(1+K_0) + 32a_1\|\partial_x^2\psi_p(t)\|^2 K_2 \\ \le& \frac{2N}{\nu^2}(I_0+1)^3(1+K_0) + 32a_1 e^{T_*(2+N(I_0+1)^2)}\left(\|\theta_1\|_2^2 + T_*L_3\right)K_2 \\ \le& \frac{2N}{\nu^2}(I_0+1)^3(1+K_0) + 64a_1(2\|v_1\|_{C^1([0,1])}^2\|w_1\|_1^2 + T_*L_3)K_2. \end{aligned} \tag{3.60}$$

Here, we used the fact that (3.52) is also valid for ψ_p in $[0,T_*]$.

¿From (3.11) and (3.40) we infer

$$128a_1\|v_1\|_{C^1([0,1])}^2\|w_1\|_1^2 = 128\|v_1\|_{C^1([0,1])}^2(1-\alpha)^{-2}w_*^{-2}\|w_1\|_1^2 < 1$$

which implies

$$1 - \frac{128\|v_1\|_{C^1([0,1])}^2\|w_1\|_1^2}{(1-\alpha)^2 w_*^2} > 0.$$

Thus, choosing

$$K_2 = \frac{8(1-\alpha)^2 w_*^2 N(I_0+1)^3(1+K_0)}{\nu^2[(1-\alpha)^2 w_*^2 - 128\|v_1\|_{C^1([0,1])}^2\|w_1\|_1^2]}, \tag{3.61}$$

where K_0 is defined by (3.57), we obtain from (3.60) and the Sobolev embedding theorem that

$$\|\partial_x^3\psi_{p+1}(t)\|^2 + \|\partial_x^4\psi_{p+1}(t)\|^2 \le K_2$$

if

$$T_* = \min\left\{\frac{1}{\sqrt{L_1}},\ \frac{B_0}{L_3},\ \frac{\ln 2}{2+N(I_0+1)},\ \frac{\ln 2}{L_4},\ \frac{I_0}{L_5}\right\}. \tag{3.62}$$

We recall that L_1, L_3, L_4, L_5, and I_0 are given by (3.51), (3.54), (3.55), (3.56), and (3.16), respectively, $N > 0$ is a generic constant, and

$$B_0 = \min\left\{\frac{(1-\alpha)^2 w_*^2 - 128\|v_1\|^2_{C^1([0,1])}\|w_1\|_1^2}{64}, \frac{\alpha^2 w_*^2 \varepsilon^2 - 16\|v_1\|^2_{C^1([0,1])}\|w_1\|_1^2}{8}\right\} > 0$$

due to (3.11). Notice that (3.62) implies (3.50) and (3.59).

Differentiating the differential equation in (3.8) with respect to t, integrating over $(0,1)$, and using (3.58), (3.41) and (3.37), we can estimate $\partial_x^4 \psi'_{p+1}$ as

$$\begin{aligned}
\|\partial_x^4 \psi'_{p+1}(t)\|^2 \leq & \frac{N}{\nu^2}(\|\psi'''_{p+1}(t)\|^2 + \|\psi''_{p+1}(t)\|^2 + \|\psi'_{p+1}(t)\|^2 + \|(k\partial_x \psi'_{p+1})'(t)\|^2) \\
& + \frac{2}{\nu^2}\|g_4'(\cdot, U^p)(t)\|^2 \\
\leq & \frac{N}{\nu^2}\left(K_0 + (\|k(\cdot, U^p)(t)\|^2 + \|k'(\cdot, U^p)(t)\|^2)K_0\right) \\
& + \frac{N}{\nu^2}(I_0 + 1 + K_0 + K_2 + K_5)^4 \\
\leq & \frac{N}{\nu^2}\left(K_0 + (I_0+1)^3 K_0 + (I_0 + 1 + K_0 + K_5)^2 K_0\right) \\
& + \frac{N}{\nu^2}(I_0 + 1 + K_0 + K_2 + K_5)^4 \\
\leq & \frac{2N}{\nu^2}(I_0 + 1 + K_0 + K_2 + K_5)^4 .
\end{aligned}$$

Thus, by the Sobolev embedding theorem, we have

$$\|\partial_x^3 \psi'_{p+1}(t)\|^2 + \|\partial_x^4 \psi'_{p+1}(t)\|^2 \leq K_1 \tag{3.63}$$

if

$$K_1 = \frac{4N}{\nu^2}(I_0 + 1 + K_0 + K_2 + K_5)^4. \tag{3.64}$$

Differentiating the differential equation in (3.8) once and twice with respect to x, integrating over $(0,1)$, and employing (3.58), (3.41), and (3.37), we can estimate $\partial_x^5 \psi_{p+1}$ and $\partial_x^6 \psi_{p+1}$ as

$$\begin{aligned}
\|\partial_x^5 \psi_{p+1}(t)\|^2 \leq & \frac{N}{\nu^2}\left(\|\partial_x \psi''_{p+1}(t)\|^2 + \|\partial_x \psi'_{p+1}(t)\|^2 + \|\partial_x \psi_{p+1}(t)\|^2\right) \\
& + \frac{N}{\nu^2}\|\partial_x(k\partial_x \psi'_{p+1})(t)\|^2 + \frac{2}{\nu^2}\|\partial_x g_4(\cdot, U^2 p)(t)\|^2 \\
\leq & \frac{N}{\nu^2}\left(1 + K_0 + (I_0 + 1 + K_0 + K_2)^5 + \|k(\cdot, U^p)(t)\|_1^2(K_0 + K_2)\right) \\
\leq & \frac{3N}{\nu^2}(I_0 + 1 + K_0 + K_2)^5 \\
\leq & \frac{1}{2}K_3,
\end{aligned} \tag{3.65}$$

$$\|\partial_x^6 \psi_{p+1}(t)\|^2 \leq \frac{N}{\nu^2}\left(\|\partial_x^2 \psi''_{p+1}(t)\|^2 + \|\partial_x^2 \psi'_{p+1}(t)\|^2 + \|\partial_x^2 \psi_{p+1}(t)\|^2\right)$$

$$
\begin{aligned}
&+\frac{N}{\nu^2}\|\partial_x^2(k\partial_x\psi'_{p+1})(t)\|^2+\frac{2}{\nu^2}\|\partial_x^2 g_4(\cdot,U^2)(t)\|^2\\
\le&\frac{N}{\nu^2}\left(1+K_0+(I_0+1+K_0+K_2)^5\right)\\
&+\frac{N}{\nu^2}\|k(\cdot,U^p)(t)\|_2^2(K_0+K_1)\\
\le&\frac{N}{\nu^2}\left(1+K_0+(I_0+1)^3(K_0+K_1)+(I_0+1+K_0+K_2)^5\right)\\
\le&\frac{3N}{\nu^2}(I_0+1+K_0+K_1+K_2)^5\\
\le&\frac{1}{2}K_3,
\end{aligned}
\tag{3.66}
$$

if we choose

$$
K_3=\frac{6N}{\nu^2}(I_0+1+K_0+K_1+K_2)^5. \tag{3.67}
$$

Now we choose the constants K_i as follows. Let K_0 be given by (3.57), K_2 by (3.61), K_5 by (3.47), K_1 by (3.64), K_3 by (3.67), K_6 by (3.48), and K_7 by (3.49) (with $\nu=\varepsilon^2/4$). The constant T_* is determined by (3.62). This shows that (ψ_{p+1},η_{p+1}) satisfies (3.27)–(3.30) for $t\in[0,T_*]$.

Step 5: end of the proof. The uniform bounds for $e_{p+1}\in C^1([0,T_*];H^4)$ of (3.9) follow from similar computations as those needed to derive (3.44), where the index p is replaced by $p+1$.

By induction, we conclude that $\{U^i\}_{p=1}^\infty$ exists uniformly in $[0,T_*]$ with T_* given by (3.62) and satisfies (3.13)–(3.14) uniformly for

$$
M_0=\max\{K_0,\ K_1,\ K_2,\ K_3,\ K_5,\ K_6,\ K_7\}.
$$

The proof of Lemma 3.2 is complete. □

Remark 3.3 It follows from the last two inequalities of (3.14) that

$$
\psi_i(x,t)+w_1(x)\ge(1-\alpha)w_*>0,\quad i\ge 1. \tag{3.68}
$$

Proof of Theorem 1.1. By Lemma 3.2, the sequence $\{U^p\}_{p=1}^\infty$ satisfies (3.13)–(3.14) and (3.68) uniformly in $[0,T_*]$. Applying the Ascoli-Arzela theorem and the Aubin lemma to $\{U^p\}_{p=1}^\infty$, it follows that there exists $U=(\psi,\eta,e)$ satisfying

$$
\begin{gathered}
\eta\in C^1([0,T_*];H^3),\quad e\in C^1([0,T_*];H^4),\\
\psi\in C^i([0,T_*];H^{6-2i}\cap H_0^2)\cap C^3([0,T_*];L^2),\quad i=0,1,2,
\end{gathered}
$$

such that (maybe after extracting a subsequence), as $p \to \infty$,

$$\begin{aligned}
\psi_p &\xrightarrow{p\to\infty} \psi \quad \text{strongly in } C^i([0,T_*];H^{6-2i-\sigma}),\ i=0,1,2,\\
\eta_p &\xrightarrow{p\to\infty} \eta \quad \text{strongly in } C^1([0,T_*];H^{3-\sigma}),\\
e_p &\xrightarrow{p\to\infty} e \quad \text{strongly in } C^1([0,T_*];H^{4-\sigma}),
\end{aligned}$$

for any $\sigma > 0$.

It is not difficult to verify that U is a solution of (3.7)–(3.9) and satisfies (3.68) where U^p is replaced by U. Setting

$$w = w_1 + \psi > 0, \quad j = j_1 + \eta, \quad \phi = \phi_1 + e,$$

we see that $j \in C([0,T_*];H^5)$ and (w,j,ϕ) is a local-in-time solution of the IBVP (1.14)–(1.19). The proof of Theorem 1.1 is complete. $\square$

Acknowledgments: A.J. acknowledges partial support from the TMR project "Asymptotic Methods in Kinetic Theory", grant ERB-FMBX-CT97-0157, from the Gerhard-Hess Award of the Deutsche Forschungsgemeinschaft (DFG), grant JU359/3-2, from the Priority Program "Multi-scale Problems" of the DFG, grant JU359/5-1, and from the AFF Project of the University of Konstanz, grant 4/00. H.L. acknowledges partial support from the Austrian–Chinese Scientific–Technical Collaboration Agreement, from CTS of Taiwan, and from the Wittgenstein Award 2000 of P. A. Markowich, funded by the Austrian FWF.

References

[1] F. Brezzi, I. Gasser, P. Markowich, and C. Schmeiser, Thermal equilibrium state of the quantum hydrodynamic model for semiconductors in one dimension, *Appl. Math. Lett.* **8** (1995), 47–52.

[2] P. Degond and P. A. Markowich, On a one-dimensional steady-state hydrodynamic model, *Appl. Math. Lett.* **3** (1990), 25–29.

[3] K. O. Friedrichs, Symmetric positive linear differential equations, *Comm. Pure Appl. Math.* **11** (1958), 333–418.

[4] I. Gamba and A. Jüngel, Positive solutions to singular second and third order differential equations for quantum fluids, *Arch. Rational Mech. Anal.* **156** (2001), 183–203.

[5] C. Gardner, Numerical simulation of a steady-state electron shock wave in a submicron semiconductor device, *IEEE Trans. El. Dev.* **38** (1991), 392–398.

[6] C. Gardner, The quantum hydrodynamic model for semiconductors devices, *SIAM J. Appl. Math.* **54** (1994), 409–427.

[7] I. Gasser and A. Jüngel, The quantum hydrodynamic model for semiconductors in thermal equilibrium, *Z. Angew. Math. Phys.* **48** (1997), 45–59.

[8] I. Gasser and P. Markowich, Quantum hydrodynamics, Wigner transforms and the classical limit, *Asymptotic Anal.* **14** (1997), 97–116.

[9] I. Gasser, P. A. Markowich, and C. Ringhofer, Closure conditions for classical and quantum moment hierarchies in the small temperature limit, *Transp. Theory Stat. Phys.* **25** (1996), 409–423.

[10] M. T. Gyi and A. Jüngel, A quantum regularization of the one-dimensional hydrodynamic model for semiconductors, *Adv. Diff. Eqs.* **5** (2000), 773–800.

[11] A. Jüngel, A steady-state potential flow Euler-Poisson system for charged quantum fluids, *Comm. Math. Phys.* **194** (1998), 463–479.

[12] A. Jüngel, *Quasi-hydrodynamic semiconductor equations*, Progress in Nonlinear Differential Equations, Birkhäuser, Basel (2001).

[13] A. Jüngel, M. C. Mariani, and D. Rial, Local existence of solutions to the transient quantum hydrodynamic equations, to appear in *Math. Models Meth. Appl. Sci.* (2002).

[14] A. Jüngel and H.-L. Li, Quantum Euler-Poisson systems: existence of stationary states, preprint, Universität Konstanz (2001). (http://www.fmi.uni-konstanz.de/Schriften)

[15] T. Kato, The Cauchy problem for quasi-linear symmetric hyperbolic systems, *Arch. Rational Mech. Anal.* **58** (1975), 181–205.

[16] L.D. Landau and E.M. Lifshitz, *Quantum Mechanics: Non-relativistic Theory*, New York, Pergamon Press (1977).

[17] H.-L. Li and C.-K. Lin, Semiclassical limit and well-posedness of nonlinear Schrödinger-Poisson, preprint, Universität Wien (2001).

[18] H.-L. Li and P. A. Markowich, Existence and asymptotic behavior of multi-dimensional quantum hydrodynamic model for semiconductors, preprint (2002).

[19] M. Loffredo and L. Morato, On the creation of quantum vortex lines in rotating HeII, *Il nouvo cimento* **108B** (1993), 205–215.

[20] E. Madelung, Quantentheorie in hydrodynamischer Form, *Z. Physik* **40** (1927), 322.

[21] A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, *Appl. Math. Sci.* **53**, Springer (1984).

[22] T. Makino and S. Ukai, Sur l'existence des solutions locales de l'équation d'Euler-Poisson pour l'évolution d'étoiles gazeuses, *J. Math. Kyoto Univ.* **27** (1987), 387–399.

[23] P. A. Markowich, C. Ringhofer, and C. Schmeiser, *Semiconductor Equations*, Springer, Wien (1990).

[24] C.-W. Shu, Essentially non-oscillatory and weighted essentially non-oscillatory schemes for hyperbolic conservation laws, ICASE Report No. 97-65, NASA Langley Research Center, Hampton, USA (1997).

[25] R. Temam, *Infinite-dimensional Dynamical Systems in Mechanics and Physics*, *Appl. Math. Sci.* **68**, Springer (1988).

[26] B. Zhang and J. Jerome, On a steady-state quantum hydrodynamic model for semiconductors, *Nonlinear Anal., TMA* **26** (1996), 845–856.

[27] E. Zeidler, *Nonlinear functional analysis and its applications. Vol. II: Nonlinear monotone operators*, Springer (1990).