Existence of solutions of a segregation model arising in population dynamics

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Abstract

A positivity-preserving numerical scheme for a strongly coupled cross-diffusion model for two competing species is presented, based on a semi-discretization in time. The unknowns are the population densities of the species. We present results on existence of strictly positive weak solutions to the semidiscrete problem and on existence of non-negative solutions of the evolution problem in one spatial dimension, which are constructed as the limit of the corresponding semidiscrete solutions. The proofs, which are only sketched, are based on a symmetrization of the problem via an exponential transformation of variables and the use of an entropy functional. Our results improves previous results in the literature, in which restrictive assumptions on the parameters of the problem had to be done, see (9) below.

We also illustrate the notion of *segregation* by studying a simplified version of the stationary one-dimensional problem and give some numerical examples.

1 Introduction

For the time evolution of two competing species with homogeneous population density, usually the Lotka-Volterra differential equations are used as an appropiate mathematical model. In the case of non-homogeneous densities, diffusion effects have to be taken into account leading to reaction-diffusion equations. Shigesada *et al.* proposed in their pioneering work [26] to introduce further so-called cross-diffusion terms modeling interspecific influence of the species. Denoting by n_i the population density of the i-th species ($i = 1, 2$) and by J_i the corresponding flows of population, the time-dependent equations can be written as

$$\partial_t n_1 + \operatorname{div} J_1 = g_1(n_1, n_2), \tag{1}$$
$$\partial_t n_2 + \operatorname{div} J_2 = g_2(n_1, n_2), \tag{2}$$
$$J_1 = -\nabla\left((c_1 + \alpha_{11} n_1 + \alpha_{12} n_2) n_1\right) + d_1 n_1 \nabla U, \tag{3}$$
$$J_2 = -\nabla\left((c_2 + \alpha_{21} n_1 + \alpha_{22} n_2) n_2\right) + d_2 n_2 \nabla U, \tag{4}$$

in the bounded domain $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) with time $t > 0$. Here, $U = U(x)$ is the (given) environmental potential, modeling areas where the environmental conditions are more or less favorable [23, 26]. The diffusion coefficients c_i and α_{ij} are non-negative, and $d_i \in \mathbb{R}$ ($i, j = 1, 2$). The source terms are in Lotka-Volterra form:

$$g_1(n_1, n_2) = (R_1 - \gamma_{11} n_1 - \gamma_{12} n_2) n_1, \tag{5}$$
$$g_2(n_1, n_2) = (R_2 - \gamma_{21} n_1 - \gamma_{22} n_2) n_2, \tag{6}$$

where $R_i \geq 0$ is the intrinsic growth rate of the i-th species ($i = 1, 2$), $\gamma_{11} \geq 0$ and $\gamma_{22} \geq 0$ are the coefficients of intra-specific competition, and $\gamma_{12} \geq 0$ and $\gamma_{21} \geq 0$ are those of interspecific competition.

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The above system of equations is completed with mixed Dirichlet-Neumann boundary conditions and initial conditions:

$$n_i = n_{D,i} \quad \text{on } \Gamma_D \times (0,\infty), \quad J_i \cdot \nu = 0 \quad \text{on } \Gamma_N \times (0,\infty), \tag{7}$$

$$n_i(\cdot,0) = n_{0,i} \quad \text{in } \Omega, \quad i = 1,2, \tag{8}$$

where ν denotes the exterior unit normal to $\partial\Omega$. This means that the population density is fixed at a part of the domain boundary (due to emigration and immigration processes), whereas no flux boundary conditions are prescribed at the remaining boundary parts.

Eqs. (1)-(4) contain various types of reaction-diffusion models. Indeed, in the case $\alpha_{ij} = 0$ for $i,j = 1,2$, they reduce to the drift-diffusion equations, which has been studied in various fields of application, e.g. electro-chemistry, [1, 3], biophysics [7] or semiconductor theory [22] . When $c_1 = c_2 = 0$ and $\alpha_{12} = \alpha_{21} = 0$, Eqs. (1)-(4) are of degenerate type. These types of problems arise, for instance, in porous media flow [16], oil-recovery [8], plasma physics [15] or semiconductor theory [14]. In chemotaxis, related models appear [9, 24]. In population dynamics, related models appear [13, 2, 17, 6].

For $\alpha_{12} > 0$ and $\alpha_{21} > 0$, the problem becomes strongly coupled with full diffusion matrix

$$A(n_1,n_2) = \begin{pmatrix} c_1 + 2\alpha_{11}n_1 + \alpha_{12}n_2 & \alpha_{12}n_1 \\ \alpha_{21}n_2 & c_2 + 2\alpha_{22}n_2 + \alpha_{21}n_1 \end{pmatrix}.$$

Nonlinear problems of this kind are quite difficult to deal with since the usual idea to apply maximum principle arguments to get a priori estimates cannot be used here. Furthermore, the diffusion matrix is not symmetric.

Up to now, only partial results are available in the literature concerning the well-posedness of the problem, and no results can be found concerning the numerical analysis. We summarize some of the available results for the time-dependent equations (see [28] for a review) and refer to [19, 20] for the stationary problem. Global existence of solutions and their qualitive behavior for $\alpha_{11} = \alpha_{22} = \alpha_{21} = 0$ have been proved in, e.g., [21, 25]. In this case, Eq. (2) is only weakly coupled. For sufficiently small cross-diffusion parameters $\alpha_{12} > 0$ and $\alpha_{21} > 0$ (or equivalently, "small" initial data) and vanishing self-diffusion coefficients $\alpha_{11} = \alpha_{22} = 0$, Deuring proved the global existence of solutions [5]. For the case $c_1 = c_2$ a global existence result in one space dimension has been obtained by Kim [18]. Furthermore, under the condition

$$8\alpha_{11} > \alpha_{12}, \qquad 8\alpha_{22} > \alpha_{21}, \tag{9}$$

Yagi [29] has shown the global existence of solutions in two space dimensions assuming $\alpha_{12} = \alpha_{21}$. A global existence result for weak solutions in any space dimension under condition (9) can be found in [10]. Condition (9) can be easily understood by observing that in this case, the diffusion matrix is positive definite:

$$\xi^T A(n_1,n_2)\xi \geq \min\{c_1,c_2\}|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^2,$$

hence yielding an elliptic operator. If the condition (9) does not hold, there are choices of $c_i, \alpha_{ij}, n_i \geq 0$ for which the matrix $A(n_1,n_2)$ is *not* positive definite, and it is therefore unclear if the problem (1)-(8) can be solved for these data.

The aim of this paper is to prove the existence of solutions of problem (1)-(8) without assuming condition (9). Observe that there already existed strong evidence of the fact that (9) was merely a technical restriction since the existence of solutions of the corresponding steady state problem was proved without using this condition, see [19].

The main features of our method of proof are the following:

- we construct a positivity-preserving numerical scheme based on a semidiscretization in time,
- no restriction on the diffusion coefficients $c_i, a_i > 0$ are needed,

- the solution of the continuous problem (as the limit of discrete solutions) is non-negative and exists globally in time, in one space dimension.

We stress once again the fact that the positivity (and non-negativity) property is obtained without the use of the maximum principle. In fact, we obtain positivity and $L^\infty(\Omega)$ estimates of the solutions of the discretized problem via $H^1(\Omega)$ regularity results for the logarithm of the solutions. This is the reason of the restriction to one space dimension since the embedding $H^1(\Omega) \subset L^\infty(\Omega)$ is crucial.

Before we state the results and sketch the method of proof, we perform (for a smoother presentation) the following change of unknowns:

$$u_1 = \alpha_{21} n_1, \quad u_2 = \alpha_{12} n_2, \quad \text{and} \quad q = -\nabla U.$$

We assume that $\alpha_{12} > 0$ and $\alpha_{21} > 0$ which is no restriction since if $\alpha_{12} = 0$ or $\alpha_{21} = 0$, at least one of the equations (1), (2) is weakly coupled, and the results of [25] apply. Eqs. (1)-(8) can be reformulated as

$$\partial_t u_i - \operatorname{div}(c_i \nabla u_i + 2a_i u_i \nabla u_i + \nabla(u_1 u_2) + d_i u_i q) = f_i(u_1, u_2), \tag{10}$$

$$u_i = u_{D,i} \quad \text{on } \Gamma_D \times (0,T), \tag{11}$$

$$(c_i \nabla u_i + 2a_i u_i \nabla u_i + \nabla(u_1 u_2) + d_i u_i q) \cdot \nu = 0 \quad \text{on } \Gamma_N \times (0,T), \tag{12}$$

$$u(\cdot, 0) = u_i^0 \quad \text{in } \Omega, \quad i = 1, 2, \tag{13}$$

where $T > 0$, $u_{D,1} = \alpha_{21} n_{D,1}, u_{D,2} = \alpha_{12} n_{D,2}, u_1^0 = \alpha_{21} n_{0,1}, u_2^0 = \alpha_{12} n_{0,2}$ and $a_1 = \alpha_{11}/\alpha_{21}, a_2 = \alpha_{22}/\alpha_{12}$. The source terms are given by $f_i(u_1, u_2) = (R_i - \beta_{i1} u_1 - \beta_{i2} u_2) u_i$, with $\beta_{1i} = \gamma_{1i}/\alpha_{21}, \beta_{2i} = \gamma_{2i}/\alpha_{12}$, $i = 1, 2$.

2 Existence of solutions of problem (10)-(13)

In this section we assume:

(A1) $\Omega \subset \mathbb{R}$ is a bounded interval, $\partial\Omega = \Gamma_D \cup \Gamma_N$, and $\Gamma_D \neq \emptyset$.

(A2) $u_i^0 \in L^\infty(\Omega)$ satisfies $u_i^0 \geq \gamma > 0$ in Ω, $u_{D,i} = \text{const.} > 0$ on Γ_D, $i = 1, 2$.

(A3) $a_i, c_i > 0$, $d_i \in \mathbb{R}$ $(i = 1, 2)$ and $q \in L^2(\Omega \times (0,T))$.

(A4) $f_i : [0,\infty)^2 \to \mathbb{R}$ $(i = 1, 2)$ is continuous and it holds for all $s, \sigma > 0$, $p, q > 0$: $f_i(s, \sigma) \leq C_1$,

$$f_1(s,\sigma) \log \frac{s}{p} + f_2(s,\sigma) \log \frac{\sigma}{q} \leq C_2, \quad \text{and} \quad f_1(s,\sigma) \left(\frac{1}{p} - \frac{1}{s} \right) + f_2(s,\sigma) \left(\frac{1}{q} - \frac{1}{\sigma} \right) \leq C_3,$$

for some C_1, $C_2(p,q)$, $C_3(p,q) > 0$.

We observe here that the Lotka-Volterra source terms satisfy condition (A4) if $\beta_{ii} > 0$, $i = 1, 2$ and $\beta_{12} = \beta_{21} \geq 0$. The main result is the following theorem.

Theorem 1 *There exists a weak solution, (u_1, u_2), of (10)-(13) satisfying $u_1, u_2 \in L^2(0,T; H^1(\Omega)) \cap W^{1,1}(0,T; H^{-1}(\Omega))$ and*

$$u_1(x,t),\ u_2(x,t) \geq 0 \quad \text{for } (x,t) \in Q_T.$$

The proof consists of several steps.

Step 1. We work with unknowns which symmetrize the elliptic operator. Introduce $w = (w_1, w_2)$ by defining $u_1 = e^{w_1}, u_2 = e^{w_2}$ and set $b(w) = (b_1(w), b_2(w)) = (e^{w_1}, e^{w_2})$. With the diffusion coefficients

$$a_{ii}(w) = c_i e^{w_i} + 2a_i e^{2w_i} + e^{w_1 + w_2}, \quad i = 1, 2, \quad a_{12}(w) = a_{21}(w) = e^{w_1 + w_2},$$

Eqs. (10)-(13) are formally equivalent to

$$\partial_t b_i(w) - \operatorname{div}\left(\sum_{j=1}^2 a_{ij}(w)\nabla w_j + d_i b_i(w) q\right) = F_i(w), \qquad \text{in } \Omega\times(0,T), \tag{14}$$

$$\left(\sum_{j=1}^2 a_{ij}(w)\nabla w_j + d_i b_i(w) q\right)\cdot\nu = 0 \qquad \text{on } \Gamma_N\times(0,T), \tag{15}$$

$$w = w_D \qquad \text{on } \Gamma_D\times(0,T), \tag{16}$$

$$w(0) = w^0 \qquad \text{in } \Omega, \tag{17}$$

where $F_i(w) = f_i(e^{w_1}, e^{w_2})$, $w_{D,i} = \log(u_{D,i})$, and $w_i^0 = \log(u_i^0)$, $i = 1, 2$.

Step 2. We introduce now the semi-discrete problem. Let $N \in \mathbb{N}$ and let $\tau = T/N$ be the time step. (We can also allow for quasi-uniform time steps). We are seeking solutions w^k, approximating w in the interval $((k-1)\tau, k\tau]$, $k = 1, \ldots, N$, of the recursive elliptic problem

$$\frac{b_i(w^k) - b_i(w^{k-1})}{\tau} - \operatorname{div}\left(\sum_{j=1}^2 a_{ij}(w^k)\nabla w_j^k + d_i b_i(w^k) q^k\right) = F_i(w^k) \ \text{ in } \Omega, \tag{18}$$

$$\left(\sum_{j=1}^2 a_{ij}(w^k)\nabla w_j^k + d_i b_i(w^k) q^k\right)\cdot\nu = 0 \ \text{ on } \Gamma_N, \tag{19}$$

$$w^k = w_D \ \text{ on } \Gamma_D, \tag{20}$$

for $k = 1, \ldots, N$. Here we defined $q^k = \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} q(\cdot, t)dt$.

Step 3. We introduce the discrete entropy (for $k = 0, \ldots, N$, including thus the entropy of the initial data) $\eta^k = \eta_1^k + \alpha\eta_2^k$,where $\alpha = 2\min\{c_1, c_2\}$, η_1^k is the discrete "physical" entropy

$$\eta_1^k = \sum_{i=1}^2 \int_\Omega (b_i(w^k)(w_i^k - w_{D,i}) - b_i(w^k) + b_i(w_D))dx,$$

and η_2^k is another discrete entropy:

$$\eta_2^k = \sum_{i=1}^2 \int_\Omega (e^{w_i^k - w_{D,i}} - (w_i^k - w_{D,i}))dx.$$

Notice that $\eta^k \geq 0$. We then prove the following entropy-type estimate, which holds in any space dimension.

Lemma 1 *Let $w^k \in H^1(\Omega;\mathbb{R}^2)$ be a weak solution of (18)-(20). Then there exists a constant $C > 0$ such that for any $k = 1, \ldots, N$ and any $\tau > 0$,*

$$\eta^k + \tau\sum_{i=1}^2 \int_\Omega \left(\frac{\alpha^2}{4}|\nabla w_i^k| + \alpha|\nabla e^{w_i^k/2}|^2 + a_i|\nabla e^{w_i^k}|^2\right) dx \leq \eta^{k-1} + C\tau. \tag{21}$$

The key of the proof of this lemma is to use $(w_i^k - w_{D,i}) + \alpha(b_i(-w_D) - b_i(-w_i)) \in H_0^1(\Omega\cup\Gamma_N) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$ as a test function in the weak formulation of (18)-(20).

Step 4. We use the Lax-Milgram's lemma together with the Leray-Schauder's fixed-point theorem to prove the existence of weak solutions of the semi-discrete problem.

Lemma 2 *Let $w^{k-1} \in L^\infty(\Omega;\mathbb{R}^2), k \geq 1$. Then there exists a weak solution $w^k \in H^1(\Omega;\mathbb{R}^2)$ of (18)-(20).*

We note that since the solution satisfies $w^k \in H^1(\Omega;\mathbb{R}^2) \hookrightarrow L^\infty(\Omega;\mathbb{R}^2)$ in one space dimension, the unknowns $u_i^k = \exp(w_i^k)$ are well defined and elements of $H^1(\Omega)$. Hence $(u_1^k, u_2^k) = (e^{w_1^k}, e^{w_2^k})$, $k = 1, \ldots, N$, is a solution of the discrete problem corresponding to (10)-(13).

Step 5. We define the piecewise constant functions $w^{(\tau)}$ by

$$w^{(\tau)}(x,t) = w^k(x) \quad \text{if } (x,t) \in \Omega \times ((k-1)\tau, k\tau]$$

and $q^{(\tau)}$ in a similar way. Observe that assumption (A3) implies

$$q^{(\tau)} \to q \quad \text{in } L^2(Q_T) \text{ as } \tau \to 0. \tag{22}$$

An immediate consequence of Lemma 1 is the following result:

Corollary 1 *Let $\tau > 0$. Then we have $\|\eta^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))} \leq C$ and*

$$\sum_{i=1}^2 \Big(\alpha \|e^{w_i^{(\tau)}/2}\|_{L^2(0,T;H^1(\Omega))} + a_i \|e^{w_i^{(\tau)}}\|_{L^2(0,T;H^1(\Omega))} \Big) \leq C,$$

where $C > 0$ is independent of τ, $\alpha = 2\min\{c_1, c_2\}$, and

$$\eta^{(\tau)}(t) = \sum_{i=1}^2 \int_\Omega \Big(b_i(w^{(\tau)})(w_i^{(\tau)} - w_D) - b_i(w^{(\tau)}) + b_i(w_D) + \alpha(b_i(w^{(\tau)}) - w_i^{(\tau)} + w_{D,i}) \Big)(t) dx.$$

We also need an estimate for the discrete time derivative. For this we define

$$\tilde{b}^{(\tau)}(\cdot,t) = \frac{k\tau - t}{\tau} \big(b(w^k) - b(w^{k-1}) \big) + b(w^k), \quad t > 0,$$

and consider σ_τ, the shift operator

$$\sigma_\tau w^{(\tau)}(\cdot,t) = w^{k-1} \quad \text{if } t \in ((k-1)\tau, k\tau], \quad k = 1, \ldots, N.$$

Then we have

Lemma 3 *It holds*

$$\|b(w^{(\tau)}) - b(\sigma_\tau w^{(\tau)})\|_{L^1(0,T;V^*)} \leq C\tau, \qquad \|\partial_t \tilde{b}^{(\tau)}\|_{L^1(0,T;V^*)} + \|\tilde{b}^{(\tau)}\|_{L^2(0,T;H^1(\Omega))} \leq C,$$

where C does not depend on τ and $V^* = (H_0^1(\Omega \cup \Gamma_N))^*$.

Step 6. We already have all the necessary estimates to pass to the limit $\tau \to 0$. Since the embedding $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ is compact in one space dimension, we can apply Aubin's lemma [27] to $\tilde{b}^{(\tau)}$, in view of the uniform bounds of Lemma 3, to obtain, up to a subsequence which is not relabeled,

$$\partial_t \tilde{b}^{(\tau)} \rightharpoonup \partial_t z \qquad \text{weakly in } L^1(0,T;V^*), \tag{23}$$

$$\tilde{b}^{(\tau)} \rightharpoonup z \qquad \text{weakly in } L^2(0,T;H^1(\Omega)), \tag{24}$$

$$\tilde{b}^{(\tau)} \to z \qquad \text{strongly in } L^2(0,T;L^\infty(\Omega)), \tag{25}$$

$$b(w^{(\tau)}) \rightharpoonup u \qquad \text{weakly in } L^2(0,T;H^1(\Omega)). \tag{26}$$

By Lemma 3 we have, as $\tau \to 0$,

$$\|\tilde{b}^{(\tau)} - b(w^{(\tau)})\|_{L^1(0,T;V^*)} \leq \|b(w^{(\tau)}) - b(\sigma_\tau w^{(\tau)})\|_{L^1(0,T;V^*)} \to 0,$$

and hence $z = u$. Finally, using the above estimates and convergences, we are able to prove that

$$b(w^{(\tau)}) \to u \quad \text{strongly in } L^2(Q_T). \tag{27}$$

Step 7. Now we can let $\tau \to 0$ in the weak formulation of (18), $i = 1, 2$, which reads for $\phi \in L^\infty(0,T;(W^{1,\infty}(\Omega))^*)$:

$$\begin{aligned}\int_0^T \langle \partial_t \tilde{b_i}^{(\tau)}, \phi\rangle dt + \int_{Q_T} \left(c_i \nabla e^{w_i^{(\tau)}} + 2a_i e^{w_i^{(\tau)}} \nabla e^{w_i^{(\tau)}} + \nabla e^{w_1^{(\tau)}+w_2^{(\tau)}}\right) \cdot \nabla\phi dxdt \\ = \quad -d_i \int_{Q_T} e^{w_i^{(\tau)}} q^{(\tau)} \cdot \nabla\phi dxdt + \int_{Q_T} f_i(e^{w_1^{(\tau)}}, e^{w_2^{(\tau)}})\phi dxdt.\end{aligned}$$

In view of (23)-(27), (22) and Assumption (A4) we obtain

$$\begin{aligned}\int_0^T \langle \partial_t u_i, \phi\rangle dt + \int_{Q_T} (c_i \nabla u_i + 2a_i u_i \nabla u_i + \nabla(u_1 u_2)) \cdot \nabla\phi dxdt \\ = \quad -d_i \int_{Q_T} u_i q \cdot \nabla\phi dxdt + \int_{Q_T} f_i(u_1, u_2)\phi dxdt,\end{aligned}$$

i.e. $u = (u_1, u_2)$ is a weak solution of (10)-(11). Moreover, the initial condition (13) is satisfied in the sense of V^*.

More detailed proofs will we published elsewhere [11]. The presented positivity-preserving scheme will be used for numerical simulations in a future work. For numerical stationary solutions, we refer to [10].

3 A simple example

We consider in this section, as a matter of example, the one-dimensional version of the stationary problem corresponding to problem (10)-(13), with $f_i(u_1, u_2) = 0$, $i = 1, 2$ and non-flux boundary conditions, as stated in [26]. Note that this choice of boundary conditions and source terms implies the conservation of the number of individuals of each specie. This may be regarded as a simple equlibrium model of a dynamics in which population pressure effects are noticed much faster than growth and competition effects. For the study of the long-term behaviour of solutions in the case in which interaction of pressure and growth-competition terms are present (no enviromental potential), the reader is refered to [23].

We shall see that this rather simple example already exhibits the phenomena of segregation of populations. We obtain necessary conditions relating transport and cross-diffusion coefficients. However, sufficient conditions either involves the size of the solution or are just a consequence of a very steep difference between the transport coefficients of both equations.

Integrating the equations that one obtains when dropping the time derivative and source terms of equations (10) in $(0, x)$, with $x \in \Omega$ and using the boundary conditions we obtain the following problem. Find $(u_1, u_2) : \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}^2_+$ such that

$$\begin{cases} \big(u_1(c_1 + a_1 u_1 + u_2)\big)' + d_1 q u_1 = 0 \\ \big(u_2(c_2 + a_2 u_2 + u_1)\big)' + d_2 q u_2 = 0 \end{cases} \quad \text{in } \Omega, \tag{28}$$

with

$$\int_\Omega u_1 = \tilde{u}_1 \quad \text{and} \quad \int_\Omega u_2 = \tilde{u}_2. \tag{29}$$

Here $\tilde{u}_1$ and $\tilde{u}_2$ stand for the *mass* of u_1 and u_2, conserved along time in the evolution problem and determined by the initial distributions: $\tilde{u}_i := \int_\Omega u_{i0}$, $i = 1, 2$. We have the following existence result, which is independent of the size of the non-negative parameters a_i. Therefore, condition (9) is not necessary in this example, too.

Proposition 1 *Let c_1 and c_2 be positive and assume $q \in L^\infty(\Omega)$. There exists a solution (u_1, u_2) of Problem (28)-(29) such that $u_1, u_2 \geq 0$ in Ω and $u_1, u_2 \in W^{1,\infty}(\Omega)$. In addition, if $q \in W^{m,\infty}(\Omega)$ then the solution (u_1, u_2) satisfies $u_1, u_2 \in W^{m+1,\infty}(\Omega)$.*

The proof is straightforward. We just rewrite equations (28)-(29) as

$$u_1' = -\frac{qh_1(u_1,u_2)}{g(u_1,u_2)}u, \quad u_2' = -\frac{qh_2(u_1,u_2)}{g(u_1,u_2)}u_2, \tag{30}$$

with

$$\begin{aligned} g(u_1,u_2) &:= (c_1 + 2a_1u_1 + u_2)(c_2 + 2a_2u_2 + u_1) - u_1u_2 > 0, \\ h_1(u_1,u_2) &:= d_1(c_2 + 2a_2u_2 + u_1) - d_2u_2, \\ h_2(u_1,u_2) &:= d_2(c_1 + 2a_1u_1 + u_2) - d_1u_1. \end{aligned} \tag{31}$$

Then, we use an interative scheme on the representation formulae of the solutions given by

$$u_1(x) = u_1(0)\exp\{\int_0^x \frac{-qh_1(u_1,u_2)}{g_1(u_1,u_2)}\} \quad u_2(x) = u_2(0)\exp\{\int_0^x \frac{-qh_2(u_1,u_2)}{g_2(u_1,u_2)}\}, \tag{32}$$

with $u_1(0), u_2(0)$ determined by (29).

We now discuss the notion of segregation in a simplified framework. We shall assume, following [26], that the environmental potential $U(x)$ is a smooth function such that the corresponding enviromental flux satisfies $q(x_0) = 0$ for a single point $x_0 \in \Omega$.

Definition 1 *We say that the stationary problem (28)-(29) has the property of* segregation *if there exists a point $x_0 \in \Omega$ such that u_1 and u_2 have a local maximum and minimum at x_0, respectively (or vice versa).*

Observe that since $q(x_0) = 0$, from (30) we find $u_1'(x_0) = u_2'(x_0) = 0$, so we deduce the existence of a critical point, x_0, both for u_1 and u_2. Taking derivatives in (30) and evaluating at x_0 we obtain

$$\begin{aligned} g(u_1(x_0),u_2(x_0))u_1''(x_0) &= -q'(x_0)h_1(u_1(x_0),u_2(x_0)), \\ g(u_1(x_0),u_2(x_0))u_2''(x_0) &= -q'(x_0)h_2(u_1(x_0),u_2(x_0)). \end{aligned}$$

Therefore, segregation will take place if $h_1(u_1(x_0),u_2(x_0))h_2(u_1(x_0),u_2(x_0)) < 0$. From (31) we have that this is equivalent to one of the following conditions

$$u_2(x_0)(d_2 - 2a_2d_1) > d_1(c_2 + u_1(x_0)), \qquad u_1(x_0)(d_1 - 2a_1d_2) > d_2(c_1 + u_2(x_0)). \tag{33}$$

In the trivial case in which $d_1d_2 < 0$ segregation does always occurr. Note that in this case x_0 is an attractive point (from the environmental point of view) for one of the populations and a repulsive point for the other. We may then analyze the case $d_i > 0$, $i = 1, 2$ (the case $d_i < 0$ is similar). From (33) we obtain a necessary condition involving the diffusion coefficients, namely $d_i > 2a_id_j$, $i, j = 1, 2$, $i \neq j$. Observe that if none of these conditions are satisfied (no segregation) then the coefficients a_1, a_2 satisfy $a_1a_2 > 1/4$, which is a condition in the same spirit than (9). On the other hand, since the L^∞ norms of u_i are bounded when $d_i \to 0$, $i = 1, 2$, see [10], we obtain that a sufficient condition for segregation to ocurr is that d_1/d_2 is large or small enough.

We discretized problem (10)-(13), but with no-flux boundary conditions and zero source terms, by a semi-implicit finite difference method. The numerical experiments conducted show that, for a wide range of model parameters and initial conditions, the numerical scheme is stable, which allows us to show some different behaviours of the two interacting species included in the model. In particular it is interesting to observe when segregation of the two species, due to habitat hetereogenity, does appear.

We also used an iterative scheme based on the formulas (32) to positively check the coincidence of the long-term solutions and the stationary solutions of the problem (28)-(29).

We show here numerical experiments corresponding to the following cases:

(a) Large and small cross-diffusion terms. $c_i = d_i = 1$ and constant initial conditions $u_{10} = 10, u_{20} = 20$ are fixed, while small a_i corresponds to large cross diffusion and viceversa. Figure 1 shows the numerical solution of model corresponding to a_i values of $0, 0.1, 10$.

(b) Segregation effects due to different transport coefficients, i.e., $d_1 \gg d_2$. $c_i = d_2 = 1$ and $u_{10} = 10, u_{20} = 10$ are fixed. Figure 1 shows numerical solutions corresponding to $a_i = 1$ and $d_1 = 4, 8, 20, 40$. As case (a) shows, large cross-diffusion enhance segregation effects, therefore we repeat last simulation with $a_i = 0.1$. Figure 2 shows same results for this case.

(c) $c_i = 1, a_1 = 1, a_2 = 0.01, d_1 = 40, d_2 = 1, u_{10} = u_{20} = 10$ (Figure 2). Here, a neat segregation of the two species can be observed.

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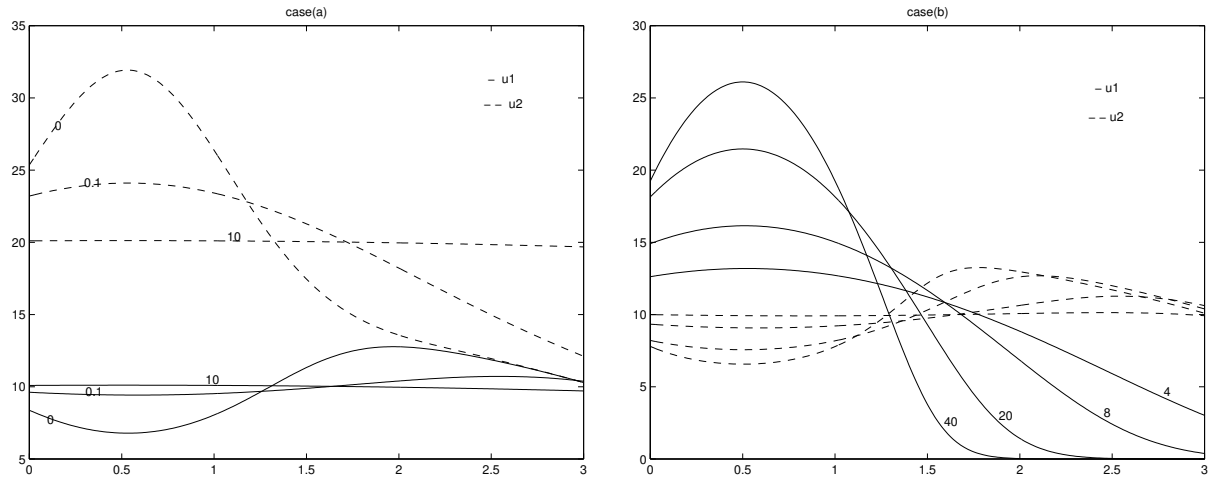

Figure 1: Stationary density distributions (u_1, u_2) corresponding to: Case (a) Curves labeled with the corresponding a_i values. Case (b) Curves labeled with the corresponding d_1 values and $a_i = 1$.

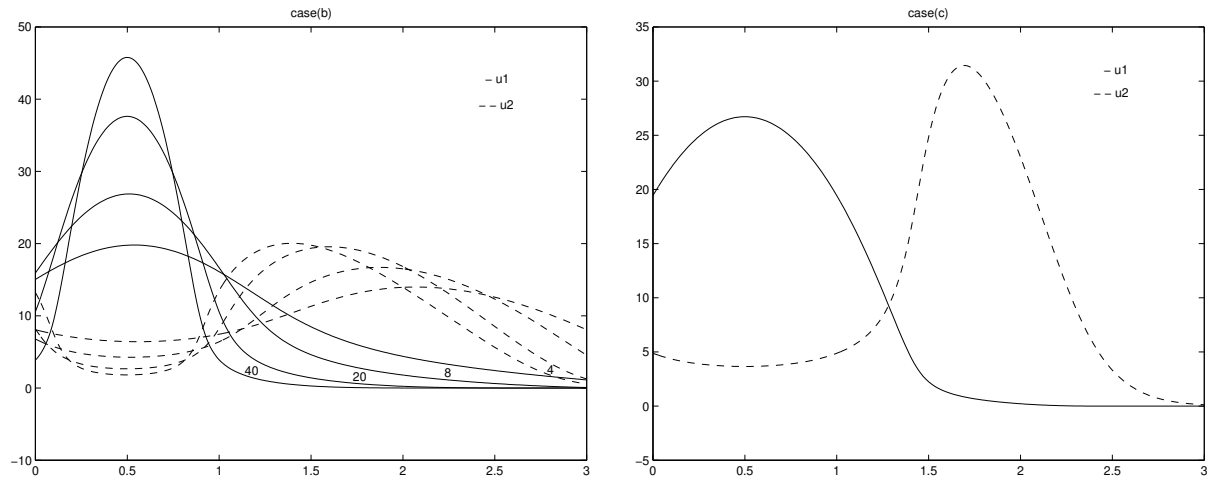

Figure 2: Stationary density distributions(u_1, u_2) corresponding to: Case (b) Curves labeled with the corresponding d_1 values and $a_i = 0.1$. Case (c).