Analysis of a Spherical Harmonics Expansion Model of Plasma Physics

Olaf Hansen and **Ansgar Jüngel**
Fachbereich Mathematik und Informatik
Johannes Gutenberg-Universität Mainz
Staudingerweg 9, 55099 Mainz
E-mail: {hansen,juengel}@mathematik.uni-mainz.de

Abstract

A spherical harmonics expansion model arising in plasma and semiconductor physics is analyzed. The model describes the distribution of particles in the position-energy space subject to a (given) electric potential and consists of a parabolic degenerate equation. The existence and uniqueness of global-in-time solutions is shown by semigroup theory if the particles are moving in a one-dimensional interval with Dirichlet boundary conditions. The degeneracy allows to show that there is no transport of particles across the boundary corresponding to zero energy. Furthermore, under certain conditions on the potential, it is proved that the solution converges in the long-time limit exponentially fast to some steady state.

Keywords. Spherical harmonics expansion model, existence of global solutions, semigroup theory, long-time behavior of solutions, plasmas, semiconductors.

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1 Introduction

The evolution of charged particles can be described at a microscopic level by the kinetic Boltzmann equation. This equation is a nonlocal integro-differential equation in the six-dimensional phase space and therefore, its numerical approximation, by using Monte-Carlo or deterministic particle methods for instance, is extremely time-consuming. Simpler macroscopic fluid models derived from the Boltzmann equation, by using moment or Hilbert expansion methods, are much easier to solve numerically but their accuracy is limited. In recent years, considerable attention has been devoted to the derivation of new macroscopic models which yield a better description of the underlying microscopic physics than the fluid models at still a moderate numerical cost. Among these models, the Spherical Harmonics Expansion (SHE) model seems to be a good compromise between physical accuracy and numerical efficiency.

The SHE model consists of a diffusion equation in the position-energy space for the (scaled) distribution function $F(x,\varepsilon,t)$ of the particles:

$$\begin{aligned} N(x,\varepsilon)\partial_t F + \mathrm{div}_x J - \nabla_x \phi \cdot \partial_\varepsilon J &= 0, \\ J &= -D(x,\varepsilon)(\nabla_x F - \nabla_x \phi \cdot \partial_\varepsilon F), \end{aligned}$$

where the (scaled) independent variables are the position $x \in \mathbb{R}^d$ ($d \le 3$), the energy $\varepsilon \ge 0$, and the time $t \ge 0$. Furthermore, $J(x,\varepsilon,t)$ is the particle current density, $\phi(x)$ the electric potential, $N(x,\varepsilon)$ the energy density of states, and $D(x,\varepsilon)$ the diffusion matrix (see below).

The SHE model has been used to model trapped particles in a surface potential $\phi(x)$, $x \in \mathbb{R}^2$, subject to collisions with the surface. Such a situation arises, for instance, on the surface of satellite panels. Electrons emitted from the metallic parts are confined along the surface of the dielectric parts by the surrounding potential and undergo many collisions with the dielectric surface possibly damaging the panel [4, 6]. Here, the variable x denotes the position coordinate along the surface and ε the kinetic energy of a particle when it hits the wall located at $z = 0$. Let $\psi(x,z)$ be the potential in the half space $z \ge 0$, related to $\phi(x)$ by the formula $\phi(x) = \psi(x,0)$, $\psi_0(x,z) = \psi(x,z) - \phi(x)$ the shifted potential which vanishes on the surface, and $E(x,\varepsilon) = \{(z,v) \in [0,\infty) \times \mathbb{R}^3 : \psi_0(x,z) + |v|^2/2 = \varepsilon\}$ the set of all points in the phase space of the perpendicular coordinate z and velocity $v = (v_1, v_2, v_3)$ with energy ε. Then the energy density of states $N(x,\varepsilon)$ and the diffusion matrix $D(x,\varepsilon)$ are given by

$$N(x,\varepsilon) = \int_{E(x,\varepsilon)} dz\, dv, \quad D(x,\varepsilon) = \int_{E(x,\varepsilon)} \chi \otimes (v_1, v_2)^\top dz\, dv,$$

where χ is the solution of a boundary-value problem involving the (elastic) operator of interactions of the particles with the solid boundary. For linear or quadratic potentials ψ_0 the functions N and D are known explicitly [6]:

$$\begin{aligned} &\psi_0(x,z) = -f_0 z: \quad N(x,\varepsilon) = \frac{8\sqrt{2}\pi}{3f_0}\varepsilon^{3/2}, \quad D(x,\varepsilon) = \frac{4\pi}{3f_0^2}\varepsilon^3 \mathrm{id}, \\ &\psi_0(x,z) = \frac{\omega^2}{2}z^2: \quad N(x,\varepsilon) = \frac{2\pi^2}{\omega}\varepsilon, \qquad D(x,\varepsilon) = \frac{\pi^3}{2\omega^2}\varepsilon^2 \mathrm{id}, \end{aligned} \tag{1}$$

where f_0 is a constant force, ω the frequency of the harmonic oscillator, and id denotes the 2×2 unit matrix. With these diffusion matrices, the SHE model is of degenerate parabolic type.

Another situation occurs in semiconductor theory. Electrons confined in the vicinity of the isolator surface of the oxide of a MOS transistor undergo many collisions with the isolator surface reducing the electron mobility and therefore the device efficiency [6]. The SHE model also appears in gas discharge physics [21] and semiconductor superlattice modeling [9].

Originally, the SHE model has been heuristically derived from a truncated expansion of the Boltzmann equation in spherical harmonics (therefore its name), first introduced by Stratton [23] and numerically realized later by Baccarani and Goldsman [11, 12] (also see [13, 18]). In plasma physics, the spherical harmonics expansion method has been used, for instance, to derive currents in weakly ionized gases [19].

The above SHE model is derived from a diffusion approximation of the Boltzmann equation, in the context of semiconductor theory [1, 10, 22] under the assumption of dominant elastic scattering (see [5] for inelastic interactions) or in the context of plasma physics [6]. Mathematically rigorous derivations of SHE models has been performed, for instance, in [3, 7, 16].

The model has also been used to derive simpler fluid-type models, for instance energy-transport [1, 8] and high-field equations [2]. However, to our knowledge, there does not exist any existence result for the SHE equations in the literature, yielding a basis for further rigorous investigations of the model. In this paper we prove the existence, uniqueness and long-time behavior of solutions to the SHE model. More precisely, we consider the one-dimensional version

$$\begin{aligned} N(\varepsilon)\partial_t F + (\partial_s - \partial_s\phi\partial_\varepsilon)J &= 0, && (2)\\ J &= -D(\varepsilon)(\partial_s - \partial_s\phi\partial_\varepsilon)F, && (3)\\ F(s,\varepsilon,0) &= F_0(s,\varepsilon), && (4) \end{aligned}$$

for $s \in I := (0,1)$, $\varepsilon > 0$, $t > 0$, corresponding to a one-dimensional "plate" of finite length (see Remark 4.2 concerning the two-dimensional case). As we are mainly interested in situations including (1), we have assumed that the functions N and D only depend on the energy ε. We choose Dirichlet boundary conditions

$$F(0,\varepsilon,t) = F_L(\varepsilon,t), \quad F(1,\varepsilon,t) = F_R(\varepsilon,t), \quad s \in I,\ \varepsilon > 0. \qquad (5)$$

The main difficulties in the mathematical treatment of the problem (2)-(5) are as follows. With a density of states and diffusion matrix as in (1), the problem is of degenerate type. Moreover, at points where $\partial_s\phi$ vanishes, we do not obtain informations about the derivative of F with respect to ε. Our main idea of the existence proof is to transform the SHE equations to a degenerate parabolic problem in symmetric form and then to apply semigroup theory. Indeed, introducing the "chemical potential" $\delta = \varepsilon + \phi(s)$, the transformed function $G(s,\delta,t) = F(s,\delta-\phi(s),t)$ satisfies $\partial_s G = \partial_s F - \partial_s\phi\partial_\varepsilon F$ and hence

$$N(\delta - \phi(s))\partial_t G = \partial_s(D(\delta - \phi(s))\partial_s G).$$

Notice that the variable δ is only a parameter in this equation. The set $(s,\varepsilon) \in I \times [0,\infty)$ transforms to $Y = \{(s,\delta) : s \in I,\ \phi(s) \le \delta\}$. Thus we do not have a boundary condition at $\{s \in I,\ \phi(s) = \delta\}$. Considering the above equation on each of the segments $Y_\Delta = \{(s,\delta) :$

$\delta = \Delta\}$, we have to solve the above equation either with no, one or two (Dirichlet) boundary conditions (see section 2 for details). This is possible since the equation is degenerate.

Transforming back the solution from the (s, δ) to the (s, ε) variables, we obtain the existence of a solution $F(s, \varepsilon, t)$ to (2)-(5) which is regular on each of the curves Z_Δ obtained by back-transforming Y_Δ. Moreover, we are able to prove that $F(s, \varepsilon, t)$ on each Z_Δ converges for $t \to \infty$ exponentially fast to a steady-state under the assumptions that the boundary functions are constant in time and that the segment Y_Δ corresponding to Z_Δ does not cross the graph of ϕ tangentially (since $\partial_s \phi = 0$ at the intersection point).

This paper is organized as follows. In section 2 we transform the SHE equations to a symmetric parabolic equation with no, one or two boundary conditions. The corresponding degenerate elliptic operators are analyzed in section 3. Finally, the existence and long-time behavior result is given in section 4.

2 Transformation of the problem

Introduce the sets $I = (0, 1)$, $Z = I \times [0, \infty)$, and $Y = \{(\tau, \delta) \in I \times \mathbb{R} : \phi(\tau) \leq \delta\}$. Defining the invertible transformation

$$\varphi : Y \to Z, \quad \varphi(\tau, \delta) = (\tau, \delta - \phi(\tau))$$

(see Figure 1) and the function $G(\tau, \delta, t) = F(\varphi(\tau, \delta), t)$, we obtain

$$\begin{aligned} D(\delta - \phi_0(\tau))\partial_\tau G(\tau, \delta) &= D(\delta - \phi_0(\tau))(\partial_s F(\varphi(\tau, \delta), t) - (\partial_\tau \phi_0(\tau))\partial_\varepsilon F(\varphi(\tau, \delta), t)) \\ &= -J(\varphi(\tau, \delta), t) \end{aligned}$$

and therefore

$$\begin{aligned} \partial_\tau(D(\delta - \phi_0(\tau))\partial_\tau G(\tau, \delta)) &= -\partial_s J(\varphi(\tau, \delta), t) + (\partial_\tau \phi_0(\tau))\partial_\varepsilon J(\varphi(\tau, \delta), t) \\ &= N(\delta - \phi_0(\tau))\partial_t F(\varphi(\tau, \delta), t) \\ &= N(\delta - \phi_0(\tau))\partial_t G(\tau, \delta, t). \end{aligned}$$

Introducing the notations $\widetilde{N}(\tau, \delta) = N(\delta - \phi_0(\tau))$ and $\widetilde{D}(\tau, \delta) = D(\delta - \phi_0(\tau))$, the function G fulfills the initial-boundary-value problem

$$\partial_t G = \frac{1}{\widetilde{N}}\partial_\tau(\widetilde{D}\partial_\tau G), \qquad (\tau, \delta) \in Y,\ t > 0, \tag{6}$$

$$G(\tau, \delta, 0) = G_0(\tau, \delta), \qquad (\tau, \delta) \in Y, \tag{7}$$

$$G(0, \delta, t) = G_L(\delta, t), \qquad \delta \geq \phi(0),\ t \geq 0, \tag{8}$$

$$G(1, \delta, t) = G_R(\delta, t), \qquad \delta \geq \phi(1),\ t \geq 0. \tag{9}$$

where

$$G_0(\tau, \delta) := F_0(\varphi(\tau, \delta)), \quad G_L(\delta, t) := F_L(\varphi(0, \delta), t), \quad G_R(\delta, t) := F_R(\varphi(1, \delta), t).$$

We recall that the variable δ is just a parameter in the above problem.

Now we are able to state our main hypotheses.

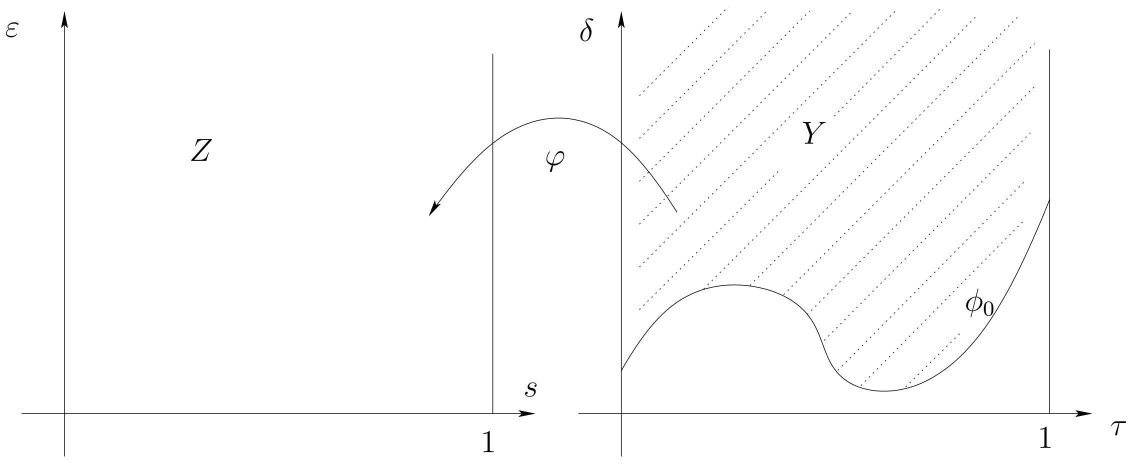

Figure 1: The sets Z and Y.

(H1) The functions N, $D \in C^2([0,\infty))$ are positive in $(0,\infty)$ and satisfy for some $\kappa > 0$

$$N(\varepsilon) \sim \varepsilon^{\kappa}, \; D(\varepsilon) \sim \varepsilon^{2\kappa} \quad (\varepsilon \to 0). \tag{10}$$

(H2) The potential $\phi \in C^2(\overline{I})$ has the property that each of the sets $Y_\Delta = \{(\tau,\delta) \in I \times \mathbb{R} : \phi(\tau) = \Delta\}$ is the union of a finite number of intervals:

$$Y_\Delta = \bigcup_{j=1}^{m_\Delta} Y_{\Delta,j} \quad \text{with} \quad Y_{\Delta,j} = [a_{\Delta,j}, b_{\Delta,j}] \times \{\Delta\}.$$

The first assumption makes precise the degeneracy of equation (2). The examples (1) are included in (H1). The second hypothesis is needed in order to solve the problem (6)-(9) on each set Y_Δ. Hypothesis (H2) is illustrated in Figure 2, where $Z_\Delta = \varphi(Y_\Delta)$.

Let $\Delta_{\max} = \max\{\phi(\tau) : \tau \in \overline{I}\}$. For all $\Delta > \Delta_{\max}$, we have $Y_\Delta = I$ and (6)-(9) is a parabolic equation for fixed $\delta = \Delta$. Since $\widetilde{N}$ and $\widetilde{D}$ are smooth and bounded away from zero for this parameter (see Hypothesis (H1)), the initial-boundary-value problem has a smooth solution $(\tau,t) \mapsto G(\tau,\Delta,t)$. In particular, the function F is smooth along the manifold $Z_\Delta \times [0,\infty)$.

For $\Delta \le \Delta_0$ the set Y_Δ is generally the union of a finite number of intervals $Y_{\Delta,j}$ (see Hypothesis (H2) and Figure 2). For each interval we obtain an initial-value problem with no, one or two boundary conditions. If $\widetilde{N}(a_{\Delta,j},\Delta) = 0$ we have no boundary condition at the left point $a_{\Delta,j}$. If $\widetilde{N}(b_{\Delta,j},\Delta) = 0$ we have no boundary condition at the right point $b_{\Delta,j}$. Moreover, $\widetilde{N}(a_{\Delta,j}) > 0$ if $a_{\Delta,j} = 0$ and similarly for the right interval point. In this case (6) has to be supplemented by the boundary conditions (8), (9), respectively.

Remark 2.1 Hypothesis (H2) and $\widetilde{N}(a_{\Delta,j},\Delta) = 0$ imply $\widetilde{N}(\tau,\Delta) \sim (\tau - a_{\Delta,j})^{m\kappa}$ for $\tau \searrow a_{\Delta,j}$, where m is the multiplicity of the root of the function $\tau \mapsto \Delta - \phi(\tau)$ at $a_{\Delta,j}$ (see Y_{Δ_3} in Figure 2). In an analogous way we conclude $\widetilde{N}(\tau,\Delta) \sim (b_{\Delta,j} - \tau)^{m\kappa}$ for $\tau \nearrow b_{\Delta,j}$ if $\widetilde{N}(b_{\Delta,j},\Delta) = 0$. The same formulas hold for $\widetilde{D}(\tau)$ but $m\kappa$ has to be replaced by $2m\kappa$. $\square$

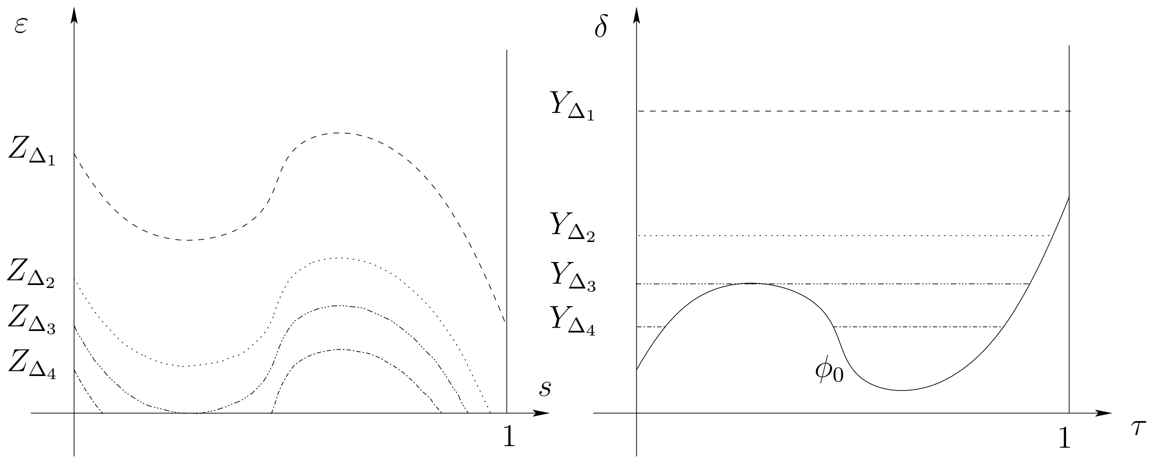

Figure 2: Illustration of Y_Δ and Z_Δ.

The discussion shows that we need to study three different initial-value problems, either with no boundary conditions (but growth conditions for $\widetilde{N}$ and $\widetilde{D}$ at the interval end points), with one boundary condition at the left or right end point, or with two boundary conditions. Clearly, the last situation immediately yields a (unique) smooth solution. Hence, it remains to analyze the following two initial-value problems. In order to simplify the presentation, we consider these problems in the interval $(a_{\Delta,j}, b_{\Delta,j})$ with $a_{\Delta,j} = 0$ and $b_{\Delta,j} = 1$. Moreover, we suppress the dependence of the functions on Δ and write $\widetilde{N}(\tau)$ instead of $\widetilde{N}(\tau,\Delta)$ etc.:

$$\partial_t G(\tau,t) = \frac{1}{\widetilde{N}(\tau)}\partial_\tau(\widetilde{D}(\tau)\partial_\tau G(\tau,t)), \qquad (\tau,t) \in I \times (0,\infty), \tag{11}$$
$$G(\tau,0) = G_0(\tau), \qquad \tau \in I, \tag{12}$$
$$\widetilde{N}(\tau) \sim \tau^{\kappa_1}, \quad \widetilde{N}(1-\tau) \sim \tau^{\kappa_2}, \qquad \tau \searrow 0, \tag{13}$$
$$\widetilde{D}(\tau) \sim \tau^{2\kappa_1}, \quad \widetilde{D}(1-\tau) \sim \tau^{2\kappa_2}, \qquad \tau \searrow 0, \tag{14}$$

where κ_1, $\kappa_2 > 0$, and

$$\partial_t G(\tau,t) = \frac{1}{\widetilde{N}(\tau)}\partial_\tau(\widetilde{D}(\tau)\partial_\tau G(\tau,t)), \qquad (\tau,t) \in I \times (0,\infty), \tag{15}$$
$$G(\tau,0) = G_0(\tau), \qquad \tau \in I, \tag{16}$$
$$G(1,t) = G_R(t), \qquad t \geq 0, \tag{17}$$
$$\widetilde{N}(\tau) \sim \tau^{\kappa_3}, \ \widetilde{D}(\tau) \sim \tau^{2\kappa_3} \qquad \tau \searrow 0, \tag{18}$$

where $\kappa_3 > 0$. To analyze these problems we study the operator $u \mapsto -(1/\widetilde{N})\partial_s(\widetilde{D}\partial_s u)$ in the next section.

3 Analysis of the degenerate elliptic operators

Define the two operators $\widetilde{A}_1$ and $\widetilde{A}_2$ by

$$(\widetilde{A}_1 u)(\tau) = -\frac{1}{\widetilde{N}(\tau)}\partial_\tau(\widetilde{D}(\tau)\partial_\tau u(\tau))$$

with domain $D(\widetilde{A}_1) = C^2(\overline{I})$ and functions $\widetilde{N}$, $\widetilde{D} \in C^2(\overline{I})$ satisfying (13)-(14), and

$$(\widetilde{A}_2 u)(\tau) = -\frac{1}{\widetilde{N}(\tau)}\partial_\tau(\widetilde{D}(\tau)\partial_\tau u(\tau))$$

with domain $D(\widetilde{A}_2) = \{u \in C^2(\overline{I}) : u(1) = 0\}$ and functions $\widetilde{N}$, $\widetilde{D} \in C^2(\overline{I})$ satisfying (18). Again we have not indicated the dependence of $\widetilde{N}$ and $\widetilde{D}$ on the parameter Δ. An obvious Hilbert space for these two operators is the weighted Lebesgue space

$$L^2_w(I) = \{u : I \to \mathbb{R} \text{ measurable} : \|u\|^2_w := \textstyle\int_I u^2(\tau)\widetilde{N}(\tau)d\tau < \infty\}.$$

The corresponding scalar product is given by

$$(u, v)_w = \int_I u(\tau)v(\tau)\widetilde{N}(\tau)d\tau \quad \text{for } u, v \in L^2_w(I).$$

The operators $\widetilde{A}_1$ and $\widetilde{A}_2$ are densely defined symmetric operators on $L^2_w(I)$. The operator $\widetilde{A}_2$ is positive definite and $\widetilde{A}_1$ is positive semi-definite with a one-dimensional kernel consisting of constant functions.

Remark 3.1 The weighted space $L^2_w(I)$ is convenient not only since $\widetilde{A}_1$ and $\widetilde{A}_2$ are becoming symmetric operators but also since the weight has a physical meaning. Indeed, let $F(x, \varepsilon, t)$ be the distribution function corresponding to (2)-(3). Then

$$\int_\Omega F(x, \varepsilon, t)N(x, \varepsilon)dxd\varepsilon$$

is the number of particles in the domain $\Omega \subset I \times [0, \infty)$. We investigate the case that Ω is given by

$$\Omega = \bigcup_{\Delta \in [a,b]} Z_\Delta = \varphi(Y \cap (I \times [a, b])).$$

This implies

$$\begin{aligned}
\int_\Omega F(x, \varepsilon, t)N(x, \varepsilon)dxd\varepsilon &= \int_{Y\cap(I\times[a,b])} F(\varphi(\tau, \delta))N(\varphi(\tau, \delta))|\det D\varphi(\tau, \delta)|d\tau d\delta \\
&= \int_a^b \int_{Y\cap(I\times\{\delta\})} F(\varphi(\tau, \delta))\widetilde{N}(\tau, \delta)d\tau d\delta,
\end{aligned}$$

since $\det D\varphi = 1$. This shows that

$$\delta \mapsto \int_{Y\cap(I\times\{\delta\})} F(\varphi(\tau,\delta))\widetilde{N}(\tau,\delta)d\tau$$

is the particle density with respect to δ. $\square$

By well known results for the Friedrich extension of operators [17, 24] we obtain our first result:

Proposition 3.2 *The operators $\widetilde{A}_1$ and $\widetilde{A}_2$ can be extended to self-adjoint operators A_1, A_2 with domains $D(A_1)$, $D(A_2) \subset L^2_w(I)$, respectively, with the same lower bound*

$$\inf_{u\in D(A_i),\ \|u\|_w=1} (A_i u, u)_w = \inf_{u\in D(\widetilde{A}_i),\ \|u\|_w=1} (\widetilde{A}_i u, u)_w \geq 0, \quad i = 1, 2.$$

The energy space $H(A_i)$ with $D(A_i) \subset H(A_i) \subset L^2_w(I)$ $(i = 1, 2)$ consists of all functions for which the norm

$$\|u\|^2_{H(A_i)} := (A_i u, u)_w + \|u\|^2_w$$

is finite.

In the following we characterize the functions of $D(A_i)$ and show that the imbedding $H(A_i) \hookrightarrow L^2_w(I)$ is compact for certain values of κ (with $\kappa \in \{\kappa_1, \kappa_2, \kappa_3\}$). We prove our result only for the operator A_2; the proof for A_1 is analogous.

Let $u \in H(A_2)$. Then there exists a sequence $u_n \in D(\widetilde{A}_2)$ such that $\|u_n - u\|_{H(A_2)} \to 0$ as $n \to \infty$. This implies that the derivative u' exists and $u(1) = 0$ [17]. By partial integration we see that

$$\|u\|^2_{H(A_2)} = \int_I \left(\widetilde{D}(\tau)u'(\tau)^2 + \widetilde{N}(\tau)u(\tau)^2\right) d\tau.$$

The functions $\widetilde{N}$ and $\widetilde{D}$ satisfy the growth conditions (18); in the following we write κ instead of κ_3. An equivalent norm to $\|\cdot\|_{H(A_2)}$ is thus given by

$$\|u\|^2_\kappa := \int_I \left(\tau^{2\kappa}u'(\tau)^2 + \tau^\kappa u(\tau)^2\right) d\tau.$$

For the operator A_1 we have to use the expressions $\tau^{2\kappa_1}(1-\tau)^{2\kappa_2}$ and $\tau^{\kappa_1}(1-\tau)^{\kappa_2}$ instead of $\tau^{2\kappa}$ and τ^κ. The following lemma is needed for the compactness result.

Lemma 3.3 *Let $0 < \kappa < 2$. Then there exists a constant $c_1 > 0$ such that for any $u \in H(A_2)$ and $\eta \in (0, 1)$ it holds*

$$\int_0^\eta \tau^\kappa u(\tau)^2 d\tau \leq c_1(\eta^{1+\kappa} + \eta^{2-\kappa})\|u\|^2_\kappa.$$

Proof. Let τ, $\sigma \in I$ and $u \in H(A_2)$. Then

$$u(\tau) = u(\sigma) + \int_{\sigma}^{\tau} u'(s)ds.$$

In order to get pointwise estimates for $u(\tau)$, we integrate with respect to σ over $(1/4, 3/4)$ (see [15]):

$$u(\tau) = 2\int_{1/4}^{3/4} u(\sigma)d\sigma + 2\int_{1/4}^{3/4}\int_{\sigma}^{\tau} u'(s)dsd\sigma$$

and use the Cauchy-Schwarz inequality:

$$\begin{aligned} |u(\tau)| &\leq 2\|u\|_{L^1(1/4,3/4)} + 2\int_{1/4}^{3/4}\left|\int_{\sigma}^{\tau} s^{-\kappa}s^{\kappa}u'(s)ds\right| d\sigma \\ &\leq c_0\left(\|u\|_{L^1(1/4,3/4)} + \tau^{1/2-\kappa}\left(\int_I s^{2\kappa}u'(s)^2ds\right)^{1/2}\right), \end{aligned}$$

for $\kappa > 1/2$, where $c_0 > 0$ does not depend on u or τ. For $\kappa < 1/2$ or $\kappa = 1/2$ the function u is bounded near the origin or has at most a logarithmic singularity, respectively. Hence we obtain for $\eta \in (0, 1)$

$$\begin{aligned} \int_0^{\eta} \tau^{\kappa}|u(\tau)|^2 d\tau &\leq 2c_0^2\left(\frac{\eta^{1+\kappa}}{1+\kappa}\|u\|^2_{L^1(1/4,3/8)} + \frac{\eta^{2-\kappa}}{2-\kappa}\|u'\|^2_w\right) \\ &\leq c_1(\eta^{1+\kappa} + \eta^{2-\kappa})\|u\|^2_{\kappa}, \end{aligned}$$

where we have used the fact that $L^2_w(I)$ has a stronger norm than $L^1(1/4, 3/4)$. The inequality also holds for $\kappa \leq 1/2$. □

Lemma 3.4 *Let κ_1, κ_2, $\kappa_3 \in (0, 2)$. Then the imbeddings of the energy spaces $H(A_1)$ and $H(A_2)$ into $L^2_w(I)$ are compact.*

Proof. Let $B \subset H(A_2)$ be a bounded set. This implies $\|u\|_{\kappa} \leq c$ for all $u \in B$, where $c > 0$ is a constant. We denote by

$$B_{\eta} = \{u\chi_{(\eta,1)} : u \in B\} \subset L^2_w(I), \quad 0 < \eta < 1,$$

the set of restrictions of $u \in B$ to the interval $(\eta, 1)$. As the imbedding $H^1(\eta, 1) \hookrightarrow L^2(\eta, 1)$ is compact, B_{η} is a compact set in $L^2_w(I)$ for any $\eta > 0$. We conclude from Lemma 3.3 that

$$\lim_{\eta\to 0} \mathrm{dist}(B_{\eta}, B) = 0,$$

where dist is the distance in $L^2_w(I)$. The lemma is proved. □

Remark 3.5 Lemma 3.4 is sharp with respect to κ. Indeed, the functions $u_n \in L^2_w(I)$, defined by

$$u_n(\tau) = n^{(3+\kappa)/2} \times \begin{cases} (\tau - 1/2n), & \tau \in [1/2n, 1/n], \\ (3/2n - \tau), & \tau \in [1/n, 3/2n], \\ 0, & \text{else}, \end{cases}$$

are bounded in the norm $\|\cdot\|_w$ and in the norms $\|\cdot\|_{H(A_i)}$ but do not have a convergent subsequence. Thus the imbedding $H(A_i) \hookrightarrow L^2_w(I)$ is *not* compact for $\kappa \geq 2$.

This observation has an important consequence. We already remarked that for special examples of the potential $\kappa = 1$ or $\kappa = 3/2$ holds (see (1)). The discussion in Remark 2.1 shows that all Δ values which are critical values for ϕ do *not* fulfill $\kappa < 2$. Therefore we expect to obtain results only for values of the parameter Δ which are not critical values of ϕ. $\square$

For the characterization of the domain $D(A_2)$ we need an additional assumption:

$$\widetilde{D}'(\tau) \sim \tau^{2\kappa-1} \quad \text{for } \tau \searrow 0. \tag{19}$$

Recall that we write κ instead of κ_3. For the operator A_1 we have $\kappa = \kappa_1$ and we also need the condition

$$\widetilde{D}'(1-\tau) \sim \tau^{2\kappa_2-1} \quad \text{for } \tau \searrow 0. \tag{20}$$

Lemma 3.6 *Let (19) hold and let $u \in D(A_2)$. Set $\beta = \min\{3/2, 1+1/\kappa\}$. Then it follows $u(1) = 0$, u' and u'' exist and*

$$\|u\|^2_{D(A_2)} := \int_I \left(\widetilde{D}(\tau)^\beta u''(\tau)^2 + \widetilde{D}(\tau) u'(\tau)^2 + \widetilde{N}(\tau) u(\tau)^2 \right) d\tau \tag{21}$$

is finite.

Proof. By the Friedrich extension we know that for any $u \in D(A_2)$ there exists a function $f \in L^2_w(I)$ such that for all $v \in C_0^\infty(I)$

$$\int_I \widetilde{D}(\tau) u'(\tau) v'(\tau) d\tau = \int_I f(\tau) \widetilde{N}(\tau) v(\tau) d\tau.$$

Therefore, $\widetilde{D}u'$ is absolutely continuous in I and

$$-\partial_\tau(\widetilde{D}(\tau)\partial_\tau u(\tau)) = f(\tau)\widetilde{N}(\tau).$$

As $\widetilde{D}$ is assumed to be smooth, we obtain for $\tau > 0$

$$u''(\tau) = \frac{\widetilde{N}(\tau)}{\widetilde{D}(\tau)} f(\tau) - \frac{\widetilde{D}'(\tau)}{\widetilde{D}(\tau)} u'(\tau) = \frac{\widetilde{N}^{1/2}(\tau)}{\widetilde{D}(\tau)} \widetilde{N}^{1/2}(\tau) f(\tau) - \frac{\widetilde{D}'(\tau)}{\widetilde{D}(\tau)^{3/2}} \widetilde{D}(\tau)^{1/2} u'(\tau).$$

Since, by (H1) and (19),

$$\frac{\widetilde{H}(\tau)^{1/2}}{\widetilde{D}(\tau)} \sim \tau^{-3\kappa/2} \quad \text{and} \quad \frac{\widetilde{D}'(\tau)}{\widetilde{D}(\tau)^{3/2}} \sim \tau^{-1-\kappa}$$

as $\tau \searrow 0$ and $\widetilde{N}^{1/2} f$, $\widetilde{D}^{1/2} u' \in L^2_w(I)$, we conclude that

$$\int_I \widetilde{D}(\tau)^\beta u''(\tau)^2 d\tau$$

is finite, where $\beta = \min\{3/2, 1 + 1/\kappa\}$. □

For the space $D(A_1)$ we obtain a similar result as Lemma 3.6: The condition $u(1) = 0$ is missing and in (21) two different values for β at the end points of I have to be used.

Finally, we estimate the singularity of u' for $u \in D(A_2)$ near the origin.

Lemma 3.7 *Let (19) hold and let $\kappa > 1/2$. Then there exists a constant $c_2 > 0$ such that for all $u \in D(A_2)$ and $\tau \in I$*

$$|u'(\tau)| \leq c_2 \tau^{1/2-\beta\kappa} \|u\|_{D(A_2)},$$

where $\beta = \min\{3/2, 1 + 1/\kappa\}$. In particular, $\widetilde{D}(\tau) u'(\tau)$ vanishes for $\tau = 0$.

Proof. Let $\tau \in (0, 1)$ and integrate the expression

$$u'(\tau) = u'(\sigma) + \int_\sigma^\tau u''(s) ds$$

over $\sigma \in (1/4, 3/4)$. Using the Cauchy-Schwarz inequality we estimate similarly as in the proof of Lemma 3.3

$$\begin{aligned} |u'(\tau)| &\leq 2\|u'\|_{L^1(1/4,3/4)} + 2\int_{1/4}^{3/4} \left(\int_\sigma^\tau \widetilde{D}(s)^{-\beta} ds\right)^{1/2} \left(\int_\sigma^\tau \widetilde{D}(s)^\beta u''(s)^2 ds\right)^{1/2} d\sigma \\ &\leq c(\|u'\|_{L^1(1/4,3/4)} + \tau^{1/2-\beta\kappa} \|u\|_{D(A_2)}) \\ &\leq c(\|u\|_{D(A_2)} + \tau^{1/2-\beta\kappa} \|u\|_{D(A_2)}) \\ &\leq c\tau^{1/2-\beta\kappa} \|u\|_{D(A_2)}, \end{aligned}$$

since $2\kappa\beta > 1$, and $c > 0$ is some constant independent of τ. □

Lemma 3.7 allows to prove a conservation law for the solutions of (11)-(14), (15)-(18), respectively. But first we have to guarantee that these problems are solvable. By the Theorem of Lumer and Philips and Theorem 2.5.2 in [20] the operators $-A_1$ and $-A_2$ are generators of elliptic semigroups (see also [14]).

Proposition 3.8 *The operators $-A_1$ and $-A_2$ generate analytic semigroups in $L^2_w(I)$. Moreover, for $f \in C^1([0,T]; L^2_w(I))$, $x_i \in D(A_i)$ $(i = 1, 2)$, the initial-value problems*

$$\begin{aligned} \frac{du_i}{dt} &= -A_i u_i + f(t), \quad t > 0, \\ u_i(0) &= x_i, \end{aligned}$$

have (unique) solutions $u_i \in C^0([0,T]; D(A_i)) \cap C^1([0,T]; L^2_w(I))$, $i = 1, 2$.

This proposition implies the existence of solutions to the problems (11)-(14) and (15)-(18).

Proposition 3.9 *Let $\widetilde{N}$ and $\widetilde{D}$ satisfy (13)-(14) for κ_1, κ_2, $\kappa_3 > 0$. Furthermore, let $G_0 \in C^2(\overline{I})$, $G_R \in C^2([0,T])$ for $T > 0$ with $G_R(0) = G_0(1)$.*

(1) *The problem (11)-(14) has a (unique) solution $G \in C^0([0,T]; D(A_1)) \cap C^1([0,T]; L^2_w(I))$. If (19) and (20) are satisfied and κ_1, $\kappa_2 > 1/2$ then the integral of G is preserved:*
$$\frac{d}{dt} \int_I G(\tau, t) \widetilde{N}(\tau) d\tau = 0.$$

(2) *The problem (15)-(18) has a (unique) solution $G \in C^0([0,T]; D(A_2)) \cap C^1([0,T]; L^2_w(I))$. If (19) is satisfied and $\kappa_3 > 1/2$ then*
$$\frac{d}{dt} \int_I G(\tau, t) \widetilde{N}(\tau) d\tau = \widetilde{D}(1) \partial_\tau G(1, t).$$

Proof. We only give a proof for (2). Defining $f(\tau, t) = G_R(t)$, we obtain from Proposition 3.8 that the initial-value problem

$$\begin{aligned} \frac{du}{dt}(t) &= -A_2 u(t) + \left(\frac{1}{\widetilde{N}(\tau)} (\partial_\tau (\widetilde{D}(\tau) \partial_\tau f(\tau, t)) - \frac{\partial f}{\partial t}(\cdot, t) \right) \\ &= -A_2 u(t) - \frac{\partial f}{\partial t}(\cdot, t), \\ u(0) &= G_0 - f(\cdot, 0), \end{aligned}$$

has a solution $u : [0,T] \to D(A_2)$. Thus, the function $G := u + f$ solves (15)-(18) and

$$\frac{d}{dt} \int_I G(\tau, t) \widetilde{N}(\tau) d\tau = \int_I \partial_\tau (\widetilde{D}(\tau) \partial_\tau G(\tau, t)) d\tau = \widetilde{D}(1) \partial_\tau G(1, t),$$

by Lemma 3.7 if $\kappa_3 > 1/2$. $\square$

Recall that we have not indicated the dependence of $\widetilde{N}$ and $\widetilde{D}$ on the parameter Δ. From the above result we obtain solutions $G = G_{\Delta,j}$ (see (H2)).

Remark 3.10 The integral

$$\int_I G(\tau, t)\widetilde{N}(\tau)d\tau$$

describes the density of particles (see Remark 3.1). Thus Proposition 3.9 (1) implies that the particle density is constant in time and (2) shows that the change of the density is caused by a flow over the boundary part on which the diffusion constant does not vanish. □

Remark 3.11 The condition $\kappa > 1/2$ can be understood as follows. Suppose that $\kappa < 1/2$. Starting with the domain

$$D(A_2) = \{u \in C^2(\overline{I}) : u(0) = u(1) = 0\}$$

of the operator A_2, a function $u \in H(A_2)$ in the energy space fulfills $u(0) = 0$ (see [17]). Hence we get a different Friedrich's extension. Only for values $\kappa > 1/2$ we do not need to prescribe a boundary condition at the left boundary point of the interval (for A_2). If $\kappa < 1/2$ some physical boundary condition has to be chosen in order to define the self-adjoint extension. □

Finally, we obtain the following result concerning the long-time behavior of the solutions to (11)-(14) and (15)-(18).

Proposition 3.12 (1) *Let G be a solution to (11)-(14) and let κ_1, $\kappa_2 \in (0, 2)$. Set*

$$\overline{G} = \left(\int_I \widetilde{N}(\tau)d\tau\right)^{-2} \int_I G_0(\tau)\widetilde{N}(\tau)d\tau.$$

Then there exists a constant $\lambda > 0$ such that for all $t > 0$

$$\|G(t) - \overline{G}\|_w \le e^{-\lambda t}\|G_0\|_w.$$

(2) *Let G be a solution to (15)-(18) and let $\kappa_3 \in (0, 2)$. Furthermore, let G_R be independent of t. Then there exists a constant $\lambda > 0$ such that for all $t > 0$:*

$$\|G(t) - G_R\|_w \le e^{-\lambda t}\|G_0 - G_R\|_w.$$

Proof. (1) By Lemma 3.4 the imbedding $H(A_1) \hookrightarrow L^2_w(I)$ is compact for κ_1, $\kappa_2 < 2$. Therefore, A_1 has a discrete spectrum with eigen values $0 = \lambda_0 < \lambda_1 \le \cdots$ and eigen functions w_0, $w_1, \ldots$ The eigen functions form a complete orthonormal system of $L^2_w(I)$. The solution of

$$\frac{du}{dt} = -A_1 u, \quad t > 0, \quad u(0) = G_0,$$

is given by

$$u(t) = \sum_{i=0}^{\infty} e^{-\lambda_i t}(G_0, w_i)_w w_i.$$

This implies

$$\|u(t) - (G_0, w_0)_w w_0\|_w \le e^{-\lambda_1 t}\|G_0\|_w.$$

Since w_0 is a constant function, it follows from $\|w_0\|_w = 1$ that

$$w_0 = \left(\int_I \widetilde{N}(\tau)d\tau\right)^{-1}$$

and hence $(G_0, w_0)_w w_0 = w_0^2 \int_I G_0\widetilde{N}d\tau = \overline{G}$.

(2) The imbedding $H(A_2) \hookrightarrow L_w^2(I)$ is compact for $\kappa_3 < 2$, by Lemma 3.4, and the operator A_2 is positive definite. Therefore, the spectrum of A_2 is discrete with eigen values $0 < \lambda_1 \le \lambda_2 \le \cdots$ and eigen functions $w_1, w_2, \ldots$ The solution of

$$\frac{du}{dt} = -A_2 u, \quad t > 0, \quad u(0) = x_0,$$

satisfies $\|u(t)\|_w \le e^{-\lambda_1 t}\|x_0\|_w$. For the problem (15)-(18) we have to find the solution of

$$\frac{du}{dt} = -A_2 u - \frac{\partial f}{\partial t}(\cdot, t),$$

where $f = G_R$ (see the proof of Proposition 3.9). The assumption that G_R does not depend on time implies $\partial f/\partial t(\cdot, t) = 0$, and since $G = u + f = u + G_R$, we conclude

$$\|G(t) - G_R\|_w \le e^{-\lambda_1 t}\|G_0 - G_R\|_w,$$

which proves the proposition. □

4 The existence result

By transforming back from the (τ, δ) to the (s, ε) variables in Proposition 3.9, we obtain an existence result for the original problem (2)-(5).

Theorem 4.1 *Let $F_0 \in C^2(\overline{I} \times [0,\infty))$, F_L, $F_R \in C^2([0,\infty)^2)$ with $F_L(\varepsilon, 0) = F_0(0, \varepsilon)$ and $F_R(\varepsilon, 0) = F_0(1, \varepsilon)$ for all $\varepsilon \ge 0$. Moreover, we assume that (H1)-(H2) hold and that $\kappa_i > 0$, $i = 1, 2, 3$. Then there exists a (unique) function $F : \overline{I} \times [0,\infty)^2 \to \mathbb{R}$ such that the initial and boundary conditions (4)-(5) are fulfilled. Furthermore, setting $Z_{\Delta,j} = \varphi(Y_{\Delta,j})$ (see (H2)), the restrictions $F_{\Delta,j} = F|_{Z_{\Delta,j}\times[0,\infty)}$ fulfill the differential equation (2)-(3) for any $\Delta \in \mathbb{R}$ and $j = 1, \ldots, m_\Delta$. The function $F_{\Delta,j}$ is given by $F_{\Delta,j} = G \circ \varphi^{-1}$, where G is the solution constructed in Proposition 3.9 for the level set $Y_{\Delta,j}$.*

Our result implies that the solution F is only differentiable with respect to one direction in the (s, ε) space: along the curves $Z_{\Delta,j}$. As

$$Z = I \times [0,\infty) = \bigcup_{\Delta\in\mathbb{R}} \bigcup_{j=1}^{m_\Delta} Z_{\Delta,j},$$

for every point (s, ε) there is one direction along which F is smooth. If F is also smooth across the curves $Z_{\Delta,j}$ remains an open problem.

Although some of the curves $Z_{\Delta,j}$ intersect the line $\varepsilon = 0$, there is no flow across this boundary if (1) is satisfied. In fact this holds true for smaller growth exponents κ of N (as long as $\kappa > 1/2$).

Concerning the long-time behavior of the solution F, we have to distinguish the following cases. Let $\kappa_i < 2$ $(i = 1, 2, 3)$ and $\Delta > 0$ such that the lines $\tau \mapsto \Delta$ and $\tau \mapsto \phi(\tau)$ in the (τ, δ) plane do not cross tangentially.

- If $Z_{\Delta,j}$ does not contain a boundary point $s \in \partial I$ then

  $$\int_I \left(F(s, \Delta - \phi(s), t) - \overline{G}\right)^2 N(\Delta - \phi(s))ds \to 0 \quad \text{as } t \to \infty.$$

  This situation corresponds, for instance, to the inner interval of Z_{Δ_4} in Figure 2.

- If $Z_{\Delta,j}$ contains exactly one boundary point $s_i \in \partial I$ $(i = L, R)$ and F_L or F_R does not depend on t, respectively, then

  $$\int_I (F(s, \Delta - \phi(s), t) - F_i(\Delta - \phi(s_i)))^2 N(\Delta - \phi(s))ds \to 0 \quad \text{as } t \to \infty.$$

  This situation with $i = L$ corresponds to Z_{Δ_2} in Figure 2.

- If $Z_{\Delta,j}$ contains two boundary points of ∂I, the problem (11)-(12) together with the boundary conditions is a non-degenerate parabolic problem and $F(t)$ converges to $G \circ \varphi^{-1}$ as $t \to \infty$, where G is the steady state of (11) along $Y_{\Delta,j} = \varphi^{-1}(Z_{\Delta,j})$ (see Z_{Δ_1} in Figure 2).

If the curves $\tau \mapsto \Delta$ and $\tau \mapsto \phi(\tau)$ cross tangentially, the multiplicity m of zero of $\tau \mapsto \Delta - \phi(\tau)$ is at least two. We conclude the same results as above only if $m\kappa < 2$. Hence, along each curve $Z_{\Delta,j}$ where the two curves cross tangentially, the long-time behavior result depends on the precise shape of the potential ϕ near points where $\partial_s \phi$ vanishes.

Remark 4.2 Some of the above results also hold for the more realistic two-dimensional plate $P \subset \mathbb{R}^2$ with Dirichlet boundary conditions. Indeed, the Hilbert space $L^2_w(P)$ can be defined similar to $L^2_w(I)$. Moreover, we can define the Friedrich extensions in the same way as in section 2. However, the properties of the energy spaces $H(A_i)$ are much more difficult to derive since the geometry of the problem is more involved. The manifolds Z_Δ are no longer one-dimensional but two-dimensional. The imbedding properties of the weighted Sobolev spaces will also change because the dimension plays an important role. These imbeddings are important for the asymptotic behavior and the conservation properties. □

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