Inviscid Limits of the Complex Ginzburg-Landau Equation

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Abstract

In the inviscid limit the generalized complex Ginzburg-Landau equation reduces to the nonlinear Schrödinger equation. This limit is proved rigorously with H^1 data in the whole space for the Cauchy problem and in the torus with periodic boundary conditions. The results are valid for nonlinearities with arbitrary growth exponent in the defocusing case and with subcritical or critical growth exponent at the level of L^2 in the focusing case, in any spatial dimension. Furthermore, optimal convergence rates are proved. The proofs are based on estimates of the Schrödinger energy functional and on Gagliardo-Nirenberg inequalities.

Keywords. Generalized complex Ginzburg-Landau equation, nonlinear Schrödinger equation, energy estimates, optimal rate of convergence, strong solutions.

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1 Introduction

The description of spatial pattern formation or chaotic dynamics in continuum systems, in particular fluid dynamical systems, is a challenging task in theoretical physics and applied mathematics. Due to the complexity of the corresponding nonlinear evolution equations, simpler model equations for which the mathematical issues can be solved with greater success, have been derived. The complex Ginzburg-Landau equation is one of these equations. It models the evolution of the amplitude of perturbations to steady-state solutions at the onset of instability [23, 31].

Originally discovered by Ginzburg and Landau for a phase transition in superconductivity [17], this equation was subsequently derived for instability waves in hydrodynamics such as the nonlinear growth of Rayleigh-Bénard convective rolls [33, 34], the appearance of Taylor vortices in the Couette flow between counter-rotating cylinders [39], and the development of Tollmien-Schlichting waves in plane Poiseuille flows [2]. The equation is also used to model the transition to turbulence in chemical reactions [19, 20, 24] or to describe pattern formation [8, 32].

The generalized complex Ginzburg-Landau (CGL) equation for the function $u(x,t)$ is given by

$$\begin{aligned} \partial_t u &= (a+i\nu)\Delta u + Ru - (b+i\mu)f(u) \quad \text{in } \Omega,\ t>0, && (1.1)\\ u(x,0) &= u_0(x) \quad \text{for } x\in\Omega. && (1.2) \end{aligned}$$

We consider this equation either in the whole space $\Omega = \mathbb{R}^d$ or in the torus $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$. In the latter case we prescribe periodic boundary conditions. The parameters a, b, ν, μ, R are real with $R \geq 0$, $a > 0$ and $b > 0$. Without loss of generality, we take $\nu > 0$. The interaction term $f(u)$ is a nonlinear function of u, a typical form of which is

$$f(u) = u|u|^{2\sigma}, \quad u \in \mathbb{C}, \tag{1.3}$$

for some $\sigma > 0$. Usually, the CGL equation with cubic nonlinearity $\sigma = 1$ has been considered in physics.

The Eq. (1.1) reduces to the nonlinear Schrödinger (NLS) equation for $a = b = 0$

$$\partial_t u = i\nu\Delta u + Ru - i\mu f(u) \quad \text{in } \Omega,\ t>0, \tag{1.4}$$

(more commonly with $R = 0$) and to the nonlinear heat equation with dissipative nonlinearity for $\nu = \mu = 0$, so that one may expect for it a mixture of the properties that occur in those two cases. Now, the self-focusing mechanism and its resulting finite-time blow-up for solutions of the NLS equation (with $R = 0$) is well understood both physically and mathematically [26, 30, 36]. More precisely, necessary conditions for the possible formation of finite-time singularities are $\mu < 0$ and $\sigma \geq d/2$. In this case, the interaction is attractive, and when $\sigma \geq d/2$ this attraction is strong enough to lead to singularities. When $\mu > 0$, the interaction is repulsive, and there is no tendency for the particles to concentrate.

Therefore, it is of particular mathematical interest to ask how "far" is the solution of the CGL equation (1.1) from the solution of the NLS equation (1.4), in order to relate the results for the NLS equation to the behavior of the solution to the CGL equation. In this paper we answer this question. More precisely, we prove rigorously the *inviscid limit* $a \to 0$, $b \to 0$ in the Eq. (1.1) and give optimal *convergence rates* for the difference of the solutions to Eqs. (1.1) and (1.4).

Optimal convergence rates for the difference of the solutions to Eqs. (1.1) and (1.4) have also been given by Wu in [41], however, his estimates do not allow the limit $a \to 0$, $b \to 0$, except in the case of linear interactions. The results of this paper are valid for any nonlinearity satisfying some structure conditions (see below).

Bartuccelli *et al.* [1] obtained asymptotic bounds for the CGL equation (with periodic boundary conditions) which suggest that large intermittent fluctuations can occur in two or three space dimensions near the inviscid limit. In one space dimension, however, only small deviations from mean quantities are possible. Numerical studies of the CGL equation near the inviscid limit have been performed in [18, 38]. Another paper which relates the CGL equation to the NLS equation is [25]. The authors show that so-called traveling holes of the CGL equation are singular perturbations of nonlinear Schrödinger dark solitons. Furthermore, Levermore studied recently the inviscid limit in terms of wavenumber modes [27].

In order to describe the main results of this paper we suppose that

$$0 < a < a_0, \quad 0 < b < b_0, \quad \nu > 0, \quad R \geq 0, \quad \mu \in \mathbb{R}, \quad d \geq 1, \tag{1.5}$$

and, for the sake of presentation, we assume that the nonlinearity $f(u)$ is given by (1.3). In Section 2 we present results for more general nonlinearities. We show the inviscid limit for nonlinearities with arbitrary growth exponent when $\mu \geq 0$ and at the level of L^2 when $\mu < 0$. Here, the term "subcritical at the level of L^2" indicates that $\sigma < 2/d$. More precisely, we assume that

$$\text{if } \mu \geq 0: \ 0 < \sigma < \infty; \tag{1.6}$$

$$\text{if } \mu < 0: \ \text{(i) } \sigma < 2/d, \ \text{or (ii) } \sigma = 2/d \text{ and } \|u_0\|_{L^2} \text{ is small enough.} \tag{1.7}$$

Let $\varepsilon = (a, b)$, let the initial datum satisfy $u_0 \in V \overset{\text{def}}{=} H^1(\Omega) \cap L^{2\sigma+2}(\Omega)$, $T > 0$, and let u_ε be the solution to the CGL equation (1.1)–(1.2) in $\Omega \times (0, T)$ (see Section 2 for a review on existence results). Furthermore, when $\mu < 0$, let $(ba^{-d\sigma/2})$ be bounded as $\varepsilon \to 0$.

Theorem 1.1 *Let the above assumptions hold. Then there exists a subsequence of* (u_ε) *(not relabeled) such that, as* $\varepsilon \to 0$, u_ε *converges to a function* u *which solves the NLS equation (1.4), (1.2). The convergence is strong in the space* $L^\infty(0, T; L^r_{loc}(\Omega))$, *where* $r < 2d/(d-2)$, *and weak-∗ in the spaces* $L^\infty(0, T; H^1(\Omega))$ *and* $W^{1,\infty}(0, T; V^*)$.

If the NLS equation is uniquely solvable (e.g., when $\sigma \leq 2/(d-2)$; cf. [6]) then the whole sequence (u_ε) converges.

For the convergence rates, let u_ε be the solution to the CGL equation (1.1), (1.2) with initial datum $u_{\varepsilon 0}$ and let u be the solution to the NLS equation (1.4), (1.2) with initial datum u_0. We assume that $u_{\varepsilon 0}$, $u_0 \in H^n(\Omega)$ with $n > d/2$ and $\|u_{\varepsilon 0} - u_0\|_{L^2} = \mathcal{O}(a) + \mathcal{O}(b)$ as $\varepsilon \to 0$.

Theorem 1.2 *Let the above assumptions hold. Then there exists a constant $T^* > 0$ independent of a, b such that for all $0 \le t < T^*$,*

$$\|(u_\varepsilon - u)(t)\|_{L^2} = \mathcal{O}(a) + \mathcal{O}(b) \qquad \text{as } \varepsilon \to 0.$$

Moreover, we can take $T^* = T$ if $d = 1$.

The growth condition for the nonlinearity in the above results is sharp. Indeed, when $\Omega = \mathbb{R}^d$ and $u_0 \in H^1(\Omega) \cap L^{2\sigma+2}(\Omega)$, there exists a global in time weak solution to the NLS equation if the conditions (1.6)–(1.7) hold. Moreover, if $\sigma \ge 2/d$ then there exist solutions that blow up in finite time (see [6, Rem. 6.5.1, Thm. 9.4.1]). Theorems 1.1 and 1.2 are corollaries from the more general results of Section 2 (see Theorems 2.1, 2.2, 2.4, and 2.5).

In the following we explain our method of proof of the inviscid limits. The basic idea is to treat the CGL equation as a perturbation of the NLS equation. Now, the NLS flow (with $R = 0$) conserves the mass M and the energy E, which are defined by

$$M(t) = \int_\Omega |u(t)|^2 \, dx, \tag{1.8}$$

$$E(t) = \frac{\nu}{2} \int_\Omega |\nabla u(t)|^2 \, dx + \frac{\mu}{2} \int_\Omega F(|u(t)|^2) \, dx, \tag{1.9}$$

and F is given by

$$F(s) = \frac{1}{\sigma + 1} s^{\sigma+1}, \quad s \ge 0.$$

One may expect that in the case of the CGL equation, these quantities are also bounded in terms of $M(0)$, $E(0)$, and Rt (if $R > 0$). This is indeed true. Let u be a weak solution to the CGL equation. A simple computation shows that, when $\mu \ge 0$ (see Section 3),

$$\begin{aligned} \partial_t M &\le 2RM, \\ \partial_t E &\le 2R(\sigma + 1)E, \end{aligned}$$

for all $t > 0$, which implies that

$$M(t) \le M(0)e^{2Rt}, \quad E(t) \le E(0)e^{2R(\sigma+1)t}.$$

When $\mu \ge 0$ we immediately obtain H^1 and $L^{2\sigma+2}$ bounds for $u(t)$ uniformly in a, b. In the case $\mu < 0$ we need, for instance, to control the integral

$$-\frac{\mu}{2} \int_\Omega F(|u(t)|^2) dx = -\frac{\mu}{2(\sigma + 1)} \int_\Omega |u(t)|^{2\sigma+2} dx$$

with the H^1 norm of $u(t)$. This is achieved through the Gagliardo-Nirenberg inequality (see Lemma 2.6)

$$\|u\|_{L^{2\sigma+2}} \le C(\sigma, d)\|u\|_{H^1}^{\theta}\|u\|_{L^2}^{1-\theta},$$

where the interpolation parameter θ satisfies $\theta = \sigma d/(2\sigma+2)$. Notice that $0 < \theta < 1$ if and only if $0 < \sigma < 2/(d-2)$. This inequality gives, for $0 \le t \le T$,

$$\begin{aligned} E(t) + \frac{\nu}{2}M(t) &= \frac{\nu}{2}\|u(t)\|_{H^1}^2 + \frac{\mu}{2(\sigma+1)}\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2} \\ &\ge \frac{\nu}{2}\|u(t)\|_{H^1}^2 + \mu C'\|u(t)\|_{H^1}^{(2\sigma+2)\theta}\|u(t)\|_{L^2}^{(2\sigma+2)(1-\theta)}. \end{aligned}$$

Because $\|u(t)\|_{L^2}$ is uniformly bounded, this estimate provides an upper bound for $\|u(t)\|_{H^1}$ uniformly in a, b, whenever $E(t)$ is uniformly bounded and the exponent of the H^1 norm of $u(t)$ in the second term of the right-hand side is less than in the first term, which is the case if and only if $\sigma < 2/d$. For the limit case $\sigma = 2/d$, the above estimate gives a uniform bound for $\|u(t)\|_{H^1}$ provided that $\|u(t)\|_{L^2}$ or, equivalently, $\|u_0\|_{L^2}$ is small enough.

It is clear that the uniform H^1 bound for $u(t)$ is the key estimate in the proof of Theorem 1.1. In a similar way, one can obtain uniform H^n bounds for $u(t)$ $(n > 1)$ by estimating the quantities $\partial_t\|u\|_{H^n}^2$ (see Sections 5 and 7). These bounds can be only derived for small enough $t > 0$, since the use of the Gagliardo-Nirenberg inequality leads to inequalities of the type

$$\partial_t\|u\|_{H^n}^2 \le C_1 + C_2\|u\|_{H^n}^{2+\alpha},$$

with positive $\alpha = \alpha(\sigma, d)$.

For the proof of Theorem 1.2 we consider the equation satisfied by the difference $u_\varepsilon - u$, where u_ε with $\varepsilon = (a, b)$ solves the CGL equation and u solves the NLS equation. In order to estimate the term $f(u_\varepsilon) - f(u)$ we need uniform L^∞ bounds for the solutions which are obtained via the Sobolev embedding theorem from the uniform H^n bounds with $n > d/2$.

This paper is organized as follows. In Section 2 we recall some existence results for the CGL equation and state our main theorems. Section 3 is concerned with the uniform H^1 estimate for the solutions of the CGL equation. The first convergence result is proved in Section 4. A uniform bound in H^2 is derived in Section 5. Finally, the proof of the optimal convergence rate is presented in Section 6 (for at most three spatial dimensions) and Section 7 (for any spatial dimension).

2 Preliminaries and main results

The (global in time) existence and uniqueness of solutions to the generalized CGL equation (1.1)–(1.2) is shown by Ginibre and Velo for $\Omega = \mathbb{R}^d$ [14, 15, 16] and by

Doering, Gibbon and Levermore for $\Omega = \mathbb{T}^d$ [9] (also see the review [28]) under some assumptions on the nonlinearity. In particular, assume the following:

$$\begin{array}{l}\text{Let } \Omega = \mathbb{R}^d \text{ or } \Omega = \mathbb{T}^d\ (d \geq 1),\ u_0 \in V \stackrel{\text{def}}{=} H^1(\Omega) \cap L^{2\sigma+2}(\Omega), \\ T > 0,\ 0 < \sigma < \infty \text{ and, in the case } \mu < 0, \text{ let either (i) } \sigma < 2/d, \text{ or} \\ \text{(ii) } \sigma = 2/d \text{ and } \|u_0\|_{L^2} \text{ be small enough.}\end{array} \tag{2.1}$$

Note that $V = H^1(\Omega)$ if $\sigma \leq 2/(d-2)$. We impose the following assumption on the non-linearity $f(u)$:

$$f(u) = ug(|u|^2), \quad u \in \mathbb{C}, \tag{2.2}$$

where the function $g(s)$ satisfies:

$$\begin{array}{l}g \in C^1([0,\infty);[0,\infty)) \text{ and there exist } \sigma, \gamma, \Gamma, \Gamma' > 0 \text{ such that for} \\ \text{all } s \geq 0 \\ \qquad\qquad \gamma s^\sigma \leq g(s) \leq \Gamma(1 + s^\sigma), \\ \qquad\qquad 0 \leq s g'(s) \leq \Gamma'(1 + s^\sigma).\end{array} \tag{2.3}$$

For instance, the functions $g(s) = s^\sigma, g(s) = s^\sigma + \log(1+s)$, or $g(s) = se^s/(1+e^s)$ satisfy the condition (2.3). In order to define the mass (1.8) and energy (1.9) we introduce

$$F(s) = \int_0^s g(\xi)d\xi, \quad s \geq 0.$$

Under the above assumptions there exists a unique solution u to (1.1)–(1.2) satisfying

$$u \in H^1(0,T;L^2(\Omega)) \cap C^0([0,T];H^1(\Omega) \cap L^{2\sigma+2}(\Omega)) \cap L^2(0,T;H^2(\Omega)).$$

This follows from the a priori estimates of Section 3 and the compactness arguments of [9, 14]. Furthermore, if additionally $u_0 \in H^n(\Omega)$ $(n \in \mathbb{N})$, then the solution u satisfies

$$u \in H^1(0,T;H^{n-1}(\Omega)) \cap C^0([0,T];H^n(\Omega)) \cap L^2(0,T;H^{n+1}(\Omega)).$$

This regularity result can be achieved by applying, for instance, Theorems II.3.2 and II.3.3 in [40, Ch. I.3.3].

Concerning the existence and uniqueness of solutions to the NLS equation (1.4), (1.2) we refer to [6, 7, 13] for $\Omega = \mathbb{R}^d$ and to [3, 4, 21] for $\Omega = \mathbb{T}^d$ (see also [22] and the reviews in [11, 12]).

Our first main result is concerned with the convergence of the solution u_ε of the CGL equation with initial datum u_0 and parameter $\varepsilon = (a, b)$ to a solution u of the NLS equation. In the case $\mu < 0$ we need an additional technical assumption on the parameters a and b:

$$\text{Let } (ba^{-d\sigma/2}) \text{ be bounded as } (a,b) \to 0. \tag{2.4}$$

Theorem 2.1 *Let the assumptions (1.5), (2.1)–(2.4) hold. Furthermore, let $\Omega = \mathbb{T}^d$. Let u_ε be the solution to the CGL equation (1.1)–(1.2). Then there exists a subsequence of u_ε (not relabeled) such that, as $\varepsilon \to 0$,*

$$\begin{aligned} u_\varepsilon &\rightharpoonup u && \text{in } L^\infty(0,T;H^1(\Omega)) \text{ weakly-}*, \\ \partial_t u_\varepsilon &\rightharpoonup \partial_t u && \text{in } L^\infty(0,T;V^*) \text{ weakly-}*, \\ u_\varepsilon &\to u && \text{in } L^\infty(0,T;L^r_{loc}(\Omega)) \text{ strongly}, \\ f(u_\varepsilon) &\to f(u) && \text{in } L^\infty(0,T;L^q_{loc}(\Omega)) \text{ strongly}, \end{aligned}$$

where $r < 2d/(d-2)$ ($r < \infty$ if $d \le 2$); $2/(2\sigma+1) < q < (2\sigma+2)/(2\sigma+1)$ when $\mu \ge 0$; $q = 2d/(d+2)$ when $\mu < 0$; and $u \in L^\infty(0,T;H^1(\Omega)) \cap W^{1,\infty}(0,T;H^{-1}(\Omega)) \cap L^{(2\sigma+1)q}(\Omega \times (0,T))$ solves the NLS equation (1.4), (1.2).

Theorem 2.2 *Let the assumptions of Theorem 2.1 hold, but assume that $\Omega = \mathbb{R}^d$ and that $g(s) \le \Gamma s^\sigma$ for $\sigma \ge 0$. Then the same conclusion as in Theorem 2.1 holds.*

The proof of this result is based on a uniform bound for u_ε in $L^\infty(0,T;H^1(\Omega))$, which allows to bound the term $f(u_\varepsilon)$. In one space dimension, the Sobolev embedding $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ therefore provides a uniform bound for u_ε in $L^\infty(\Omega \times (0,T))$ from which we can conclude a $\mathcal{O}(\sqrt{a}) + \mathcal{O}(\sqrt{b})$ convergence rate for the difference $u_\varepsilon - u$ in $L^\infty(0,T;L^2(\Omega))$ under the condition that the initial data satisfies

$$\|u_{\varepsilon 0} - u_0\|_{L^2} = \mathcal{O}(\sqrt{a}) + \mathcal{O}(\sqrt{b}) \qquad \text{as } a,b \to 0. \tag{2.5}$$

Theorem 2.3 *Let $d = 1$ and assume that the hypotheses (1.5), (2.1)–(2.5) hold. Then we have for all $0 \le t \le T$,*

$$\|(u_\varepsilon - u)(t)\|_{L^2} = \mathcal{O}(\sqrt{a}) + \mathcal{O}(\sqrt{b}) \qquad \textit{as } a,b \to 0.$$

More precisely, there exist two constants $M_1(T), M_2(T) > 0$ depending on the data but not on a, b such that for all $0 \le t \le T$,

$$\|(u_\varepsilon - u)(t)\|_{L^2} \le \|u_{\varepsilon 0} - u_0\|_{L^2} + M_1(T)\sqrt{a} + M_2(T)\sqrt{b}.$$

In order to get the optimal convergence rate $\mathcal{O}(a) + \mathcal{O}(b)$ for $(u_\varepsilon - u)(t)$ in $L^2(\Omega)$, we have to impose stronger assumptions on the nonlinearity g. Suppose that

$$\begin{gathered} g \in C^2([0,\infty);[0,\infty)) \text{ and there exist } \sigma \ge 1/2, \Gamma_1, \Gamma_1' > 0 \text{ such that} \\ \text{for all } s \ge 0 \\ 0 \le s g'(s) \le \Gamma_1(s^{1/2} + s^\sigma), \\ 0 \le s^2 g''(s) \le \Gamma_1'(s^{1/2} + s^\sigma). \end{gathered} \tag{2.6}$$

It can be easily checked that the condition on g' in (2.6) implies that

$$0 \le s g'(s) \le \Gamma_2(1 + s^\sigma) \quad \text{for all } s \ge 0,$$

for some $\Gamma_2 > 0$. Thus the condition on g' in (2.6) is stronger than the corresponding condition in (2.3).

Theorem 2.4 *Let $1 \leq d \leq 3$ and $u_{\varepsilon 0}, u_0 \in H^2(\Omega)$ and assume the conditions (1.5), (2.1)–(2.6). Then, if $\|u_{\varepsilon 0} - u_0\|_{L^2} = \mathcal{O}(a) + \mathcal{O}(b)$, there exists a constant $T^* > 0$ depending on the data but not on a, b such that for all $0 \leq t < T^*$,*

$$\|(u_\varepsilon - u)(t)\|_{L^2} = \mathcal{O}(a) + \mathcal{O}(b) \qquad \text{as } a, b \to 0.$$

Moreover, we can take $T^* = T$ if $d = 1$.

More precisely, there exist two constants $M_3(t), M_4(t) > 0$ which are independent of a, b such that for all $0 \leq t < T^*$,

$$\|(u_\varepsilon - u)(t)\|_{L^2} \leq \|u_{\varepsilon 0} - u_0\|_{L^2} + M_3(t)a + M_4(t)b.$$

The constant $T^* > 0$ depends on $d, \sigma, R, \Gamma_1, |b_0 + i\mu|, C_4, C_6$ and $\|\Delta u_0\|_{L^2}$ (see (5.8)). See (5.3) and (5.6) for the definitions of C_4 and C_6.

Our last result is concerned with the optimal convergence rate in any space dimension. In order to avoid cumbersome assumptions on the function g, we suppose that

$$g(s) = s^\sigma \qquad \text{for some } \sigma > 0. \tag{2.7}$$

Theorem 2.5 *Let $d \geq 1$, $u_{\varepsilon 0}, u_0 \in H^n(\Omega)$ with $n \in \mathbb{N}$, $n > d/2$, and suppose that (1.5), (2.1)–(2.4) and (2.7) hold. Then, if $\|u_{\varepsilon 0} - u_0\|_{L^2} = \mathcal{O}(a) + \mathcal{O}(b)$, there exists a constant $T^* > 0$ depending on the data but not on a, b such that for all $0 \leq t < T^*$,*

$$\|(u_\varepsilon - u)(t)\|_{L^2} = \mathcal{O}(a) + \mathcal{O}(b) \qquad \text{as } a, b \to 0.$$

In the proofs of the main theorems we employ the Gagliardo-Nirenberg inequality (see, e.g., [6, p. 21] or [10, p. 242]):

Lemma 2.6 *Let $\Omega = \mathbb{R}^d$, or $\Omega = \mathbb{T}^d$, or let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $\partial\Omega \in C^{0,1}$, and $d, m \geq 1$. Furthermore, let $1 \leq p, q, r \leq \infty$, $j \in \mathbb{N} \cup \{0\}$ with $j < m$ and $\theta \in [j/m, 1]$ such that*

$$\frac{1}{p} = \frac{j}{d} + \theta\Big(\frac{1}{r} - \frac{m}{d}\Big) + (1-\theta)\frac{1}{q},$$

provided that $m - j - d/r$ is not a nonnegative integer (else take $\theta = j/m$). Then there exists $G = G(d, m, j, \theta, q, r)$ such that for all $u \in W^{m,r}(\Omega) \cap L^q(\Omega)$

$$\sum_{|\alpha|=j} \|D^\alpha u\|_{L^p} \leq G\|u\|^\theta_{W^{m,r}} \|u\|^{1-\theta}_{L^q}.$$

Frequently, we use the inequality

$$\|v\|_{H^2} \leq C_0(\|v\|_{L^2} + \|\Delta v\|_{L^2}) \tag{2.8}$$

which is valid for all $v \in H^2(\Omega)$ with $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$.

3 A uniform bound in H^1

We show that the mass M and the energy E (see (1.8), (1.9), respectively) of the solution u to the CGL equation (1.1)–(1.2) is bounded uniformly in a, b. In the following we assume that (1.5), (2.2)–(2.3) hold.

Lemma 3.1 *Let $u_0 \in L^2(\Omega)$. For all $0 \le t \le T$, it holds*

$$M(t) \le K_0(T),$$

where $K_0(T) = e^{2RT}\|u_0\|_{L^2}^2$.

Proof. The proof of this lemma is standard. Indeed, multiply the equation (1.1) by u^*, integrate over Ω, integrate by parts and take the real part to get

$$\frac{1}{2}\partial_t \|u(t)\|_{L^2}^2 + a\|\nabla u\|_{L^2}^2 + b\int_\Omega |u|^2 g(|u|^2)dx = R\|u\|_{L^2}^2.$$

An application of Gronwall's lemma gives the assertion.

Lemma 3.2 *Let $u_0 \in H^1(\Omega) \cap L^{2\sigma+2}(\Omega)$, $\mu \ge 0$, and $\sigma < \infty$. Then, for all $0 \le t \le T$,*

$$E(t) \le K_1(T),$$

where

$$K_1(T) = (E(0) + \mu R\Gamma T K_0(T))e^{2R\max(1,(\sigma+1)\Gamma\gamma^{-1})T}.$$

Proof. For every solution u to (1.1), (1.2), we get

$$\begin{aligned}
E(t) - E(0) &= \mathrm{Re}\int_0^t\int_\Omega(-\nu\Delta u + \mu f(u))\partial_t u^*\,dxd\tau \\
&= -\nu a\int_0^t \|\Delta u\|_{L^2}^2\,d\tau - \mu b\int_0^t \|f(u)\|_{L^2}^2\,d\tau \\
&\quad + \nu R\int_0^t \|\nabla u\|_{L^2}^2\,d\tau + \mu R\mathrm{Re}\int_0^t\int_\Omega f(u)u^*\,dxd\tau \\
&\quad - \mu\mathrm{Re}\int_0^t\int_\Omega (a - i\nu)\nabla f(u)\cdot\nabla u^*\,dxd\tau \\
&\quad - \nu\mathrm{Re}\int_0^t\int_\Omega (b - i\mu)\nabla f(u)^*\cdot\nabla u\,dxd\tau \\
&= -\nu a\int_0^t \|\Delta u\|_{L^2}^2\,d\tau - \mu b\int_0^t\int_\Omega |u|^2 g(|u|^2)\,dxd\tau \\
&\quad + \nu R\int_0^t \|\nabla u\|_{L^2}^2\,d\tau + \mu R\int_0^t\int_\Omega |u|^2 g(|u|^2)\,dxd\tau \qquad (3.1) \\
&\quad - (a\mu + b\nu)\mathrm{Re}\int_0^t\int_\Omega \nabla f(u)\cdot\nabla u^*\,dxd\tau.
\end{aligned}$$

The fourth term on the right-hand side of (3.1) is estimated as follows. We use (2.3) to obtain

$$\begin{aligned}
&\mu R \int_0^t \int_\Omega |u|^2 g(|u|^2)\, dx d\tau \\
&\le \quad \mu R \Gamma \int_0^t \int_\Omega (|u|^2 + |u|^{2\sigma+2})\, dx d\tau \\
&\le \quad \mu R \Gamma \int_0^t \|u\|_{L^2}^2\, d\tau + \mu R \Gamma \gamma^{-1}(\sigma+1) \int_0^t \int_\Omega F(|u|^2)\, dx d\tau.
\end{aligned}$$

For the last integral we use $g'(s) \ge 0$ and $\mathrm{Re}((\nabla u^*)^2 u^2) \le |\nabla u|^2 |u|^2$:

$$\begin{aligned}
&-(a\mu + b\nu)\mathrm{Re} \int_0^t \int_\Omega \nabla f(u) \cdot \nabla u^*\, dx d\tau \\
&\quad = \quad -(a\mu + b\nu) \int_0^t \int_\Omega \Big[g(|u|^2)|\nabla u|^2 + g'(|u|^2)|u|^2 |\nabla u|^2 \\
&\qquad\qquad + g'(|u|^2)\mathrm{Re}((\nabla u^*)^2 u^2) \Big]\, dx d\tau \\
&\quad \le \quad -(a\mu + b\nu) \int_0^t \int_\Omega g(|u|^2)|\nabla u|^2\, dx d\tau \\
&\quad \le \quad 0.
\end{aligned}$$

Therefore, we get from (3.1)

$$\begin{aligned}
E(t) \quad &\le \quad E(0) + \nu R \int_0^t \|\nabla u\|_{L^2}^2\, d\tau + \mu R \Gamma \int_0^t \|u\|_{L^2}^2\, d\tau \\
&\qquad + \mu R \Gamma \gamma^{-1}(\sigma+1) \int_0^t \int_\Omega F(|u|^2)\, dx d\tau \\
&\le \quad E(0) + \mu R \Gamma K_0(T) T + 2R \max(1, \Gamma \gamma^{-1}(\sigma+1)) \int_0^t E(\tau)\, d\tau,
\end{aligned}$$

using Lemma 3.1. From Gronwall's lemma, we conclude

$$E(t) \le K_1(T).$$

The lemma is proved.

Remark 3.3 The condition (2.3) on g' can be weakened. Indeed, instead of assuming $s \cdot g'(s) \ge 0$ for $s \ge 0$ it suffices to assume that

$$s g'(s) \ge -\frac{1}{2} g(s) \quad \text{for } s \ge 0. \tag{3.2}$$

In order to see this we have to estimate the last term on the right-hand side of (3.1):

$$-(a\mu + b\nu)\mathrm{Re} \int_0^t \int_\Omega \nabla f(u) \cdot \nabla u^*\, dx d\tau$$

$$
\begin{aligned}
&\le (a\mu + b\nu) \int_0^t \int_\Omega |\nabla u|^2 \Big[-g(|u|^2) + |u|^2 (|g'(|u|^2)| - g'(|u|^2)) \Big] \\
&\le (a\mu + b\nu) \int_0^t \int_\Omega |\nabla u|^2 [-g(|u|^2) + g(|u|^2)]\, dx d\tau \\
&= 0.
\end{aligned}
$$

In the case $\mu < 0$ we need more restrictive conditions on σ:

Lemma 3.4 *Let $u_0 \in H^1(\Omega) \cap L^{2\sigma+2}(\Omega), \mu < 0$, and either (i) $\sigma < 2/d$, or (ii) $\sigma = 2/d$ and $\|u_0\|_{L^2}$ is small enough. Then, for all $0 \le t \le T$,*

$$\|\nabla u(t)\|_{L^2}^2 \le K_2(T),$$

where $K_2(T)$ depends on the given data, on T and on $ba^{-d\sigma/2}$.

We use the Gagliardo-Nirenberg inequality (see Lemma 2.6) in some special cases: There exist $G_1, G_2, G_3, G_4 > 0$ such that for all $u \in H^2(\Omega)$

$$\|u\|_{L^{2\sigma+2}} \le G_1 \|u\|_{H^1}^{\theta 1} \|u\|_{L^2}^{1-\theta_1}, \quad \theta_1 = \frac{d\sigma}{2\sigma+2} \in (0,1), \tag{3.3}$$

$$\|u\|_{L^{2\sigma+2}} \le G_2 \|u\|_{H^2}^{\theta_2} \|u\|_{L^2}^{1-\theta_2}, \quad \theta_2 = \frac{d\sigma}{4\sigma+4} \in \Big(0, \frac{1}{2}\Big], \tag{3.4}$$

$$\|u\|_{L^{4\sigma+2}} \le G_3 \|u\|_{H^2}^{\theta_3} \|u\|_{L^2}^{1-\theta_3}, \quad \theta_3 = \frac{d\sigma}{4\sigma+2} \in (0,1), \tag{3.5}$$

$$\|\nabla u\|_{L^{2\sigma+2}} \le G_4 \|u\|_{H^2}^{\theta_4} \|u\|_{L^2}^{1-\theta_4}, \quad \theta_4 = \frac{\sigma(2+d)+2}{4\sigma+4} \in \Big[0, \frac{1}{2}\Big). \tag{3.6}$$

These inequalities are valid for all $\sigma \le 2/(d-2)$ if $d > 2$ and $\sigma < \infty$ if $d \le 2$.

Proof of Lemma 3.4. Again we start from the relation (3.1). Since $\mu < 0$, we get

$$
\begin{aligned}
&\frac{\nu}{2} \|\nabla u(t)\|_{L^2}^2 + a\nu \int_0^t \|\Delta u\|_{L^2}^2\, d\tau \\
&\quad \le \frac{\nu}{2} \|\nabla u_0\|_{L^2}^2 + \frac{|\mu|}{2} \int_\Omega F(|u(t)|^2)\, dx \\
&\qquad + |\mu| b \int_0^t \int_\Omega |u|^2 g(|u|^2)^2\, dx d\tau + \nu R \int_0^t \|\nabla u\|_{L^2}^2\, d\tau \\
&\qquad - (a\mu + b\nu) \mathrm{Re} \int_0^t \int_\Omega \nabla f(u) \cdot \nabla u^*\, dx d\tau.
\end{aligned} \tag{3.7}
$$

We have to estimate the second, third and fifth term on the right-hand side. For the second term we use the hypothesis (2.3), the Gagliardo-Nirenberg inequality (3.3) and Lemma 3.1:

$$\frac{|\mu|}{2} \int_\Omega F(|u(t)|^2\, dx$$

$$
\begin{aligned}
&\le \frac{|\mu|}{2}\Gamma \int_\Omega (|u(t)|^2 + (\sigma+1)^{-1}|u(t)|^{2\sigma+2})\,dx \\
&\le \frac{|\mu|}{2}\Gamma K_0(T) + \frac{|\mu|\Gamma}{2\sigma+2} G_1^{2\sigma+2}\|u(t)\|_{H^1}^{(2\sigma+2)\theta_1}\|u(t)\|_{L^2}^{(2\sigma+2)(1-\theta_1)} \\
&\le \frac{|\mu|}{2}\Gamma K_0(T) + \frac{|\mu|\Gamma G_1^{2\sigma+2}}{\sigma+1}(\|u(t)\|_{L^2}^{d\sigma} + \|\nabla u(t)\|_{L^2}^{d\sigma})\|u(t)\|_{L^2}^{2\sigma+2-d\sigma}.
\end{aligned}
$$

In the last step we have used the inequality

$$
(x+y)^{(2\sigma+2)\theta_1} = (x+y)^{d\sigma} \le 2(x^{d\sigma} + y^{d\sigma}) \quad \text{for } x, y \ge 0,
$$

since $d\sigma \le 2$. Again using Lemma 3.1 we get

$$
\begin{aligned}
\frac{|\mu|}{2}\int_\Omega F(|u(t)|^2)\,dx &\le \frac{|\mu|}{2}\Gamma K_0(T) + \frac{|\mu|\Gamma G_1^{2\sigma+2}}{\sigma+1} K_0(T)^{\sigma+1} \\
&\quad + \frac{|\mu|\Gamma G_1^{2\sigma+2}}{\sigma+1}\|\nabla u(t)\|_{L^2}^{d\sigma}\|u(t)\|_{L^2}^{2\sigma+2-d\sigma}.
\end{aligned} \tag{3.8}
$$

Now we have to distinguish the cases $d\sigma < 2$ and $d\sigma = 2$. Let first $d\sigma < 2$. Then, by Young's inequality,

$$
\begin{aligned}
&\frac{|\mu|\Gamma G_1^{2\sigma+2}}{\sigma+1}\|\nabla u(t)\|_{L^2}^{d\sigma}\|u(t)\|_{L^2}^{2\sigma+2-d\sigma} \\
&\quad \le \varepsilon \frac{d\sigma}{2}\frac{|\mu|\Gamma G_1^{2\sigma+2}}{\sigma+1}\|\nabla u(t)\|_{L^2}^2 \\
&\quad\quad + \Big(\frac{1}{\varepsilon}\Big)^{d\sigma/(2-d\sigma)} \frac{2-d\sigma}{2}\frac{|\mu|\Gamma G_1^{2\sigma+2}}{\sigma+1}\|u(t)\|_{L^2}^{(4\sigma+4-2d\sigma)/(2-d\sigma)}
\end{aligned}
$$

and taking $\varepsilon = \nu(\sigma+1)/(2d\sigma|\mu|\Gamma G_1^{2\sigma+2})$, we conclude

$$
\frac{|\mu|}{2}\int_\Omega F(|u(t)|^2)\,dx \le C_1 + \frac{\nu}{4}\|\nabla u(t)\|_{L^2}^2, \tag{3.9}
$$

where

$$
\begin{aligned}
C_1 &\stackrel{\text{def}}{=} \frac{|\mu|}{2}\Gamma K_0(T) + \frac{|\mu|\Gamma G_1^{2\sigma+2}}{\sigma+1}K_0(T)^{\sigma+1} \\
&\quad + \Big(\frac{2d\sigma|\mu|\Gamma G_1^{2\sigma+2}}{\nu(\sigma+1)}\Big)^{d\sigma/(2-d\sigma)} \frac{(2-d\sigma)|\mu|\Gamma G_1^{2\sigma+2}}{2\sigma+2}K_0(T)^{(2\sigma+2-d\sigma)/(2-d\sigma)}.
\end{aligned} \tag{3.10}
$$

In the case $d\sigma = 2$ we choose $\|u_0\|_{L^2}$ small enough such that

$$
\frac{|\mu|\Gamma G_1^{2\sigma+2}}{\sigma+1}e^{2\sigma RT}\|u_0\|_{L^2}^{2\sigma} \le \frac{\nu}{4}.
$$

Hence, by (3.8),

$$
\begin{aligned}
\frac{|\mu|}{2}\int_\Omega F(|u(t)|^2)\,dx &\le \frac{|\mu|}{2}\Gamma K_0(T) + \frac{|\mu|\Gamma G_1^{2\sigma+2}}{\sigma+1}K_0(T)^{\sigma+1} + \frac{\nu}{4}\|\nabla u(t)\|_{L^2}^2 \\
&\le C_1 + \frac{\nu}{4}\|\nabla u(t)\|_{L^2}^2,
\end{aligned}
$$

and $C_1 > 0$ is given by (3.10).

The third term on the right-hand side of (3.7) is estimated by employing the assumption (2.3), the Gagliardo-Nirenberg inequality (3.5), and the relation (2.8)

$$
\begin{aligned}
& b|\mu| \int_0^t \int_\Omega |u|^2 g(|u|^2)^2 \, dx d\tau \\
& \leq \; 2b|\mu|\Gamma^2 \int_0^t \int_\Omega (|u|^2 + |u|^{4\sigma+2}) \, dx d\tau \\
& \leq \; 2b|\mu|\Gamma^2 T K_0(T) + 2b|\mu|\Gamma^2 G_3^{4\sigma+2} \int_0^t \|u\|_{H^2}^{(4\sigma+2)\theta_3} \|u\|_{L^2}^{(4\sigma+2)(1-\theta_3)} \, d\tau \\
& \leq \; 2b|\mu|\Gamma^2 T K_0(T) + 4b|\mu|\Gamma^2 (G_3 C_0^{\theta_3})^{4\sigma+2} \int_0^t (\|u\|_{L^2}^{d\sigma} + \|\Delta u\|_{L^2}^{d\sigma}) \|u\|_{L^2}^{4\sigma+2-d\sigma} \, d\tau \\
& \leq \; 2b|\mu|\Gamma^2 T K_0(T) + 4b|\mu|\Gamma^2 (G_3 C_0^{\theta_3})^{4\sigma+2} T K_0(T)^{2\sigma+1} \\
& \quad + 4b|\mu|\Gamma^2 (G_3 C_0^{\theta_3})^{4\sigma+2} \int_0^t \|\Delta u\|_{L^2}^{d\sigma} \|u\|_{L^2}^{4\sigma+2-d\sigma} \, d\tau. \qquad (3.11)
\end{aligned}
$$

Again, we have to distinguish the cases $d\sigma < 2$ and $d\sigma = 2$. If $d\sigma < 2$ we use Young's inequality

$$
\begin{aligned}
\int_0^t \|\Delta u\|_{L^2}^{d\sigma} \|u\|_{L^2}^{4\sigma+2-d\sigma} \, d\tau \;\; \leq \;\; & \varepsilon \frac{d\sigma}{2} \int_0^t \|\Delta u\|_{L^2}^2 \, d\tau \\
& + \left(\frac{1}{\varepsilon}\right)^{d\sigma/(2-d\sigma)} \frac{2-d\sigma}{2} \int_0^t \|u\|_{L^2}^{(8\sigma+4-2d\sigma)/(2-d\sigma)} \, d\tau
\end{aligned}
$$

and choose

$$
\varepsilon = \frac{a}{b} \frac{\nu}{4|\mu|\Gamma^2 d\sigma (G_3 C_0^{\theta_3})^{4\sigma+2}}
$$

to get

$$
b|\mu| \int_0^t \int_\Omega |u|^2 g(|u|^2)^2 \, dx d\tau \leq C_2 + \frac{a\nu}{2} \int_0^t \|\Delta u\|_{L^2}^2 \, d\tau, \qquad (3.12)
$$

where

$$
\begin{aligned}
C_2 \;\; \stackrel{\text{def}}{=} \;\; & 2b_0|\mu|\Gamma^2 T K_0(T) + 4b_0|\mu|\Gamma^2 (G_3 C_0^{\theta_3})^{4\sigma+2} T K_0(T)^{2\sigma+1} \\
& + 2(ba^{-d\sigma/2})^{2/(2-d\sigma)} |\mu|\Gamma^2 (G_3 C_0^{\theta_3})^{4\sigma+2} (2-d\sigma) \qquad (3.13) \\
& \times \left(4\nu^{-1}|\mu|\Gamma^2 d\sigma (G_3 C_0^{\theta_3})^{4\sigma+2}\right)^{d\sigma/(2-d\sigma)} T K_0(T)^{(4\sigma+2-d\sigma)/(2-d\sigma)}.
\end{aligned}
$$

In the case $d\sigma = 2$ choose $\|u_0\|_{L^2}$ such that

$$
4|\mu|\Gamma^2 (G_3 C_0^{\theta_3})^{4\sigma+2} e^{(4\sigma+2-d\sigma)RT} \|u_0\|_{L^2}^{4\sigma+2-d\sigma} \leq \frac{\nu}{2} \frac{a}{b}.
$$

Clearly, we have to assume that the quotient b/a remains bounded as $a, b \to 0$. Thus, we conclude from (3.11)

$$
b|\mu| \int_0^t \int_\Omega |u|^2 g(|u|^2)^2 \, dx d\tau
$$

$$
\begin{aligned}
&\leq\ 2b|\mu|\Gamma^2 T K_0(T) + 4b|\mu|\Gamma^2 (G_3 C_0^{\theta_3})^{4\sigma+2} T K_0(T)^{2\sigma+1} + \frac{a\nu}{2}\int_0^t \|\Delta u\|_{L^2}^2\, d\tau \\
&\leq\ C_2 + \frac{a\nu}{2}\int_0^t \|\Delta u\|_{L^2}^2\, d\tau, \qquad (3.14)
\end{aligned}
$$

and C_2 is given by (3.13).

It remains to estimate the last term on the right-hand side of (3.7):

$$
\begin{aligned}
&-(a\mu + b\nu)\mathrm{Re}\int_0^t\int_\Omega \nabla f(u)\cdot\nabla u^*\, dx d\tau \\
&\leq\ a|\mu|\int_0^t\int_\Omega [g(|u|^2)|\nabla u|^2 + 2g'(|u|^2)|u|^2|\nabla u|^2]\, dx d\tau \\
&\leq\ a|\mu|(\Gamma + 2\Gamma')\int_0^t\int_\Omega (1+|u|^{2\sigma})|\nabla u|^2\, dx d\tau \qquad (3.15) \\
&=\ a|\mu|(\Gamma + 2\Gamma')\int_0^t \|\nabla u\|_{L^2}^2\, d\tau + a|\mu|(\Gamma + 2\Gamma')\int_0^t\int_\Omega |u|^{2\sigma}|\nabla u|^2\, dx d\tau.
\end{aligned}
$$

The last integral is estimated by employing the Hölder inequality and the Gagliardo-Nirenberg inequalities (3.4) and (3.6):

$$
\begin{aligned}
\int_0^t\int_\Omega |u|^{2\sigma}|\nabla u|^2\, dx d\tau &\leq \int_0^t \|u\|_{L^{2\sigma+2}}^{2\sigma}\|\nabla u\|_{L^{2\sigma+2}}^2\, d\tau \\
&\leq G_2^{2\sigma} G_4^2 \int_0^t \|u\|_{H^2}^{1+\alpha_1}\|u\|_{L^2}^{1+\alpha_2}\, d\tau,
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_1 &\stackrel{\mathrm{def}}{=} 2\sigma\theta_2 + 2\theta_4 - 1 = \frac{d\sigma}{2} \in (0,1], \\
\alpha_2 &\stackrel{\mathrm{def}}{=} 2\sigma(1-\theta_2) + 2(1-\theta_4) - 1 = \frac{(4-d)\sigma}{2} \in (-1,3],
\end{aligned}
$$

since $\sigma \leq 2/d$. The inequalities (2.8) and

$$
(x+y)^{1+\alpha_1} \leq 2(x^{1+\alpha_1} + y^{1+\alpha_1}) \quad \text{for all } x, y \geq 0
$$

imply

$$
\begin{aligned}
&a|\mu|(\Gamma + 2\Gamma')\int_0^t\int_\Omega |u|^{2\sigma}|\nabla u|^2\, dx d\tau \\
&\leq\ 2a|\mu|(\Gamma + 2\Gamma')G_2^{2\sigma}G_4^2\int_0^t (\|u\|_{L^2}^{1+\alpha_1} + \|\Delta u\|_{L^2}^{1+\alpha_1})\|u\|_{L^2}^{1+\alpha_2}\, d\tau \\
&\leq\ 2a|\mu|(\Gamma + 2\Gamma')G_2^{2\sigma}G_4^2 T K_0(T)^{\sigma+1} \qquad (3.16) \\
&\quad + 2a|\mu|(\Gamma + 2\Gamma')G_2^{2\sigma}G_4^2\int_0^t \|\Delta u\|_{L^2}^{1+\alpha_1}\|u\|_{L^2}^{1+\alpha_2}\, d\tau.
\end{aligned}
$$

If $\sigma < 2/d$ then $1+\alpha_1 < 2$, and we can apply Young's inequality to the last integral:

$$\begin{aligned}
\int_0^t \|\Delta u\|_{L^2}^{1+\alpha_1} \|u\|_{L^2}^{1+\alpha_2}\, d\tau &\le \varepsilon \frac{1+\alpha_1}{2} \int_0^t \|\Delta u\|_{L^2}^2\, d\tau \\
&\quad + \left(\frac{1}{\varepsilon}\right)^{(1+\alpha_1)/(1-\alpha_1)} \frac{1-\alpha_1}{2} \int_0^t \|u\|_{L^2}^{(2+2\alpha_1)/(1-\alpha_1)}\, d\tau \\
&\le \varepsilon \frac{1+\alpha_1}{2} \int_0^t \|\Delta u\|_{L^2}^2\, d\tau \\
&\quad + \left(\frac{1}{\varepsilon}\right)^{(1+\alpha_1)/(1-\alpha_1)} \frac{1-\alpha_1}{2} T K_0(T)^{(1+\alpha_1)/(1-\alpha_1)},
\end{aligned}$$

and taking

$$\varepsilon = \frac{\nu}{2|\mu|(1+\alpha_1)(\Gamma+2\Gamma')G_2^{2\sigma}G_4^2},$$

we obtain from (3.16)

$$a|\mu|(\Gamma+2\Gamma') \int_0^t \int_\Omega |u|^{2\sigma} |\nabla u|^2\, dx d\tau \le C_3 + \frac{a\nu}{2} \int_0^t \|\Delta u\|_{L^2}^2\, d\tau,$$

where

$$\begin{aligned}
C_3 &\stackrel{\text{def}}{=} a_0|\mu|(\Gamma+2\Gamma')G_1^{2\sigma}G_4^2 T \Big(2K_0(T)^{\sigma+1} \\
&\quad + (1-\alpha_1)\varepsilon^{-(1+\alpha_1)/(1-\alpha_1)} K_0(T)^{(1+\alpha_1)/(1-\alpha_1)}\Big).
\end{aligned} \tag{3.17}$$

In the case $\sigma = 2/d$ we get $1+\alpha_1 = 2$. Choose $\|u_0\|_{L^2}$ small enough such that

$$2a|\mu|(\Gamma+2\Gamma')G_2^{2\sigma}G_4^2 \int_0^t \|\Delta u\|_{L^2}^{1+\alpha_1} \|u\|_{L^2}^{1+\alpha_2} d\tau \le \frac{a\nu}{2} \int_0^t \|\Delta u\|_{L^2}^2 d\tau.$$

Hence, the estimate (3.15) becomes

$$\begin{aligned}
&-(a\mu + b\nu)\mathrm{Re} \int_0^t \int_\Omega \nabla f(u) \cdot \nabla u^*\, dx d\tau \\
&\le a_0|\mu|(\Gamma+2\Gamma') \int_0^t \|\nabla u\|_{L^2}^2\, d\tau + C_3 + \frac{a\nu}{2} \int_0^t \|\Delta u\|_{L^2}^2\, d\tau.
\end{aligned} \tag{3.18}$$

Now we insert the inequalities (3.9), (3.12), and (3.18) into (3.7) to get

$$\begin{aligned}
\frac{\nu}{4} \|\nabla u(t)\|_{L^2}^2 &\le \frac{\nu}{2} \|\nabla u_0\|_{L^2}^2 + C_1 + C_2 + C_3 \\
&\quad + (\nu R + a_0|\mu|(\Gamma+2\Gamma')) \int_0^t \|\nabla u\|_{L^2}^2\, d\tau.
\end{aligned}$$

Notice that C_2 depends on $ba^{-d\sigma/2}$. An application of Gronwall's inequality yields

$$\|\nabla u(t)\|_{L^2}^2 \le K_2(T), \quad 0 \le t \le T,$$

where

$$K_2(T) \stackrel{\text{def}}{=} \Big(2\|\nabla u_0\|_{L^2}^2 + 4\nu^{-1}(C_1 + C_2 + C_3)\Big) \times \exp\Big(4T(R + \nu^{-1}a_0|\mu|(\Gamma + 2\Gamma'))\Big). \tag{3.19}$$

This proves the lemma.

The Lemmas 3.1, 3.2 and 3.4 yield a uniform H^1-bound for the solutions to (1.1),(1.2), provided that $ba^{-d\sigma/2}$ remains bounded as $a, b \to 0$ in the case $\mu < 0$:

Corollary 3.5 *Let (2.1)–(2.3) hold. In the case $\mu < 0$ we also have to require that $ba^{-d\sigma/2}$ remains bounded as $a, b \to 0$. Then there exists a constant $K_3(T) > 0$, only depending on the data and on T, but not on a, b, such that for every solution u to (1.1),(1.2) it holds*

$$\|u(t)\|_{H^1}^2 \le K_3(T) \quad \textit{for all } 0 \le t \le T.$$

Remark 3.6 The constant $K_3(T)$ is defined by

$$K_3(T) = \begin{cases} K_0(T) + K_1(T) & \text{if } \mu \ge 0 \\ K_0(T) + K_2(T) & \text{if } \mu < 0. \end{cases}$$

The precise values of $K_0(T)$ and $K_1(T)$ are given in Lemma 3.1, 3.2, respectively. The constant $K_2(T)$ is defined in (3.19), with constants C_1, C_2, C_3 which are introduced in (3.10), (3.13) and (3.17), respectively.

Remark 3.7 The above corollary also holds true if $\mu \ge 0$ and the second chain of inequalities in condition (2.3) is replaced by

$$-\frac{1}{2}g(s) \le s \cdot g'(s) \le \Gamma'(1 + s^\sigma) \quad \text{for } s \ge 0$$

(see Remark 3.3).

Remark 3.8 The estimates of Lemma 3.1 and 3.2 are rather standard, whereas the estimate of Lemma 3.4 seems to be new, in the sense that the particular estimate and the dependence on the parameters have not been presented before.

4 Proof of Theorems 2.1 and 2.2

Let u_ε be a solution to (1.1)–(1.2). Let $p = 2d/(d-2)$ ($p < \infty$ if $d \le 2$). When $\mu \ge 0$, we choose $2/(2\sigma+1) < q < (2\sigma+2)/(2\sigma+1)$. When $\mu < 0$, take $q = 2d/(d+2)$; then q is the conjugate exponent to p, i.e. $1/p + 1/q = 1$, and the condition $\sigma < 2/d$ implies $(2\sigma + 1)q < p$, showing that the embedding $H^1(\Omega) \hookrightarrow L^{(2\sigma+1)q}(\Omega)$ is continuous.

In order to give uniform bounds for $f(u_\varepsilon)$ we have to consider the cases $\Omega = \mathbb{R}^d$ and $\Omega = \mathbb{T}^d$ separately. First let $\Omega = \mathbb{T}^d$ and $\mu \geq 0$. From (2.3) we get, for all $0 \leq t \leq T$,

$$\begin{aligned}
\|f(u_\varepsilon(t))\|_{L^q} &\leq \Gamma\left(\int_\Omega |u_\varepsilon(t)|^q(1+|u_\varepsilon(t)|^{2\sigma})^q dx\right)^{1/q} \\
&\leq \Gamma\left(\|u_\varepsilon(t)\|_{L^q}^q + \|u_\varepsilon(t)\|_{L^{(2\sigma+1)q}}^{(2\sigma+1)q}\right)^{1/q} \\
&\leq c\left(\|u_\varepsilon(t)\|_{L^2} + \|u_\varepsilon(t)\|_{L^{2\sigma+2}}^{2\sigma+1}\right),
\end{aligned}$$

where here and in the following we denote by c, c_i positive constants independent of $\varepsilon = (a, b)$ and t. In the last step we have used that $q < 2$, hence $L^2(\Omega) \hookrightarrow L^q(\Omega)$, and $L^{2\sigma+2}(\Omega) \hookrightarrow L^{(2\sigma+1)q}(\Omega)$, because Ω is bounded. When $\mu < 0$, we obtain as above (since $q = 2d/(d+2) < 2$)

$$\begin{aligned}
\|f(u_\varepsilon(t))\|_{L^q} &\leq \Gamma\left(\|u_\varepsilon(t)\|_{L^q}^q + \|u_\varepsilon(t)\|_{L^{(2\sigma+1)q}}^{(2\sigma+1)q}\right)^{1/q} \\
&\leq c\left(\|u_\varepsilon(t)\|_{L^2} + \|u_\varepsilon(t)\|_{H^1}^{2\sigma+1}\right).
\end{aligned}$$

Applying Lemma 3.2 and Corollary 3.5, we conclude that $(f(u_\varepsilon))$ is uniformly bounded in $L^\infty(0, T;\, L^q(\Omega))$ when $\Omega = \mathbb{T}^d$.

Now let $\Omega = \mathbb{R}^d$. Then the above arguments do not apply since the injection $L^q(\Omega) \hookrightarrow L^2(\Omega)$ is generally not valid. Here we need the condition $g(s) \leq \Gamma s^\sigma$. Then, if $\mu \geq 0$,

$$\|f(u_\varepsilon(t))\|_{L^q} \leq \Gamma\|u_\varepsilon(t)\|_{L^{(2\sigma+1)q}}^{2\sigma+1} \leq c.$$

The last inequality follows from the fact that the uniform bounds on $u_\varepsilon(t)$ in $L^2(\Omega)$ and $L^{2\sigma+2}(\Omega)$ imply a uniform bound in $L^{(2\sigma+1)q}(\Omega)$ since $2 < (2\sigma+1)q < 2\sigma+2$, by interpolation. When $\mu < 0$, we obtain

$$\|f(u_\varepsilon(t))\|_{L^q} \leq \Gamma\|u_\varepsilon(t)\|_{L^{(2\sigma+1)q}}^{2\sigma+1} \leq c\|u_\varepsilon(t)\|_{H^1}^{2\sigma+1} \leq c.$$

Applying Corollary 3.5, we conclude that $(f(u_\varepsilon))$ is uniformly bounded in $L^\infty(0, T;$ $L^q(\Omega))$. Since $V^* = H^{-1}(\Omega) + L^{(2\sigma+2)/(2\sigma+1)}(\Omega)$, this implies that (see, e.g., [6, p. 217])

$$\|\partial_t u_\varepsilon(t)\|_{V^*} \leq c\|u_\varepsilon(t)\|_V \leq c.$$

This shows that the sequence (u_ε) is uniformly bounded in $L^\infty(0, T; H^1(\Omega))$ and $W^{1,\infty}(0, T; V^*)$.

From the above estimates we get the following weak convergence results for a subsequence of (u_ε), which is not relabeled:

$$\begin{aligned}
u_\varepsilon &\rightharpoonup u && \text{in } L^\infty(0, T; H^1(\Omega)) \text{ weakly-}*, \\
\partial_t u_\varepsilon &\rightharpoonup \partial_t u && \text{in } L^\infty(0, T; V^*) \text{ weakly-}*, \\
f(u_\varepsilon) &\rightharpoonup \overline{f} && \text{in } L^\infty(0, T; L^q(\Omega)) \text{ weakly-}*.
\end{aligned}$$

It remains to identify the limit $\overline{f}$. For this, let $\omega \subset \Omega$ be an arbitrary bounded open set. Then (u_ε) is also bounded in $L^\infty(0,T;\ H^1(\omega))$ and $W^{1,\infty}(0,T;V^*(\omega))$, where $V^*(\omega) = H^{-1}(\omega) + L^{(2\sigma+2)/(2\sigma+1)}(\omega)$. By Lemma 9.4.6 of [6], there exists a subsequence, still denoted by (u_ε), such that for every $t > 0$,

$$u_\varepsilon(x,t) \to u(x,t) \qquad \text{for almost every } x \in \omega,$$

and $u \in L^\infty(0,T;H^1(\omega)) \cap W^{1,\infty}(0,T;V^*(\omega))$. Applying the mean value theorem to both $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$, we get for all $u,v \in \mathbb{C}$,

$$f(u) - f(v) = \int_0^1 \left((u-v)(g(|w_\lambda|^2) + |w_\lambda|^2 g'(|w_\lambda|^2)) + (u-v)^* w_\lambda^2 g'(|w_\lambda|^2)\right) d\lambda, \tag{4.1}$$

where $w_\lambda = \lambda u + (1-\lambda)v$. Thus, using Hölder's inequality and condition (2.3), for all $0 \le s,t \le T$,

$$\begin{aligned}
&\|f(u_\varepsilon(t)) - f(u(t))\|_{L^q(\omega)} \\
&\le (\Gamma + 2\Gamma') \left[\int_\omega |u_\varepsilon(t) - u(t)|^q \left(\int_0^1 (1 + |\lambda u_\varepsilon(t) + (1-\lambda)u(t)|^{2\sigma}) d\lambda\right)^q dx\right]^{1/q} \\
&\le (\Gamma + 2\Gamma') \left[\int_\omega |u_\varepsilon(t) - u(t)|^q \left(1 + \max(1, 2^{2\sigma-1})(|u_\varepsilon(t)|^{2\sigma} + |u(t)|^{2\sigma})\right)^q dx\right]^{1/q} \\
&\le c_1 \left[\int_\omega |u_\varepsilon(t) - u(t)|^q (1 + |u_\varepsilon(t)|^{2\sigma} + |u(t)|^{2\sigma})^q dx\right]^{1/q} \\
&\le c_2 \|u_\varepsilon(t) - u(t)\|_{L^q(\omega)} + c_2 \left[\int_\omega |u_\varepsilon(t) - u(t)|^q (|u_\varepsilon(t)|^{2\sigma q} + |u(t)|^{2\sigma q}) dx\right]^{1/q} \\
&\le c_2 \|u_\varepsilon(t) - u(t)\|_{L^q(\omega)} \\
&\quad + c_2 \left(\|u_\varepsilon(t)\|^{2\sigma}_{L^{(2\sigma+1)q}} + \|u(t)\|^{2\sigma}_{L^{(2\sigma+1)q}}\right) \|u_\varepsilon(t) - u(t)\|_{L^{(2\sigma+1)q}(\omega)},
\end{aligned}$$

where $c_1 = (\Gamma + 2\Gamma')\max(1, 2^{2\sigma-1})$ and $c_2 = 3^{1-1/q} c_1$. First, consider $\mu \ge 0$. Then, since $(u_\varepsilon(t))$ is bounded in $L^{(2\sigma+1)q}(\Omega)$ (see above),

$$\begin{aligned}
&\|f(u_\varepsilon(t)) - f(u(t))\|_{L^q(\omega)} \\
&\quad \le c_2 \|u_\varepsilon(t) - u(t)\|_{L^q(\omega)} + c_3 \|u_\varepsilon(t) - u(t)\|_{L^{(2\sigma+1)q}(\omega)}.
\end{aligned}$$

We know that $u_\varepsilon(t) \to u(t)$ a.e. in ω and $(u_\varepsilon(t))$ is uniformly bounded in $L^{2\sigma+2}(\omega)$. Hence, by Lemma 1.3 of [29, Ch. 1], $u_\varepsilon(t) \to u(t)$ in $L^s(\omega)$ strongly for all $s < 2\sigma+2$. In particular, we can choose $s = q$ and $s = (2\sigma+1)q$. Hence,

$$f(u_\varepsilon) \to f(u) \quad \text{in } L^\infty(0,T;L^q(\omega)) \text{ strongly.} \tag{4.2}$$

Now, let $\mu < 0$. Then, by Corollary 3.5,

$$\begin{aligned}
&\|f(u_\varepsilon(t)) - f(u(t))\|_{L^q(\omega)} \\
&\le c_2 \|u_\varepsilon(t) - u(t)\|_{L^q(\omega)} + 2c_2 \sup_{0<t<T} \|u_\varepsilon(t)\|^{2\sigma}_{H^1(\Omega)} \|u_\varepsilon(t) - u(t)\|_{L^{(2\sigma+1)q}(\omega)} \\
&\le c_2 \|u_\varepsilon(t) - u(t)\|_{L^q(\omega)} + c_4 \|u_\varepsilon(t) - u(t)\|_{L^r(\omega)}.
\end{aligned}$$

To see the last step observe that, since $(2\sigma+1)q < p$, there exists $r < p$ such that $(2\sigma+1)q \le r$. The pointwise a.e. convergence of $(u_\varepsilon(x,t))$ and the uniform bound on (u_ε) in $L^\infty(0,T;H^1(\omega))$ imply the strong convergence of (u_ε) to u in $L^\infty(0,T;L^r(\omega))$ with $r < p = 2d/(d-2)$, and hence the convergence (4.2) holds.

Therefore, we get

$$f(u) = \overline{f} \quad \text{in } \omega \times (0,T)$$

for every bounded $\omega \subset \Omega$, hence almost everywhere in $\Omega \times (0,T)$.

Now, the limit in the equation satisfied for u_ε can be performed and we see that u solves the nonlinear Schrödinger equation (1.4). Finally, notice that the initial value is satisfied in the sense of V^*.

5 A uniform bound in $H^2(\Omega)$

In this section we derive a uniform bound in $H^2(\Omega)$ for the solution u of the CGL equation for space dimensions $d \le 3$. First we prove the existence of a uniform bound in one space dimension.

Lemma 5.1 *Let the conditions (2.1)–(2.2), (2.4), (2.6) hold. Furthermore, let $d = 1$ and $u_0 \in H^2(\Omega)$. Then there exists a constant $K_4(T) > 0$, only depending on the data and on T, but not on a, b such that*

$$\|\partial_{xx} u(t)\|_{L^2}^2 \le K_4(T) \quad \textit{for all } 0 \le t \le T.$$

Proof. The first part of the proof is valid for any space dimension. We impose the restriction $d = 1$ later for the estimation of the L^∞ bound of u.

Taking the Laplacian of Eq. (1.1) we obtain

$$\begin{aligned}\partial_t \Delta u &= (a+i\nu)\Delta^2 u + R\Delta u - (b+i\mu)\Big(g(|u|^2)\Delta u \\ &\quad + 2g'(|u|^2)\nabla u \cdot \nabla |u|^2 + u g''(|u|^2)(\nabla|u|^2)^2 \\ &\quad + u g'(|u|^2)\Delta|u|^2\Big).\end{aligned}$$

Multiplying this equation by Δu^*, taking the real part and integrating over Ω yields

$$\begin{aligned}\frac{1}{2}\partial_t \|\Delta u\|_{L^2}^2 &= -a\|\nabla\Delta u\|_{L^2}^2 + R\|\Delta u\|_{L^2}^2 - b\int_\Omega g(|u|^2)|\Delta u|^2 dx \\ &\quad - 2\mathrm{Re}\int_\Omega (b+i\mu) g'(|u|^2)\nabla u \cdot \nabla|u|^2 \Delta u^* dx \\ &\quad - \mathrm{Re}\int_\Omega (b+i\mu) u g''(|u|^2)(\nabla|u|^2)^2 \Delta u^* dx \\ &\quad - \mathrm{Re}\int_\Omega (b+i\mu) u g'(|u|^2)\Delta|u|^2 \Delta u^* dx. \end{aligned} \tag{5.1}$$

The fourth term on the right-hand side can be estimated by using (2.6) and the inequality $|\nabla|u|^2| \leq 2|\nabla u||u|$:

$$\begin{aligned}
&-2\mathrm{Re}\int_\Omega (b+i\mu)g'(|u|^2)\nabla u\cdot\nabla|u|^2\Delta u^* dx\\
&\leq\ 4\Gamma_1|b+i\mu|\int_\Omega(1+|u|^{2\sigma-1})|\nabla u|^2|\Delta u|dx\\
&\leq\ 4\Gamma_1|b+i\mu|(1+\|u\|_{L^\infty}^{2\sigma-1})\|\nabla u\|_{L^4}^2\|\Delta u\|_{L^2}.
\end{aligned}$$

The fifth term on the right-hand side of (5.1) is treated similarly

$$\begin{aligned}
&-\mathrm{Re}\int_\Omega (b+i\mu)ug''(|u|^2)(\nabla|u|^2)^2\Delta u^* dx\\
&\leq\ 4\Gamma_1'|b+i\mu|\int_\Omega(1+|u|^{2\sigma-1})|\nabla u|^2|\Delta u|dx\\
&\leq\ 4\Gamma_1'|b+i\mu|(1+\|u\|_{L^\infty}^{2\sigma-1})\|\nabla u\|_{L^4}^2\|\Delta u\|_{L^2}.
\end{aligned}$$

In the last term of (5.1) we use the inequality $|\Delta|u|^2| \leq 2|\Delta u||u| + |\nabla u|^2$ to get

$$\begin{aligned}
&-\mathrm{Re}\int_\Omega (b+i\mu)ug'(|u|^2)\Delta|u|^2\Delta u^* dx\\
&\leq\ 2\Gamma_1|b+i\mu|\int_\Omega(1+|u|^{2\sigma-1})(2|\Delta u||u|+|\nabla u|^2)|\Delta u|dx\\
&\leq\ 2\Gamma_1|b+i\mu|(1+\|u\|_{L^\infty}^{2\sigma-1})(2\|\Delta u\|_{L^2}\|u\|_{L^\infty}+\|\nabla u\|_{L^4}^2)\|\Delta u\|_{L^2}.
\end{aligned}$$

Hence, we obtain from Eq. (5.1)

$$\begin{aligned}
\partial_t\|\Delta u\|_{L^2}^2 \ \leq\ & 2R\|\Delta u\|_{L^2}^2 + 4\Gamma_1|b+i\mu|(1+\|u\|_{L^\infty}^{2\sigma-1})\|u\|_{L^\infty}\|\Delta u\|_{L^2}^2\\
&+8(2\Gamma_1+\Gamma_1')|b+i\mu|(1+\|u\|_{L^\infty}^{2\sigma-1})\|\nabla u\|_{L^4}^2\|\Delta u\|_{L^2}.
\end{aligned}$$

The following Gagliardo-Nirenberg inequality holds for $d \leq 4$:

$$\|\nabla u\|_{L^4} \leq G_5\|\nabla u\|_{H^1}^{d/4}\|\nabla u\|_{L^2}^{1-d/4}.$$

By employing Lemma 3.2 and Corollary 3.5 this gives

$$\begin{aligned}
\|\nabla u\|_{L^4}^2 \ &\leq\ G_5^2\|\nabla u\|_{H^1}^{d/2}\|\nabla u\|_{L^2}^{2-d/2}\\
&\leq\ G_5^2(\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2)^{d/4}\|\nabla u\|_{L^2}^{2-d/2}\\
&\leq\ G_5^2K_3^{1-d/4}\max(1,K_3^{d/4})(1+\|\Delta u\|_{L^2}^{d/2}),
\end{aligned}$$

and we can continue to estimate

$$\begin{aligned}
\partial_t\|\Delta u\|_{L^2}^2 \ \leq\ & [2R+4\Gamma_1|b_0+i\mu|(1+\|u\|_{L^\infty}^{2\sigma-1})\|u\|_{L^\infty}]\|\Delta u\|_{L^2}^2\\
&+C_4(1+\|u\|_{L^\infty}^{2\sigma-1})(1+\|\Delta u\|_{L^2}^{d/2})\|\Delta u\|_{L^2},
\end{aligned}\tag{5.2}$$

where we have set

$$C_4 \stackrel{\text{def}}{=} 8(2\Gamma_1 + \Gamma_1')|b_0 + i\mu| G_5^2 K_3^{1-d/4} \max(1, K_3^{d/4}). \tag{5.3}$$

In order to bound the L^∞ norm of u, we have to impose the condition $d = 1$. Then it holds

$$\|u\|_{L^\infty} \leq G\|u\|_{H^1}^{1/2}\|u\|_{L^2}^{1/2} \tag{5.4}$$

with $G = \sqrt{2}$ if $\Omega = \mathbb{R}$ and $G = 1/\sqrt{\pi}$ if $\Omega = \mathbb{T}$. Thus, by Lemma 3.1 and Corollary 3.5,

$$\|u\|_{L^\infty} \leq (4K_0K_3)^{1/4},$$

and using the elementary inequalities

$$\begin{aligned}
\|\partial_{xx}u\|_{L^2} &\leq \frac{1}{2}(1 + \|\partial_{xx}u\|_{L^2}^2), \\
\|\partial_{xx}u\|_{L^2}^{3/2} &\leq \frac{1}{4}(1 + 3\|\partial_{xx}u\|_{L^2}^2),
\end{aligned}$$

we obtain from (5.2)

$$\begin{aligned}
\partial_t\|\partial_{xx}u\|_{L^2}^2 &\leq [2R + 4\Gamma_1|b + i\mu|(1 + (4K_0K_3)^{\sigma/2-1/4})(4K_0K_3)^{1/4}]\|\partial_{xx}u\|_{L^2}^2 \\
&\quad + C_4(1 + (4K_0K_3)^{\sigma/2-1/4})\Big(\frac{3}{4} + \frac{5}{4}\|\partial_{xx}u\|_{L^2}^2\Big) \\
&\leq \frac{3}{4}C_4(1 + (4K_0K_3)^{\sigma/2-1/4}) + C_5\|\partial_{xx}u\|_{L^2}^2,
\end{aligned}$$

where

$$\begin{aligned}
C_5 &\stackrel{\text{def}}{=} 2R + 4\Gamma_1|b + i\mu|(1 + (4K_0K_3)^{\sigma/2-1/4})(4K_0K_3)^{1/4} \\
&\quad + \frac{5}{4}C_4(1 + (4K_0K_3)^{\sigma/2-1/4}).
\end{aligned}$$

By Gronwall's lemma, we conclude that

$$\|\partial_{xx}u(t)\|_{L^2}^2 \leq K_4(T) \quad \text{for all } 0 \leq t \leq T,$$

where

$$K_4(T) \stackrel{\text{def}}{=} \|\partial_{xx}u_0\|_{L^2}^2 + \frac{3}{4}C_4(1 + (4K_0K_3)^{\sigma/2-1/4})e^{C_5T}.$$

This finishes the proof.

In two and three space dimensions we have the following result:

Lemma 5.2 *Let the conditions (2.1)–(2.2), (2.4), (2.6) hold. Furthermore, let* $1 \leq d \leq 3$ *and* $u_0 \in H^2(\Omega)$. *Then there exist constants* $K_5(t)$, $T^* > 0$ *depending on the data but not on* a, b *such that*

$$\|\Delta u(t)\|_{L^2}^2 \leq K_5(t) \quad \textit{for all } 0 \leq t < T^*.$$

Remark 5.3 *The constant* $K_5(t)$ *satisfies* $\lim_{t\to T^*} K_5(t) = +\infty$.

Proof. We start from the inequality (5.2) which holds for $1 \le d \le 4$. For space dimensions $d \ge 2$, we cannot use the H^1 norm for estimating the L^∞ norm of u as in the proof of Lemma 5.1. Instead we employ the Gagliardo-Nirenberg inequality

$$\|u\|_{L^\infty} \le G_6 \|u\|_{H^2}^{d/4} \|u\|_{L^2}^{1-d/4}, \tag{5.5}$$

which holds for $d \le 3$. Thus, using the inequality (2.8) we get

$$\begin{aligned} \|u\|_{L^\infty} &\le G_6 C_0^{d/4} (K_0^{d/8} + \|\Delta u\|_{L^2}^{d/4}) K_0^{1/2-d/8} \\ &\le C_6 (1 + \|\Delta u\|_{L^2}^{d/4}), \end{aligned}$$

where we have set

$$C_6 \stackrel{\text{def}}{=} G_6 C_0^{d/4} K_0^{1/2-d/8} \max(1, K_0^{d/8}). \tag{5.6}$$

The above estimate together with the inequality (5.2) yields

$$\begin{aligned} \partial_t \|\Delta u\|_{L^2}^2 \le\ & 2R\|\Delta u\|_{L^2}^2 + 4\Gamma_1 |b_0 + i\mu| C_6 (1 + C_6^{2\sigma-1}(1 + \|\Delta u\|_{L^2}^{d/4})^{2\sigma-1}) \\ & \times (1 + \|\Delta u\|_{L^2}^{d/4}) \|\Delta u\|_{L^2}^2 \\ & + C_4 (1 + C_6^{2\sigma-1}(1 + \|\Delta u\|_{L^2}^{d/4})^{2\sigma-1})(1 + \|\Delta u\|_{L^2}^{d/2}) \|\Delta u\|_{L^2}. \end{aligned}$$

Since $d < 4$ the largest exponent of $\|\Delta u\|_{L^2}$ is $2 + \sigma d/2$, and therefore, there exist two constants $L_1, L_2 > 0$ only depending on $d, \sigma, R, \Gamma_1, |b_0 + i\mu|, C_4$, and C_6 such that

$$\partial_t \|\Delta u\|_{L^2}^2 \le L_1 + L_2 \|\Delta u\|_{L^2}^{2+\sigma d/2}. \tag{5.7}$$

From the Gronwall-type inequality of the appendix we conclude that

$$\|\Delta u(t)\|_{L^2}^2 \le K_5(t) \quad \text{for all } 0 \le t < T^*,$$

where

$$K_5(t) \stackrel{\text{def}}{=} \left\{ \left[\|\Delta u_0\|_{L^2}^2 + \left(\frac{L_1}{L_2}\right)^{4/(4+\sigma d)} \right]^{-\sigma d/4} - \frac{\sigma d}{4} L_2 t \right\}^{-4/\sigma d} - \left(\frac{L_1}{L_2}\right)^{4/(4+\sigma d)}$$

and

$$T^* \stackrel{\text{def}}{=} \frac{4}{\sigma d L_2} \left[\|\Delta u_0\|_{L^2}^2 + \left(\frac{L_1}{L_2}\right)^{4/(4+\sigma d)} \right]^{-\sigma d/4}. \tag{5.8}$$

This proves the lemma.

Remark 5.4 Lemma 5.2 can be improved slightly by replacing the Gagliardo-Nirenberg inequality (5.5) by the inequality [40]

$$\|u\|_{L^\infty} \le B_0 \|u\|_{H^1}^{1/2} \|u\|_{H^2}^{1/2} \tag{5.9}$$

if $d = 3$, and by the Brézis-Gallouet inequality [5, 35]

$$\|u\|_{L^\infty} \leq B_1 \|u\|_{H^1} \Big(1 + (\, \ln(1 + B_2 \|u\|_{H^2}))^{1/2} \Big) \tag{5.10}$$

if $d = 2$, in the sense that the final time $T^* > 0$ can be chosen larger than in (5.8). Indeed, from (5.2) we obtain for $d = 3$, using (5.9),

$$\partial_t \|\Delta u\|_{L^2}^2 \leq B_3 + B_4 \|\Delta u\|_{L^2}^{2+\sigma},$$

and for $d = 2$, employing the elementary inequality

$$\ln(1 + x) \leq C_\delta (1 + x^\delta) \qquad \text{for } x \geq 0,\ \delta > 0,$$

and the Brézis-Gallouet inequality (5.10),

$$\partial_t \|\Delta u\|_{L^2}^2 \leq B_5 + B_\delta \|\Delta u\|_{L^2}^{2+\delta},$$

for any $\delta > 0$. Now Lemma 8.1 of the Appendix applies. Since the exponents of the right-hand sides of the above two estimates are smaller than the corresponding exponent in (5.7) (if $\delta < \sigma$), we can choose $T^* > 0$ larger than in (5.8).

Remark 5.5 The estimates of Lemmas 5.1 and 5.2 seem to be new in the presented form.

6 Proof of Theorems 2.3 and 2.4

Proof of Theorem 2.3. The difference $w = u_\varepsilon - u$ satisfies the equation

$$\partial_t w = i\nu \Delta w - i\mu (f(u_\varepsilon) - f(u)) + Rw + a\Delta u_\varepsilon - bf(u_\varepsilon).$$

Multiplying this equation by w^*, integrating over Ω, taking the real part and finally, using the formula (4.1) gives

$$\begin{aligned} \frac{1}{2}\partial_t \|w\|_{L^2}^2 &= \mu \mathrm{Im} \int_\Omega \int_0^1 w_\lambda^2 g'(|w_\lambda|^2)(w^*)^2 d\lambda dx + R\|w\|_{L^2}^2 \\ &\quad + a\mathrm{Re} \int_\Omega \Delta u_\varepsilon w^* dx - b\mathrm{Re} \int_\Omega f(u_\varepsilon) w^* dx, \end{aligned}$$

where $w_\lambda = \lambda u_\varepsilon + (1 - \lambda) u$. Therefore, by assumption (2.3),

$$\begin{aligned} \partial_t \|w\|_{L^2}^2 &\leq 2|\mu|\Gamma' \int_\Omega (1 + \max(1, 2^{2\sigma-1})(|u_\varepsilon|^{2\sigma} + |u|^{2\sigma}))|w|^2 dx \\ &\quad + 2R\|w\|_{L^2}^2 + 2a\Big| \int_\Omega \Delta u_\varepsilon w^* dx \Big| \\ &\quad + 2b\Gamma \int_\Omega (1 + |u_\varepsilon|^{2\sigma})|u^\varepsilon||w| dx \\ &\leq 2|\mu|\Gamma'(1 + \max(1, 2^{2\sigma-1})(\|u_\varepsilon\|_{L^\infty}^{2\sigma} + \|u\|_{L^\infty}^{2\sigma}))\|w\|_{L^2}^2 \\ &\quad + 2R\|w\|_{L^2}^2 + 2a\Big| \int_\Omega \Delta u_\varepsilon w^* dx \Big| \\ &\quad + 2b\Gamma(1 + \|u_\varepsilon\|_{L^\infty}^{2\sigma})\|u^\varepsilon\|_{L^2}\|w\|_{L^2}. \end{aligned} \tag{6.1}$$

Now, let $d = 1$. Then, using the inequality (5.4) and observing that $G \le \sqrt{2}$, we get

$$\begin{aligned} \|u_\varepsilon\|_{L^\infty}^2 \le \ & G^2 (K_0 K_3)^{1/2} & \le (4K_0K_3)^{1/2}, \\ \|u\|_{L^\infty}^2 \le \ & G^2 (K_0 K_3)^{1/2} & \le (4K_0K_3)^{1/2}, \end{aligned}$$

where we also have used Lemma 3.1 and Corollary 3.5. Hence, since

$$\begin{aligned} \|w\|_{L^2} \le \ & \|u_\varepsilon\|_{L^2} + \|u\|_{L^2} & \le 2K_0^{1/2}, \\ \|w\|_{H^1} \le \ & \|u_\varepsilon\|_{H^1} + \|u\|_{H^1} & \le 2K_3^{1/2}, \end{aligned}$$

we obtain from (6.1)

$$\begin{aligned} \partial_t \|w\|_{L^2}^2 \ & \le \ 2|\mu|\Gamma'(1 + \max(2, 2^{2\sigma})(4K_0K_3)^{\sigma/2})\|w\|_{L^2}^2 + 2R\|w\|_{L^2}^2 \\ & \quad + 2a\|\partial_x u_\varepsilon\|_{L^2}\|\partial_x w\|_{L^2} + 4b\Gamma(1 + (4K_0K_3)^{\sigma/2})K_0 \\ & \le \ C_7\|w\|_{L^2}^2 + a \cdot 4K_3 + b \cdot 4\Gamma K_0(1 + (4K_0K_3)^{1/2}), \end{aligned}$$

where we have set

$$C_7 \stackrel{\text{def}}{=} 2R + 2|\mu|\Gamma'(1 + \max(2, 4^\sigma)(4K_0K_3)^{\sigma/2}). \tag{6.2}$$

Applying Gronwall's lemma, we conclude that

$$\|w(t)\|_{L^2}^2 \le \|w(0)\|_{L^2}^2 + a \cdot 4K_3 e^{C_7 T} + b \cdot 4\Gamma K_0(1 + (4K_0K_3)^{\sigma/2})e^{C_7 T},$$

which proves the theorem.

Proof of Theorem 2.4. We start from the inequality (6.1) which is valid for any $d \ge 1$. In order to bound the L^∞ norm of u_ε and u we use the Gagliardo-Nirenberg inequality (5.5), the inequality (2.8) and Lemmas 5.1 and 5.2:

$$\begin{aligned} \|u_\varepsilon\|_{L^\infty} \ & \le \ G_6 \|u_\varepsilon\|_{H^2}^{d/4} \|u_\varepsilon\|_{L^2}^{1-d/4} \\ & \le \ G_6 C_0^{d/4} (\|u_\varepsilon\|_{L^2}^{d/4} + \|\Delta u_\varepsilon\|_{L^2}^{d/4}) \|u_\varepsilon\|_{L^2}^{1-d/4} \\ & \le \ C_8, \end{aligned}$$

where

$$C_8 \stackrel{\text{def}}{=} G_6 C_0^{d/4} K_0^{(4-d)/8} (K_0^{d/8} + \max(K_4, K_5)^{d/8}). \tag{6.3}$$

Moreover,

$$\|u\|_{L^\infty} \le C_8.$$

Then the estimate (6.1) yields

$$\begin{aligned} \partial_t \|w\|_{L^2}^2 \ & \le \ [2R + 2|\mu|\Gamma'(1 + \max(2, 4^\sigma)C_8^{2\sigma})]\|w\|_{L^2}^2 \\ & \quad + 2a\|\Delta u_\varepsilon\|_{L^2}\|w\|_{L^2} + 2b\Gamma(1 + C_8^{2\sigma})K_0^{1/2}\|w\|_{L^2} \\ & \le \ 2C_9\|w\|_{L^2}^2 + 2[a\max(K_4, K_5)^{1/2} + b\Gamma(1 + C_8^{2\sigma})K_0^{1/2}]\|w\|_{L^2}, \end{aligned}$$

where

$$C_9 \stackrel{\text{def}}{=} R + |\mu|\Gamma'(1 + \max(2, 4^\sigma)C_8^{2\sigma}). \tag{6.4}$$

Therefore,

$$\partial_t \|w\|_{L^2} \le C_9 \|w\|_{L^2} + a \cdot \max(K_4, K_5)^{1/2} + b \cdot \Gamma(1 + C_8^{2\sigma})K_0^{1/2},$$

and we obtain from Gronwall's lemma

$$\begin{aligned} \|w(t)\|_{L^2} &\le \|w(0)\|_{L^2} + a \cdot \max(K_4, K_5)^{1/2} e^{C_9 t} \\ &\quad + b \cdot \Gamma(1 + C_8^{2\sigma})K_0^{1/2} e^{C_9 t}, \end{aligned}$$

for all $0 \le t < T^*$ (and $0 \le t \le T$ if $d = 1$). This finishes the proof.

7 Proof of Theorem 2.5

First we prove the following uniform bound in $H^n(\Omega)$ for the solution u to (1.1)–(1.2).

Lemma 7.1 *Under the assumptions of Theorem 2.5 there exist constants $T^* > 0$ and $K_6(t) > 0$ independent of a, b such that*

$$\|u(t)\|_{H^n} \le K_6(t) \quad \text{for all } \; 0 \le t < T^*.$$

Proof. Let $D^\alpha u$ denote the partial derivative of u according to the multi-index α. Applying D^α with $|\alpha| = n$ to the CGL equation (1.1), multiplying this equation by $D^\alpha u^*$, taking the real part and integrating over Ω yields

$$\begin{aligned} \partial_t \sum_{|\alpha|=n} \int_\Omega |D^\alpha u|^2 dx &= -2a \sum_{|\alpha|=n+1} \int_\Omega |D^\alpha u|^2 dx + 2R \sum_{|\alpha|=n} \int_\Omega |D^\alpha u|^2 dx \\ &\quad - 2\mathrm{Re} \sum_{|\alpha|=n} (b + i\mu) \int_\Omega D^\alpha(|u|^{2\sigma} u) D^\alpha u^* dx. \end{aligned}$$

Using the notations

$$\begin{aligned} \nabla^n u &= (D^\alpha u)_{|\alpha|=n}, \\ \|\nabla^n u\|_{L^2}^2 &= \sum_{|\alpha|=n} \int_\Omega |D^\alpha u|^2 dx, \end{aligned}$$

we obtain

$$\partial_t \|\nabla^n u\|_{L^2}^2 \le 2R \|\nabla^n u\|_{L^2}^2 + 2|b_0 + i\mu| \Big| \int_\Omega \nabla^n(|u|^{2\sigma} u) \cdot \nabla^n u^* dx \Big|. \tag{7.1}$$

From [28, p. 183] we get the existence of a constant $c(d) > 0$ only depending on the space dimension, such that the last term of the above inequality is majorized by

$$2|b_0 + i\mu| c(d) \|\nabla^n u\|_{L^2}^2 \|u\|_{L^\infty}^{2\sigma}.$$

The L^∞ norm of u is estimated by using the Gagliardo-Nirenberg inequality

$$\|u\|_{L^\infty} \le G_7 \|u\|_{H^n}^{\theta} \|u\|_{L^2}^{1-\theta}, \tag{7.2}$$

with $\theta = d/2n < 1$, so that we conclude from (7.1)

$$\partial_t \|\nabla^n u\|_{L^2}^2 \le 2R\|\nabla^n u\|_{L^2}^2 + 2G_7^{2\sigma} c(d)|b_0 + i\mu| K_0^{\sigma(1-\theta)} \|u\|_{H^n}^{2\sigma\theta} \|\nabla^n u\|_{L^2}^2.$$

There exists a constant $\widetilde{C}_0 > 0$ such that

$$\|u\|_{H^n} \le \widetilde{C}_0 (\|u\|_{L^2} + \|\nabla^n u\|_{L^2}).$$

Hence, introducing $y(t) = \|\nabla^n u(t)\|_{L^2}^2$, this gives

$$\partial_t y \le (2R + C_{10} K_0^{\sigma\theta}) y + C_{10} y^{1+\sigma\theta},$$

where

$$C_{10} \stackrel{\text{def}}{=} 2C_7^{2\sigma} c(d)|b_0 + i\mu| K_0^{\sigma(1-\theta)} \widetilde{C}_0^{2\sigma\theta} \max(1, 2^{2\sigma\theta - 1}).$$

Employing Hölder's inequality, there exist two constants D_1, $D_2 > 0$ such that

$$\partial_t y \le D_1 + D_2 y^{1+\sigma\theta}.$$

Lemma 8.1 of the Appendix implies that for $0 \le t < T^*$,

$$y(t) \le K_6(t) \stackrel{\text{def}}{=} \left(\left(y(0) + \left(\frac{D_1}{D2}\right)^{1/(1+\sigma\theta)} \right)^{-\sigma\theta} - \sigma\theta D_2 t \right)^{-1/\sigma\theta} - \left(\frac{D_1}{D_2}\right)^{1/(1+\sigma\theta)},$$

where $T^* > 0$ is given by

$$T^* \stackrel{\text{def}}{=} \left[\sigma\theta D_2 \left(y(0) + \left(\frac{D_1}{D_2}\right)^{1/(1+\sigma\theta)} \right)^{\sigma\theta} \right]^{-1}.$$

The lemma is proved.

Proof of Theorem 2.5. The proof is very similar to the proof of Theorem 2.4. We only have to use the Gagliardo-Nirenberg inequality (7.2) instead of (5.5) and to apply Lemma 7.1 in order to obtain a uniform L^∞ bound on u_ε.

8 Appendix

Lemma 8.1 *Let the function $y : (0, T) \to \mathbb{R}$ be absolutely continuous and satisfy the following differential inequality:*

$$y' \le A y^\alpha + B \qquad \textit{for} \ \ t > 0,$$

with $\alpha > 1$*,* $A, B > 0$ *and* $y(0) > 0$*. Then there exists a constant* $T^* > 0$ *such that for all* $0 \leq t < T^*$*,*

$$y(t) \leq \left[\left(y(0) + \left(\frac{B}{A}\right)^{1/\alpha} \right)^{1-\alpha} + (1-\alpha)At \right]^{-1/(\alpha-1)} - \left(\frac{B}{A}\right)^{1/\alpha},$$

where $T^* > 0$ *is given by*

$$T^* = \left[(\alpha - 1)A\Big(y(0) + (B/A)^{1/\alpha}\Big)^{\alpha-1} \right]^{-1}.$$

Proof. Via the change of variable $z = y + C$, where $C = (B/A)^{1/\alpha}$, we get from

$$(y + C)^\alpha \geq y^\alpha + C^\alpha$$

the differential inequality

$$z' - Az^\alpha \leq 0.$$

This inequality can be integrated:

$$z(t) \leq \Big(z(0)^{1-\alpha} + (1-\alpha)At\Big)^{-1/(\alpha-1)},$$

for all $0 \leq t < T^*$. The lemma follows.

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