ANALYSIS OF A MULTI-DIMENSIONAL PARABOLIC POPULATION MODEL WITH STRONG CROSS-DIFFUSION*

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Abstract. The global existence of a non-negative weak solution to a multi-dimensional parabolic strongly coupled model for two competing species is proved. The main feature of the model is that the diffusion matrix is non-symmetric and generally not positive definite and that the non-diagonal matrix elements (the cross-diffusion terms) are allowed to be "large". The ideas of the existence proof are a careful approximation of the cross-diffusion terms using finite differences and the use of an entropy inequality yielding a priori estimates.

Key words. Cross-diffusion system, entropy functional, existence of weak solutions, Orlicz space.

AMS subject classifications. 35K55, 35D05, 92D25.

1. Introduction.

For the time evolution of two competing species with homogeneous population density, usually the Lotka-Volterra differential equations are used as an appropiate mathematical model. In the case of non-homogeneous densities, diffusion effects have to be taken into account leading to reaction-diffusion equations. Shigesada *et al.* proposed in their pioneering work [24] to introduce further so-called cross-diffusion terms modeling segregation phenomena of the competing species. Denoting by $u_i(x,t)$ the population density of the i-th species and by $J_i(x,t)$ the corresponding population flows, the time-dependent equations can be written as

$$\partial_t u_i - \mathrm{div} J_i = f_i(u_1,u_2), \quad J_i = \nabla(c_i u_i + a_i u_i^2 + u_1 u_2) + d_i u_i q, \tag{1.1}$$

where $i = 1,2$. The equations are solved in the bounded domain $\Omega \subset \mathbb{R}^N$ $(N \le 3)$ with time $t > 0$. The function q is given by $q = \nabla U$ and $U = U(x,t)$ is a prescribed environmental potential, modeling areas where the environmental conditions are more or less favorable [19, 24]. The diffusion coefficients c_i and a_i are non-negative, and $d_i \in \mathbb{R}$ $(i = 1,2)$. The source terms are in Lotka-Volterra form:

$$f_i(u_1,u_2) = (R_i - \beta_{i1}u_1 - \beta_{i2}u_2)u_i, \quad i = 1,2, \tag{1.2}$$

where $R_i \ge 0$ is the intrinsic growth rate of the i-th species, $\beta_{ii} > 0$ are the coefficients of intra-specific competition, and $\beta_{12} \ge 0$ and $\beta_{21} \ge 0$ are those of interspecific competition. The above system of equations is supplemented with (biologically motivated) homogeneous Neumann boundary conditions and initial conditions:

$$J_i \cdot \gamma = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{1.3}$$
$$u_i(\cdot,0) = u_i^0 \quad \text{in } \Omega, \quad i = 1,2, \tag{1.4}$$

and γ denotes the exterior unit normal to $\partial\Omega$, which is assumed to exist almost everywhere. The subsequent analysis is also valid for mixed Dirichlet-Neumann boundary conditions (see, e.g., [9]).

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Notice that the above system is scaled in such a way that the coefficient of the *cross-diffusion term* $\nabla(u_1u_2)$ is equal to one (see [9] for details).

The problem (1.1)-(1.4) is strongly coupled with full diffusion matrix

$$A(u_1,u_2) = \begin{pmatrix} c_1 + 2a_1u_1 + u_2 & u_1 \\ u_2 & c_2 + 2a_2u_2 + u_1 \end{pmatrix}.$$

Nonlinear problems of this kind are quite difficult to deal with since the usual idea of applying maximum principle arguments to get a priori estimates cannot be used here. Furthermore, the diffusion matrix is not symmetric and of degenerate type if $c_1 = c_2 = 0$.

Up to now, only partial results are available in the literature concerning the well-posedness of the above problem. We summarize some of the available results for the time-dependent equations (see [28] for a review) and refer to [16, 17, 22, 23] for the stationary problem. Global existence of solutions and their qualitive behavior for $a_1 = a_2 = 0$ and no cross-diffusion for the second species have been proved in, e.g., [4, 18, 20, 21, 27]. In this case, eq. (1.1) for $i = 2$ is only weakly coupled. The existence of an attractor has been studied in [15, 21]. Notice that in chemotaxis, related models appear [8, 10].

For sufficiently small cross-diffusion terms (or "small" initial data) and vanishing self-diffusion coefficients $a_1 = a_2 = 0$, Deuring proved the global existence of solutions in [7]. For the case $c_1 = c_2$ a global existence result in one space dimension has been obtained by Kim [12]. Furthermore, under the condition

$$2a_1 > 1, \quad 2a_2 > 1, \tag{1.5}$$

Yagi [26] has shown the global existence of solutions in two space dimensions. A global existence result for weak solutions in any space dimension under assumption (1.5) can be found in [9]. Condition (1.5) can be easily understood by observing that in this case, the diffusion matrix is positive definite:

$$\xi^T A(u_1,u_2)\xi \geq \min\{c_1,c_2\}|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^2,$$

hence yielding an elliptic operator. If the condition (1.5) does not hold, there are choices of c_i, a_i, $u_i \geq 0$ for which the matrix $A(u_1,u_2)$ is *not* positive definite. Finally, Galiano *et al.* [9] proved the existence of global weak solutions for *any* a_1, $a_2 > 0$. However, the proof uses the embedding $H^1(\Omega) \subset L^\infty(\Omega)$ in a crucial way such that the result is restricted to one space dimension only.

In this paper we solve the problem (1.1)-(1.4) for (up to) *three* space dimensions without any restriction on the diffusion coefficients. More precisely, we prove the following result.

THEOREM 1.1. *Let $T > 0$ and assume that*

- $\Omega \subset \mathbb{R}^N$ $(N \leq 3)$ *is a bounded domain with boundary* $\partial\Omega \in C^{0,1}$*;*
- *the parameters satisfy* $c_i \geq 0$, $a_i > 0$*;* $R_i \geq 0$, $\beta_{ii} > 0$ $(i = 1,2)$, $\beta_{12} = \beta_{21} \geq 0$*;* $q \in (L^2(Q_T))^N$, *where* $Q_T = \Omega \times (0,T)$*;*
- *the initial data satisfy* $u_i^0 \in L_\Psi(\Omega)$ *and* $u_i^0 \geq 0$ *in* Q_T $(i = 1,2)$.

Then problem (1.1)-(1.4) has a weak solution (u_1,u_2) *satisfying* $u_i \geq 0$ *in* Q_T *and*

$$u_i \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L_\Psi(\Omega)) \cap W^{1,r}(0,T;(W^{1,r'}(\Omega))'), \quad i = 1,2,$$

where $r = (2N+2)/(2N+1)$ *and* $r' = r/(r-1) = 2N+2$, *in the sense that for all* $\varphi \in L^{r'}(0,T;W^{1,r'}(\Omega))$, $i=1,2$,

$$\int_0^T \langle \partial_t u_i, \varphi\rangle \, dt + \int_{Q_T} (c_i \nabla u_i + 2a_i u_i \nabla u_i + \nabla(u_1 u_2) + d_i u_i q) \cdot \nabla \varphi dx dt$$
$$= \int_{Q_T} f_i(u_1,u_2)\varphi dx dt,$$

and $\langle \cdot,\cdot \rangle$ *denotes the dual product between* $W^{1,r'}(\Omega)$ *and its dual* $(W^{1,r'}(\Omega))'$.

Here, $L_\Psi(\Omega)$ denotes the Orlicz space for $\Psi(s) = (1+s)\ln(1+s) - s$, $s \geq 0$. We refer to the appendix for its definition and some properties.

In order to explain the method of our proof it is convenient to recall the ideas of [9]. By using the exponential transformation of variables $u_1 = \exp(w_1)$, $u_2 = \exp(w_2)$, eqs. (1.1) transform into

$$\partial_t \begin{pmatrix} e^{w_1} \\ e^{w_2} \end{pmatrix} - \operatorname{div}\left(B(w_1,w_2)\nabla \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} d_1 e^{w_1} \\ d_2 e^{w_2} \end{pmatrix} q \right) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

and the new diffusion matrix

$$B(w_1,w_2) = \begin{pmatrix} c_1 e^{w_1} + 2a_1 e^{2w_1} + e^{w_1+w_2} & e^{w_1+w_2} \\ e^{w_1+w_2} & c_2 e^{w_2} + 2a_2 e^{2w_2} + e^{w_1+w_2} \end{pmatrix},$$

is symmetric and positive definite:

$$\det B(w_1,w_2) \geq (c_1 e^{w_1} + 2a_1 e^{2w_1})(c_2 e^{w_2} + 2a_2 e^{2w_2}) > 0.$$

In this formulation the matrix B provides an elliptic operator for all $c_i > 0$, $a_i \geq 0$ or $c_i \geq 0$, $a_i > 0$ $(i=1,2)$. In this sense, the system (1.1)-(1.2) is called *parabolic*. The above change of unknowns symmetrizing the problem implies the existence of an *entropy functional*

$$E(t) = \sum_{i=1}^{2} \int_\Omega (u_i(\ln u_i - 1) + 1) dx \geq 0,$$

with the corresponding entropy inequality

$$E(t) + 2\int_0^t \int_\Omega \Big(\sum_{i=1}^{2} (2c_i |\nabla \sqrt{u_i}|^2 + a_i |\nabla u_i|^2) + 2|\nabla \sqrt{u_1 u_2}|^2 \Big) dx dt \leq E(0) + C, \quad (1.6)$$

for $0 < t < T$ and any $T > 0$, where the constant $C > 0$ depends on T, q, and the source terms. It can be formally derived by using $\ln u_i$ as a test function in the weak formulation of (1.1)-(1.4). This inequality provides an $L^2(0,T;H^1(\Omega))$ estimate for u_1 and u_2 if a_1, $a_2 > 0$. We notice that the existence of a symmetric formulation of the problem is even equivalent to the existence of an entropy functional [6, 11].

However, the entropy inequality can be made rigorous only if $u_i \geq 0$, which cannot be easily obtained from the minimum principle. The non-negativity of the solutions is obtained in [9] by proving that the transformed variable satisfies $w_i \in L^2(0,T;H^1(\Omega))$. As $H^1(\Omega)$ embeddes continuously into $L^\infty(\Omega)$ in one space dimension, this implies $w_i(\cdot,t) \in L^\infty(\Omega)$ for almost every $t > 0$, and hence, $u_i(\cdot,t) = \exp(w_i(\cdot,t)) > 0$ in Ω. This method, clearly, cannot be used in several space dimensions.

The main idea of our proof is to *discretize* the cross-diffusion term $\nabla(u_1u_2)$ by *finite differences* and first to prove the existence of solutions to the approximate problem, which is now only weakly coupled. The precise approximation has to be chosen in such a way that the above entropy inequality also holds for the approximate problem. This provides the a priori estimates necessary to perform the limit of vanishing approximation parameters. The idea is inspired from [13] where a different problem is studied.

One possibility is to approximate the cross-diffusion term $\Delta(u_1u_2) = \mathrm{div}(u_1u_2 \nabla \ln(u_1u_2))$ by the finite differences

$$D^{-h}[\chi_h u_1u_2 D^h(\ln(u_1u_2))],$$

where D^h is an approximation of the gradient,

$$D^h f = (D_1^h f, \ldots, D_N^h f) \quad \text{and} \quad D_j^h f(x,t) = \frac{f(x+he_j,t) - f(x,t)}{h}, \tag{1.7}$$

D^{-h} is an approximation of the divergence,

$$D^{-h}F(x,t) = \sum_{j=1}^N \frac{F_j(x-he_j,t) - F_j(x,t)}{-h}, \tag{1.8}$$

with the j-th unit vector e_j of $\mathbb{R}^N$, $j = 1,\ldots,N$, and χ_h is the characteristic function of $\{x \in \Omega : \mathrm{dist}(x,\partial\Omega) > h\}$. It can be shown formally that the problem with this discrete cross-diffusion term possesses the entropy inequality

$$E(t) + \int_0^t \int_\Omega \Big(\sum_{i=1}^2 (4c_i|\nabla\sqrt{u_i}|^2 + 2a_i|\nabla u_i|^2) + \chi_h u_1u_2|D^h \ln(u_1u_2)|^2 \Big) dtdx \le E(0) + C,$$

for some constant $C > 0$.

However, this estimate is only valid for *positive* population densities u_i. In order to deal with this difficulty, we employ Stampacchia's truncation method, i.e., we replace u_i by $(u_i)_+ + \eta$, where $(u_i)_+ = \max\{0,u_i\}$ and $\eta > 0$. This allows to define the expression $\ln\big(((u_1)_+ + \eta)((u_2)_+ + \eta)\big)$, for instance.

The above estimate is formally derived by employing $\ln((u_i)_+ + \eta)$ as a test function in the weak formulation. Therefore, we only obtain estimates for $(u_i)_+$. In order to derive estimates also for $(u_i)_- = \min\{0,u_i\}$, we employ $(u_i)_-$ as a test function. This yields, for instance, an estimate of the type $\|(u_i)_-\|_{L^\infty(0,T;L^2(\Omega))} \le C/|\ln\eta|$ for some constant $C > 0$ which is independent of η. In the limit $\eta \to 0$ this gives $(u_i)_- = 0$ in Q_T and hence the non-negativity of the population densities.

The entropy inequality (1.6) is the key estimate for proving results on the long-time behavior of the solutions by employing entropy dissipation methods [3]. However, the obtained regularity of solution is not strong enough to define the continuous entropy inequality (1.6) rigorously. For results in this direction we refer to the forthcoming work [2].

We notice that our strategy can be also applied to general systems of the type

$$\partial_t u - \mathrm{div}(A(u)\nabla u) = f(u),$$

where $u = u(x,t) \in \mathbb{R}^n$, $f(u) \in \mathbb{R}^n$ satisfies some growth condition, and $A(u) \in \mathbb{R}^{n\times n}$ is a diffusion matrix, maybe non-symmetric and not positive definite, provided that the system is symmetrizable in the sense given above and that the a priori estimates derived from the entropy inequality (which exists due to the symmetrizability) are sufficient to define a weak solution.

Let us summarize the main features of the presented method of proof:

- No restriction on the diffusion coefficients c_i and a_i are needed.
- The global existence result holds in up to three space dimensions.
- The method provides the non-negativity of the solutions.
- The degenerate case $c_i = 0$ can be also treated.

The idea of discretizing the cross-diffusion term by finite differences can be used for numerical purposes. We will exploit this idea in [2].

This paper is organized as follows. In section 2 we define and solve an approximate problem yielding an discrete entropy inequality. The key estimates are contained in Lemma 2.5. The limit of vanishing approximation parameters is then performed in section 3. Finally, in the appendix we recall the definition of Orlicz spaces and some of its properties.

2. Existence of solutions to an approximate problem. We use semi-discretization in time to construct the approximate problem. Moreover, as explained in the introduction, we also discretize the cross-diffusion terms by finite differences. For this, we decompose $(0,T] = \cup_{k=1}^{K}((k-1)\tau, k\tau]$ for some $\tau > 0$ such that $\tau = T/K$. Furthermore, let $h > 0$ and let χ_h be the characteristic function of $\{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > h\}$. Finally, let $0 < \eta < 1$ and set $\overline{s} = s/(1 + \eta(s)_+)$.

As the proof of Theorem 1.1 is highly technical, it is convenient, for the sake of a smoother presentation, to assume in this section the regularity $u_i^0 \in L^2(\Omega)$ and $q \in (L^\infty(Q_T))^N$ instead of the weaker conditions $u_i^0 \in L_\Psi(\Omega)$ and $q \in (L^2(Q_T))^N$. The general result can be proved by using appropriate smooth approximations and passing to the limit. Details are left to the reader. In fact, we simplify further and assume that $q \in (L^\infty(\Omega))^N$. The time-dependence can be treated as in [6], for instance, by averaging $q(x,t)$ over $((k-1)\tau, k\tau]$. Moreover, we assume that c_1, c_2 are *positive* numbers. We refer to Remark 3.5 for the case $c_1 = 0$ or $c_2 = 0$.

For given u_1^{k-1}, $u_2^{k-1} \in L^2(\Omega)$, we solve recursively the following problem

$$\begin{aligned}
&\frac{u_i^k - u_i^{k-1}}{\tau} - \mathrm{div}\big(c_i \nabla u_i^k + 2a_i((u_i^k)_+ + \eta)\nabla u_i^k + d_i(u_i^k)_+ q\big) \\
&= D^{-h}\Big[\chi_h \overline{u_1^k}\,\overline{u_2^k} D^h \ln\big(((u_1^k)_+ + \eta)((u_2^k)_+ + \eta)\big)\Big] + f_i\big((u_1^k)_+ + \eta, (u_2^k)_+ + \eta\big) \text{ in } \Omega, \\
&\big(c_i \nabla u_i^k + 2a_i((u_i^k)_+ + \eta)\nabla u_i^k + d_i(u_i^k)_+ q\big) \cdot \gamma = 0 \quad \text{on } \partial\Omega,
\end{aligned} \tag{2.1}$$

where $i = 1, 2$. The finite difference operators are defined in (1.7) and (1.8).

The existence of solutions to the approximate system (2.1) is proved in two steps. In order to apply Lax-Milgram's lemma we need bounded diffusion coeffcients. Therefore, we approximate the diffusion coefficients $2a_i((u_i^k)_+ + \eta)$ by

$$2a_i \frac{(u_i^k)_+ + \eta}{1 + \nu((u_i^k)_+ + \eta)}$$

for some $\nu > 0$ and prove the existence of solutions to the resulting system. Then we derive uniform bounds with respect to ν which allows to pass to the limit $\nu \to 0$. The second approximate system reads as follows:

$$\frac{u_i^k - u_i^{k-1}}{\tau} - \operatorname{div}\Big(c_i \nabla u_i^k + 2a_i \frac{(u_i^k)_+ + \eta}{1+\nu((u_i^k)_+ + \eta)} \nabla u_i^k + d_i (u_i^k)_+ q\Big)$$
$$= D^{-h}\Big[\chi_h \overline{u_1^k}\ \overline{u_2^k} D^h \ln\big(((u_1^k)_+ + \eta)((u_2^k)_+ + \eta)\big)\Big] + f_i\big((u_1^k)_+ + \eta, (u_2^k)_+ + \eta\big) \text{ in } \Omega,$$
$$\Big(c_i \nabla u_i^k + 2a_i \frac{(u_i^k)_+ + \eta}{1+\nu((u_i^k)_+ + \eta)} \nabla u_i^k + d_i (u_i^k)_+ q\Big) \cdot \gamma = 0 \quad \text{on } \partial\Omega,\ i = 1, 2. \tag{2.2}$$

In subsection 2.1 we prove some bounds uniform in ν and the existence of weak solutions to (2.2). Then by letting $\nu \to 0$ in subsection 2.2 we conclude the solvability of (2.1).

In the following, C and $C(\cdots)$ denote positive constants with values varying from occurence to occurence and depending on the quantities indicated in the brackets.

2.1. Existence of solutions to the second approximate problem (2.2).
LEMMA 2.1. *Assume that the time discretization parameter $\tau > 0$ is so small that*

$$\frac{3}{16\tau} \geq \max_{i=1,2}\Big\{\frac{d_i^2}{2c_i}\|q\|^2_{L^\infty(\Omega)} + 2(R_i + \beta_{i1} + \beta_{i2})\Big\} \quad \textit{and} \quad 32\tau \leq h^2\eta^2. \tag{2.3}$$

Then there exists a solution $(u_1, u_2) \in (H^1(\Omega))^2$ of problem (2.2) satisfying the following estimate:

$$\int_\Omega \sum_{i=1}^2 \Big(\frac{c_i}{2}|\nabla u_i|^2 + \frac{u_i^2}{4\tau} + 2a_i \frac{(u_i)_+ + \eta}{1+\nu((u_i)_+ + \eta)}|\nabla u_i|^2\Big)dx \leq C(\tau), \tag{2.4}$$

where the constant $C(\tau) > 0$ depends on τ but not on ν.

The above estimate is only used to pass to the limit $\nu \to 0$ for *fixed* parameters τ, h, and η. For the limits $\tau, h \to 0$ and $\eta \to 0$ we need other estimates.

REMARK 2.2. The second restriction on the time discretization parameter τ in (2.3) is similar to the well-known condition $\tau/h^2 \leq$ const. needed for explicit finite difference approximations of parabolic equations since we treat the discrete cross-diffusion term in an "explicit" way. Clearly, this condition has no importance for the existence result.

Proof. Construct a mapping

$$\mathcal{T} : (\sigma, v_1, v_2) \in [0,1] \times (L^4(\Omega))^2 \to (L^4(\Omega))^2$$

by solving the following linear problem:

$$-\operatorname{div}(c_i \nabla u_i) + \frac{u_i}{\tau} - \sigma \operatorname{div}\Big(2a_i \frac{(v_i)_+ + \eta}{1+\nu((v_i)_+ + \eta)} \nabla u_i\Big) - \sigma \operatorname{div}(d_i (v_i)_+ q)$$
$$= \sigma \frac{u_i^{k-1}}{\tau} + \sigma F_i(v_1, v_2) \quad \text{in } \Omega, \tag{2.5}$$
$$\Big(c_i \nabla u_i + 2\sigma a_i \frac{(v_i)_+ + \eta}{1+\nu((v_i)_+ + \eta)} \nabla u_i + \sigma d_i (v_i)_+ q\Big) \cdot \gamma = 0 \quad \text{on } \partial\Omega,$$

where

$$F_i(v_1, v_2) = D^{-h}\left[\chi_h \overline{v_1}\ \overline{v_2} D^h \ln\big(((v_1)_+ + \eta)((v_2)_+ + \eta)\big)\right] + f_i((v_1)_+ + \eta, (v_2)_+ + \eta)$$

and $i = 1, 2$. The functionals F_i satisfy the estimate $\|F_i(v_1, v_2)\|_{L^2(\Omega)} \leq C(1 + \|v_1\|^2_{L^4(\Omega)} + \|v_2\|^2_{L^4(\Omega)})$ for $i = 1, 2$. The above problem has a unique solution (by Lax-Milgram's lemma) since the diffusion coefficients are bounded. Thus, the mapping $\mathcal{T}$ is well defined. It is not difficult to prove the continuity of $\mathcal{T}$. Moreover, since the embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ is compact, for every $\sigma \in [0, 1]$, the mapping $\mathcal{T}$ is compact. Here, we use the restriction $N \leq 3$ of the space dimension (see Remark 2.3). When $\sigma = 0$, the equation $\mathcal{T}(0, u_1, u_2) = (u_1, u_2)$ immediately yields $u_1 = u_2 = 0$ in Ω.

It remains to establish uniform estimates for every fixed point of $\mathcal{T}$. Any fixed point (u_1, u_2) satisfies the equation

$$\begin{aligned} &-\operatorname{div}(c_i \nabla u_i) + \frac{u_i}{\tau} - \sigma \operatorname{div}\Big(2a_i \frac{(u_i)_+ + \eta}{1 + \nu((u_i)_+ + \eta)} \nabla u_i\Big) - \sigma \operatorname{div}(d_i (u_i)_+ q) \\ &\qquad = \sigma \frac{u_i^{k-1}}{\tau} + \sigma F_i(u_1, u_2) \quad \text{in } \Omega,\ i = 1, 2, \end{aligned} \tag{2.6}$$

together with homogeneous Neumann boundary conditions. We use $u_i \in H^1(\Omega)$ as a test function in the weak formulation of (2.6) for $i = 1, 2$ and add the resulting equations:

$$\begin{aligned} &\sum_{i=1}^2 \int_\Omega \Big(c_i |\nabla u_i|^2 + \frac{u_i^2}{\tau} + 2\sigma a_i \frac{(u_i)_+ + \eta}{1 + \nu((u_i)_+ + \eta)} |\nabla u_i|^2\Big) dx \\ &= -\sum_{i=1}^2 \int_\Omega \sigma d_i (u_i)_+ q \cdot \nabla u_i dx + \frac{\sigma}{\tau} \sum_{i=1}^2 \int_\Omega u_i u_i^{k-1} dx \\ &\quad + \sum_{i=1}^2 \int_\Omega \sigma D^{-h} \left[\chi_h \overline{u_1}\, \overline{u_2} D^h \ln\big(((u_1)_+ + \eta)((u_2)_+ + \eta)\big)\right] u_i dx \\ &\quad + \sum_{i=1}^2 \int_\Omega \sigma \big[R_i - \beta_{i1}((u_1)_+ + \eta) - \beta_{i2}((u_2)_+ + \eta)\big]((u_i)_+ + \eta) u_i dx \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{2.7}$$

We estimate the right-hand side term by term. First, Young's inequality allows to estimate

$$\begin{aligned} I_1 + I_2 &\leq \sum_{i=1}^2 \int_\Omega \Big(\frac{c_i}{2} |\nabla u_i|^2 + \frac{u_i^2}{4\tau}\Big) dx + \|q\|^2_{L^\infty(\Omega)} \sum_{i=1}^2 \frac{d_i^2}{2c_i} \int_\Omega u_i^2 dx \\ &\quad + \frac{1}{\tau} \sum_{i=1}^2 \int_\Omega (u_i^{k-1})^2 dx. \end{aligned}$$

The third integral is estimated by using Young's inequality and the elementary inequalities $|\overline{u_i}| \leq 1/\eta$ and $|\ln(x + \eta)| \leq x + |\ln \eta|$ for all $x \geq 0$ and $0 < \eta < 1$:

$$\begin{aligned} I_3 &\leq \sum_{i=1}^2 \int_\Omega \frac{u_i^2}{4\tau} dx + 2\tau \int_\Omega \left|D^{-h}\left[\chi_h \overline{u_1}\, \overline{u_2} D^h \ln\big(((u_1)_+ + \eta)((u_2)_+ + \eta)\big)\right]\right|^2 dx \\ &\leq \sum_{i=1}^2 \int_\Omega \frac{u_i^2}{4\tau} dx + \frac{16\tau}{h^4 \eta^4} \sum_{i=1}^2 \int_\Omega \left|\ln\big(((u_1)_+ + \eta)((u_2)_+ + \eta)\big)\right|^2 dx \end{aligned}$$

$$\leq \sum_{i=1}^2 \int_\Omega \frac{u_i^2}{4\tau} dx + \frac{64\tau}{h^4\eta^4} \sum_{i=1}^2 \int_\Omega (u_i^2 + |\ln \eta|^2) dx,$$

For the last integral we employ the special structure of the source term f_i:

$$\begin{aligned} I_4 &\leq \sum_{i=1}^2 \int_\Omega \sigma R_i (u_i)_+^2 + \sum_{i=1}^2 \int_\Omega \sigma \big[R_i - \beta_{i1}((u_1)_+ + \eta) - \beta_{i2}((u_2)_+ + \eta)\big] \eta u_i dx \\ &\leq \sum_{i=1}^2 \int_\Omega 2(R_i + \beta_{i1} + \beta_{i2})(u_i^2 + 1) dx, \end{aligned} \tag{2.8}$$

since $\eta < 1$. Putting together the above estimates, we can estimate (2.7) as follows:

$$\begin{aligned} &\sum_{i=1}^2 \int_\Omega \Big(\frac{c_i}{2} |\nabla u_i|^2 + \frac{u_i^2}{4\tau} + 2\sigma a_i \frac{(u_i)_+ + \eta}{1 + \nu((u_i)_+ + \eta)} |\nabla u_i|^2 \Big) dx \\ &\leq \frac{1}{\tau} \sum_{i=1}^2 \int_\Omega (u_i^{k-1})^2 dx + 2|\Omega| \sum_{i=1}^2 (R_i + \beta_{i1} + \beta_{i2}) + \frac{128\tau}{h^4\eta^4} |\ln \eta|^2 |\Omega| \\ &\quad + \sum_{i=1}^2 \int_\Omega u_i^2 \Big(-\frac{1}{4\tau} + \frac{d_i^2}{2c_i} \|q\|_{L^\infty}^2 + \frac{64\tau}{h^4\eta^4} + 2(R_i + \beta_{i1} + \beta_{i2}) \Big) dx, \end{aligned}$$

and $|\Omega|$ is the measure of Ω. By choosing τ so small that (2.3) is satisfied, in particular $64\tau/(h^4\eta^4) \leq 1/16\tau$, we obtain

$$\begin{aligned} &\sum_{i=1}^2 \int_\Omega \Big(\frac{c_i}{2} |\nabla u_i|^2 + \frac{u_i^2}{4\tau} + 2\sigma a_i \frac{(u_i)_+ + \eta}{1 + \nu((u_i)_+ + \eta)} |\nabla u_i|^2 \Big) dx \\ &\leq \frac{1}{\tau} \sum_{i=1}^2 \int_\Omega (u_i^{k-1})^2 dx + \sum_{i=1}^2 \int_\Omega u_i^2 \Big(-\frac{3}{16\tau} + \frac{d_i^2}{2c_i} \|q\|_{L^\infty}^2 + 2(R_i + \beta_{i1} + \beta_{i2}) \Big) dx \\ &\quad + C(\tau) \\ &\leq \frac{1}{\tau} \sum_{i=1}^2 \int_\Omega (u_i^{k-1})^2 dx + C(\tau) \\ &\leq C(\tau). \end{aligned}$$

By the Leray-Schauder theorem, $\mathcal{T}(1, \cdot)$ has a fixed point. Thus we conclude the existence of a weak solution of problem (2.2). The inequality (2.4) follows from the above estimate with $\sigma = 1$. □

REMARK 2.3. From the proof of the above lemma we see that if $\beta_{ij} = 0$ for all $i, j = 1, 2$, then the fixed-point mapping $\mathcal{T}$ can be defined on $[0, 1] \times (L^2(\Omega))^2$. This allows to prove the above result for *any* space dimension N (see Remark 3.6).

2.2. The limit $\nu \to 0$. We show in the following that the limit $\nu \to 0$ can be performed in (2.2).

LEMMA 2.4. *There exists a weak solution* $(u_1, u_2) \in (H^1(\Omega))^2$ *of problem (2.1) in the sense that for all* $\varphi \in W^{1,2\cdot 2^*/(2^*-2)}(\Omega)$,

$$\begin{aligned} &\int_\Omega \frac{u_i - u_i^{k-1}}{\tau}\varphi dx + \int_\Omega (c_i \nabla u_i + 2a_i((u_i)_+ + \eta)\nabla u_i + d_i(u_i)_+ q)\cdot\nabla\varphi dx \\ &\quad - \int_\Omega D^{-h}\left[\chi_h \overline{u_1}\,\overline{u_2} D^h \ln\big(((u_1)_+ + \eta)((u_2)_+ + \eta)\big)\right]\varphi dx \\ &= \int_\Omega f_i\big((u_1)_+ + \eta, (u_2)_+ + \eta\big)\varphi dx, \end{aligned} \tag{2.9}$$

where $2^* = \infty$ *if* $N = 1$, $2^* < \infty$ *if* $N = 2$, *and* $2^* = 2N/(N-2)$ *if* $N \geq 3$.

Proof. Let $(u_1^\nu, u_2^\nu) \in (H^1(\Omega))^2$ be a weak solution of (2.2). From the uniform estimate (2.4) we conclude the existence of a subsequence of (u_1^ν, u_2^ν) (not relabeled) such that, as $\nu \to 0$,

$$\begin{aligned} \nabla u_i^\nu &\rightharpoonup \nabla u_i \quad \text{weakly in } (L^2(\Omega))^N, \\ u_i^\nu &\to u_i \quad \text{strongly in } L^r(\Omega),\ 1 \leq r < 2^*,\ i = 1, 2, \end{aligned} \tag{2.10}$$

The last convergence result follows from the compactness of the embedding $H^1(\Omega) \hookrightarrow L^r(\Omega)$ for all $r < 2^*$. In particular, we have $(u_i^\nu)_+ \to (u_i)_+$ strongly in $L^r(\Omega)$ and

$$((u_i^\nu)_+ + \eta)\nabla u_i^\nu \rightharpoonup ((u_i)_+ + \eta)\nabla u_i \quad \text{weakly in } (L^s(\Omega))^N \text{ for all } 1 \leq s \leq \frac{2r}{r+2}.$$

Since (u_i^ν) is uniformly bounded in $H^1(\Omega)$, Hölder's inequality implies

$$\|((u_i^\nu)_+ + \eta)\nabla u_i^\nu\|_{L^{2\cdot 2^*/(2+2^*)}(\Omega)} \leq \|(u_i^\nu)_+ + \eta\|_{L^{2^*}(\Omega)} \|\nabla u_i^\nu\|_{L^2(\Omega)} \leq C,$$

where $C > 0$ is independent of ν. Thus, the above weak convergence also holds for $s = 2\cdot 2^*/(2+2^*)$.

Now we use the following result: Let $(v_\nu) \subset L^\infty(\Omega)$ and $(w_\nu) \subset L^s(\Omega)$ with $s \geq 1$ be two sequences such that (v_ν) is bounded in $L^\infty(\Omega)$, $v_\nu \to v$ pointwise almost everywhere in Ω as $\nu \to 0$, and $w_\nu \rightharpoonup w$ weakly in $L^s(\Omega)$. Then, as $\nu \to 0$, $v_\nu w_\nu \rightharpoonup vw$ weakly in $L^s(\Omega)$. Applying this result to $v_\nu = 1/(1 + \nu((u_i^\nu)_+ + \eta))$ and $w_\nu = ((u_i^\nu)_+ + \eta)\nabla u_i^\nu$ with $s = 2\cdot 2^*/(2+2^*)$ yields

$$\frac{(u_i^\nu)_+ + \eta}{1 + \nu((u_i^\nu)_+ + \eta)}\nabla u_i^\nu \rightharpoonup ((u_i)_+ + \eta)\nabla u_i \quad \text{weakly in } (L^s(\Omega))^N,\ s = \frac{2\cdot 2^*}{2+2^*}.$$

Moreover, by similar arguments as above, as $\nu \to 0$,

$$\begin{aligned} &f_i((u_1^\nu)_+ + \eta, (u_2^\nu)_+ + \eta) \\ &= \big(R_i - \beta_{i1}((u_1^\nu)_+ + \eta) - \beta_{i2}((u_2^\nu)_+ + \eta)\big)((u_i^\nu)_+ + \eta) \\ &\rightharpoonup \big(R_i - \beta_{i1}((u_1)_+ + \eta) - \beta_{i2}((u_2)_+ + \eta)\big]((u_i)_+ + \eta) \quad \text{weakly in } L^{2^*/2}(\Omega), \\ &D^{-h}\left[\chi_h \overline{u_1^\nu}\,\overline{u_2^\nu} D^h \ln(((u_1^\nu)_+ + \eta)((u_2^\nu)_+ + \eta))\right] \\ &\rightharpoonup D^{-h}\left[\chi_h \overline{u_1}\,\overline{u_2} D^h \ln(((u_1)_+ + \eta)((u_2)_+ + \eta))\right] \quad \text{weakly in } L^s(\Omega), \end{aligned}$$

for all $1 < s < \infty$. These convergence results allow to pass to the limit $\nu \to 0$ in the weak formulation of (2.2) which yields (2.9) and hence the conclusion. ∎

2.3. Uniform estimates with respect to τ and h. The following entropy inequality is the key estimate of this paper providing uniform bounds in τ, h, and η.

LEMMA 2.5. *Let $(u_1, u_2) \in (H^1(\Omega))^2$ be a solution of (2.1). Then the following estimates hold:*

$$
\begin{aligned}
&\int_\Omega \Bigg[\sum_{i=1}^2 \Big(c_i \frac{|\nabla (u_i)_+|^2}{(u_i)_+ + \eta} + a_i |\nabla (u_i)_+|^2\Big) \\
&\qquad + \chi_h \overline{u_1}\,\overline{u_2} \left|D^h \ln\left(((u_1)_+ + \eta)((u_2)_+ + \eta)\right)\right|^2\Bigg] dx \\
&\qquad + \frac{1}{\tau}\sum_{i=1}^2 \int_\Omega [((u_i)_+ + \eta)\left(\ln((u_i)_+ + \eta) - 1\right) + (u_i)_- \ln \eta]\,dx \\
&\le \frac{1}{\tau}\sum_{i=1}^2 \int_\Omega \left[((u_i^{k-1})_+ + \eta)\left(\ln((u_i^{k-1})_+ + \eta) - 1\right) + (u_i^{k-1})_- \ln\eta\right] dx + C
\end{aligned} \tag{2.11}
$$

and

$$
\begin{aligned}
&\sum_{i=1}^2 \int_\Omega \Big(\frac{c_i}{2}|\nabla(u_i)_-|^2 + 2a_i\eta|\nabla(u_i)_-|^2\Big) dx + \frac{1}{2\tau}\sum_{i=1}^2 \int_\Omega |(u_i)_-|^2 dx \\
&\le \frac{1}{2\tau}\sum_{i=1}^2 \int_\Omega |(u_i^{k-1})_-|^2 dx + \eta C \int_\Omega \sum_{i=1}^2 (|(u_i)_+|^2 + |(u_i)_-|^2) dx \\
&\qquad + \frac{C(c_1, c_2)}{\eta^2} \int_\Omega \chi_h \overline{u_1}\,\overline{u_2} \left|D^h \ln\left(((u_1)_+ + \eta)((u_2)_+ + \eta)\right)\right|^2 dx + C,
\end{aligned} \tag{2.12}
$$

where $C > 0$ depends only on R_i, β_{ij} $(i, j = 1, 2)$, and $\|q\|_{L^2(\Omega)}$.

Proof. Let (u_1, u_2) be a solution of (2.1), i.e. $u_i \in H^1(\Omega)$ satisfies (2.9), $i = 1, 2$. As $\ln((u_i)_+ + \eta) \notin W^{1,2\cdot 2^*/(2^*-2)}(\Omega)$ in general, we cannot use this function as a test function in the weak formulation (2.9). Therefore, we choose a sequence (v^ε) of smooth functions satisfying $v^\varepsilon \to (u_i)_+$ in $H^1(\Omega)$ (for some fixed i) and $v^\varepsilon \ge 0$ in Ω, and use $\varphi = \ln(v^\varepsilon + \eta)$ as a test function in (2.9):

$$
\begin{aligned}
&\int_\Omega \frac{u_i - u_i^{k-1}}{\tau} \ln(v^\varepsilon + \eta) dx \\
&\qquad + \int_\Omega \left(c_i \nabla u_i + 2a_i((u_i)_+ + \eta)\nabla u_i + d_i (u_i)_+ q\right) \cdot \nabla \ln(v^\varepsilon + \eta) dx \\
&\qquad - \int_\Omega D^{-h}\left[\chi_h \overline{u_1}\,\overline{u_2} D^h \ln\left(((u_1)_+ + \eta)((u_2)_+ + \eta)\right)\right] \ln(v^\varepsilon + \eta) dx \\
&= \int_\Omega \left[R_i - \beta_{i1}((u_1)_+ + \eta) - \beta_{i2}((u_2)_+ + \eta)\right]((u_i)_+ + \eta)\ln(v^\varepsilon + \eta) dx.
\end{aligned} \tag{2.13}
$$

We claim that, as $\varepsilon \to 0$,

$$
\int_\Omega ((u_i)_+ + \eta)\nabla u_i \cdot \nabla \ln(v^\varepsilon + \eta) dx \to \int_\Omega |\nabla (u_i)_+|^2 dx. \tag{2.14}
$$

In order to prove this claim we observe that

$$
((u_i)_+ + \eta)\nabla \ln(v^\varepsilon + \eta) \rightharpoonup ((u_i)_+ + \eta)\nabla \ln((u_i)_+ + \eta) = \nabla (u_i)_+
$$

weakly in $L^{2\cdot 2^*/(2+2^*)}(\Omega)$ and

$$\|((u_i)_+ + \eta)\nabla \ln(v^\varepsilon + \eta)\|_{L^2(\Omega)} \le \left\| \frac{(u_i)_+ + \eta}{v^\varepsilon + \eta} \right\|_{L^\infty(\Omega)} \|\nabla v^\varepsilon\|_{L^2(\Omega)} \le C,$$

where $C > 0$ is a constant independent of ε. Therefore, the above weak convergence holds also in $L^2(\Omega)$. Since $\nabla u_i \in L^2(\Omega)$, the claim follows.

As $\ln(v^\varepsilon + \eta) \to \ln((u_i)_+ + \eta)$ in $H^1(\Omega)$, we can pass to the limit $\varepsilon \to 0$ in (2.13). Adding the equations (2.13) for $i = 1$ and $i = 2$ and using (2.14) then gives in the limit $\varepsilon \to 0$

$$\begin{aligned} &\int_\Omega \Big[\sum_{i=1}^2 \Big(c_i \frac{|\nabla (u_i)_+|^2}{(u_i)_+ + \eta} + 2a_i |\nabla (u_i)_+|^2 \Big) \\ &\quad + \chi_h \overline{u_1}\, \overline{u_2} \left| D^h \ln \left(((u_1)_+ + \eta)((u_2)_+ + \eta) \right) \right|^2 \Big] dx \end{aligned} \tag{2.15}$$

$$\begin{aligned} &+ \sum_{i=1}^2 \int_\Omega \frac{u_i - u_i^{k-1}}{\tau} \ln((u_i)_+ + \eta) dx - \sum_{i=1}^2 \int_\Omega |d_i q \nabla (u_i)_+| dx \\ &\le \sum_{i=1}^2 \int_\Omega \big[R_i - \beta_{i1}((u_1)_+ + \eta) - \beta_{i2}((u_2)_+ + \eta) \big] ((u_i)_+ + \eta) \ln((u_i)_+ + \eta) dx. \end{aligned}$$

In the following we estimate the terms of the above inequality. With the elementary inequality $x(\ln x - \ln y) \ge x - y$ for all $x, y > 0$ (which is a consequence of the convexity of $x \mapsto \ln x$), we obtain

$$\begin{aligned} &\int_\Omega \frac{u_i - u_i^{k-1}}{\tau} \ln((u_i)_+ + \eta) dx \\ &= \frac{1}{\tau} \int_\Omega \big[((u_i)_+ + \eta) \ln((u_i)_+ + \eta) - ((u_i^{k-1})_+ + \eta) \ln((u_i^{k-1})_+ + \eta) \\ &\quad + ((u_i^{k-1})_+ + \eta)(\ln((u_i^{k-1})_+ + \eta) - \ln((u_i)_+ + \eta)) \big] dx \\ &\quad + \frac{1}{\tau} \int_\Omega ((u_i)_- - (u_i^{k-1})_-) \ln((u_i)_+ + \eta) dx \\ &\ge \frac{1}{\tau} \int_\Omega \big[((u_i)_+ + \eta) \ln((u_i)_+ + \eta) - ((u_i^{k-1})_+ + \eta) \ln((u_i^{k-1})_+ + \eta) \\ &\quad + ((u_i^{k-1})_+ - (u_i)_+) \big] dx + \frac{1}{\tau} \int_\Omega ((u_i)_- - (u_i^{k-1})_-) \ln \eta dx \\ &\ge \frac{1}{\tau} \int_\Omega [((u_i)_+ + \eta)(\ln((u_i)_+ + \eta) - 1) + (u_i)_- \ln \eta] dx \\ &\quad - \frac{1}{\tau} \int_\Omega [((u_i^{k-1})_+ + \eta)(\ln((u_i^{k-1})_+ + \eta) - 1) + (u_i^{k-1})_- \ln \eta] dx. \end{aligned} \tag{2.16}$$

The last term on the left-hand side in (2.15) is estimated by employing Young's inequality:

$$\sum_{i=1}^2 \int_\Omega |d_i q \nabla (u_i)_+| dx \le \sum_{i=1}^2 a_i \int_\Omega |\nabla (u_i)_+|^2 dx + C(a_1, a_2, d_1, d_2, \|q\|_{L^2(\Omega)}). \tag{2.17}$$

Finally, we use the assumptions $\beta_{ii} > 0$ and $\beta_{12} = \beta_{21}$ to estimate the right-hand side of (2.15):

$$\begin{aligned}
&\sum_{i=1}^{2} \big[R_i - \beta_{i1}((u_1)_+ + \eta) - \beta_{i2}((u_2)_+ + \eta)\big]((u_i)_+ + \eta)\ln((u_i)_+ + \eta) \\
&\le \sum_{i=1}^{2}[R_i - \beta_{ii}((u_i)_+ + \eta)]((u_i)_+ + \eta)\ln((u_i)_+ + \eta) \qquad (2.18)\\
&\quad - \beta_{12}((u_1)_+ + \eta)((u_2)_+ + \eta)\ln[((u_1)_+ + \eta)((u_2)_+ + \eta)] \\
&\le C.
\end{aligned}$$

Putting the above estimates (2.16)-(2.18) together, the first inequality (2.11) follows from (2.15).

In order to derive the second inequality (2.12), we take a sequence (v^ε) of smooth functions satisfying $v^\varepsilon \to (u_i)_-$ in $H^1(\Omega)$ and $v^\varepsilon = 0$ in $\{u_i \ge 0\}$, and we choose $\varphi = v^\varepsilon$ as a test function in the weak formulation (2.9):

$$\begin{aligned}
&\sum_{i=1}^{2}\int_\Omega \Big(c_i\nabla(u_i)_- \cdot \nabla v^\varepsilon + 2a_i\eta\nabla(u_i)_- \cdot \nabla v^\varepsilon\Big)dx + \sum_{i=1}^{2}\int_\Omega \frac{u_i - u_i^{k-1}}{\tau}v^\varepsilon dx \\
&\le -\sum_{i=1}^{2}\int_\Omega \chi_h\overline{u_1}\,\overline{u_2}D^h\ln\big(((u_1)_+ + \eta)((u_2)_+ + \eta)\big)\cdot D^h v^\varepsilon dx \\
&\quad + \sum_{i=1}^{2}\int_\Omega \big[R_i - \beta_{i1}((u_1)_+ + \eta) - \beta_{i2}((u_2)_+ + \eta)\big]\eta v^\varepsilon dx.
\end{aligned}$$

As above we can let $\varepsilon \to 0$ to obtain

$$\begin{aligned}
&\sum_{i=1}^{2}\int_\Omega \Big(c_i|\nabla(u_i)_-|^2 + 2a_i\eta|\nabla(u_i)_-|^2\Big)dx + \sum_{i=1}^{2}\int_\Omega \frac{u_i - u_i^{k-1}}{\tau}(u_i)_- dx \\
&\le -\sum_{i=1}^{2}\int_\Omega \chi_h\overline{u_1}\,\overline{u_2}D^h\ln\big(((u_1)_+ + \eta)((u_2)_+ + \eta)\big)\cdot D^h(u_i)_- dx \qquad (2.19)\\
&\quad + \sum_{i=1}^{2}\int_\Omega \big[R_i - \beta_{i1}((u_1)_+ + \eta) - \beta_{i2}((u_2)_+ + \eta)\big]\eta(u_i)_- dx.
\end{aligned}$$

The second term on the left-hand side can be estimated as follows:

$$\begin{aligned}
\int_\Omega \frac{u_i - u_i^{k-1}}{\tau}(u_i)_- dx &= \frac{1}{\tau}\int_\Omega \big(|(u_i)_-|^2 - (u_i^{k-1})_+(u_i)_- - (u_i^{k-1})_-(u_i)_-\big)\,dx \\
&\ge \frac{1}{\tau}\int_\Omega \big(|(u_i)_-|^2 - (u_i^{k-1})_-(u_i)_-\big)\,dx \qquad (2.20)\\
&\ge \frac{1}{2\tau}\int_\Omega \big(|(u_i)_-|^2 - |(u_i^{k-1})_-|^2\big)\,dx.
\end{aligned}$$

For the first term on the right-hand side of (2.19) we employ Young's inequality:

$$\begin{aligned}
&-\int_\Omega \chi_h\overline{u_1}\,\overline{u_2}D^h\ln\big(((u_1)_+ + \eta)((u_2)_+ + \eta)\big)\cdot D^h(u_i)_- dx \qquad (2.21)\\
&\le \frac{c_i}{2}\int_\Omega |\nabla(u_i)_-|^2 dx + \frac{C(c_i)}{\eta^2}\int_\Omega \chi_h\overline{u_1}\,\overline{u_2}\,\big|D^h\ln\big(((u_1)_+ + \eta)((u_2)_+ + \eta)\big)\big|^2 dx + C.
\end{aligned}$$

Finally, for the last term on the right-hand side of (2.19) follows

$$\sum_{i=1}^{2}\int_\Omega \big[R_i - \beta_{i1}((u_1)_+ + \eta) - \beta_{i2}((u_2)_+ + \eta)\big]\eta(u_i)_- dx \\ \le \eta C \sum_{i=1}^{2}\int_\Omega (|(u_i)_+|^2 + |(u_i)_-|^2) dx. \tag{2.22}$$

Hence, (2.12) is a consequence of (2.19)-(2.22). □

3. Proof of Theorem 1.1. Let $(u_1^k, u_2^k) \in (H^1(\Omega))^2$ be a solution to (2.1). We set $u_i^{(\tau)}(x,t) = u_i^k(x)$ if $(x,t) \in \Omega \times ((k-1)\tau, k\tau]$. With the discrete time derivative

$$D_t^\tau v(x,t) := \frac{v(x,t+\tau) - v(x,t)}{\tau}, \quad (x,t) \in \Omega \times [0,\infty),$$

we can rewrite the approximate problem (2.1) as

$$\begin{aligned} &D_t^\tau u_i^{(\tau)} - \operatorname{div}\Big(c_i \nabla u_i^{(\tau)} + 2a_i((u_i^{(\tau)})_+ + \eta)\nabla u_i^{(\tau)} + d_i (u_i^{(\tau)})_+ q\Big) \\ &\quad - D^{-h}\Big[\chi_h \overline{u_1^{(\tau)}}\,\overline{u_2^{(\tau)}} D^h \ln\big(((u_1^{(\tau)})_+ + \eta)((u_2^{(\tau)})_+ + \eta)\big)\Big] \\ &= f_i\big((u_1^{(\tau)})_+ + \eta, (u_2^{(\tau)})_+ + \eta\big) \quad \text{in } \Omega, \\ &\Big(c_i \nabla u_i^{(\tau)} + 2a_i((u_i^{(\tau)})_+ + \eta)\nabla u_i^{(\tau)} + d_i (u_i^{(\tau)})_+ q\Big)\cdot\gamma = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

together with the initial conditions corresponding to (1.4).

The proof of Theorem 1.1 is divided into two parts. In subsection 3.1, we assume that $\eta > 0$ is fixed and perform the limit $\tau, h \to 0$. In subsection 3.2, we prove the limit $\eta \to 0$. At this step we show the non-negativity of the solution.

3.1. The limit $\tau, h \to 0$. The problem (2.1) has a solution under the condition that the parameters τ and h are related by the inequality $32\tau \le h^2\eta^2$. Therefore we let τ and h tend to zero simultanously in such a way that the inequality $32\tau \le h^2\eta^2$ is satisfied (for fixed $\eta > 0$).

LEMMA 3.1. *Let $T > 0$. The following estimates hold for $i = 1,2$,*

$$\|\nabla(u_i^{(\tau)})_+\|_{L^2(Q_T)} + \|(u_i^{(\tau)})_+\|_{L^\infty(0,T;L_\Psi(\Omega))} \le C, \tag{3.2}$$

$$\Big\|\chi_h \sqrt{\overline{u_1^{(\tau)}}\,\overline{u_2^{(\tau)}}} D^h \ln\Big((u_1^{(\tau)})_+ + \eta)((u_2^{(\tau)})_+ + \eta)\Big)\Big\|_{L^2(Q_T)} \le C, \tag{3.3}$$

where $C > 0$ is independent of c_1, c_2, h, τ, and η. Furthermore,

$$\|\nabla(u_i^{(\tau)})_-\|_{L^2(Q_T)} + \|(u_i^{(\tau)})_-\|_{L^\infty(0,T;L^2(\Omega))} \le C(c_1,c_2,\eta), \tag{3.4}$$

$$\|u_i^{(\tau)}\|_{L^2(0,T;H^1(\Omega))} + \|u_i^{(\tau)}\|_{L^p(Q_T)} \le C(c_1,c_2,\eta), \tag{3.5}$$

$$\|D_t^\tau u_i^{(\tau)}\|_{L^r(0,T;(W^{1,r'}(\Omega))')} \le C(c_1,c_2,\eta), \tag{3.6}$$

where $p = (2N+2)/N$, $r = (2N+2)/(2N+1)$, $r' = r/(r-1) = 2N+2$, and $C(c_1,c_2,\eta) > 0$ does not depend on τ or h.

Proof. The estimates (3.2)-(3.5) are consequences of the key inequalities (2.11) and (2.12). First we prove (3.2) and (3.3). Let $K \in \mathbb{N}$ and set $\tau = T/K$. The estimate (2.11) can be rewritten at $t_k = k\tau$ as

$$\begin{aligned}
&\int_0^{t_k}\int_\Omega \Big[\sum_{i=1}^2 \Big(4c_i\Big|\nabla\sqrt{(u_i^{(\tau)})_+ + \eta}\Big|^2 + a_i|\nabla(u_i^{(\tau)})_+|^2\Big) \\
&\quad + \chi_h\overline{u_1^{(\tau)}}\;\overline{u_2^{(\tau)}}\Big|D^h\ln\big(((u_1^{(\tau)})_+ + \eta)((u_2^{(\tau)})_+ + \eta)\big)\Big|^2\Big]dxdt \\
&\quad + \sum_{i=1}^2\int_\Omega \Big(((u_i^{(\tau)})_+ + \eta)(\ln((u_i^{(\tau)})_+ + \eta) - 1) + \ln\eta(u_i^{(\tau)})_-\Big)\Big|_{t=t_k}dx \\
&\le C(T, \|u_i^0\|_{L_\Psi(\Omega)}).
\end{aligned}$$

From the elementary inequalities $x \le x(\ln x - 1) + C$ and $(1+x)\ln(1+x) - x \le x(\ln x - 1) + x + C$ for all $x \ge 0$ for some $C > 0$, we obtain at $t = t_k$,

$$\begin{aligned}
\int_\Omega (u_i^{(\tau)})_+dx &\le \int_\Omega((u_i^{(\tau)})_+ + \eta)(\ln((u_i^{(\tau)})_+ + \eta) - 1)dx + C|\Omega| \\
&\le C(|\Omega|, T, \|u_i^0\|_{L_\Psi(\Omega)})
\end{aligned} \tag{3.7}$$

and (see (4.1))

$$\begin{aligned}
\|(u_i^{(\tau)})_+\|_{L_\Psi(\Omega)} &\le 1 + \int_\Omega \Psi\big((u_i^{(\tau)})_+\big)dx \\
&\le \int_\Omega((u_i^{(\tau)})_+ + \eta)(\ln((u_i^{(\tau)})_+ + \eta) - 1)dx + C \\
&\le C(|\Omega|, T, \|u_i^0\|_{L_\Psi(\Omega)}).
\end{aligned}$$

Since the functions $u_i^{(\tau)}$ are piecewise constant with respect to t, we have

$$\begin{aligned}
&\int_0^T\int_\Omega \Big[\sum_{i=1}^2 \Big(4c_i\Big|\nabla\sqrt{(u_i^{(\tau)})_+ + \eta}\Big|^2 + a_i|\nabla(u_i^{(\tau)})_+|^2\Big) \\
&\quad + \chi_h\overline{u_1^{(\tau)}}\;\overline{u_2^{(\tau)}}\Big|D^h(\ln\big(((u_1^{(\tau)})_+ + \eta)((u_2^{(\tau)})_+ + \eta)\big)\Big|^2\Big]dxdt \\
&\quad + \sum_{i=1}^2 \sup_{0<t<T}\Big(\|(u_i^{(\tau)})_+(\cdot,t)\|_{L_\Psi(\Omega)} + \|\ln\eta(u_i^{(\tau)})_-(\cdot,t)\|_{L^1(\Omega)}\Big) \\
&\le C(|\Omega|, T, \|u_i^0\|_{L_\Psi(\Omega)}).
\end{aligned} \tag{3.8}$$

This gives a uniform bound for $\|\nabla(u_i^{(\tau)})_+\|_{L^2(Q_T)}$ and shows (3.2)-(3.3). An L^2 bound for $(u_i^{(\tau)})_+$ can be derived from this estimate, the Poincaré inequality, and (3.7):

$$\begin{aligned}
\int_0^T \|(u_i^{(\tau)})_+\|_{L^2(\Omega)}^2dt &\le 2\int_0^T\Big(\|(u_i^{(\tau)})_+ - [(u_i^{(\tau)})_+]_\Omega\|_{L^2(\Omega)}^2 + \|[(u_i^{(\tau)})_+]_\Omega\|_{L^2(\Omega)}^2\Big)dt \\
&\le C(|\Omega|, T)\int_0^T \|\nabla(u_i^{(\tau)})_+\|_{L^2(\Omega)}^2dt + C(|\Omega|, T),
\end{aligned} \tag{3.9}$$

where $[(u_i^{(\tau)})_+]_\Omega$ denotes the average of $(u_i^{(\tau)})_+$, i.e.

$$[(u_i^{(\tau)})_+]_\Omega = \frac{1}{|\Omega|}\int_\Omega (u_i^{(\tau)})_+dx.$$

For the proof of (3.4) we employ the estimate (2.12), rewritten at $t_k = k\tau$ as

$$\int_0^{t_k}\int_\Omega \sum_{i=1}^2 \Big(\frac{c_i}{2}|\nabla(u_i^{(\tau)})_-|^2 + 2a_i\eta|\nabla(u_i^{(\tau)})_-|^2\Big)dxdt + \frac{1}{2}\sum_{i=1}^2\int_\Omega |(u_i^{(\tau)})_-(\cdot,t_k)|^2dx$$
$$\leq C + \eta C\sum_{i=1}^2\int_0^{t_k}\int_\Omega (|(u_i^{(\tau)})_+|^2 + |(u_i^{(\tau)})_-|^2)dxdt$$
$$+ \frac{C(c_1,c_2)}{\eta^2}\int_0^{t_k}\int_\Omega \chi_h\overline{u_1^{(\tau)}}\;\overline{u_2^{(\tau)}}\left|D^h \ln\big(((u_1^{(\tau)})_+ + \eta)((u_2^{(\tau)})_+ + \eta)\big)\right|^2 dxdt.$$

Taking into account (3.3) and (3.9) and applying Gronwall's inequality, this proves (3.4).

Next we show the estimate (3.5). As the functions $u_i^{(\tau)}$ are piecewise constant with respect to t, we obtain, with the help of (3.8) and (3.9),

$$\int_0^t\int_\Omega \sum_{i=1}^2 \Big(\frac{c_i}{2}|\nabla(u_i^{(\tau)})_-|^2 + 2a_i\eta|\nabla(u_i^{(\tau)})_-|^2\Big)dxdt + \frac{1}{2}\sum_{i=1}^2\int_\Omega |(u_i^{(\tau)})_-(\cdot,t)|^2dx$$
$$\leq C(|\Omega|,T,\|u_i^0\|_{L_\Psi(\Omega)},c_1,c_2,\eta) + C\int_0^t\int_\Omega\sum_{i=1}^2 |(u_i^{(\tau)})_-|^2dxdt.$$

Thus, by Gronwall's inequality,

$$\sum_{i=1}^2\int_0^T\int_\Omega \Big(\frac{c_i}{2}|\nabla(u_i^{(\tau)})_-|^2 + 2a_i\eta|\nabla(u_i^{(\tau)})_-|^2\Big)dxdt$$
$$+ \frac{1}{2}\sum_{i=1}^2\sup_{0<t<T}\int_\Omega |(u_i^{(\tau)})_-(\cdot,t)|^2dx \leq C(|\Omega|,T,\|u_i^0\|_{L_\Psi(\Omega)},c_1,c_2,\eta). \tag{3.10}$$

This provides a uniform bound for $(u_i^{(\tau)})_-$ in $L^2(0,T;H^1(\Omega))$, and from (3.2), (3.8), (3.9), and (3.10) we infer

$$\|u_i^{(\tau)}\|_{L^2(0,T;H^1(\Omega))} + \|u_i^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))} \leq C(c_1,c_2,\eta).$$

Applying the Gagliardo-Nirenberg inequality with $p = (2N+2)/N$ and $\theta = 2N(p-1)/(p(N+2))$ (and thus $\theta p = 2$) yields

$$\|u_i^{(\tau)}\|_{L^p(Q_T)} \leq \Big(\int_0^T \|u_i^{(\tau)}\|_{L^1(\Omega)}^{(1-\theta)p}\|u_i^{(\tau)}\|_{L^2(0,T;H^1(\Omega))}^{\theta p}dt\Big)^{1/p}$$
$$\leq \|u_i^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))}^{1-\theta}\Big(\int_0^T \|u_i^{(\tau)}\|_{L^2(0,T;H^1(\Omega))}^{\theta p}dt\Big)^{1/p}$$
$$= \|u_i^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))}^{1-\theta}\|u_i^{(\tau)}\|_{L^2(0,T;H^1((\Omega))}^{\theta}$$
$$\leq C(c_1,c_2,\eta).$$

Finally, we derive a bound for the discrete time derivative $D_t^\tau u_i^{(\tau)}$. Using (3.1), we obtain, for $r = (2N+2)/(2N+1)$, since $p > r$,

$$
\begin{aligned}
&\|D_t^\tau u_i^{(\tau)}\|_{L^r(0,T;(W^{1,r'}(\Omega))')} \\
&\le \|c_i \nabla u_i^{(\tau)} + 2a_i((u_i^{(\tau)})_+ + \eta)\nabla u_i^{(\tau)}\|_{L^r(Q_T)} \\
&\quad + \left\| \chi_h \overline{u_1^{(\tau)}}\ \overline{u_2^{(\tau)}} D^h \ln\big(((u_1^{(\tau)})_+ + \eta)((u_2^{(\tau)})_+ + \eta)\big) + d_i (u_i^{(\tau)})_+ q \right\|_{L^r(Q_T)} \\
&\quad + \big\| f_i\big((u_1^{(\tau)})_+ + \eta, (u_2^{(\tau)})_+ + \eta\big)\big\|_{L^r(Q_T)} \\
&\le C(c_i, |\Omega|, T)\|\nabla u_i^{(\tau)}\|_{L^2(Q_T)} + 2a_i \|u_i^{(\tau)} + \eta\|_{L^p(Q_T)} \|\nabla u_i^{(\tau)}\|_{L^2(Q_T)} \\
&\quad + \frac{1}{2}(\|u_1^{(\tau)}\|_{L^p(Q_T)} + \|u_2^{(\tau)}\|_{L^p(Q_T)}) \\
&\quad \times \left\| \chi_h \sqrt{\overline{u_1^{(\tau)}}\ \overline{u_2^{(\tau)}}} D^h \ln\big(((u_1^{(\tau)})_+ + \eta)((u_2^{(\tau)})_+ + \eta)\big) \right\|_{L^2(Q_T)} \\
&\quad + |d_i|\, \|u_i^{(\tau)}\|_{L^p(Q_T)} \|q\|_{L^2(Q_T)} + C(T, |\Omega|)(\|u_i^{(\tau)}\|_{L^p(Q_T)} + \|u_i^{(\tau)}\|_{L^p(Q_T)}).
\end{aligned}
$$

Then (3.6) follows from (3.3) and (3.5). □

Now we are able to perform the limit τ, $h \to 0$.

LEMMA 3.2. *As $\tau, h \to 0$ such that $32\tau \le h^2\eta^2$, there exists a pair (u_1^η, u_2^η) satisfying (up to a subsequence which is not relabeled), for $i = 1, 2$,*

$$
\nabla u_i^{(\tau)} \rightharpoonup \nabla u_i^\eta \quad \text{weakly in } (L^2(Q_T))^N, \tag{3.11}
$$

$$
((u_i^{(\tau)})_+ + \eta)\nabla u_i^{(\tau)} \rightharpoonup ((u_i^\eta)_+ + \eta)\nabla u_i^\eta \quad \text{weakly in } (L^r(Q_T))^N, \tag{3.12}
$$

$$
\begin{aligned}
&\chi_h \overline{u_1^{(\tau)}}\ \overline{u_2^{(\tau)}} D^h \ln\big(((u_1^{(\tau)})_+ + \eta)((u_2^{(\tau)})_+ + \eta)\big) \\
&\quad \rightharpoonup \overline{u_1^\eta}\ \overline{u_2^\eta} \nabla \ln\big(((u_1^\eta)_+ + \eta)((u_2^\eta)_+ + \eta)\big) \quad \text{weakly in } (L^2(Q_T))^N,
\end{aligned} \tag{3.13}
$$

$$
d_i (u_i^{(\tau)})_+ q \rightharpoonup d_i (u_i^\eta)_+ q \quad \text{weakly in } (L^r(Q_T))^N, \tag{3.14}
$$

$$
\begin{aligned}
&f_i\big((u_1^{(\tau)})_+ + \eta, (u_2^{(\tau)})_+ + \eta\big) \\
&\quad \rightharpoonup f_i\big((u_1^\eta)_+ + \eta, (u_2^\eta)_+ + \eta\big) \quad \text{weakly in } L^{p/2}(\Omega),
\end{aligned} \tag{3.15}
$$

$$
D_t^\tau u_i^{(\tau)} \rightharpoonup \partial_t u_i^\eta \quad \text{weakly in } L^r(0,T;(W^{1,r'}(\Omega))'), \tag{3.16}
$$

where $p = (2N+2)/N$, $r = (2N+2)/(2N+1)$, and $r' = 2N+2$.

Proof. The first and last convergences are direct consequences of (3.2) and (3.5). In order to treat the nonlinear terms, we need a strong convergence result. Taking into account (3.5) and (3.6), we can apply the version of Aubin's lemma in [25, Thm. 6] to obtain, for a subsequence which is not relabeled, as $\tau, h \to 0$,

$$
u_i^{(\tau)} \to u_i^\eta \quad \text{strongly in } L^q(0,T;L^2(\Omega)), \quad 1 < q < 2. \tag{3.17}
$$

In particular, (a subsequence of) $(u_i^{(\tau)})$ converges pointwise almost everywhere in Q_T to u_i^η. This, together with the bound $\|u_i^{(\tau)}\|_{L^p(Q_T)} \le C$ (which comes from (3.5)), implies

$$
u_i^{(\tau)} \to u_i^\eta \quad \text{strongly in } L^\alpha(Q_T), \quad 2 < \alpha < p, \tag{3.18}
$$

and

$$
(u_i^{(\tau)})_+ \to (u_i^\eta)_+ \quad \text{strongly in } L^\alpha(Q_T).
$$

From (3.11) we obtain for $s = 2\alpha/(2+\alpha) < r$

$$(u_i^{(\tau)})_+\nabla u_i^{(\tau)} \rightharpoonup (u_i^\eta)_+\nabla u_i^\eta \quad \text{weakly in } (L^s(Q_T))^N.$$

Since

$$\|(u_i^{(\tau)})_+\nabla u_i^{(\tau)}\|_{L^r(Q_T)} \le \|u_i^{(\tau)}\|_{L^p(Q_T)}\|\nabla u_i^{(\tau)}\|_{L^2(Q_T)} \le C,$$

the above weak convergence also holds for $s = r$.

In a similar way, since $q \in (L^2(Q_T))^N$, the convergences (3.14) and (3.15) can be proved.

Finally, we show (3.13). Using

$$\|\ln((u_i^{(\tau)})_+ + \eta)\|_{L^2(Q_T)} \le |\ln\eta| + \|(u_i^{(\tau)})_+ + \eta\|_{L^2(Q_T)} \le C(\eta),$$
$$\|D^h\ln((u_i^{(\tau)})_+ + \eta)\|_{L^2(Q_T)} \le \frac{C}{\eta}\|\nabla(u_i^{(\tau)})_+\|_{L^2(Q_T)} \le C(\eta),$$

and $(u_i^{(\tau)})_+ \to (u_i^\eta)_+$ almost everywhere in Q_T, we conclude

$$D^h\ln((u_i^{(\tau)})_+ + \eta) \rightharpoonup \nabla\ln((u_i^\eta)_+ + \eta) \quad \text{weakly in } (L^2(Q_T))^N.$$

Then (3.13) follows from

$$\left\|\chi_h\overline{u_1^{(\tau)}}\;\overline{u_2^{(\tau)}}\right\|_{L^\infty(Q_T)} \le \frac{1}{\eta^2}.$$

This proves Lemma 3.2. ◻

Letting $\tau, h \to 0$ in the weak version of (3.1) such that $32\tau \le h^2\eta^2$, we obtain for all $\varphi \in L^{r'}(0,T;W^{1,r'}(\Omega))$

$$\begin{aligned}\int_0^T \langle\partial_t u_i^\eta,\varphi\rangle_{(W^{1,r'}(\Omega))',W^{1,r'}(\Omega)}\,dt + \int_{Q_T}(c_i\nabla u_i^\eta + 2a_i((u_i^\eta)_+ + \eta)\nabla u_i^\eta)\cdot\nabla\varphi dxdt \\ + \int_{Q_T}\Big[\overline{u_1^\eta}\;\overline{u_2^\eta}\nabla\ln\big(((u_1^\eta)_+ + \eta)((u_2^\eta)_+ + \eta)\big) + d_i(u_i)_+q\Big]\cdot\nabla\varphi dxdt \\ = \int_{Q_T} f_i\big((u_1^\eta)_+ + \eta, (u_2^\eta)_+ + \eta\big)\varphi dxdt.\end{aligned} \tag{3.19}$$

By Lemma 3.2, the functions u_1^η and u_2^η are satisfying the properties

$$\begin{aligned}u_i^\eta \in L^2(0,T;H^1(\Omega)) \cap L^p(Q_T),\\ (u_i^\eta)_+ \in L^\infty(0,T;L_\Psi(\Omega)), \quad (u_i^\eta)_- \in L^\infty(0,T;L^2(\Omega)), \quad i = 1,2.\end{aligned} \tag{3.20}$$

3.2. The limit $\eta \to 0$. The last step in the proof of Theorem 1.1 is to perform the limit $\eta \to 0$. First we need some a priori estimates.

LEMMA 3.3. *Let $T > 0$. The following estimates hold for $i = 1,2$:*

$$\|\nabla(u_i^\eta)_+\|_{L^2(Q_T)} + \|(u_i^\eta)_+\|_{L^\infty(0,T;L_\Psi(\Omega))} \le C, \tag{3.21}$$
$$\|\ln\eta(u_i^\eta)_-\|_{L^\infty(0,T;L^1(\Omega))} \le C, \tag{3.22}$$
$$\left\|\sqrt{\overline{u_1^\eta}\;\overline{u_2^\eta}}\nabla\ln\big(((u_1^\eta)_+ + \eta)((u_2^\eta)_+ + \eta)\big)\right\|_{L^2(Q_T)} \le C, \tag{3.23}$$
$$\|\nabla(u_i^\eta)_-\|_{L^2(Q_T)} + \|(u_i^\eta)_-\|_{L^\infty(0,T;L^2(\Omega))} \le C, \tag{3.24}$$
$$\|u_i^\eta\|_{L^2(0,T;H^1(\Omega))} + \|u_i^\eta\|_{L^p(Q_T)} \le C, \tag{3.25}$$
$$\|\partial_t u_i^\eta\|_{L^r(0,T;(W^{1,r'}(\Omega))')} \le C, \tag{3.26}$$

where $p = (2N+2)/N$, $r = (2N+2)/(2N+1)$, $r' = 2N+2$, *and* $C > 0$ *is a constant independent of* c_1, c_2, *and* η.

Proof. Let $i \in \{1,2\}$. Choose a sequence (v^ε) of smooth functions such that, as $\varepsilon \to 0$,

$$v^\varepsilon \to (u_i^\eta)_+ \quad \text{in } L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L_\Psi(\Omega)) \cap L^p(Q_T) \tag{3.27}$$

and $v^\varepsilon = 0$ on $\{u_i \le 0\}$. Such a choice is possible in view of the regularity (3.20). We claim that

$$\begin{aligned}\int_0^t \langle \partial_t u_i^\eta, \ln(v^\varepsilon + \eta)\rangle dt \to & \int_\Omega \big((u_i^\eta)_+ + \eta)(\ln(u_i^\eta)_+ + \eta) - 1) + \ln\eta (u_i^\eta)_-\big)dx \\ & - \int_\Omega (u_i^0 + \eta)(\ln(u_i^0 + \eta) - 1)dx.\end{aligned} \tag{3.28}$$

and

$$\int_{Q_T} a_i((u_i^\eta)_+ + \eta)\nabla u_i^\eta \cdot \nabla \ln(v^\varepsilon + \eta)dxdt \to \int_{Q_T} a_i|\nabla (u_i^\eta)_+|^2 dxdt. \tag{3.29}$$

In fact, in order to show the second claim (3.29), we only need to show

$$((u_i^\eta)_+ + \eta)\nabla \ln(v^\varepsilon + \eta) \rightharpoonup \nabla (u_i^\eta)_+ \quad \text{weakly in } (L^2(Q_T))^N.$$

This convergence follows from $((u_i^\eta)_+ + \eta)/(v^\varepsilon + \eta) \to 1$ almost everywhere in Q_T and $\nabla v^\varepsilon \to \nabla (u_i^\eta)_+$ in $(L_2(Q_T))^N$.

The proof of the first claim (3.28) is more delicate. By integration by parts, we have

$$\begin{aligned}&\int_0^t \langle \partial_t u_i^\eta, \ln(v^\varepsilon + \eta)\rangle dt \\ &= \int_0^t \langle \partial_t (u_i^\eta)_+, \ln(v^\varepsilon + \eta)\rangle dt + \int_0^t \langle \partial_t (u_i^\eta)_-, \ln\eta\rangle dt \\ &= \int_\Omega \left[((u_i^\eta)_+ + \eta)\ln(v^\varepsilon + \eta)\right]_0^t dx - \int_{Q_t} \frac{(u_i^\eta)_+ + \eta}{v^\varepsilon + \eta}\partial_t v^\varepsilon dxdt + \ln\eta \int_\Omega \left[(u_i^\eta)_-\right]_0^t dx.\end{aligned} \tag{3.30}$$

We consider the first term on the right-hand side. It holds for all $t \in (0,T)\backslash\mathcal{N}$, where $\mathcal{N}$ is a set of measure zero,

$$\|\ln(v^\varepsilon(\cdot,t) + \eta) - \ln((u_i^\eta)_+(\cdot,t) + \eta)\|_{L^\infty(\Omega)} = \left\| \ln \frac{v^\varepsilon(\cdot,t) + \eta}{(u_i^\eta)_+(\cdot,t) + \eta}\right\|_{L^\infty(\Omega)} \le C$$

for some $C > 0$ and, as $\varepsilon \to 0$,

$$\ln(v^\varepsilon(\cdot,t) + \eta) - \ln((u_i^\eta)_+(\cdot,t) + \eta) \to 0 \quad \text{strongly in } L^1(\Omega),$$

uniformly in $t \in (0,T)\backslash\mathcal{N}$. In particular, this sequence converges in measure. Now let $\Phi(s) = e^s - s - 1$ be the complementary Young function to Ψ and define $\Phi_2(s) = \exp(s^2) - 1$, $s \ge 0$. Then Φ_2 is a Young function and

$$\lim_{t\to\infty} \frac{\Phi(kt)}{\Phi_2(t)} = 0 \quad \text{for all } k > 0.$$

Thus, by Theorem 4.1 of the Appendix,

$$\ln(v^\varepsilon(\cdot,t)+\eta) - \ln((u_i^\eta)_+(\cdot,t)+\eta) \to 0 \quad \text{strongly in } L_\Phi(\Omega),$$

uniformly in $t \in (0,T)\backslash\mathcal{N}$. Therefore, as $(u_i^\eta)_+ + \eta \in L^\infty(0,T;L_\Psi(\Omega))$, Young's inequality (4.2) implies, for $t \in (0,T)\backslash\mathcal{N}$,

$$\begin{aligned}
&\int_\Omega ((u_i^\eta)_+(\cdot,t)+\eta)\big(\ln(v^\varepsilon(\cdot,t)+\eta) - \ln((u_i^\eta)_+(\cdot,t)+\eta)\big)dx \\
&\le 2\|(u_i^\eta)_+(\cdot,t)+\eta\|_{L_\Psi(\Omega)}\|\ln(v^\varepsilon(\cdot,t)+\eta) - \ln((u_i^\eta)_+(\cdot,t)+\eta)\|_{L_\Phi(\Omega)} \\
&\to 0.
\end{aligned}$$

We conclude that, for almost every $t \in (0,T)$, as $\varepsilon \to 0$,

$$\int_\Omega [((u_i^\eta)_+ + \eta)\ln(v^\varepsilon+\eta)]_0^t dx \to \int_\Omega [((u_i^\eta)_+ + \eta)\ln((u_i^\eta)_+ + \eta)]_0^t\, dx.$$

It remains to treat the second term in (3.30). Let $(0,t] = \cup_{k=0}^{K-1}(t_k,t_{k+1}]$, where $t_k \in (0,t]\backslash\mathcal{N}$ and $t_K := t$, be a partition of the interval $(0,t]$. Then we can write the term as follows:

$$\begin{aligned}
&\lim_{\varepsilon\to0}\int_{Q_t}\frac{(u_i^\eta)_+ + \eta}{v^\varepsilon+\eta}\partial_t v^\varepsilon\, dxdt \\
&= \lim_{\varepsilon\to0}\lim_{K\to\infty}\sum_{k=0}^{K-1}\int_\Omega \frac{(u_i^\eta)_+(x,t_k)+\eta}{v^\varepsilon(x,t_k)+\eta}(v^\varepsilon(x,t_{k+1}) - v^\varepsilon(x,t_k))dx
\end{aligned}$$

The sequence $((u_i^\eta)_+(\cdot,t_k)+\eta)/(v^\varepsilon(\cdot,t_k)+\eta)$ converges to one weakly* in $L^\infty(\Omega)$ as $\varepsilon \to 0$ and $v^\varepsilon(\cdot,t_{k+1}) - v^\varepsilon(\cdot,t_k)$ converges to $(u_i^\eta)_+(\cdot,t_{k+1}) - (u_i^\eta)_+(\cdot,t_k)$ strongly in $L^1(\Omega)$, uniformly in $t \in (0,T)\backslash\mathcal{N}$. Hence, we can exchange the limits $\varepsilon \to 0$ and $K \to \infty$ to obtain

$$\begin{aligned}
&\lim_{\varepsilon\to0}\int_{Q_t}\frac{(u_i^\eta)_+ + \eta}{v^\varepsilon+\eta}\partial_t v^\varepsilon\, dxdt \\
&= \lim_{K\to\infty}\lim_{\varepsilon\to0}\sum_{k=0}^{K-1}\int_\Omega \frac{(u_i^\eta)_+(x,t_k)+\eta}{v^\varepsilon(x,t_k)+\eta}(v^\varepsilon(x,t_{k+1}) - v^\varepsilon(x,t_k))dx \\
&= \lim_{K\to\infty}\sum_{k=0}^{K-1}\int_\Omega ((u_i^\eta)_+(x,t_{k+1}) - (u_i^\eta)_+(x,t_k))dx \\
&= \int_\Omega ((u_i^\eta)_+(x,t) - u_i^0(x))dx,
\end{aligned}$$

This proves (3.28).

Now we use $\varphi = \ln(v^\varepsilon+\eta)$ as a test function in (3.19) and perform the limit $\varepsilon \to 0$ by employing the above claims (3.28) and (3.29). This implies, after addition of the two equations for $i=1,2$ and estimating as above,

$$\begin{aligned}
&\sum_{i=1}^{2}\int_\Omega \big((u_i^\eta)_+ + \eta)(\ln(u_i^\eta)_+ + \eta) - 1) + \ln\eta (u_i^\eta)_-\big)dx \\
&\quad + \int_{Q_T}\sum_{i=1}^{2}\Big(c_i\frac{|\nabla (u_i^\eta)_+|^2}{(u_i^\eta)_+ + \eta} + a_i|\nabla (u_i^\eta)_+|^2\Big)dxdt \\
&\quad + \int_{Q_T}\overline{u_1^\eta}\,\overline{u_2^\eta}\,\big|\nabla \ln\big(((u_1^\eta)_+ + \eta)((u_2^\eta)_+ + \eta)\big)\big|^2\,dxdt \\
&\le C,
\end{aligned} \tag{3.31}$$

where $C>0$ depends only on a_1, a_2, $\|q\|_{L^2(Q_T)}$, and $\|u_i^0\|_{L_\Psi(\Omega)}$. This shows (3.21)-(3.23).

In the next step, we choose $\varphi = w^\varepsilon$ as a test function in (3.19), where (w^ε) is a smooth sequence such that, as $\varepsilon\to 0$,

$$w^\varepsilon \to (u_i^\eta)_- \quad \text{in } L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega)) \tag{3.32}$$

and $w^\varepsilon = 0$ in $\{u_i^\eta \ge 0\}$. Then we have

$$\begin{aligned}
&\int_0^t \langle \partial_t (u_i^\eta)_-, w^\varepsilon\rangle dt + \int_{Q_t}(c_i\nabla (u_i^\eta)_-\cdot\nabla w^\varepsilon + 2a_i\eta\nabla (u_i^\eta)_-\cdot\nabla w^\varepsilon)dxdt \\
&\qquad = \int_{Q_t}\big[R_i - \beta_{i1}((u_1^\eta)_+ + \eta) - \beta_{i2}((u_2^\eta)_+ + \eta)\big]\eta w^\varepsilon dxdt.
\end{aligned} \tag{3.33}$$

We infer from (3.32):

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^t \langle \partial_t (u_i^\eta)_-, w^\varepsilon\rangle dt \\
&= \lim_{\varepsilon\to 0}\frac12\int_\Omega [|w^\varepsilon|^2]_0^t dx + \lim_{\varepsilon\to 0}\int_0^t\langle \partial_t((u_i^\eta)_- - w^\varepsilon), w^\varepsilon\rangle dt \\
&= \frac12\int_\Omega [|(u_i^\eta)_-|^2]_0^t dx + \lim_{\varepsilon\to 0}\int_\Omega [((u_i^\eta)_- - w^\varepsilon)w^\varepsilon]_0^t dx + \lim_{\varepsilon\to 0}\int_{Q_t}\partial_t w^\varepsilon((u_i^\eta)_- - w^\varepsilon)dxdt \\
&= \frac12\int_\Omega [|(u_i^\eta)_-|^2]_0^t dx \\
&\quad + \lim_{\varepsilon\to 0}\lim_{K\to\infty}\sum_{k=0}^{K-1}\int_\Omega (w^\varepsilon(x,t_{k+1}) - w^\varepsilon(x,t_k))((u_i^\eta)_-(x,t_k) - w^\varepsilon(x,t_k))dx \\
&= \frac12\int_\Omega [|(u_i^\eta)_-|^2]_0^t dx,
\end{aligned}$$

where similarly as above $(0,t] = \cup_{k=0}^{K-1}(t_k,t_{k+1}]$. The convergence of the other terms in (3.33) as $\varepsilon\to 0$ follows directly from (3.32). This yields

$$\begin{aligned}
&\frac12\sum_{i=1}^2\int_\Omega |(u_i^\eta)_-(\cdot,t)|^2 dx + \sum_{i=1}^2\int_{Q_t}(c_i|\nabla (u_i^\eta)_-|^2 + 2a_i\eta|\nabla (u_i^\eta)_-|^2)dxdt \\
&= \sum_{i=1}^2\int_{Q_t}\big[R_i - \beta_{i1}((u_1^\eta)_+ + \eta) - \beta_{i2}((u_2^\eta)_+ + \eta)\big]\eta (u_i^\eta)_- dxdt \\
&\le C\sum_{i=1}^2\int_{Q_t}(|(u_i^\eta)_+|^2 + |(u_i^\eta)_-|^2) + C,
\end{aligned} \tag{3.34}$$

where $C > 0$ is independent of η. The estimate (3.21) and Gronwall's inequality then imply (3.24).

Finally, the inequalities (3.25) and (3.26) can be derived similarly as in the proof of Lemma 3.1. □

LEMMA 3.4. *As $\eta \to 0$, there exist functions u_1, $u_2 \geq 0$ such that the following convergences hold (up to subsequences which are not relabeled), for $i = 1, 2$,*

$$\begin{aligned} c_i \nabla u_i^\eta &\rightharpoonup c_i \nabla u_i \quad \textit{weakly in } (L^2(Q_T))^N, \\ 2a_i((u_i^\eta)_+ + \eta)\nabla u_i^\eta &\rightharpoonup 2a_i u_i \nabla u_i \quad \textit{weakly in } (L^r(Q_T))^N, \\ \overline{u_1^\eta}\; \overline{u_2^\eta} \nabla \ln((u_1^\eta)_+ + \eta)((u_2^\eta)_+ + \eta) &\rightharpoonup \nabla(u_1 u_2) \quad \textit{weakly in } (L^r(Q_T))^N, \\ d_i (u_i^\eta)_+ q &\rightharpoonup d_i u_i q \quad \textit{weakly in } (L^r(Q_T))^N, \\ f_i\big((u_1^\eta)_+ + \eta, (u_2^\eta)_+ + \eta\big) &\rightharpoonup f_i(u_1, u_2) \quad \textit{weakly in } L^{p/2}(Q_T), \\ \partial_t u_i^\eta &\rightharpoonup \partial_t u_i \quad \textit{weakly in } L^r(0, T; (W^{1,r'}(\Omega))'). \end{aligned}$$

Proof. Similar to the discussion in the proof of Lemma 3.2, we conclude that there exist functions u_1 and u_2 such that

$$u_i^\eta \to u_i \quad \text{in } L^\alpha(Q_T) \quad \text{for all } 2 \leq \alpha < p.$$

The estimate (3.22) implies

$$\|(u_i^\eta)_-\|_{L^\infty(0,T;L^1(\Omega))} \leq \frac{C}{|\ln \eta|} \to 0,$$

from which we obtain $u_i \geq 0$ in Q_T, $i = 1, 2$.

Except the third convergence, the discussion of the remaining convergence results are similar to those in the proof of Lemma 3.2. Observe that

$$\overline{u_1^\eta}\; \overline{u_2^\eta} \nabla \ln((u_1^\eta)_+ + \eta) = \frac{1}{1 + \eta(u_2^\eta)_+} \frac{1}{1 + \eta(u_1^\eta)_+} \frac{(u_1^\eta)_+}{(u_1^\eta)_+ + \eta} (u_2^\eta)_+ \nabla (u_1^\eta)_+.$$

By similar arguments as above, it holds

$$(u_2^\eta)_+ \nabla (u_1^\eta)_+ \rightharpoonup u_2 \nabla u_1 \quad \text{weakly in } (L^r(Q_T))^N.$$

Taking into account

$$\frac{1}{1 + \eta(u_2^\eta)_+} \frac{1}{1 + \eta(u_1^\eta)_+} \frac{(u_1^\eta)_+}{(u_1^\eta)_+ + \eta} \leq 1,$$

we infer the desired convergence. □

Now, Theorem 1.1 is a consequence of the convergence results of Lemma 3.4 applied to (3.19).

REMARK 3.5. Since the estimates in Lemma 3.3 are independent of c_1 and c_2, we obtain the existence of a weak solution even in the case $c_1 = 0$ or $c_2 = 0$. Indeed, we first obtain a weak solution for $c_1 > 0$, $c_2 > 0$, respectively. The a priori estimates of Lemma 3.3 allow to perform the limit $c_1, c_2 \to 0$.

REMARK 3.6. All the above estimates are true in *any* space dimension. The restriction $N \leq 3$ is only used in the proof of Lemma 2.1. As mentioned in Remark 2.3, Lemma 2.1 holds in any space dimension if $\beta_{ij} = 0$ for all $i, j = 1, 2$. Therefore, Theorem 1.1 is true in *any* space dimension provided that $\beta_{ij} = 0$ for all $i, j = 1, 2$.

4. Appendix. We recall the definition of an Orlicz space and some of its properties. For details, we refer, e.g., to [1, 14].

A real valued function $\Psi : [0,\infty) \to \mathbb{R}$ is called *Young function* if $\Psi(t) = \int_0^t \psi(s)ds$ and $\psi : [0,\infty) \to [0,\infty)$ has the properties

- $\psi(0) = 0$, $\psi > 0$ on $(0,\infty)$, $\psi(t) \to \infty$ as $t \to \infty$;
- ψ is non-decreasing and right continuous at any point $s \geq 0$.

The function $\Phi(t) = \int_0^t \phi(s)ds$ with $\phi(s) = \sup_{\psi(t)\leq s} t$ is called the *complementary Young function* of Ψ. For instance, $\Psi(s) = (1+s)\ln(1+s) - s$ and $\Phi(s) = e^s - s - 1$ are a pair of complementary Young functions.

Let Ψ be a Young function. The *Orlicz class* $K_\Psi(\Omega)$ is the set of (equivalence classes of) real-valued measurable functions u on Ω satisfying $\int_\Omega \Psi(|u(x)|)dx < \infty$. Then the *Orlicz space* $L_\Psi(\Omega)$ is the linear hull of $K_\Psi(\Omega)$ supplemented with the *Luxemburg norm*

$$\|u\|_{L_\Psi(\Omega)} := \inf\Big\{k > 0 : \int_\Omega \Psi\Big(\frac{|u(x)|}{k}\Big) \leq 1\Big\}.$$

With this norm, the Orlicz space $L_\Psi(\Omega)$ is a Banach space.

We need some properties of Orlicz spaces. The first is the inequality [14, 3.6.3 and 3.8.5]

$$\|u\|_{L_\Psi(\Omega)} \leq 1 + \int_\Omega \Psi(|u(x)|)dx, \quad u \in L_\Psi. \tag{4.1}$$

The second property is the *Hölder inequality* [14, 3.8.5, 3.8.6]: Let Ψ and Φ be a pair of complementary Young functions and $u \in L_\Psi(\Omega)$, $v \in L_\Phi(\Omega)$. Then

$$\Big|\int_\Omega uv dx\Big| \leq 2\|u\|_{L_\Psi(\Omega)}\|v\|_{L_\Phi(\Omega)}. \tag{4.2}$$

Finally, we need the following theorem [1, Thm. 8.22]:

THEOREM 4.1. *Let $\Omega \in \mathbb{R}^N$ be bounded and let Φ_1 and Φ_2 be two Young functions such that for all $k > 0$,*

$$\lim_{t\to\infty} \frac{\Phi_1(kt)}{\Phi_2(t)} = 0.$$

Then, any (u_n) sequence which is bounded in $L_{\Phi_2}(\Omega)$ and convergent in measure, is convergent in $L_{\Phi_1}(\Omega)$.

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