Global Smooth Solutions to the Multi-dimensional Hydrodynamic Model for Two-carrier Plasmas

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Abstract

The existence of global smooth solutions to the multi-dimensional hydrodynamic model for plasmas of electrons and positively charged ions is shown under the assumption that the initial densities are close to a constant. The model consists of the conservation laws for the particle densities and the current densities, coupled to the Poisson equation for the electrostatic potential. Furthermore, it is proved that the particle densities converge exponentially fast to the (constant) steady state. The proof uses a higher-order energy method inspired from extended thermodynamics.

Keywords. Hydrodynamic model, global existence of smooth solutions, exponential stability of steady states, long-time behavior of solutions, plasmas, thermodynamical energy.

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1 Introduction

Recently, Peng and one of the authors started a program to derive rigorously a hierarchy of macroscopic models for plasmas. The models are derived from the hydrodynamic equations for two charge carriers (usually, electrons and positively charged ions) by performing various asymptotic limits in which a small parameter tends to zero. The limit of vanishing (scaled) relaxation time and, under some conditions, the limit of vanishing (scaled) electron mass has been performed [11, 20, 22] (also see [2, 3, 18, 31]).

The limit of vanishing Debye length (the so-called quasineutral limit) has been studied by Cordier and Grenier for the transient model [6] and by Slemrod and Sternberg for the stationary equations [35]. A combined relaxation-quasineutral limit has been performed by Gasser and Marcati [9].

These results hold for weak entropy solutions or smooth solutions to the hydrodynamic model in *one* space dimension. No results are available in several space dimensions (however, see [25]). The reason is that there is no global existence theory in several space dimensions as in the one-dimensional case. The main goal of this paper is to prove the existence of smooth solutions to the multi-dimensional hydrodynamic equations for electrons and ions. This is a first step in showing the asymptotic limits explained above also in the multi-dimensional case. For the vanishing relaxation-time limit, see Remark 3 at the end of Section 4.

More specifically, we consider an unmagnetized plasma consisting of electrons with (scaled) mass m_e and charge $q_e = -1$ and of a single species of ions with mass m_i and charge $q_i = +1$ [19]. We denote by $n_e = n_e(x,t)$, $\mathbf{j}_e = \mathbf{j}_e(x,t)$ (n_i, $\mathbf{j}_i$, respectively) the scaled particle density and current density of the electrons (ions, respectively) and by $\phi = \phi(x,t)$ the electrostatic potential. These variables satisfy the following scaled Euler-Poisson system:

$$
\begin{aligned}
&\frac{\partial}{\partial t} n_a + \nabla \cdot \mathbf{j}_a = 0, \\
&m_a \frac{\partial}{\partial t} \mathbf{j}_a + m_a \nabla \cdot \left(\frac{\mathbf{j}_a \otimes \mathbf{j}_a}{n_a} \right) + \nabla p_a(n_a) = -q_a n_a \nabla \phi - m_a \frac{\mathbf{j}_a}{\tau_a}, \\
&-\lambda^2 \Delta \phi = n_i - n_e,
\end{aligned}
\tag{1.1}
$$

where $a = e, i$, $(x,t) \in \mathbb{R}^d \times (0,\infty)$, $d \geq 1$, and the k-th component of the term $\nabla \cdot (\mathbf{j}_a \otimes \mathbf{j}_a / n_a)$ equals

$$
\sum_{\ell=1}^{d} \frac{\partial}{\partial x^\ell} \Big(\frac{\mathbf{j}_a^\ell \mathbf{j}_a^k}{n_a} \Big).
$$

Here, the pressure functions are regular functions satisfying

$$p \in C^s(0,\infty), \qquad p_a'(n_a) > 0 \quad \text{for all } n_a > 0, \tag{1.2}$$

with $s > d/2+1$. In particular, this condition is satisfied if

$$p_a(n_a) = c_a n_a^{\gamma_a},$$

where $c_a > 0$ and $\gamma_a \geq 1$ are constants. In this case, the plasma is called *isothermal* if $\gamma_a = 1$ and *adiabatic* if $\gamma_a > 1$.

The physical parameters are the (scaled) electron and ion mass m_e and m_i, the momentum relaxation time constants τ_e, $\tau_i > 0$ of the electrons and ions, respectively, and the Debye length $\lambda > 0$. For the precise scaling of the equations, we refer to [19, 21]. For simplicity, in the following we will set $m_e = m_i = 1$ and $\tau_e = \tau_i = \tau$.

The system (1.1) is supplemented by initial conditions for n_a and $\mathbf{j}_a$ ($a = e, i$), and by boundary conditions for ϕ:

$$n_a(x,0) = n_{a0}(x), \qquad \mathbf{j}_a(x,0) = \mathbf{j}_{a0}(x), \qquad x \in \mathbb{R}^d, \tag{1.3}$$

$$\lim_{|x|\to\infty} \phi(x,t) = 0, \qquad \text{a.e. } t > 0. \tag{1.4}$$

The homogeneous boundary condition for ϕ means that the plasma is in equilibrium at infinity.

The equations (1.1) have been mainly studied only in one space dimension and usually only for one carrier type. The stationary one-dimensional model has been first studied by Degond and Markowich [7]. They proved the existence and uniqueness of subsonic solutions. The extension to the transonic case has been done by Gamba [5]. Global existence of classical or weak entropy solutions to the one-dimensional hydrodynamic model has been shown by Zhang [37], Poupaud, Rascle and Vila [34], Marcati and Natalini [31], Luo, Natalini and Xin [29], Hsiao and Yang [16], for the whole-space or the initial-boundary value problem, and for isothermal or isentropic pressure-density relations. In some of these papers, also the large-time behavior of the solutions has been analyzed and the exponential (local) stability of the steady states is proved. For more references, we refer to the review paper [26].

The hydrodynamic equations for two carrier types (the so-called bipolar model) have been considered only recently. Natalini [33], Hsiao and Zhang [17], and Zhang [38] obtained global weak entropy solutions for the whole space or initial-boundary value problem. Moreover, Hattori and Zhu [12] proved the stability of steady states. Gasser, Hsiao and Li [8] showed that the electron and ion densities tend both to the same nonlinear diffusive wave in the long-time limit.

In the case of several space dimensions (for one carrier type), there are only some partial results concerning the existence of solutions to (1.1). This comes from the fact that the theory of weak entropy solutions for isentropic gas dynamics is only well developed in one space dimension (see [27, 28]). The existence of global solutions to the multi-dimensional problem has been studied in the literature, but only under additional assumptions (see [24, 30] for local classical solutions): Chen and Wang [4] proved the existence of (global) weak solutions when the system (1.1) possesses a geometric structure. Spherically symmetrical smooth solutions are considered by Hsiao and Wang [14, 15]. Small smooth irrotational solutions have been found by Guo [10]. The existence of small smooth solutions without any geometric structure assumption has been shown by Hsiao, Markowich and Wang [13]. Finally, Alì [1] proved the existence of smooth solutions, close to the thermal equilibrium state, to the *full* hydrodynamic model, consisting of the balance laws of mass, momentum and energy. These results refer to the hydrodynamic model for one carrier type only.

In this paper we prove that the problem (1.1), (1.3)-(1.4) possesses a global smooth solution if the initial densities are close enough to a constant (which is the same for both the electron and the ion density) and if the initial current densities are small enough in some norm. This means that we are able to prove the existence of *small* solutions. As explained above, a proof of the existence of *large* solutions without any additional assumption cannot be expected with the actual methods of hyperbolic theory.

This paper generalizes the results of [13] since we consider the hydrodynamic model for *two* carrier types and since our results are valid in *any* space dimension. (The paper [13] deals with the model for one carrier type in at most three space dimensions.) A consequence of this fact is that we obtain *arbitrarily smooth* solutions if the initial data is smooth enough (in the sense of $H^s(\mathbb{R}^d)$ for any $s > d/2+1$). As a byproduct of the existence proof, we can show the exponential stability of the steady states $n_e = n_i =$ const. and $\mathbf{j}_e = \mathbf{j}_i = 0$, i.e., the particle densities converge exponentially fast to the constant steady state. Moreover, our method allows to extend the existence result easily to the full hydrodynamic model consisting of the conservation laws of mass, momentum and energy (see Remark 2 at the end of Section 4).

The main results of this paper are comprised in the following theorem.

Theorem 1 (Global existence and asymptotic decay) *Let (1.2) hold and let $d \geq 1$, $n^* > 0$ and $s \in \mathbb{N}$ with $s > d/2 + 1$. There exists $\epsilon > 0$ such that, if*

$$\|(n_a - n^*, \mathbf{j}_a, \nabla\phi)(\cdot, 0)\|_{H^s} \leq \epsilon,$$

then the equations (1.1), (1.3), (1.4) have a unique classical solution $(n_a, \mathbf{j}_a, \nabla\phi)(\mathbf{x}, t) \in C^1(\mathbb{R}^d \times [0, \infty))$. Furthermore,

$$(n_a - n^*, \mathbf{j}_a, \nabla\phi) \in C^0([0, \infty), H^s) \cap C^1([0, \infty), H^{s-1}),$$

and

$$\|(n_a - n^*, \mathbf{j}_a, \nabla\phi)(\cdot, t)\|_{H^s}^2 \leq ce^{-Kt},$$

where $c = c(\epsilon)$ and K are positive constants.

The proof is based on an energy method in order to obtain a priori bounds which allow to extend the local solution. (The existence of a unique local solution follows from standard arguments; see Section 3). Actually, the physical energy only provides L^2 estimates which do not allow to extend the local solution to a global one. In order to get higher order estimates in H^s we use ideas from extended thermodynamics [32]. Specifically, the existence of an additional conservation law, such as (in our case) the thermodynamical energy or (in general) the entropy, is equivalent to the symmetrizability of the system. In extended thermodynamics the system is symmetrized by using Lagrange multipliers which arise from the existence of the conservation of the entropy. Here, we use the Lagrange multipliers in order to derive higher order estimates for the fluiddynamical variables. Then, the final estimates are obtained by combining these results with correction terms which account for the electric field.

The paper is organized as follows. In Section 2 some notations are introduced and two Moser-type calculus lemmas are recalled. The local existence result is stated in Section 3. Finally, in Section 4 our main result is proved.

2 Notations and basic lemmas

We start this section by introducing a notation for the components of the k-th derivative operator [36]. The derivative operator is defined by

$$D = (\partial_1, \partial_2, \ldots, \partial_d), \quad \partial_i = \frac{\partial}{\partial x^i}, \quad i = 1, \ldots, d.$$

Sometimes we use the symbol ∇ instead of D. For any $k \geq 0$, the k-th derivative operator is defined by

$$D^0 \stackrel{\text{def}}{=} I, \qquad D^k \stackrel{\text{def}}{=} DD^{k-1}, \qquad k \geq 1.$$

Let $J = (j_1, \ldots, j_k)$ be a k-tuple of integers between 1 and d. We set

$$\partial_J = \partial_{j_1} \partial_{j_2} \cdots \partial_{j_k},$$

and $|J| = k$, the total order of differentiation.

For any $f, g \in C^k(\mathbb{R}^d)$, we consider $D^k f$, $D^k g$ as vectors in $(C^0)^{d^k}$ and define the Euclidean norm and scalar product

$$|D^k f| = \left(\sum_{|J|=k} (\partial_J f)^2 \right)^{1/2}, \qquad D^k f \cdot D^k g = \sum_{|J|=k} (\partial_J f)(\partial_J g).$$

We can extend these definitions to a vector-valued function $\boldsymbol{v} = (v^1, \ldots, v^d) \in (C^k(\mathbb{R}^d))^d$ by

$$|D^k \boldsymbol{v}|^2 = \sum_{r=1}^{d} \sum_{|J|=k} (\partial_J v^r)^2, \qquad D^k \boldsymbol{v} \cdot D^k \boldsymbol{w} = \sum_{r=1}^{d} \sum_{|J|=k} (\partial_J v^r)(\partial_J w^r).$$

The symbol $\otimes$ denotes the tensorial product, which combines two symmetric tensors with k and l indices to give a symmetric tensor with $k + l$ indices. In particular, we have the following formula for the k-th derivative of the product of two functions,

$$D^k(fg) = \sum_{r=0}^{k} \binom{k}{r} (D^r f) \otimes (D^{k-r} g). \tag{2.1}$$

Remark 1 *The most common notation for the derivatives makes use of a d-tuple* $\alpha = (\alpha_1, \ldots, \alpha_d)$ *of nonnegative integers, and of the symbol*

$$\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d}.$$

The order of derivation is $|\alpha| := \sum_{i=1}^d \alpha_i$. *If* $|J| = |\alpha| = k$, *and* $f \in C^k(\mathbb{R}^d)$, *then* $\partial_J f = \partial^\alpha f$, *with* $\alpha_i = \#\{l : j_l = i\}$. *It is simple to prove that the Euclidean norm* $\left|D^k f\right|_*$ *of the independent components of* $D^k f$, *defined by*

$$\left|D^k f\right|_* = \left(\sum_{|\alpha|=k} (\partial^\alpha f)^2 \right)^{1/2},$$

is equivalent to $\left|D^k f\right|$, *since [36]*

$$|D^k f| = \left(\sum_{|\alpha|=k} \frac{k!}{\alpha!} (\partial^\alpha f)^2 \right)^{1/2}, \quad \alpha! = \alpha_1! \cdots \alpha_d!.$$

All the functional spaces will be considered on $\mathbb{R}^d$, so we will omit the argument. We will use the following norms:

$$\begin{aligned}
\|f\| = \|f\|_{L^2} &= \left(\int |f(\mathbf{x})|^2 d\mathbf{x} \right)^{1/2}, \\
\|f\|_{H^k} &= \sum_{i=0}^{k} \left\| D^i f \right\|, \\
\|f\|_{L^\infty} &= \sup_{\mathbf{x}\in\mathbb{R}^d} |f(\mathbf{x})|, \\
\|f\|_{C^k} &= \sum_{i=0}^{k} \left\| D^i f \right\|_{L^\infty}.
\end{aligned}$$

In the following, the symbol C will denote a generic positive constant. Sometimes we will write $C(a_1, \dots, a_m)$, to signify that the generic constant C depends on the arguments $a_1, \dots, a_m$. We will use the following classical Lemma [30, 36].

Lemma 1 (Moser–type calculus) *If* f, $g \in H^k \cap L^\infty$, *then we have*

$$\left\| D^k (fg) \right\| \leq C(k) \left(\|f\|_{L^\infty} \left\| D^k g \right\| + \|g\|_{L^\infty} \left\| D^k f \right\| \right). \qquad (2.2)$$

If $f \in H^k$, $Df \in L^\infty$, $g \in H^{k-1} \cap L^\infty$, *then we have*

$$\left\| D^k (fg) - f D^k (g) \right\| \leq C(k) \left(\|Df\|_{L^\infty} \left\| D^{k-1} g \right\| + \|g\|_{L^\infty} \left\| D^k f \right\| \right). \qquad (2.3)$$

If $F(w)$ *is a smooth vector-valued function and* $f(\mathbf{x})$ *is a continuous function which takes values into a compact subset of the domain* Ω *of* F, *with* $f \in H^k \cap L^\infty$, *then we have*

$$\left\| D^k F(f) \right\| \leq C(k, F, \|f\|_{L^\infty}) \left\| D^k f \right\|, \qquad (2.4)$$

where

$$C(k, F, \|f\|_{L^\infty}) = C(k) \left| \frac{\partial F}{\partial w} \right|_{C^{k-1}} \left(\sum_{\mu=1}^{k} \|f\|_{L^\infty}^{\mu-1} \right),$$

with

$$\left|\frac{\partial F}{\partial w}\right|_{C^k} = \sup_{w\in\Omega} \sum_{i=0}^{k} \left| \left(\frac{\partial}{\partial w}\right)^i F(w) \right|.$$

We use also the following result [1].

Lemma 2 *If $F(w)$ is a smooth vector-valued function and $f(\boldsymbol{x})$, $g(\boldsymbol{x})$ are continuous functions which take values into a compact subset of the domain Ω of F, with $g \in L^\infty$, $Dg \in H^{k-1}$ and $f - g \in H^k \cap L^\infty$, then we have*

$$\begin{aligned}&\left\| D^k(F(f) - F(g)) - \frac{\partial F}{\partial w}(g) D^k(f-g) \right\| \\ &\le C(F,f,g) \left(\|f-g\|_{L^\infty} \left\| D^k(f-g) \right\| + \|Dg\|_{H^{k-1}} \|f-g\|_{H^{k-1}} \right), \quad (2.5)\end{aligned}$$

where the constant C depends on the C^k-norm of the derivative of F, on $\|g\|_{L^\infty}$ and on $\|f-g\|_{L^\infty}$.

All the results of this section are still true when f and g are vector-valued functions.

3 Local existence

We consider an equilibrium solution of (1.1), that is a stationary solution with zero current density,

$$\nabla p_a(n_a^*) = -q_a n_a^* \nabla \phi^*, \qquad -\lambda^2 \Delta \phi^* = n_i^* - n_e^*.$$

Specifically, we take the constant solution

$$n_i^*(\mathbf{x}) = n_e^*(\mathbf{x}) =: n^* = \text{constant}, \qquad \phi^*(\mathbf{x}) = 0.$$

We introduce the electric field $\mathbf{E} = -\nabla\phi$ and the notation

$$\mathbf{F} = (n_i, \mathbf{j}_i, n_e, \mathbf{j}_e)^\top, \qquad \mathbf{F}^* = (n^*, 0, n^*, 0)^\top.$$

For any function $f(\mathbf{F},\mathbf{E})$, we introduce the symbol $\delta f = f(\mathbf{F},\mathbf{E}) - f(\mathbf{F}^*, 0)$. We want to study the existence and the long-time behavior of a solution $(\mathbf{F},\mathbf{E})$ of (1.1) which is close enough to an equilibrium solution $(\mathbf{F}^*, 0)$, in an appropriate norm. More precisely, we assume that the H^s norm of the initial value of $(\delta\mathbf{F},\mathbf{E})$ is sufficiently small, with $s > d/2 + 1$.

By using a contraction mapping argument, it is simple to establish the local existence of a unique solution to the Cauchy problem (1.1), (1.3), (1.4).

Theorem 2 *Let s be an integer greater than $d/2+1$ and n^* a strictly positive constant. Under the assumption that $\delta\mathbf{F}(\mathbf{x},0)$ and $\mathbf{E}(\mathbf{x},0)$ belong to $H^s(\mathbb{R}^d)$, there exists a unique smooth solution $\delta\mathbf{F}(\mathbf{x},t)$, $\mathbf{E}(\mathbf{x},t)$ to the Cauchy problem (1.1), (1.3), (1.4) defined in the maximal interval of existence $[0,T_{\max})$. The solution satisfies*

$$(\delta\mathbf{F},\mathbf{E})(\cdot,t) \in C([0,T_{\max}),H^s) \cap C^1([0,T_{\max}),H^{s-1}). \tag{3.1}$$

Moreover, if $T_{\max}$ is finite, then

$$\|(\delta\mathbf{F}(\cdot,t),\mathbf{E}(\cdot,t))\|^2_{H^s} \to \infty$$

as $t \to T_{\max}$.

The occurrence of the electric field $\mathbf{E}$ in the Euler equations deserves a further comment. As noted in [13], using Green's formulation, we can express $\mathbf{E}$ in terms of the electric current $\mathbf{j}_i - \mathbf{j}_e$,

$$\lambda^2\mathbf{E} = \nabla\Delta^{-1}\left(n_{i0}(\mathbf{x}) - n_{e0}(\mathbf{x})\right) - \nabla\Delta^{-1}\nabla\cdot\int_0^t \left(\mathbf{j}_i - \mathbf{j}_e\right)(\mathbf{x},\tau)\,d\tau. \tag{3.2}$$

The symbol $\nabla\Delta^{-1}\nabla\cdot$ can be written as a sum of products of Riesz's transforms. Then, by the L^2 boundedness of the Riesz transform, for any function $\mathbf{w}$ in $(H^s(\mathbb{R}^d))^d$, $s \geq 0$, we have

$$\left\|\nabla\Delta^{-1}\nabla\cdot\mathbf{w}\right\|_{H^s} \leq C\,\|\mathbf{w}\|_{H^s}\,, \tag{3.3}$$

for some positive constant C. Using this estimate, the proof of Theorem 2 is easily achieved.

As an additional consequence of (3.2), we note that

$$\lambda^2\frac{\partial}{\partial t}\mathbf{E} = -\nabla\Delta^{-1}\nabla\cdot\left(\mathbf{j}_i - \mathbf{j}_e\right). \tag{3.4}$$

It is simple to see that the Poisson equation $(1.1)_3$ can be replaced by the nonlocal evolutionary equation (3.4), provided that $(1.1)_3$ is satisfied at the initial time.

It is convenient to introduce the electric field $\mathbf{E}_a$ and the electric potential ϕ_a generated by the charge $q_a(n_a - n^*)$. They are defined by the Poisson equation

$$\lambda^2\nabla\cdot\mathbf{E}_a \equiv -\lambda^2\Delta\phi_a = q_a(n_a - n^*). \tag{3.5}$$
$$\lim_{|x|\to\infty}\phi_a(x,t) = 0, \qquad \text{a.e. } t > 0.$$

Clearly, we have

$$\mathbf{E} = \mathbf{E}_e + \mathbf{E}_i,$$

and

$$\lambda^2 \frac{\partial}{\partial t} \mathbf{E}_a = -q_a \nabla \Delta^{-1} \nabla \cdot \mathbf{j}_a. \tag{3.6}$$

Moreover, as a consequence of the definition of ϕ_a and $\mathbf{E}_a$, we have

$$\|\mathbf{E}_a\|_{H^{s+1}} \leq \|\phi_a\|_{H^{s+2}} \leq C \|\delta n_a\|_{H^s}. \tag{3.7}$$

In the following, we will regard $\mathbf{E}_i$ and $\mathbf{E}_e$ as auxiliary evolutionary variables, which satisfy the nonlocal equation (3.6), with the index $a = i, e$.

4 A priori estimates

In this section we derive the a priori estimate necessary to extend the local solutions. The basic estimate is proved by using, as a starting point, the physical energy and correcting it with additional terms which exploit the coupling with the electric field.

We introduce the notation

$$\mathbf{U} = (\mathbf{F}, \mathbf{E})^\top, \qquad \mathbf{V} = (\mathbf{F}, \mathbf{E}_i, \mathbf{E}_e)^\top,$$

where $\mathbf{F} = (n_i, \mathbf{j}_i, n_e, \mathbf{j}_e)^\top$. Since $|\mathbf{E}|^2 \leq 2\left(|\mathbf{E}_i|^2 + |\mathbf{E}_e|^2\right)$, we have

$$|\delta \mathbf{U}|^2 \leq 2 |\delta \mathbf{V}|^2. \tag{4.1}$$

For some fixed positive number T, we assume that a solution $\mathbf{U}$ of (1.1) exists and that, for all times t in the interval $[0, T]$, the function $\delta \mathbf{U}(\mathbf{x}, t)$ belongs to $H^s(\mathbb{R}^d)$, for an integer $s > d/2 + 1$. Then, $\mathbf{E}_i$ and $\mathbf{E}_e$ exist in the same time interval and belong to H^s. We define

$$\mathcal{U}(t) = \sup_{0<\sigma<t} \|\delta \mathbf{U}(\cdot, \sigma)\|_{H^s}. \tag{4.2}$$

Using standard Sobolev inequalities and the elliptic regularity estimate (3.7), there exists a positive constant C_U such that

$$\sup_{0<\sigma<t} \|\delta \mathbf{V}(\cdot, \sigma)\|_{C^r} \leq C_U \mathcal{U}(t), \qquad r < s - \frac{d}{2}. \tag{4.3}$$

Theorem 1 follows from the subsequent energy estimate.

Lemma 3 *Let* $\mathbf{U}$ *be a solution of (1.1), (1.3), (1.4) with the regularity (3.1). There exist positive constants* C *and* K *such that, if the solution is so small that* $\mathcal{U}(T) \le \epsilon$*, then the following a priori estimate holds for all* $t \in [0, T]$*:*

$$\|\delta \mathbf{U}(\cdot, t)\|_{H^s}^2 \le C e^{-Kt} \|\delta \mathbf{U}(\cdot, 0)\|_{H^s}^2 . \tag{4.4}$$

For clarity of exposition, the proof of Lemma 3 will be organized in three steps, stated in three separate lemmas.

Lemma 4 (L^2-regularity) *Let* $\mathbf{U}$ *be a differentiable solution of (1.1), (1.3), (1.4) in the time interval* $[0, T]$*. Then, in the same time interval, we have*

$$\|\delta \mathbf{U}(\cdot, t)\|^2 \le C \|\delta \mathbf{U}(\cdot, 0)\|^2 , \tag{4.5}$$

for some positive constant C*.*

Proof. We introduce the thermodynamical energy

$$e_a(n) = n\left(e_a(1) + \int_1^n \frac{p_a(\nu)}{\nu^2} d\nu\right), \qquad e_a(1) > 0.$$

The energy and the pressure are related by

$$p_a(n) = n e_a'(n) - e_a(n).$$

We can write the equations (1.1) as a hyperbolic system of conservation laws, using Einstein's summation convention,

$$\partial_t \mathbf{F} + \partial_r \mathbf{F}^r = \mathbf{P}(\mathbf{F}, \mathbf{E}), \tag{4.6}$$

with $\mathbf{F}^r = \mathbf{F}^r(\mathbf{F})$ and

$$\mathbf{P}(\mathbf{F}, \mathbf{E}) = \left(0, n_i \mathbf{E} - \frac{\mathbf{j}_i}{\tau}, 0, -n_e \mathbf{E} - \frac{\mathbf{j}_e}{\tau}\right)^\top . \tag{4.7}$$

The system (4.6) admits the additional balance law

$$\frac{\partial \tilde{h}}{\partial t}(\mathbf{F}) + \partial_r \tilde{h}^r(\mathbf{F}) = g(\mathbf{F}, \mathbf{E}),$$

which holds for all smooth solutions of (4.6). Here, the thermodynamic energy $\tilde{h} = \tilde{h}(\mathbf{F})$ and its flux $\tilde{h}^r = \tilde{h}^r(\mathbf{F})$ are given by

$$\begin{aligned}
\tilde{h} &= \frac{|\mathbf{j}_i|^2}{2n_i} + e_i(n_i) + \frac{|\mathbf{j}_e|^2}{2n_e} + e_e(n_e), \\
\tilde{h}^r &= \left(\frac{|\mathbf{j}_i|^2}{2n_i} + n_i e_i'(n_i)\right)\frac{j_i^r}{n_i} + \left(\frac{|\mathbf{j}_e|^2}{2n_e} + n_e e_e'(n_e)\right)\frac{j_e^r}{n_e},
\end{aligned}$$

and the energy production $g = g(\mathbf{F}, \mathbf{E})$ is defined as

$$g = -\frac{1}{\tau}\left(\frac{|\mathbf{j}_i|^2}{n_i} + \frac{|\mathbf{j}_e|^2}{n_e}\right) + \mathbf{E} \cdot (\mathbf{j}_i - \mathbf{j}_e).$$

We observe that the function $\tilde{h}$ coincides with the physical energy when $\mathbf{E}$ vanishes. It is simple to see that the energy density is a convex function of $\mathbf{F}$, meaning that

$$\frac{\partial^2 \tilde{h}}{\partial \mathbf{F} \partial \mathbf{F}} \quad \text{is positive definite.}$$

We can "center" the energy around the constant value $n_i = n_e = n^*$. In other words, we replace the pair $\tilde{h}$, $\tilde{h}^r$ with the shifted pair h, h^r defined by

$$\begin{aligned} h &= \tilde{h} - e_i(n^*) - e_i'(n^*)\delta n_i - e_e(n^*) - e_e'(n^*)\delta n_e, \\ h^r &= \tilde{h}^r - e_i'(n^*) j_i^r - e_e'(n^*) j_e^r, \end{aligned}$$

where $\delta n_a = n_a - n^*$. We recall the notation $\delta\mathbf{F} = \mathbf{F} - \mathbf{F}^*$. It is immediate to see that

$$\frac{\partial h}{\partial t}(\mathbf{F}) + \partial_r h^r(\mathbf{F}) = g(\mathbf{F}, \mathbf{E}), \tag{4.8}$$

and $h(\mathbf{F})$ is a convex function. Moreover, $h(\mathbf{F}) \equiv h(\mathbf{F}^* + \delta\mathbf{F})$ is equivalent to $|\delta\mathbf{F}|^2$. Using (4.8), we find

$$\frac{d}{dt}\int h \, d\mathbf{x} = -\int \frac{1}{\tau}\left(\frac{|\mathbf{j}_e|^2}{n_e} + \frac{|\mathbf{j}_i|^2}{n_i}\right) d\mathbf{x} + \int \mathbf{E} \cdot (\mathbf{j}_i - \mathbf{j}_e) \, d\mathbf{x}. \tag{4.9}$$

Next, from (3.4), after integrating twice by parts, we obtain

$$\begin{aligned} &\frac{d}{dt}\int \frac{\lambda^2}{2}|\mathbf{E}|^2 \, d\mathbf{x} = -\int \mathbf{E} \cdot \nabla\Delta^{-1}\nabla \cdot (\mathbf{j}_i - \mathbf{j}_e) \, d\mathbf{x} \\ &= -\int \phi \nabla \cdot (\mathbf{j}_i - \mathbf{j}_e) \, d\mathbf{x} = -\int \mathbf{E} \cdot (\mathbf{j}_i - \mathbf{j}_e) \, d\mathbf{x}. \end{aligned} \tag{4.10}$$

Combining (4.9) and (4.10), we arrive to the estimate

$$\frac{d}{dt}\int \left(h + \frac{\lambda^2}{2}|\mathbf{E}|^2\right) d\mathbf{x} = -\int \frac{1}{\tau}\left(\frac{|\mathbf{j}_e|^2}{n_e} + \frac{|\mathbf{j}_i|^2}{n_i}\right) d\mathbf{x} \leq 0. \tag{4.11}$$

This estimate is sufficient to establish the L^2 stability stated by the Lemma. □

In order to prove the asymptotic stability we need some additional estimates.

Lemma 5 (Zero order estimates) *Under the assumptions of Lemma 4, there exists $\epsilon' > 0$ such that, if*

$$\sup_{0<\sigma<T} \|\delta \mathbf{V}(\cdot,\sigma)\|_{C^1} \leq \epsilon', \tag{4.12}$$

then for all $t \in [0,T]$ we have

$$\|\delta \mathbf{U}(\cdot,t)\|^2 \leq Ce^{-Kt} \|\delta \mathbf{U}(\cdot,0)\|^2, \tag{4.13}$$

for some positive constants C and K.

Proof. Starting from (3.6) and proceding as in the derivation of (4.10), we obtain

$$\frac{d}{dt}\int \frac{\lambda^2}{2}|\mathbf{E}_a|^2\, d\mathbf{x} = -q_a \int \mathbf{E}_a \cdot \mathbf{j}_a \, d\mathbf{x}. \tag{4.14}$$

Also, we have

$$\begin{aligned}
&-q_a \frac{d}{dt}\int \mathbf{E}_a \cdot \mathbf{j}_a \, d\mathbf{x} = \frac{1}{\lambda^2}\int \mathbf{j}_a \cdot \nabla\Delta^{-1}\nabla \cdot \mathbf{j}_a \, d\mathbf{x} \\
&+q_a \int \mathbf{E}_a \cdot \left\{ \nabla \cdot \left(\frac{\mathbf{j}_a \otimes \mathbf{j}_a}{n_a} \right) + \nabla p_a(n_a) + q_a n_a \nabla\phi + \frac{\mathbf{j}_a}{\tau} \right\} d\mathbf{x} \\
&= \frac{1}{\lambda^2}\int \mathbf{j}_a \cdot \nabla\Delta^{-1}\nabla \cdot \mathbf{j}_a \, d\mathbf{x} - \int \mathbf{E}_a \cdot \left(n_a \mathbf{E} - \frac{q_a}{\tau}\mathbf{j}_a \right) d\mathbf{x} \\
&-q_a \int \frac{1}{n_a}(\mathbf{j}_a \cdot \nabla \mathbf{E}_a) \cdot \mathbf{j}_a \, d\mathbf{x} - \frac{1}{\lambda^2}\int \delta n_a (p_a(n_a) - p_a(n^*))\, d\mathbf{x}.
\end{aligned} \tag{4.15}$$

Combining (4.14) and (4.15), and recalling the estimate (3.3), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int \left\{ \frac{\lambda^2}{2\tau}|\mathbf{E}_i|^2 - \mathbf{E}_i \cdot \mathbf{j}_i + \frac{\lambda^2}{2\tau}|\mathbf{E}_e|^2 + \mathbf{E}_e \cdot \mathbf{j}_e \right\} d\mathbf{x} \\
&\leq \frac{C_0}{\lambda^2}(\|\mathbf{j}_i\|^2 + \|\mathbf{j}_e\|^2) - \frac{1}{\lambda^2}(p_i'(n^*)\, \|\delta n_i\|^2 + p_e'(n^*)\, \|\delta n_e\|^2) \\
&-n^* \|\mathbf{E}\|^2 + C \|\delta \mathbf{V}\|_{C^1} \|\delta \mathbf{U}\|^2.
\end{aligned} \tag{4.16}$$

We define

$$h_0 = h + \frac{\lambda^2}{2}|\mathbf{E}|^2 + \mu\tau \left\{ \frac{\lambda^2}{2\tau}|\mathbf{E}_i|^2 - \mathbf{E}_i \cdot \mathbf{j}_i + \frac{\lambda^2}{2\tau}|\mathbf{E}_e|^2 + \mathbf{E}_e \cdot \mathbf{j}_e \right\}, \tag{4.17}$$

where μ and ν are real numbers. Using (4.11) and (4.16), we obtain

$$\frac{d}{dt}\int h_0 \, d\mathbf{x} \leq -\tau \int g_0 \, d\mathbf{x} + C' \|\delta \mathbf{V}\|_{C^1} \|\delta \mathbf{U}\|^2, \tag{4.18}$$

for some positive constant C', with

$$g_0 = \left(\frac{1}{n^*} - \frac{C_0\tau^2\mu}{\lambda^2}\right)\left(\left|\frac{\mathbf{j}_i}{\tau}\right|^2 + \left|\frac{\mathbf{j}_e}{\tau}\right|^2\right)$$
$$+n^*\mu|\mathbf{E}|^2 + \frac{\mu}{\lambda^2}\left(p_i'(n^*)\delta n_i^2 + p_e'(n^*)\delta n_e^2\right).$$

We prove that it is possible to choose μ such that both h_0 and g_0 are positive definite quadratic forms.

First, we show that there exist a positive value $\bar{\mu}$ such that if $0 < \mu < \bar{\mu}$, then h_0 is positive definite around $\mathbf{V}^* = (\mathbf{F}^*, 0, 0)$ as a function of $\delta\mathbf{V} = (\delta\mathbf{F}, \mathbf{E}_i, \mathbf{E}_e)$. It suffices to consider the part of h_0 which depends on $\mathbf{j}_a$ and $\mathbf{E}_a$, which is a quadratic form in $(\mathbf{j}_i, \mathbf{E}_i, \mathbf{j}_e, \mathbf{E}_e)^\top$, with matrix

$$\frac{1}{2}\begin{pmatrix} \frac{1}{n_i} & -\tau\mu & 0 & 0 \\ -\tau\mu & \lambda^2(1+\mu) & 0 & \lambda^2 \\ 0 & 0 & \frac{1}{n_e} & \tau\mu \\ 0 & \lambda^2 & \tau\mu & \lambda^2(1+\mu) \end{pmatrix}.$$

This matrix is positive definite if its principal minors are positive definite, that is,

$$\frac{1}{n_i} > 0, \tag{4.19}$$
$$\frac{\lambda^2}{n_i} + \mu\nu_i > 0, \tag{4.20}$$
$$\frac{1}{n_e}\left(\frac{\lambda^2}{n_i} + \mu\nu_i\right) > 0, \tag{4.21}$$
$$\left[\frac{\lambda^2}{n_i} + \mu\nu_i\right]\left[\frac{\lambda^2}{n_e} + \mu\nu_e\right] - \frac{\lambda^4}{n_i n_e} > 0, \tag{4.22}$$

with

$$\nu_a = \frac{\lambda^2}{n_a} - \tau^2\mu, \qquad a = i, e.$$

For positive values of μ, the inequalities (4.19)–(4.22) are satisfied all together if ν_i and ν_e are positive, that is

$$\mu < \min\left\{\frac{\lambda^2}{\tau^2 n_i}, \frac{\lambda^2}{\tau^2 n_e}\right\}. \tag{4.23}$$

This inequality if meaningful if we choose $\|\delta n_i\|_{C^0}$ and $\|\delta n_e\|_{C^0}$ smaller than $n^*/2$, so that both n_i and n_e are strictly positive.

Next, we prove that it is possible to choose μ so that also g_0 is a definite positive quadratic form as a function of $\delta\mathbf{U}$. It is sufficient to restrict the choise of the parameter μ in order to satisfy the additional condition

$$\mu < \frac{\lambda^2}{C_0 \tau^2 n^*}. \tag{4.24}$$

With this choice of μ, there exist positive constants K_h, K_g such that

$$2K_h \|\delta\mathbf{V}\|^2 \leq \int h_0 \, d\mathbf{x}, \qquad 2K_g \|\delta\mathbf{U}\|^2 \leq \int \tau g_0 \, d\mathbf{x}. \tag{4.25}$$

Therefore, choosing $\epsilon' = \min\{K_g/C', n^*/2\}$ and employing the assumption (4.12), the estimate (4.18) implies

$$\frac{d}{dt}\int h_0 \, d\mathbf{x} \leq -K_g \|\delta\mathbf{U}\|^2 . \tag{4.26}$$

Integrating both sides of the estimate (4.18) with respect to time and using (4.1), we get

$$K_h \|\delta\mathbf{U}(\cdot,t)\|^2 \leq \int h_0(\mathbf{x},0) \, d\mathbf{x} - K_g \int_0^t \left\|\delta\mathbf{U}(\cdot,t')\right\|^2 \, dt'. \tag{4.27}$$

Finally, using Gronwall's lemma, the estimate (4.27) provides a control on the L^2 norm of $\delta\mathbf{U}$, and the assertion of the lemma follows, with $K = K_g/K_h$. □

To extend this approach to the derivatives of $\delta\mathbf{U}$, we show that the existence of an additional conservation law for the variables of the system (4.6) implies the symmetrizability of the system.

To see this, we observe that equation (4.6) can be viewed as a constraint for the functions that satisfy the balance equation (4.8). Using Liu's lemma [32], there exist Lagrange multipliers $\mathbf{\Lambda} = (\Lambda_{n_i}, \Lambda_{\mathbf{j}_i}, \Lambda_{n_e}, \Lambda_{\mathbf{j}_e})^\top$ such that the new unconstrained equality

$$\frac{\partial h}{\partial t} + \partial_r h^r - g - \mathbf{\Lambda} \cdot \left(\frac{\partial \mathbf{F}}{\partial t} + \partial_r \mathbf{F}^r - \mathbf{P} \right) = 0, \tag{4.28}$$

holds for all continuous differentiable fields $\mathbf{F}$. Then, it follows

$$\frac{\partial h}{\partial \mathbf{F}} = \mathbf{\Lambda}, \qquad \frac{\partial h^r}{\partial \mathbf{F}} = \mathbf{\Lambda} \cdot \frac{\partial \mathbf{F}^r}{\partial \mathbf{F}}, \tag{4.29}$$

and the residual equality

$$g = \mathbf{\Lambda} \cdot \mathbf{P}. \tag{4.30}$$

The Lagrange multipliers are explicitly given by

$$\Lambda_{n_a} = e_a'(n_a) - e_a'(n^*) - \frac{|\mathbf{j}_a|^2}{2n_a^2}, \qquad \Lambda_{\mathbf{j}_a} = \frac{\mathbf{j}_a}{n_a}, \qquad a = i, e.$$

In the following, we use the index notation $\mathbf{\Lambda} = (\Lambda_A)^\top$, where $A, B, \ldots$ are indices corresponding to $n_i, \mathbf{j}_i, n_e, \mathbf{j}_e$. In the same way, we write $\mathbf{F} = (F^A)^\top$, $\mathbf{F}^r = (F^{Ar})^\top$, and use the implicit summation convention, so that, for instance,

$$\begin{aligned}\Lambda_A F^A = \sum_A \Lambda_A F^A &= \Lambda_{n_i} F^{n_i} + \Lambda_{\mathbf{j}_i} \cdot F^{\mathbf{j}_i} + \Lambda_{n_e} F^{n_e} + \Lambda_{\mathbf{j}_i} \cdot F^{\mathbf{j}_i} \\ &= \Lambda_{n_i} n_i + \Lambda_{\mathbf{j}_i} \cdot \mathbf{j}_i + \Lambda_{n_e} n_e + \Lambda_{\mathbf{j}_i} \cdot \mathbf{j}_i.\end{aligned}$$

The relations (4.29) imply that the system (4.6) is symmetric in the sense of Friedrichs [30], and the symmetrizer is the symmetric, positive definite matrix $\partial \mathbf{\Lambda} / \partial \mathbf{F}$. In fact, using the relation (4.29), it is simple to see that

$$\frac{\partial \Lambda_A}{\partial F^B} \frac{\partial F^{Br}}{\partial F^C} = \frac{\partial \Lambda_C}{\partial F^B} \frac{\partial F^{Br}}{\partial F^A}. \tag{4.31}$$

Now we are ready to state the third step needed for the proof of Lemma 3.

Lemma 6 (Higher order estimates) *Let $\mathbf{U}$ be a k times differentiable solution of (1.1), (1.3), (1.4) in the time interval $[0, T]$, with $k \geq 1$. There exists $\epsilon'' > 0$ such that, if*

$$\sup_{0<\sigma<T} \|\delta \mathbf{V}(\cdot, \sigma)\|_{C^1} \leq \epsilon'', \tag{4.32}$$

then for all $t \in [0, T]$ we have

$$\|\delta \mathbf{U}(\cdot, t)\|_{H^k}^2 \leq C e^{-Kt} \|\delta \mathbf{U}(\cdot, 0)\|_{H^k}^2, \tag{4.33}$$

for some positive constants K and C.

Proof. Recalling the notation introduced in Section 2, for any integer $k \geq 1$, it is immediate to see that the quantity

$$\sum_{|J|=k} \int \frac{1}{2} \frac{\partial \Lambda_A}{\partial F^B} \partial_J F^A \partial_J F^B \, d\mathbf{x}$$

is equivalent to $\left\|D^k\mathbf{F}\right\|^2$. We can compute

$$\begin{aligned}
&\frac{d}{dt}\sum_{|J|=k}\int\frac{1}{2}\frac{\partial\Lambda_A}{\partial F^B}\partial_J F^A\partial_J F^B\,d\mathbf{x}\\
&=\sum_{|J|=k}\int\left\{\frac{1}{2}\frac{\partial}{\partial t}\frac{\partial\Lambda_A}{\partial F^B}\partial_J F^A\partial_J F^B+\frac{\partial\Lambda_A}{\partial F^B}\partial_J F^A\partial_J\frac{\partial F^B}{\partial t}\right\}\,d\mathbf{x}\\
&=\sum_{|J|=k}\frac{1}{2}\int\frac{\partial^2\Lambda_A}{\partial F^B\partial F^C}\left(P^C-\partial_r F^{Cr}\right)\partial_J F^A\partial_J F^B\,d\mathbf{x}\\
&+\sum_{|J|=k}\int\frac{\partial\Lambda_A}{\partial F^B}\partial_J F^A\left(\partial_J P^B-\partial_J\left(\frac{\partial F^{Br}}{\partial F^C}\partial_r F^C\right)\right)\,d\mathbf{x}. \qquad (4.34)
\end{aligned}$$

Using the symmetry relation (4.31), integrating by parts, and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\sum_{|J|=k}\int\frac{\partial\Lambda_A}{\partial F^B}\partial_J F^A\partial_J\left(\frac{\partial F^{Br}}{\partial F^C}\partial_r F^C\right)\,d\mathbf{x}\\
&\leq-\frac{1}{2}\int\partial_r\left(\frac{\partial\Lambda_A}{\partial F^B}\frac{\partial F^{Br}}{\partial F^C}\right)\partial_J F^A\partial_J F^C\,d\mathbf{x}\\
&+C_{AB}\sum_{|J|=k}\left\|\partial_J F^A\right\|\cdot\left\|\partial_J\left(\frac{\partial F^{Br}}{\partial F^C}\partial_r F^C\right)-\frac{\partial F^{Br}}{\partial F^C}\partial_J\left(\partial_r F^C\right)\right\|. \qquad (4.35)
\end{aligned}$$

Recalling Lemma 1, we have

$$\begin{aligned}
&\left\|\partial_J\left(\frac{\partial F^{Br}}{\partial F^C}\partial_r F^C\right)-\frac{\partial F^{Br}}{\partial F^C}\partial_J\left(\partial_r F^C\right)\right\|\\
&\leq C\left\|D\frac{\partial F^{Br}}{\partial F^C}\right\|_{L^\infty}\left\|D^{k-1}\partial_r F^C\right\|+C\left\|\partial_r F^C\right\|_{L^\infty}\left\|D^k\frac{\partial F^{Br}}{\partial F^C}\right\|\\
&\leq C\left\|D\delta\mathbf{F}\right\|_{L^\infty}\left\|D^k\delta\mathbf{F}\right\|. \qquad (4.36)
\end{aligned}$$

We also have

$$\|\mathbf{P}\|_{L^\infty}\leq C\|\delta\mathbf{U}\|_{L^\infty}. \qquad (4.37)$$

Using (4.35), (4.36) and (4.37) in (4.34), we arrive to the estimate

$$\begin{aligned}
&\frac{d}{dt}\sum_{|J|=k}\int\frac{1}{2}\frac{\partial\Lambda_A}{\partial F^B}\partial_J F^A\partial_J F^B\,d\mathbf{x}\\
&\leq\sum_{|J|=k}\int\frac{\partial\Lambda_A}{\partial F^B}\partial_J F^A\partial_J P^B\,d\mathbf{x}+C\|\delta\mathbf{U}\|_{C^1}\left\|D^k\delta\mathbf{F}\right\|^2. \qquad (4.38)
\end{aligned}$$

The estimate (4.38) is the analogue of the energy estimate (4.9). It is simple to prove the analogue of (4.10),

$$
\begin{aligned}
&\frac{d}{dt}\sum_{|J|=k}\int \frac{\lambda^2}{2}\partial_J\mathbf{E}\cdot\partial_J\mathbf{E}\,d\mathbf{x} \\
&= -\sum_{|J|=k}\int \partial_J\mathbf{E}\cdot\partial_J(\mathbf{j}_i-\mathbf{j}_e)\,d\mathbf{x}.
\end{aligned}
\tag{4.39}
$$

Finally, combining (4.38) and (4.39), and recalling the definition of $\mathbf{P}$ (4.7),we obtain

$$
\begin{aligned}
&\frac{d}{dt}\sum_{|J|=k}\int \left(\frac{1}{2}\frac{\partial\Lambda_A}{\partial F^B}\partial_J F^A\partial_J F^B+\frac{\lambda^2}{2}\partial_J\mathbf{E}\cdot\partial_J\mathbf{E}\right)\,d\mathbf{x} \\
&\le -\sum_{|J|=k}\int\left(\frac{1}{\tau n_i}\partial_J\mathbf{j}_i\cdot\partial_J\mathbf{j}_i+\frac{1}{\tau n_e}\partial_J\mathbf{j}_e\cdot\partial_J\mathbf{j}_e\right)\,d\mathbf{x} \\
&+\sum_{|J|=k}\int\frac{1}{n_i}\partial_J\mathbf{j}_i\cdot\left[\partial_J\left(n_i\mathbf{E}\right)-n_i\partial_J\mathbf{E}\right]\,d\mathbf{x} \\
&-\sum_{|J|=k}\int\frac{1}{n_e}\partial_J\mathbf{j}_e\cdot\left[\partial_J\left(n_e\mathbf{E}\right)-n_e\partial_J\mathbf{E}\right]\,d\mathbf{x} \\
&+C\left\|\delta\mathbf{U}\right\|_{C^1}\left\|D^k\delta\mathbf{U}\right\|^2.
\end{aligned}
\tag{4.40}
$$

Using Moser's calculus, we find

$$
\begin{aligned}
&\frac{d}{dt}\sum_{|J|=k}\int \left(\frac{1}{2}\frac{\partial\Lambda_A}{\partial F^B}\partial_J F^A\partial_J F^B+\frac{\lambda^2}{2}\partial_J\mathbf{E}\cdot\partial_J\mathbf{E}\right)\,d\mathbf{x} \\
&\le -\frac{1}{\tau n^*}\left(\left\|D^k\mathbf{j}_i\right\|^2+\left\|D^k\mathbf{j}_e\right\|^2\right)+C\left\|\delta\mathbf{U}\right\|_{C^1}\left\|D^k\delta\mathbf{U}\right\|^2.
\end{aligned}
\tag{4.41}
$$

Now we evaluate

$$
\begin{aligned}
&\frac{d}{dt}\sum_{|J|=k}\int\partial_J\mathbf{E}_a\cdot\left\{\frac{\lambda^2}{2\tau}\partial_J\mathbf{E}_a-q_a\partial_J\mathbf{j}_a\right\}\,d\mathbf{x} \\
&=\frac{1}{\lambda^2}\sum_{|J|=k}\int\partial_J\mathbf{j}_a\cdot\partial_J(\nabla\Delta^{-1}\nabla\cdot\mathbf{j}_a)\,d\mathbf{x} \\
&+q_a\sum_{|J|=k}\int\partial_J\mathbf{E}_a\cdot\partial_J\left\{\nabla\cdot\left(\frac{\mathbf{j}_a\otimes\mathbf{j}_a}{n_a}\right)\right\}\,d\mathbf{x}
\end{aligned}
$$

$$+q_a \sum_{|J|=k} \int \partial_J \mathbf{E}_a \cdot \partial_J \left\{ \nabla \left[p_a(n_a) - p'_a(n^*)\delta n_a \right] \right\} \, d\mathbf{x}$$

$$+q_a \sum_{|J|=k} \int \partial_J \mathbf{E}_a \cdot \partial_J \left\{ p'_a(n^*)\delta n_a - q_a n_a \mathbf{E} \right\} \, d\mathbf{x}. \tag{4.42}$$

We estimate the right-hand side term by term. Recalling the definition of ∂_J, we can write this operator as the composition of the operator ∂_s with a derivative ∂'_J of order $k-1$. Integrating two times by parts, for the first integral at the right-hand side of (4.42) we find

$$\begin{aligned}
&\sum_{|J'|=k-1} \int \partial_s \partial_{J'} \mathbf{j}_a \cdot \partial_s \partial_{J'} (\nabla \Delta^{-1} \nabla \cdot \mathbf{j}_a) \, d\mathbf{x} \\
&= - \sum_{|J'|=k-1} \int \partial_{J'} \mathbf{j}_a \cdot \Delta \partial_{J'} (\nabla \Delta^{-1} \nabla \cdot \mathbf{j}_a) \, d\mathbf{x} \\
&= \sum_{|J'|=k-1} \int (\partial_{J'} \nabla \cdot \mathbf{j}_a)^2 \, d\mathbf{x} \le C_k \left\| D^k \mathbf{j}_a \right\|^2 .
\end{aligned} \tag{4.43}$$

Here, we are implicitely summing over s.

For the second term, decomposing the operator ∂_J as above, integrating by parts and using the Poisson equation (3.5), we obtain

$$\begin{aligned}
&q_a \sum_{|J'|=k-1} \int \partial_s \partial_{J'} E^r_a \partial_s \partial_{J'} \nabla \cdot \left(\frac{j^r_a}{n_a} \mathbf{j}_a \right) \, d\mathbf{x} \\
&= -q_a \sum_{|J'|=k-1} \int \partial_{J'} \Delta E^r_a \partial_{J'} \nabla \cdot \left(\frac{j^r_a}{n_a} \mathbf{j}_a \right) \, d\mathbf{x} \\
&= -\frac{1}{\lambda^2} \sum_{|J'|=k-1} \int \partial_{J'} \partial_r \delta n_a \partial_{J'} \nabla \cdot \left(\frac{j^r_a}{n_a} \mathbf{j}_a \right) \, d\mathbf{x} \\
&\le C \left\| \delta \mathbf{F} \right\|_{C^1} \left\| D^k \delta \mathbf{F} \right\|^2 .
\end{aligned}$$

The last inequality is obtained by employing Moser's calculus. The third term at the right-hand side of (4.42) is dealt in a similar way, by using Lemma 2.

Finally, integrating by parts and using Moser's calculus and the Poisson equation (3.5) again, for the last term in (4.42) we get

$$q_a \sum_{|J|=k} \int \partial_J \mathbf{E}_a \cdot \partial_J \left\{ \nabla p'_a(n^*)\delta n_a - q_a n_a \mathbf{E} \right\} \, d\mathbf{x}$$

$$
\begin{aligned}
&= -\frac{1}{\lambda^2} p_a'(n^*) \left\| D^k \delta n_a \right\|^2 - \sum_{|J|=k} \int n^* \partial_J \mathbf{E}_a \cdot \partial_J \mathbf{E} \, d\mathbf{x} \\
&+ \frac{q_a}{\lambda^2} \sum_{|J'|=k-1} \int \partial_{J'} \nabla \delta n_a \cdot \partial_{J'} \left(\delta n_a \mathbf{E} \right) \, d\mathbf{x} \\
&\leq -\frac{1}{\lambda^2} p_a'(n^*) \left\| D^k \delta n_a \right\|^2 - \sum_{|J|=k} \int n^* \partial_J \mathbf{E}_a \cdot \partial_J \mathbf{E} \, d\mathbf{x} \\
&+ C \left\| \delta \mathbf{U} \right\|_{C^0} \left\| D^{k-1} \delta \mathbf{U} \right\| \left\| D^k \delta \mathbf{U} \right\| . \qquad (4.44)
\end{aligned}
$$

In conclusion, using these results in (4.42), we arrive to

$$
\frac{d}{dt} \int h_k \, d\mathbf{x} \leq -\tau \int g_k \, d\mathbf{x} + C \left\| \delta \mathbf{U} \right\|_{C^1} \left(\left\| D^{k-1} \delta \mathbf{U} \right\|^2 + \left\| D^k \delta \mathbf{U} \right\|^2 \right), \quad (4.45)
$$

with

$$
\begin{aligned}
h_k &= \sum_{|J|=k} \frac{1}{2} \frac{\partial \Lambda_A}{\partial F^B} \partial_J F^A \partial_J F^B + \frac{\lambda^2}{2} \left| D^k \mathbf{E} \right|^2 \\
&+ \mu \tau \left\{ \frac{\lambda^2}{2\tau} \left| D^k \mathbf{E}_i \right|^2 - \sum_{J=k} \partial_J \mathbf{E}_i \cdot \partial_J \mathbf{j}_i + \frac{\lambda^2}{2\tau} \left| D^k \mathbf{E}_e \right|^2 + \sum_{J=k} \partial_J \mathbf{E}_e \cdot \partial_J \mathbf{j}_e \right\} ,
\end{aligned}
$$

and

$$
\begin{aligned}
g_k &= \left(\frac{1}{n^*} - \frac{C_k \tau^2 \mu}{\lambda^2} \right) \left(\left| D^k \frac{\mathbf{j}_i}{\tau} \right|^2 + \left| D^k \frac{\mathbf{j}_e}{\tau} \right|^2 \right) \\
&+ n^* \mu \left| D^k \mathbf{E} \right|^2 + \frac{\mu}{\lambda^2} \left(p_i'(n^*) \left| D^k \delta n_i \right|^2 + p_e'(n^*) \left| D^k \delta n_e \right|^2 \right) .
\end{aligned}
$$

Proceeding similarly as in the proof of Lemma 5, it is not difficult to prove that the constant μ can be chosen so that h_k is equivalent to $\left\| D^k \delta \mathbf{V} \right\|^2$, and g_k is equivalent to $\left\| D^k \delta \mathbf{U} \right\|^2$, provided $\left\| \delta \mathbf{F} \right\|_{L^\infty}$ is smaller than an appropriate constant $\bar{n}$. Then, we can estimate

$$
2K_h \left\| D^k \delta \mathbf{V} \right\|^2 \leq \int h_k \, d\mathbf{x}, \qquad 2K_g \left\| D^k \delta \mathbf{U} \right\|^2 \leq \int \tau g_k \, d\mathbf{x}, \qquad (4.46)
$$

with the same constants K_h, K_g as in (4.25).

Combining the zero order estimate (4.18) with the higher order estimates (4.45), up to the k-th order, we find

$$
\frac{d}{dt} \sum_{r=0}^{k} \int h_r \, d\mathbf{x} \leq -\left(2K_g - C'' \left\| \delta \mathbf{V} \right\|_{C^1} \right) \sum_{r=0}^{k} \left\| D^r \delta \mathbf{U} \right\|^2 .
$$

Similar to the conclusion of the previous Lemma, setting $\epsilon'' = \min\{K_g/C'', \bar{n}\}$ and using the assumption (4.32), we integrate with respect to time and use (4.46), to obtain

$$K_h \|\delta\mathbf{U}(\cdot,t)\|^2_{H^k} \leq \sum_{r=0}^{k} \int h_r(\mathbf{x},0)\, d\mathbf{x} - K_g \|\delta\mathbf{U}(\cdot,t)\|^2_{H^k}. \tag{4.47}$$

The assertion follows immediately by invoking Gronwall's lemma. □

Lemma 3 is a simple consequence of Lemmas 5 and 6, recalling that the C^1 norm of $\delta\mathbf{V}$ is controlled by $\mathcal{U}(T)$, employing the Sobolev estimate (4.3) and the elliptic regularity estimate (3.7).

Remark 2 *The estimate (4.38) is valid also for more general plasma models consisting of the balance laws for mass, momentum and energy, since its proof requires only the existence of an additional balance law, such as the entropy equation. Besides (4.38), the proofs of Lemmas 5 and 6 reside heavily on the structure of the production term* $\mathbf{P}$*. Comforted by the results obtained in [1], we expect a priori estimates such as in Lemma 3 to be valid also in the presence of thermal effects, at least when the production term is of relaxation type around equilibrium.*

Remark 3 *Our estimates do not allow the quasineutral limit* $\lambda \to 0$*. For instance, the quadratic form* h_0 *(recall the definition (4.17)) cannot be positive definite in the limit* $\lambda \to 0$ *(see (4.23)) since we loose the electric energy part* $\lambda^2|\mathbf{E}|/2$ *in (4.17).*

Anyway, our estimates do allow the limit $\tau = \tau_e = \tau_i \to 0$*. We introduce the scaled time and current densities*

$$t' = \tau t, \qquad \mathbf{j}'_a(\mathbf{x},t') = \frac{1}{\tau}\mathbf{j}_a\left(\mathbf{x}, \frac{t'}{\tau}\right).$$

At the same way, we introduce the functions

$$n'_a(\mathbf{x},t') = n_a\left(\mathbf{x}, \frac{t'}{\tau}\right). \qquad \mathbf{E}'(\mathbf{x},t') = \mathbf{E}\left(\mathbf{x}, \frac{t'}{\tau}\right).$$

Keeping track of τ *in the proofs of Lemmas 5 and 6, it is simple to see that the constant* K *is proportional to* τ*,* $K = \tau K'$*. Moreover, we have*

$$\delta\mathbf{U}(\mathbf{x},t) = (\delta n_i, \mathbf{j}_i, \delta n_e, \mathbf{j}_e, \mathbf{E})^\top(\mathbf{x},t) = (\delta n'_i, \tau\mathbf{j}'_i, \delta n'_e, \tau\mathbf{j}'_e, \mathbf{E}')^\top(\mathbf{x},\tau t).$$

Thus, in terms of the rescaled variables, the final a priori estimate (4.4) in Lemma 3 becomes

$$\left\|(\delta n'_i, \tau \mathbf{j}'_i, \delta n'_e, \tau \mathbf{j}'_e, \mathbf{E}')(\cdot, t')\right\|^2_{H^s} \le C e^{-K't'} \left\|(\delta n'_i, \tau \mathbf{j}'_i, \delta n'_e, \tau \mathbf{j}'_e, \mathbf{E}')(\cdot, 0)\right\|^2_{H^s}.$$

This estimate is valid also as τ tends to zero. In the limit $\tau = 0$, the primed variables satisfy the diffusion system

$$\begin{aligned}
&\frac{\partial}{\partial t} n'_a + \nabla \cdot \mathbf{j}'_a = 0, \\
&\nabla p_a(n'_a) = -q_a n'_a \nabla \phi' - \mathbf{j}'_a, \\
&-\lambda^2 \Delta \phi' = n'_i - n'_e.
\end{aligned}$$

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