Global Non–Negative Solutions of a Nonlinear Fourth–Order Parabolic Equation for Quantum Systems

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Abstract

The existence of non–negative weak solutions globally in time of a nonlinear fourth–order parabolic equation in one space dimension is shown. This equation arises in the study of interface fluctuations in spin systems and in quantum semiconductor modeling. The problem is considered on a bounded interval subject to initial and Dirichlet and Neumann boundary conditions. Further, the initial datum is only assumed to be non–negative and to satisfy a weak integrability condition. The main difficulty of the existence proof is to ensure that the solutions stay *non–negative* and exist *globally* in time. The first property is obtained by an exponential transformation of variables. Moreover, entropy–type estimates allow for the proof of the second property. Results concerning the uniqueness and long–time behaviour are given and the multi–dimensional problem is discussed. Finally, numerical experiments underlining the preservation of positivity are presented.

Key words. Higher order parabolic PDE, global solution, existence, uniqueness, positivity, entropy.

AMS(MOS) subject classification. 35K35, 35B99, 35G30

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1 Introduction

In the last years, the study of *non–negative* or *positive* solutions to parabolic fourth–order equations has attracted a lot of attention in the mathematical literature (see [Ber98], [BP98], [dPGG98], [Grü95] and the references therein). In particular, it was shown that certain degenerate equations of the form

$$h_t = -(f(h)h_{xxx})_x + (g(h)h_x)_x \tag{1.1}$$

allow for positive solutions, if the functions f and g satisfy certain growth conditions [dPGG98]. Equation (1.1) appears in the context of surface dominated motion of thin viscous films and spreading droplets or plasticity (for an overview see [Ber98] and the references therein). When $f(h) = h,\ g(h) \equiv 0$, this equation especially arises in the modeling of droplet breakup in a Hele-Shaw cell where the variable h describes the thickness of a neck between two masses of fluid.

Clearly, maximum principles are in general not available for fourth–order equations such that the positivity or non–negativity property has to be proved by other techniques. The main ingredient is to exploit the special nonlinear structure of (1.1) introduced by the degenerate mobility $f(h)$, i.e. $f(h) = h^\alpha$ as $h \to 0$ for some $\alpha > 0$. This allows for nonlinear entropy dissipation, which is essential for the positivity of solutions [dPGG98].

In this paper we show that the fourth–order equation

$$n_t = -(n(\log(n))_{xx})_{xx} \tag{1.2a}$$

for $t > 0$, subject to the initial condition

$$n(0, x) = n_0(x) \tag{1.2b}$$

allows for *non–negative* solutions.

This equation, which can be equivalently written as

$$n_t = -n_{xxxx} + \left(\frac{n_x^2}{n}\right)_{xx},$$

arises as a scaling limit in the study of interface fluctuations in a certain spin system [DLSS91]. The variable n describes the scaling limit of probabilities for a random variable. Problem (1.2a)–(1.2b) with periodic boundary conditions was first studied by Bleher *et al.* in [BLS94]. Assuming (strictly) positive $H^1(\Omega)$–data, they showed that there exists a unique positive classical solution locally in time. For "small" initial data, the solution is even global in time. However, the problem whether non–negative solutions for general (non–negative) initial data exist *globally* in time remained open. In this paper we solve this problem. Note that the equivalent formulation of (1.2a) is not degenerate such that the concept of nonlinear entropy dissipation is not applicable.

More specifically, we consider equation (1.2a) in the bounded domain $\Omega = (0,1)$ subject to the boundary conditions

$$n(0) = n(1) = 1, \quad n_x(0) = n_x(1) = 0. \tag{1.2c}$$

Our results extend to Dirichlet boundary conditions $n(0) \neq n(1)$, but we use (1.2c) for the sake of a smoother presentation. Although equation (1.2a) is (formally) derived for $\Omega = \mathbb{R}$, we study the problem in a bounded domain subject to the conditions (1.2c) for the following reason:

Equation (1.2a) also arises in the modeling of quantum semiconductor devices [Pin99a]. More precisely, the so–called quantum drift diffusion model ([PU99], [AI89]) simplifies to equation (1.2a) in the case of zero temperature and zero (or negligible) electric field (see also [GJ99a, Jün98, Jün97]). In several space dimensions the simplified and scaled equation reads

$$n_t = -2 \text{ div} \left(n \nabla \left(\frac{\Delta \sqrt{n}}{\sqrt{n}} \right) \right)$$

or, equivalently (assuming smooth non–vacuum solutions),

$$n_t = - \sum_{i,j} \partial_i \partial_j \left(n \, \partial_i \partial_j \log(n) \right).$$

In this context, n denotes the density of electrons in the semiconductor crystal. The expression $\Delta\sqrt{n}/\sqrt{n}$ is the so–called quantum Bohm potential. Now, in quantum semiconductor modeling, usually the boundary conditions (1.2c) are used (see [Gar94],[Pin99b]). Note that our arguments also apply to the case of periodic boundary conditions.

We show that for non–negative initial data satisfying a certain integrability condition, there exists a generalized *non–negative* solution *globally* in time. We stress the fact that we do not assume (strictly) positive initial data. As we impose only weak assumptions on the data, we can a priori not expect that our solutions have $L^2_{loc}(0,\infty; H^2(\Omega))$–regularity (see Proposition 3.3). On account of this fact we have to weaken our solution concept. Our main result is as follows.

Theorem 1.1. *Assume that the initial datum n_0 is measurable and satisfies the condition*

$$\int_\Omega n_0 - \log(n_0) \, dx < +\infty. \tag{1.3}$$

Then there exists a solution n of (1.2a)–(1.2c) *satisfying*

$$n(x,t) \geq 0 \quad a.e. \ in \ (0,\infty) \times \Omega, \tag{1.4a}$$
$$n \in L^2_{loc}(0,\infty; W^{1,1}(\Omega)), \quad n_t \in L^1_{loc}(0,\infty; H^{-2}(\Omega)), \tag{1.4b}$$
$$\log(n) \in L^2_{loc}(0,\infty; H^2(\Omega)) \cap L^\infty(0,\infty; L^1(\Omega)). \tag{1.4c}$$

Further, $n(\cdot,0) = n_0$ in the sense of $H^{-2}(\Omega)$ and it holds for any $T > 0$ and any smooth test function $\phi \in C_c^\infty((0,\infty) \times \Omega)$,

$$\int_0^T \langle n_t, \phi \rangle_{H^{-2},H_0^2} \, dt = -\int_0^T \int_\Omega n \, (\log(n))_{xx} \, \phi_{xx} \, dx \, dt.$$

Remark 1.2.

a) From condition (1.3) one can readily deduce that $n_0 \in L^1(\Omega)$ (see Corollary 2.7). Thus, we only impose very weak regularity assumptions on the initial data.

b) Starting from smooth, positive initial data Bleher *et al.* used the stronger concept of mild solutions [BLS94], employing results from semigroup theory.

c) Considering Eq. (1.1) with $g(h) \equiv 0$, Bernis and Friedman [BF90] established a very weak solution concept, basically saying that h is a solution if for all $\phi \in L^2(0,T; H^1(\Omega))$ it holds

$$\int_0^T \langle h_t, \phi \rangle \, dt - \int_{\{|h|>0\}} f(h) \, h_{xxx} \, \phi_x \, dx \, dt = 0.$$

The proof of Theorem 1.1 is based on two ideas. The first one is to perform an exponential transformation of variables. Setting $n = e^{2u}$, equation (1.2a) reads in the new variable:

$$\left(e^{2u}\right)_t = -2 \left(e^{2u} \, u_{xx}\right)_{xx}. \tag{1.5}$$

Hence, the existence of a (generalized) solution u of (1.5) implies the existence of a non–negative solution n of (1.2a). Exponential transformations were already successfully employed in the study of the stationary quantum hydrodynamic equations [GJ99b, BGMS95].

Clearly, a solution $u \in L^\infty((0,\infty) \times \Omega)$ to (1.5) provides a *positive* solution n to (1.2a). However, we only get the regularity $u \in L^2_{loc}(0,\infty; L^\infty(\Omega))$ (see (1.4)) such that we can only conclude the existence of *non–negative* solutions to (1.2a). This is in contrast to the stationary problem, where the positivity property immediately follows from an $H^s(\Omega)$ bound for the corresponding stationary variable u and the Sobolev embedding $H^s(\Omega) \hookrightarrow L^\infty(\Omega)$ when $s > d/2$, d being the space dimension (see [GJ99b]).

This observation motivated us to discretize (1.5) in time, which is the second main idea for the proof, yielding a sequence of elliptic problems. We show the existence of solutions $u(t_k, \cdot)$ in $H^2(\Omega)$ to the resulting elliptic problems. Hence, the approximate solutions $u(t_k, \cdot)$ are in $L^\infty(\Omega)$ and expressions like $e^{u(t_k,x)}$ are well defined.

It is worth noting that equation (1.2a) possesses several Lyapunov functionals which provide a priori estimates in the existence proof. It can be easily seen that the *entropy*

$$S(t) = \int_\Omega n(t)\left(\log(n(t)) - 1\right) + 1 \, dx$$

is (formally) non–increasing in time. This has also been observed in [BLS94]. In the case of periodic boundary conditions, also the *Fisher information*

$$\int_\Omega \left|(\sqrt{n})_x\right|^2 \, dx$$

is non–increasing in time. In addition we prove that the quantity

$$\int_\Omega n(t) - \log(n(t)) \, dx$$

is non–increasing in time. More precisely, we show that

$$\int_\Omega n(t) - \log(n(t)) \, dx + \int_0^t \int_\Omega |(\log(n(t)))_{xx}|^2 \, dx\, dt \le \int_\Omega n_0 - \log(n_0) \, dx$$

(in a sense to be precised later).

As already mentioned, the regularity of the solutions provided by Theorem 1.1 is quite weak. In particular, it is not clear whether uniqueness of solutions holds in the class of functions satisfying (1.4). Bleher *et al.* proved in [BLS94] the uniqueness of solutions to (1.2a)–(1.2b) with periodic boundary conditions, assuming (strictly) positive $H^1(\Omega)$ initial data. On the other hand, for equations of type (1.1) uniqueness is still an open problem, due to the lack of a suitable comparison principle.

We employ the following monotonicity property, which implies the uniqueness of solutions in a larger class of functions, only assuming non–negative initial data. The idea is the following: First, divide equation (1.2a) by $\sqrt{n}$. Then we obtain (formally)

$$(\sqrt{n})_t = A(\sqrt{n}) \stackrel{\text{def}}{=} -\frac{1}{2\sqrt{n}}(n(\log(n))_{xx})_{xx}.$$

A formal computation shows that the operator $-A(\sqrt{n})$ is monotone:

$$-\left\langle A(\sqrt{n_1}) - A(\sqrt{n_2}), \sqrt{n_1} - \sqrt{n_2}\right\rangle \ge 0, \qquad \text{for suitable } \sqrt{n_1}, \sqrt{n_2}.$$

This property will be made precise in Section 2. We notice that the monotonicity of the operator $-A(\sqrt{n})$ was already used in the analysis of the stationary quantum drift diffusion equations [PU99], as well as for the investigation of stability properties of the linearized transient model [Pin99a]. However, in order to make the monotonicity property rigorous, we need (strictly) positive solutions (see Theorem 3.1). Clearly, for more regular initial data, we obtain the same uniqueness result as in [BLS94].

The paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.1. In Section 3 we prove the uniqueness of (strictly positive) generalized solutions, give a result on the long–time behavior of solutions and consider regularity questions. Further, we discuss the multi–dimensional problem and present some numerical experiments underlining the preservation of positivity for $t > 0$.

1.1 Notation and Auxiliary Results

We use the standard notation for Sobolev spaces (see [Ada75]), denoting the norm of $W^{m,p}(\Omega)$ ($m \in \mathbb{N}, p \in [1,\infty]$) by $\|\cdot\|_{W^{m,p}(\Omega)}$. In the special case $p = 2$ we use $H^m(\Omega)$ instead of $W^{m,2}(\Omega)$. Further, let $H_0^m(\Omega)$ be the closure of $C_c^\infty(\Omega)$ with respect to the $H^m(\Omega)$–norm. Its dual space $(H_0^m(\Omega))^*$ is denoted by $H^{-m}(\Omega)$ and the duality pairing of $H_0^m(\Omega)$ with its dual space is given by $\langle\cdot,\cdot\rangle_{H^{-m},H_0^m}$. Moreover, for any Banach space B we define the space $L^p(0,T;B)$ with $p \in [1,\infty]$ consisting of all measurable functions $\varphi : (0,T) \to B$ for which the norm

$$\|\varphi\|_{L^p(0,T;B)} \stackrel{\text{def}}{=} \left(\int_0^T \|\varphi\|_B^p \; dt \right)^{1/p}, \quad p \in [1,\infty),$$
$$\|\varphi\|_{L^\infty(0,T;B)} \stackrel{\text{def}}{=} \sup_{t\in(0,T)} \|\varphi(t)\|_B\,, \quad p = \infty,$$

is finite. If the time interval is clear we shortly write $\|\cdot\|_{L^p(B)}$.

In the forthcoming analysis we make frequent use of the Gagliardo–Nirenberg inequality [GT83]:

Lemma 1.3. *Let $\Omega \in \mathbb{R}$ be a bounded domain and $m \geq 1$. Furthermore, let $1 \leq p,q,r \leq \infty$, $j \in \mathbb{N}_0$ with $j < m$ and $\theta \in [j/m, 1]$ such that*

$$\frac{1}{p} = j + \theta \left(\frac{1}{r} - m \right) + (1-\theta)\frac{1}{q}$$

provided that $m - j - 1/r$ is a non–negative integer (else take $\theta = j/m$). Then, there exists a constant $C = C(\Omega, m, j, \theta, p, q, r) > 0$ such that for all $\varphi \in W^{m,r}(\Omega) \cap L^q(\Omega)$

$$\left\|D^j\varphi\right\|_{L^p(\Omega)} \leq C \, \|\varphi\|_{W^{m,r}(\Omega)}^\theta \, \|\varphi\|_{L^q(\Omega)}^{1-\theta}.$$

2 Existence

In this section we provide the proof of Theorem 1.1, which is done in several steps. First, we introduce an exponential transformation of variables. Setting $n = e^{2u}$ we get Eq. (1.5), which will be investigated in the following. Instead of Theorem 1.1, we prove the following result, which yields Theorem 1.1 by back transforming the variables.

Proposition 2.1. *Assume that the initial datum u_0 is measurable and satisfies*

$$\int_\Omega e^{2u_0} - 2u_0 \, dx < \infty. \tag{2.1}$$

Then there exists a solution $u \in H_0^2(\Omega)$ of

$$\left(e^{2u}\right)_t = -2 \left(e^{2u} u_{xx}\right)_{xx}$$

satisfying $e^{2u(\cdot,0)} = n_0$ in the sense of $H^{-2}(\Omega)$ and

$$u \in L^2_{loc}(0,\infty; H_0^2(\Omega)) \cap L^\infty(0,\infty; L^1(\Omega)), \tag{2.2a}$$

$$e^{2u} \in L^2_{loc}(0,\infty; W^{1,1}(\Omega)), \quad \left(e^{2u}\right)_t \in L^r_{loc}(0,\infty; H^{-2}(\Omega)), \tag{2.2b}$$

for $r \in [1, 10/9)$. Further, it holds for each $T > 0$ and each $\phi \in C_c^\infty((0,\infty)\times\Omega)$

$$\int_0^T \left\langle \left(e^{2u}\right)_t, \phi \right\rangle_{H^{-2},H_0^2} dt + 2\int_0^T \int_\Omega e^{2u} u_{xx} \phi_{xx} \, dx\, dt = 0. \tag{2.3}$$

Remark 2.2. Note that this result is even stronger than Theorem 1.1, since it implies $n_t \in L^r_{loc}(0,\infty; H^{-2}(\Omega))$ for $r \in [1, 10/9)$. This gain of regularity is significantly simplifying the proof due to the reflexivity of $L^r_{loc}(0,\infty; H^{-2}(\Omega))$ for $r > 0$.

Secondly, to prove Proposition 2.1 we employ a vertical line method [Rek82], i.e. we use a semidiscretization in time. This yields a sequence of elliptic problems, for which we show that each possesses a unique positive solution.

Thirdly, we derive a priori estimates on the sequence of approximating solutions, which allow to perform the limit in the weak formulation. They are of energy type and of entropy type as well.

2.1 Semidiscretization

We divide the interval $[0, T]$ into N subintervals by introducing the partition $0 = t_0 < t_1 < \ldots < t_N = T$. Setting $\tau_k \stackrel{\text{def}}{=} t_k - t_{k-1}$ we define the maximal subinterval length $\tau \stackrel{\text{def}}{=} \max_{k=1,\ldots,N} \tau_k$. We assume that the partition fulfills

$$\tau \to 0, \qquad \text{as } N \to \infty. \tag{2.4}$$

Remark 2.3. Certainly, uniform partitions satisfy (2.4) and are sufficient for the analytical investigations. But as the method is also of great numerical interest as it provides a positivity preserving scheme, we allow for variable time–steps, which increases the flexibility of the method [JP99].

For any Banach space B we define

$$PC_N(0,T;B) \stackrel{\text{def}}{=} \left\{ u^{(N)} : (0,T] \to B : u^{(N)}|_{(t_{k-1},t_k]} \equiv const. \text{ for } k = 1,\dots,N \right\}$$

and introduce the short–cut $u_k = u^{(N)}(t)$ for $t \in (t_{k-1}, t_k]$ and $k = 1,\dots,N$. Further, let $\tilde{u}^{(N)}$ denote the linear interpolant of $u^{(N)} \in PC_N(0,T;L^2(\Omega))$ given by

$$\tilde{u}^{(N)}(t,x) = \frac{t-t_{k-1}}{\tau_k}(u_k - u_{k-1}) + u_{k-1}, \qquad \text{for } x \in \Omega, \quad t \in (t_{k-1}, t_k].$$

Now, we discretize (1.5) in the following way:

For $k = 1,\dots,N$ solve recursively the elliptic equations

$$\frac{1}{\tau_k}\left(e^{2\,u_k} - e^{2\,u_{k-1}}\right) = -2\left(e^{2\,u_k}\,u_{k,xx}\right)_{xx} \tag{2.5}$$

subject to $u_k \in H_0^2(\Omega)$ and get an approximate solution $u^{(N)} \in PC_N(0,T;H_0^2(\Omega))$. We set $e^{2\,u_0} = n_0$. Then problem (2.5) possesses a unique solution, which is the content of the following result.

Proposition 2.4. *Let u_0 satisfy (2.1). For each $k = 1,\dots,N$, there exists a unique weak solution $u_k \in H_0^2(\Omega)$ of (2.5) fulfilling*

$$2\int_\Omega e^{2\,u_k}\,u_{k,xx}\,\phi_{xx}\,dx + \frac{1}{\tau_k}\int_\Omega e^{2\,u_k}\,\phi\,dx = \frac{1}{\tau_k}\int_\Omega e^{2\,u_{k-1}}\,\phi\,dx, \tag{2.6}$$

for all $\phi \in H_0^2(\Omega)$.

For the proof of Proposition 2.4, especially for the uniqueness part we need the following result. It states the monotonicity of the nonlinear elliptic operator, which has been already successfully employed for the investigation of stability properties of stationary states [Pin99a].

Lemma 2.5. *Let $u, v \in H_0^2(\Omega)$. Then the operator $A : H_0^2(\Omega) \to H^{-2}(\Omega)$ given by*

$$A(u) = -e^{-u}\left(e^{2\,u}\,u_{xx}\right)_{xx}$$

is well defined and $-A$ is monotone in the following sense

$$-\left\langle A(u) - A(v), e^u - e^v \right\rangle_{H^{-2},H_0^2} \geq 0. \tag{2.7}$$

Proof. Since $u \in H_0^2(\Omega)$ we have by Sobolev's embedding theorems [Ada75] that $u \in C^{1,\alpha}(\bar{\Omega})$ for $0 \leq \alpha \leq 1/2$, which implies $e^u > 0$ in $\bar{\Omega}$. For $\phi \in H_0^2(\Omega)$ it holds

$$
\begin{aligned}
\langle A(u), \phi \rangle_{H^{-2},H_0^2} &= - \int_\Omega e^{2\,u}\, u_{xx}\, \left(e^{-u}\, \phi\right)_{xx}\, dx \\
&= - \int_\Omega e^{2\,u}\, u_{xx}\, \left(-u_{xx}\, \phi + u_x^2\, \phi - 2\, u_x\, \phi_x + \phi_{xx}\right)\, dx \\
&\leq \left\| e^{2\,u} \right\|_{L^\infty(\Omega)} \left\| u_{xx} \right\|_{L^2(\Omega)} \Big(\left\| u_{xx} \right\|_{L^2(\Omega)} \left\| \phi \right\|_{L^\infty(\Omega)} \\
&\quad + \left\| u_x \right\|_{L^4(\Omega)}^2 \left\| \phi \right\|_{L^\infty(\Omega)} + 2 \left\| u_x \right\|_{L^2(\Omega)} \left\| \phi_x \right\|_{L^\infty(\Omega)} + \left\| \phi_{xx} \right\|_{L^2(\Omega)} \Big) \\
&\leq c\left(\Omega, \|u\|_{H^2(\Omega)}\right) \|\phi\|_{H^2(\Omega)}\,,
\end{aligned}
$$

where we used the embedding $H^2(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ in one space dimension. Hence, A is well defined.

Now, we prove the monotonicity of A. Let $u, v \in H_0^2(\Omega)$ be given. Then, $e^u \in H^2(\Omega)$ since

$$
\begin{aligned}
\left\| (e^u)_{xx} \right\|_{L^2(\Omega)} &\leq \left\| e^u \right\|_{L^\infty(\Omega)} \left(\left\| u_{xx} \right\|_{L^2(\Omega)} + \left\| u_x \right\|_{L^4(\Omega)}^2 \right) \\
&\leq c(\Omega)\, e^{\|u\|_{L^\infty(\Omega)}} \, \|u\|_{H^2(\Omega)} \left(1 + \|u\|_{H^2(\Omega)} \right),
\end{aligned}
$$

for some positive constant $c(\Omega)$, depending only on Ω by Sobolev's embedding theorems. Now consider

$$
\begin{aligned}
&- \langle A(u) - A(v)\,, e^u - e^v \rangle_{H^{-2},H_0^2} \\
&\qquad = \left\langle \left(e^{-u} \left(e^{2\,u}\, u_{xx} \right) - e^{-v} \left(e^{2\,v}\, v_{xx} \right) \right)_{xx}, e^u - e^v \right\rangle_{H^{-2},H_0^2} \\
&\qquad = \left\langle (e^u)_{xxxx} - e^{-u}\, (e^u)_{xx}^2 - (e^v)_{xxxx} + e^{-v}\, (e^v)_{xx}^2, e^u - e^v \right\rangle_{H^{-2},H_0^2} \\
&\qquad = \int_\Omega (e^u - e^v)_{xx}^2 - (e^u)_{xx}^2 - (e^v)_{xx}^2 + e^{v-u}\, (e^u)_{xx}^2 + e^{u-v}\, (e^v)_{xx}^2 \, dx \\
&\qquad = \int_\Omega \left(e^{\frac{v-u}{2}}\, (e^u)_{xx} - e^{\frac{u-v}{2}}\, (e^v)_{xx} \right)^2 \, dx \\
&\qquad \geq 0,
\end{aligned}
$$

which yields the assertion. □

Now we are in the position to prove the existence and uniqueness result for the elliptic problem (2.6).

Proof of Proposition 2.4. We employ Schauder's fixed point theorem to show that there exists at least one solution. Let $k \in \{1, \ldots, N\}$ be fixed and assume that $e^{2\,u_{k-1}} \in L^1(\Omega)$. Further, let $w \in H^1(\Omega)$ be given and consider the problem: Find $u \in H_0^2(\Omega)$ with

$$2 \int_\Omega e^{2\,w}\, u_{xx}\, \phi_{xx}\, dx + \frac{1}{\tau_k} \int_\Omega e^{2\,u}\, \phi\, dx = \frac{1}{\tau_k} \int_\Omega e^{2\,u_{k-1}}\, \phi\, dx, \tag{2.8}$$

for all $\phi \in H_0^2(\Omega)$.

On account of standard results from the theory of monotone operators [Zei90], there exists a unique weak solution $u \in H_0^2(\Omega)$ of (2.8). Thus, the fixed point map $T : H^1(\Omega) \to H^1(\Omega)$, given by $T(w) = u$, is well defined.

Using $\phi = u$ as test function in (2.8) and employing Hölder's inequality yields the estimate

$$\begin{aligned} 2\, \exp\Big(-2\, \|w\|_{L^\infty(\Omega)}\Big) \int_\Omega u_{xx}^2\, dx + \frac{1}{\tau_k} \int_\Omega \underbrace{e^{2\,u}\, u + 1}_{\geq 0}\, dx & \\ \leq \frac{1}{\tau_k}\, \big\|e^{2\,u_{k-1}}\big\|_{L^1(\Omega)}\, \|u\|_{L^\infty(\Omega)} + \frac{1}{\tau_k} & \end{aligned}$$

and by Young's inequality

$$\leq \frac{1}{2\,\eta\,\tau_k^2}\, \big\|e^{2\,u_{k-1}}\big\|_{L^1(\Omega)}^2 + \frac{\eta}{2}\, \|u\|_{L^\infty(\Omega)}^2 + \frac{1}{\tau_k},$$

for $\eta \in \mathbb{R}^+$. Employing the embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, Poincaré's inequality and choosing η small enough we get

$$\|u\|_{H^2(\Omega)} \leq C\, \exp\Big(2\, \|w\|_{L^\infty(\Omega)}\Big), \tag{2.9}$$

for some positive constant $C = C\Big(\Omega, \tau_k, \|e^{2\,u_{k-1}}\|_{L^1(\Omega)}\Big)$. Due to the embedding $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ and the compact embedding $H^2(\Omega) \hookrightarrow H^1(\Omega)$, we get the compactness of T. Note that Eq. (2.9) readily implies the continuity of T, since for some sequence $(w_n)_{n\in\mathbb{N}}$ converging to w in $H^1(\Omega)$ as $n \to \infty$, we have that $\|w_n\|_{L^\infty(\Omega)}$ is uniformly bounded. Hence, the sequence $(u_n)_{n\in\mathbb{N}}$ defined by $u_n = T(w_n)$ is bounded in $H^2(\Omega)$ such that there exists a weakly convergent subsequence, again denoted by $(u_n)_{n\in\mathbb{N}}$, with $u_n \rightharpoonup u$ in $H^2(\Omega)$ as $n \to \infty$. This is sufficient to pass to the limit in (2.8) by which we identify $u = T(w)$. Now the existence of at least one solution follows from Schauder's fixed point theorem.

To prove uniqueness of solutions we make use of Lemma 2.5. Assume that there exist two solutions $u, v \in H_0^2(\Omega)$ of

$$2\, \left(e^{2\,u}\, u_{xx}\right)_{xx} + \frac{1}{\tau_k}\, e^{2\,u} = \frac{1}{\tau_k}\, e^{2\,u_k}.$$

Since u and v are bounded in $L^\infty(\Omega)$ we can divide the corresponding equations for u and v by e^u and e^v, respectively. Using $\phi = e^u - e^v \in H_0^2(\Omega)$ (see the proof of Lemma 2.5) as test function for the difference of the equations yields

$$\begin{aligned} -2\,\langle A(u) - A(v), e^u - e^v\rangle_{H^{-2},H_0^2} + \frac{1}{\tau_k}\int_\Omega (e^u - e^v)^2\,dx \\ &= \frac{1}{\tau_k}\int_\Omega e^{2\,u_k}\left(e^{-u} - e^{-v}\right)(e^u - e^v)\,dx \\ &\le 0, \end{aligned}$$

due to $(1/x - 1/y)(x-y) \le 0$ for $x, y \in \mathbb{R}^+$. Further, the monotonicity property (2.7) implies the non–negativity of the first term on the left–hand side. Hence, we obtain

$$\int_\Omega (e^u - e^v)^2\,dx \le 0.$$

Thus, $e^u = e^v$ in $L^2(\Omega)$ and finally $u \equiv v$, which settles the uniqueness of solutions. □

2.2 A Priori Estimates

In this section we derive a priori estimates on the sequence of approximate solutions $\left(u^{(N)}\right)_{N\in\mathbb{N}}$, which is generated by the semidiscretization. First, we show an energy type inequality and and some bound on the entropy.

Lemma 2.6. *Let u_0 satisfy (2.1). For $k = 1, \ldots, N$ let $u_k \in H_0^2(\Omega)$ be the recursively defined solution of (2.6) and $u^{(N)} \in PC_N(0,T; H_0^2(\Omega))$. Then $u^{(N)} \in L^2(0,T; H_0^2(\Omega))$ and there exists a positive constant c, independent of k and N, such that*

$$\int_\Omega e^{2\,u_k} - 2\,u_k\,dx + 4\,\tau_k\,\|u_{k,xx}\|^2_{L^2(\Omega)} \le c, \tag{2.10a}$$

$$\sup_{t\in[0,T]}\int_\Omega e^{2\,u^{(N)}(t,x)} - 2\,u^{(N)}(t,x)\,dx + 4\,\left\|u^{(N)}\right\|^2_{L^2(H^2)} \le c. \tag{2.10b}$$

Additionally, let u_0 satisfy $\int_\Omega e^{2\,u_0}\,(2\,u_0 - 1) + 1\,dx < +\infty$. Then it also holds

$$\int_\Omega e^{2\,u_k}\,(2\,u_k - 1) + 1\,dx \le \int_\Omega e^{2\,u_{k-1}}\,(2\,u_{k-1} - 1) + 1\,dx, \tag{2.10c}$$

for $k = 1, \ldots, N$.

Proof. Let $k \in \{1, \dots, N\}$ be fixed and use $\phi = 1 - e^{-2\,u_k}$ as test function in (2.5). Note that ϕ is suitable, since $u_k \in L^\infty(\Omega)$. Integration by parts yields

$$\frac{1}{\tau_k} \int_\Omega e^{2\,u_k} - e^{2\,u_{k-1}} + e^{2(u_{k-1}-u_k)} - 1 \; dx = 2 \int_\Omega e^{2\,u_k}\, u_{k,xx}\, \left(e^{-2\,u_k}\right)_{xx} \; dx.$$

Using the well known inequality $e^x \geq 1 + x$, for $x \in \mathbb{R}$, we get

$$\frac{1}{\tau_k} \int_\Omega 2\,u_{k-1} - 2\,u_k + e^{2\,u_k} - e^{2\,u_{k-1}} \; dx \\ \leq -4 \int_\Omega |u_{k,xx}|^2 \; dx + 8 \int_\Omega u_{k,xx}\, u_{k,x}^2 \; dx.$$

By Young's inequality and the fact that

$$\int_\Omega u_{k,xx}\, u_{k,x}^2 \; dx = \frac{1}{3} \int_\Omega \left(u_{k,x}^3\right)_x \; dx = \frac{1}{3} \left(u_{k,x}^3(1) - u_{k,x}^3(0)\right) = 0$$

we derive

$$\int_\Omega e^{2\,u_k} - 2\,u_k \; dx + 4\,\tau_k \int_\Omega |u_{k,xx}|^2 \; dx \leq \int_\Omega e^{2\,u_{k-1}} - 2\,u_{k-1} \; dx.$$

Thus, consecutively we get

$$\int_\Omega e^{2\,u_k} - 2\,u_k \; dx \leq \int_\Omega e^{2\,u_{k-1}} - 2\,u_{k-1} \; dx \leq \ldots \leq \int_\Omega e^{2\,u_0} - 2\,u_0 \; dx,$$

from which (2.10a) follows. Furthermore, summation with respect to k yields

$$\int_\Omega e^{2\,u_k} - 2\,u_k \; dx + 4 \sum_{l=1}^{k} \tau_l \int_\Omega |u_{l,xx}|^2 \; dx \leq \int_\Omega e^{2\,u_0} - 2\,u_0 \; dx,$$

which gives the desired estimate on $u^{(N)}$.
Now, choosing $\phi = u_k$ as test function in (2.6) we obtain

$$\frac{1}{\tau_k} \int_\Omega \left(e^{2\,u_k} - e^{2\,u_{k-1}}\right)\, u_k \; dx = -2 \int_\Omega e^{2\,u_k}\, u_{k,xx}^2 \; dx. \tag{2.11}$$

Again, employing $e^x \geq 1 + x$ for $x \in \mathbb{R}$, we deduce

$$\begin{aligned} \frac{2}{\tau_k} \int_\Omega \left(e^{2\,u_k} - e^{2\,u_{k-1}}\right)\, u_k \; dx &= \frac{2}{\tau_k} \int_\Omega e^{2\,u_k} \left(u_k - \frac{1}{2}\right) - e^{2\,u_{k-1}} \left(u_{k-1} - \frac{1}{2}\right) \; dx \\ &\quad + \frac{1}{\tau_k} \int_\Omega e^{2\,u_{k-1}}\, \underbrace{\left(e^{2\,(u_k - u_{k-1})} - 1 - 2\,(u_k - u_{k-1})\right)}_{\geq 0} \; dx \\ &\geq \frac{2}{\tau_k} \int_\Omega e^{2\,u_k} \left(u_k - \frac{1}{2}\right) - e^{2\,u_{k-1}} \left(u_{k-1} - \frac{1}{2}\right) \; dx. \end{aligned}$$

This inequality together with (2.11) immediately implies

$$\int_\Omega e^{2\,u_k}\,(2\,u_k - 1)\;dx \le \int_\Omega e^{2\,u_{k-1}}\,(2\,u_{k-1} - 1)\;dx,$$

from which we obtain (2.10c). $\square$

As a consequence of interpolation theory we derive the following estimates.

Corollary 2.7. *Let u_0 satisfy (2.1) and for $N \in \mathbb{N}$ let $u^{(N)} \in PC_N(0,T;H_0^2(\Omega))$ be the approximate solution. Then*

$$u^{(N)} \in L^\infty(0,T;L^1(\Omega)) \cap L^{5/2}(0,T;W^{1,\infty}(\Omega)), \quad e^{2\,u^{(N)}} \in L^{5/2}(0,T,W^{1,1}(\Omega))$$

and there exists a constant $c > 0$, independent of N, such that

$$\left\|u^{(N)}\right\|_{L^\infty(L^1)} \le c, \quad \left\|u^{(N)}\right\|_{L^{5/2}(W^{1,\infty})} \le c, \quad \left\|e^{2\,u^{(N)}}\right\|_{L^{5/2}(W^{1,1})} \le c.$$

Proof. Using Taylor's expansion we have $e^x \ge 1 + x + x^2$ for $x \ge 0$, which yields

$$\begin{aligned}\int_\Omega e^{2\,u^{(N)}} - 2\,u^{(N)}\;dx &\ge \int_\Omega e^{2\left(u^{(N)}\right)^+} - 2\,\left(u^{(N)}\right)^+ + 2\,\left(u^{(N)}\right)^-\;dx\\ &= \int_\Omega 4\,\left(\left(u^{(N)}\right)^+\right)^2 + 2\,\left(u^{(N)}\right)^- + 1\;dx,\end{aligned}$$

where $u^+ = \max(0,u)$ and $u^- = -\min(0,u)$. This estimate immediately implies $(u^{(N)})^-(t) \in L^1(\Omega)$ and $(u^{(N)})^+(t) \in L^2(\Omega)$ for any $t > 0$. Thus, $u^{(N)}(t) \in L^1(\Omega)$ and it holds

$$\left\|u^{(N)}\right\|_{L^\infty(L^1)} \le c,$$

where $c = c(\Omega, u_0) > 0$ is independent of N.

To show the second inequality we use Lemma 1.3 with $m = r = 2$, $j = 1$, $p = \infty$ and $q = 1$, yielding

$$\left\|u_x^{(N)}\right\|_{L^\infty(\Omega)} \le c\,\left\|u^{(N)}\right\|_{H^2(\Omega)}^{4/5}\,\left\|u^{(N)}\right\|_{L^1(\Omega)}^{1/5},$$

which gives

$$\left\|u_x^{(N)}\right\|_{L^{5/2}(L^\infty)} \le c\,\left\|u^{(N)}\right\|_{L^2(H^2)}^{4/5}\,\left\|u^{(N)}\right\|_{L^\infty(L^1)}^{1/5}.$$

Now the assertion follows from Lemma 2.6 and the previous inequality.

From Lemma 2.6 and the first inequality we get

$$\left\|e^{2\,u^{(N)}}\right\|_{L^\infty(L^1)} \le c.$$

Hence,

$$\left\| \left(e^{2\,u^{(N)}} \right)_x \right\|_{L^{5/2}(L^1)} \leq \left\| e^{2\,u^{(N)}} \right\|_{L^\infty(L^1)} \|u_x\|_{L^{5/2}(L^\infty)}$$

and the third inequality follows from the second one. $\square$

The next two lemmata provide the estimates, which are necessary for the compactness arguments.

Lemma 2.8. *Let u_0 satisfy (2.1) and for $N \in \mathbb{N}$ let $u^{(N)} \in PC_N(0,T;H_0^2(\Omega))$ be the approximate solution. Choose $p \in (1, 4/3)$ and fix $q \in (2, 5/2)$ such that $1/q = 2(2-1/p)/5$. Then $e^{2\,u^{(N)}} \in L^q(0,T;W^{1,p}(\Omega))$ and there exists a constant $c > 0$, independent of N, such that*

$$\left\| e^{2\,u^{(N)}} \right\|_{L^q(W^{1,p})} \leq c.$$

Proof. Again, employing Lemma 1.3 we derive

$$\left\| e^{2\,u^{(N)}} \right\|_{L^p(\Omega)} \leq c \left\| e^{2\,u^{(N)}} \right\|_{W^{1,1}(\Omega)}^{\frac{p-1}{p}} \left\| e^{2\,u^{(N)}} \right\|_{L^1(\Omega)}^{\frac{1}{p}}.$$

Furthermore, it holds

$$\frac{1}{q} = \frac{2\,(p-1)}{5\,p} + \frac{2}{5}$$

and thus Hölder's inequality implies

$$\left\| \left(e^{2\,u^{(N)}} \right)_x \right\|_{L^q(L^p)} \leq c \left\| e^{2\,u^{(N)}} \right\|_{L^\infty(L^1)}^{\frac{1}{p}} \left\| e^{2\,u^{(N)}} \right\|_{L^{5/2}(W^{1,1})}^{\frac{p-1}{p}} \left\| u_x^{(N)} \right\|_{L^{5/2}(L^\infty)}.$$

Hence, the assertion follows from Corollary 2.7 and (2.10c). $\square$

As we want to employ compactness results, we also need some regularity on the time derivative. We introduce the linear interpolant of $e^{2\,u^{(N)}} \in PC_N(0,T,L^2(\Omega))$, defined by

$$\tilde{e}^{(N)}(t,x) \stackrel{\text{def}}{=} \frac{t - t_{k-1}}{\tau_k} \left(e^{2\,u_k(x)} - e^{2\,u_{k-1}(x)} \right) + e^{2\,u_{k-1}(x)}, \quad x \in \Omega, \quad t \in (t_{k-1}, t_k].$$

Lemma 2.9. *Let the assumptions of Lemma 2.8 hold. Choose $r \in (1, 10/9)$ such that $1/r = 1/2 + 1/q$. Then*

$$\tilde{e}_t^{(N)} \in L^r(0,T;H^{-2}(\Omega)), \quad \tilde{e}^{(N)} \in L^q(0,T;W^{1,p}(\Omega))$$

and there exists a constant $c > 0$, independent of N, such that

$$\left\| \tilde{e}_t^{(N)} \right\|_{L^r(H^{-2})} + \left\| \tilde{e}^{(N)} \right\|_{L^q(W^{1,p})} \leq c.$$

Proof. We introduce the solution operator $\Phi : H^{-2}(\Omega) \to H_0^2(\Omega)$, $f \mapsto \Phi[f]$ by $\Phi[f]_{xxxx} = f$. Then $\|\Phi[f]_{xx}\|_{L^2(\Omega)}$ defines a norm on $H^{-2}(\Omega)$ [Zei90]. Further, $\Phi\left[\frac{e^{2u_k}-e^{2u_{k-1}}}{\tau_k}\right]$ is an appropriate test function in (2.6), which yields

$$\int_\Omega \frac{e^{2\,u_k} - e^{2\,u_{k-1}}}{\tau_k}\,\Phi\left[\frac{e^{2\,u_k} - e^{2\,u_{k-1}}}{\tau_k}\right]\,dx = -2\int_\Omega e^{2\,u_k}\,u_{k,xx}\,\Phi\left[\frac{e^{2\,u_k} - e^{2\,u_{k-1}}}{\tau_k}\right]_{xx}\,dx$$

and using integration by parts

$$\int_\Omega \left|\Phi\left[\frac{e^{2\,u_k} - e^{2\,u_{k-1}}}{\tau_k}\right]_{xx}\right|^2\,dx \le 2\left\|e^{2\,u_k}\right\|_{L^\infty(\Omega)}\left\|u_{k,xx}\right\|_{L^2(\Omega)}\left\|\Phi\left[\frac{e^{2\,u_k} - e^{2\,u_{k-1}}}{\tau_k}\right]_{xx}\right\|_{L^2(\Omega)}.$$

Thus, we finally can estimate

$$\left\|\Phi\left[\frac{e^{2\,u_k} - e^{2\,u_{k-1}}}{\tau_k}\right]_{xx}\right\|_{L^2(\Omega)} \le 2\left\|e^{2\,u_k}\right\|_{L^\infty(\Omega)}\left\|u_k\right\|_{H^2(\Omega)}.$$

Now, we deduce from Hölder's inequality

$$\begin{aligned}\left\|\tilde{e}_t^{(N)}\right\|^r_{L^r(H^{-2})} &= \sum_{k=1}^N \tau_k \left\|\frac{e^{2\,u_k} - e^{2\,u_{k-1}}}{\tau_k}\right\|^r_{H^{-2}(\Omega)} \\ &\le 2^r \sum_{k=1}^N \tau_k \left\|e^{2\,u_k}\right\|^r_{L^\infty(\Omega)}\left\|u_k\right\|^r_{H^2(\Omega)} \\ &\le 2^r \left\|e^{2\,u^{(N)}}\right\|^r_{L^q(L^\infty)}\left\|u^{(N)}\right\|^r_{L^2(H^2)},\end{aligned}$$

from which we get the uniform boundedness of $\tilde{e}_t^{(N)}$ in $L^r(0,T;H^{-2}(\Omega))$ by Lemma 2.8 together with the embedding $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ and Lemma 2.6.

Further, we note that for arbitrary $t \in (t_{k-1}, t_k]$ it holds

$$0 \le \frac{t - t_{k-1}}{\tau_k} \le 1$$

such that (see Lemma 2.8)

$$\begin{aligned}\left\|\tilde{e}^{(N)}(t)\right\|_{W^{1,p}(\Omega)} &\le \left(1 - \frac{t - t_{k-1}}{\tau_k}\right)\left\|e^{2\,u_{k-1}}\right\|_{W^{1,p}(\Omega)} + \frac{t - t_{k-1}}{\tau_k}\left\|e^{2\,u_k}\right\|_{W^{1,p}(\Omega)} \\ &\le c.\end{aligned}$$

Thus, we obtain

$$\int_0^T \left\|\tilde{e}^{(N)}(t)\right\|_{W^{1,p}(\Omega)}^q \; dt \leq T\, c^q,$$

finishing the proof. □

2.3 Proof of the Existence Result

Now we are in the position to prove Proposition 2.1.

Proof. We choose a sequence of partitions of $[0,T]$ satisfying (2.4). Taking into account Lemma 2.6, it follows that the sequence $\left(u^{(N)}\right)_{N\in\mathbb{N}}$ is bounded in $L^2(0,T;H^2(\Omega))$. Thus, there exists a subsequence, again denoted by $\left(u^{(N)}\right)_{N\in\mathbb{N}}$, such that

$$u^{(N)} \rightharpoonup u \quad \text{weakly in } L^2(0,T;H^2(\Omega)) \text{ as } N\to\infty.$$

Furthermore, from Lemma 2.8 and Lemma 2.9 we deduce the boundedness of $\left(\tilde{e}^{(N)}\right)_{N\in\mathbb{N}}$ in $L^q(0,T;W^{1,p}(\Omega))\cap W^{1,r}(0,T;H^{-2}(\Omega))$, where p,q,r are specified therein. Since the embedding $W^{1,p}(\Omega)\hookrightarrow L^\infty(\Omega)$ is compact for $p\in(1,4/3)$, it follows from Aubin's Lemma [Sim87] that

$$L^q(0,T;W^{1,p}(\Omega))\cap W^{1,r}(0,T;H^{-2}(\Omega)) \hookrightarrow L^q(0,T;L^\infty(\Omega)) \quad \text{compactly.}$$

Hence, there exists a subsequence, not relabeled, such that

$$\tilde{e}^{(N)} \to \rho \quad \text{strongly in } L^q(0,T;L^\infty(\Omega)) \text{ for } N\to\infty.$$

As $q>2$ it also holds $\tilde{e}^{(N)}\to\rho$ in $L^2(0,T;L^2(\Omega))$ for $N\to\infty$. Note that $\tilde{e}^{(N)}\to\rho$ in $L^2(0,T;L^2(\Omega))$ implies $e^{2\,u^{(N)}}\to\rho$ in $L^2(0,T;L^2(\Omega))$ as $N\to\infty$ (see [Rek82, pp. 205]). Due to the monotonicity of the exponential function we have for all $v\in L^\infty((0,\infty)\times\Omega)$

$$\int_\Omega \left(e^{2\,u^{(N)}} - e^{2\,v}\right)\left(u^{(N)} - v\right)\; dx \geq 0.$$

The derived convergence properties are by far sufficient to pass to the limit in this inequality, which yields

$$\int_\Omega \left(\rho - e^{2\,v}\right)(u-v)\; dx \geq 0,$$

for all $v\in L^\infty((0,\infty)\times\Omega)$. Again, the monotonicity of the exponential implies $\rho = e^{2\,u}$.

After this identification we can perform the limit in the weak formulation, which reads

$$\int_0^T \left\langle \tilde{e}_t^{(N)}, \phi \right\rangle_{H^{-2},H_0^2} dt = -2 \int_0^T \int_\Omega e^{2\,u^{(N)}}\, u_{xx}^{(N)}\, \phi_{xx}\, dx\, dt, \tag{2.12}$$

for all $\phi \in L^{r'}(0,T;H_0^2(\Omega))$ with $1/r + 1/r' = 1$. One easily verifies that the following convergence properties are sufficient to pass to the limit in (2.12)

$$\begin{aligned}
\tilde{e}_t^{(N)} &\rightharpoonup \left(e^{2\,u}\right)_t && \text{weakly in } L^r(0,T;H^{-2}(\Omega)),\\
e^{2\,u^{(N)}} &\to e^{2\,u} && \text{strongly in } L^q(0,T;L^\infty(\Omega)),\\
u_{xx}^{(N)} &\rightharpoonup u_{xx} && \text{weakly in } L^2(0,T;L^2(\Omega)),
\end{aligned}$$

as $N \to \infty$, which proves our main result. $\square$

3 Additional Results and Discussion

In this section we present some additional results, concerning the uniqueness of solutions to (1.2), their long–time behaviour and regularity properties. Further, we shortly discuss the multi–dimensional problem and give some numerical examples.

3.1 Uniqueness

As already pointed out, Bleher *et al.* proved uniqueness of solutions to (1.2) assuming strictly positive $H^1(\Omega)$ initial data [BLS94]. We extend this result to non–smooth positive data.

Proposition 3.1. *Let* $\log(n_0) \in L^\infty(\Omega)$. *Then there exists a unique solution to* (1.2) *in the class of functions satisfying* (1.4) *and*

$$n \in L^2_{loc}(0,\infty;H^2(\Omega)), \quad \log(n) \in L^\infty_{loc}(0,\infty;L^\infty(\Omega)). \tag{3.1}$$

Proof. Let n_1, n_2 be two solutions to (1.2) satisfying the regularity conditions (1.4) and (3.1). Then, n_i solves

$$\left(\sqrt{n_i}\right)_t = -\frac{1}{2\sqrt{n_i}} \left(n_i \left(\log(n_i)\right)_{xx}\right)_{xx}, \quad i = 1, 2. \tag{3.2}$$

Multiplying the difference of the equations (3.2) for $i = 1, 2$ by $\phi = \sqrt{n_1} - \sqrt{n_2}$, we obtain for $t_1 > 0$

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \left|\left(\sqrt{n_1} - \sqrt{n_2}\right)(t_1)\right|^2\, dx\\
&\qquad = \int_0^{t_1} \int_\Omega \langle A(\sqrt{n_1}) - A(\sqrt{n_2}), \sqrt{n_1} - \sqrt{n_2} \rangle_{H^{-2},H_0^2}\, dx\, dt,
\end{aligned}$$

where

$$A(\rho) = -\frac{1}{2\,\rho}\left(\rho^2\,(\log(\rho))_{xx}\right)_{xx}, \qquad \text{for appropriate } \rho.$$

Due to Lemma 2.5, the right–hand side of the above equation is non–positive. This implies $\sqrt{n_1(t)} = \sqrt{n_2(t)}$ in Ω, for $t > 0$. $\square$

3.2 Long–Time Behaviour

The next result states that the solution converges for $t \to \infty$ to the stationary state $n_\infty \equiv 1$ in some weak sense.

Proposition 3.2. *Assume* (1.3) *and let* $n \in L^2_{loc}(0,\infty,W^{1,1}(\Omega))$ *be a solution to* (1.2)*. Then there exists a sequence* $(t_m)_{m\in\mathbb{N}}$ *with* $t_m \to \infty$ *such that*

$$\|\log\left(n\left(t_m\right)\right)\|_{H^2(\Omega)} \to 0, \qquad \text{as } t_m \to \infty.$$

Moreover, if it holds $\int_\Omega n_0\,(\log(n_0)-1)+1\;dx < +\infty$ *then we have*

$$\log(n) \in L^2_{loc}(0,\infty,H^2_0(\Omega)) \cap L^\infty(0,\infty;L^2(\Omega))$$

and it holds

$$\|\log\left(n\left(t\right)\right)\|_{L^2(\Omega)} \to 0, \qquad \text{as } t \to \infty.$$

Proof. Let $n = e^{2\,u}$. From the proof of Lemma 2.6 follows that

$$\int_0^\infty \int_\Omega u_{xx}^2\;dx\,dt \le \int_\Omega e^{2\,u_0} - 2\,u_0\;dx.$$

Therefore, using Poincaré's inequality, there exists a sequence $(t_m)_{m\in\mathbb{N}}$ with $t_m \to \infty$ such that

$$\|u(t_m)\|_{H^2(\Omega)} \to 0 \quad \text{ as } t_m \to \infty. \tag{3.3}$$

Now let $u_0 = \log\left(\sqrt{n_0}\right)$ satisfy $\int_\Omega e^{2\,u_0}\,(2\,u_0-1)+1\;dx < +\infty$. We introduce the discrete entropy

$$S^{(N)}(t) = \int_\Omega e^{2\,u^{(N)}(t)}\left(2\,u^{(N)}(t)-1\right)+1\;dx, \quad t \ge 0.$$

This function is well defined for all $t \ge 0$, non–negative and uniformly bounded from above. From (2.10c) immediately follows for $t_2 \ge t_1 \ge 0$ that

$$S^{(N)}(t_2) \le S^{(N)}(t_1),$$

which shows that $S^{(N)}$ is non–increasing with respect to t. As in the proof of the main theorem we can pass to the limit and get the *entropy*

$$S(t) = \lim_{N\to\infty} S^{(N)}(t) = \int_\Omega e^{2\,u(t)}\,(2\,u(t) - 1) + 1\;dx, \quad t \geq 0.$$

The inequality $x \leq e^x - 1$ for all $x \in \mathbb{R}$ implies on the one hand

$$S(t) \leq \int_\Omega e^{2\,u(t)}\left(e^{2\,u(t)} - 2\right) + 1\;dx = \int_\Omega \left(e^{2\,u(t)} - 1\right)^2\;dx,$$

on the other hand

$$S(t) \geq \int_\Omega (2\,u(t) + 1)\,(2\,u(t) - 1) + 1\;dx = 4\,\|u(t)\|^2_{L^2(\Omega)}\,. \tag{3.4}$$

The result (3.3) implies convergence of $u(t_m)$ to zero in $L^\infty(\Omega)$ and hence

$$0 \leq S(t_m) \leq \int_\Omega \left(e^{2\,u(t_m)} - 1\right)^2\;dx \to 0 \quad \text{ as } t_m \to \infty.$$

Thus, $S_\infty \overset{\text{def}}{=} \lim_{t\to\infty} S(t) = 0$. Finally, by (3.4) we get

$$\lim_{t\to\infty} \|u(t)\|^2_{L^2(\Omega)} \leq \frac{1}{4} \lim_{t\to\infty} S(t) = 0,$$

which proves the proposition. □

3.3 Regularity

Now we investigate the regularity of solutions.

Proposition 3.3. *Assume* (1.3). *If it holds*

$$\int_\Omega n_0\,(\log(n_0) - 1) + 1\;dx < +\infty$$

then any solution $n \in L^2_{loc}(0,\infty;W^{1,1}(\Omega))$ *to* (1.2) *with*

$$\log(n) \in L^2_{loc}(0,\infty;H^2_0(\Omega)) \cap L^\infty(0,\infty;L^2(\Omega))$$

even fulfills

$$n \in L^{16/15}_{loc}(0,\infty;H^2(\Omega)).$$

Remark 3.4. Notice that Theorem 1.1 and Proposition 3.2 ensure the existence of a solution to (1.2) with the desired regularity properties.

Proof. The proof is an easy consequence of the results derived so far combined with the Gagliardo–Nirenberg inequality. Let $n = e^{2u}$. In the following c denotes positive, but not necessarily identical, constants. We estimate

$$\begin{aligned}\left\| \left(e^{2u}\right)_{xx} \right\|_{L^2(\Omega)} &\le c \left(\left\| e^{2u} \right\|_{L^\infty(\Omega)} \|u\|_{H^2(\Omega)} + \left\| e^{2u}\, u_x^2 \right\|_{L^2(\Omega)} \right) \\ &= c \left(\left\| e^{2u} \right\|_{L^\infty(\Omega)} \|u\|_{H^2(\Omega)} + \left\| (e^u)_x \right\|_{L^4(\Omega)}^2 \right).\end{aligned}$$

Due to $u \in L^2_{loc}(0,\infty; H^2_0(\Omega)) \cap L^\infty(0,\infty; L^2(\Omega))$ and $e^{2u} \in L^2_{loc}(0,\infty; W^{1,1}(\Omega))$ it holds (compare Lemma 2.7)

$$\left\| e^{2u} \right\|_{L^{8/3}(L^\infty)} \le c \quad \text{ and } \|u\|_{L^{8/3}(W^{1,\infty})} \le c.$$

Further, we have $e^u \in L^{4/3}_{loc}(0,\infty; H^2(\Omega))$, since

$$\|(e^u)_{xx}\|_{L^2(\Omega)} \le \|e^u\|_{L^\infty(\Omega)} \|u_{xx}\|_{L^2(\Omega)} + \|e^u\|_{L^2(\Omega)} \|u_x\|_{L^\infty(\Omega)}^2$$

and by multiple use of Hölder's inequality

$$\|(e^u)_{xx}\|_{L^{4/3}(L^2)} \le \|e^u\|_{L^4(L^\infty)} \|u_{xx}\|_{L^2(L^2)} + \|e^u\|_{L^\infty(L^2)} \|u_x\|_{L^{8/3}(L^\infty)}^2,$$

which is finite. Now we deduce by

$$\|e^u\|_{W^{1,4}(\Omega)} \le c \, \|e^u\|_{L^2(\Omega)}^{3/8} \, \|e^u\|_{H^2(\Omega)}^{5/8}$$

that $e^u \in L^{32/15}_{loc}(0,\infty; W^{1,4}(\Omega))$. Hence, we get finally

$$\left\| e^{2u} \right\|_{L^{16/15}(H^2)} \le c \left(\left\| e^{2u} \right\|_{L^{16/7}(L^\infty)} \|u\|_{L^2(H^2)} + \|e^u\|_{L^{32/15}(W^{1,4})}^2 \right).$$

□

3.4 The Multi–dimensional Problem: Discussion

It seems difficult to extend our technique to the multi–dimensional problem introduced in Section 1 and which reads after the exponential transformation of variables:

$$\begin{gathered}\left(e^{2u}\right)_t = -2 \sum_{i,j=1}^{d} \partial_i \partial_j \left(e^{2u}\, \partial_i \partial_j u \right) \quad \text{ in } \Omega, \\ u = \nabla u \cdot \nu = 0 \quad \text{ on } \partial\Omega, \\ u(\cdot, 0) = u_0 \quad \text{ in } \Omega,\end{gathered}$$

where $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, is a bounded domain.

Indeed, our key estimate (2.10a) is obtained by using the test function $\phi = 1 - e^{-u_k}$ in the weak formulation of the discrete problem

$$\frac{1}{\tau_k}\left(e^{2\,u_k} - e^{2\,u_{k-1}}\right) = -2\sum_{i,j=1}^{d} \partial_i\partial_j\left(e^{2\,u_k}\,\partial_i\partial_j u_k\right) \qquad \text{in } \Omega,$$
$$u_k = \nabla u_k \cdot \nu = 0 \qquad \text{on } \partial\Omega.$$

This leads to

$$\frac{1}{\tau_k}\int_\Omega \left(e^{2\,u_k} - 2\,u_k\right)\,dx + 2\sum_{i,j=1}^{d}\int_\Omega \left|\partial_i\partial_j u_k\right|^2\,dx$$
$$\le \frac{1}{\tau_k}\int_\Omega \left(e^{2\,u_{k-1}} - 2\,u_{k-1}\right)\,dx + 4\sum_{i,j=1}^{d}\int_\Omega \partial_i\partial_j u_k\,\partial_i u_k\,\partial_j u_k\,dx.$$

The last integral can be reformulated by the integration by parts formula of [dPGG98, Lemma 2.3], employing the Neumann condition $\nabla u_k \cdot \nu = 0$ on $\partial\Omega$:

$$\begin{aligned}
4\sum_{i,j=1}^{d}\int_\Omega \partial_i\partial_j u_k\,\partial_i u_k\,\partial_j u_k\,dx &= 2\sum_{i,j=1}^{d}\int_\Omega \partial_j\left(\partial_i u_k\right)^2\partial_j u_k\,dx \\
&= -2\int_\Omega \left|\nabla u_k\right|^2\Delta u_k\,dx \\
&= -\frac{4}{3}\int_\Omega u_k\left(\left|D^2 u_k\right|^2 - \left|\Delta u_k\right|^2\right)\,dx \\
&\quad -\frac{4}{3}\int_{\partial\Omega} u_k\,II\left(\nabla u_k\right)\,ds,
\end{aligned}$$

where $II\left(\nabla u_k\right)$ denotes the second fundamental form of $\partial\Omega$. Notice that in the one–dimensional case, the right–hand side vanishes. In the general case, however, it is not clear how to estimate the above two integrals. In context of the thin film equations (1.1) they can be handled, which is due to the fact that these equations are of degenerate type. But in our case the estimates of [dPGG98] cannot be used.

In a forthcoming paper [JP99] we study the multi–dimensional problem employing different techniques, applied to the so–called quantum drift diffusion model [Pin99a].

3.5 Numerical Examples

After the analytical discussion of problem (1.2) we present some numerical results that do not only underline the preservation of non–negativity by the solution n.

They are also indicating that the solution is *positive* for $t > 0$, even for initial data, which vanishes at some point $x_0 \in \Omega = (0,1)$. This behaviour was already pointed out in [BLS94], but only for strictly positive initial data.

For the numerical experiments we chose the initial datum

$$n_0(x) = \cos^{2m}(\pi\, x), \quad x \in (0,1), \tag{3.5}$$

with $m = 1$ or 8 and which is compatible with the boundary data. Note that n_0 vanishes at $x_0 = 1/2$ such that $\log(n_0)$ has there a singularity. But still it holds $\log(n_0) \in L^1(\Omega)$.

For the computations we employed the iteration defined by the semidiscretization in Proposition 2.4 with a uniform time step $\tau = 10^{-8}$. Moreover, we chose a suitable space discretization such that $x_0 = 1/2$ is included in the set of nodes. The discretized nonlinear equations were solved by a Newton–iteration, which proved to be very robust such that no damping was necessary.

Figure 3.1 shows the evolution of the initial datum with $m = 1$. Note, that we use a logarithmic scale for the ordinate such that we cut–off all values less than 10^{-8}. Here, the solution moves very fast away from zero and converges monotonically to the stationary state $n_\infty \equiv 1$. To contrast this behaviour we refer to Figure 3.2, which shows the evolution for $m = 8$. Starting with *one* higher order extremum the minimum bifurcates and reduces to one extremum again. We emphasize that also in this case the solution stays strictly positive for $t > 0$, although the evolution is not monotone anymore. Again, analogous results are reported in [BLS94] for strictly positive initial data.

From a numerical point of view the preservation of positivity for $t > 0$ is clear, since —for a *fixed* maximal time step τ— Proposition 2.4 assures the positivity at each time step. But also for decreasing $\tau \to 0^+$ this behaviour did not change.

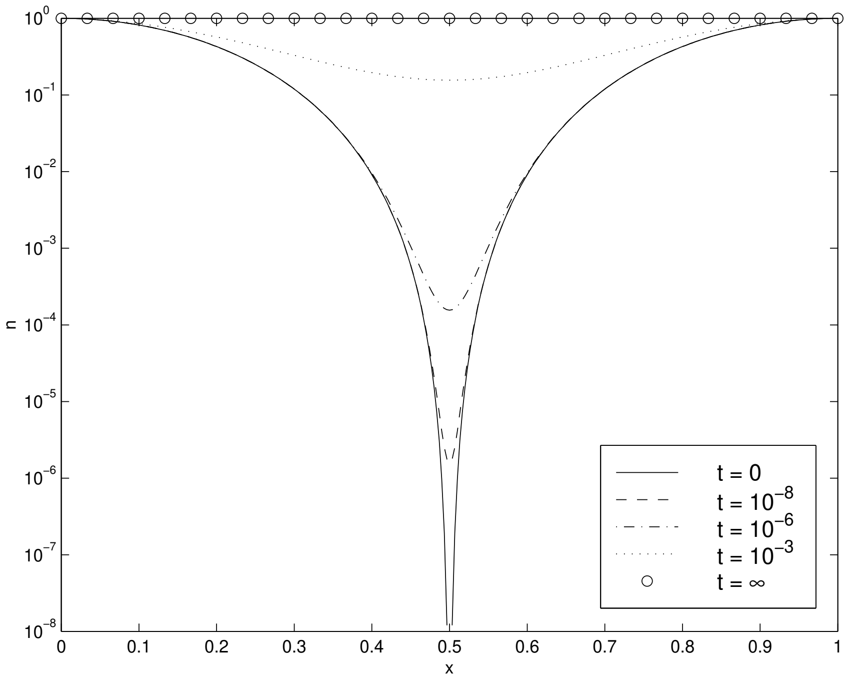

Figure 3.1: Evolution for $m = 1$

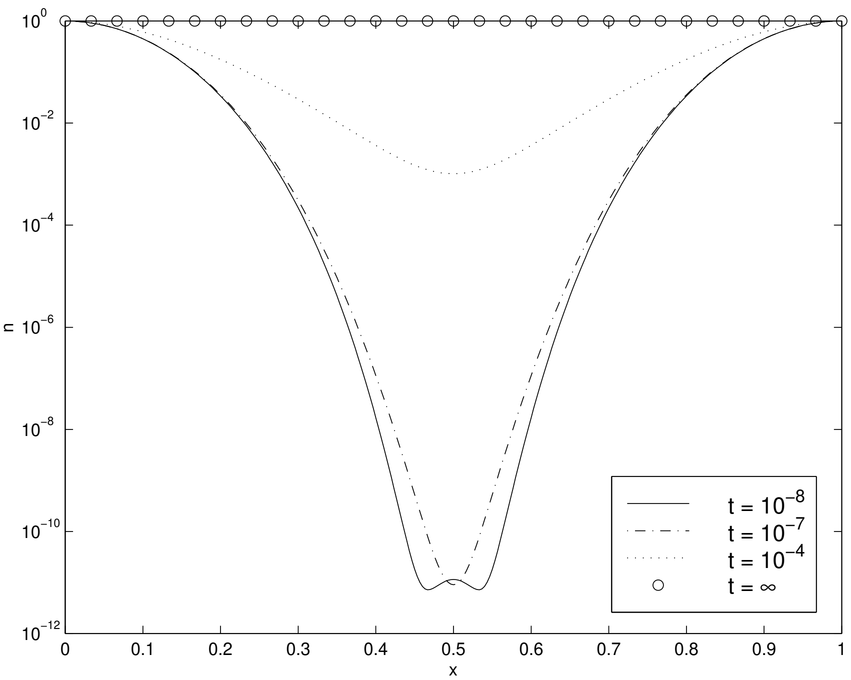

Figure 3.2: Evolution for $m = 8$

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