Local Existence of Solutions to the Transient Quantum Hydrodynamic Equations

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Abstract

The existence of weak solutions locally in time to the quantum hydrodynamic equations in bounded domains is shown. These Madelung-type equations consist of the Euler equations, including the quantum Bohm potential term, for the particle density and the particle current density and are coupled to the Poisson equation for the electrostatic potential. This model has been used in the modeling of quantum semiconductors and superfluids. The proof of the existence result is based on a formulation of the problem as a nonlinear Schrödinger-Poisson system and uses semigroup theory and fixed-point techniques.

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1 Introduction

In 1927, Madelung gave a fluiddynamical description of quantum systems governed by the Schrödinger equation for the wave function ϕ :

$$
\begin{aligned}
i\varepsilon\partial_t\phi &= -\frac{\varepsilon^2}{2}\Delta\phi - V\phi \quad \text{in } \mathbb{R}^d \times (0,T), \\
\phi(\cdot,0) &= \phi_0 \quad \text{in } \mathbb{R}^d,
\end{aligned}
$$

where $T > 0$, $d \geq 1$, $\varepsilon > 0$ is the scaled Planck constant, and $V = V(x,t)$ is some (given) potential. Separating the amplitude and phase of $\phi = |\phi|\exp(iS/\varepsilon)$, the particle density $n = |\phi|^2$ and the particle current density $J = n\nabla S$ satisfy the so-called *Madelung equations* [17]

$$
\partial_t n + \mathrm{div} J = 0, \tag{1}
$$

$$
\partial_t J + \mathrm{div}\left(\frac{J \otimes J}{n}\right) - n\nabla V - \frac{\varepsilon^2}{2} n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right) = 0 \quad \text{in } \mathbb{R}^d \times (0,T), \tag{2}
$$

where the i-th component of the convective term $\mathrm{div}(J \otimes J/n)$ equals

$$
\sum_{j=1}^{d} \frac{\partial}{\partial x_j}\left(\frac{J_i J_j}{n}\right).
$$

The equations (1)-(2) can be interpreted as the pressureless Euler equations including the quantum Bohm potential

$$
\frac{\varepsilon^2}{2}\frac{\Delta\sqrt{n}}{\sqrt{n}}.
$$

They have been used for the modeling of superfluids like Helium II [15, 16].

Recently, Madelung-type equations have been derived for the modeling of quantum semiconductor devices, like resonant tunneling diodes, starting from the Wigner-Boltzmann equation [5] or from a mixed-state Schrödinger-Poisson system [8, 9]. There are several advantages of the fluiddynamical description of semiconductors. First, kinetic equations, like the Wigner equation, or Schrödinger systems are computationally very expensive, whereas for Euler-type equations efficient numerical algorithms are available [4, 6]. Second, the macroscopic description allows for a coupling of classical and quantum models. Indeed, setting the Planck constant ε in (2) equal to zero, we obtain the classical pressureless equations, so in both pictures, the same (macroscopic) variables can be used. Finally, as semiconductor devices are modeled in bounded domains, it is easier to find physically relevant boundary conditions for the macroscopic variables than for the Wigner function or for the wave function.

The Madelung-type equations derived by Gardner [5] and Gasser et al. [8] also include a pressure term and a momentum relaxation term taking into account interactions of the electrons in the semiconductor crystal, and are self-consistently coupled to the Poisson equation for the electrostatic potential V:

$$\partial_t n + \mathrm{div} J = 0, \tag{3}$$

$$\partial_t J + \mathrm{div}\left(\frac{J \otimes J}{n}\right) + \nabla p(n) - n\nabla V - \frac{\varepsilon^2}{2} n \nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right) = -\frac{J}{\tau}, \tag{4}$$

$$\lambda^2 \Delta V = n - C(x) \qquad \text{in } \Omega \times (0,T), \tag{5}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain, $\tau > 0$ is the (scaled) momentum relaxation time constant, $\lambda > 0$ the (scaled) Debye length, and $C(x)$ is the doping concentration modeling the semiconductor device under consideration [14, 18]. The pressure is assumed to depend only on the particle density and, like in classical fluiddynamics, often the expression

$$p(n) = Tn^\alpha, \quad n \geq 0,$$

with the temperature constant $T > 0$ is employed [5, 12]. *Isothermal* fluids correspond to $\alpha = 1$, *isentropic* fluids to $\alpha > 1$. Notice that the particle temperature is $T(n) = Tn^{\alpha-1}$. In this paper, the pressure p is assumed to be an arbitrary smooth function of n.

The equations (3)-(5) are referred to as the *quantum hydrodynamic equations*.

The boundary values are assumed to be the superposition of the thermal equilibrium state $J = 0$ and the applied potential $U = U(x) \in \mathbb{R}$. Our main assumption in this paper is that we consider *irrotational* flows only, i.e. we suppose that the velocity J/n has a potential S, satisfying $J = n\nabla S$. Notice that this condition is consistent with the above formal equivalence between the Madelung equations and the Schrödinger equation. Then the boundary conditions read as follows (see [14] for details):

$$n = n_D, \quad S = S_D, \quad V = V_D \qquad \text{on } \partial\Omega \times (0,T), \tag{6}$$

where

$$n_D = C, \quad S_D = V_D = U.$$

Clearly, mixed Dirichlet-Neumann boundary conditions would be physically more relevant [14]. However, we need some regularity results for solutions of elliptic problems which are generally not available in the case of mixed Dirichlet-Neumann boundary conditions. Finally, we impose the initial conditions

$$n(\cdot,0) = n_0, \quad S(\cdot,0) = S_0 \qquad \text{in } \Omega, \tag{7}$$

which yields $J(\cdot,0) = n_0 \nabla S_0$ in Ω.

The stationary equations corresponding to (3)-(5) have been investigated in [3, 12, 21]. In [12] it has been shown that there exists a weak irrotational solution of the *subsonic* state (i.e. $|\nabla S| \ll 1$) with boundary conditions (6). For this solution, the electron density is strictly positive. In the general (possibly transonic) case, but for a different set of boundary conditions, the *non-existence* of weak solutions could be proved [3]. Moreover, the semi-classical limit $\varepsilon \to 0$ in one space dimension has been studied in [2, 11]. For numerical solutions of these equations, we refer to [5, 20]. An overview of different macroscopic quantum models has been given in [13, 14].

For the transient equations (3)-(5) much less analytical work has been published up to now. For instance, Gasser [7] has examined traveling-wave solutions. However, no existence results are available for the problem (3)-(7).

In this paper we show the existence of H^2-solutions to the potential-flow formulation of (3)-(7) *locally* in time. The main difficulties in the mathematical treatment of quantum hydrodynamic equations are the third derivatives in equation (4) and the proof of the positivity of the electron density in order to define the quotients. Actually, it is possible to give a meaning to these terms even for $n = 0$ through a formulation for the wave function [8]. However, in the case $n = 0$ the phase S is undefined and moreover, a Schrödinger formulation of (3)-(4) is not obvious. The difficulty of the third derivatives has been solved in [11] by differentiating the equation (4), yielding a fourth-order differential operator. This method works well for the stationary problem, but not for the multi-dimensional transient equations.

The main idea of our existence result is the (formal) equivalence of the system (3)-(7) to the following nonlinear Schrödinger-Poisson problem:

$$
\begin{aligned}
i\varepsilon\partial_t\phi &= -\frac{\varepsilon^2}{2}\Delta\phi - V\phi + h(|\phi|^2)\phi + \frac{1}{\tau}\arg(\phi)\phi, && (8)\\
\lambda^2\Delta V &= |\phi|^2 - C(x) \quad \text{in } \Omega\times(0,T), && (9)
\end{aligned}
$$

with boundary and initial conditions

$$
\begin{aligned}
\phi &= \phi_D && \text{on } \partial\Omega\times(0,T), && (10)\\
\phi(\cdot,0) &= \phi_0 && \text{in } \Omega. && (11)
\end{aligned}
$$

Here, the function h is the *enthalpy* defined by $sh'(s) = p'(s)$, $s > 0$, and $h(1) = 0$, $\arg(\phi) = S$ is the argument or phase function, and

$$
\phi_D = n_D \exp(iS_D/\varepsilon), \quad \phi_0 = n_0 \exp(iS_0/\varepsilon). \tag{12}
$$

In this formulation, however, the phase is also not defined if $\phi = 0$. Therefore, we suppose *positive* particle densities on $\partial\Omega$ and at $t = 0$, and solve the equations (8)-(11) for *small* time, assuring $|\phi| > 0$ in the corresponding time interval. Hence

$n = |\phi|^2 > 0$, and the problem (8)-(11) is rigorously equivalent to the potential-flow quantum hydrodynamic equations:

$$\partial_t n + \operatorname{div}(n\nabla S) = 0, \tag{13}$$

$$\partial_t S + \frac{1}{2}|\nabla S|^2 + h(n) - V - \frac{\varepsilon^2}{2}\frac{\Delta\sqrt{n}}{\sqrt{n}} = -\frac{S}{\tau}, \tag{14}$$

$$\lambda^2 \Delta V = n - C(x) \qquad \text{in } \Omega \times (0,T), \tag{15}$$

with initial and boundary conditions (6)-(7). If S is smooth enough, we can define $J = n\nabla S$ and obtain a weak solution (n, J, V) to (3)-(7).

The argument function in equation (8) will be treated mathematically as follows. The function $S = \arg(\phi) = -i\varepsilon \ln(\phi/|\phi|^2)$ satisfies

$$\Delta S = \varepsilon F(\phi, \nabla\phi, \Delta\phi),$$

where

$$F(u,v,z) = |u|^{-4}(|u|^2 \operatorname{Im}(u^* z) - 2\operatorname{Re}(u^* v)\cdot \operatorname{Im}(u^* v)) \tag{16}$$

for $u \in \mathbb{C}\backslash\{0\}$, $v \in \mathbb{C}^d$, $z \in \mathbb{C}$. Hence we replace the term $\arg(\phi)\phi/\tau$ in (8) by $S\phi/\tau$.

This paper is organized as follows. In Section 2 we state our assumptions and the main result. Section 3 is devoted to the proof of some auxiliary lemmas. The main theorem is proved in Section 4. Finally, we end this paper by some comments on generalizations of the main result.

2 Assumptions and main results

We impose the following assumptions:

(H1) $\Omega \subset \mathbb{R}^d$ $(1 \le d \le 3)$ is a bounded domain with $\partial\Omega \in C^2$.

(H2) $p \in C^{2,1}(0,\infty)$; $\varepsilon, \tau, \lambda > 0$; $C \in L^2(\Omega)$.

(H3) n_D, S_D, $V_D \in H^{3/2}(\partial\Omega)$, $n_D > \delta^2 > 0$ on $\partial\Omega$; n_0, $S_0 \in H^2(\Omega)$, $n_0 > \delta^2 > 0$ in Ω, and $n_0 = n_D$, $S_0 = S_D$ on $\partial\Omega$.

Our main result is the following theorem:

Theorem 1. *Let (H1)-(H3) hold. Then there exist $T^* > 0$ and a unique solution (n,S,V) to (13)-(15), (6)-(7) satisfying*

$$n \in C^1([0,T^*]; L^2(\Omega)), \quad n > \delta^2 > 0 \text{ in } \Omega \times (0,T^*),$$
$$n, S, V \in C^0([0,T^*]; H^2(\Omega)).$$

Moreover, if $S \in C^1([0,T^*]; H^1(\Omega))$ then (n, J, V) with $J = n\nabla S$ is a solution to (3)-(7) with $J \in C^1([0,T^*]; L^{3/2}(\Omega)) \cap C^0([0,T^*]; H^1(\Omega))$ and

$$\begin{aligned} J &= n_D \nabla S_D \qquad \text{on } \partial\Omega \times (0, T^*), \\ J(\cdot, 0) &= n_0 \nabla S_0 \qquad \text{in } \Omega. \end{aligned}$$

Let $\phi_0 = \sqrt{n_0}\exp(iS_0/\varepsilon) \in H^2(\Omega)$ and let $\phi_D \in H^2(\Omega)$ be the harmonic extension of $\sqrt{n_D}\exp(iS_D/\varepsilon)$ to Ω, i.e.

$$\Delta\phi_D = 0 \quad \text{in } \Omega, \qquad \phi_D = \sqrt{n_D}\exp(iS_D/\varepsilon) \quad \text{on } \partial\Omega.$$

Then, by (H3),

$$|\phi_0| > \delta > 0 \text{ in } \Omega \quad \text{ and } \quad \phi_0 = \phi_D \qquad \text{on } \partial\Omega. \tag{17}$$

The above theorem is a consequence of the following proposition.

Proposition 2. *Let (H1)-(H3) hold. Then there exist $T^* > 0$ and a unique solution (ϕ, S, V) to*

$$i\varepsilon\partial_t\phi = -\frac{\varepsilon^2}{2}\Delta\phi - V\phi + h(|\phi|^2)\phi + \frac{1}{\tau}S\phi, \tag{18}$$

$$\Delta S = \varepsilon F(\phi, \nabla\phi, \Delta\phi), \tag{19}$$

$$\lambda^2 \Delta V = |\phi|^2 - C(x) \qquad \text{in } \Omega \times (0, T^*), \tag{20}$$

with initial and boundary conditions (10)-(11) satisfying (12). The solution (ϕ, V) satisfies

$$\phi \in C^1([0,T^*]; L^2(\Omega)), \quad |\phi| > \delta > 0 \quad \text{in } \Omega \times (0,T^*),$$
$$\phi, S, V \in C^0([0,T^*]; H^2(\Omega)).$$

Recall the definition (16) of F in Section 1.

Remark 3. We have assumed time-independent boundary data for simplicity. The results are also valid for more general boundary data.

Remark 4. From the proof of Proposition 2 it can be seen that $T^* > 0$ depends on δ and ε such that $T^* \to 0$ as $\delta \to 0$ or $\varepsilon \to 0$.

For more remarks, see Section 4.

3 Auxiliary lemmas

For the proof of Theorem 1 and Proposition 2 we need some auxiliary results. Throughout this section, we assume that (H1)-(H3) hold. Let $T > 0$.

It is well known that $\Delta = \operatorname{div}\operatorname{grad}$ is a self-adjoint and dissipative operator on its domain $D(\Delta) = H_0^1(\Omega) \cap H^2(\Omega)$ [19]. Thus, by the Theorem of Stone, $i(\varepsilon/2)\Delta$ generates a unitary group of isometries $e^{i(\varepsilon/2)t\Delta}$ on $L^2(\Omega)$ [19]. The restriction of $e^{i(\varepsilon/2)t\Delta}$ on $D(\Delta)$ is also a group of isometries on $D(\Delta)$ and will be denoted again by $e^{i(\varepsilon/2)t\Delta}$.

Define the space

$$X_\delta = \{\phi \in H^2(\Omega) : \inf_\Omega |\phi| > \delta > 0\},$$

let $\phi \in X_\delta$ and let $S = S[\phi] \in H^2(\Omega)$ be the unique solution to

$$\Delta S = \varepsilon F(\phi, \nabla\phi, \Delta\phi) \text{ in } \Omega, \quad S = S_D \text{ on } \partial\Omega.$$

The operator $S : X_\delta \to H^2(\Omega)$, $\phi \mapsto S[\phi]$, is well defined. Indeed, the function F satisfies the estimate

$$|F(u,v,z)| \le \frac{1}{\delta}|z| + \frac{2}{\delta^2}|v|^2 \qquad \text{for } u \in \mathbb{C},\ |u| \ge \delta,\ v \in \mathbb{C}^d,\ z \in \mathbb{C}.$$

Hence, using the embedding $H^2(\Omega) \hookrightarrow W^{1,4}(\Omega)$, $F(\phi, \nabla\phi, \Delta\phi) \in L^2(\Omega)$. Then $S[\phi] \in H^2(\Omega)$ follows from elliptic regularity [10], and it holds

$$\|S[\phi]\|_{H^2(\Omega)} \le c(\delta, \|\phi\|_{H^2(\Omega)}). \tag{21}$$

Next, let $V[\phi]$ be the unique solution to

$$\lambda^2 \Delta V[\phi] = |\phi|^2 - C(x) \quad \text{in } \Omega, \qquad V[\phi] = 0 \quad \text{on } \partial\Omega.$$

This defines an operator $V : H^2(\Omega) \to H^2(\Omega)$ with

$$\|V[\phi]\|_{H^2(\Omega)} \le c(\|\phi\|_{L^4(\Omega)}). \tag{22}$$

Let $\bar{V}_D \in H^2(\Omega)$ be the harmonic extension of V_D to Ω, and define the operator

$$R : X_\delta \to H^2(\Omega), \quad R(\phi) = h(|\phi|^2) - V[\phi] - \bar{V}_D + \tau^{-1} S[\phi].$$

Then R is well defined since $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, $h \in C^2(0,\infty)$, and

$$\|h(|\phi|^2)\|_{H^2(\Omega)} \le c(\delta, \|\phi\|_{L^\infty(\Omega)})(1 + \|\phi\|_{H^2(\Omega)}) \le c(\delta, \|\phi\|_{H^2(\Omega)}). \tag{23}$$

In order to solve the problem (18)-(20), (10)-(11), we will use a fixed-point argument. For this, we define the operator $U : D(\Delta) \to H^2(\Omega)$:

$$U(\phi) = \theta(t) - \frac{i}{\varepsilon}\int_0^t e^{i(\varepsilon/2)(t-s)\Delta} R(\phi)\phi(s)\, ds,$$

where

$$\theta(t) = \phi_D + e^{i(\varepsilon/2)t\Delta}(\phi_0 - \phi_D).$$

Then $\theta(t)$ is well defined since $\phi_0 - \phi_D \in D(\Delta)$ (see (H3) and (12)) and $\theta \in C^0([0,T]; H^2(\Omega))$. Clearly, any fixed point is a (mild) solution to (18)-(20), (10)-(11).

Lemma 5. *Let $\phi, \phi_i \in X_\delta$ with $\|\phi\|_{H^2(\Omega)} \le M,\ \|\phi_i\|_{H^2(\Omega)} \le M,\ i = 1,2$. Then*

$$\begin{aligned} \|R(\phi)\|_{H^2(\Omega)} &\le C_0(\delta, M), &(24)\\ \|R(\phi_1) - R(\phi_2)\|_{H^2(\Omega)} &\le C_0(\delta, M)\|\phi_1 - \phi_2\|_{H^2(\Omega)}, &(25)\end{aligned}$$

where the constant $C_0(\delta, M) > 0$ depends on δ and M.

Proof. The first estimate follows from (21), (22) and (23):

$$\begin{aligned} \|R(\phi)\|_{H^2(\Omega)} &\le \|h(|\phi|^2)\|_{H^2(\Omega)} + \|V[\phi]\|_{H^2(\Omega)} + \|\bar{V}_D\|_{H^2(\Omega)} + \tau^{-1}\|S[\phi]\|_{H^2(\Omega)} \\ &\le c(\delta, \|\phi\|_{H^2(\Omega)}) \le c(\delta, M), \end{aligned}$$

where $c(\delta, M)$ also depends on $h, \lambda, C(x), \varepsilon, V_D$, and τ. For the second estimate we use the fact that $h,\ h'$ and h'' are locally Lipschitz continuous on $[\delta, \infty)$ and that F is locally Lipschitz continuous on $\{z \in \mathbb{C} : |z| \ge \delta\} \times \mathbb{C}^d \times \mathbb{C}$:

$$\begin{aligned} \|R(\phi_1) - R(\phi_2)\|_{H^2(\Omega)} &\le \|h(|\phi_1|^2) - h(|\phi_2|^2)\|_{H^2(\Omega)} + \|V[\phi_1] - V[\phi_2]\|_{H^2(\Omega)} \\ &\quad + \|S[\phi_1] - S[\phi_2]\|_{H^2(\Omega)} \\ &\le c(\delta, \|\phi_1\|_{H^2(\Omega)}, \|\phi_2\|_{H^2(\Omega)})\|\phi_1 - \phi_2\|_{H^2(\Omega)} \\ &\quad + c(\lambda)\||\phi_1|^2 - |\phi_2|^2\|_{L^2(\Omega)} \\ &\quad + c(\varepsilon)\|F(\phi_1, \nabla\phi_1, \Delta\phi_1) - F(\phi_2, \nabla\phi_2, \Delta\phi_2)\|_{L^2(\Omega)} \\ &\le c(\delta, M)\|\phi_1 - \phi_2\|_{H^2(\Omega)}. \end{aligned}$$

This proves the lemma. □

Now, define the set

$$B_{r,T} = \{\phi \in C^0([0,T]; H^2(\Omega)) : \|\phi - \theta\|_{L^\infty(0,T;H^2(\Omega))} \le r\}.$$

The functions in $B_{r,T}$ have the following property.

Lemma 6. *There exist $r_0,\ T_0 > 0$ such that for all $r \in (0, r_0),\ T \in (0, T_0)$ and $\phi \in B_{r,T}$ it holds*

$$\inf_\Omega |\phi| > \delta > 0.$$

Proof. Since $\inf_\Omega |\phi_0| > \delta$ (see (17)), there exists $0 < \eta < \inf_\Omega |\phi_0| - \delta$. By the semigroup property of $e^{i(\varepsilon/2)t\Delta}$, there exists $T_0 > 0$ such that for all $0 < t \le T_0$ it holds

$$\|\theta(t) - \phi_0\|_{H^2(\Omega)} = \|(e^{i(\varepsilon/2)t\Delta} - \mathrm{id})(\phi_0 - \phi_D)\|_{H^2(\Omega)} \le \frac{\eta}{2}.$$

Set $r_0 = \eta/2$. Then, for $0 < r \le r_0,\ 0 < T \le T_0,\ 0 < t \le T$, and $\phi \in B_{r,T}$,

$$\|\phi(t) - \phi_0\|_{H^2(\Omega)} \le \|\phi(t) - \theta(t)\|_{H^2(\Omega)} + \|\theta(t) - \phi_0\|_{H^2(\Omega)} \le r + \frac{\eta}{2} \le \eta.$$

Thus, for $(x,t) \in \Omega \times (0, T_0)$,

$$-|\phi(x,t)| + |\phi_0(x)| \le |\,|\phi(x,t)| - |\phi_0(x)|\,| \le |\phi(x,t) - \phi_0(x)| \le \eta$$

and

$$|\phi(x,t)| \ge |\phi_0(x)| - \eta > \delta.$$

The lemma is proved. □

4 Proof of the main result

Proof of Proposition 2. We show that the operator U, defined in Section 3, is a contraction in $B_{r,T}$ for small enough $r > 0$ and $T > 0$. Then, by Banach's fixed-point theorem, we obtain a (unique) solution locally in time to (18)-(20), (10)-(11).

Let $0 < r \le r_0$ and $0 < T \le T^* = \min\{T_0, T_1, T_2\}$, where r_0 and T_0 are as in Lemma 6 and

$$T_1 = \frac{\varepsilon r}{C_0(\delta, M)M}, \quad T_2 = \frac{\varepsilon}{2C_0(\delta, M)(M+1)},$$

where $M = r_0 + \|\theta\|_{L^\infty(0,T_0;H^2(\Omega))}$. Then any $\phi \in B_{r,T}$ satisfies

$$\|\phi\|_{L^\infty(0,T;H^2(\Omega))} \le \|\phi - \theta\|_{L^\infty(0,T;H^2(\Omega))} + \|\theta\|_{L^\infty(0,T;H^2(\Omega))} \le M.$$

We show first that $U : B_{r,T} \to B_{r,T}$. Let $\phi \in B_{r,T}$. Then, by Lemma 5, since $e^{i(\varepsilon/2)t\Delta}$ is an isometry,

$$\begin{aligned}
\|U(\phi) - \theta\|_{L^\infty(0,T;H^2(\Omega))} &\le \frac{1}{\varepsilon} \sup_{0<t<T} \int_0^t \|e^{i(\varepsilon/2)(t-s)\Delta} R(\phi)\phi(s)\|_{H^2(\Omega)} ds \\
&\le \frac{1}{\varepsilon} \sup_{0<t<T} \int_0^T \|R(\phi)\phi(s)\|_{H^2(\Omega)} ds \\
&\le \frac{1}{\varepsilon} \sup_{0<t<T} \int_0^T \|R(\phi(s))\|_{H^2(\Omega)} \|\phi(s)\|_{H^2(\Omega)} ds \\
&\le \frac{1}{\varepsilon} T C_0(\delta, M) M \\
&\le \frac{1}{\varepsilon} T_1 C_0(\delta, M) M = r.
\end{aligned}$$

Now let $\phi_1, \phi_2 \in B_{r,T}$. Then

$$\begin{aligned}
&\|U(\phi_1) - U(\phi_2)\|_{L^\infty(0,T;H^2(\Omega))} \\
&\le \frac{1}{\varepsilon} \sup_{0<t<T} \int_0^t \|e^{i(\varepsilon/2)(t-s)\Delta} (R(\phi_1)\phi_1(s) - R(\phi_2)\phi_2(s))\|_{H^2(\Omega)} \, ds \\
&\le \frac{1}{\varepsilon} \sup_{0<t<T} \int_0^t (\|R(\phi_1) - R(\phi_2)\|_{H^2(\Omega)} \|\phi_1(s)\|_{H^2(\Omega)} \\
&\qquad + \|\phi_1 - \phi_2\|_{H^2(\Omega)} \|R(\phi_2(s))\|_{H^2(\Omega)}) \, ds \\
&\le \frac{1}{\varepsilon} \sup_{0<t<T} \int_0^t C_0(\delta, M)(M+1) \|\phi_1(s) - \phi_2(s)\|_{H^2(\Omega)} \, ds \\
&\le \frac{1}{\varepsilon} C_0(\delta, M)(M+1) T_2 \|\phi_1 - \phi_2\|_{L^\infty(0,T;H^2(\Omega))} \\
&\le \frac{1}{2} \|\phi_1 - \phi_2\|_{L^\infty(0,T;H^2(\Omega))}.
\end{aligned}$$

Hence, we can apply the Banach fixed-point theorem. □

Proof of Theorem 1. An elementary computation shows that $n = |\phi|^2 \in C^0([0,T^*];H^2(\Omega)) \cap C^1([0,T^*];L^2(\Omega))$, $S \in C^0([0,T^*];H^2(\Omega))$ solves the equations (13)-(15). Moreover, $n = |\phi|^2 > \delta^2 > 0$ in $\Omega \times (0,T^*)$.

Suppose that $S \in C^1([0,T^*];H^1(\Omega))$. Equation (14) implies that $\Delta\sqrt{n}/\sqrt{n} \in C^0([0,T^*];W^{1,1}(\Omega))$. Moreover, by equation (13), $\partial_t n \in C^0([0,T^*];L^2(\Omega))$. Then, taking the gradient of (14), multiplying the resulting equation by n, and adding the equation (13), which has been multiplied by ∇S, we see that $J = n\nabla S \in C^1([0,T^*];L^{3/2}(\Omega)) \cap C^0([0,T^*];\, H^1(\Omega))$ satisfies the equation (4) in the sense of $L^2(\Omega)$. $\square$

Remark 7. In the modeling of quantum semiconductors, usually an external potential V_{ext} is added to the electrostatic potential, modeling heterostructures [5], i.e. the term $-n\nabla V$ in (4) is replaced by $-n\nabla(V + V_{ext})$. Our results are also valid for equations including this term if $V_{ext} \in H^2(\Omega)$.

Remark 8. The restriction to at most three space dimensions comes from the fact that we used the embeddings $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ and $H^2(\Omega) \hookrightarrow W^{1,4}(\Omega)$, in particular for the definition of $h(|\phi|^2)$. Whereas this can be weakened to any space dimension by imposing appropriate growth conditions on h (or in p) [1], we also need solutions with positive particle density $|\phi|^2$, so we need to solve the equations for L^∞-solutions.

Remark 9. In order to obtain the *global* existence of solutions, appropriate a priori estimates are necessary. Actually, it is not difficult to show that the *energy*

$$E(t) = \int_\Omega \left(\frac{\varepsilon^2}{2}|\nabla\phi|^2 + H(|\phi|^2) + \frac{\lambda^2}{2}|\nabla V|^2 \right)(t)\ dx,$$

where H is a primitive of h, is bounded by $E(0)$:

$$E(t) + \frac{1}{\tau}\int_0^t\int_\Omega |\phi|^2|\nabla S|^2\ dx\ dt = E(0).$$

(Here we use that the boundary data are time-independent; otherwise, terms depending on appropriate norms of the boundary functions have to be added on the right-hand side of the energy equation.) However, we also need an estimate for the phase S. The energy production term does not help since $|\nabla S|$ may be unbounded as $|\phi| \to 0$. The same difficulty arises in the stationary equations [12].

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