Regularity and Uniqueness of Solutions to a Parabolic System in Nonequilibrium Thermodynamics

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Abstract

The time-dependent equations for a charged gas or fluid consisting of several components, exposed to an electric field, are considered. These equations for the chemical potentials and the gas temperature form a system of nonlinear parabolic equations with full diffusion matrix, coupled to the Poisson equation for the electric field. The model can be derived in some situations from the Boltzmann equation. The main feature is that the variables of the time differential operator and the space differential operator are different and related by an algebraic relation.

The existence of a solution satisfying some regularity properties is shown. In particular, the time derivative of the solution is an L^2 function both in time and space and the space derivative is an L^q function in time and space, where $q > 2$. In the proof the method of semidiscretization of time and regularity properties for elliptic systems are used. Furthermore, the uniqueness of weak solutions is proved. The proof is based on an elliptic dual method and uses the existence of a solution with the above regularity.

Keywords. Parabolic systems, spatial and temporal regularity, uniqueness of weak solutions, semidiscretization of time, elliptic dual method.

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1 Introduction

We consider a gas or fluid consisting of n components (species), which is influenced by thermal, electrical and diffusive forces. Each component, labeled by i and with density ρ_i, has the charge e_i per particle. The evolution of the particles is governed by the conservation laws of the mass and of the energy (see, e.g., [22, 28]) :

$$\frac{\partial}{\partial t}\rho_i(u) + \operatorname{div} J_i = W_i(u,V), \qquad i = 1,\dots,n, \tag{1.1}$$

$$\frac{\partial}{\partial t}\rho_{n+1}(u) + \operatorname{div} J_{n+1} = W_{n+1}(u,V) - \sum_{k=1}^{n} e_k \nabla V \cdot J_k \quad \text{in } \Omega \times (0,\tau), \tag{1.2}$$

where $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ is the (bounded) domain occupied by the gas or fluid and $\tau > 0$. The carrier flux density of the i-th component is denoted by J_i, the energy flux density or heat flux is denoted by J_{n+1}, and ρ_{n+1} is the internal energy density. The functions W_i are source terms. We assume that external forces are due to the electric potential V which is coupled selfconsistently to the Poisson equation

$$\Delta V = -\sum_{k=1}^{n} e_k \rho_k - C(x) \quad \text{in } \Omega \times (0,\tau). \tag{1.3}$$

The given function $C(x)$ models fixed charged background particles.

The thermodynamic fluxes J_i $(i = 1,\dots,n+1)$ are assumed to depend on the thermodynamic forces $X_i = \nabla(\mu_i/T) + e_i \nabla V/T$ $(i = 1,\dots,n)$ and $X_{n+1} = \nabla(-1/T)$ where μ_i denotes the chemical potential and T the temperature of the gas or fluid. For sufficiently small deviations from the thermal equilibrium state $J_i = 0$ $(i = 1,\dots,n+1)$, the phenomenological relations between the fluxes and the forces are given by

$$J_i = -\sum_{k=1}^{n+1} L_{ik} X_k, \quad i = 1,\dots,n+1.$$

Setting $e_{n+1} = 0$ and

$$u = (u_1,\dots,u_n,u_{n+1}) = (\mu_1/T,\dots,\mu_n/T,-1/T)$$

this relation can be written as

$$J_i = -\sum_{k=1}^{n+1} L_{ik}(x)(\nabla u_k - e_k u_{n+1} \nabla V), \quad i = 1,\dots,n+1. \tag{1.4}$$

From Onsager's principle, the symmetry of the coefficient matrix $L = (L_{ik})$ can be concluded. The second law of thermodynamics requires that (L_{ik}) is positive definite

[28]. We refer to the monographs [22, 23, 28] for a discussion of the choice of the fluxes and forces and the relation between them. Furthermore, we mention that recently, in the context of semiconductor theory, the model (1.1)-(1.4) is derived from the Boltzmann equation [4, 6]. This justifies the relation between J_i and X_k.

The particle densities ρ_i and the internal energy density ρ_{n+1} are assumed to depend on u: $\rho_i = \rho_i(u)$, $i = 1, \ldots, n+1$. The main assumption on $\rho = (\rho_1, \ldots, \rho_{n+1})$ is that ρ is strongly monotone and a gradient, i.e.

$$(\rho(u) - \rho(v)) \cdot (u - v) \geq c_0|u - v|^2 \quad \text{for some } c_0 > 0 \text{ and all } u, v \in \mathbb{R}^{n+1} \text{ and}$$
$$\exists \chi \in C^1(\mathbb{R}^{n+1}) \, : \, \rho = \nabla_u \chi.$$

The equations have to be supplemented with physically motivated initial and boundary conditions. The boundary $\partial\Omega$ of the domain Ω consists of two disjoint subsets Γ_D and Γ_N. The electric potential, the chemical potentials, and the temperature are fixed at Γ_D, whereas Γ_N models the union of insulating boundary segments (zero outflow):

$$u = u_D, \quad V = V_D \qquad \text{on } \Gamma_D \times (0, \tau), \tag{1.5}$$
$$J_i \cdot \nu = \nabla V \cdot \nu = 0 \qquad \text{on } \Gamma_N \times (0, \tau), \quad i = 1, \ldots, n+1, \tag{1.6}$$

where ν is the exterior normal vector which is assumed to exist a.e. Thanks to the relation between ρ and u, the carrier densities and the temperature instead of u can be prescribed at at the boundary. Thus the boundary condition

$$\rho = \rho_D, \quad T = T_{D,} \quad V = V_D \qquad \text{on } \Gamma_D \times (0, \tau),$$

is equivalent to (1.5) (note that $\rho(u)$ is invertible since it is strongly monotone). The initial condition is given by

$$\rho(u(0)) = \rho(u^0) \qquad \text{in } \Omega. \tag{1.7}$$

The thermodynamic model (1.1)-(1.7) occurs in many applications of charged particle transport, for instance in semiconductor theory [4, 27], in electro-chemistry [9, 14], and alloy solidification processes [7, 24]. We only present the energy-transport model for semiconductors as an important example for (1.1)-(1.7) and refer to [12] for the details of other examples.

Example: energy-transport model for semiconductors. This model consists of equations for the electron density ρ_1 with charge $e_1 = -1$, the electron temperature $T > 0$, and the electric potential V:

$$\begin{aligned}
\frac{\partial \rho_1}{\partial t} + \operatorname{div} J_1 &= 0, \\
\frac{\partial \rho_2}{\partial t} + \operatorname{div} J_2 &= J_1 \cdot \nabla V + W(\mu_1, T), \\
\Delta V &= \rho_1 - C(x),
\end{aligned}$$

where the electron and energy current densities are given by

$$\begin{aligned}
J_1 &= -L_{11}\Big(\nabla(\frac{\mu_1}{T}) - \frac{\nabla V}{T}\Big) - L_{12}\nabla\Big(-\frac{1}{T}\Big), \\
J_2 &= -L_{21}\Big(\nabla(\frac{\mu_1}{T}) - \frac{\nabla V}{T}\Big) - L_{22}\nabla\Big(-\frac{1}{T}\Big).
\end{aligned}$$

The model describes the flow of electrons through a doped semiconductor crystal which is characterized by the doping profile $C(x)$. It can be derived rigorously from the Boltzmann equation incorporating electron-electron and elastic collisions [4, 5, 17]. The energy relaxation term $W(\mu_1, T)$ satisfies

$$W(\mu_1, T)(T - T_L) \leq 0, \tag{1.8}$$

where T_L is the lattice temperature. The variables ρ_1 and ρ_2 depend on μ_1 and T. In the case of Boltzmann statistics it holds

$$\rho_1 = T^{3/2} \exp(\mu_1/T), \quad \rho_2 = \frac{3}{2}\rho_1 T.$$

The energy-transport model corresponds to (1.1)-(1.4) for $n = 1$. Notice that

$$\rho(u) = \Big(-\frac{1}{u_2}\Big)^{3/2} \exp(u_1) \cdot \begin{pmatrix} 1 \\ -(3/2)u_2 \end{pmatrix}$$

is strongly monotone and a gradient, i.e. $\chi(u) = (-1/u_2)^{3/2} \exp(u_1)$ satisfies $\nabla_u \chi = \rho$.

This paper is a continuation of [11] where the existence of weak solutions to (1.1)-(1.7) is proved. Here we investigate the regularity and uniqueness of solutions to (1.1)-(1.7).

Mathematically, there arise two main difficulties in the analysis of the system (1.1)-(1.7). First, the variable of the time differential operator, i.e. ρ, and the variable of the space differential operator, i.e. u, are *different* and related by the algebraic relation $\rho = \rho(u)$. For more than one equation, this type of equation with $V = 0$ is studied in, e.g., [2, 16]. The uniqueness of solutions for a single equation with monotone ρ is proved in [30]. Secondly, the equations form a *system* of nonlinear parabolic equations. For systems of elliptic or parabolic equations, it is well known that there does not exist a regularity theory as for single elliptic or parabolic equations. For instance, there exist examples of linear elliptic systems whose weak solution is not bounded if $d \geq 3$ [18, Ch. II]. Counterexamples to the regularity of weak solutions to parabolic systems can be found in, e.g., [26, 32].

One main idea to overcome the above problems is to use another set of variables, called the *entropic variables*, which symmetrizes the problem (see (2.10)). In particular, the gradient terms in (1.2), (1.4) vanish. A second idea is to discretize the equations in time (method of line), to solve the corresponding elliptic problems, and then to pass to the limit in the discretization.

We summarize the main results of this paper: Under some assumptions on the nonlinear functions and the initial and boundary data, we prove that there exists a number $q > 2$ and a solution (u, V) to (1.1)-(1.7) satisfying

$$\partial u/\partial t \in L^2(\Omega \times (0,\tau)), \quad \nabla u \in L^q(\Omega \times (0,\tau)) \cap L^\infty(0,T; L^2(\Omega)),$$
$$\nabla V \in L^\infty(\Omega \times (0,\tau)).$$

This solution is obtained as the L^2 limit of approximate solutions solving a semi-discretized version of (1.1)-(1.7). Appropriate *a priori* estimates independent of the discretization and Sobolev embeddings allow to conclude the claimed regularity.

We only get *a priori* estimates if the second order operator is linear, i.e. we need to assume that L_{ik} only depends on the space variable (and not on u). Alt and Luckhaus proved in [2] the regularity $\rho_i(u) \in H^1(0,T; L^2(\Omega))$ for solutions to systems of the type

$$\partial_t \rho_i(u) - \operatorname{div} a_i(u, \nabla u) = f_i(u), \qquad i = 1, \ldots, n+1, \tag{1.9}$$

which corresponds to (1.1)-(1.2) in the case of vanishing electric potential $V = 0$.

The uniqueness of solutions to equations or systems of the type (1.9) subject to Dirichlet or mixed Dirichlet-Neumann boundary conditions and initial conditions is a delicate problem. In the case where $\rho(u)$ is a scalar-valued function, the uniqueness of solutions satisfying $\partial\rho(u)/\partial t \in L^1(\Omega \times (0,\tau))$ is shown in [2]. The function $\rho(u)$ is only assumed to be monotone. Recently, the uniqueness of *weak* solutions is obtained in [30] by using the method of time doubling. For *vector-valued* functions $\rho(u)$, there exists to our knowledge no general result. In [2] the uniqueness in the class of weak solutions could be proved for linear elliptic operators, i.e. $a(x,t,u,p) = A(x,t)p$.

For the system (1.1)-(1.7), we encounter additional difficulties due to the gradient terms and the electric potential. We prove uniqueness in the class of weak solutions satisfying additionally $u \in L^\infty(0,T;W^{1,3/2}(\Omega))$, $V \in L^\infty(0,T;W^{1,\infty}(\Omega))$. In the case of one space dimension we show the uniqueness of solution in the class of weak solutions. We have to assume as in [2] that the diffusion matrix does not depend on u. The uniqueness proof is performed by using an *elliptic* dual method, i.e. we take a test function in the weak formulation of the equations, which satisfies an elliptic problem. The existence of a solution with the above stated regularity properties is used in the proof.

The plan of the paper is as follows. In Section 2 we state the assumptions and recall some results from [10, 11] on the semidiscretized problem. The regularity properties are investigated in Section 3. Section 4 is devoted to the proof of the uniqueness of solutions.

2 Assumptions and the semidiscretized problem

We start with some notations. Denote by $W^{s,q}(\Omega)$, $W^{s,q}(\Omega;\mathbb{R}^{n+1})$ the usual Sobolev space of real-valued, vector-valued functions, respectively. The norms of $W^{s,q}(\Omega)$ and $W^{s,q}(\Omega;\mathbb{R}^{n+1})$ are denoted in both cases by $\|\cdot\|_{s,q,\Omega}$. We abreviate $W^{s,q}(X) \stackrel{\text{def}}{=} W^{s,q}(0,\tau;X)$ if X is a Banach space. Furthermore, set $\mathcal{V} = H_0^1(\Omega\cup\Gamma_N) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_N\}$ (see [33]) and $\mathcal{V}^{n+1} = H_0^1(\Omega\cup\Gamma_N;\mathbb{R}^{n+1})$, and denote by $\mathcal{V}^*$ the dual space to $\mathcal{V}$. Finally, we set $Q_\tau = \Omega\times(0,\tau)$.

We impose the following hypotheses:

(A1) $\Omega\subset\mathbb{R}^d$ $(1\le d\le 3)$ is a bounded domain with boundary $\partial\Omega = \Gamma_D\cup\Gamma_N \in C^{1,1}$, $\Gamma_D\cap\Gamma_N = 0$, $\text{meas}_{d-1}(\Gamma_D) > 0$, and Γ_N is open and closed in $\partial\Omega$, i.e. Γ_N coincides with connected components of $\partial\Omega$.

(A2) $\rho = (\rho_1,\ldots,\rho_{n+1}) \in W^{1,\infty}(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ is strongly monotone and a gradient, i.e. there exists a constant $c_0 > 0$ such that $(\rho(u)-\rho(v))\cdot(u-v) \ge c_0|u-v|^2$ for all $u,v\in\mathbb{R}^{n+1}$, and there exists a function $\chi\in C^1(\mathbb{R}^{n+1})$ such that $\nabla_u\chi = \rho$.

(A3) $L = (L_{ik})$ with $L_{ik}\in W^{1,r}(\Omega)\cap L^\infty(\Omega)$, where $r = 2$ if $d = 1$, $r > 2$ if $d = 2$, and $r = 3$ if $d = 3$, is a symmetric uniformly positive definite matrix satisfying, for some $\lambda,\Lambda > 0$,

$$\lambda|\xi|^2 \le \sum_{i,k=1}^{n+1} L_{ik}(x)\xi_i\xi_k \le \Lambda|\xi|^2 \qquad \text{for } \xi\in\mathbb{R}^{n+1}.$$

(A4) $W_i : Q_\tau \times \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}$ $(i = 1, \ldots, n+1)$ are Carathéodory functions satisfying

$$\sum_{k=1}^{n}(W_k(u,V) - W_k(\hat{u},\hat{V}))(u_k - \hat{u}_k) \le 0,$$

$$e \cdot W(u,V) = 0, \quad W_{n+1}(x,t,u,V)(u_{n+1} - \overline{u}) \le 0,$$

$$|W_i(x,t,u,V)| \le c(1 + |u| + |V|), \quad i = 1, \ldots, n+1,$$

for all u, V, $\hat{u}$, $\hat{V}$, with $\overline{u} < 0$ and $e = (e_1, \ldots, e_n, 0)$.

(A5) $u_D \in H^1(0,\tau; H^1(\Omega;\mathbb{R}^{n+1}))$, $V_D \in C^0([0,\tau]; L^\infty(\Omega)) \cap L^\infty(0,\tau;\, W^{2,p}(\Omega)) \cap H^1(0,\tau;\, W^{1,p_0}(\Omega))$ with $p_0 = 2$ if $d = 1$, $p_0 > 2$ if $d = 2$, $p_0 = 3$ if $d = 3$, and $p > d$, $u^0 \in L^\infty(\Omega;\mathbb{R}^{n+1}) \cap H^1(\Omega;\mathbb{R}^{n+1})$, $u_{D,n+1} = \overline{u} = \text{const}$.

We discuss the assumptions. The condition that Γ_N is relatively open *and* closed means that the intersection of $\overline{\Gamma}_D$ and Γ_N is empty. This condition is needed to conclude that the electric field $-\nabla V(t)$ is an essentially bounded function. For general mixed boundary conditions, this regularity may fail at points where the Dirichlet and Neumann boundary segments meet. Indeed, for general "smooth" boundaries, it is well known that we can only expect the regularity $\nabla V(t) \in W^{1,4/3-\varepsilon}(\Omega)$ for all $\varepsilon > 0$ [3]. For polygonal domains in $\mathbb{R}^2$, we get $\nabla V(t) \in W^{1,s}(\Omega) \hookrightarrow L^\infty(\Omega)$ for some $s > 2$ if the angle between a Dirichlet and Neumann boundary side is smaller than $\pi/2$ (see [20]). The monotonicity condition for W_{n+1} in assumption (A4) means that W_{n+1} is a "relaxation" term. For the energy-transport model for semiconductors, this condition is verified (see (1.8)). Furthermore, for the *bipolar* energy-transport model where $n = 2$, the source terms are monotone and satisfy $e_1 W_1 + e_2 W_2 = 0$ (see [11] for details). Therefore, the hypothesis (A4) is satisfied in this case. Finally, we assume in (A5) that the temperature u_{n+1} is constant at the contacts, i.e. we neglect surface thermal effects. The term W_{n+1} relaxes to the constant temperature at the boundary (assumption (A4)).

For the method of lines, introduce uniform partitions $0 = t_0 < t_1 < \cdots < t_N = \tau$ of $[0,\tau]$ such that $h \stackrel{\text{def}}{=} t_j - t_{j-1} < 1$. We denote by $C_N(0,\tau;X)$ (for a Banach space X) the space of all functions $u^{(N)} : (0,\tau] \to X$ which are constant on $(t_{j-1}, t_j]$, and we set $u^{(N)}(t) = u^j$ for $t \in (t_{j-1}, t_j]$, $j = 1, \ldots, N$.

Let $u_D^{(N)}$, $V_D^{(N)} \in C_N(0,\tau; H^1(\Omega))$ be defined by

$$u_D^j \stackrel{\text{def}}{=} h^{-1}\int_{t_{j-1}}^{t_j} u_D(\tau)d\tau,$$

$$V_D^j \stackrel{\text{def}}{=} h^{-1}\int_{t_{j-1}}^{t_j} V_D(\tau)d\tau$$

for $j = 1, \ldots, N$. Furthermore, define the linear interpolation of $u^{(N)} \in C_N(0,\tau; L^2(\Omega))$ by

$$\tilde{u}^{(N)}(x,t) = h^{-1}(t_j - t)(u^j(x) - u^{j-1}(x)) + u^j(x) \quad \text{for } x \in \Omega,\ t \in (t_{j-1}, t_j],$$

for $j = 1, \ldots, N$, where u^0 is the initial value of (A5). Then (see [25, Lemma 5.2.5]),

$$\begin{aligned} \tilde{u}_D^{(N)} &\to u_D && \text{in } H^1(H^1), && (2.1)\\ V_D^{(N)} &\to V_D && \text{in } L^2(W^{2,p}) \quad \text{as } N \to \infty. && (2.2) \end{aligned}$$

The time discretization of (1.1)-(1.7) is, for $j = 1, \ldots, N$,

$$\begin{aligned} \frac{\partial \tilde{\rho}_i^{(N)}}{\partial t} + \operatorname{div} J_i^{(N)} &= W_i(u^{(N)}, V^{(N)}), \qquad i = 1, \ldots, n, && (2.3)\\ \frac{\partial \tilde{\rho}_{n+1}^{(N)}}{\partial t} + \operatorname{div} J_{n+1}^{(N)} &= W_{n+1}(u^{(N)}, V^{(N)}) - \sum_{k=1}^n e_k J_k^{(N)} \cdot V^{(N)}, && (2.4)\\ \Delta V^{(N)} &= -\sum_{k=1}^n e_k \rho_k(u^{(N)}) - C(x), && (2.5)\\ J_i^{(N)} &= -\sum_{k=1}^{n+1} L_{ik}(x)(\nabla u_k^{(N)} - e_k u_{n+1}^{(N)} \nabla V^{(N)}),\\ &\quad\ i = 1, \ldots, n+1, && (2.6) \end{aligned}$$

subject to the boundary and initial conditions

$$\begin{aligned} u^{(N)} = u_D^{(N)}, \quad V^{(N)} = V_D^{(N)} \quad & \text{on } \Gamma_D \times (0,\tau), && (2.7)\\ J_i^{(N)} \cdot \nu = \nabla V^{(N)} \cdot \nu = 0 \quad & \text{on } \Gamma_N \times (0,\tau), \quad i = 1, \ldots, n+1, && (2.8)\\ \rho(u^{(N)}(\cdot, 0)) = \rho(u^0) \quad & \text{in } \Omega, && (2.9) \end{aligned}$$

where

$$\tilde{\rho}_i^{(N)}(\cdot, t) = h^{-1}(t_j - t)(\rho_i(u^j) - \rho_i(u^{j-1})) + \rho_i(u^j) \quad \text{for } t \in (t_{j-1}, t_j].$$

For given $\rho(u^0)$ and V^0 (computed from $\Delta V^0 = -e \cdot \rho(u^0) - C$ with boundary conditions $V^0 = V_D(0)$ on $\partial\Omega$), the equations (2.3)-(2.9) define recursively (u^j, V^j), for $j = 1, \ldots, N$.

Using the discrete *entropic variables*

$$w^j = u^j - e u_{n+1}^j V^j, \qquad j \geq 0, \tag{2.10}$$

the above discrete system is equivalent, for $j = 1, \ldots, N$, to

$$\begin{aligned}
\frac{\partial \tilde{b}_i^{(N)}}{\partial t} + \operatorname{div} I_i^{(N)} &= Q_i(w^{(N)}, V^{(N)}), \quad i = 1, \ldots, n+1, && (2.11)\\
\Delta V^{(N)} &= -e \cdot b(w^{(N)}, V^{(N)}) - C(x), && (2.12)\\
I_i^{(N)} &= -\sum_{k=1}^{n+1} D_{ik}(x, V^{(N)}) \nabla w_k^{(N)}, \qquad i = 1, \ldots, n+1, && (2.13)
\end{aligned}$$

subject to the boundary and initial conditions

$$\begin{aligned}
w^{(N)} = w_D^{(N)}, \quad V^{(N)} = V_D^{(N)} &\qquad \text{on } \Gamma_D \times (0, \tau), && (2.14)\\
I_i^{(N)} \cdot \nu = \nabla V^{(N)} \cdot \nu = 0 &\qquad \text{on } \Gamma_N \times (0, \tau), && (2.15)\\
\tilde{b}_i^{(N)}(0) = b_i(w^0, V^0) &\qquad \text{in } \Omega. && (2.16)
\end{aligned}$$

where $w_D^j = u_D^j - e u_{D,n+1}^j V_D^j \in H^1(\Omega; \mathbb{R}^{n+1})$ thanks to (A5), $w^0 = u^0 - e u_{n+1}^0 V^0$, and, for $t \in (t_{j-1}, t_j]$,

$$\frac{\partial \tilde{b}_i^{(N)}}{\partial t}(t) = h^{-1}(b_i(w^j, V^j) - b_i(w^{j-1}, V^{j-1})), \quad i = 1, \cdots, n+1.$$

We have used the definitions:

$$\begin{aligned}
b_i(w, V) &= \rho_i(u), \quad i = 1, \ldots, n,\\
b_{n+1}(w, V) &= \rho_{n+1}(u) + V e \cdot \rho(u),\\
Q_i(w, V) &= W_i(u, V), \quad i = 1, \ldots, n+1,
\end{aligned}$$

and the new diffusion coefficients are given by

$$\begin{aligned}
D_{ik} &= L_{ik}, \quad i, k = 1, \ldots, n,\\
D_{i,n+1} &= D_{n+1,i} = L_{i,n+1} + V \sum_{k=1}^{n} e_k L_{ik}, \quad i = 1, \ldots, n, && (2.17)\\
D_{n+1,n+1} &= L_{n+1,n+1} + 2V \sum_{k=1}^{n} e_k L_{n+1,k} + V^2 \sum_{i,k=1}^{n} e_i e_k L_{ik}.
\end{aligned}$$

The diffusion matrix $D = (D_{ik})$ is symmetric and positive definite. Indeed, it holds $D = P^T L P$, where $P = (P_{ik})$ is the regular matrix defined by

$$P_{ik} = \begin{cases} 1 & \text{if } i = k \in \{1, \ldots, n+1\} \\ e_i V & \text{if } k = n+1, \ i \in \{1, \ldots, n\} \\ 0 & \text{else.} \end{cases}$$

Thus, if L is symmetric, positive definite, then D is symmetric, positive definite, too.

We note that the transformation (2.10) is well known in nonequilibrium thermodynamics [22, §53] and is also used in the drift-diffusion model for semiconductors (i.e. $T = \text{const.}$), where w_i is called quasi-Fermi potential. Recently, this transformation appears naturally in the derivation of an energy-transport model for semiconductor heterostructures [13]. The symmetrization property of the transformation (2.10) is also observed by Albinus [1] in the case of the energy-transport model.

Now it follows from [11]:

Theorem 2.1 *Let the assumptions (A1)-(A5) hold. Then:*

(i) There exists a solution $(u^{(N)}, V^{(N)})$ *to (2.3)-(2.9) satisfying* $u^{(N)} \in L^\infty(L^2) \cap L^2(H^1)$, $\tilde{\rho}_i^{(N)} \in H^1(\mathcal{V}^*)$ $(i = 1, \ldots, n+1)$, *and* $V^{(N)} \in L^\infty(L^\infty) \cap L^\infty(W^{2,p})$.

(ii) There exists a solution $(w^{(N)}, V^{(N)})$ *to (2.11)-(2.16) satisfying* $w^{(N)} \in L^\infty(L^2) \cap L^2(H^1)$, $\tilde{b}_i^{(N)} \in H^1(\mathcal{V}^*)$ $(i = 1, \ldots, n+1)$, *and* $V^{(N)} \in L^\infty(L^\infty) \cap L^\infty(W^{2,p})$.

(iii) There exists a constant $c > 0$ *independent of* N *such that*

$$\begin{aligned} \|u^{(N)}\|_{L^\infty(L^2)} + \|u^{(N)}\|_{L^2(H^1)} + \|J_i^{(N)}\|_{L^2(L^2)} + \|\tilde{\rho}_i^{(N)}\|_{H^1(\mathcal{V}^*)} \\ + \|V^{(N)}\|_{L^\infty(L^\infty)} \le c. \end{aligned} \tag{2.18}$$

(iv) There exists a subsequence of $(u^{(N)}, V^{(N)})$ *(not relabeled) such that*

$$\begin{aligned} u^{(N)} &\to u && \text{in } L^2(L^2), \\ \tilde{\rho}^{(N)} &\to \rho(u) && \text{in } L^2(L^2), \\ V^{(N)} &\to V && \text{in } L^2(H^1) \quad \text{as } N \to \infty, \end{aligned}$$

and (u, V) *is a solution to (1.1)-(1.7).*

3 Regularity of solutions

Our first regularity result is as follows:

Theorem 3.1 *Let the hypotheses (A1)-(A5) hold. Then there exists a solution* (u, V) *to (1.1)-(1.7) satisfying*

$$u \in L^\infty(H^1) \cap H^1(L^2), \qquad V \in L^\infty(W^{2,p}),$$

where $p > 2/\alpha \ge d$. *In particular,* $\nabla V \in L^\infty(Q_\tau)$.

At first sight, it may appear advantageously to use the formulation (2.11)-(2.13) in the entropic variables $(w^{(N)}, V^{(N)})$ to prove the above regularity result, since the gradient terms $J_k^{(N)} \cdot \nabla V^{(N)}$ and $\mathrm{div}\,(u_{n+1}^{(N)} \nabla V^{(N)})$ disappear. However, in this formulation the problem becomes quasilinear with a diffusion matrix depending on $V^{(N)}(x,t)$ (see (2.17)). Using our technique of proof, it can be seen that we would need $L^\infty(Q_\tau)$ estimates for $\partial V^{(N)}/\partial t$ which are not available. Therefore, we use the formulation in the conservative variables $(u^{(N)}, V^{(N)})$.

Proof of Theorem 3.1. We show that $u^{(N)}$ and $V^{(N)}$ are uniformly bounded in $L^\infty(H^1) \cap H^1(L^2)$, $L^\infty(W^{2,p})$, respectively. Then the conclusion of the theorem follows after letting $N \to \infty$.

First step: an estimate for $\nabla V^{(N)}$. Thanks to elliptic regularity theory and Sobolev embeddings (see, e.g., [19, 33]) it holds, since $p > d$,

$$\begin{aligned} \|\nabla V^{(N)}\|_{L^\infty(L^\infty)} &\leq c\|V^{(N)}\|_{L^\infty(W^{2,p})} \leq c(1 + \|\rho(u^{(N)})\|_{L^\infty(L^p)}) \\ &\leq c(1 + \|u^{(N)}\|_{L^\infty(L^p)}), \end{aligned}$$

where we have used (2.2) and the hypothesis (A2). Employing the Gagliardo-Nirenberg inequality [8] we get

$$\|u^j\|_{0,p,\Omega} \leq c\|u^j\|_{1,2,\Omega}^\theta \|u^j\|_{0,2,\Omega}^{1-\theta}, \qquad j = 1, \ldots, N,$$

where $\theta \in [0,1]$ if $d = 1$ and $\theta = d(p-2)/2p$ if $d = 2, 3$. Young's inequality yields for every $\varepsilon > 0$

$$\|u^j\|_{0,p,\Omega} \leq \varepsilon\|u^j\|_{1,2,\Omega} + c(\varepsilon)\|u^j\|_{0,2,\Omega},$$

such that we obtain for every $\delta > 0$

$$\|\nabla V^{(N)}\|_{L^\infty(L^\infty)}^2 \leq \delta\|\nabla u^{(N)}\|_{L^\infty(L^2)}^2 + c(\delta), \tag{3.1}$$

taking into account (2.18). The constant $c(\delta) > 0$ is independent of N.

Second step: estimates for $u^{(N)}$. Let $m \in \{1, \ldots, N\}$ and use $(\partial/\partial t)\ (\tilde{u}^{(N)} - \tilde{u}_D^{(N)}) \in L^2(\mathcal{V})$ as test function in the weak formulation of (2.3)-(2.4):

$$\begin{aligned} \int_{Q_{t_m}} \frac{\partial \tilde{\rho}^{(N)}}{\partial t} \cdot \frac{\partial}{\partial t}(\tilde{u}^{(N)} - \tilde{u}_D^{(N)}) - \int_{Q_{t_m}} \sum_{k=1}^{n+1} J_k^{(N)} \cdot \nabla \frac{\partial}{\partial t}(\tilde{u}_k^{(N)} - \tilde{u}_{D,k}^{(N)}) & \\ = \int_{Q_{t_m}} \sum_{k=1}^{n+1} W_k(u^{(N)}, V^{(N)}) \frac{\partial}{\partial t}(\tilde{u}_k^{(N)} - \tilde{u}_{D,k}^{(N)}) & \\ - \int_{Q_{t_m}} \sum_{k=1}^{n} e_k J_k^{(N)} \cdot \nabla V^{(N)} \frac{\partial}{\partial t}(\tilde{u}_{n+1}^{(N)} - \tilde{u}_{D,n+1}^{(N)}). & \end{aligned} \tag{3.2}$$

We estimate both sides term by term:

$$\begin{aligned}
&\int_{Q_{t_m}} \frac{\partial \tilde\rho^{(N)}}{\partial t} \cdot \frac{\partial}{\partial t}(\tilde u^{(N)} - \tilde u_D^{(N)}) \\
&= \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \int_\Omega h^{-2}(\rho(u^j) - \rho(u^{j-1})) \cdot ((u^j - u^{j-1}) - (u_D^j - u_D^{j-1})) \\
&\geq \frac{c_0}{2} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \int_\Omega h^{-2}|u^j - u^{j-1}|^2 - c \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \int_\Omega h^{-2}|u_D^j - u_D^{j-1}|^2 \\
&\geq \frac{c_0}{2} \int_{Q_{t_m}} \left|\frac{\partial \tilde u^{(N)}}{\partial t}\right|^2 - c,
\end{aligned}$$

where we have used the condition (A2) and (2.1). The second integral on the left hand side of (3.2) is splitted into several integrals:

$$\begin{aligned}
&-\int_{Q_{t_m}} \sum_{\ell=1}^{n+1} J_\ell^{(N)} \cdot \nabla \frac{\partial}{\partial t}(\tilde u_\ell^{(N)} - \tilde u_{D,\ell}^{(N)}) \\
&= \int_{Q_{t_m}} \sum_{k,\ell=1}^{n+1} L_{k\ell}(x) \nabla \tilde u_k^{(N)} \cdot \frac{\partial}{\partial t} \nabla \tilde u_\ell^{(N)} \\
&\quad + \int_{Q_{t_m}} \sum_{k,\ell=1}^{n+1} L_{k\ell}(x) \nabla (u_k^{(N)} - \tilde u_k^{(N)}) \cdot \frac{\partial}{\partial t} \nabla \tilde u_\ell^{(N)} \\
&\quad + \int_{Q_{t_m}} \sum_{k,\ell=1}^{n+1} L_{k\ell}(x) \nabla u_k^{(N)} \cdot \frac{\partial}{\partial t} \nabla \tilde u_{D,\ell}^{(N)} \\
&\quad - \int_{Q_{t_m}} \sum_{k,\ell=1}^{n+1} L_{k\ell}(x) e_k u_{n+1}^{(N)} \nabla V^{(N)} \cdot \nabla \frac{\partial}{\partial t}(\tilde u_\ell^{(N)} - \tilde u_{D,\ell}^{(N)}) \\
&= K_1 + \cdots + K_4.
\end{aligned}$$

For the estimate of the integral K_1 we use the facts that L_{ik} is symmetric and does not depend on $u^{(N)}$ or $V^{(N)}$:

$$\begin{aligned}
K_1 &= \frac{1}{2} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \frac{\partial}{\partial t} \int_\Omega \sum_{k,\ell=1}^{n+1} L_{k\ell}(x) \nabla \tilde u_k^j \cdot \nabla \tilde u_\ell^j \\
&= \frac{1}{2} \int_\Omega \sum_{k,\ell=1}^{n+1} L_{k\ell}(x) (\nabla u_k^m \cdot \nabla u_\ell^m - \nabla u_k^0 \cdot \nabla u_\ell^0) \\
&\geq \frac{\lambda}{2} \int_\Omega |\nabla u^{(N)}(t_m)|^2 - \frac{\Lambda}{2} \int_\Omega |\nabla u^0|^2 \\
&\geq \frac{\lambda}{2} \int_\Omega |\nabla u^{(N)}(t_m)|^2 - c.
\end{aligned}$$

The integral K_2 is estimated by employing again the assumption (A3):

$$
\begin{aligned}
K_2 &= \sum_{j=1}^m \int_{t_{j-1}}^{t_j} h^{-2}(t_j - t) \int_\Omega \sum_{k,\ell=1}^{n+1} L_{k\ell}(x) \nabla(u_k^j - u_k^{j-1}) \cdot \nabla(u_\ell^j - u_\ell^{j-1}) \\
&\geq \lambda \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - t) \int_\Omega \Big| \frac{\partial}{\partial t} \nabla \tilde{u}^{(N)} \Big|^2 \\
&= \frac{\lambda}{2} h \int_{Q_{t_m}} \Big| \frac{\partial}{\partial t} \nabla \tilde{u}^{(N)} \Big|^2.
\end{aligned}
$$

For every $\varepsilon > 0$ we have, using (2.1),

$$
\begin{aligned}
K_3 &\geq -\frac{\varepsilon}{t_m} \int_{Q_{t_m}} |\nabla u^{(N)}|^2 - c(\varepsilon) \int_{Q_{t_m}} \Big| \frac{\partial}{\partial t} \nabla \tilde{u}_D^{(N)} \Big|^2 \\
&\geq -\varepsilon \|\nabla u^{(N)}\|^2_{L^\infty(0,t_m;L^2)} - c(\varepsilon).
\end{aligned}
$$

For the last integral K_4 we employ Green's formula:

$$
\begin{aligned}
K_4 &= \int_{Q_{t_m}} \sum_{k,\ell=1}^{n+1} e_k \Big(\nabla L_{k\ell}(x) \cdot \nabla V^{(N)} u_{n+1}^{(N)} + L_{k\ell}(x) \nabla u_{n+1}^{(N)} \cdot \nabla V^{(N)} \\
&\quad + L_{k\ell}(x) u_{n+1}^{(N)} \Delta V^{(N)} \Big) \frac{\partial}{\partial t} (\tilde{u}_\ell^{(N)} - \tilde{u}_{D,\ell}^{(N)}) \\
&\geq -\frac{c_0}{8} \int_{Q_{t_m}} \Big| \frac{\partial \tilde{u}^{(N)}}{\partial t} \Big|^2 - c \int_{Q_{t_m}} \Big| \frac{\partial \tilde{u}_D^{(N)}}{\partial t} \Big|^2 \\
&\quad - c \int_{Q_{t_m}} \sum_{k,\ell=1}^{n+1} \Big(|\nabla L_{k\ell}|^2 |\nabla V^{(N)}|^2 |u_{n+1}^{(N)}|^2 + |L_{k\ell}|^2 |\nabla u_{n+1}^{(N)}|^2 |\nabla V^{(N)}|^2 \\
&\quad + |L_{k\ell}|^2 |u_{n+1}^{(N)}|^2 |\Delta V^{(N)}|^2 \Big).
\end{aligned}
$$

The last integral is estimated as follows: Let $q = r/(r-2)$ (see condition (A3) for the definition of r). Then the embedding $H^1(\Omega) \hookrightarrow L^{2q}(\Omega)$ is continuous and, by Hölder's inequality,

$$
\begin{aligned}
\int_{Q_{t_m}} |\nabla L_{k\ell}|^2 |\nabla V^{(N)}|^2 |u_{n+1}^{(N)}|^2 &\leq \|\nabla L_{k\ell}\|^2_{L^\infty(L^r)} \|\nabla V^{(N)}\|^2_{L^\infty(L^\infty)} \|u_{n+1}^{(N)}\|^2_{L^2(L^{2q})} \\
&\leq c \|L_{k\ell}\|^2_{L^\infty(W^{1,r})} \|\nabla V^{(N)}\|^2_{L^\infty(L^\infty)} \|u_{n+1}^{(N)}\|^2_{L^2(H^1)} \\
&\leq c\delta \|\nabla u^{(N)}\|^2_{L^\infty(0,t_m;L^2)} + c(\delta),
\end{aligned}
$$

using (3.1). Furthermore, we have

$$
\begin{aligned}
\int_{Q_{t_m}} |L_{k\ell}|^2 |\nabla u_{n+1}^{(N)}|^2 |\nabla V^{(N)}|^2 &\leq \|L_{k\ell}\|^2_{L^\infty(L^\infty)} \|\nabla u_{n+1}^{(N)}\|^2_{L^2(L^2)} \|\nabla V^{(N)}\|^2_{L^\infty(L^\infty)} \\
&\leq c\delta \|\nabla u^{(N)}\|^2_{L^\infty(0,t_m;L^2)} + c(\delta).
\end{aligned}
$$

In this estimate, the boundedness of $\nabla V^{(N)}$ in $L^\infty(L^\infty)$ is needed. Finally,

$$\begin{aligned}\int_{Q_{t_m}} |L_{k\ell}|^2 |u_{n+1}^{(N)}|^2 |\Delta V^{(N)}|^2 &\le c\|L_{k\ell}\|^2_{L^\infty(L^\infty)} \int_{Q_{t_m}} (1+|u^{(N)}|^\alpha)^2 |u_{n+1}^{(N)}|^2 \\ &\le c\int_{Q_{t_m}} |u^{(N)}|^{2(\alpha+1)} + c.\end{aligned}$$

For the estimate of the last integral, observe that we have $2(\alpha+1)\le 10/3$ if $d=3$ and $2(\alpha+1)=4$ if $d\le 2$. Therefore, using [15, Prop. I.3.2], we get

$$\|u^{(N)}\|_{0,2(\alpha+1),Q_{t_m}} \le c(\|u^{(N)}\|_{L^\infty(L^2)} + \|u^{(N)}\|_{L^2(H^1)}).$$

Taking into account the a priori estimate (2.18), we conclude

$$\int_{Q_{t_m}} |L_{k\ell}|^2 |u_{n+1}^{(N)}|^2 |\Delta V^{(N)}|^2 \le c,$$

which implies, together with the other above estimates,

$$K_4 \ge -\frac{c_0}{8}\int_{Q_{t_m}} \left|\frac{\partial \tilde u^{(N)}}{\partial t}\right|^2 - c\delta\|\nabla u^{(N)}\|^2_{L^\infty(0,t_m;L^2)} - c(\delta).$$

Now we turn to the terms on the right hand side of (3.2):

$$\begin{aligned}&\int_{Q_{t_m}} \sum_{k=1}^{n+1} W_k(u^{(N)},V^{(N)})\frac{\partial}{\partial t}(\tilde u_k^{(N)} - \tilde u_{D,k}^{(N)}) \\ &\le \frac{c_0}{8}\int_{Q_{t_m}} \left|\frac{\partial \tilde u^{(N)}}{\partial t}\right|^2 + c\int_{Q_{t_m}} \left|\frac{\partial \tilde u_D^{(N)}}{\partial t}\right|^2 + c\int_{Q_{t_m}} (1+|u^{(N)}|^2+|V^{(N)}|^2) \\ &\le \frac{c_0}{8}\int_{Q_{t_m}} \left|\frac{\partial \tilde u^{(N)}}{\partial t}\right|^2 + c,\end{aligned}$$

using (2.18) and (2.1). Finally, again employing (2.18) and (3.1),

$$\begin{aligned}&-\int_{Q_{t_m}} \sum_{k=1}^{n} e_k J_k^{(N)}\cdot\nabla V^{(N)}\frac{\partial}{\partial t}(\tilde u_{n+1}^{(N)} - \tilde u_{D,n+1}^{(N)}) \\ &\le \frac{c_0}{8}\int_{Q_{t_m}} \left|\frac{\partial \tilde u_{n+1}^{(N)}}{\partial t}\right|^2 + c\int_{Q_{t_m}} \left|\frac{\partial \tilde u_{D,n+1}^{(N)}}{\partial t}\right|^2 \\ &\quad + c\sum_{k=1}^{n} \|J_k^{(N)}\|^2_{L^2(L^2)} \|\nabla V^{(N)}\|^2_{L^\infty(L^\infty)} \\ &\le \frac{c_0}{8}\int_{Q_{t_m}} \left|\frac{\partial \tilde u^{(N)}}{\partial t}\right|^2 + c\delta\|\nabla u^{(N)}\|^2_{L^\infty(0,t_m;L^2)} + c(\delta).\end{aligned}$$

Hence we get from (3.2):

$$\frac{c_0}{8}\int_{Q_{t_m}}\Big|\frac{\partial\tilde{u}^{(N)}}{\partial t}\Big|^2+\frac{\lambda}{2}\int_\Omega|\nabla u^{(N)}(t_m)|^2+\frac{\lambda}{2}h\int_{Q_{t_m}}\Big|\nabla\frac{\partial\tilde{u}^{(N)}}{\partial t}\Big|^2$$
$$\leq\ c(\varepsilon,\delta)+(\varepsilon+c\delta)\|\nabla u^{(N)}\|^2_{L^\infty(0,t_m;L^2)}.$$

This implies

$$\frac{c_0}{8}\int_{Q_\tau}\Big|\frac{\partial\tilde{u}^{(N)}}{\partial t}\Big|^2+(\lambda/2-\varepsilon-c\delta)\sup_{t\in(0,\tau)}\int_\Omega|\nabla u^{(N)}(t)|^2\leq c(\varepsilon,\delta).$$

Therefore, choosing $\varepsilon>0$ and $\delta>0$ small enough, we obtain

$$\int_{Q_\tau}\Big|\frac{\partial\tilde{u}^{(N)}}{\partial t}\Big|^2+\sup_{t\in(0,\tau)}\int_\Omega|\nabla u^{(N)}(t)|^2\leq c,$$

where $c>0$ does not depend on N. The conclusion of the theorem follows. $\square$

Using a regularity result for *elliptic* systems due to [21, 29], we can improve the regularity of the transient solution with respect to the space variable:

Theorem 3.2 *Let the hypotheses (A1)-(A5) hold and let $u_D\in L^2(W^{1,q_0})$ for some $q_0>2$. Then there exists $q\in(2,q_0]$ and a solution (u,V) of (1.1)-(1.7) such that $\nabla u\in L^q(Q_\tau)$. Moreover, $u\in C^0(\overline{Q}_\tau)$ if $d=1$ and $u\in L^s(Q_\tau)$ for all $s<\infty$ if $d=2$.*

Proof. We show that the sequence of approximate solutions $(u^{(N)})$ (see Theorem 2.1) is uniformly bounded in the space $L^q(W^{1,q})$. We rewrite the problem (2.11)-(2.12) in the entropic variables $(w^{(N)},V^{(N)})$:

$$\operatorname{div}\sum_{k=1}^{n+1}D_{ik}(x,V^j)\nabla w_k^j=f_i^j\qquad\text{in }\Omega,\quad i=1,\ldots,n+1,$$
$$w^j=w_D^j\ \text{ on }\Gamma_D,\quad\sum_{k=1}^{n+1}D_{ik}(x,V^j)\nabla w_k^j\cdot\nu=0\ \text{ on }\Gamma_N,\qquad j=1,\ldots,N,$$

where f_i^j is defined by

$$f_i^j=\frac{\partial\tilde{\rho}_i^{(N)}(t_j)}{\partial t}-W_i(u^j,V^j)\quad\text{if }i=1,\ldots,n,$$
$$f_{n+1}^j=\frac{\partial\tilde{\rho}_{n+1}^{(N)}(t_j)}{\partial t}-W_i(u^j,V^j)-V^je\cdot\frac{\partial\tilde{\rho}^{(N)}(t_j)}{\partial t}\quad\text{if }i=n+1.$$

Thanks to the a priori estimate (2.18), the diffusion matrix (D_{ik}) is uniformly positive definite, and there exist $\delta_0, \delta_1 > 0$ independent of j and N such that:

$$\delta_0|\xi|^2 \le \sum_{i,k=1}^{n+1} D_{ik}(x, V^j)\xi_i\xi_k \le \delta_1|\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^{n+1}.$$

We claim that $f_i^{(N)} = (f_i^1, \ldots, f_i^N)$ is uniformly bounded in $L^2(L^2)$. Indeed, thanks to Theorem 3.1 and assumption (A4) it holds

$$\begin{aligned}
\|f_i^{(N)}\|_{L^2(L^2)} &\le c\Big(1 + \Big\|\frac{\partial \tilde{\rho}_i^{(N)}}{\partial t}\Big\|_{L^2(L^2)} + \|u^{(N)}\|_{L^2(L^2)} + \|V^{(N)}\|_{L^2(L^2)}\Big) \le c \\
&\quad \text{if } i = 1, \ldots, n, \\
\|f_{n+1}^{(N)}\|_{L^2(L^2)} &\le c\Big(1 + \Big\|\frac{\partial \tilde{\rho}_{n+1}^{(N)}}{\partial t}\Big\|_{L^2(L^2)} + \|u^{(N)}\|_{L^2(L^2)} + \|V^{(N)}\|_{L^2(L^2)} \\
&\quad + \|V^{(N)}\|_{L^\infty(L^\infty)}\Big\|\frac{\partial \tilde{\rho}^{(N)}}{\partial t}\Big\|_{L^2(L^2)}\Big) \le c \quad \text{if } i = n+1.
\end{aligned}$$

The regularity result in [21, 29] now implies the existence of $s \in (2, q_0]$ such that $w_i^j \in W^{1,s}(\Omega)$ for all i, j. Furthermore,

$$\|w^j\|_{1,s,\Omega} \le c(1 + \|f_i^j\|_{0,2,\Omega} + \|w^j\|_{1,2,\Omega}),$$

which yields

$$\|w^{(N)}\|_{L^2(W^{1,s})} \le c(1 + \|f^{(N)}\|_{L^2(L^2)} + \|w^{(N)}\|_{L^2(H^1)}) \le c,$$

using (2.18), where $c > 0$ is independent of N. Therefore, letting $N \to \infty$, we get $w \in L^2(W^{1,s})$. It follows $u \in L^2(W^{1,s})$. Furthermore, since u lies in the space $L^\infty(H^1)$, by Theorem 3.1, and applying Hölder's inequality, we get for $q = s/2+1 > 2$:

$$\begin{aligned}
\int_0^\tau \|\nabla u\|_{0,q,\Omega}^q &\le \int_0^\tau \|\nabla u\|_{0,2,\Omega}\|\nabla u\|_{0,s,\Omega}^{s/2} \le \|\nabla u\|_{L^\infty(L^2)}\|\nabla u\|_{L^{s/2}(L^s)}^{s/2} \\
&\le \|u\|_{L^\infty(H^1)}\|u\|_{L^{s/2}(W^{1,s})}^{s/2} < \infty,
\end{aligned}$$

which implies $\nabla u \in L^q(Q_\tau)$. We conclude from $u^{(N)} \in L^\infty(H^1)\cap H^1(L^2)$ and Aubin's Lemma [31] that $u \in C^0([0,\tau]; C^0(\overline{\Omega})) = C^0(\overline{Q}_\tau)$ if $d = 1$, since the embedding $H^1(\Omega) \hookrightarrow C^0(\overline{\Omega})$ is compact. The last regularity result for $d = 2$ follows from $u \in L^\infty(H^1)$ and the Sobolev embedding $H^1(\Omega) \hookrightarrow L^s(\Omega)$ for all $s < \infty$. □

Remark 3.3 *(i)* It is not difficult to see that we even get the regularity $\nabla u \in L^\sigma(L^q)$ with $\sigma = 2q/(q-1) < 4$.

(ii) The continuity of u in $\overline{Q}_\tau$ implies that the temperature $T = -1/u_{n+1}$ is strictly bounded away from zero in $\overline{Q}_\tau$.

4 Uniqueness of solutions

We first show the uniqueness of solutions in a class slightly larger than the class of weak solutions.

Theorem 4.1 *Let the hypotheses (A1)-(A5) hold. Furthermore, let $W_i(x,t,\ u,V)$ be globally Lipschitz continuous in (u,V) uniformly in (x,t) for $i=1,\ldots,n+1$. Then there exists a unique solution (u,V) of (1.1)-(1.7) in the class of functions satisfying $u \in L^\infty(W^{1,3/2})$, $V \in L^\infty(W^{1,\infty})$.*

Proof. Let (u^1,V^1), (u^2,V^2) be two solutions to (1.1)-(1.7) satisfying the above regularity properties. Set $r=\rho(u^1)-\rho(u^2) \in L^2((\mathcal{V}^*)^{n+1})$. By Lax-Milgram's lemma, there exists $y \in L^2(\mathcal{V}^{n+1})$ such that for all $\phi \in L^2(\mathcal{V}^{n+1})$

$$\int_{Q_\tau} \sum_{k,\ell=1}^{n+1} L_{k\ell}(x)\nabla y_k \cdot \nabla \phi_\ell = \int_0^\tau \langle r,\phi\rangle, \tag{4.1}$$

where $\langle\cdot,\cdot\rangle$ is the usual duality pairing between $(\mathcal{V}^*)^{n+1}$ and $\mathcal{V}^{n+1}$. Taking the difference of the equations satisfied by (u^1,V^1) and (u^2,V^2), respectively, we get, for $t \in (0,\tau)$,

$$\begin{aligned} &\int_0^t \langle \partial_t r, y\rangle - \int_{Q_t} \sum_{k=1}^{n+1} (J_k^1 - J_k^2)\cdot\nabla y_k \\ &= \int_{Q_t} \sum_{k=1}^{n+1} (W_k(u^1,V^1) - W_k(u^2,V^2)) y_k \\ &\quad - \int_{Q_t} \sum_{k=1}^{n} e_k (J_k^1\cdot\nabla V^1 - J_k^2\cdot\nabla V^2) y_{n+1}. \end{aligned} \tag{4.2}$$

In [2] the following equality is shown:

$$\int_0^t \langle \partial_t r, y\rangle = \frac{1}{2}\int_\Omega \sum_{k,\ell=1}^{n+1} L_{k\ell}(x)\nabla y_k(t)\cdot\nabla y_\ell(t)$$

for a.e. $t \in (0,\tau)$, which implies, thanks to the uniform positive definiteness of $(L_{k\ell})$,

$$\int_0^t \langle \partial_t r, y\rangle \geq \frac{\lambda}{2}\int_\Omega |\nabla y(t)|^2. \tag{4.3}$$

Furthermore, in view of (4.1) and assumption (A2) we have

$$\begin{aligned} \int_{Q_t} \sum_{k,\ell=1}^{n+1} L_{k\ell}(x)\nabla(u_\ell^1 - u_\ell^2)\cdot\nabla y_k &= \int_{Q_t} (\rho(u^1)-\rho(u^2))\cdot(u^1-u^2) \\ &\geq c_0 \int_{Q_t} |u^1-u^2|^2. \end{aligned}$$

Hence, we can rewrite (4.2) using this estimate together with (4.3):

$$
\begin{aligned}
&\frac{\lambda}{2}\int_\Omega |\nabla y(t)|^2 + c_0\int_{Q_t}|u^1-u^2|^2 \\
&\quad\le \int_{Q_t}\sum_{k,\ell=1}^n e_k L_{k\ell}(x)(u^1_{n+1}\nabla V^1 - u^2_{n+1}\nabla V^2)\cdot\nabla y_\ell \\
&\qquad + \int_{Q_t}\sum_{k=1}^{n+1}(W_k(u^1,V^1)-W_k(u^2,V^2))y_k \\
&\qquad - \int_{Q_t}\sum_{k,\ell=1}^n e_k L_{k\ell}(x)(\nabla u^1_\ell\cdot\nabla V^1 - \nabla u^2_\ell\cdot\nabla V^2)y_{n+1} \\
&\qquad + \int_{Q_t}\sum_{k,\ell=1}^n e_k e_\ell L_{k\ell}(x)(u^1_{n+1}|\nabla V^1|^2 - u^2_{n+1}|\nabla V^2|^2)y_{n+1} \\
&\quad= K_1+\cdots+K_4.
\end{aligned}
\tag{4.4}
$$

Now we estimate the integrals $K_1,\ldots,K_4$. Since it holds for $d\le 3$

$$
\begin{aligned}
\|\nabla(V^1-V^2)\|_{L^2(L^6)} &\le c\|V^1-V^2\|_{L^2(H^2)} \le c\|\rho(u^1)-\rho(u^2)\|_{L^2(L^2)} \\
&\le c\|u^1-u^2\|_{L^2(L^2)},
\end{aligned}
\tag{4.5}
$$

we get, using Hölder's inequality,

$$
\begin{aligned}
K_1 &= \int_{Q_t}\sum_{k,\ell=1}^n e_k L_{k\ell}(x)((u^1_{n+1}-u^2_{n+1})\nabla V^1 + u^2_{n+1}\nabla(V^1-V^2))\cdot\nabla y_\ell \\
&\le c\|u^1_{n+1}-u^2_{n+1}\|_{L^2(L^2)}\|\nabla V^1\|_{L^\infty(L^\infty)}\|\nabla y\|_{L^2(L^2)} \\
&\quad + c\|u^2_{n+1}\|_{L^\infty(L^3)}\|\nabla(V^1-V^2)\|_{L^2(L^6)}\|\nabla y\|_{L^2(L^2)} \\
&\le c\|u^1_{n+1}-u^2_{n+1}\|_{L^2(L^2)}\|\nabla y\|_{L^2(L^2)} \\
&\le \varepsilon\int_{Q_t}|u^1-u^2|^2 + c(\varepsilon)\int_{Q_t}|\nabla y|^2,
\end{aligned}
$$

where $\varepsilon>0$ and $c(\varepsilon)>0$ depends on the $L^\infty(\Omega)$ norm of $L_{k\ell}$, the $L^\infty(L^\infty)$ norm of ∇V^1, and the $L^\infty(L^3)$ norm of u^2_{n+1}. Note that for $d\le 3$ the space $W^{1,3/2}(\Omega)$ is embedded into $L^3(\Omega)$. The Lipschitz continuity of W_i and Poincaré's inequality imply

$$
\begin{aligned}
K_2 &\le c\int_{Q_t}(|u^1-u^2|+|V^1-V^2|)|y| \\
&\le c(\|u^1-u^2\|_{L^2(L^2)} + \|\nabla(V^1-V^2)\|_{L^2(L^2)})\|\nabla y\|_{L^2(L^2)} \\
&\le \varepsilon\int_{Q_t}|u^1-u^2|^2 + c(\varepsilon)\int_{Q_t}|\nabla y|^2.
\end{aligned}
$$

For the estimate of K_3 we use Green's formula:

$$
\begin{aligned}
K_3 \;&=\; \int_{Q_t} \sum_{k,\ell=1}^{n} e_k \Big(\nabla L_{k\ell}(x) \cdot (\nabla V^1 u_\ell^1 - \nabla V^2 u_\ell^2) y_{n+1} \\
&\quad + L_{k\ell}(x)(u_\ell^1 \Delta V^1 - u_\ell^2 \Delta V^2) y_{n+1} \\
&\quad + L_{k\ell}(x)(u_\ell^1 \nabla V^1 - u_\ell^2 \nabla V^2) \cdot \nabla y_{n+1} \Big) \\
&\le\; c \sum_{k,\ell=1}^{n} \Big(\|\nabla L_{k\ell}\|_{L^\infty(L^r)} \|y_{n+1}\|_{L^2(L^s)} \|\nabla V^1\|_{L^\infty(L^\infty)} \|u_\ell^1 - u_\ell^2\|_{L^2(L^2)} \\
&\quad + \|\nabla L_{k\ell}\|_{L^\infty(L^r)} \|y_{n+1}\|_{L^2(L^s)} \|u_\ell^2\|_{L^\infty(L^3)} \|\nabla(V^1 - V^2)\|_{L^2(L^6)} \\
&\quad + \|L_{k\ell}\|_{L^\infty(L^\infty)} \|y_{n+1}\|_{L^2(L^6)} \|u_\ell^1 - u_\ell^2\|_{L^2(L^2)} \|\Delta V^1\|_{L^\infty(L^3)} \\
&\quad + \|L_{k\ell}\|_{L^\infty(L^\infty)} \|y_{n+1}\|_{L^2(L^6)} \|u_\ell^2\|_{L^\infty(L^3)} \|\Delta(V^1 - V^2)\|_{L^2(L^2)} \\
&\quad + \|L_{k\ell}\|_{L^\infty(L^\infty)} \|\nabla y_{n+1}\|_{L^2(L^2)} \|u_\ell^1 - u_\ell^2\|_{L^2(L^2)} \|\nabla V^1\|_{L^\infty(L^\infty)} \\
&\quad + \|L_{k\ell}\|_{L^\infty(L^\infty)} \|\nabla y_{n+1}\|_{L^2(L^2)} \|u_\ell^2\|_{L^\infty(L^3)} \|\nabla(V^1 - V^2)\|_{L^2(L^6)} \Big),
\end{aligned}
$$

where $r \ge 2$ is defined in hypothesis (A3) and $s = 2r/(r-2)$. Observing that $H^1(\Omega) \hookrightarrow L^s(\Omega)$ and

$$
\|\Delta V^1\|_{L^\infty(L^3)} \le c(1 + \|u^1\|_{L^\infty(L^3)}) \le c,
$$

we conclude, employing (4.5),

$$
K_3 \le c\|u^1 - u^2\|_{L^2(L^2)} \|y\|_{L^2(H^1)} \le \varepsilon \int_{Q_t} |u^1 - u^2|^2 + c(\varepsilon) \int_{Q_t} |\nabla y|^2,
$$

where $c(\varepsilon) > 0$ depends on $L_{k\ell}$, the $L^\infty(L^3)$ norms of u^1 and u^2, and on the $L^\infty(L^\infty)$ norm of ∇V^1. Finally, using again (4.5),

$$
\begin{aligned}
K_4 \;&=\; \int_{Q_\tau} \sum_{k,\ell=1}^{n} e_k e_\ell L_{k\ell}(x) \Big((u_{n+1}^1 - u_{n+1}^2) |\nabla V^1|^2 \\
&\quad + u_{n+1}^2 \nabla(V^1 + V^2) \cdot \nabla(V^1 - V^2) \Big) y_{n+1} \\
&\le\; c\|u_{n+1}^1 - u_{n+1}^2\|_{L^2(L^2)} \|\nabla V^1\|_{L^\infty(L^\infty)}^2 \|y_{n+1}\|_{L^2(L^2)} \\
&\quad + c\|u_{n+1}^2\|_{L^\infty(L^2)} \|\nabla(V^1 + V^2)\|_{L^\infty(L^\infty)} \|\nabla(V^1 - V^2)\|_{L^2(L^6)} \\
&\quad \times \|y_{n+1}\|_{L^2(L^3)} \\
&\le\; c\|u^1 - u^2\|_{L^2(L^2)} \|y\|_{L^2(H^1)} \\
&\le\; \varepsilon \int_{Q_t} |u^1 - u^2|^2 + c(\varepsilon) \int_{Q_t} |\nabla y|^2.
\end{aligned}
$$

Therefore, we get from (4.4), for a.e. $t \in (0, \tau)$,

$$
\frac{\lambda}{2} \int_\Omega |\nabla y(t)|^2 + (c_0 - 4\varepsilon) \int_{Q_t} |u^1 - u^2|^2 \le c(\varepsilon) \int_{Q_t} |\nabla y|^2.
$$

Choosing $\varepsilon > 0$ small enough and employing Gronwall's lemma, we conclude

$$\int_\Omega |\nabla y(t)|^2 = 0 \qquad \text{for a.e. } t \in (0, \tau),$$

which implies $u^1 - u^2 = 0$ and $V^1 - V^2 = 0$ a.e. in Q_τ. □

From Theorem 4.1 we cannot exclude the existence of a (second) weak solution (in the sense $u \in L^2(H^1) \cap H^1((\mathcal{V}^*)^{n+1})$, $V \in L^\infty(W^{1,\infty})$). However, in one space dimension, we can prove that the solution constructed in the proof of Theorem 3.1 as the limit of approximate solutions is the unique solution of the system:

Theorem 4.2 *Let the assumptions of Theorem 4.1 hold and let $d = 1$. Then there exists a unique weak solution (u, V) to (1.1)-(1.7). From Theorem 3.1 follows that this solution satisfies $u \in L^\infty(H^1) \cap H^1(L^2)$ and $V \in L^\infty(W^{1,\infty})$.*

Proof. Let (u^1, V^1) be an arbitrary weak solution to (1.1)-(1.7). Furthermore, let (u^2, V^2) be a solution satisfying $u^2 \in L^\infty(H^1)$ whose existence is assured by Theorem 3.1. Since $d = 1$, the embedding $H^2 \hookrightarrow W^{1,\infty}$ is continuous. Then, for $i = 1, 2$,

$$\|\nabla V^i\|_{L^\infty(L^\infty)} \le c\|V^i\|_{L^\infty(H^2)} \le c(1 + \|u^i\|_{L^\infty(L^2)}) < \infty.$$

We can proceed as in the proof of Theorem 4.1 since the constants $c(\varepsilon)$ only depend on the $L^\infty(L^\infty)$ norms on ∇V^1, $i = 1, 2$, and on the $L^\infty(L^3)$ norm of u^2, except in the estimate of K_3. Therefore, it remains to consider the term

$$\begin{aligned}
&\int_{Q_t} L_{k\ell}(x) y_{n+1}(u^1_\ell - u^2_\ell)\Delta V^1 \\
&\quad \le \|L_{k\ell}\|_{L^\infty(L^\infty)} \|y_{n+1}\|_{L^2(L^\infty)} \|u^1_\ell - u^2_\ell\|_{L^2(L^2)} \|\Delta V^1\|_{L^\infty(L^2)} \\
&\quad \le c\|y_{n+1}\|_{L^2(H^1)} \|u^1_\ell - u^2_\ell\|_{L^2(L^2)} (1 + \|u^1\|_{L^\infty(L^2)}),
\end{aligned}$$

since $H^1 \hookrightarrow L^\infty$ in one space dimension. Hence

$$\int_{Q_t} L_{k\ell}(x) y_{n+1}(u^1_\ell - u^2_\ell)\Delta V^1 \le c\|\nabla y_{n+1}\|_{L^2(L^2)} \|u^1_\ell - u^2_\ell\|_{L^2(L^2)},$$

and we can estimate as in the proof of Theorem 4.1. The conclusion of the theorem follows. □

References

[1] G. Albinus. A thermodynamically motivated formulation of the energy model of semiconductor devices. Preprint No. 210, Weierstrass-Institut, Berlin, Germany, 1995.

[2] H. Alt and S. Luckhaus. Quasilinear elliptic-parabolic differential equations. *Math. Z.*, 183:311–341, 1983.

[3] H. Beirao da Veiga. On the $W^{2,p}$-regularity for solutions of mixed problems. *J. Math. Pures Appl.*, 53:279–290, 1974.

[4] N. Ben Abdallah and P. Degond. On a hierarchy of macroscopic models for semiconductors. *J. Math. Phys.*, 37:3308–3333, 1996.

[5] N. Ben Abdallah, P. Degond, L. Desvillettes, and S. Génieys. On the convergence of the Boltzmann equation for semiconductors towards an energy transport model. Submitted for publication, 1997.

[6] N. Ben Abdallah, P. Degond, and S. Génieys. An energy-transport model for semiconductors derived from the Boltzmann equation. *J. Stat. Phys.*, 84:205–231, 1996.

[7] A. Bermudez and C. Saguez. Mathematical formulation and numerical solution of an alloy solidification problem. In A. Fasano, editor, *Free Boundary Problems: Theory and Applications*, volume 1, pages 237–247. Pitman, 1983.

[8] H. Brézis. *Analyse fonctionelle.* Masson, Paris, 1994.

[9] Y. Choi and R. Lui. Multi-dimensional electrochemistry model. *Arch. Rat. Mech. Anal.*, 130:315–342, 1995.

[10] P. Degond, S. Génieys, and A. Jüngel. An existence result for a strongly coupled parabolic system arising in nonequilibrium thermodynamics. *C. R. Acad. Sci. Paris*, 325:227–232, 1997.

[11] P. Degond, S. Génieys, and A. Jüngel. A system of parabolic equations in nonequilibrium thermodynamics including thermal and electrical effects. *J. Math. Pures Appl.*, 76:991–1015, 1997.

[12] P. Degond, S. Génieys, and A. Jüngel. A steady-state system in nonequilibrium thermodynamics including thermal and electrical effects. To appear in *Math. Meth. Appl. Sci.*, 1998.

[13] P. Degond and C. Schmeiser. Macroscopic models for semiconductor heterostructures. Preprint No. 97.25, Université Paul Sabatier, Toulouse, France, 1997.

[14] Z. Deyl, editor. *Electrophoresis: A Survey of Techniques and Applications*, Amsterdam, 1979. Elsevier.

[15] E. DiBenedetto. *Degenerate Parabolic Equations.* Springer, New York, 1993.

[16] J. Filo and J. Kačur. Local existence of general nonlinear parabolic systems. *Nonlin. Anal.*, 24:1597–1618, 1995.

[17] S. Génieys. Energy-transport model for a non degenerate semiconductor. Convergence of the Hilbert expansion in the linearized case. To appear in *Asympt. Anal.*, 1997.

[18] M. Giaquinta. *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems.* Princeton University Press, New Jersey, 1983.

[19] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer, Berlin, 2nd edition, 1983.

[20] P. Grisvard. *Elliptic Problems in Nonsmooth Domains.* Pitman, Boston, 1985.

[21] K. Gröger. A $W^{1,p}$-estimate for solutions to mixed boundary value problems for second order elliptic differential equations. *Math. Ann.*, 283:679–687, 1989.

[22] S. De Groot. *Thermodynamik irreversibler Prozesse.* Bibliographisches Institut, Mannheim, 1960.

[23] S. De Groot and P. Mazur. *Non-equilibrium thermodynamics.* Dover, 1984.

[24] R. Hills, D. Loper, and P. Roberts. A thermodynamically consistent model of a mushy zone. *Q. J. Mech. Appl. Math.*, 36:505–539, 1983.

[25] J. Jerome. *Approximation of Nonlinear Evolution Systems.* Academic Press, New York, 1983.

[26] O. John and J. Stará. On the regularity and non-regularity of elliptic and parabolic systems. In J. Kurzweil, editor, *Equadiff 7*, Teubner-Texte Math. 118, pages 28–36, Leipzig, 1990. Teubner.

[27] A. Jüngel. Stationary transport equations for charge carriers in semiconductors including electron-hole scattering. *Appl. Anal.*, 62:53–69, 1996.

[28] H. Kreuzer. *Nonequilibrium Thermodynamics and its Statistical Foundation.* Clarendon Press, Oxford, 1981.

[29] J. Naumann and M. Wolff. On a global L^q estimate on weak solutions of nonlinear elliptic systems. Preprint, Dipartimento di Matematica, Università di Catania, Italy, 1991.

[30] F. Otto. L^1-contraction and uniqueness for quasilinear elliptic-parabolic equations. *J. Diff. Eqs.*, 131:20–38, 1996.

[31] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Math. Pura Appl.*, 146:65–96, 1987.

[32] J. Stará, O. John, and J. Malý. Counterexample to the regularity of weak solutions of the quasilinear parabolic system. *Comment. Math. Univ. Carolin.*, 27:123–136, 1986.

[33] G. M. Troianiello. *Elliptic Differential Equations and Obstacle Problems.* Plenum Press, New York, 1987.