Semi-discretization in time and numerical convergence of solutions of a nonlinear cross-diffusion population model

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Abstract. A positivity-preserving numerical scheme for a strongly coupled cross-diffusion model for two competing species is presented, based on a semi-discretization in time. The variables are the population densities of the species. Existence of strictly positive weak solutions to the semidiscrete problem is proved. Moreover, it is shown that the semidiscrete solutions converge to a non-negative solution of the continuous system in one space dimension. The proofs are based on a symmetrization of the problem via an exponential transformation of variables and the use of an entropy functional.

Keywords. Positivity-preserving numerical scheme, implicit Euler discretization, population dynamics, full diffusion matrix, entropy method, global existence of solutions.

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1 Introduction

For the time evolution of two competing species with homogeneous population density, usually the Lotka-Volterra differential equations are used as an appropiate mathematical model. In the case of non-homogeneous densities, diffusion effects have to be taken into account leading to reaction-diffusion equations. Shigesada *et al.* proposed in their pioneering work [27] to introduce further so-called cross-diffusion terms modeling interspecific influence of the species. Denoting by n_i the population density of the i-th species ($i = 1, 2$) and by J_i the corresponding flows of population, the time-dependent equations can be written as

$$\begin{aligned} \partial_t n_1 + \mathrm{div} J_1 &= g_1(n_1, n_2), && (1.1)\\ \partial_t n_2 + \mathrm{div} J_2 &= g_2(n_1, n_2), && (1.2)\\ J_1 &= -\nabla\left((c_1 + \alpha_{11} n_1 + \alpha_{12} n_2) n_1\right) + \delta_1 n_1 \nabla U, && (1.3)\\ J_2 &= -\nabla\left((c_2 + \alpha_{21} n_1 + \alpha_{22} n_2) n_2\right) + \delta_2 n_2 \nabla U, && (1.4) \end{aligned}$$

in the bounded domain $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) with time $t > 0$. Here, $U = U(x)$ is the (given) environmental potential, modeling areas where the environmental conditions are more or less favorable [22, 27]. The diffusion coefficients c_i and α_{ij} are non-negative, and $\delta_i \in \mathbb{R}$ ($i, j = 1, 2$). The source terms are in Lotka-Volterra form:

$$\begin{aligned} g_1(n_1, n_2) &= (R_1 - \gamma_{11} n_1 - \gamma_{12} n_2) n_1, && (1.5)\\ g_2(n_1, n_2) &= (R_2 - \gamma_{21} n_1 - \gamma_{22} n_2) n_2, && (1.6) \end{aligned}$$

where $R_i \geq 0$ is the intrinsic growth rate of the i-th species ($i = 1, 2$), $\gamma_{11} \geq 0$ and $\gamma_{22} \geq 0$ are the coefficients of intra-specific competition, and $\gamma_{12} \geq 0$ and $\gamma_{21} \geq 0$ are those of interspecific competition.

The above system of equations is completed with mixed Dirichlet-Neumann boundary conditions and initial conditions:

$$\begin{aligned} n_i = n_{D,i} \quad \text{on } \Gamma_D \times (0, \infty), \quad J_i \cdot \nu = 0 \quad \text{on } \Gamma_N \times (0, \infty), && (1.7)\\ n_i(\cdot, 0) = n_{0,i} \quad \text{in } \Omega, \quad i = 1, 2, && (1.8) \end{aligned}$$

where ν denotes the exterior unit normal to $\partial\Omega$. This means that the population density is fixed at a part of the domain boundary (due to emigration and immigration processes), whereas no flux boundary conditions are prescribed at the remaining boundary parts.

Eqs. (1.1)-(1.4) contain various types of reaction-diffusion models. Indeed, in the case $\alpha_{ij} = 0$ for $i, j = 1, 2$, they reduce to the drift-diffusion equations, which has been studied in various fields of application, e.g. electro-chemistry [2, 3], biophysics [7] or semiconductor theory [21]. When $c_1 = c_2 = 0$ and $\alpha_{12} = \alpha_{21} = 0$, Eqs. (1.1)-(1.4) are of degenerate type. These types of problems

arise, for instance, in porous media flow [17], oil-recovery [8], plasma physics [14] or semiconductor theory [13]. In chemotaxis, related models appear [9, 23].

For $\alpha_{12} > 0$ and $\alpha_{21} > 0$, the problem becomes strongly coupled with full diffusion matrix

$$A(n_1, n_2) = \begin{pmatrix} c_1 + 2\alpha_{11} n_1 + \alpha_{12} n_2 & \alpha_{12} n_1 \\ \alpha_{21} n_2 & c_2 + 2\alpha_{22} n_2 + \alpha_{21} n_1 \end{pmatrix}.$$

Nonlinear problems of this kind are quite difficult to deal with since the usual idea to apply maximum principle arguments to get a priori estimates cannot be used here. Furthermore, the diffusion matrix is not symmetric and of degenerate type if $c_1 = c_2 = 0$.

Up to now, only partial results are available in the literature concerning the well-posedness of the problem, and no results can be found concerning the numerical analysis. We summarize some of the available results for the time-dependent equations (see [31] for a review) and refer to [19, 26] for the stationary problem. Global existence of solutions and their qualitive behavior for $\alpha_{11} = \alpha_{22} = \alpha_{21} = 0$ have been proved in, e.g., [20, 24, 25, 30]. In this case, Eq. (1.2) is only weakly coupled. For sufficiently small cross-diffusion parameters $\alpha_{12} > 0$ and $\alpha_{21} > 0$ (or equivalently, "small" initial data) and vanishing self-diffusion coefficients $\alpha_{11} = \alpha_{22} = 0$, Deuring proved the global existence of solutions [6]. For the case $c_1 = c_2$ a global existence result in one space dimension has been obtained by Kim [18]. Furthermore, under the condition

$$8\alpha_{11} > \alpha_{12}, \qquad 8\alpha_{22} > \alpha_{21}, \tag{1.9}$$

Yagi [32] has shown the global existence of solutions in two space dimensions assuming $\alpha_{12} = \alpha_{21}$. A global existence result for weak solutions in any space dimension under condition (1.9) can be found in [10].

Condition (1.9) can be easily understood by observing that in this case, the diffusion matrix is positive definite:

$$\xi^T A(n_1, n_2)\xi \geq \min\{c_1, c_2\}|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^2,$$

hence yielding an elliptic operator. If the condition (1.9) does not hold, there are choices of $c_i, \alpha_{ij}, n_i \geq 0$ for which the matrix $A(n_1, n_2)$ is *not* positive definite, and it is therefore unclear if the problem (1.1)-(1.8) can be solved for these data.

In this paper we provide the tools to treat this problem both analytically and numerically. More precisely,

- we construct a positivity-preserving numerical scheme based on a semidiscretization in time, and
- we show the numerical convergence of the semidiscrete solutions in one space dimension.

Moreover, we obtain *global* existence of non-negative solutions in one space dimension. For these results we do not need any restriction on the diffusion coefficients (except positivity; see Section 2). Our results are valid only for the case of one space dimension since the continuous embedding $H^1(\Omega) \subset L^\infty(\Omega)$ is crucial.

Before we introduce the method of proof, we perform (for a smoother presentation) the following change of unknowns:

$$u_1 = \alpha_{21} n_1, \quad u_2 = \alpha_{12} n_2, \quad \text{and} \quad q = -\nabla U.$$

We assume that $\alpha_{12} > 0$ and $\alpha_{21} > 0$ which is no restriction since if $\alpha_{12} = 0$ or $\alpha_{21} = 0$, at least one of the equations (1.1), (1.2) is weakly coupled, and the results of [24, 25, 30] apply. Eqs. (1.1)-(1.8) can be reformulated as

$$\partial_t u_i - \operatorname{div}(c_i \nabla u_i + 2a_i u_i \nabla u_i + \nabla(u_1 u_2) + d_i u_i q) = f_i(u_1, u_2), \tag{1.10}$$
$$u_i = u_{D,i} \quad \text{on } \Gamma_D \times (0,T), \tag{1.11}$$
$$(c_i \nabla u_i + 2a_i u_i \nabla u_i + \nabla(u_1 u_2) + d_i u_i q) \cdot \nu = 0 \quad \text{on } \Gamma_N \times (0,T), \tag{1.12}$$
$$u(\cdot, 0) = u_i^0 \quad \text{in } \Omega, \quad i = 1, 2, \tag{1.13}$$

where $T > 0$,

$$u_{D,1} = \alpha_{21} n_{D,1}, \quad u_{D,2} = \alpha_{12} n_{D,2}, \quad u_1^0 = \alpha_{21} n_{0,1}, \quad u_2^0 = \alpha_{12} n_{0,2}$$

and

$$a_1 = \alpha_{11}/\alpha_{21}, \quad a_2 = \alpha_{22}/\alpha_{12}, \quad d_1 = \alpha_{21}\delta_1, \quad d_2 = \alpha_{12}\delta_2.$$

The source terms are given by

$$f_i(u_1, u_2) = (R_i - \beta_{i1} u_1 - \beta_{i2} u_2) u_i,$$

with

$$\beta_{1i} = \gamma_{1i}/\alpha_{21}, \quad \beta_{2i} = \gamma_{2i}/\alpha_{12}, \quad i = 1, 2.$$

The key for understanding the problem (1.10)-(1.13) mathematically is based on two observations. First, Eqs. (1.10)-(1.13) admit the *entropy*

$$\eta_1(t) = \sum_{i=1}^{2} \int_\Omega (u_i(\log u_i - \log u_{D.i}) - u_i + u_{D,i})\, dx \geq 0$$

with the entropy inequality

$$\eta_1(t) + 2 \int_0^t \int_\Omega \left\{ \sum_{i=1}^{2} (2c_i |\nabla \sqrt{u_i}|^2 + a_i |\nabla u_i|^2) + 2 |\nabla \sqrt{u_1 u_2}|^2 \right\} dx dt \leq \eta_1(0) + C(T), \tag{1.14}$$

where $C(T) > 0$ depends on T, q, the boundary data and the source terms. By Poincaré's inequality, this estimate provides $L^2(0,T; H^1(\Omega))$ estimates for u_i (if

$a_i > 0$). However, the entropy inequality can be made rigorous only if $u_i \geq 0$, which cannot be easily obtained from the minimum principle.

The second observation is that the existence of an entropy allows for a transformation of variables which symmetrizes the problem (cf. [4]). This transformation reads

$$u_1 = e^{w_1}, \quad u_2 = e^{w_2},$$

and then Eqs. (1.10) transform into

$$\partial_t \begin{pmatrix} e^{w_1} \\ e^{w_2} \end{pmatrix} - \operatorname{div} \left(B(w_1, w_2) \nabla \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} d_1 e^{w_1} \\ d_2 e^{w_2} \end{pmatrix} q \right) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

with the new diffusion matrix

$$B(w_1, w_2) = \begin{pmatrix} c_1 + 2a_1 e^{2w_1} + e^{w_1+w_2} & e^{w_1+w_2} \\ e^{w_1+w_2} & c_2 + 2a_2 e^{2w_1} + e^{w_1+w_2} \end{pmatrix},$$

which is symmetric and positive definite:

$$\det B(w_1, w_2) \geq (c_1 + 2a_1 e^{2w_1})(c_2 + 2a_2 e^{2w_2}). \tag{1.15}$$

In this formulation the matrix B provides an elliptic operator for all $c_i > 0$, $a_i \geq 0$, $i = 1, 2$.

Moreover, if L^∞ bounds for w_i are available, we obtain a strictly positive solution u_i to the original problem. In order to obtain these L^∞ bounds for $w_i = \log u_i$, the entropy estimate (1.14) is not sufficient since this estimate does not exclude the case $u_i = 0$ locally. We use another 'entropy' to derive a priori bounds:

$$\eta(t) = \eta_1(t) + \alpha \eta_2(t),$$

where

$$\eta_2(t) = \sum_{i=1}^{2} \int_\Omega (u_i - u_{D,i} - \log(u_i/u_{D,i}))dx.$$

For appropriate $\alpha > 0$ (see Section 3), we can show that, in addition to (1.14), it holds

$$\sum_{i=1}^{2} \int_0^t \int_\Omega |\nabla \log u_i|^2 dxdt \leq c(\alpha),$$

where $c(\alpha)$ depends on α, but not on u_i. Using Poincaré's and Sobolev's inequalities, we can see that this provides a bound for $\log u_i$ in $L^2(0, T; L^\infty(\Omega))$. Here, the assumption of one space dimension becomes crucial.

The idea of employing an *exponential* transformation of variables has been sucessfully used to obtain non-negative or positive solutions to elliptic and parabolic equations of fourth order in [11, 15, 16].

Finally, we remind here that parabolic systems of the type

$$\partial_t b_i(w) - \operatorname{div} a_i(x, b(w), \nabla w) = f_i(b(w)), \quad i = 1, \ldots, n,$$

have been studied by Alt and Luckhaus [1] assuming a monotone function $b = (b_1, \ldots, b_n) : \mathbb{R}^n \to \mathbb{R}^n$ and a uniform elliptic operator $a_i(x, b(w), \nabla w)$. However, no positivity or non-negativity results have been obtained.

Let us summarize the main features of the presented method of proof:

- The numerical scheme preserves the positivity of the solution.
- No restriction on the diffusion coefficients $c_i, a_i > 0$ are needed.
- The solution of the continuous problem (as the limit of discrete solutions) is non-negative and exists globally in time.

We stress once again the fact that the positivity (and non-negativity) property is obtained without the use of the maximum principle.

This paper is organized as follows. In Section 2 we make precise the semidiscretization in time and state the main results. Section 3 is devoted to the proof of the existence of positive semidiscrete solutions. Finally, in Section 4 the continuous limit is performed.

2 Semi-discretization in time and main results

We consider the following assumptions:

(A1) $\Omega \subset \mathbb{R}$ is a bounded interval, $\partial\Omega = \Gamma_D \cup \Gamma_N$, and $\Gamma_D \neq \emptyset$.

(A2) $u_i^0 \in L^\infty(\Omega)$ satisfies $u_i^0 \geq \gamma > 0$ in Ω, $u_{D,i} = \text{const.} > 0$ on Γ_D, $i = 1, 2$.

(A3) $a_i, c_i > 0$, $d_i \in \mathbb{R}$ $(i = 1, 2)$ and $q \in L^2(\Omega \times (0, T))$.

(A4) $f_i : [0, \infty)^2 \to \mathbb{R}$ $(i = 1, 2)$ is continuous and it holds for all $u_1, u_2 > 0$, $p, q > 0$:

$$\begin{aligned} f_i(u_1, u_2) &\leq C_1, \\ f_1(u_1, u_2) \log \frac{u_1}{p} + f_2(u_1, u_2) \log \frac{u_2}{q} &\leq C_2(p, q), \\ f_1(u_1, u_2) \left(\frac{1}{p} - \frac{1}{u_1} \right) + f_2(u_1, u_2) \left(\frac{1}{q} - \frac{1}{u_2} \right) &\leq C_3(p, q), \end{aligned}$$

for some C_1, $C_2(p, q)$, $C_3(p, q) > 0$.

Remark 2.1 The Lotka-Volterra source terms

$$\begin{aligned} f_1(u_1,u_2) &= (R_1-\beta_{11}u_1-\beta_{12}u_2)u_1, \\ f_2(u_1,u_2) &= (R_2-\beta_{21}u_1-\beta_{22}u_2)u_2 \end{aligned}$$

satisfy condition (A4) if $\beta_{ii}>0$, $i=1,2$ and $\beta_{12}=\beta_{21}\geq 0$. Indeed, we obtain for $u_1,u_2>0$,

$$\begin{aligned} &f_1(u_1,u_2)\log\frac{u_1}{p}+f_2(u_1,u_2)\log\frac{u_2}{q} \\ &= (R_1-\beta_{11}u_1)u_1\log\frac{u_1}{p}+(R_2-\beta_{22}u_2)u_2\log\frac{u_2}{q}-\beta_{12}u_1u_2\log\left(\frac{u_1u_2}{pq}\right) \\ &\leq C, \end{aligned}$$

and

$$\begin{aligned} &f_1(u_1,u_2)\left(\frac{1}{p}-\frac{1}{u_1}\right)+f_2(u_1,u_2)\left(\frac{1}{q}-\frac{1}{u_2}\right) \\ &= (R_1-\beta_{11}u_1-\beta_{12}u_2)\left(\frac{u_1}{p}-1\right)+(R_2-\beta_{21}u_1-\beta_{22}u_2)\left(\frac{u_2}{q}-1\right) \\ &= (R_1-\beta_{11}u_1)\left(\frac{u_1}{p}-1\right)+(R_2-\beta_{22}u_2)\left(\frac{u_2}{q}-1\right) \\ &\quad +\beta_{12}\left(u_1+u_2-\left(\frac{1}{p}+\frac{1}{q}\right)u_1u_2\right) \\ &= \left(\frac{R_1}{p}+\beta_{11}+\beta_{12}-\frac{\beta_{11}}{p}u_1\right)u_1+\left(\frac{R_2}{q}+\beta_{22}+\beta_{12}-\frac{\beta_{22}}{q}u_2\right)u_2 \\ &\quad -R_1-R_2-\beta_{12}\left(\frac{1}{p}+\frac{1}{q}\right)u_1u_2 \\ &\leq C, \end{aligned}$$

for an appropriate $C>0$.

We introduce now the semi-discrete problem. Since some of our results also holds for the multi-dimensional problem we keep the notation using div and ∇. Let $N\in\mathbb{N}$ and let $\tau=T/N$ be the time step. (We can also allow for quasi-uniform time steps; see [5] for details.) We are seeking solutions u_1^k,u_2^k, approximating u,v, respectively, in the interval $((k-1)\tau,k\tau]$, $k=1,\ldots,N$, of the recursive elliptic problem

$$\frac{1}{\tau}(u_i^k-u_i^{k-1})-\mathrm{div}J_i^k=f_i(u_1^k,u_2^k)\quad\text{in }\Omega, \tag{2.1}$$

$$u_i^k=u_{D,i}\quad\text{on }\Gamma_D,\quad J_i^k\cdot\nu=0\quad\text{on }\Gamma_N, \tag{2.2}$$

where

$$J_i^k = c_i \nabla u_i^k + 2a_i u_i^k \nabla u_i^k + \nabla(u_1^k u_2^k) + d_i q^k u_i^k, \quad i = 1, 2,$$

$u_i^0 = u_0$ in Ω, for $i = 1, 2$, and

$$q^k = \frac{1}{\tau} \int_{(k-1)\tau}^{k\tau} q(\cdot, t) dt.$$

Define the (in time) piecewise constant functions $u^{(\tau)}, v^{(\tau)}$ and $q^{(\tau)}$ by

$$u_i^{(\tau)} = u_i^k \quad (i = 1, 2), \quad q^{(\tau)} = q^k \quad \text{in } \Omega \times ((k-1)\tau, k\tau],$$

for $k = 1, \ldots, N$. Then it holds (see, e.g., [12])

$$q^{(\tau)} \to q \quad \text{in } L^2(Q_T) \text{ as } \tau \to 0. \tag{2.3}$$

Our main results are the following.

Theorem 2.2 *Let (A1)-(A4) hold. Then there exists solutions* $(u_1^k, u_2^k) \in H^1(\Omega; \mathbb{R}^2)$ *of (2.1)-(2.2) satisfying*

$$0 < \gamma_k \le u_1^k(x), u_2^k(x) \le \Gamma_k, \quad x \in \Omega,$$

for some $\gamma_k, \Gamma_k > 0$, $k = 1, \ldots, N$.

The following theorem concerns the convergence of the discrete solutions $(u_1^{(\tau)}, u_2^{(\tau)})$ to a solution (u_1, u_2) of the continuous problem:

Theorem 2.3 *Let (A1)-(A4) hold. Then there exists a subsequence of* $(u_1^{(\tau)}, u_2^{(\tau)})$ *(not relabeled) such that as* $\tau \to 0$,

$$\begin{aligned} u_i^{(\tau)} &\to u_i \qquad \text{strongly in } L^2(Q_T), \\ u_i^{(\tau)} &\rightharpoonup u_i \qquad \text{weakly in } L^2(0, T; H^1(\Omega)), \\ \partial_t u_i^{(\tau)} &\rightharpoonup \partial_t u_i \qquad \text{weakly in } L^1(0, T; H^{-1}(\Omega)) \end{aligned}$$

and (u_1, u_2) *is a weak solution of (1.10)-(1.13) satisfying* $u_i \in L^2(0, T; H^1(\Omega)) \cap W^{1,1}(0, T; H^{-1}(\Omega))$ *and*

$$u_1(x, t), \ u_2(x, t) \ge 0 \quad \text{for } (x, t) \in Q_T.$$

3 Proof of Theorem 2.2

As explained in the introduction, we have to work with variables which symmetrize the elliptic operator. Introduce the new variables $w = (w_1, w_2)$ by defining

$$u_1 = e^{w_1}, \quad u_2 = e^{w_2}$$

and set

$$b(w) = (b_1(w), b_2(w)) = (e^{w_1}, e^{w_2}).$$

With the diffusion coefficients

$$\begin{aligned} a_{ii}(w) &= c_i e^{w_i} + 2a_i e^{2w_i} + e^{w_1+w_2}, \quad i = 1, 2, \\ a_{12}(w) &= a_{21}(w) = e^{w_1+w_2}, \end{aligned}$$

Eqs. (1.10)-(1.13) are formally equivalent to

$$\partial_t b_i(w) - \operatorname{div}\left(\sum_{j=1}^{2} a_{ij}(w)\nabla w_j + d_i b_i(w) q\right) = F_i(w), \qquad \text{in } \Omega \times (0,T), \quad (3.1)$$

$$\left(\sum_{j=1}^{2} a_{ij}(w)\nabla w_j + d_i b_i(w) q\right) \cdot \nu = 0 \qquad \text{on } \Gamma_N \times (0,T), (3.2)$$

$$w = w_D \qquad \text{on } \Gamma_D \times (0,T), (3.3)$$

$$w(0) = w^0 \qquad \text{in } \Omega, \quad (3.4)$$

where $F_i(w) = f_i(e^{w_1}, e^{w_2})$, $w_{D,i} = \log(u_{D,i})$, and $w_i^0 = \log(u_i^0)$, $i = 1, 2$. Moreover, Eqs. (2.1)-(2.2) can be rewritten as

$$\frac{b_i(w^k) - b_i(w^{k-1})}{\tau} - \operatorname{div}\left(\sum_{j=1}^{2} a_{ij}(w^k)\nabla w_j^k + d_i b_i(w^k) q^k\right) = F_i(w^k) \text{ in } \Omega, \quad (3.5)$$

$$\left(\sum_{j=1}^{2} a_{ij}(w^k)\nabla w_j^k + d_i b_i(w^k) q^k\right) \cdot \nu = 0 \text{ on } \Gamma_N, (3.6)$$

$$w^k = w_D \text{ on } \Gamma_D, (3.7)$$

for $k = 1, \ldots, N$. We introduce the discrete entropy (for $k = 0, \ldots, N$, including thus the entropy of the initial data)

$$\eta^k = \eta_1^k + \alpha\eta_2^k, \quad \text{where} \quad \alpha = 2\min\{c_1, c_2\},$$

η_1^k is the discrete "physical" entropy

$$\eta_1^k = \sum_{i=1}^{2} \int_\Omega (b_i(w^k)(w_i^k - w_{D,i}) - b_i(w^k) + b_i(w_D))dx,$$

and η_2^k is another discrete entropy:

$$\eta_2^k = \sum_{i=1}^{2} \int_\Omega (e^{w_i^k - w_{D,i}} - (w_i^k - w_{D,i}))dx.$$

Notice that $\eta^k \geq 0$.

First we prove the discrete analogue of an entropy-type estimate which holds in any space dimension.

Lemma 3.1 *Let (A1)-(A4) hold and let $w^k \in H^1(\Omega;\mathbb{R}^2)$ be a weak solution of (3.5)-(3.7). Then there exists a constant $C>0$ such that for any $k=1,\ldots,N$ and any $\tau>0$,*

$$\eta^k + \tau\sum_{i=1}^2\int_\Omega\left(\frac{\alpha^2}{4}|\nabla w_i^k| + \alpha|\nabla e^{w_i^k/2}|^2 + a_i|\nabla e^{w_i^k}|^2\right)\,dx \le \eta^{k-1} + C\tau. \quad (3.8)$$

Proof. The key of the proof is to use $(w_i^k - w_{D,i}) + \alpha(b_i(-w_D) - b_i(-w_i)) \in H_0^1(\Omega\cup\Gamma_N) = \{v\in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$ as a test function in the weak formulation of (3.5)-(3.7). Adding the corresponding equations for $i=1$ and $i=2$ gives

$$\begin{aligned}
&\frac{1}{\tau}\sum_{i=1}^2\int_\Omega (b_i(w^k) - b_i(w^{k-1}))\left[(w_i^k - w_{D,i}) + \alpha(b_i(-w_D) - b_i(-w^k))\right]dx \\
&\quad + \sum_{i,j=1}^2\int_\Omega a_{ij}(w^k)\nabla w_j^k\cdot\nabla\left[(w_i^k - w_{D,i}) + \alpha(b_i(-w_D) - b_i(-w^k))\right]dx \\
&= -\sum_{i=1}^2\int_\Omega d_i b_i(w^k)q^k\cdot\nabla\left[(w_i^k - w_{D,i}) + \alpha(b_i(-w_D) - b_i(-w^k))\right]dx \\
&\quad + \sum_{i=1}^2\int_\Omega F_i(w^k)\left[(w_i^k - w_{D,i}) + \alpha(b_i(-w_D) - b_i(-w^k))\right]dx. \qquad (3.9)
\end{aligned}$$

In order to estimate the first term on the left-hand side of (3.9), we use the convexity of $x\mapsto b_i(x)$ and the elementary inequality $e^x \ge 1+x$ for all $x\in\mathbb{R}$:

$$\begin{aligned}
&\frac{1}{\tau}\sum_{i=1}^2\int_\Omega (b_i(w^k) - b_i(w^{k-1}))[(w_i^k - w_{D,i}) + \alpha(b_i(-w_D) - b_i(-w^k))]dx \\
&= \frac{1}{\tau}(\eta_1^k - \eta_1^{k-1}) + \frac{1}{\tau}\sum_{i=1}^2\int_\Omega (b_i(w^k) - b_i(w^{k-1}) - b_i(w^{k-1})(w_i^k - w_i^{k-1}))dx \\
&\quad + \frac{\alpha}{\tau}(\eta_2^k - \eta_2^{k-1}) + \frac{\alpha}{\tau}\sum_{i=1}^2\int_\Omega (e^{w_i^{k-1}-w_i^k} - w_i^{k-1} + w_i^k - 1)dx \\
&\ge \frac{1}{\tau}(\eta^k - \eta^{k-1}).
\end{aligned}$$

We rewrite the second term on the left-hand side of (3.9), using $w_{D,i} = \text{const.}$, as follows:

$$\sum_{i,j=1}^2\int_\Omega a_{ij}(w^k)\nabla w_j^k\cdot(\nabla w_i^k + \alpha e^{-w_i^k}\nabla w_i^k)dx$$

$$
\begin{aligned}
&= \sum_{i=1}^2 \int_\Omega (c_i e^{w_i^k} + 2a_i e^{2w_i^k} + \alpha c_i + 2\alpha a_i e^{w_i^k}) |\nabla w_i^k|^2 dx \\
&\quad + \alpha \int_\Omega (e^{w_2^k} |\nabla w_1^k|^2 + e^{w_1^k} |\nabla w_2^k|^2 + (e^{w_1^k} + e^{w_2^k}) \nabla w_1^k \cdot \nabla w_2^k) dx \\
&\quad + 2 \int_\Omega |\nabla e^{w_1^k + w_2^k}|^2 dx.
\end{aligned}
$$

For the first term on the right-hand side of (3.9) we employ Young's inequality:

$$
\begin{aligned}
&- \sum_{i=1}^2 \int_\Omega d_i b_i(w^k) q^k (\nabla w_i^k + \alpha e^{-w_i^k} \nabla w_i^k) dx \\
&\leq \sum_{i=1}^2 \int_\Omega \left(a_i e^{2w_i^k} |\nabla w_i^k|^2 + \frac{\alpha c_i}{2} |\nabla w_i^k|^2 + \left(\frac{1}{4} + \frac{1}{2\alpha c_i} \right) d_i^2 |q_k|^2 \right) dx.
\end{aligned}
$$

Finally, by Assumption (A4), we obtain for the last term of (3.9):

$$
\begin{aligned}
&\sum_{i=1}^2 \int_\Omega F_i(w^k)[(w_i^k - w_{D,i}) + \alpha(b_i(-w_{D,i}) - b_i(-w_i^k))] dx \\
&\qquad = \sum_{i=1}^2 \int_\Omega (f_i(e^{w_1^k}, e^{w_2^k})(w_i^k - w_{D,i}) + \alpha f_i(e^{w_1^k}, e^{w_2^k})(e^{-w_{D,i}} - e^{-w_i^k})) dx \\
&\leq \sum_{i=1}^2 \int_\Omega (C_2(w_{D,1}, w_{D,2}) + C_3(u_{D,1}, u_{D,2})) dx \\
&\leq C,
\end{aligned}
$$

where here and in the following $C > 0$ denotes a constant independent of w_i^k and τ with values varying from occurence to occurence.

Putting the above estimates together, we infer from (3.9):

$$
\begin{aligned}
&\frac{1}{\tau}(\eta^k - \eta^{k-1}) + \sum_{i=1}^2 \int_\Omega \left(\frac{\alpha c_i}{2} + (c_i + 2\alpha a_i) e^{w_i^k} + a_i e^{2w_i^k} \right) |\nabla w_i^k|^2 dx \\
&\leq C + \sum_{i=1}^2 \int_\Omega \left(\frac{1}{4} + \frac{1}{2\alpha c_i} \right) d_i^2 |q_k|^2 dx \\
&\qquad - \alpha \int_\Omega (e^{w_2^k} |\nabla w_1^k|^2 + e^{w_1^k} |\nabla w_2^k|^2 + (e^{w_1^k} + e^{w_2^k}) \nabla w_1^k \cdot \nabla w_2^k) \, dx \\
&\leq C + \frac{\alpha}{4} \int_\Omega (e^{w_1^k} |\nabla w_1^k|^2 + e^{w_2^k} |\nabla w_2^k|^2) dx.
\end{aligned}
$$

The last integral can be absorbed by the second term on the left-hand side since $\alpha = 2 \min\{c_1, c_2\}$:

$$
\frac{1}{\tau}(\eta^k - \eta^{k-1}) + \sum_{i=1}^2 \int_\Omega \left(\frac{\alpha^2}{4} + \frac{\alpha}{4} e^{w_i^k} + a_i e^{2w_i^k} \right) |\nabla w_i^k|^2 dx \leq C,
$$

from which we deduce (3.8) and hence the assertion of the lemma. □

Remark 3.2 The a priori estimate of Lemma 3.1 can only be obtained if $c_1, c_2 > 0$. If $c_i = 0$ and $a_1, a_2 > 0$, we get uniform estimates only for $\nabla e^{w_i^k/2}$ and $\nabla e^{w_i^k}$ which is not enough to control w_i^k. If $a_i = 0$ and $c_1, c_2 > 0$ we control w_i^k in $H^1(\Omega)$, by Poincaré's inequality and therefore, in the one-dimensional case, also $e^{w_i^k}$ in $H^1(\Omega)$. Finally, notice that the a priori estimate (3.8) holds for any $\alpha > 0$ if $a_i > \frac{1}{8}$ which is exactly the condition needed in [32].

Lemma 3.3 *Let (A1)-(A4) hold and let $w^{k-1} \in L^\infty(\Omega;\mathbb{R}^2), k \geq 1$. Then there exists a solution $w^k \in H^1(\Omega;\mathbb{R}^2)$ of (3.5)-(3.7).*

Remark 3.4 Since the solution satisfies $w^k \in H^1(\Omega;\mathbb{R}^2) \hookrightarrow L^\infty(\Omega;\mathbb{R}^2)$ in one space dimension, the unknowns $u_i^k = \exp(w_i^k)$ are well defined and elements of $H^1(\Omega)$. Hence (u_1^k, u_2^k), $k = 1, \ldots, N$, is a solution of (2.1)-(2.2), and Theorem 2.2 is a consequence of Lemma 3.3.

Proof. We use Leray-Schauder's fixed-point theorem. For this, let $z = (z_1, z_2) \in L^\infty(\Omega;\mathbb{R}^2)$ be given and consider the linear system

$$-\partial_x\left(\sum_{j=1}^2 a_{ij}(z)\partial_x w_j^k + d_i b_i(z) q^k\right) = \frac{1}{\tau}(b_i(w^{k-1}) - b_i(z)) + F_i(z) \qquad (3.10)$$

in $\Omega, i = 1, 2$, together with the boundary conditions

$$\left(\sum_{j=1}^2 a_{ij}(z)\partial_x w_j^k + d_i b_i(z) q^k\right)\cdot \nu = 0 \quad \text{on } \Gamma_N, \qquad (3.11)$$

$$w^k = w_D^k \quad \text{on } \Gamma_D. \qquad (3.12)$$

Since

$$\sum_{i,j=1}^2 a_{ij}(z)\xi_i\xi_j \geq \gamma(\|z\|_{L^\infty(\Omega)})|\xi|^2$$

for all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and some $\gamma = \gamma(\|z\|_{L^\infty(\Omega)}) > 0$, we can apply Lax-Milgram's lemma to get the existence of a unique solution $w^k = (w_1^k, w_2^k) \in H^1(\Omega;\mathbb{R}^2)$ to (3.10)-(3.12). Since $d = 1$, Sobolev's embedding theorem implies $w^k \in L^\infty(\Omega;\mathbb{R}^2)$. This defines the fixed-point operator

$$S : L^\infty(\Omega;\mathbb{R}^2) \to L^\infty(\Omega;\mathbb{R}^2), \quad z \mapsto w^k.$$

The continuity of S follows from standard arguments. Indeed, let $z_n \to z$ in $L^\infty(\Omega)$ as $n \to \infty$ and $w_n^k = S(z_n)$. Using $w_n^k - w_D^k$ as a test function in the weak

formulation of (3.10), we obtain from standard elliptic estimates and Poincaré's inequality the bound

$$\|w_n^k\|_{H^1(\Omega)} \le c(\|z_n\|_{L^\infty(\Omega)}),$$

where the constant $c = c(\|z_n\|_{L^\infty(\Omega)})$ also depends on w^{k-1} and the data. Since $H^1(\Omega;\mathbb{R}^2)$ embeds in $L^\infty(\Omega;\mathbb{R}^2)$ compactly in one space dimension, there is a subsequence $(w_{n'}^k)$ of (w_n^k) such that

$$\begin{aligned} w_{n'}^k &\to w^k \quad \text{strongly in } L^\infty(\Omega), \\ w_{n'}^k &\rightharpoonup w^k \quad \text{weakly in } H^1(\Omega), \end{aligned}$$

as $n' \to \infty$. Performing the limit $n' \to \infty$ in the weak formulation of (3.10) shows that $w^k = S(z)$. Since the limit w^k is unique,

$$w_n^k \to w^k \quad \text{strongly in } L^\infty(\Omega),$$

for the whole sequence (w_n^k). The compactness of the embedding implies the compactness of the operator S.

Now let $w^k \in L^\infty(\Omega;\mathbb{R}^2)$ and $\sigma \in [0,1]$ be such that $w^k = \sigma S(w^k)$. An estimate very similar to the estimate of Lemma 3.1 (which settles the case $\sigma = 1$) gives the existence of w^k and σ such that

$$\sum_{i=1}^2 \int_\Omega |\nabla w_i^k|^2 \le C\tau + \eta^{k-1} \le C.$$

By Poincaré's and Sobolev's inequality, this implies

$$\|w^k\|_{L^\infty(\Omega)} \le C\|w^k\|_{H^1(\Omega)} \le C,$$

which is the desired uniform bound. Therefore, we can apply the Leray-Schauder fixed-point theorem to deduce the existence of a fixed-point of S and thus a solution of (3.5)-(3.7). □

4 Proof of Theorem 2.3

For the proof of Theorem 2.3 we need a priori estimates uniformly in τ. Let us define the piecewise constant functions $w(\tau)$ by

$$w^{(\tau)}(x,t) = w^k(x) \quad \text{if } (x,t) \in \Omega \times ((k-1)\tau, k\tau].$$

An immediate consequence of Lemma 3.1 is the following result:

Corollary 4.1 *It holds for $\tau > 0$,*

$$\begin{aligned} \|\eta^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))} &\le C, \\ \sum_{i=1}^2 \left(\alpha \|e^{w_i^{(\tau)}/2}\|_{L^2(0,T;H^1(\Omega))} + a_i \|e^{w_i^{(\tau)}}\|_{L^2(0,T;H^1(\Omega))}\right) &\le C, \end{aligned}$$

where $C > 0$ is independent of τ and

$$\begin{aligned}\eta^{(\tau)}(t) &= \sum_{i=1}^{2}\int_\Omega \Big(b_i(w^{(\tau)})(w_i^{(\tau)} - w_D) - b_i(w^{(\tau)}) + b_i(w_D) \\ &\quad + \alpha(b_i(w^{(\tau)}) - w_i^{(\tau)} + w_{D,i})\Big)(t)dx.\end{aligned}$$

Recall that $\alpha = 2\min\{c_1, c_2\}$.

Proof of Corollary 4.1. We obtain from the entropy inequality (3.8) for $1 \le m \le N$,

$$\begin{aligned}\eta^m - \eta^0 &= \sum_{k=1}^{m}(\eta^k - \eta^{k-1}) \\ &\le -\sum_{k=1}^{m}\tau\int_\Omega\sum_{i=1}^{2}(\alpha|\nabla e^{w_i^k/2}|^2 + a_i|\nabla e^{w_i^k}|^2)dx + Cm\tau.\end{aligned}$$

Applying the maximum over $m = 1, \ldots, N$ and using $m\tau \le N\tau = T$ gives

$$\|\eta^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))} + \sum_{i=1}^{2}\Big(\alpha\|\nabla e^{w_i^{(\tau)}/2}\|_{L^2(Q_T)} + a_i\|\nabla e^{w_i^{(\tau)}}\|_{L^2(Q_T)}\Big) \le \eta^0 + CT.$$

Thus, Poincaré's inequality gives the conclusion. □

We also need an estimate for the discrete time derivative. For this we define

$$\tilde{b}^{(\tau)}(\cdot,t) = \frac{k\tau - t}{\tau}\big(b(w^k) - b(w^{k-1})\big) + b(w^k), \quad t > 0.$$

Furthermore, let σ_τ be the shift operator

$$\sigma_\tau w^{(\tau)}(\cdot,t) = w^{k-1} \quad \text{if } t \in ((k-1)\tau, k\tau], \quad k = 1, \ldots, N.$$

Then we have

Lemma 4.2 *It holds*

$$\begin{aligned}\|b(w^{(\tau)}) - b(\sigma_\tau w^{(\tau)})\|_{L^1(0,T;V^*)} &\le C\tau, \\ \|\partial_t\tilde{b}^{(\tau)}\|_{L^1(0,T;V^*)} + \|\tilde{b}^{(\tau)}\|_{L^2(0,T;H^1(\Omega))} &\le C,\end{aligned}$$

where C does not depend on τ and $V^* = (H_0^1(\Omega \cup \Gamma_N))^*$.

Proof. The usual idea is to use $w_i^k - w_i^{k-1}$ as a test function in (3.5) for $i = 1, 2$ and to obtain the estimate

$$\int_\Omega \big(b(w^k) - b(w^{k-1})\big) \cdot (w^k - w^{k-1}) dx \le C\tau,$$

from which it follows that $w^{(\tau)} - \sigma_\tau w^{(\tau)} \to 0$ in $L^2(Q_T)$. The above bound can only be achieved if an L^∞ bound on w^k independent of k is available. However, by Corollary 4.1, we only have a uniform H^1 bound on e^{w^k}, i.e. we do not control any lower bound of w^k. (Here we can allow for $c_i \ge 0$, $i = 1, 2$.)

Therefore we compute a bound for $b(w^{(\tau)}) - b(\sigma_\tau w^{(\tau)})$ in a larger space than $L^2(Q_T)$ and use the assumption $d = 1$ in order to obtain a uniform L^∞ bound for e^{w^k}. Indeed, from the weak formulation of Eqs. (3.5) for $i = 1, 2$ we obtain for $V = H_0^1(\Omega \cup \Gamma_N)$, $i = 1, 2$,

$$\begin{aligned}
&\tau^{-1} \|b_i(w^{(\tau)}) - b_i(\sigma_\tau w^{(\tau)})\|_{L^1(0,T;V^*)} \\
&\quad \le \|c_i \nabla e^{w_i^{(\tau)}} + 2a_i e^{w_i^{(\tau)}} \nabla e^{w_i^{(\tau)}} + d_i e^{w_i^{(\tau)}} q^{(\tau)} + F_i(w^{(\tau)})\|_{L^1(0,T;L^2(\Omega))} \\
&\qquad + \|e^{w_1^{(\tau)}} \nabla e^{w_2^{(\tau)}} + e^{w_2^{(\tau)}} \nabla e^{w_1^{(\tau)}}\|_{L^1(0,T;L^2(\Omega))} \\
&\quad \le c_i \|\nabla e^{w_i^{(\tau)}}\|_{L^1(0,T;L^2(\Omega))} + 2a_i \|e^{w_i^{(\tau)}}\|_{L^2(0,T;L^\infty(\Omega))} \|\nabla e^{w_i^{(\tau)}}\|_{L^2(Q_T)} \\
&\qquad + d_i \|e^{w_i^{(\tau)}}\|_{L^2(0,T;L^\infty(\Omega))} \|q^{(\tau)}\|_{L^2(Q_T)} + C_f \\
&\qquad + \|e^{w_1^{(\tau)}}\|_{L^2(0,T;L^\infty(\Omega))} \|e^{w_2^{(\tau)}}\|_{L^2(Q_T)} + \|e^{w_2^{(\tau)}}\|_{L^2(0,T;L^\infty(\Omega))} \|e^{w_1^{(\tau)}}\|_{L^2(Q_T)}.
\end{aligned}$$

Since $a_i > 0$ and $d = 1$, we obtain

$$\|e^{w_i^{(\tau)}}\|_{L^2(0,T;L^\infty(\Omega))} \le C \|e^{w_i^{(\tau)}}\|_{L^2(0,T;H^1(\Omega))} \le C,$$

in view of Corollary 4.1, and therefore

$$\|b_i(w^{(\tau)}) - b_i(\sigma_\tau w^{(\tau)})\|_{L^1(0,T;V^*)} \le C\tau.$$

Furthermore,

$$\|\partial_t \tilde{b}^{(\tau)}\|_{L^1(0,T,V^*)} \le \tau^{-1} \|b(w^{(\tau)}) - b(\sigma_\tau w^{(\tau)})\|_{L^1(0,T;V^*)} \le C$$

and

$$\|\tilde{b}^{(\tau)}\|_{L^2(0,T;H^1(\Omega))} \le 2\|b(w^{(\tau)})\|_{L^2(0,T;H^1(\Omega))} + \|b(\sigma_\tau w^{(\tau)})\|_{L^2(0,T;H^1(\Omega))} \le C,$$

by Corollary 4.1. □

Proof of Theorem 2.3. Since the embedding $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ is compact in one space dimension, we can apply Aubin's lemma [28] to $\tilde{b}^{(\tau)}$, in view of the uniform

bounds of Lemma 4.2, to obtain, up to a subsequence which is not relabeled,

$$\partial_t \tilde{b}^{(\tau)} \rightharpoonup \partial_t z \qquad \text{weakly in } L^1(0,T;V^*), \tag{4.1}$$

$$\tilde{b}^{(\tau)} \rightharpoonup z \qquad \text{weakly in } L^2(0,T;H^1(\Omega)), \tag{4.2}$$

$$\tilde{b}^{(\tau)} \to z \qquad \text{strongly in } L^2(0,T;L^\infty(\Omega)), \tag{4.3}$$

$$b(w^{(\tau)}) \rightharpoonup u \qquad \text{weakly in } L^2(0,T;H^1(\Omega)). \tag{4.4}$$

By Lemma 4.2 we have, as $\tau \to 0$,

$$\|\tilde{b}^{(\tau)} - b(w^{(\tau)})\|_{L^1(0,T;V^*)} \le \|b(w^{(\tau)}) - b(\sigma_\tau w^{(\tau)})\|_{L^1(0,T;V^*)} \to 0,$$

and hence $z = u$.

We claim now that

$$b(w^{(\tau)}) \to u \quad \text{strongly in } L^2(Q_T). \tag{4.5}$$

Indeed, by (4.2)-(4.4),

$$\begin{aligned}
&\|b(w^{(\tau)}) - u\|_{L^1(0,T;L^2(\Omega))} \\
&\quad\le \|b(w^{(\tau)}) - \tilde{b}^{(\tau)}\|_{L^1(0,T;L^2(\Omega))} + \|\tilde{b}^{(\tau)} - u\|_{L^1(0,T;L^2(\Omega))} \\
&\quad\le \|b(w^{(\tau)}) - \tilde{b}^{(\tau)}\|^{1/2}_{L^1(0,T;V^*)} \|b(w^{(\tau)}) - \tilde{b}^{(\tau)}\|^{1/2}_{L^1(0,T;V)} + \|\tilde{b}^{(\tau)} - u\|_{L^1(0,T;L^2(\Omega))} \\
&\quad\le \|b(w^{(\tau)}) - \tilde{b}^{(\tau)}\|_{L^1(0,T;V^*)} \cdot C + \|\tilde{b}^{(\tau)} - u\|_{L^1(0,T;L^2(\Omega))} \\
&\quad\to 0 \text{ as } \tau \to 0,
\end{aligned}$$

and thus

$$b(w^{(\tau)}) \to u \quad \text{strongly in } L^1(0,T;L^2(\Omega)). \tag{4.6}$$

Then Corollary 4.1 and (4.6) give

$$\begin{aligned}
\|b(w^{(\tau)}) - u\|_{L^2(0,T;L^1(\Omega))} &\le \|b(w^{(\tau)}) - u\|^{1/2}_{L^\infty(0,T;L^1(\Omega))} \|b(w^{(\tau)}) - u\|^{1/2}_{L^1(Q_T)} \\
&\le C\|b(w^{(\tau)}) - u\|^{1/2}_{L^1(Q_T)} \\
&\to 0 \quad \text{as } \tau \to 0,
\end{aligned}$$

and Gagliardo-Nirenberg's inequality yields

$$\begin{aligned}
\|b(w^{(\tau)}) - u\|_{L^2(Q_T)} &\le C\|b(w^{(\tau)}) - u\|^{1/2}_{L^2(0,T;H^1(\Omega))} \|b(w^{(\tau)}) - u\|^{1/2}_{L^2(0,T;L^1(\Omega))} \\
&\le C\|b(w^{(\tau)}) - u\|^{1/2}_{L^2(0,T;L^1(\Omega))} \\
&\to 0 \quad \text{as } \tau \to 0,
\end{aligned}$$

which is (4.5).

Now we can let $\tau \to 0$ in the weak formulation of (3.5), $i = 1, 2$, which reads for $\phi \in L^\infty(0,T;(W^{1,\infty}(\Omega))^*)$:

$$\int_0^T \langle \partial_t \tilde{b}_i^{(\tau)}, \phi \rangle dt + \int_{Q_T} \left(c_i \nabla e^{w_i^{(\tau)}} + 2a_i e^{w_i^{(\tau)}} \nabla e^{w_i^{(\tau)}} + \nabla e^{w_1^{(\tau)} + w_2^{(\tau)}} \right) \cdot \nabla \phi dx dt$$
$$= -d_i \int_{Q_T} e^{w_i^{(\tau)}} q^{(\tau)} \cdot \nabla \phi dx dt + \int_{Q_T} f_i(e^{w_1^{(\tau)}}, e^{w_2^{(\tau)}}) \phi dx dt.$$

In view of (4.1)-(4.5), (2.3) and Assumption (A4) we obtain

$$\int_0^T \langle \partial_t u_i, \phi \rangle dt + \int_{Q_T} (c_i \nabla u_i + 2a_i u_i \nabla u_i + \nabla(u_1 u_2)) \cdot \nabla \phi dx dt$$
$$= -d_i \int_{Q_T} u_i q \cdot \nabla \phi dx dt + \int_{Q_T} f_i(u_1, u_2) \phi dx dt,$$

i.e. $u = (u_1, u_2)$ is a weak solution of (1.10)-(1.11). Moreover, the initial condition (1.13) is satisfied in the sense of V^*. □

Remark 4.3 The presented positivity-preserving scheme will be used for numerical simulations in a future work. For numerical stationary solutions, we refer to [10].

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