A Hierarchy of Hydrodynamic Models for Plasmas. Quasi-Neutral Limits in the Drift-Diffusion Equations

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Abstract

This paper is a continuation of a series of papers in which (quasi-) hydrodynamic models for plasmas are rigorously derived by means of asymptotic analysis. Here, the quasi-neutral limit (zero-Debye-length limit) in the drift-diffusion equations is performed in the two cases: weakly ionized plasmas and not weakly ionized plasmas.

The model consists of the continuity equations for the electrons and ions, the constitutive relations for the particle current densities, and the Poisson equation for the electrostatic potential in a bounded domain. In the case of a weakly ionized plasma, the continuity equation for the electrons is replaced by a relation between the electrostatic potential and the electron density such that the Poisson equation becomes nonlinear. The equations are complemented by mixed Dirichlet-Neumann boundary conditions and initial conditions.

The quasi-neutral limits are shown without assuming compatibility conditions on the boundary densities. The proofs rely on the use of the so-called entropy functional which yields appropriate uniform estimates, and compensated compactness methods.

Keywords. Asymptotic analysis, singular perturbation, drift-diffusion equations, plasmas, entropy method, boundary layers.

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1 Introduction

Recently, we started a program to derive rigorously a hierarchy of macroscopic models for plasmas. In particular, we are interested in justifying mathematically various asymptotic limits in the hydrodynamic and the drift-diffusion equations used in plasma physics, namely (i) the zero-relaxation-time limit, (ii) the zero-electron-mass limit, and (iii) the quasi-neutral limit. In this paper we continue our program by studying the quasi-neutral limit in the drift-diffusion equations. We refer to the papers [14, 18, 19, 20] for the limits (i) and (ii).

Consider an unmagnetized plasma consisting of electrons with density n_e^ε and of a single species of positively charged ions with density n_i^ε. We suppose that the particle densities satisfy the drift-diffusion equations coupled self-consistently to the Poisson equation for the electrostatic potential ϕ^ε (system **(DD-EI)**):

$$
\begin{aligned}
\partial_t n_i^\varepsilon - \operatorname{div}\left(\nabla p_i(n_i^\varepsilon) + \mu_i n_i^\varepsilon \nabla\phi^\varepsilon\right) &= 0, && (1.1)\\
\partial_t n_e^\varepsilon - \operatorname{div}\left(\nabla p_e(n_e^\varepsilon) - \mu_e n_e^\varepsilon \nabla\phi^\varepsilon\right) &= 0, && (1.2)\\
-\varepsilon^2 \Delta\phi^\varepsilon &= n_i^\varepsilon - n_e^\varepsilon && (1.3)
\end{aligned}
$$

in an open and bounded domain $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$. The boundary $\partial\Omega$ of the domain is supposed to consist of two disjoint sets Γ_D and Γ_N. The system (DD-EI) is complemented by mixed Dirichlet-Neumann boundary conditions and initial conditions:

$$
\begin{aligned}
n_\alpha^\varepsilon = n_{D,\alpha}, \quad \phi^\varepsilon = \phi_D & \qquad \text{on } (0,T)\times\Gamma_D, && (1.4)\\
\nabla p_\alpha(n_\alpha^\varepsilon)\cdot\nu = \nabla\phi^\varepsilon\cdot\nu = 0 & \qquad \text{on } (0,T)\times\Gamma_N, && (1.5)\\
n_\alpha^\varepsilon(0) = n_{I,\alpha} & \qquad \text{in } \Omega, && (1.6)
\end{aligned}
$$

for $\alpha = e, i$. Here, $p_e(n_e^\varepsilon)$, $p_i(n_i^\varepsilon)$ denote the pressure functions of the electrons and ions, respectively. The function ν is the normal unit vector of $\partial\Omega$ which is assumed to exist almost everywhere. The physical parameters are the (constant) mobilities $\mu_e > 0$, $\mu_i > 0$ and the (scaled) Debye length $\varepsilon > 0$. The model (1.1)–(1.3) can be derived from the hydrodynamic plasma equations in the zero-relaxation-time limit [18, 19]. Our first main goal is to perform the limit $\varepsilon \to 0$ in the system (DD-EI) rigorously.

When the kinetic energy of the electrons is much smaller than their thermal energy, the electron continuity equation (1.1) reduces asymptotically to the relation (see [20] for details)

$$0 = h_e(n_e^\varepsilon) - \mu_e\phi^\varepsilon,$$

where h_α is the enthalpy function defined by

$$s h_\alpha'(s) = p_\alpha'(s) \ \text{ for } \ s > 0 \ \text{ and } \ h_\alpha(1) = 0, \ \ \alpha = e, i.$$

Hence the electron density is given by the relation $n_e^\varepsilon = f_e(\phi^\varepsilon)$, where

$$f_e(s) = h_e^{-1}(\mu_e s)$$

and ϕ^ε solves the nonlinear Poisson equation

$$-\varepsilon^2 \Delta \phi^\varepsilon = n_i - f_e(\phi^\varepsilon).$$

In this situation ('weakly ionized plasma'), the ion density and the electrostatic potential solve the problem **(DD-I)**:

$$\begin{aligned} \partial_t n_i^\varepsilon - \mathrm{div}\,(\nabla p_i(n_i^\varepsilon) + \mu_i n_i^\varepsilon \nabla \phi^\varepsilon) &= 0, && (1.7) \\ -\varepsilon^2 \Delta \phi^\varepsilon &= n_i^\varepsilon - f_e(\phi^\varepsilon) && (1.8) \end{aligned}$$

subject to the boundary and initial conditions (1.4)–(1.6) for $\alpha = i$. Our second main goal is to study rigorously the limit $\varepsilon \to 0$ in (DD-I).

Before stating our main results, we perform the quasi-neutral limit $\varepsilon \to 0$ formally in the systems (DD-EI) and (DD-I) to derive the limit problems. In (DD-EI) we get for $\varepsilon = 0$:

$$\begin{aligned} n \stackrel{\text{def}}{=} n_e &= n_i, \\ \partial_t n - \mathrm{div}\,(\nabla p_i(n) + \mu_i n \nabla \phi) &= 0, \\ \partial_t n - \mathrm{div}\,(\nabla p_e(n) - \mu_e n \nabla \phi) &= 0. \end{aligned}$$

Adding and subtracting these equations leads to

$$\begin{aligned} 2\partial_t n - \mathrm{div}\,(\nabla(p_i(n) + p_e(n)) + (\mu_i - \mu_e) n \nabla \phi) &= 0, \\ -\mathrm{div}\,((\mu_i + \mu_e) n \nabla \phi) &= \Delta(p_i(n) - p_e(n)). \end{aligned}$$

Inserting the second equation in the first, we can eliminate the drift term, and the final system **(QN-EI)** becomes

$$\begin{aligned} \partial_t n - \Delta p(n) &= 0, && (1.9) \\ -\mathrm{div}\,((\mu_i + \mu_e) n \nabla \phi) &= \Delta(p_i(n) - p_e(n)), && (1.10) \end{aligned}$$

where

$$p(n) = \frac{\mu_e}{\mu_i + \mu_e} p_i(n) + \frac{\mu_i}{\mu_i + \mu_e} p_e(n). \tag{1.11}$$

As an example, let us assume that the electrons and ions are described by the same density-pressure relation, but with different diffusivities, which accounts for the different masses, i.e.

$$p_\alpha(s) = D_\alpha p_0(s), \quad \alpha = e, i.$$

Then Eqs. (1.9)–(1.10) become

$$\begin{aligned} \partial_t n - D_a \Delta p_0(n) &= 0, \\ -\mathrm{div}\,(n \nabla \phi) &= D_b \Delta p_0(n), \end{aligned}$$

where

$$D_a = \frac{D_e \mu_i + D_i \mu_e}{\mu_i + \mu_e}, \quad D_b = \frac{D_i - D_e}{\mu_i + \mu_e}.$$

This means that the new diffusivity of the quasi-neutral plasma is given by the so-called *ambipolar* diffusion coefficient D_a. This relation is well-known in plasma physics [7, p. 160].

The quasi-neutral limit in the system (DD-I) gives formally

$$\begin{aligned} \partial_t n_i - \operatorname{div}(\nabla p_i(n_i) + \mu_i n_i \nabla\phi) &= 0, \\ n_i &= f_e(\phi). \end{aligned}$$

Hence, we get the limiting system **(QN-I)** for $n \stackrel{\text{def}}{=} n_i$:

$$\partial_t n - \Delta(p_i(n) + (\mu_i/\mu_e)p_e(n)) = 0, \tag{1.12}$$

$$\phi = h_e(n)/\mu_e, \tag{1.13}$$

where the functions p_e and h_e are defined, for given f_e, by $p_e'(s) = sh_e'(s)$, $p_e(1) = 0$, and $h_e(s) = \mu_e f_e^{-1}(s)$, $s > 0$.

In order to make the quasi-neutral limits in the systems (DD-EI) and (DD-I) rigorously, we assume that at $t = 0$, the electron and ion density are equal:

$$n_{I,e} = n_{I,i}. \tag{1.14}$$

This assumption avoids the formation of initial layers (see also [13]). Furthermore, we assume that the boundary and initial densities are strictly positive. Then we can show that in the system (DD-EI) the sequences n_e^ε and n_i^ε converge locally in some Sobolev spaces to a function n satisfying the limit equation (1.9) (see Theorem 2.1). In general we cannot expect to get global convergence due to boundary layer phenomena. However, if the electron and ion densities coincide on the Dirichlet boundary, then we have strong convergence. Moreover, if the boundary and initial densities are only non-negative, we can also perform the quasi-neutral limit (see Theorem 3.6).

For the quasi-neutral limit in the system (DD-I) we have a stronger result. Indeed, we do not need compatibility conditions on the boundary or initial functions in order to get strong global convergence of the sequences. However, we need some regularity for the electrostatic potential, i.e., we need to require that the Dirichlet and Neumann boundary segments do not meet (see Theorem 2.2). For the quasi-neutral limit in the case of only non-negative densities, we refer to Theorem 4.6.

The quasi-neutral limit in macroscopic models for plasmas has been first studied by Brézis *et al.* in [4] and by Cordier *et al.* in [8, 9]. In the first mentioned paper the limit $\varepsilon \to 0$ is rigorously performed for the nonlinear Poisson equation (the ion density being fixed). In the second article the authors study the quasi-neutral limit for the traveling wave solutions of the hydrodynamic plasma equations. The quasi-neutral limit in the hydrodynamic equations for smooth solutions is rigorously performed by Cordier and Grenier in [10]. Asymptotic expansions in powers of ε to the one-dimensional drift-diffusion equations for semiconductors are derived by Markowich and Ringhofer [22]. Recently, Gasser *et al.* [13] also studied the quasi-neutral limit in the drift-diffusion equations for semiconductors. They used homogeneous Neumann boundary conditions which simplifies the analysis, since no boundary layers are expected. In this paper, however, the main difficulty comes from the possible presence of boundary layers, and the analysis has to be more careful on this subject.

The proofs of the Theorems 2.1 and 2.2 are based on the entropy method. The entropy (or free energy or entropy functional) gives additional a priori estimates via the entropy inequality. It is defined essentially (up to the contribution of the boundary functions) by

$$\int_\Omega \left(P_e(n_e^\varepsilon) + P_i(n_i^\varepsilon) + \frac{1}{2}|\nabla\phi^\varepsilon|^2 \right) dx,$$

where $P'_\alpha(s) = p_\alpha(s)$ $(\alpha = e, i)$. The entropy method has been used very successfully in the analysis of parabolic equations. First used by Alt and Luckhaus [1] to obtain the existence of solutions to elliptic-parabolic systems, this method has been subsequently employed for the analysis of the long-time behavior of solutions [12, 16] and their decay rates as $t \to \infty$ [2, 5, 6, 23] and for singular perturbation analysis [13, 20].

First we prove uniform L^∞ bounds for n_e^ε, n_i^ε by means of the maximum principle. Then the entropy inequality shows that the entropy and the entropy production

$$\int_0^t \int_\Omega \left\{ \left|\nabla\Big(h_e(n_e^\varepsilon) - \mu_e\phi^\varepsilon\Big)\right|^2 + \left|\nabla\Big(h_i(n_i^\varepsilon) + \mu_i\phi^\varepsilon\Big)\right|^2 \right\} dx d\tau$$

are bounded uniformly in ε, for $t > 0$. Away from the boundary, using the Poisson equation and the above estimates, we get uniform bounds for $\nabla n_\alpha^\varepsilon$ and $\partial_t n_\alpha^\varepsilon$ in appropriate Sobolev spaces such that the Aubin compactness result can be used to get the compactness of n_α^ε in the Lebesgue space L^2_{loc} $(\alpha = e, i)$. This allows us to identify the nonlinearities in the limit.

We stress the fact that in Theorem 2.2, we do not suppose any compatibility condition on the boundary or initial data but nevertheless, we get strong convergence of the sequences. In order to get this result, we first prove that the normal derivative $\nabla\phi^\varepsilon \cdot \nu$ on the Dirichlet boundary is of order $\mathcal{O}(\varepsilon^{-1})$, using the idea in [4]. This shows that $n_i^\varepsilon - f_e(\phi^\varepsilon) \to 0$ strongly in L^2 as $\varepsilon \to 0$. Finally, in order to identify the nonlinear functions we employ compensated compactness and compactness-by-convexity tools.

The plan of this paper is as follows. In Section 2 we recall the existence results and state the main results for the quasi-neutral limits in the systems (DD-EI) and (DD-I). The limit $\varepsilon \to 0$ for the system (DD-EI) is proved in Section 3, whereas the quasi-neutral limit in (DD-I) is shown in Section 4.

2 Main results

In this section we recall some existence results for the systems (DD-EI) and (DD-I) and state the main results for the quasi-neutral limit for general pressure functions.

Let $\Omega \subset \mathbb{R}^d$ with $d \geq 1$ be an open and bounded domain with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$ satisfying $\Gamma_D \cap \Gamma_N = \emptyset$, $\text{meas}(\Gamma_D) > 0$, and Γ_N is open in $\partial\Omega$. We introduce the notations

$$\Omega_T = (0,T) \times \Omega, \quad \Sigma_D = (0,T) \times \Gamma_D, \quad \Sigma_N = (0,T) \times \Gamma_N,$$

and impose the following assumptions for $\alpha = e, i$:

(A1) $n_{I,\alpha} \in L^\infty(\Omega)$, $n_{D,\alpha} \in C^0([0,T]; L^\infty(\Omega)) \cap H^1(\Omega_T)$, $\phi_D \in H^1(0,T; H^1(\Omega)) \cap L^\infty(\Omega_T)$, $\mu_\alpha > 0$.

(A2) There exist two constants $\underline{n}, \overline{n} > 0$ such that $0 < \underline{n} \leq n_{I,\alpha} \leq \overline{n}$ in Ω and $0 < \underline{n} \leq n_{D,\alpha} \leq \overline{n}$ on Σ_D.

(A3) $p_\alpha \in C^1([0,+\infty))$ is strictly increasing on $(0,+\infty)$.

Notice that the case $p'_\alpha(0) = 0$ is allowed in hypothesis (A3), and then the Eqs. (1.1)–(1.2) are of degenerate type. Under the assumptions (A1) and (A3) it has been proved in [15, 17] that there exists a solution $(n_e^\varepsilon, n_i^\varepsilon, \phi^\varepsilon)$ to the problem (1.1)–(1.6) satisfying

$$n_\alpha^\varepsilon \in H^1(0,T; V^*) \cap L^\infty(\Omega_T) \cap L^2(0,T; H^1(\Omega)), \tag{2.1}$$
$$\phi^\varepsilon \in L^\infty(0,T; H^1(\Omega)), \tag{2.2}$$

where V^* is the dual space of

$$V = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_D\}.$$

Moreover, under the assumptions (A1)–(A3), $\partial\Omega = \Gamma_D$, and some regularity conditions on the boundary functions, the solution is unique. The assumption (A2) implies that the densities n_α^ε are strictly positive in Ω_T (see Lemma 3.1). The quasi-neutral limit without assuming (A2) will be studied in Theorems 3.6 and 4.6.

For the quasi-neutral limit we also use the compatibility conditions:

(A4) $n_{I,e} = n_{I,i}$ in Ω.

(A5) $n_{D,e} = n_{D,i}$ on Σ_D.

Then it holds:

Theorem 2.1 (Quasi-neutral limit in the system (DD-EI))
Let $T > 0$ and the hypotheses (A1)–(A4) hold. Let $(n_e^\varepsilon, n_i^\varepsilon, \phi^\varepsilon)$ be a weak solution to (1.1)-(1.6). Furthermore, let $\omega \subset \Omega$ be such that $\overline{\omega} \subset \Omega$. Then there is a subsequence (not relabeled) such that for $\alpha = e, i$, as $\varepsilon \to 0$,

$$\begin{aligned} n_\alpha^\varepsilon &\to n && \text{in } L^p(\omega_T) \text{ strongly, for all } p \in [1,+\infty), \\ n_\alpha^\varepsilon &\rightharpoonup n && \text{in } L^2(0,T; H^1(\omega)) \text{ weakly}, \\ \phi^\varepsilon &\rightharpoonup \phi && \text{in } L^2(0,T; H^1(\omega)) \text{ weakly}, \end{aligned}$$

and (n,ϕ) is a solution of Eqs. (1.9)–(1.10) and $n(0) = n_{I,e} = n_{I,i}$ in the sense of $H^{-1}(\omega)$. Moreover, it holds

$$\|n_e^\varepsilon - n_i^\varepsilon\|_{L^2(\omega_T)} = \mathcal{O}(\varepsilon),$$

with $\omega_T = (0,T)\times\omega$. If additionally (A5) holds, then the above convergence results hold for $\omega = \Omega$ and for the whole sequence $(n_e^\varepsilon, n_i^\varepsilon, \phi^\varepsilon)$, and the corresponding boundary conditions for the limit problem (1.9)–(1.10) are satisfied.

Concerning the system (DD-I) we have to specify our assumption on the nonlinearity f_e. From the derivation of the problem (DD-I), $f_e(s)$ is equal to $h_e^{-1}(\mu_e s)$. Here, we consider general functions f_e:

(A6) $f_e \in C^1(\mathbb{R})$ is increasing on $\mathbb{R}$ and strictly increasing on $(0,+\infty)$, $f_e(0) > 0$, $f_e(+\infty) = +\infty$.

(A7) The function $s \mapsto s(h_i(s)+h_e(s))$ is of class C^2 and strictly convex on $(0,+\infty)$.

The assumptions (A6) and (A7) are satisfied for isothermal or adiabatic pressure functions of the form $p_\alpha(s) = a_\alpha s^{\gamma_\alpha}$ with $\gamma_\alpha \geq 1$ and $a_\alpha > 0$ since then (if f_e is defined via h_e)

$$\begin{aligned} h_\alpha(s) &= a_\alpha \log(n), & f_e(s) &= e^{\mu_e s/a_e} & &\text{if } \gamma_\alpha = 1; \\ h_\alpha(s) &= \tfrac{a_\alpha\gamma_\alpha}{\gamma_\alpha - 1}(s^{\gamma_\alpha - 1} - 1), & f_e(s) &= \left(\tfrac{\mu_e(\gamma_e-1)}{a_e\gamma_e}s + 1\right)_+^{1/(\gamma_e-1)} & &\text{if } \gamma_\alpha > 1. \end{aligned}$$

Since we have to estimate $|\nabla\phi^\varepsilon\cdot\nu|$ on Γ_D, we need some regularity for ϕ^ε. Therefore we suppose

(A8) $\partial\Omega \in C^{1,1}$, $\overline{\Gamma}_D \cap \overline{\Gamma}_N = \emptyset$, and $\phi_D \in L^2(0,T;H^2(\Omega))$.

Then the electric potential satisfies $\phi^\varepsilon \in L^2(0,T;H^2(\Omega))$ [25].

The existence of weak solutions $(n_i^\varepsilon, \phi^\varepsilon)$ to (1.7)–(1.8), (1.4)–(1.6) satisfying (2.1)–(2.2) for $\alpha = i$ has been proved in [20] under the assumptions (A1), (A3), and (A6). Uniqueness of solutions follows under the additional hypothesis (A2), $\partial\Omega = \Gamma_D$ and regularity assumptions on the boundary functions. Under the conditions (A1)–(A3) for $\alpha = i$ and (A6), (A8) we are able to show show that

$$\|n_i^\varepsilon - f_e(\phi^\varepsilon)\|_{L^2(\Omega_T)} = \mathcal{O}(\sqrt{\varepsilon}) \quad \text{as } \varepsilon \to 0.$$

In order to obtain the strong compactness of the sequence $(n_i^\varepsilon, \phi^\varepsilon)$, we need the hypothesis (A7).

Theorem 2.2 (Quasi-neutral limit in the system (DD-I))
Let the hypotheses (A1)–(A3) for $\alpha = i$ and (A6)–(A8) hold. Furthermore, let $\omega \subset \Omega$ be such that $\overline{\omega} \subset \Omega$. Then there is a sequence of weak solutions $(n_i^\varepsilon, \phi^\varepsilon)$ to the problem (DD-I) such that, as $\varepsilon \to 0$,

$$\begin{aligned} n_i^\varepsilon &\to n & &\text{in } L^2(\Omega_T) \text{ strongly}, \\ \phi^\varepsilon &\to \phi & &\text{in } L^2(\Omega_T) \text{ strongly}, \end{aligned}$$

and (n,ϕ) solves Eqs. (1.12)–(1.13) and $n(0)=n_{I,i}$ in the sense of $H^{-1}(\omega)$. Moreover, we have

$$\|n_i^\varepsilon - f_e(\phi^\varepsilon)\|_{L^2(\Omega_T)} = \mathcal{O}(\varepsilon^{1/2}), \quad \|\nabla\phi^\varepsilon\|_{L^2(\Omega_T)} = \mathcal{O}(\varepsilon^{-1/2}), \tag{2.3}$$

$$\|n_i^\varepsilon - f_e(\phi^\varepsilon)\|_{L^2(\omega_T)} = \mathcal{O}(\varepsilon), \quad \|\nabla\phi^\varepsilon\|_{L^2(\omega_T)} = \mathcal{O}(1). \tag{2.4}$$

If additionally, $n_{D,i} = f_e(\phi_D)$, then (n,ϕ) also satisfies the boundary conditions (1.4)–(1.5) for $\alpha = i$.

3 The quasi-neutral limit in (DD-EI)

In this section, we suppose that the assumptions (A1)–(A3) hold. Before proving Theorem 2.1 we show some auxiliary lemmas. For this purpose, let $(n_e^\varepsilon, n_i^\varepsilon, \phi^\varepsilon)$ be a weak solution to (1.1)–(1.6).

Lemma 3.1 *It holds*

$$0 < \underline{n} \le n_\alpha^\varepsilon(t,x) \le \overline{n} \quad \text{in } \Omega_T, \quad \alpha = e, i.$$

Proof. Use $\mu_e(n_i^\varepsilon - \overline{n})^+, \mu_i(n_e^\varepsilon - \overline{n})^+$ as test functions in Eqs. (1.1), (1.2) respectively and add both equations to get (omitting the index ε)

$$\begin{aligned}
&\frac{1}{2}\int_\Omega [\mu_e(n_i-\overline{n})^+(t)^2 + \mu_i(n_e-\overline{n})^+(t)^2]dx \\
&\quad + \int_{\Omega_t} [\mu_e p_i'(n_i)|\nabla(n_i-\overline{n})^+|^2 + \mu_i p_e'(n_e)|\nabla(n_e-\overline{n})^+|^2]dxd\tau \\
= &\quad \mu_e\mu_i \int_{\Omega_t} (n_e-\overline{n})\nabla\phi\cdot\nabla(n_e-\overline{n})^+ + \mu_e\mu_i\int_{\Omega_t} \overline{n}\nabla\phi\cdot\nabla(n_e-\overline{n})^+ dxd\tau \\
&\quad - \mu_e\mu_i \int_{\Omega_t} (n_i-\overline{n})\nabla\phi\cdot\nabla(n_i-\overline{n})^+ - \mu_e\mu_i\int_{\Omega_t} \overline{n}\nabla\phi\cdot\nabla(n_i-\overline{n})^+ dxd\tau \\
= &\quad \frac{\mu_i\mu_e}{2}\int_{\Omega_t}\nabla\phi\cdot\nabla(n_e-\overline{n})^{+2} + \frac{\mu_i\mu_e\overline{n}}{\varepsilon^2}\int_{\Omega_t}(n_i-n_e)(n_e-\overline{n})^+ dxd\tau \\
&\quad - \frac{\mu_e\mu_i}{2}\int_{\Omega_t}\nabla\phi\cdot\nabla(n_i-\overline{n})^{+2} - \frac{\mu_e\mu_i\overline{n}}{\varepsilon^2}\int_{\Omega_t}(n_i-n_e)(n_i-\overline{n})^+ dxd\tau \\
= &\quad -\frac{\mu_e\mu_i}{2\varepsilon^2}\int_{\Omega_t}(n_i-n_e)((n_i-\overline{n})^{+2} - (n_e-\overline{n})^{+2})dxd\tau \\
&\quad - \frac{\mu_e\mu_i\overline{n}}{\varepsilon^2}\int_{\Omega_t}(n_i-n_e)((n_i-\overline{n})^+ - (n_e-\overline{n})^+)dxd\tau \\
\le &\quad 0,
\end{aligned}$$

since the functions $s\mapsto s^+$ and $s\mapsto (s^+)^2$ are non-decreasing. Thus $n_\alpha \le \overline{n}$ in Ω_T.

Using $\mu_e(n_i-\underline{n})^-, \mu_i(n_e-\underline{n})^-$ as test functions in Eqs. (1.1), (1.2) respectively and employing the Poisson equation (1.3) as above leads to

$$\begin{aligned}
&\frac{1}{2}\int_\Omega [\mu_e(n_i-\underline{n})^-(t)^2+\mu_i(n_e-\underline{n})^-(t)^2]dx \\
&\qquad +\int_{\Omega_t}[\mu_e p_i'(n_i)|\nabla(n_i-\underline{n})^-|^2+\mu_i p_e'(n_e)|\nabla(n_e-\underline{n})^-|^2]dxd\tau \\
&= -\frac{\mu_e\mu_i}{2\varepsilon^2}\int_{\Omega_t}(n_i-n_e)((n_i-\underline{n})^{-2}-(n_e-\underline{n})^{-2})dxd\tau \\
&\quad -\frac{\mu_e\mu_i\underline{n}}{\varepsilon^2}\int_{\Omega_t}(n_i-n_e)((n_i-\underline{n})^{-}-(n_e-\underline{n})^{-})dxd\tau \\
&\le -\frac{\mu_e\mu_i}{2\varepsilon^2}\int_{\Omega_t}(n_i-n_e)((n_i-\underline{n})^{-2}-(n_e-\underline{n})^{-2})dxd\tau,
\end{aligned}$$

since the functions $s\mapsto s^-=\min(0,s)$ is non-decreasing. Thus

$$\begin{aligned}
&\min(\mu_e,\mu_i)\int_\Omega[(n_i-\underline{n})^-(t)^2+(n_e-\underline{n})^-(t)^2]dx \\
&\le \frac{\mu_e\mu_i\overline{n}}{\varepsilon^2}\int_{\Omega_t}[(n_i-\underline{n})^{-2}+(n_e-\underline{n})^{-2}]dxd\tau.
\end{aligned}$$

From Gronwall's lemma we conclude that $n_\alpha\ge\underline{n}$ in Ω_T, $\alpha=e,i$.

Lemma 3.2 *It holds*

$$\phi^\varepsilon\in H^1(0,T;H^1(\Omega)).$$

Proof. For almost all $t>0$, the function $\phi^\varepsilon(t)\in H^1(\Omega)$ solves

$$\begin{aligned}
-\varepsilon^2\Delta\phi^\varepsilon(t) &= (n_i-n_e)(t) \quad \text{in } \Omega, \\
\phi^\varepsilon(t)=\phi_D(t) \ \text{ on } \Gamma_D, &\quad \nabla\phi^\varepsilon(t)\cdot\nu=0 \ \text{ on } \Gamma_N.
\end{aligned}$$

Since $n_i-n_e\in L^2(0,T;H^1(\Omega)), \phi^\varepsilon$ is Bochner-measurable. Furthermore,

$$\varepsilon\|\phi^\varepsilon(t)\|_V\le c(\|(n_i-n_e)(t)\|_{V^*}+\varepsilon\|\phi_D(t)\|_V),$$

which yields $\phi^\varepsilon\in L^2(0,T;H^1(\Omega))$. Finally, for $h>0$ and almost all $t<T-h$,

$$\begin{aligned}
&\varepsilon\Big\|\frac{1}{h}\Big(\phi^\varepsilon(t+h)-\phi^\varepsilon(t)\Big)\Big\|_V \\
&\le c\Big(\Big\|\frac{1}{h}\Big((n_i-n_e)(t+h)-(n_i-n_e)(t)\Big)\Big\|_{V^*} \\
&\qquad +\varepsilon\Big\|\frac{1}{h}\Big(\phi_D(t+h)-\phi_D(t)\Big)\Big\|_V\Big),
\end{aligned}$$

so that the regularity properties $n_i,n_e\in H^1(0,T;V^*), \phi_D\in H^1(0,T;H^1(\Omega))$ imply that $\partial_t\phi^\varepsilon\in L^2(0,T;H^1(\Omega))$. The assertion of the lemma follows.

For general pressure functions, the entropy is defined by

$$\begin{aligned} H_\varepsilon(t) &= \int_\Omega \Big[\frac{1}{\mu_e}\int_{n_{D,e}(t)}^{n_e^\varepsilon(t)} (h_e(s) - h_e(n_{D,e}(t)))ds \\ &\quad + \frac{1}{\mu_i}\int_{n_{D,i}(t)}^{n_i^\varepsilon(t)} (h_i(s) - h_i(n_{D,i}(t)))ds\Big]dx + \frac{\varepsilon^2}{2}\int_\Omega |\nabla(\phi^\varepsilon(t) - \phi_D(t))|^2 dx. \end{aligned}$$

This function is non-negative and well defined for $t = 0$ and for almost all $t > 0$. Notice that under the assumption (A4), $H_\varepsilon(0)$ is bounded uniformly in $\varepsilon > 0$. Indeed, $\phi^\varepsilon(0)$ solves

$$\begin{gathered} -\varepsilon^2\Delta\phi^\varepsilon(0) = (n_i^\varepsilon - n_e^\varepsilon)(0) = 0 \quad \text{in } \Omega, \\ \phi^\varepsilon(0) = \phi_D(0) \text{ on } \Gamma_D, \quad \nabla\phi^\varepsilon(0)\cdot\nu = 0 \text{ on } \Gamma_N, \end{gathered}$$

and the initial and boundary data do not depend on $\varepsilon > 0$.

The key result is the following lemma:

Lemma 3.3 *Assume that (A4) hold. Then the sequences*

$$(F_i^\varepsilon)_{\varepsilon>0} \stackrel{\text{def}}{=} (h_i(n_i^\varepsilon) + \mu_i\phi^\varepsilon)_{\varepsilon>0}, \quad (F_e^\varepsilon)_{\varepsilon>0} \stackrel{\text{def}}{=} (h_e(n_e^\varepsilon) - \mu_e\phi^\varepsilon)_{\varepsilon>0} \tag{3.1}$$

are bounded in $L^2(0,T;H^1(\Omega))$. Furthermore, $(\varepsilon\phi^\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty(0,T;$ $H^1(\Omega))$.

Proof. Omitting the index ε, we obtain for almost all $t > 0$,

$$\begin{aligned} H_\varepsilon(t) - H_\varepsilon(0) &= \int_0^t \sum_{\alpha=e}^{i} \frac{1}{\mu_\alpha}\Big[\langle\partial_t n_\alpha, h_\alpha(n_\alpha) - h_\alpha(n_{D,\alpha})\rangle_{V^*,V} \\ &\quad - \frac{1}{\mu_\alpha}\int_{\Omega_t} h'_\alpha(n_{D,\alpha})\partial_t n_{D,\alpha}(n_\alpha - n_{D,\alpha})dx\Big]d\tau \\ &\quad + \varepsilon^2\int_{\Omega_t}\nabla\partial_t(\phi - \phi_D)\cdot\nabla(\phi - \phi_D)dxd\tau. \end{aligned} \tag{3.2}$$

From the Poisson equation

$$-\varepsilon^2\Delta\partial_t\phi = \partial_t n_i - \partial_t n_e$$

we get, using (1.1), (1.2),

$$\begin{aligned} \varepsilon^2\int_{\Omega_t}\nabla\partial_t\phi\cdot\nabla(\phi - \phi_D)dxd\tau &= -\int_{\Omega_t} n_i\nabla(h_i(n_i) + \mu_i\phi)\cdot\nabla(\phi - \phi_D)dxd\tau \\ &\quad + \int_{\Omega_t} n_e\nabla(h_e(n_e) - \mu_e\phi)\cdot\nabla(\phi - \phi_D)dxd\tau. \end{aligned}$$

Inserting this relation into (3.2) and introducing the quasi-Fermi potentials on the Dirichlet boundary

$$F_{D,i} = h_i(n_{D,i}) + \mu_i\phi_D, \quad F_{D,e} = h_e(n_{D,e}) - \mu_e\phi_D, \tag{3.3}$$

we infer

$$
\begin{aligned}
H_\varepsilon(t) - H_\varepsilon(0) &= -\int_{\Omega_t} \frac{1}{\mu_e} n_e \nabla F_e \cdot \nabla(F_e - F_{D,e}) dx d\tau \\
&\quad - \int_{\Omega_t} \frac{1}{\mu_i} n_i \nabla F_i \cdot \nabla(F_i - F_{D,i}) dx d\tau \\
&\quad - \sum_{\alpha=e}^{i} \frac{1}{\mu_\alpha} \int_{\Omega_t} h'_\alpha(n_{D,\alpha}) \partial_t n_{D,\alpha}(n_\alpha - n_{D,\alpha}) dx d\tau \\
&\quad - \varepsilon^2 \int_{\Omega_t} \nabla \partial_t \phi_D \cdot \nabla(\phi - \phi_D) dx d\tau.
\end{aligned}
\tag{3.4}
$$

Then, with the use of Young's and Hölder's inequalities we can estimate:

$$
\begin{aligned}
H_\varepsilon(t) &\le H_\varepsilon(0) - \int_{\Omega_t} \Big(\frac{\underline{n}}{2\mu_e} |\nabla F_e|^2 + \frac{\underline{n}}{2\mu_i} |\nabla F_i|^2 \Big) dx d\tau \\
&\quad + \int_{\Omega_t} \Big(\frac{\overline{n}}{2\mu_e} |\nabla F_{D,e}|^2 + \frac{\overline{n}}{2\mu_i} |\nabla F_{D,i}|^2 \Big) dx d\tau \\
&\quad + \sum_{\alpha=e}^{i} \|h'_\alpha(n_{D,\alpha})\|_{L^\infty(\Omega_t)} \|\partial_t n_{D,\alpha}\|_{L^1(\Omega_t)} \|n_\alpha - n_{D,\alpha}\|_{L^\infty(\Omega_t)} \\
&\quad + \frac{\varepsilon^2}{2} \int_{\Omega_t} |\nabla(\phi - \phi_D)|^2 dx d\tau + \frac{\varepsilon^2}{2} \int_{\Omega_t} |\nabla \partial_t \phi_D|^2 dx d\tau.
\end{aligned}
\tag{3.5}
$$

In the following, $c > 0$ denotes a constant independent of $\varepsilon \in (0,1)$ with different values at different occurrences. We conclude from the above inequality

$$
\frac{\varepsilon^2}{2} \int_\Omega |\nabla(\phi - \phi_D)(t)|^2 dx \le c + \frac{\varepsilon^2}{2} \int_{\Omega_t} |\nabla(\phi - \phi_D)|^2 dx d\tau.
$$

By Gronwall's lemma, for almost all $t \in (0,T)$,

$$
\varepsilon^2 \|\nabla(\phi - \phi_D)(t)\|^2_{L^2(\Omega)} \le 2ce^T.
$$

In view of Poincaré's inequality we obtain the second assertion of the lemma. The first assertion then follows directly from (3.5).

Corollary 3.4 *Let (A4) hold. Then*

$$
\|n_i^\varepsilon - n_e^\varepsilon\|_{L^\infty(0,T;V^*)} = \mathcal{O}(\varepsilon) \quad \text{as } \varepsilon \to 0.
$$

Proof. From the Poisson equation (1.3) and Lemma 3.3 follows that

$$
\|n_i^\varepsilon - n_e^\varepsilon\|_{L^\infty(0,T;V^*)} = \varepsilon^2 \|\Delta \phi^\varepsilon\|_{L^\infty(0,T;V^*)} \le \varepsilon^2 \|\phi^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \le c\varepsilon.
$$

Proof of Theorem 2.1. Let $\omega \subset \Omega$ be a domain such that $\overline{\omega} \subset \Omega$ and set $\omega_T = (0,T) \times \omega$. We claim that there exists a constant independent of $\varepsilon > 0$ such that

$$
\|n_i^\varepsilon - n_e^\varepsilon\|_{L^2(\omega_T)} \le c\sqrt{\varepsilon}, \quad \sqrt{\varepsilon}\|\nabla \phi^\varepsilon\|_{L^2(\omega_T)} \le c.
$$

For $\alpha = e, i$, we denote by $\widetilde{f}_\alpha$ the inverse function of h_α, and we write

$$n_i^\varepsilon = \widetilde{f}_i(F_i^\varepsilon - \mu_i\phi^\varepsilon), \quad n_e^\varepsilon = \widetilde{f}_e(F_e^\varepsilon + \mu_e\phi^\varepsilon). \tag{3.6}$$

Let $v \in \mathcal{D}(\Omega_T)$ be satisfying $v \geq 0$ and $v = 1$ in ω. From the Poisson equation we have

$$\begin{aligned}
\|n_i^\varepsilon - n_e^\varepsilon\|^2_{L^2(\omega_T)} &\leq \int_{\Omega_T} (n_i^\varepsilon - n_e^\varepsilon)^2 v dxdt \\
&= -\varepsilon^2 \int_{\Omega_T} \Delta\phi^\varepsilon (n_i^\varepsilon - n_e^\varepsilon) v dxdt \qquad (3.7)\\
&= \varepsilon^2 \int_{\Omega_T} \nabla\phi^\varepsilon \cdot \nabla(n_i^\varepsilon - n_e^\varepsilon) v dxdt + \varepsilon^2 \int_{\Omega_T} \nabla\phi^\varepsilon \cdot \nabla v (n_i^\varepsilon - n_e^\varepsilon) dxdt \\
&= \varepsilon^2 \int_{\Omega_T} \nabla\phi^\varepsilon \cdot [\widetilde{f}_i'(F_i^\varepsilon - \mu_i\phi^\varepsilon)\nabla(F_i^\varepsilon - \mu_i\phi^\varepsilon) \\
&\quad - \widetilde{f}_e'(F_e^\varepsilon + \mu_e\phi^\varepsilon)\nabla(F_e^\varepsilon + \mu_e\phi^\varepsilon)] v dxdt \\
&\quad + \varepsilon^2 \int_{\Omega_T} \nabla\phi^\varepsilon \cdot \nabla v (n_i^\varepsilon - n_e^\varepsilon) dxdt \\
&= \varepsilon^2 \int_{\Omega_T} [\widetilde{f}_i'(F_i^\varepsilon - \mu_i\phi^\varepsilon)\nabla\phi^\varepsilon \cdot \nabla F_i^\varepsilon \\
&\quad - \widetilde{f}_e'(F_e^\varepsilon + \mu_e\phi^\varepsilon)\nabla\phi^\varepsilon \cdot \nabla F_e^\varepsilon] v dxdt \\
&\quad - \varepsilon^2 \int_{\Omega_T} (\widetilde{f}_i'(F_i^\varepsilon - \mu_i\phi^\varepsilon) + \widetilde{f}_e'(F_e^\varepsilon + \mu_e\phi^\varepsilon))|\nabla\phi^\varepsilon|^2 v dxdt \\
&\quad + \varepsilon^2 \int_{\Omega_T} \nabla\phi^\varepsilon \cdot \nabla v (n_i^\varepsilon - n_e^\varepsilon) dxdt,
\end{aligned}$$

where we have used the expressions (3.6). In view of the identities

$$\widetilde{f}_i'(F_i^\varepsilon - \mu_i\phi^\varepsilon) = \widetilde{f}_i'(h_i(n_i^\varepsilon)) = \frac{1}{h_i'(n_i^\varepsilon)} = \frac{n_i^\varepsilon}{p_i'(n_i^\varepsilon)}, \quad \widetilde{f}_e'(F_e^\varepsilon + \mu_e\phi^\varepsilon) = \frac{n_e^\varepsilon}{p_e'(n_e^\varepsilon)}$$

and the uniform lower and upper bounds for n_α^ε ($\alpha = e, i$), there exist two constants $f_0,\ f_1 > 0$ such that

$$f_0 \leq \widetilde{f}_i'(F_i^\varepsilon - \mu_i\phi^\varepsilon) \leq f_1, \quad f_0 \leq \widetilde{f}_e'(F_e^\varepsilon + \mu_e\phi^\varepsilon) \leq f_1 \quad \text{in } \Omega_T.$$

Therefore, by Young's inequality,

$$\begin{aligned}
\|n_i^\varepsilon - n_e^\varepsilon\|^2_{L^2(\omega_T)} &\leq \frac{\varepsilon^2}{2} \int_{\Omega_T} (\widetilde{f}_i'(F_i^\varepsilon - \mu_i\phi^\varepsilon)|\nabla F_i^\varepsilon|^2 + \widetilde{f}_e'(F_e^\varepsilon + \mu_e\phi^\varepsilon)|\nabla F_e^\varepsilon|^2) v dxdt \\
&\quad - \frac{\varepsilon^2}{2} \int_{\Omega_T} (\widetilde{f}_i'(F_i^\varepsilon - \mu_i\phi^\varepsilon) + \widetilde{f}_e'(F_e^\varepsilon + \mu_e\phi^\varepsilon))|\nabla\phi^\varepsilon|^2 v dxdt \\
&\quad + \varepsilon^2 \|\nabla\phi^\varepsilon\|_{L^2(\sigma_T)} \|n_i^\varepsilon - n_e^\varepsilon\|_{L^2(\sigma_T)} \|\nabla v\|_{L^\infty(\Omega_T)}, \qquad (3.8)
\end{aligned}$$

where we have set $\sigma = \mathrm{supp}(v)$ and $\sigma_T = (0,T) \times \sigma^o$. Taking into account the uniform estimate for $\varepsilon\nabla\phi$ in $L^2(\Omega_T)$, this yields

$$\|n_i^\varepsilon - n_e^\varepsilon\|^2_{L^2(\omega_T)} + \varepsilon^2 f_0 \|\nabla\phi^\varepsilon\|^2_{L^2(\omega_T)} \le c(\varepsilon^2 f_1 \textstyle\int_{\Omega_T}(|\nabla F_i^\varepsilon|^2 + |\nabla F_e^\varepsilon|^2)dxdt + \varepsilon).$$

Hence we get the estimates

$$\|n_i^\varepsilon - n_e^\varepsilon\|_{L^2(\omega_T)} \le c\sqrt{\varepsilon}, \quad \sqrt{\varepsilon}\|\nabla\phi^\varepsilon\|_{L^2(\omega_T)} \le c. \tag{3.9}$$

These estimates are valid for any domain $\omega \subset \Omega$ with $\overline{\omega} \subset \Omega$. In particular, since $\overline{\sigma} \subset \Omega$, the inequalities (3.9) also hold for σ in the place of ω. Going back to (3.8), we obtain for all $\omega \subset \Omega$ with $\overline{\omega} \subset \Omega$,

$$\|n_i^\varepsilon - n_e^\varepsilon\|^2_{L^2(\omega_T)} + \varepsilon^2 f_0 \|\nabla\phi\|_{L^2(\omega_T)} \le c\varepsilon^2,$$

which improves (3.9).

Now we prove the strong compactness of the sequences $(n_\alpha^\varepsilon)_{\varepsilon>0}$ in L^2_{loc}. First we claim that $(n_\alpha^\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^2(0,T;H^1(\omega))$ $(\alpha = e, i)$. Indeed,

$$\begin{aligned}
\int_0^T \|\nabla n_i^\varepsilon\|_{L^2(\omega)}dt &= \int_0^T \|\nabla \widetilde{f}_i(F_i^\varepsilon - \mu_i\phi^\varepsilon)\|_{L^2(\omega)}dt \\
&= \int_0^T \|\widetilde{f}_i'(F_i^\varepsilon - \mu_i\phi^\varepsilon)\|_{L^\infty(\omega)} \|\nabla(F_i^\varepsilon - \mu_i\phi^\varepsilon)\|_{L^2(\omega)}dt \\
&\le \left\|\frac{n_i^\varepsilon}{p_i'(n_i^\varepsilon)}\right\|_{L^\infty(\Omega_T)} \Big(\|\nabla F_i^\varepsilon\|_{L^2(\omega_T)} + |\mu_i|\|\nabla\phi^\varepsilon\|_{L^2(\omega_T)}\Big) \\
&\le c.
\end{aligned}$$

A similar estimate holds for ∇n_e^ε. Furthermore, the sequences $(\partial_t n_\alpha^\varepsilon)_{\varepsilon>0}$ $(\alpha = e, i)$ are uniformly bounded in $L^2(0,T;H^{-1}(\omega))$ since

$$\begin{aligned}
\int_0^T \|\partial_t n_\alpha^\varepsilon\|_{H^{-1}(\omega)}dt &= \int_0^T \|\mathrm{div}\,(n_\alpha^\varepsilon \nabla F_\alpha^\varepsilon)\|_{H^{-1}(\omega)}dt \\
&\le \int_0^T \|n_\alpha^\varepsilon \nabla F_\alpha^\varepsilon\|_{L^2(\omega)}dt \\
&\le c.
\end{aligned}$$

From Aubin's lemma [3, 21] we conclude the existence of a subsequence (not relabeled) such that, as $\varepsilon \to 0$,

$$n_i^\varepsilon \to n, \quad n_e^\varepsilon \to n \quad \text{in } L^2(\omega_T) \text{ strongly}.$$

The strong convergence in any $L^p(\omega_T)$ $(1 \le p < \infty)$ follows from the uniform L^∞ bound for n_i^ε and n_e^ε.

Thanks to the above uniform estimates it can be easily seen that (n, ϕ) solves the equations (1.9)–(1.10) locally. From the uniform bounds of n_α^ε in $H^1(0,T;H^{-1}(\omega))$

and $L^\infty(0,T;L^\infty(\omega))$ we get, applying Aubin's lemma again, that up to a subsequence,

$$n_\alpha^\varepsilon \to n \quad \text{in } C^0([0,T];H^{-1}(\omega)) \text{ strongly.}$$

Hence the initial condition for n is satisfied in the sense of $H^{-1}(\omega)$.

Now let the compatibility condition (A5) hold. Then we choose $v=1$ in Ω_T in the relation (3.7). Performing integration by parts in (3.7), the boundary integral vanishes due to condition (A5). Thus the above convergence results also hold for $\omega=\Omega$. In view of the uniform estimates the corresponding boundary conditions for the limit problem (1.9)–(1.10) are satisfied. In this case, the convergence of the whole sequence $(n_e^\varepsilon, n_i^\varepsilon, \phi^\varepsilon)$ follows from the unique solvability of the limit equations. This proves the theorem.

Remark 3.5 If the compatibility condition (A5) is *not* satisfied, we cannot expect that, for instance, $(\nabla\phi^\varepsilon)_{\varepsilon>0}$ is bounded in $L^2(\Omega_T)$. Indeed, if this would be the case then also the sequences $\nabla n_\alpha^\varepsilon$ are uniformly bounded in $L^2(\Omega_T)$, thanks to the relations $n_i^\varepsilon = \widetilde{f}_i(F_i^\varepsilon - \mu_i\phi^\varepsilon)$ and $n_e^\varepsilon = \widetilde{f}_e(F_e^\varepsilon + \mu_e\phi^\varepsilon)$, and we obtain, up to subsequences,

$$n_\alpha^\varepsilon \rightharpoonup n \quad \text{in } L^2(0,T;H^1(\Omega)) \text{ weakly, } \alpha = e,i.$$

Therefore (see, e.g., [25, p. 88, 1.27])

$$n_\alpha^\varepsilon|_{\Gamma_D} \rightharpoonup n|_{\Gamma_D} \quad \text{in } L^2(0,T;H^{1/2}(\Gamma_D)) \text{ weakly, } \alpha = e,i.$$

Since $n_\alpha^\varepsilon|_{\Gamma_D} = n_{D,\alpha}$ do not depend on ε, we get $n|_{\Gamma_D} = n_{D,e} = n_{D,i}$, contradiction.

However, we can replace (A5) by the weaker assumption

$$\|n_{D,e}^\varepsilon - n_{D,i}^\varepsilon\|_{L^2(0,T;H^{1/2}(\Gamma_D))} = \mathcal{O}(\varepsilon) \quad \text{as } \varepsilon \to 0.$$

We can prove the quasi-neutral limit in the problem (DD-EI) without assuming the positivity of the densities (A2) under a stronger assumption on the pressure functions:

(A2)' $p \stackrel{\text{def}}{=} p_e = p_i$, $Q(0)=0$, and Q^{-1} is Hölder continuous,

where Q is defined as $Q(s) = \int_0^s \sqrt{y}h'(y)dy$, $s \geq 0$, and $h \stackrel{\text{def}}{=} h_e = h_i$. For instance, the function $p(s) = s^\gamma$ with $\gamma \geq 1$ satisfies the assumption (A2)'.

Theorem 3.6 *Let (A1), (A2)' and (A3)–(A5) hold. Then, as* $\varepsilon \to 0$,

$$n_i^\varepsilon \to n, \quad n_e^\varepsilon \to n \qquad \text{in } L^2(\Omega_T) \text{ strongly,}$$

the limit n *solves the nonlinear heat equation*

$$\partial_t n - \Delta p(n) = 0 \quad \text{in } \Omega_T,$$

and the initial and boundary conditions (1.4)–(1.6) are satisfied for n.

Proof. We first notice that under the assumptions of Theorem 3.6, Lemmas 3.1 and 3.2 hold and the densities n_α^ε are uniformly bounded in $L^\infty(\Omega_T)$ independently of the validity of (A2). However, we only get the non-negativity of n_α^ε in Ω_T, i.e., $n_\alpha^\varepsilon \geq 0$. Omitting the index ε, we get from (3.4)

$$\begin{aligned} H_\varepsilon(t) &\leq H_\varepsilon(0) - \sum_{\alpha=e}^{i} \int_{\Omega_t} \mu_\alpha^{-1} n_\alpha \nabla F_\alpha \cdot \nabla(F_\alpha - F_{D,\alpha}) dx d\tau \\ &\quad + \frac{\varepsilon^2}{2} \int_{\Omega_t} |\nabla(\phi - \phi_D)|^2 dx d\tau + c, \end{aligned}$$

where $c > 0$ depends on the L^∞ bounds for n_α and on the data. Using the definitions of F_α and $F_{D,\alpha}$ and employing Young's inequality, we have

$$\begin{aligned} &-\sum_{\alpha=e}^{i} \int_{\Omega_t} \mu_\alpha^{-1} n_\alpha \nabla F_\alpha \cdot \nabla(F_\alpha - F_{D,\alpha}) dx d\tau \\ &\leq -\int_{\Omega_t} \frac{1}{4}\Big(\sum_{\alpha=e}^{i} \mu_\alpha^{-1} n_\alpha |\nabla h_\alpha(n_\alpha)|^2 + (\mu_i n_i + \mu_e n_e)|\nabla\phi|^2 \Big) dx d\tau \\ &\quad - \int_{\Omega_t} \nabla(p_i(n_i) - p_e(n_e)) \cdot \nabla(\phi - \phi_D) dx d\tau \\ &\quad + \int_{\Omega_t} \Big(2\sum_{\alpha=e}^{i} \mu_\alpha^{-1} n_\alpha |\nabla h_\alpha(n_{D,\alpha})|^2 + \frac{1}{2}(\mu_i n_i + \mu_e n_e)|\nabla\phi_D|^2 \Big) dx d\tau. \end{aligned}$$

The last integral is uniformly bounded. For the second integral we use the assumptions (A5) and (A2)' and the monotonicity of p to obtain

$$\begin{aligned} &-\int_{\Omega_t} \nabla(p_i(n_i) - p_e(n_e)) \cdot \nabla(\phi - \phi_D) dx d\tau \\ &= -\varepsilon^{-2} \int_{\Omega_t} (p(n_i) - p(n_e))(n_i - n_e) dx d\tau \\ &\leq 0. \end{aligned}$$

Therefore

$$\begin{aligned} H_\varepsilon(t) + c \int_{\Omega_t} (n_e|\nabla h_e(n_e)|^2 + n_i|\nabla h_i(n_i)|^2 + (n_e + n_i)|\nabla\phi|^2) dx d\tau \\ \leq c + \frac{\varepsilon^2}{2} \int_{\Omega_t} |\nabla(\phi - \phi_D)|^2 dx d\tau \end{aligned} \tag{3.10}$$

and, by Gronwall's lemma, the left-hand side of this inequality is uniformly bounded. In particular, we get

$$\|Q(n_e)\|_{L^2(0,T;H^1(\Omega))} + \|Q(n_i)\|_{L^2(0,T;H^1(\Omega))} \leq c.$$

By assumption (A2)', the function Q^{-1} is Hölder continuous with, say, exponent θ. Thus we can apply a result due to Chavent (see [11]) which gives, for all $0 < s < 1$,

$$\int_0^T \|n_\alpha\|^{2/\theta}_{W^{\theta s,2/\theta}(\Omega)} dt \le c \int_0^T \|Q(n_\alpha)\|^2_{H^s(\Omega)} dt \le c \int_0^T \|Q(n_\alpha)\|^2_{H^1(\Omega)} dt \le c,$$

where $\alpha = e, i$. The above estimates show that $(n^\varepsilon_\alpha)_{\varepsilon>0}$ is bounded in $H^1(0,T;V^*)$ and $L^{2/\theta}(0,T;W^{\theta s,2/\theta}(\Omega))$. Since $W^{\theta s,2/\theta}(\Omega)$ is compactly embedded into $L^2(\Omega)$, we obtain from Aubin's lemma [3, 21] the existence of a subsequence (not relabeled) such that, as $\varepsilon \to 0$,

$$n^\varepsilon_\alpha \to n \quad \text{in } L^2(\Omega_T) \text{ strongly.}$$

Moreover, we get weak convergence of $p_\alpha(n^\varepsilon_\alpha)$ to $p_\alpha(n_\alpha)$ in $L^2(0,T;H^1(\Omega))$ and strong convergence of (a subsequence of) n^ε_α to n_α in $C^0([0,T];V^*)$ which shows that n_α satisfies the initial and boundary conditions (1.4)–(1.6) in the weak sense.

We prove now that the limit n solves the nonlinear heat equation. Adding the Eqs. (1.1) and (1.2) (divided by μ_α) we have

$$\partial_t(\mu_i^{-1} n^\varepsilon_i + \mu_e^{-1} n^\varepsilon_e) - \operatorname{div}(\mu_i^{-1}\nabla p(n^\varepsilon_i) + \mu_e^{-1}\nabla p(n^\varepsilon_e)) = \operatorname{div}((n^\varepsilon_e - n^\varepsilon_i)\nabla\phi^\varepsilon).$$

The left-hand side of this equation converges (in the sense of distributions, for instance) to

$$2(\mu_i^{-1} + \mu_e^{-1})(\partial_t n - \Delta p(n)).$$

It remains to show that the right-hand side converges to zero. To this end, we use the estimate (3.10) and Hölder's inequality:

$$\begin{aligned}
&\int_{\Omega_T} |(n^\varepsilon_i - n^\varepsilon_e)\nabla\phi^\varepsilon| dxdt \\
&\le \left(\int_{\Omega_T} \frac{(n^\varepsilon_i - n^\varepsilon_e)^2}{n^\varepsilon_i + n^\varepsilon_e} dxdt\right)^{1/2} \left(\int_{\Omega_T} (n^\varepsilon_i + n^\varepsilon_e)|\nabla\phi^\varepsilon|^2 dxdt\right)^{1/2} \\
&\le c\Big(\int_{\Omega_T} |n^\varepsilon_i - n^\varepsilon_e| dxdt\Big)^{1/2},
\end{aligned}$$

since the function $(x,y) \mapsto (x-y)/(x+y)$ is bounded in $(0,+\infty)^2$. Hence $\operatorname{div}((n^\varepsilon_e - n^\varepsilon_i)\nabla\phi^\varepsilon) \to 0$ in the sense of distributions. This shows the desired result.

4 The quasi-neutral limit in (DD-I)

We suppose that (A1)–(A3), (A6) hold. Again, we first show some auxiliary results for the system (DD-I).

Lemma 4.1 *There exists a solution $(n^\varepsilon_i, \phi^\varepsilon)$ such that*

$$k \le \phi^\varepsilon \le K, \quad 0 < m \le n^\varepsilon_i \le M \quad \text{in } \Omega_T, \tag{4.1}$$

where

$$m = \min(\underline{n}, f_e(0)), \quad M = \max(\overline{n}, f_e(\|\phi_D\|_{L^\infty(\Sigma_D)})),$$

and k, $K > 0$ are such that $f_e(k) = m$ and $f_e(K) = M$.

Notice that $m > 0$ in view of hypothesis (A6).

Proof. In the following we omit the index $\varepsilon > 0$. Since the electric potential is only defined up to an additive constant, we can choose ϕ_D such that $\phi_D \geq 0$ on Σ_D. Define $s^{(M,m)} = \max(m, \min(s, M))$ for $s \in \mathbb{R}$ and let $\delta > 0$. In [20] the existence of a solution (n_i, ϕ) to the truncated problem

$$\begin{aligned} \partial_t n_i - \operatorname{div}\left(\nabla(p_i(n_i^{(M,m)}) + \delta n_i) + \mu_i n_i^{(M,m)} \nabla\phi\right) &= 0, && (4.2) \\ -\varepsilon^2 \Delta\phi &= n_i^{(M,m)} - f_e(\phi), && (4.3) \end{aligned}$$

subject to the initial and boundary conditions (1.4)–(1.6) for $\alpha = i$ has been proved. We show that the above lower and upper bounds hold for the approximate solution to (4.2)–(4.3) and pass then to the limit $\delta \to 0$. In [20] it is shown that the limit $\delta \to 0$ can be performed in $L^2(\Omega_T)$ strongly. Therefore, the lower and upper bounds hold also for the limit solution which solves the problem (1.7)–(1.8), (1.4)–(1.6) for $\alpha = i$.

In order to prove the upper bound for ϕ, we use $(\phi - K)^+ = \max(0, \phi - K)$ as test function in (4.3). This is possible since the inequalities $f_e(\phi) \leq f_e(\|\phi_D\|_{L^\infty(\Sigma_D)}) \leq M = f_e(K)$ on Σ_D and the strict monotonicity of f_e on $(0, +\infty)$ imply that $(\phi - K)^+$ vanishes on Σ_D. We get

$$\begin{aligned} \varepsilon^2 \int_{\Omega_T} |\nabla(\phi - K)^+|^2 dx d\tau &= \int_{\Omega_T} (n_i^{(M,m)} - f_e(\phi))(\phi - K)^+ dx d\tau \\ &\leq \int_{\Omega_T} (M - f_e(K)) dx d\tau \\ &= 0, \end{aligned}$$

and hence $\phi \leq K$ in Ω_T. Since $f_e(\phi) \geq f_e(0) \geq m = f_e(k)$ on Σ_D, we get $(\phi - k)^- = \min(0, \phi - k) \in L^2(0, T; V)$. Thus, using $(\phi - k)^-$ as test function in (4.3) gives

$$\begin{aligned} \varepsilon^2 \int_{\Omega_T} |\nabla(\phi - k)^-|^2 dx d\tau &= \int_{\Omega_T} (n_i^{(M,m)} - f_e(\phi))(\phi - k)^- dx d\tau \\ &\leq \int_{\Omega_T} (m - f_e(k))(\phi - k)^- dx d\tau \\ &= 0. \end{aligned}$$

Thus $\phi \geq k$ in Ω_T.

In view of the definition of M we have $(n_i - M)^+ \in L^2(0, T; V)$. Using this function as a test function in (4.2), observing that $n_i^{(M,m)} = M$ in $\{n_i > M\}$ and employing (4.3) we obtain

$$\begin{aligned} &\frac{1}{2} \int_\Omega (n_i - M)^+(t)^2 dx + \delta \int_{\Omega_t} |\nabla(n_i - M)^+|^2 dx d\tau \\ &\leq \; -\mu_i \int_{\Omega_t} n_i^{(M,m)} \nabla\phi \cdot \nabla(n_i - M)^+ dx d\tau \end{aligned}$$

$$
\begin{aligned}
&= -\frac{\mu_i M}{\varepsilon^2}\int_{\Omega_t}(n_i^{(M,m)} - f_e(\phi))(n_i - M)^+ dx d\tau \\
&\le -\frac{\mu_i M}{\varepsilon^2}\int_{\Omega_t}(M - f_e(K))(n_i - M)^+ dx d\tau \\
&= 0,
\end{aligned}
$$

which implies that $n_i \le M$ in Ω_T. Finally we employ the test function $(n_i - m)^- \in L^2(0,T;V)$ in (4.2):

$$
\begin{aligned}
\frac{1}{2}\int_\Omega (n_i - m)^-(t)^2 dx &\le -\frac{\mu_i m}{\varepsilon^2}\int_{\Omega_t}(n_i^{(M,m)} - f_e(\phi))(n_i - m)^- dx d\tau \\
&\le -\frac{\mu_i m}{\varepsilon^2}\int_{\Omega_t}(m - f_e(k))(n_i - m)^- dx d\tau \\
&= 0.
\end{aligned}
$$

Hence $n_i \ge m$ in Ω_T. The lemma is proved.

We fix now the solution $(n_i^\varepsilon, \phi^\varepsilon)$ to (1.7)–(1.8) of Lemma 4.1 with the properties (4.1).

Lemma 4.2 *The function ϕ^ε lies in the space $H^1(0,T;H^1(\Omega))$ and satisfies the estimate*

$$
\varepsilon^2\|\partial_t\phi^\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le c_1\|\partial_t\phi_D\|_{L^2(0,T;H^1(\Omega))} + \|\partial_t n_i^\varepsilon\|_{L^2(0,T;V^*)}, \tag{4.4}
$$

where the constant $c_1 > 0$ only depends on f_e, k, and K.

Proof. Set, for $h > 0$ and almost all $0 < t < T - h$, omitting the index $\varepsilon > 0$,

$$
n_i^h(t) = \frac{1}{h}(n_i(t+h) - n_i(t)) \quad \text{and} \quad g^h(t) = \frac{f_e(\phi(t+h)) - f_e(\phi(t))}{\phi(t+h) - \phi(t)}
$$

if $\phi(t+h,x) \ne \phi(t,x)$ and $g^h(t,x) = 0$ else, and

$$
\phi^h(t) = \frac{1}{h}(\phi(t+h) - \phi(t)), \quad \phi_D^h(t) = \frac{1}{h}(\phi_D(t+h) - \phi_D(t)).
$$

Then ϕ^h solves

$$
-\varepsilon^2\Delta\phi^h + g^h\phi^h = n_i^h \quad \text{in } \Omega_T.
$$

Employing $\phi^h - \phi_D^h \in L^2(0,T;V)$ as test function yields

$$
\begin{aligned}
&\varepsilon^2\int_{\Omega_T}|\nabla(\phi^h - \phi_D^h)|^2 dx dt + \int_{\Omega_T} g^h(\phi^h - \phi_D^h)^2 dx dt \\
&\quad = -\varepsilon^2\int_{\Omega_T}\nabla\phi_D^h\cdot\nabla(\phi^h - \phi_D^h) dx dt - \int_{\Omega_T} g^h\phi_D^h(\phi^h - \phi_D^h) dx dt \\
&\qquad + \int_{\Omega_T} n_i^h(\phi^h - \phi_D^h) dx dt.
\end{aligned}
$$

Observing that $g^h \geq 0$ in Ω_T and that g^h is bounded in $L^\infty(\Omega_T)$ uniformly in ε, thanks to the L^∞ bounds for ϕ, we obtain

$$\begin{aligned}\varepsilon^2 \|\nabla(\phi^h - \phi_D^h)\|^2_{L^2(\Omega_T)} &\leq (\varepsilon^2 \|\nabla \phi_D^h\|_{L^2(\Omega_T)} + c\|g^h\|_{L^\infty(\Omega_T)} \|\phi_D^h\|_{L^2(\Omega_T)} \\ &\quad + \|n_i^h\|_{L^2(0,T;V^*)}) \|\phi^h - \phi_D^h\|_{L^2(0,T,V)}\end{aligned}$$

and

$$\begin{aligned}\varepsilon^2 \|\nabla(\phi^h - \phi_D^h)\|_{L^2(\Omega_T)} &\leq \varepsilon^2 \|\nabla \phi_D^h\|_{L^2(\Omega_T)} + c\|g^h\|_{L^\infty(\Omega_T)} \|\phi_D^h\|_{L^2(\Omega_T)} \\ &\quad + \|n_i^h\|_{L^2(0,T;V^*)}.\end{aligned}$$

Performing the limit $h \to 0$ we see that $\partial_t \phi \in L^2(0,T;H^1(\Omega))$, by Poincaré's inequality, and that the estimate (4.4) holds. The lemma is proved.

The entropy for the system (DD-I) can be obtained by setting formally $n_e^\varepsilon = f_e(\phi^\varepsilon)$ in the entropy for the system (DD-EI):

$$\begin{aligned}H_\varepsilon(t) &= \int_\Omega \Big(\frac{1}{\mu_i} \int_{n_{D,i}(t)}^{n_i^\varepsilon(t)} (h_i(s) - h_i(n_{D,i}(t)))ds \\ &\quad + \int_{f_e(\phi_D(t))}^{f_e(\phi^\varepsilon(t))} (h_e(s) - h_e(f_e(\phi_D(t))))ds \Big) dx + \frac{\varepsilon^2}{2} \int_\Omega |\nabla(\phi^\varepsilon - \phi_D)(t)|^2 dx.\end{aligned}$$

The entropy is well defined for $t = 0$ and for almost all $t > 0$. Moreover, $H_\varepsilon(t)$ is non-negative and $H_\varepsilon(0)$ is bounded uniformly in ε, since $\phi^\varepsilon(0)$ solves

$$\begin{aligned}-\varepsilon^2 \Delta \phi^\varepsilon(0) &= n_{I,i} - f_e(\phi^\varepsilon(0)) \quad \text{in } \Omega, \\ \phi^\varepsilon(0) &= \phi_D(0) \quad \text{on } \Gamma_D, \quad \nabla \phi^\varepsilon(0) \cdot \nu = 0 \quad \text{on } \Gamma_N.\end{aligned}$$

It follows that $(\phi^\varepsilon(0))_{\varepsilon>0}$ is bounded in $L^\infty(\Omega)$. Multiplying the equation by $\phi^\varepsilon(0) - \phi_D(0)$ and integrating over Ω we obtain

$$\varepsilon^2 \int_\Omega \nabla \phi^\varepsilon(0) \cdot \nabla(\phi^\varepsilon(0) - \phi_D(0))dx = \int_\Omega (n_{I,i} - f_e(\phi^\varepsilon(0)))(\phi^\varepsilon(0) - \phi_D(0))dx.$$

The right-hand side of the above relation is bounded uniformly in ε. Therefore, by Young's inequality, the sequence $(\varepsilon \nabla \phi^\varepsilon(0))_{\varepsilon>0}$ is bounded in $L^2(\Omega)$, which implies the uniform bound for $H_\varepsilon(0)$.

Lemma 4.3 *For almost all $t > 0$ it holds*

$$H_\varepsilon(t) + \frac{1}{2\mu_i} \int_{\Omega_t} n_i^\varepsilon |\nabla F_i^\varepsilon|^2 dx d\tau \leq c. \tag{4.5}$$

In particular, we have the uniform bounds

$$\|F_i^\varepsilon\|_{L^2(0,T;H^1(\Omega))} \leq c, \quad \varepsilon \|\nabla \phi^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq c, \tag{4.6}$$

where F_i^ε is defined by (3.1) and $c > 0$ does not depend on ε.

Proof. It holds (omitting the index ε again)

$$
\begin{aligned}
H_\varepsilon(t) &= H_\varepsilon(0) + \frac{1}{\mu_i}\int_0^t \langle \partial_t n_i, h_i(n_i) - h_i(n_{D,i})\rangle d\tau \\
&\quad -\frac{1}{\mu_i}\int_{\Omega_t} h_i'(n_{D,i})\partial_t n_{D,i}(n_i - n_{D,i})dxd\tau + \int_{\Omega_t} f_e'(\phi)\partial_t\phi(\phi-\phi_D)dxd\tau \\
&\quad -\int_{\Omega_t}\partial_t\phi_D(f_e(\phi) - f_e(\phi_D))dxd\tau + \varepsilon^2\int_{\Omega_t}\nabla\partial_t(\phi-\phi_D)\cdot\nabla(\phi-\phi_D)dxd\tau.
\end{aligned}
$$

From the Poisson equation

$$
-\varepsilon^2\Delta\partial_t\phi = \partial_t n_i - f_e'(\phi)\partial_t\phi,
$$

we get

$$
\begin{aligned}
\varepsilon^2\int_{\Omega_t}\nabla\partial_t\phi\cdot\nabla(\phi-\phi_D)dxd\tau &= \int_0^t \langle\partial_t n_i, \phi-\phi_D\rangle_{V^*,V}d\tau \\
&\quad -\int_{\Omega_t} f_e'(\phi)\partial_t\phi(\phi-\phi_D)dxd\tau.
\end{aligned}
$$

Therefore we have from Eq. (1.7)

$$
\begin{aligned}
H_\varepsilon(t) &= H_\varepsilon(0) - \frac{1}{\mu_i}\int_{\Omega_t} n_i\nabla F_i\cdot\nabla(F_i - F_{D,i})dxd\tau \\
&\quad -\frac{1}{\mu_i}\int_{\Omega_t} h_i'(n_{D,i})\partial_t n_{D,i}(n_i-n_{D,i})dxd\tau - \int_{\Omega_t}\partial_t\phi_D(f_e(\phi)-f_e(\phi_D))dxd\tau \\
&\quad -\varepsilon^2\int_{\Omega_t}\nabla\partial_t\phi_D\cdot\nabla(\phi-\phi_D)dxd\tau \\
&\le H_\varepsilon(0) - \frac{1}{2\mu_i}\int_{\Omega_t} n_i|\nabla F_i|^2dxd\tau + \frac{1}{2\mu_i}\int_{\Omega_t} n_i|\nabla F_{D,i}|^2dxd\tau \\
&\quad +\frac{1}{\mu_i}\|h_i'(n_{D,i})\|_{L^\infty(\Omega_T)}\|\partial_t n_{D,i}\|_{L^2(\Omega_T)}\|n_i-n_{D,i}\|_{L^2(\Omega_T)} \\
&\quad +\|\partial_t\phi_D\|_{L^2(\Omega_T)}\|f_e(\phi)-f_e(\phi_D)\|_{L^2(\Omega_T)} \\
&\quad +\frac{\varepsilon^2}{2}\int_{\Omega_t}|\nabla(\phi-\phi_D)|^2dxd\tau + \frac{\varepsilon^2}{2}\int_{\Omega_t}|\nabla\partial_t\phi_D|^2dxd\tau \\
&\le c - \frac{1}{2\mu_i}\int_{\Omega_t} n_i|\nabla F_i|^2dxd\tau + \frac{\varepsilon^2}{2}\int_{\Omega_t}|\nabla(\phi-\phi_D)|^2dxd\tau. \qquad (4.7)
\end{aligned}
$$

In particular we have

$$
\frac{\varepsilon^2}{2}\int_\Omega|\nabla(\phi-\phi_D)(t)|^2dx \le c + \frac{\varepsilon^2}{2}\int_{\Omega_t}|\nabla(\phi-\phi_D)|^2dxd\tau,
$$

which yields, by the Gronwall inequality,

$$
\frac{\varepsilon^2}{2}\int|\nabla(\phi-\phi_D)(t)|^2dx \le ce^T.
$$

Hence we can bound the last term in (4.7) uniformly in ε, and we conclude the estimate (4.5). Finally, the estimates (4.6) follow from the uniform lower bound for n^ε and the definition of $H_\varepsilon(t)$ which ends the proof of the lemma.

Lemma 4.4 *For almost all $t \in (0,T)$ it holds*

$$\varepsilon \int_{\Gamma_D} |\nabla \phi^\varepsilon(t) \cdot \nu| d\gamma = \mathcal{O}(1) \quad \textit{as } \varepsilon \to 0.$$

Proof. Let $\phi_1 \in L^\infty(\Omega_T) \cap L^\infty(0,T;H^2(\Omega))$ be such that

$$\Delta\phi_1 \in L^\infty(\Omega_T), \quad \phi_1 = \phi_D - 1 \text{ on } \Sigma_D, \quad \nabla\phi_1 \cdot \nu = 0 \text{ on } \Sigma_N.$$

This function exists in view of hypothesis (A8). Then $\phi_2^\varepsilon = \phi^\varepsilon - \phi_1$ satisfies

$$\begin{aligned} L_\varepsilon \phi_2^\varepsilon &\stackrel{\text{def}}{=} -\varepsilon^2 \Delta\phi_2^\varepsilon + \phi_2^\varepsilon = f_1^\varepsilon \text{ in } \Omega_T, \\ \phi_2^\varepsilon &= 1 \text{ on } \Sigma_D, \quad \nabla\phi_2^\varepsilon \cdot \nu = 0 \text{ on } \Sigma_N, \end{aligned}$$

where $f_1^\varepsilon \stackrel{\text{def}}{=} n_i^\varepsilon + \phi^\varepsilon - f_e(\phi^\varepsilon) - L_\varepsilon\phi_1 = L_\varepsilon(\phi^\varepsilon - \phi_1)$. By Lemma 4.1, $(f_1^\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty(\Omega_T)$, i.e. there exist two constants $M_+ \geq M_-$ such that

$$M_- \leq f_1^\varepsilon \leq M_+ \quad \text{in } \Omega_T.$$

Now consider the following two problems:

$$L_\varepsilon \phi_\pm^\varepsilon = M_\pm \text{ in } \Omega, \quad \phi_\pm^\varepsilon = 1 \text{ on } \Gamma_D, \quad \nabla\phi_\pm^\varepsilon \cdot \nu = 0 \text{ on } \Gamma_N.$$

From the comparison principle we obtain

$$\phi_-^\varepsilon \leq \phi_2^\varepsilon(t) \leq \phi_+^\varepsilon.$$

The functions ϕ_+^ε, ϕ_-^ε, and $\phi_2^\varepsilon(t)$ have the same trace on Γ_D, thus if

$$\varepsilon \int_{\Gamma_D} |\nabla\phi_\pm^\varepsilon \cdot \nu| d\gamma = \mathcal{O}(1), \tag{4.8}$$

then

$$\varepsilon \int_{\Gamma_D} |\nabla\phi_2^\varepsilon(t) \cdot \nu| d\gamma = \mathcal{O}(1),$$

and hence, by the definition of ϕ_2^ε,

$$\varepsilon \int_{\Gamma_D} |\nabla\phi^\varepsilon(t) \cdot \nu| d\gamma = \mathcal{O}(1).$$

From Proposition 3 in [4], we know that the solution ϕ_0^ε of

$$L_\varepsilon \phi_0^\varepsilon = 0 \text{ in } \Omega, \quad \phi_0^\varepsilon = 1 \text{ on } \Gamma_D, \quad \nabla\phi_0^\varepsilon \cdot \nu = 0 \text{ on } \Gamma_N$$

satisfies

$$\varepsilon \int_{\Gamma_D} |\nabla \phi_0^\varepsilon \cdot \nu| d\gamma = \mathcal{O}(1).$$

Therefore, the relation (4.8) follows from

$$\phi_\pm^\varepsilon = M_\pm + (1 - M_\pm)\phi_0^\varepsilon.$$

This finishes the proof.

Proposition 4.5 *There exists a constant $c > 0$, independent of ε, such that*

$$\begin{aligned}
&\|n_i^\varepsilon - f_e(\phi^\varepsilon)\|_{L^2(\Omega_T)} \le c\varepsilon^{1/2}, \quad \|\nabla \phi^\varepsilon\|_{L^2(\Omega_T)} \le c\varepsilon^{-1/2},\\
&\|n_i^\varepsilon - f_e(\phi^\varepsilon)\|_{L^2(0,T;L^2_{\mathrm{loc}}(\Omega))} \le c\varepsilon, \quad \|\nabla \phi^\varepsilon\|_{L^2(0,T;L^2_{\mathrm{loc}}(\Omega))} \le c.
\end{aligned}$$

Proof. It follows from the definition $n_i^\varepsilon = \widetilde{f}_i(F_i^\varepsilon - \mu_i \phi^\varepsilon)$ where $\widetilde{f}_i$ is the inverse of h_i, and the Poisson equation that

$$\begin{aligned}
\|n_i^\varepsilon - f_e(\phi^\varepsilon)\|^2_{L^2(\Omega_T)} &= -\varepsilon^2 \int_{\Omega_T} \Delta\phi^\varepsilon (n_i^\varepsilon - f_e(\phi^\varepsilon)) dxdt\\
&= -\varepsilon^2 \int_0^T \int_{\Gamma_D} \nabla \phi^\varepsilon \cdot \nu (n_{D,i} - f_e(\phi_D)) d\gamma dt\\
&\quad + \varepsilon^2 \int_{\Omega_T} \nabla\phi^\varepsilon \cdot \nabla(\widetilde{f}_i(F_i^\varepsilon - \mu_i\phi^\varepsilon) - f_e(\phi^\varepsilon)) dxdt.
\end{aligned}$$

The last term on the right-hand side can be estimated as follows:

$$\begin{aligned}
&\int_{\Omega_T} \nabla\phi^\varepsilon \cdot \nabla(f_i(F_i^\varepsilon - \mu_i\phi^\varepsilon) - f_e(\phi^\varepsilon)) dxdt\\
&= \int_{\Omega_T} [\widetilde{f}_i'(F_i^\varepsilon - \mu_i\phi^\varepsilon)\nabla\phi^\varepsilon \cdot (\nabla F_i^\varepsilon - \mu_i \nabla\phi^\varepsilon) - f_e'(\phi^\varepsilon)|\nabla\phi^\varepsilon|^2] dxdt \qquad (4.9)\\
&= \int_{\Omega_T} \widetilde{f}_i'(h_i(n_i^\varepsilon))\nabla\phi^\varepsilon \cdot \nabla F_i^\varepsilon dxdt - \int_{\Omega_T} (\mu_i \widetilde{f}_i'(h_i(n_i^\varepsilon)) + f_e'(\phi^\varepsilon))|\nabla\phi^\varepsilon|^2 dxdt\\
&\le \frac{1}{2\mu_i}\int_{\Omega_T} \frac{n_i^\varepsilon}{p_i'(n_i^\varepsilon)}|\nabla F_i^\varepsilon|^2 dxdt - \min(1, \mu_i/2)\int_{\Omega_T} \Big(\frac{n_i^\varepsilon}{p_i'(n_i^\varepsilon)} + f_e'(\phi^\varepsilon)\Big)|\nabla\phi^\varepsilon|^2 dxdt.
\end{aligned}$$

By Lemma 4.1, there exist two constants c_2, $c_3 > 0$ such that

$$\begin{aligned}
&\|n_i^\varepsilon - f_e(\phi^\varepsilon)\|^2_{L^2(\Omega_T)} + c_2\varepsilon^2\|\nabla\phi^\varepsilon\|^2_{L^2(\Omega_T)}\\
&\quad\le c_3\varepsilon^2\|\nabla F_i^\varepsilon\|^2_{L^2(\Omega_T)} - \varepsilon^2\int_0^T\int_{\Gamma_D} \nabla\phi^\varepsilon \cdot \nu(n_{D,i} - f_e(\phi_D)) d\gamma dt. \qquad (4.10)
\end{aligned}$$

Thus, the first two estimates of the proposition follow from Lemmas 4.3 and 4.4. The remaining two estimates can be proved similar as in the proof of Theorem 2.1.

Proof of Theorem 2.2. First we prove that there is a subsequence of $(n_i^\varepsilon, \phi^\varepsilon)$ (not relabeled) such that

$$n_i^\varepsilon \to n,\ \phi^\varepsilon \to \phi \quad \text{in } L^2(\Omega_T) \text{ strongly as } \varepsilon \to 0,$$

where (n, ϕ) satisfies Eqs. (1.12)–(1.13).

We define the two sequences

$$U_\varepsilon = (n_i^\varepsilon, n_i^\varepsilon \nabla F_i^\varepsilon), \quad V_\varepsilon = (F_i^\varepsilon, 0, \ldots, 0).$$

From the L^∞ bound for $(n_i^\varepsilon)_{\varepsilon>0}$ and the $L^2(0,T;H^1)$ bound for $(F_i^\varepsilon)_{\varepsilon>0}$, we deduce that

$$\begin{gathered}(U^\varepsilon)_{\varepsilon>0},\ (V^\varepsilon)_{\varepsilon>0} \text{ are bounded in } (L^2(\Omega_T))^{d+1}, \text{ and} \\ \operatorname{div}_{t,x} U^\varepsilon = 0, \quad (\operatorname{rot}_{t,x} V^\varepsilon)_{\varepsilon>0} \text{ is bounded in } (L^2(\Omega_T))^{(d+1)^2}.\end{gathered}$$

Therefore, the div-curl lemma [24] can be applied to $(U^\varepsilon, V^\varepsilon)$, and we have in the sense of distributions (after passing, if necessary, to a subsequence):

$$\lim_{\varepsilon\to0} n_i^\varepsilon F_i^\varepsilon = \lim_{\varepsilon\to0} (U^\varepsilon \cdot V^\varepsilon) = \lim_{\varepsilon\to0} n_i^\varepsilon \cdot \lim_{\varepsilon\to0} F_i^\varepsilon,$$

or, equivalently,

$$\lim_{\varepsilon\to0} n_i^\varepsilon (h_i(n_i^\varepsilon) + \mu_i \phi^\varepsilon) = n \lim_{\varepsilon\to0} (h_i(n_i^\varepsilon) + \mu_i \phi^\varepsilon) \quad \text{in } \mathcal{D}'(\Omega_T), \tag{4.11}$$

where

$$n = \lim_{\varepsilon\to0} n_i^\varepsilon \quad \text{in } L^\infty(\Omega_T) \text{ weakly} - * .$$

Now, let us introduce $h_e(s) = \mu_e f_e^{-1}(s)$, $s > 0$, and

$$\alpha^\varepsilon \stackrel{\text{def}}{=} n_i^\varepsilon - f_e(\phi^\varepsilon), \quad \delta^\varepsilon \stackrel{\text{def}}{=} \mu_i(\phi^\varepsilon - f_e^{-1}(n_i^\varepsilon)) = \mu_i \phi^\varepsilon - h_e(n_i^\varepsilon).$$

Then from Proposition 4.5,

$$\lim_{\varepsilon\to0} \alpha^\varepsilon = 0 \quad \text{in } L^2(\Omega_T) \text{ strongly.}$$

We deduce that, since (n_i^ε) and (ϕ^ε) are bounded in $L^\infty(\Omega_T)$,

$$\lim_{\varepsilon\to0} \delta^\varepsilon = \lim_{\varepsilon\to0} (h_e(n_i^\varepsilon - \alpha^\varepsilon) - h_e(n_i^\varepsilon)) = -\lim_{\varepsilon\to0} h_e'(\xi^\varepsilon)\alpha^\varepsilon = 0$$

in $L^2(\Omega_T)$ strongly, where ξ^ε lies between $n_i^\varepsilon - \alpha^\varepsilon$ and n_i^ε. Hence, we obtain

$$\lim_{\varepsilon\to0} (n_i^\varepsilon \cdot \delta^\varepsilon) = \lim_{\varepsilon\to0} n_i^\varepsilon \cdot \lim_{\varepsilon\to0} \delta^\varepsilon = 0 \quad \text{in } L^2(Q_T) \text{ strongly.}$$

This gives

$$\lim_{\varepsilon\to0} n_i^\varepsilon (h_i(n_i^\varepsilon) + \mu_i \phi^\varepsilon) = \lim_{\varepsilon\to0} n_i^\varepsilon (h_i(n_i^\varepsilon) + h_e(n_i^\varepsilon)) \quad \text{in } \mathcal{D}'(\Omega_T),$$

and

$$n \lim_{\varepsilon\to 0}(h_i(n_i^\varepsilon) + \mu_i \phi^\varepsilon) = n \lim_{\varepsilon\to 0}(h_i(n_i^\varepsilon) + h_e(n_i^\varepsilon)) \quad \text{in } \mathcal{D}'(\Omega_T).$$

Let $G \stackrel{\text{def}}{=} h_i + h_e$. From (4.11) we obtain

$$\lim_{\varepsilon\to 0} n_i^\varepsilon G(n_i^\varepsilon) = \lim_{\varepsilon\to 0} n_i^\varepsilon \cdot \lim_{\varepsilon\to 0} G(n_i^\varepsilon) \quad \text{in } \mathcal{D}'(\Omega_T).$$

Since h_i and h_e are strictly increasing, so is G. Thus, we have the identification

$$\lim_{\varepsilon\to 0} G(n_i^\varepsilon) = G(n) \quad \text{in } L^\infty(\Omega_T) \text{ weakly}-*,$$

which yields

$$\lim_{\varepsilon\to 0} n_i^\varepsilon G(n_i^\varepsilon) = nG(n) \quad \text{in } L^\infty(\Omega_T) \text{ weakly}- * .$$

Finally, we conclude the strong convergence of $(n_i^\varepsilon)_{\varepsilon>0}$ from the L^∞ bound and the strict convexity of the function $s \mapsto sG(s)$, see [26]. The strong convergence of $(\phi^\varepsilon)_{\varepsilon>0}$ follows immediately. It is clear that in this situation, we can pass to the limit in the system (at least locally in Ω) and (n,ϕ) satisfies (1.12)–(1.13). The local estimates for n_i^ε imply as in the proof of Theorem 2.1 that the initial condition is satisfied in the sense of $H^{-1}_{\text{loc}}(\Omega)$. Finally, if $n_{D,i} = f_e(\phi_D)$, it is easy to see as in the proof of Theorem 2.1 that (n,ϕ) also satisfies the boundary conditions (1.4)–(1.5) for $\alpha = i$. This proves the theorem.

One may ask if the quasi-neutral limit in (DD-I) can be proved without assumption (A2), similar to Theorem 3.6. The answer is affirmative under the additional assumptions:

$$p_i(s) = s, \qquad n_{D,i} = f_e(\phi_D) \quad \text{on } \Sigma_D. \tag{4.12}$$

Theorem 4.6 *Let (A1), (A6) and (4.12) hold. Then, as $\varepsilon \to 0$,*

$$n_i^\varepsilon \to n, \quad \phi^\varepsilon \to \phi \qquad \text{in } L^2(\Omega_T) \text{ strongly,} \tag{4.13}$$

$$\nabla\phi^\varepsilon \rightharpoonup \nabla\phi \qquad \text{in } L^2(\Omega_T) \text{ weakly,} \tag{4.14}$$

and the limit (n,ϕ) solves the problem (1.12)–(1.13), (1.4)–(1.6) for $\alpha = i$.

Proof. First we notice that Lemmas 4.1 and 4.2 and the estimate (4.5) are valid without assumption (A2), where only $m \geq 0$ holds (see Lemma 4.1). Then, the conditions (4.12) together with the relation (4.9) in the proof of Proposition 4.5, imply that

$$\|n_i^\varepsilon - f_e(\phi^\varepsilon)\|^2_{L^2(\Omega_T)} + c_1\varepsilon^2 \|\nabla\phi^\varepsilon\|^2_{L^2(\Omega_T)} \leq c_2 \varepsilon^2 \|\sqrt{n_i^\varepsilon}\nabla F_i^\varepsilon\|^2_{L^2(\Omega_T)}.$$

We can conclude, using the estimate (4.5) that

$$\|n_i^\varepsilon - f_e(\phi^\varepsilon)\|_{L^2(\Omega_T)} \leq c\varepsilon, \quad \|\nabla\phi^\varepsilon\|_{L^2(\Omega_T)} \leq c.$$

Therefore, using the uniform L^∞ bound on n_i^ε it can be easily seen that n_i^ε is uniformly bounded in $L^2(0,T;H^1(\Omega))$. Hence, as in the proofs of the previous theorems, we get the convergence results (4.13) and (4.14) and finally, the conclusion of the theorem.

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