Asymptotic limits for quantum trajectory models

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Abstract. The semi-classical and the inviscid limit in quantum trajectory models given by a one-dimensional steady-state hydrodynamic system for quantum fluids are rigorously performed. The model consists of the momentum equation for the particle density in a bounded domain, with prescribed current density, and the Poisson equation for the electrostatic potential. The momentum equation can be written as a dispersive third-order differential equation which may include viscous terms. It is shown that the semi-classical and inviscid limit commute for sufficiently small data (i.e. current density) corresponding to subsonic states, were the inviscid non-dispersive solution is regular. In addition, we show these limits do *not* commute in general. The proofs are based on a reformulation of the problem as a singular second-order elliptic system and on elliptic and $W^{1,1}$ estimates.

Keywords. Semi-classical limit, inviscid limit, quantum hydrodynamic equations, semiconductors.

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1 Introduction

This paper is a continuation of the work by the authors [7] where quantum trajectory models given by steady-state hydrodynamic equations for quantum fluids have been studied. More precisely, existence of classical solutions to the viscous equations and, under some conditions, non-existence of weak solutions to the non-viscous equations have been shown.

In this paper we perform rigorously the semi-classical and inviscid limits of these models.

We consider an isothermal or isentropic quantum fluid of charged particles whose particle density n and electrostatic potential V is given by the stationary viscous quantum hydrodynamic equations in one space dimension, referred to as the model **(vQHD)**:

$$\left(\frac{J^2}{n} + Tp(n)\right)_x - nV_x - \delta^2 n \left(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\right)_x = -\frac{J}{\tau} - \nu n(\sigma(n))_{xx}, \qquad (1)$$

$$\lambda^2 V_{xx} = n - C(x), \qquad (2)$$

in the interval $\Omega = (0,1)$, with prescribed current density $J > 0$. Here, $T > 0$ is the temperature constant, $\delta > 0$ the (scaled) Planck constant, $\tau = \tau(x) > 0$ the momentum relaxation time, $\nu \geq 0$ the viscosity, $\lambda > 0$ the (scaled) Debye length, and $C(x)$ models fixed charged background ions. We refer to [16] for the choice of the scaling. The pressure function p is given by the relation $p(n) = n$ in the *isothermal* case and $p(n) = n^\alpha$ with $\alpha > 1$ in the *isentropic* case. The choice of the viscous term $\sigma(n)$ will be made explicit below.

The above system of equations is completed by the following boundary conditions:

$$n(0) = n_0, \quad n(1) = n_1, \quad V(0) = V_0, \quad V_x(0) = -E_0, \qquad (3)$$

$$\delta^2 \frac{(\sqrt{n})_{xx}(0)}{\sqrt{n_0}} - \nu\sigma'(n_0)n_x(0) = \frac{J^2}{2n_0^2} + Th(n_0) - V_0 + K, \qquad (4)$$

where h is the *enthalpy* function defined by $h'(s) = p'(s)/s$ for $s > 0$ and $h(1) = 0$, and $K > 0$ is a constant whose value is given below (see (10)). In the isothermal case, the enthalpy equals $h(s) = \log(s)$, in the isentropic case one gets $h(s) = \alpha/(\alpha-1) \cdot (s^{\alpha-1} - 1)$, $s > 0$, with $\alpha > 1$.

Notice that we need three boundary conditions for n, for Eq. (1) is of third oder. We do not prescribe the potential V at both $x = 0$ and $x = 1$ (but only at $x = 0$), since the current density is given. Then we compute the applied voltage $V(1) - V(0)$ from the solution of (1)-(4), obtaining a well-defined current-voltage characteristic. The third condition (4) can be interpreted as a boundary condition for the quantum Fermi potential (cf. [17]).

The dispersive, non-viscous quantum hydrodynamic equations (i.e. $\delta > 0$ and $\nu = 0$) arise in semiconductor modeling where it has been used for analyzing the flow of electrons in quantum semiconductor devices, like resonant tunneling diode models of Gardner and Ringhofer [8, 9]. Recent quantum chemistry calculations using Quantum Trajectories Methods of Lopreore and Wyatt [20] have been proposed to study resonant scattering with one-dimensional double barrier potentials [21] in order to obtain properties of transmission probabilities. In addition, these quantum trajectory models have been used in the modeling of collinear chemical reactions [23] and in models for photo-dissociation of molecules by Sales-Mayor *et al.* [22]. Very similar model equations have been employed in other areas of physics, e.g. in super fluidity [19] and in super conductivity [4].

We refer to [3, 8, 9, 12, 13] for a justification and derivation of the quantum hydrodynamic equations.

Mathematically, the dispersive non-viscous quantum models have been studied by Jerome and Zhang [24] and Gyi and Jüngel [15] in one space dimension and by Jüngel [17] in several space dimensions.

Recently, the authors have shown in [7] that the problem (1)-(4) with $\nu = 0$ admits a (classical) solution only for sufficiently small current densities $J > 0$, and this solution is *unique* for $\alpha > 2$, the pressure law exponent. Moreover it was shown that, for sufficiently large $J > 0$ (and some structure condition on the enthalpy), no weak solution can exist.

We recall that models with the classical hydrodynamic momentum flux structure yield *subsonic* states satisfying the condition $v^2 < dp/dn$ (v being the velocity) for sufficiently small current densities $J > 0$. Moreover, when the relaxation parameter τ becomes very small, the corresponding asymptotic model is the classical drift-diffusion-Poisson system.

Viscous or diffusive terms in the quantum hydrodynamic equations are recently derived by Arnold *et al.* [1] from the Wigner-Fokker-Planck equation via a moment method, and by Gardner and Ringhofer [9] from a Wigner-relaxation model, via a Chapman-Enskog expansion method based on scaling arguments.

It has been shown in [7] that for a special class of viscous terms, namely

$$\sigma(n) = -\frac{1}{\gamma - 1}\frac{1}{n^{(\gamma-1)/2}} \qquad \text{with } \gamma > 4, \tag{5}$$

the problem (1)-(4) admits classical solutions with positive particle density for *all* values of the current density. This means that the *ultra-diffusive* term given by σ prevents the solution from cavitating and also provides the necessary growth condition to obtain a uniform bound in the dispersive parameter δ. In this sense, this term corresponds to *anomalous diffusion* or *hyper-diffusive* corrections used for stable numerical approximations of the Navier-Stokes equations.

Hence, our goal is to perform the semi-classical limit $\delta \to 0$ and the inviscid limit $\nu \to 0$ in the model (vQHD) and to see under what kind of conditions these limits commute.

First, when $\delta \to 0$, we obtain (formally) the viscous hydrodynamic equations **(vHD)**:

$$\left(\frac{J^2}{n} + Tp(n)\right)_x - nV_x = -\frac{J}{\tau} - \nu n(\sigma(n))_{xx},$$

together with the Poisson equation (2). This model (with appropriate boundary conditions) has been studied by Gamba [5] using the viscosity term νn_{xx}. It is shown that for isentropic pressure functions, there exists a em classical solution with positive particle density, for all prescribed data. Moreover, the lower and upper bounds for this density do not depend on the viscosity parameter, and it is possible to perform rigorously the limit $\nu \to 0$. More general viscosity terms have been examined in [6]. However, the term (5) is not included in this class of viscosities. The limit $\nu \to 0$ in the model **(vHD)** yields a *weak entropic* of non-dispersive inviscid hydrodynamic model **(HD)**:

$$\left(\frac{J^2}{n} + Tp(n)\right)_x - nV_x = -\frac{J}{\tau},$$

together with Eq. (2). In addition, the solutions are smooth (*classical*) for a small data corresponding to *subsonic* states satisfying the condition $v^2 < dp/dn$.

Next, when $\nu \to 0$, the inviscid limit in the model (vQHD) leads to the dispersive model **(QHD)**:

$$\left(\frac{J^2}{n} + Tp(n)\right)_x - nV_x - \delta^2 n \left(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\right)_x = -\frac{J}{\tau},$$

with Eq. (2). The limit $\delta \to 0$ then gives the model (HD). These asymptotic limits are summarized in Figure 1. We study here under what conditions these diagram commutes.

The semi-classical limit in the model (QHD) has also been studied in [15] using different boundary conditions and in [12] in a whole-space setting.

The limits $\delta \to 0$ and $\nu \to 0$ in the above problems have not been considered in the mathematical literature.

The main results of this paper are that

$$\begin{tikzcd}
(\text{vQHD}) \arrow[r, "\delta\to 0"] \arrow[d, "\nu\to 0"'] & (\text{vHD}) \arrow[d, "\nu\to 0"] \\
(\text{QHD}) \arrow[r, "\delta\to 0"] & (\text{HD})
\end{tikzcd}$$

Figure 1: Semi-classical and inviscid limits in the (quantum) hydrodynamic equations.

- the limits $\delta \to 0$, $\nu \to 0$, and $\nu \to 0$, $\delta \to 0$ both exist if $J > 0$ is small enough corresponding to subsonic states $v^2 < dp/dn$. In addition, they commute for values of the pressure law for $\alpha > 2$, since all three models, (HD), (QHD), and (vHD), have *unique* solutions for $\alpha > 2$;

- the limits $\delta \to 0$ and $\nu \to 0$ do *not* commute if $J > 0$ is sufficiently large, corresponding to *transonic* states. Only the limit $\delta \to 0$ first, and then $\nu \to 0$ can be performed. Under this regime, the (weak) solution of the (HD) model may be discontinuous.

Remark 1 The proof of uniqueness for the (vHD) is presented in the appendix of this paper. The one for the (QHD) model was shown by the authors in [7].

The main reason for these results, obtained in [7], lies in the fact that there is a lower bound for the solution n of (vQHD), which does not depend on δ or ν if the current density is small enough, but the lower bound depends in general on ν and tends to zero as $\nu \to 0$. In fact, the limit $\nu \to 0$ in the model (vQHD) cannot be performed for sufficiently large $J > 0$ due to the non-existence result for the model (QHD) (see above). We notice that the limit $\delta \to 0$ in the model (vQHD) can be performed for *any* value of $J > 0$.

The proofs of our asymptotic limits are based on a reformulation of Eq. (1) as a singular second-order elliptic equation, on uniform a priori estimates, obtained from elliptic estimates and the $W^{1,1}(\Omega)$-technique of Gamba [5], and on standard compactness arguments.

Another interesting limit is the zero-Deby-length limit $\lambda \to 0$ usually referred as the *quasi-neutral* limit. This limit has been proved for the model (QHD) for sufficiently small $J > 0$ corresponding to subsonic states [15]. Although we do not study this limit here, as in quantum semiconductor devices the parameter λ needs not to be small, we conjecture that in the case of subsonic regime the three singular perturbations will commute. Further results on the quasi-neutral limit can be found in [10, 11, 18] in the case of drift-diffusion equations and in [2] for the hydrodynamic model.

The paper is organized as follows. In Section 2 we derive the system of second-order equations, state the main assumptions and recall the existence results of [7]. Section 3 is devoted to the limit $\delta \to 0$ in (QHD). Then we prove the semi-classical limit in (vQHD) in Section 4. In Section 5 the limit $\nu \to 0$ in (vQHD) is performed. Finally, the inviscid limit in (vHD) is studied in Section 6.

2 Reformulation of the Equations and Main Assumptions

In order to derive the system of second-order equations to be analyzed, we rewrite Eq. (1) as

$$n\left(\frac{J^2}{2n^2}+Th(n)-V-\delta^2\frac{(\sqrt{n})_{xx}}{\sqrt{n}}+J\int_0^x\frac{ds}{\tau n}+\nu\sigma(n)_x\right)_x=0.$$

This implies, if $n>0$ in Ω,

$$\frac{J^2}{2n^2}+Th(n)-V-\delta^2\frac{(\sqrt{n})_{xx}}{\sqrt{n}}+J\int_0^x\frac{ds}{\tau n}+\nu\sigma(n)_x=-K,$$

where K is an integration constant coming from (4) and defined in (10) below. Setting $w=\sqrt{n}$, we obtain

$$\begin{aligned}\delta^2 w_{xx} &= \frac{J^2}{2w^3}+Twh(w^2)-Vw+Kw+Jw\int_0^x\frac{ds}{\tau w^2}+\nu\beta(w)w_x, &(6)\\ \lambda^2 V_{xx} &= w^2-C(x) \qquad \text{in } \Omega=(0,1), &(7)\end{aligned}$$

where

$$\beta(w)=2w\sigma'(w)=w^{-\gamma}, \qquad w>0. \tag{8}$$

These equations are solved subject to the boundary conditions

$$w(0)=w_0,\quad w(1)=w_1,\quad V(0)=V_0,\quad V_x(0)=-E_0, \tag{9}$$

where $w_0=\sqrt{n_0}$, $w_1=\sqrt{n_1}$. The constant K is defined by

$$\begin{aligned}K &\stackrel{\text{def}}{=} V_0+\max(-E_0,0)+\lambda^{-2}M^2, \quad \text{where} &(10)\\ M &\stackrel{\text{def}}{=} \max(w_0,w_1,M_0), &(11)\end{aligned}$$

and M_0 is such that $h(M_0^2)\geq 0$. Notice that every solution (w,V) to (6)-(9) satisfying $w>0$ in Ω defines a solution (n,V) to (1)-(4) via $n=w^2$.

For the existence results we impose the following assumptions:

(H1) $h\in C^1(0,\infty)$ and p' (defined by $p'(s)=sh'(s)$, $s>0$) are non-increasing, and h satisfies

$$\lim_{s\to\infty}h(s)>0,\quad \lim_{s\to 0+}h(s)<0,\quad \lim_{s\to 0+}\sqrt{s}h(s)>-\infty. \tag{12}$$

(H2) $C\in L^2(\Omega)$, $C\geq 0$ in Ω; $\tau\in L^\infty(\Omega)$, $\tau(x)\geq\tau_0>0$ in Ω.

(H3) $J, w_0, w_1, \delta, \lambda, T > 0$; $\gamma > 4$; $\nu \geq 0$; $V_0, E_0 \in R$.

We recall two existence results for the problem (6)-(9), which are proved in [7, Theorems 2.1 and 3.1]:

Theorem 2 *Let (H1)-(H3) hold and let $\nu > 0$, $J > 0$. Then there exists a classical solution $(w, V) \in (C^2(\overline{\Omega}))^2$ to (6)-(9) satisfying*

$$0 < m(\nu) \leq w(x) \leq M, \quad k \leq V(x) \leq K \quad \text{in } \Omega.$$

Remark 3 The constant M is defined in (11); the constant $m(\nu)$ is given by

$$m(\nu) = \min\{w_0, w_1, m_1, m_2\}, \tag{13}$$

where

$$\begin{aligned} h(4m_1^2) &\leq 0, \\ m_2 &= \left(\frac{1}{2^{\gamma+1}} \frac{\nu}{J^2/2 + J/\tau_0 + \max(0, K-k)} \right)^{(1/\gamma-4)}, \\ k &= V_0 - \max(E_0, 0) - \lambda^{-2} \|C\|_{L^1(\Omega)}. \end{aligned}$$

Theorem 4 *Let (H1)-(H3) hold and let $\nu \geq 0$. Furthermore, suppose that there exists $m_0 > 0$ such that*

$$\frac{1}{2} T p'(m_0^2) + T h(m_0^2) + \frac{1}{\tau_0} \sqrt{T p'(m_0^2)} + K - k \leq 0, \tag{14}$$

and let $J > 0$ be such that

$$J \leq J_0 \stackrel{\text{def}}{=} m^2 \sqrt{T p'(m^2)}, \tag{15}$$

where $m = \min\{w_0, w_1, m_0\}$. Then there exists a classical solution $(w, V) \in (C^2(\overline{\Omega}))^2$ to (6)-(9) satisfying

$$0 < m \leq w(x) \leq M, \quad k \leq V(x) \leq K \quad \text{in } \Omega.$$

Remark 5 The condition (14) is satisfied if, for instance (see [7]),

(i) $\lim_{s \to 0+} h(s) = -\infty$, or

(ii) $E_0 > 0$ is sufficiently large.

The isothermal enthalpy $h(s) = \log(s)$ satisfies (i).

Condition (15) implies $v = J/w^2 \leq \sqrt{T p'(w^2)}$ in Ω. In particular, solutions to (6)-(9) are *subsonic*. If this inequality is not satisfied a.e. in Ω, we call the solution *transonic*. In this sense, Theorem 4 provides the existence of subsonic solutions to (vQHD) and (QHD) since

$$v = \frac{J}{w^2} \leq \frac{J_0}{m^2} = \sqrt{T p'(m^2)} \leq \sqrt{T p'(w^2)} \qquad \text{in } \Omega.$$

Theorem 4 is proved in [7] only for $\nu = 0$. For the proof of the case $\nu > 0$ we only need to show the lower bound for w. This can be achieved by using the test function $(w - r)^-$ with $r(x) = m > 0$ in the weak formulation of (6) and estimating as in the proof of Theorem 2.1 of [7], using the condition (14).

3 The Semi-Classical Limit in (QHD) in Subsonic Regimes

In this section we perform rigorously the limit $\delta \to 0$ in the Eqs. (6)-(7) with $\nu = 0$. For this, define the function

$$F(s) = \frac{J^2}{2s^4} + Th(s^2), \quad s > 0. \tag{16}$$

Our main result is the following theorem.

Theorem 6 *Let $\nu = 0$. Let the assumptions of Theorem 4 hold and let (w^δ, V^δ) be a classical solution to (6)-(9) for $\delta > 0$. Assume that*

$$J < \min\{J_0, m^2\tau_0 F_0/2\}, \tag{17}$$

where $F_0 = \min\{F'(w^\delta(x)) : \ x \in \overline{\Omega}, \ \delta > 0\} > 0$. Then, for all $\Gamma \subset\subset \Omega$, there exists a subsequence (not relabeled) such that, as $\delta \to 0$,

$$\begin{aligned}
w^\delta &\rightharpoonup w \qquad \text{weakly in } H^1(\Gamma) \text{ and weakly}^* \text{ in } L^\infty(\Omega),\\
w^\delta &\to w \qquad \text{strongly in } L^\infty(\Gamma),\\
V^\delta &\to V \qquad \text{strongly in } W^{2,\infty}(\Omega),
\end{aligned}$$

and (w, V) solves the inviscid *equations*

$$\frac{J^2}{2w^4} + Th(w^2) - V + K = -J\int_0^x \frac{ds}{\tau w^2}, \tag{18}$$

$$\lambda^2 V_{xx} = w^2 - C(x) \quad \text{in } \Omega, \tag{19}$$

$$V(0) = V_0, \quad V_x(0) = -E_0, \tag{20}$$

where the bounds for the solution w of the limiting problem, i.e. $0 < m \le w(x) \le M$ still holds, where m and M are defined in Theorem 4.

Remark 7 Taking the derivate of (18) and setting $n = w^2$, we see that (n, V) solves the hydrodynamic equations

$$\begin{aligned}
\left(\frac{J^2}{n} + Tp(n^2)\right)_x - nV_x &= -\frac{J}{\tau},\\
\lambda^2 V_{xx} &= n - C(x) \quad \text{in } \Omega.
\end{aligned}$$

Remark 8 Notice that w does not necessarily satisfy the boundary conditions (9).

Proof. Notice that by Theorem 4, the lower and upper bounds for w^δ and V^δ do not depend on δ. Moreover, since V^δ is given by

$$V^\delta(x) = V_0 - E_0 x + \lambda^{-2} \int_0^x \int_0^y (w^\delta(z)^2 - C(z))dz\ dy,\ x \in \Omega, \tag{21}$$

V^δ is uniformly bounded in $W^{2,\infty}(\Omega)$. Recalling the definition (16) of the function F, it holds, in view of (17),

$$\begin{aligned} F'(w^\delta) &= \frac{2}{(w^\delta)^5}\Big(- J^2 + T(w^\delta)^6 h'((w^\delta)^2)\Big) \\ &= \frac{2}{(w^\delta)^5}\Big(- J^2 + T(w^\delta)^4 p'((w^\delta)^2)\Big) \\ &\geq \frac{2}{(w^\delta)^5}\Big(- J^2 + Tm^4 p'(m^2)\Big) && (22) \\ &\geq \frac{2}{M^5}\Big(- J^2 + Tm^4 p'(m^2)\Big) = F_0 > 0 \quad \text{in } \Omega. && (23) \end{aligned}$$

Let $\chi \in C_0^\infty(\Omega)$ be such that $0 \leq \chi \leq 1$ in Ω, $\sqrt{\chi} \in H^1(\Omega)$ and $\chi = 1$ in $\Gamma \subset\subset \Omega$. Furthermore, set $S^\delta(x) = \int_0^x ds/\tau(w^\delta)^2$. Using the test function $[F(w^\delta) - V^\delta + K + Jw^\delta S^\delta]\chi \in H_0^1(\Omega)$ in the weak formulation of (6), we obtain

$$\begin{aligned} &\delta^2 \int_\Omega F'(w^\delta)\chi(w^\delta)_x^2 dx + \int_\Omega w^\delta \big[F(w^\delta) - V^\delta + K + Jw^\delta S^\delta\big]^2 \chi dx \\ &= \delta^2 \int_\Omega \chi w_x^\delta V_x^\delta dx - \delta^2 J \int_\Omega \chi w_x^\delta (w^\delta S^\delta)_x dx \qquad (24) \\ &\quad - \delta^2 \int_\Omega \chi_x w_x^\delta \big[F(w^\delta) - V^\delta + K + Jw^\delta S^\delta\big] dx. \end{aligned}$$

In order to estimate the first term on the right-hand side, we use $w^\delta\chi \in H_0^1(\Omega)$ as a test function in the weak formulation of (7) to get

$$\int_\Omega \chi V_x^\delta w_x^\delta dx = - \int_\Omega \chi_x V_x^\delta w^\delta dx - \lambda^{-2} \int_\Omega (C - (w^\delta)^2) w^\delta \chi dx$$

and

$$\begin{aligned} \left| \int_\Omega \chi V_x^\delta w_x^\delta dx \right| &\leq \|w^\delta\|_{L^\infty} \|V_x^\delta\|_{L^2} \|\chi_x\|_{L^2} \\ &\quad + \lambda^{-2} \|C - (w^\delta)^2\|_{L^1} \|w^\delta\|_{L^\infty} \|\chi\|_{L^\infty} \\ &\leq c, \end{aligned}$$

where $c > 0$ denotes here and in the following a constant independent of $\delta > 0$ with values varying from occurrence to occurrence.

The second term on the right-hand side of (24) is estimated as follows, using Young's inequality:

$$\begin{aligned}
\left| J \int_\Omega \chi w_x^\delta (w^\delta S^\delta)_x dx \right| &= \left| J \int_\Omega \chi (w^\delta)_x^2 S^\delta dx + J \int_\Omega \chi \frac{w_x^\delta}{\tau w^\delta} dx \right| \\
&\le \frac{J}{\tau_0 m^2 F_0} \int_\Omega \chi F'(w^\delta)(w^\delta)_x^2 dx + \frac{\varepsilon}{4} \int_\Omega \chi F'(w^\delta)(w^\delta)_x^2 dx \\
&\quad + \frac{J^2}{\varepsilon \tau_0^2} \int_\Omega \frac{\chi}{F'(w^\delta)(w^\delta)^2} dx \\
&\le \left(\frac{J}{\tau_0 m^2 F_0} + \frac{\varepsilon}{4} \right) \int_\Omega \chi F'(w^\delta)(w^\delta)_x^2 dx + c(\varepsilon),
\end{aligned}$$

where $F_0 = \min\{F'(w^\delta(x)) : x \in \overline{\Omega},\ \delta > 0\}$ from (23).

It remains to bound the last term on the right-hand side of (24):

$$\begin{aligned}
&\left| \int_\Omega \chi_x w_x^\delta \left[F(w^\delta) - V^\delta + K + J w^\delta S^\delta \right] dx \right| \\
&\le \frac{\varepsilon}{4} \int_\Omega \chi F'(w^\delta)(w^\delta)_x^2 dx + \frac{1}{\varepsilon F_0} \int_\Omega \left[F(w^\delta) - V^\delta + K + J w^\delta S^\delta \right]^2 \frac{\chi_x^2}{\chi} dx \\
&\le \frac{\varepsilon}{4} \int_\Omega \chi F'(w^\delta)(w^\delta)_x^2 dx + c(\varepsilon).
\end{aligned}$$

Thus we obtain from (24):

$$\begin{aligned}
&\left(1 - \frac{J}{\tau_0 m^2 F_0} - \frac{\varepsilon}{2} \right) \int_\Omega \chi F'(w^\delta)(w^\delta)_x^2 dx \\
&\qquad + \delta^{-2} \int_\Omega w^\delta \left[F(w^\delta) - V^\delta + K - J w^\delta S^\delta \right]^2 \chi dx \\
&\le c(\varepsilon).
\end{aligned}$$

Choosing $\varepsilon = 1 - 2J/\tau_0 m^2 F_0 > 0$ (see (17)), we get

$$\frac{F_0}{2} \int_\Omega (w^\delta)_x^2 dx \le \frac{1}{2} \int_\Omega F'(w^\delta)(w^\delta)_x^2 dx \le c.$$

Hence, w^δ is uniformly bounded in $H^1(\Gamma)$, which yields the existence of a subsequence (not relabeled) such that, as $\delta \to 0$,

$$\begin{aligned}
w^\delta &\rightharpoonup w && \text{weakly* in } L^\infty(\Omega), \\
w^\delta &\rightharpoonup w && \text{weakly in } H^1(\Gamma), \\
w^\delta &\to w && \text{strongly in } L^\infty(\Gamma).
\end{aligned}$$

This yields

$$\begin{aligned}
V^\delta &\to V && \text{strongly in } W^{2,\infty}(\Gamma), \\
V^\delta &\rightharpoonup V && \text{weakly* in } L^\infty(\Omega).
\end{aligned}$$

We claim that

$$(w^\delta)^2 \rightharpoonup w^2 \qquad \text{weakly* in } L^\infty(\Omega).$$

Indeed, from $w^\delta \to w$ pointwise a.e. in Γ we obtain $(w^\delta)^2 \to w^2$ pointwise a.e. in Γ. Moreover, $(w^\delta)^2 \rightharpoonup z$ in $L^\infty(\Gamma)$. This implies that $w^2 = z$ in Γ for all $\overline{\Gamma} \subset \Omega$, and hence, $w^2 = z$ in Ω.

Therefore, V solves the Poisson equation (19) and

$$V^\delta \to V \qquad \text{strongly in } W^{2,\infty}(\Omega).$$

Hence V satisfies the boundary conditions (20). Moreover, due to the above convergence results for w^δ, we see that w solves the equation (18). This finishes the proof.

4 The Semi-classical Limit in (vQHD)

This section is devoted to the proof of the limit $\delta \to 0$ in (vQHD) for all $J > 0$. This means that we can perform the semi-classical limit also for viscous transonic solutions. The main result is the following theorem.

Theorem 9 *Let the assumptions (H1)-(H3) hold and let $\nu > 0$, $J > 0$. Let (w^δ, V^δ) be a classical solution to (6)-(9) for $\delta > 0$. Then there exists a subsequence (not relabeled) such that, as $\delta \to 0$,*

$$\begin{aligned} w^\delta &\to w \qquad \textit{strongly in } L^p(\Omega), \quad p < \infty, && (25)\\ w^\delta &\rightharpoonup w \qquad \textit{weakly in } W^{1,1}(\Omega), && (26)\\ V^\delta &\to V \qquad \textit{strongly in } W^{2,\infty}(\Omega), && (27)\end{aligned}$$

and (w, V) solves the equations

$$\begin{aligned} &\frac{J^2}{2w^3} + Twh(w^2) - wV + wK + \nu\beta(w)w_x = -Jw\int_0^x \frac{ds}{\tau w^2}, && (28)\\ &\lambda^2 V_{xx} = w^2 - C(x) \qquad \textit{in } \Omega, && (29)\\ &V(0) = V_0, \quad V_x(0) = -E_0, && (30)\end{aligned}$$

$\beta(w)$ being defined in (8). Moreover, it holds $w \geq m(\nu) > 0$ in Ω, where $m(\nu)$ is defined in (13).

Remark 10 Dividing Eq. (28) by w, setting $n = w^2$ and differentiating, we see that (n, v) is a solution of (vHD).

Proof. First, we show that there exists a constant $c = c(\nu) > 0$ independent of δ, such that

$$\delta^2 \|w_x^\delta\|_{L^\infty(\Omega)} \leq c(\nu). \tag{31}$$

This estimate can be proved using the method in [5, Lemma 4] or employing standard elliptic estimates and the Gagliardo-Nirenberg inequality. Indeed, with the test function $w - w_D$ (where $w_D(x) = w_0(1-x) + w_1 x$, $x \in (0,1)$) in (6) and standard estimates we obtain

$$\delta \|w_x^\delta\|_{L^2(\Omega)} \leq c(\nu),$$

where $c(\nu) > 0$ depends on ν through $m(\nu)$ (see Theorem 2). Since (w^δ) is uniformly bounded in $L^\infty(\Omega)$ and hence in $L^2(\Omega)$ and

$$\delta^2 \|w_{xx}^\delta\|_{L^2(\Omega)} \leq c(\nu)(1 + \|w_x^\delta\|_{L^2(\Omega)}) \leq c(\nu)\delta^{-1},$$

we obtain

$$\|w^\delta\|_{H^2(\Omega)} \leq c(\nu)\delta^{-3}.$$

Thus, employing the Gagliardo-Nirenberg inequality [14, p. 242],

$$\|w_x^\delta\|_{L^\infty(\Omega)} \leq c(\nu) \|w^\delta\|_{H^2(\Omega)}^{2/3} \|w^\delta\|_{L^\infty(\Omega)}^{1/3} \leq c(\nu)\delta^{-2},$$

from which (31) follows.

Now we need to obtain a δ-uniform bound in the total variation norm of w^δ in Ω. We proceed similarly as in [5]. Let $I_i = (a_i, b_i)$, $i = 1, \ldots, k$, be a family of intervals with the property that w_x^δ does not change sign in I_i, and that $w_x^\delta(a_i) = w_x^\delta(b_i) = 0$ or $a_1 = 0$ or $b_k = 1$. Set

$$\Gamma_k = \bigcup_{i=1}^k I_i.$$

Next, take $g(s) = -(1/(\gamma-1))s^{1-\gamma}$, $s > 0$. Then $g(w^\delta)_x = \beta(w^\delta)w_x^\delta$ is a monotone function of w^δ, and the integration of (6) over I_i gives

$$\nu \int_{a_i}^{b_i} g(w^\delta)_x dx = \delta^2 (w_x^\delta(b_i) - w_x^\delta(a_i)) - \int_{a_i}^{b_i} Z^\delta(x) dx,$$

where

$$Z^\delta(x) = \frac{J^2}{2(w^\delta)^3} + T w^\delta h((w^\delta)^2) + (K - V^\delta) w^\delta + J w^\delta \int_o^x \frac{ds}{\tau(w^\delta)^2},$$

for $x \in \Omega$.

We claim that $g(w^\delta)_x$ is uniformly bounded in $L^1(\Omega)$. Indeed, since w^δ_x, and consequently $g(w^\delta)$ do not change sign in I_i, it holds

$$\begin{aligned}
\int_{\Gamma_k} |g(w^\delta)_x| dx &= \sum_{i=1}^k \int_{I_i} |g(w^\delta)_x| dx \\
&= \sum_{i=1}^k \left| \int_{I_i} g(w^\delta)_x dx \right| \\
&\leq \frac{\delta^2}{\nu} \sum_{i=1}^k \left| w^\delta_x(b_i) - w^\delta_x(a_i) \right| + \frac{1}{\nu} \sum_{i=1}^k \int_{a_i}^{b_i} \left| Z^\delta(x) \right| dx.
\end{aligned}$$

Now, $w^\delta_x(b_i) = w^\delta_x(a_i) = 0$ except maybe $w^\delta_x(a_1) \neq 0$, $w^\delta_x(b_k) \neq 0$. Therefore, taking into account the uniform upper and lower bounds for w^δ and V^δ, and the estimate (31),

$$\begin{aligned}
\int_{\Gamma_k} \left| g(w^\delta)_x \right| dx &\leq \frac{\delta^2}{\nu} \left(|w^\delta_x(b_k)| + |w^\delta_x(a_1)| \right) + \frac{1}{\nu} \int_\Omega \left| Z^\delta(x) \right| dx \\
&\leq \frac{2\delta^2}{\nu} \|w^\delta_x\|_{L^\infty} + c(\nu) \\
&\leq c(\nu).
\end{aligned}$$

Construct a sequence of unions of intervals (Γ_k) with

$$\Gamma_k = \bigcup_{i=1}^k I_i,$$

I_i as above, and $\Gamma_k \subset \Gamma_{k+1}$, $\lim_{k\to\infty} \Gamma_k = \Omega$. Then

$$B_k \stackrel{\text{def}}{=} \int_{\Gamma_k} \left| g(w^\delta)_x \right| dx \leq B_{k+1} \leq c(\nu).$$

Hence the sequence (B_k) admits a limit and

$$\int_\Omega \left| g(w^\delta)_x \right| dx = \lim_{k\to\infty} B_k \leq c(\nu).$$

In view of the uniform lower bound for w^δ this implies

$$\|g(w^\delta)\|_{W^{1,1}(\Omega)} \leq c(\nu).$$

Since the embedding $W^{1,1}(\Omega) \hookrightarrow L^p(\Omega)$ is compact for any $p < \infty$, we conclude that there is a subsequence (not relabeled) such that, as $\delta \to 0$,

$$\begin{aligned}
w^\delta &\rightharpoonup w && \text{weakly}^* \text{ in } L^\infty(\Omega), \\
g(w^\delta) &\rightharpoonup z && \text{weakly in } W^{1,1}(\Omega), \\
g(w^\delta) &\to z && \text{strongly in } L^p(\Omega), \quad p < \infty.
\end{aligned}$$

Since g is monotone in its argument, then $z = g(w)$ in Ω. The uniform pointwise lower bound for w^δ gives

$$w^\delta \rightharpoonup w \qquad \text{weakly in } W^{1,1}(\Omega)$$

and hence,

$$w^\delta \to w \qquad \text{strongly in } L^p(\Omega), \quad p < \infty.$$

Then the formula (21) for V^δ gives the convergence

$$V^\delta \to \ V \quad \text{strongly in } W^{2,\infty}(\Omega).$$

These convergence results are sufficient to pass to the limit $\delta \to 0$ in the equations (6)-(7). Since V^δ converges in $W^{2,\infty}(\Omega)$, we see that V satisfies the boundary conditions (9). Theorem 9 is proved.

5 The Inviscid Limit in (vQHD)

The inviscid limit in the model (vQHD) for subsonic solutions can be performed using standard methods since it is not a singular limit. We state and prove this result for completeness. Notice that the subsonic condition is crucial. Indeed, the limit $\nu \to 0$ is generally wrong, since under appropriate assumptions on the pressure function, the problem (6)-(9) with $\nu > 0$ cannot have a weak solution if $J > 0$ is large enough [7, Theorem 4.1].

Theorem 11 *Let the assumptions of Theorem 4 hold and let (n^ν, V^ν) be a classical solution to (6)-(9). (In particular, we assume that $0 < J \leq J_0$.) Then there exists a subsequence (not relabeled) such that, as $\nu \to 0$,*

$$w^\nu \to w \qquad \textit{weakly in } H^1(\Omega) \textit{ and strongly in } L^\infty(\Omega), \tag{32}$$

$$V^\nu \to V \qquad \textit{strongly in } W^{2,\infty}(\Omega), \tag{33}$$

and (w, V) solves the equations (6)-(7) with $\nu = 0$ and the boundary conditions (9).

Proof. We set $S^\nu(x) = \int_0^x ds/\tau(w^\nu)^2$ and define $w_D(x) = xw_1 + (1-x)w_0$, $x \in (0,1)$. Then use $w^\nu - w_D$ as a test function in the weak formulation of (6) to obtain

$$\begin{aligned}\delta^2 \int_\Omega \ (w_x^\nu)^2 dx &= \delta^2 \int_\Omega w_x^\nu w_{D,x} dx - \nu \int_\Omega \beta(w^\nu) w_x^\nu (w^\nu - w_D) dx \\ &\quad - \int_\Omega \Big[\frac{J^2}{2(w^\nu)^3} + Th((w^\nu)^2) + (K - V^\nu)w^\nu + Jw^\nu S^\nu\Big](w^\nu - w_D) dx.\end{aligned}$$

The last term on the right-hand side can be majorized by a constant independent of ν, using the uniform lower and upper bounds for w^ν and V^ν provided by Theorem 4. For the first two terms we employ Young's inequality:

$$\begin{aligned} \delta^2 \int_\Omega w_x^\nu w_{D,x} dx &\le \frac{\delta^2}{4}\int_\Omega (w_x^\nu)^2 dx + \frac{1}{\delta^2}\int_\Omega w_{D,x}^2 dx, \\ -\nu \int_\Omega \beta(w^\nu) w_w^\nu (w^\nu - w_D) dx &\le \frac{\delta^2}{4}\int_\Omega (w_x^\nu)^2 dx + \frac{\nu^2}{\delta^2}\int_\Omega \beta(w^\nu)^2 (w^\nu - w_D)^2 dx. \end{aligned}$$

Therefore

$$\frac{\delta^2}{2}\int_\Omega (w_x^\nu)^2 dx \le c,$$

and $c > 0$ is independent of ν.

The above estimate implies the existence of a subsequence of (w^ν, V^ν) (not relabeled) such that, as $\nu \to 0$, (32)-(33) hold and $w \ge m > 0$ in Ω. Hence, it is possible to perform the limit $\nu \to 0$ in the equations (6)-(7) to obtain the assertion of the theorem.

6 The Inviscid Limit in (vHD)

We show the limit $\nu \to 0$ for subsonic solutions to (6)-(7) with $\delta = 0$.

Theorem 12 *Let the assumptions of Theorem 4 hold and let $(w^\nu, V^\nu) \in W^{1,1}(\Omega)$ $\times W^{2,\infty}(\Omega)$ be a solution to (6)-(7) with $\delta = 0$. Then there exists a subsequence of (w^ν, V^ν) (not relabeled) such that, as $\nu \to 0$,*

$$\begin{aligned} w^\nu &\to w \quad \text{strongly in } L^p(\Omega), \quad p < \infty, \\ V^\nu &\to V \quad \text{strongly in } W^{2,\infty}(\Omega), \end{aligned}$$

and $(w, V) \in L^\infty(\Omega) \times W^{2,\infty}(\Omega)$ is a solution of (6)-(7) with $\delta = \nu = 0$. Moreover, $w \ge m > 0$ in Ω, where m is defined in Theorem 4.

Proof. Notice that, by Theorem 4, there is a constant independent of $\nu > 0$ such that

$$\|w^\nu\|_{L^\infty(\Omega)} \le c, \qquad \|V^\nu\|_{W^{2,\infty}(\Omega)} \le c. \tag{34}$$

Since $\gamma > 4$, these bounds imply immediately

$$\begin{aligned} \nu \|w_x^\nu\|_{L^\infty(\Omega)} &= \Big\| \frac{J^2}{2}(w^\nu)^{\gamma-3} + T(w^\nu)^{\gamma+1} h((w^\nu)^2) + (K - V^\nu)(w^\nu)^{\gamma+1} \\ &\quad + J(w^\nu)^{\gamma+1} S^\nu \Big\|_{L^\infty(\Omega)} \\ &\le c. \end{aligned} \tag{35}$$

The functions w^ν, V^ν are continuous in $\overline{\Omega}$, so, by Eq. (6) for $\delta = 0$ and the uniform lower and upper bounds for w^ν, it holds $w_x^\nu \in C^0(\overline{\Omega})$. Hence, there is a constant $c > 0$, independent of ν, such that

$$\nu |w_x^\nu(x)| \le c \qquad \text{for } x = 0, 1. \tag{36}$$

From (6) with $\delta = 0$ follows after differentiation with respect to x:

$$\begin{aligned} -\nu w_{xx}^\nu \;=\; & \Big(\frac{J^2}{2}(\gamma - 3)(w^\nu)^{\gamma-4} + T(\gamma+1)(w^\nu)^\gamma h((w^\nu)^2) \\ & + 2T(w^\nu)^{\gamma+1} h'((w^\nu)^2) + (\gamma+1)(K - V^\nu)(w^\nu)^\gamma \\ & + (\gamma+1)J(w^\nu)^\gamma \Big) w_x^\nu - \frac{J}{\tau}(w^\nu)^{\gamma-1} - (w^\nu)^{\gamma+1} V_x^\nu. \end{aligned} \tag{37}$$

Since $w_x^\nu \in L^1(\Omega)$ we conclude that $w_{xx}^\nu \in L^1(\Omega)$.

In oder to pass to the limit in ν it is sufficient to obtain a uniform bound in the total variation norm of w^ν. We proceed as in the proof of Theorem 9. Let $I_i = (a_i, b_i)$, $i = 1, \ldots, k$, be a family of intervals with the property that w_x^δ does not change sign in I_i, and that $w_x^\delta(a_i) = w_x^\delta(b_i) = 0$ or $a_1 = 0$ or $b_k = 1$. Set

$$\Gamma_k = \bigcup_{i=1}^k I_i.$$

Suppose that $w_x^\nu > 0$ in I_i. Then we obtain after integration of (37), since $K - V \ge 0$ in Ω and $\gamma > 4$,

$$\begin{aligned} \nu(w_x^\nu(a_i) - w_x^\nu(b_i)) \;&=\; -\nu \int_{I_i} w_{xx}^\nu dx \\ &\ge\; A \int_{I_i} w_x^\nu dx - \int_{I_i} \Big(\frac{J}{\tau}(w^\nu)^{\gamma-1} + (w^\nu)^{\gamma+1} V_x^\nu \Big) dx, \end{aligned}$$

where

$$A = \frac{J^2}{2}(\gamma-3)m^{\gamma-4} + T \inf_{0<s<M} \big((\gamma+1)s^\gamma h(s^2) + 2s^{\gamma+1} h'(s^2) \big) > 0.$$

Notice that the lower bound m on w^ν (and therefore A) does not depend on ν since we consider only subsonic solutions. If $w_x^\nu \le 0$ in I_i, we obtain

$$\nu(w_x^\nu(b_i) - w_x^\nu(a_i)) \ge A \int_{I_i} (-w_x^\nu) dx + \int_{I_i} \Big(\frac{J}{\tau}(w^\nu)^{\gamma-1} + (w^\nu)^{\gamma+1} V_x^\nu \Big) dx.$$

Therefore total variation of w^ν in each I_i is given by

$$A \int_{I_i} |w_x^\nu| dx \le \nu |w_x^\nu(a_i) - w_x^\nu(b_i)| + \int_{I_i} \Big| \frac{J}{\tau}(w^\nu)^{\gamma-1} + (w^\nu)^{\gamma+1} V_x^\nu \Big| dx.$$

Summation over $i = 1, \dots, k$ yields

$$\begin{aligned} \int_{\Gamma_k} |w_x^\nu| dx &= \sum_{i=1}^k \int_{I_i} |w_x^\nu| dx \\ &\leq \frac{\nu}{A} \sum_{i=1}^k |w_x^\nu(a_i) - w_x^\nu(b_i)| + \frac{1}{A} \int_{\Gamma_k} \Big| \frac{J}{\tau}(w^\nu)^{\gamma-1} + (w^\nu)^{\gamma+1} V_x^\nu \Big| dx. \end{aligned}$$

We infer from the definition of I_i and the bounds (34) and (36) that

$$\int_{\Gamma_k} |w_x^\nu| dx \leq A^{-1} \nu (|w_x^\nu(a_1)| + |w_x^\nu(b_k)|) + c \leq c_0.$$

Construct a sequence of unions of intervals $(\Gamma_k)_k$ such that

$$\Gamma_k = \bigcup_{i=1}^k I_i,$$

I_i is as above, and $\Gamma_k \subset \Gamma_{k+1}$, $\lim_{k\to\infty} \Gamma_k = \Omega$. Then

$$A_k \stackrel{\text{def}}{=} \int_{\Gamma_k} |w_x^\nu| dx \leq A_{k+1} \leq c_0.$$

Hence the sequence $(A_k)_k$ admits a limit and

$$\int_\Omega |w_x^\nu| dx = \lim_{k\to\infty} A_k \leq c_0.$$

Therefore, (w^ν, V^ν) is uniformly bounded in $W^{1,1}(\Omega) \times W^{2,\infty}(\Omega)$, which implies the existence of a subsequence (not relabeled) such that

$$\begin{aligned} w^\nu \to w &\quad \text{strongly in } L^p(\Omega), \quad p < \infty, \\ V^\nu \to V &\quad \text{strongly in } W^{2,\infty}(\Omega). \end{aligned}$$

The limit $\nu \to 0$ can now be performed in the weak formulation of Eqs. (6)-(7) with $\delta = 0$.

7 Appendix: A Uniqueness Result for the (vHD) Model

Under similar conditions as in Theorem 5.1 of [7], we prove the following uniqueness result for 'subsonic' solutions of the viscous hydrodynamic model (vHD) (1)-(2) for $\delta = 0$.

Theorem 13 *Let (H1)–(H3) hold and let $\delta = 0$ and $1/\tau \equiv 0$. Moreover, let h be isentropic or isothermal and let $\varepsilon \in (0,1)$.*

Then there exists $T_0 > 0$ such that for all $T \geq T_0$ there is uniqueness of classical solutions to (6)-(9) (and to (1)-(4)) in the class of positive densities satisfying the 'subsonic' condition

$$\frac{J}{n(x)} \leq \sqrt{(1-\varepsilon)Tp'(n(x))} \quad \textit{for all} \;\; x \in \Omega. \tag{38}$$

Proof. Let n_1 and n_2 be two solutions of

$$\frac{J^2}{2n^2} + Th(n) - V - \nu\sigma(n)_x = -K, \quad \lambda^2 V_{xx} = n - C(x) \quad \text{in } \Omega,$$
$$n(0) = n_0, \quad V(0) = V_0, \quad V_x(0) = -E_0.$$

Then

$$0 < m \leq n_1, n_2 \leq M \quad \text{in } \Omega,$$

where $m, M > 0$ are independent of T. Introducing the function $\chi(n) = J^2/2n^2 + (1-\varepsilon)Th(n)$, we obtain

$$\nu\big(\sigma(n_1) - \sigma(n_2)\big)_x = -\chi(n_1) + \chi(n_2) - \varepsilon T(h(n_1) - h(n_2)) + V_1 - V_2.$$

Denoting $\rho_i = -\sigma(n_i)$, $i = 1, 2$, multiplying the above equation by $\rho_1 - \rho_2$, and integrating over $(0, x)$, we get

$$\begin{aligned}
\frac{\nu}{2}(\rho_1 - \rho_2)^2(x) &= -\int_0^x (\hat\chi(\rho_1) - \hat\chi(\rho_2))(\rho_1 - \rho_2)dx \\
&\quad - \varepsilon T \int_0^x (\hat h(\rho_1) - \hat h(\rho_2))(\rho_1 - \rho_2)dx \\
&\quad + \int_0^x (V_1 - V_2)(\rho_1 - \rho_2)dx \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where $\hat\chi(\rho) = \chi(n)$ and $\hat h(\rho) = h(n)$.

For the first integral on the right-hand side we use the mean-value theorem:

$$I_1 = \int_0^x \frac{d\hat\chi}{d\rho}(\xi)(\rho_1 - \rho_2)^2 dx.$$

In view of condition (38) we get from

$$\frac{d\hat\chi}{d\rho}(\rho) = \frac{1}{2}\Big(-\frac{J^2}{n^2} + (1-\varepsilon)Tnh'(n)\Big)n^{-(\gamma+3)/2} \geq 0,$$

where $\rho = -\sigma(n)$, the inequality

$$I_1 \leq 0.$$

The second integral I_2 is estimated as follows:

$$\begin{aligned} I_2 &= -\varepsilon T \int_0^x \frac{d\hat{h}}{d\rho}(\xi)(\rho_1-\rho_2)^2 dx \\ &= -\varepsilon T \int_0^x \frac{h'(\eta)}{\sigma'(\eta)}(\rho_1-\rho_2)^2 dx \\ &\le -2\alpha\varepsilon T m^{\alpha-2+(\gamma+1)/2} \int_0^x (\rho_1-\rho_2)^2 dx, \end{aligned}$$

where $\xi = -\sigma(\eta)$. For the last inequality we have used the fact that $\alpha - 2 + (\gamma + 1)/2 > 0$. (This inequality shows that the uniqueness proof also holds for more general functions h and σ such that h'/σ' is strictly increasing.)

For the third integral I_3 we multiply

$$(V_1 - V_2)(\eta) = \lambda^{-2} \int_0^\eta \int_0^y (n_1 - n_2)(z) dz dy$$

with $(\rho_1 - \rho_2)(\eta)$ and integrate over $(0, x)$:

$$\begin{aligned} I_3 &= \int_0^x (V_1-V_2)(\rho_1-\rho_2) d\eta \\ &= \lambda^{-2} \int_0^x \int_0^\eta \int_0^y (n_1-n_2) dz dy (\rho_1-\rho_2) d\eta \\ &\le \lambda^{-2} \Big(\int_0^x (n_1-n_2)^2 dx \Big)^{1/2} \Big(\int_0^x (\rho_1-\rho_2)^2 dx \Big)^{1/2} \\ &= \lambda^{-2} \Big(\int_0^x (\sigma^{-1}(-\rho_1) - \sigma^{-1}(-\rho_2))^2 dx \Big)^{1/2} \Big(\int_0^x (\rho_1-\rho_2)^2 dx \Big)^{1/2} \\ &= \lambda^{-2} \Big(\int_0^x \sigma'(\eta)^{-2} (\rho_1-\rho_2)^2 dx \Big)^{1/2} \Big(\int_0^x (\rho_1-\rho_2)^2 dx \Big)^{1/2} \\ &\le 2\lambda^{-2} M^{(\gamma+1)/2} \int_0^x (\rho_1-\rho_2)^2 dx. \end{aligned}$$

Putting together the above estimates, we obtain

$$(\rho_1-\rho_2)^2(x) \le \frac{4}{\nu} \Big(\lambda^{-2} M^{(\gamma+1)/2} - \alpha\varepsilon T m^{\alpha-2+(\gamma+1)/2} \Big) \int_0^x (\rho_1-\rho_2)^2 dx.$$

Choosing $T > 0$ large enough, this yields

$$(\rho_1-\rho_2)^2(x) \le 0 \quad \text{for } x \in (0,1),$$

and thus $\rho_1 = \rho_2$ in Ω. This implies $n_1 = n_2$ and $V_1 = V_2$ in Ω.

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