Analysis of the viscous quantum hydrodynamic equations for semiconductors*

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Abstract

The steady-state viscous quantum hydrodynamic model in one space dimension is studied. The model consists of the continuity equations for the particle and current densities, coupled to the Poisson equation for the electrostatic potential. The equations are derived from a Wigner-Fokker-Planck model and they contain a third-order quantum correction term and second-order viscous terms. The existence of classical solutions is proved for "weakly supersonic" quantum flows. This means that a smallness condition on the particle velocity is still needed but the bound is allowed to be larger than for classical subsonic flows. Furthermore, the uniqueness of solutions and various asymptotic limits (semiclassical and inviscid limits) are investigated. The proofs are based on a reformulation of the problem as a fourth-order elliptic equation by using an exponential variable transformation.

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1 Introduction

In semiconductor device simulation, the modeling of quantum effects becomes more and more important as the characteristic device lengths can nowadays be smaller than about 100 nm. Usually, quantum effects are modeled by using *microscopic* equations, like the Schrödinger or Wigner equation [16, 21]. In recent years, *macroscopic* quantum equations have been developed and used in quantum device simulation [1, 8, 11, 13, 17, 23, 26]. There are several advantages of a macroscopic description of semiconductors. First, the Wigner or Schrödinger equation is computationally very expensive, whereas for fluid-type models efficient numerical algorithms are available. Second, as semiconductor devices are modeled in bounded domains, it is easier to find physically relevant boundary conditions for the macroscopic variables than for the Wigner function or the wave function.

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It is well known since 1927 that there exists a fluiddynamical formulation of the Schrödinger equation [20]. In fact, by separating the real and the complex part of the single-state Schrödinger equation, the electron density $n(x,t)$ and current density $J(x,t)$ are satisfying (formally) the scaled Madelung equations

$$\begin{aligned} \partial_t n + \operatorname{div} J &= 0, \\ \partial_t J + \operatorname{div}\left(\frac{J\otimes J}{n}\right) - n\nabla V - \frac{\varepsilon^2}{2} n \nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right) &= 0, \end{aligned}$$

where $V(x,t)$ is the electrostatic potential, satisfying the Poisson equation

$$\lambda^2 \Delta V = n - C(x),$$

with the concentration of fixed background charges $C(x)$. The physical constants are the (scaled) Planck constant ε and the Debye length λ. The symbol $J\otimes J$ denotes the tensor with components $J_i J_k$. The Madelung equations can be also derived from the Wigner equation by a moment method [11]. This allows to incorporate temperature effects (for many-electron ensembles) and a relaxation-time term. Temperature effects can be also obtained from a mixed-state Schrödinger approach [12]. This yields the equation

$$\partial_t J + \operatorname{div}\left(\frac{J\otimes J}{n}\right) + T\nabla n - n\nabla V - \frac{\varepsilon^2}{2} n \nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right) = -\frac{J}{\tau},$$

where T is the (scaled) temperature and τ the (scaled) momentum relaxation time. This equation, together with the equation for the electron density and the Poisson equation, is called the quantum hydrodynamic model. For vanishing scaled Planck constant $\varepsilon = 0$ (and vanishing relaxation-time term), the above equation is equal to the classical Euler momentum equation for charged particles. The relaxation term models interactions of the electrons with the phonons of the semiconductor crystal lattice.

By assuming a Fokker-Planck collision operator in the Wigner equation [3, 4], the moment method of Gardner [11] yields viscous terms in the conservation equations:

$$\begin{aligned} \partial_t n + \operatorname{div} J &= \nu \Delta n, \\ \partial_t J + \operatorname{div}\left(\frac{J\otimes J}{n}\right) + T\nabla n - n\nabla V - \frac{\varepsilon^2}{2} n \nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right) &= -\frac{J}{\tau} + \nu\Delta J, \end{aligned}$$

where $\nu > 0$ is the viscosity constant. We call these equations, coupled to the Poisson equation, the *viscous quantum hydrodynamic model*. We refer to the appendix for details on the derivation (and scaling) of this model. For vanishing scaled Planck constant $\varepsilon = 0$, we obtain a parabolic regularization of the hyperbolic hydrodynamic equations. This regularization has been employed to prove the existence of solutions to the one-dimensional Euler equations [18].

The objective of this paper is to analyze the one-dimensional stationary version of the viscous quantum hydrodynamic model:

$$J_x = \nu n_{xx}, \tag{1}$$

$$\left(\frac{J^2}{n}\right)_x + T n_x - n V_x - \frac{\varepsilon^2}{2} n \left(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\right)_x = -\frac{J}{\tau} + \nu J_{xx}, \tag{2}$$

$$\lambda^2 V_{xx} = n - C(x) \quad \text{in } (0,1). \tag{3}$$

We choose the physically motivated boundary conditions

$$n(0) = n(1) = 1, \quad n_x(0) = n_x(1) = 0, \quad V(0) = V_0, \quad J(0) = J_0, \tag{4}$$

$$V_0 = -\left(2\nu^2 + \frac{\varepsilon^2}{2}\right)(\sqrt{n})_{xx}(0) + \frac{J_0^2}{2}. \tag{5}$$

The last boundary condition can be interpreted as a Dirichlet condition for the Bohm potential at $x = 0$. Indeed, as the electrostatic potential is only defined up to an additive constant, this constant can be choosen such that $(\sqrt{n})_{xx}(0) = \alpha$ holds for any $\alpha \in \mathbb{R}$ (often $\alpha = 0$, see, e.g., [17]).

Notice that we prescribe the current density but *not* the applied voltage $V(1) - V(0)$. Given J_0, the applied voltage can be computed from the solution of the above boundary-value problem, which gives a well-defined current-voltage characteristic.

Concerning the mathematical analysis of the (inviscid) quantum hydrodynamic model, only partial results are available. It has been shown (in one or several space dimensions) that there exists a weak solution to (1)-(3) with $\nu = 0$ (and for various choices of the boundary conditions), if a subsonic-type condition of the form

$$\frac{J_0}{n} < \sqrt{T + \frac{\varepsilon^2}{4}} \quad \text{in } (0, 1) \tag{6}$$

is satisfied, i.e., if the current density is small enough [10, 14, 15, 25]. (Recall that an Euler flow is called subsonic if $J_0/n < \sqrt{T}$.) Moreover, for special boundary conditions, it has been proved [10] that the quantum hydrodynamic equations do *not* possess a weak solution if the current density is sufficiently large. The main difficulty in the existence analysis (besides of the mathematical treatment of the third-order quantum term) is the convection term $(J^2/n)_x$. In fact, in [10] it has been shown that this term forces the particle density to cavitate if the current density is large enough. Without this term, the stationary equations (still with $\nu = 0$) become the quantum drift-diffusion model for which a solution exists for any data [5]. The third-order quantum term possesses a regularizing effect since the condition (6) allows for slightly "supersonic" flows [14].

The question arises if, as in the case of the *classical* hydrodynamic equations, the viscous terms regularize the equations in such a way that the existence of solutions can be proved for all values of the current densities. In this paper we give a partial answer to this question.

More precisely, we prove the existence of classical solutions if the following "weakly supersonic" condition holds:

$$\frac{J_0}{n} < \frac{1}{\sqrt{2}}\sqrt{T + \frac{\varepsilon^2}{16} + \frac{\nu}{\tau}} \quad \text{in } (0, 1). \tag{7}$$

Thus, the current density is allowed to be large if either the viscosity ν is large or the (scaled) relaxation time is small enough. The reason for this restriction comes from the fact that, roughly speaking, the viscous term νJ_{xx} can be reformulated (up to a factor) as the third-order quantum term. In fact, integrating (1) and using the boundary condition for $J(0)$ we obtain

$$J = \nu n_x + J_0,$$

which gives

$$\left(\frac{J^2}{n}\right)_x - \nu J_{xx} = -2\nu^2 n\left(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\right)_x + \left(\frac{J_0^2}{n}\right)_x + 2\nu J_0(\log n)_{xx},$$

and therefore, we can reformulate equation (2) formally as

$$\left(\frac{J_0^2}{n}\right)_x + \left(T + \frac{\nu}{\tau}\right) n_x - nV_x - \left(2\nu^2 + \frac{\varepsilon^2}{2}\right) n \left(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\right)_x = -\frac{J_0}{\tau} - 2J_0\nu(\log n)_{xx}. \quad (8)$$

This formulation shows that the viscous terms indeed regularize the equations (as the coefficient of the quantum term becomes larger) but there is still a convection term which may force the solutions to cavitate for large values of the current density J_0. (Unfortunately, the method in [10] cannot be applied to prove this conjecture rigorously.) Thus we expect a similar restriction on the current density as (6), but allowing for larger current densities.

We notice that the factor $1/\sqrt{2}$ in (7) is needed in order to estimate the last term in (8). However, for sufficiently large ν/τ, we can allow for "supersonic" current densities $J_0/n > \sqrt{T + \varepsilon^2/4}$. We also remark that the above argument only holds in one space dimension. For the multi-dimensional problem, no results are available. This situation is similar to the inviscid quantum hydrodynamic equations, where mathematical results are essentially only available in one space dimension (except [15]).

The above reformulation (8) is the main idea of this paper, together with the key estimate (9) below.

In order to prove the existence of solutions to (8) and (3)-(5), we rewrite (8) as a fourth-order equation and employ the technique of exponential transformation of variables $n = e^u$ as in [14] (first used in [6]). The existence of a weak solution $u \in H^2(0,1)$ provides a weak solution $n = e^u$ which is *strictly positive*. Notice that maximum principle arguments can generally be *not* applied to third- or fourth-order equations, and therefore, the exponential transformation of variables circumvents this fact to prove positive lower bounds for the particle density. Our results can be easily extended to Dirichlet boundary conditions $n(0) \neq n(1)$, following the technique used in [14], but we use (4) for the sake of a smoother presentation. The existence of solutions is proved in Section 3.

As a second result we prove in Section 4 the uniqueness of solutions of (1)-(3) for sufficiently small parameters ν, ε and J_0. It is well known in semiconductor problems that uniqueness of solutions can only be expected for sufficiently small current densities since there are devices based on multiple solutions.

Using $u = \log n$ as a test function in the weak formulation of the fourth-order equation, we obtain the estimate

$$\left(\nu^2 + \frac{\varepsilon^2}{4}\right)\|u_{xx}\|_{L^2} + \left(T + \frac{\nu}{\tau}\right)\|u_x\|_{L^2} \le K, \quad (9)$$

where $K > 0$ is a constant which does not depend on u, ν or ε (see Lemma 3.1). This inequality is the key estimate of this paper. It provides an H^1 bound for u independently of ν and ε. This allows to perform the inviscid limit $\nu \to 0$ and the semiclassical limit $\varepsilon \to 0$. These limits as well as the combined limit $\nu^2 + \varepsilon^2 \to 0$ are shown in Section 5.

Finally, the Appendix is concerned with a sketch of the derivation of the model and its scaling.

2 Reformulation of the equations and statement of the main results

We reformulate the system (1)-(2) as an elliptic fourth-order equation. After integration of (1) and substitution into (2) we obtain the expression (8). When we divide (8) by n and differentiate with respect to x, this equation is formally equivalent to

$$\begin{aligned} & -\left(2\nu^2+\frac{\varepsilon^2}{2}\right)\left(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\right)_{xx}+\left(T+\frac{\nu}{\tau}\right)(\log n)_{xx} \\ & = \frac{n-C}{\lambda^2}+J_0^2\left(\frac{n_x}{n^3}\right)_x-2J_0\nu\left(\frac{1}{n}(\log n)_{xx}\right)_x-\frac{J_0}{\tau}\left(\frac{1}{n}\right)_x, \end{aligned} \tag{10}$$

where we have used the Poisson equation (3). The electrostatic potential can be recovered from (8), after division by n and integration:

$$\begin{aligned} V(x) &= -\left(2\nu^2+\frac{\varepsilon^2}{2}\right)\frac{(\sqrt{n})_{xx}}{\sqrt{n}}(x)+\left(T+\frac{\nu}{\tau}\right)\log n(x)+\frac{J_0^2}{2n(x)^2} \\ &\quad +\frac{J_0}{\tau}\int_0^x\frac{ds}{n(s)}+2J_0\nu\frac{n_x}{n^2}(x)+2J_0\nu\int_0^x\frac{n_x^2}{n^3}ds. \end{aligned} \tag{11}$$

The integration constant vanishes due to the boundary condition (5). Now we rewrite the fourth-order term as

$$n\left(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\right)_{xx}=\frac{1}{2}(n(\log n)_{xx})_{xx}$$

and introduce as in [14] the exponential variable $n=e^u$ to arrive to the problem

$$\begin{aligned} & -\left(\nu^2+\frac{\varepsilon^2}{4}\right)\left(u_{xx}+\frac{u_x^2}{2}\right)_{xx}+\left(T+\frac{\nu}{\tau}\right)u_{xx} \\ & = \lambda^{-2}(e^u-C)+J_0^2(e^{-2u}u_x)_x-2J_0\nu(e^{-u}u_{xx})_x-\frac{J_0}{\tau}(e^{-u})_x, \end{aligned} \tag{12}$$

$$\begin{aligned} V(x) &= -\left(\nu^2+\frac{\varepsilon^2}{4}\right)\left(u_{xx}+\frac{u_x^2}{2}\right)(x)+\left(T+\frac{\nu}{\tau}\right)u(x)+\frac{J_0^2}{2}e^{-2u(x)} \\ &\quad +\frac{J_0}{\tau}\int_0^x e^{-u(s)}ds+2J_0\nu e^{-u(x)}u_x(x)+2J_0\nu\int_0^x e^{-u}u_x^2ds. \end{aligned} \tag{13}$$

Equation (12) has to be solved in the interval $(0,1)$ with the boundary conditions

$$u(0)=u(1)=0, \quad u_x(0)=u_x(1)=0. \tag{14}$$

The problems (1)-(5) and (12)-(14) are equivalent for classical solutions if $n>0$ in $(0,1)$. Indeed, we have already shown that a classical solution to (1)-(5) with $n>0$ in $(0,1)$ provides via $u=\log n$ a classical solution to (12)-(14). Conversely, let (u,V) be a classical solution to (12)-(14). Setting $n=e^u$ gives $n>0$ in $(0,1)$, and the equations (10) and (11) hold. Differentiating (11) twice, multiplying by n and comparing with (10) yields the Poisson equation (3). Then, differentiating (11) once and multiplying the resulting equation by n, we obtain (2). Finally, the boundary condition (5) follows from (11) using (4). Thus it is sufficient to prove the existence of solutions to (12)-(14).

Our existence result is as follows.

Theorem 2.1 (Existence and uniqueness) *Let $C \in L^\infty(0,1)$, $C > 0$ in $(0,1)$, $0 < \gamma < 1$, and*

$$0 < J_0 \le \frac{\gamma}{\sqrt{2}} e^{-M(\gamma)} \sqrt{T + \frac{\varepsilon^2}{16} + \frac{\nu}{\tau}}, \tag{15}$$

where the constant $M(\gamma) > 0$ is defined in (25). Then there exists a classical solution (n, J, V) to (1)-(5) such that $n(x) \ge e^{-M(\gamma)} > 0$ for $x \in (0,1)$. Furthermore, if $J_0 > 0$ and $\nu^2 + \varepsilon^2$ are sufficiently small, the problem (1)-(5) has a unique solution.

The restriction (15) implies (7) since

$$\frac{J_0}{n} \le J_0 e^{M(\gamma)} < \frac{1}{\sqrt{2}} \sqrt{T + \frac{\varepsilon^2}{16} + \frac{\nu}{\tau}}.$$

We are able to prove the semiclassical limit $\varepsilon \to 0$, the inviscid limit $\nu \to 0$ and the combined semiclassical-inviscid limit $\varepsilon \to 0$ and $\nu \to 0$. We refer to the appendix for the physical assumptions on the parameters, which correspond to such limits.

Theorem 2.2 (Inviscid limit) *Let (n_ν, J_ν, V_ν) be a solution to (1)-(5) and assume that condition (15) holds for $\nu = 0$. Then, as $\nu \to 0$, maybe for a subsequence,*

$$\begin{aligned} n_\nu &\rightharpoonup n && \text{weakly in } H^2(0,1), \\ V_\nu &\rightharpoonup V && \text{weakly in } H^4(0,1), \\ J_\nu &\rightharpoonup J && \text{weakly in } H^1(0,1), \end{aligned}$$

and (n, J, V) is a (classical) solution of the quantum hydrodynamic equations

$$\begin{aligned} &\left(\frac{J^2}{n} + Tn\right)_x - nV_x - \frac{\varepsilon^2}{2} n \left(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\right)_x = -\frac{J}{\tau}, \\ &J = J_0, \quad \lambda^2 V_{xx} = n - C \quad \text{in } (0,1), \\ &n(0) = n(1) = 1, \quad n_x(0) = n_x(1) = 0, \quad V(0) = V_0. \end{aligned}$$

Theorem 2.3 (Semiclassical limit) *Let $(n_\varepsilon, J_\varepsilon, V_\varepsilon)$ be a solution to (1)-(5) and assume that condition (15) holds for $\varepsilon = 0$. Then, as $\varepsilon \to 0$, maybe for a subsequence,*

$$\begin{aligned} n_\varepsilon &\rightharpoonup n && \text{weakly in } H^2(0,1), \\ V_\varepsilon &\rightharpoonup V && \text{weakly in } H^4(0,1), \\ J_\varepsilon &\rightharpoonup J && \text{weakly in } H^1(0,1), \end{aligned}$$

and (n, J, V) is a (classical) solution of

$$\begin{aligned} &\left(\frac{J^2}{n} + Tn\right)_x - nV_x = \nu J_{xx} - \frac{J}{\tau}, \quad J_x = \nu n_{xx}, \quad \lambda^2 V_{xx} = n - C \quad \text{in } (0,1), \\ &n(0) = n(1) = 1, \quad n_x(0) = n_x(1) = 0, \quad V(0) = V_0, \quad J(0) = J_0. \end{aligned}$$

Theorem 2.4 (Semiclassical-inviscid limit) *Let $\delta = \nu^2 + \varepsilon^2/4$, $V_0 = J_0^2/2$, let $(n_\delta, J_\delta, V_\delta)$ be a solution to (1)-(5), and assume that condition (15) holds for $\delta = 0$. Then, as $\delta \to 0$, maybe for a subsequence (see Remark 5.2),*

$$n_\delta \rightharpoonup n \qquad \text{weakly in } H^1(0,1), \tag{16}$$
$$V_\delta \rightharpoonup V \qquad \text{weakly in } H^3(0,1), \tag{17}$$
$$J_\delta \rightharpoonup J \qquad \text{weakly in } H^1(0,1), \tag{18}$$

and (n, J, V) is a (classical) solution of the hydrodynamic equations

$$\left(\frac{J^2}{n} + Tn\right)_x - nV_x = -\frac{J}{\tau}, \quad J = J_0, \quad \lambda^2 V_{xx} = n - C \quad \text{in } (0,1), \tag{19}$$
$$n(0) = n(1) = 1, \quad V(0) = V_0. \tag{20}$$

3 Existence of solutions

As usual, we call $u \in H_0^2(0,1)$ a *weak solution* of (12), (14) if for all $\psi \in H_0^2(0,1)$ it holds

$$\begin{aligned} &-\Big(\nu^2 + \frac{\varepsilon^2}{4}\Big) \int_0^1 \Big(u_{xx} + \frac{1}{2}u_x^2\Big)\psi_{xx}dx - \Big(T + \frac{\nu}{\tau}\Big) \int_0^1 u_x\psi_x dx \\ &= \ 2J_0\nu \int_0^1 u_{xx}e^{-u}\psi_x dx - J_0^2 \int_0^1 u_x e^{-2u}\psi_x dx + \frac{J_0}{\tau}\int_0^1 e^{-u}\psi_x dx + \frac{1}{\lambda^2}\int_0^1 (e^u - C)\psi dx. \end{aligned} \tag{21}$$

In order to prove Theorem 2.1 we consider the following truncated problem:

$$\begin{aligned} &-\Big(\nu^2 + \frac{\varepsilon^2}{4}\Big) \int_0^1 \Big(u_{xx} + \frac{1}{2}u_x^2\Big)\psi_{xx}dx - \Big(T + \frac{\nu}{\tau}\Big) \int_0^1 u_x\psi_x dx \\ &= \ 2J_0\nu \int_0^1 u_{xx}e^{-u_M}\psi_x dx - J_0^2 \int_0^1 u_x e^{-2u_M}\psi_x dx + \frac{J_0}{\tau}\int_0^1 e^{-u}\psi_x dx + \frac{1}{\lambda^2}\int_0^1 (e^u - C)\psi dx, \end{aligned} \tag{22}$$

where $M = M(\gamma) > 0$ is the constant from (15) defined in (25) below and $u_M = \min\{M, \max\{-M, u\}\}$. The following lemma is the key result of this paper.

Lemma 3.1 (H^2-Estimate). *Let $u \in H_0^2(0,1)$ be a solution of (22) and let (15) hold for some $0 < \gamma < 1$. Then*

$$\Big(\nu^2 + \frac{\varepsilon^2}{4}\Big)\|u_{xx}\|_{L^2}^2 + \Big(T + \frac{\nu}{\tau}\Big)\|u_x\|_{L^2}^2 \le K(\gamma), \tag{23}$$

where $K(\gamma) > 0$ is independent of u, ν, and ε (see (27) for its definition). In particular, it follows

$$\|u\|_{L^\infty} \le M(\gamma), \tag{24}$$

where

$$M(\gamma) = \sqrt{\frac{K(\gamma)}{T}}. \tag{25}$$

Proof. We use $\psi = u$ as a test function in the weak formulation of (22) to obtain

$$\begin{aligned}
&\Big(\nu^2+\frac{\varepsilon^2}{4}\Big)\int_0^1\Big(u_{xx}^2+\frac{1}{2}u_x^2u_{xx}\Big)dx+\Big(T+\frac{\nu}{\tau}\Big)\int_0^1 u_x^2dx\\
&= -2J_0\nu\int_0^1 u_{xx}e^{-u_M}u_xdx+J_0^2\int_0^1 u_x^2e^{-2u_M}dx-\frac{J_0}{\tau}\int_0^1 e^{-u}u_xdx-\frac{1}{\lambda^2}\int_0^1 u(e^u-C)dx\\
&= I_1+I_2+I_3+I_4. \qquad (26)
\end{aligned}$$

We estimate the right-hand side term by term. By Young's inequality,

$$\begin{aligned}
I_1 &= -2J_0\nu\int_0^1 u_{xx}e^{-u_M}u_xdx \le 2J_0e^M\nu\int_0^1|u_{xx}||u_x|dx\\
&\le (1-\eta)\nu^2\int_0^1 u_{xx}^2dx+\frac{J_0^2e^{2M}}{1-\eta}\int_0^1 u_x^2dx,
\end{aligned}$$

where $0<\eta<(1-\gamma^2)/(1-\gamma^2/2)$. Furthermore,

$$I_2 = J_0^2\int_0^1 u_x^2e^{-2u_M}dx \le J_0^2e^{2M}\int_0^1 u_x^2dx.$$

Due to the boundary conditions (14), the third integral vanishes: $I_3=0$. It is not difficult to see that $1/e+\|C\log C\|_{L^\infty}$ is an upper bound for the function $u\mapsto -u(e^u-C(x))$, $u\in\mathbb{R}$, for any $x\in(0,1)$. Here we use the assumption that the concentration $C(x)$ is positive. Therefore,

$$I_4 \le \lambda^{-2}(e^{-1}+\|C\log C\|_{L^\infty}).$$

Noticing that the integral

$$\int_0^1 u_x^2u_{xx}dx = \frac{1}{3}(u_x^3(1)-u_x^3(0)) = 0$$

vanishes, due to the boundary conditions (14), we conclude that (26) can be estimated as

$$\Big(\eta\nu^2+\frac{\varepsilon^2}{4}\Big)\|u_{xx}\|_2^2+\Big(T+\frac{\nu}{\tau}-\frac{2-\eta}{1-\eta}J_0^2e^{2M}\Big)\|u_x\|_2^2 \le \lambda^{-2}(e^{-1}+\|C\log C\|_{L^\infty}).$$

In view of condition (15) we obtain

$$T+\frac{\nu}{\tau}-\frac{2-\eta}{1-\eta}J_0^2e^{2M} \ge \Big(1-\frac{2-\eta}{1-\eta}\frac{\gamma^2}{2}\Big)\Big(T+\frac{\nu}{\tau}\Big)-\frac{2-\eta}{1-\eta}\frac{\gamma^2}{2}\frac{\varepsilon^2}{16}.$$

Using the Poincaré inequality

$$\frac{\varepsilon^2}{16}\int_0^1 u_x^2dx \le \frac{\varepsilon^2}{4}\int_0^1 u_{xx}^2dx,$$

this gives

$$\Big[\eta\nu^2+\Big(1-\frac{\gamma^2(2-\eta)}{2(1-\eta)}\Big)\frac{\varepsilon^2}{4}\Big]\|u_{xx}\|_{L^2}^2+\Big(1-\frac{\gamma^2(2-\eta)}{2(1-\eta)}\Big)\Big(T+\frac{\nu}{\tau}\Big)\|u_x\|_{L^2}^2 \le \lambda^{-2}(e^{-1}+\|C\log C\|_{L^\infty})$$

or

$$\Big(\nu^2+\frac{\varepsilon^2}{4}\Big)\|u_{xx}\|_{L^2}^2+\Big(T+\frac{\nu}{\tau}\Big)\|u_x\|_{L^2}^2\le K(\gamma),$$

where

$$K(\gamma)=\frac{1}{\lambda^2}\Big(\frac{1}{e}+\|C\log C\|_{L^\infty}\Big)\min\Big\{\eta,1-\frac{\gamma^2(2-\eta)}{2(1-\eta)}\Big\}^{-1}. \tag{27}$$

Notice that $1-\gamma^2(2-\eta)/(2(1-\eta))>0$ due to the choice of η. Finally, from the Poincaré-Sobolev estimate,

$$\|u\|_{L^\infty}\le\|u_x\|_{L^2}\le M(\gamma),$$

where $M(\gamma)=\sqrt{K(\gamma)/T}$. This proves the lemma. $\square$

Lemma 3.2 *Under the assumptions of Lemma 3.1, there exists a solution $u\in H_0^2(0,1)$ of (21).*

Proof. The existence of a solution of the problem (21) is shown by using the Leray-Schauder fixed-point theorem. For this, we consider the following linear problem for given $w\in H_0^1(0,1)$ with test functions $\psi\in H_0^2(0,1)$:

$$\begin{aligned}
&-\Big(\nu^2+\frac{\varepsilon^2}{4}\Big)\int_0^1 u_{xx}\psi_{xx}dx-\sigma\Big(\nu^2+\frac{\varepsilon^2}{4}\Big)\int_0^1\frac{1}{2}w_x^2\psi_{xx}dx-\Big(T+\frac{\nu}{\tau}\Big)\int_0^1 u_x\psi_x dx\\
&= -2J_0\sigma\nu\int_0^1 w_x(e^{-w}\psi_x)_x dx-J_0^2\sigma\int_0^1 w_xe^{-2w}\psi_x dx+\frac{J_0}{\tau}\sigma\int_0^1 e^{-w}\psi_x dx\\
&\quad+\frac{\sigma}{\lambda^2}\int_0^1(e^w-C)\psi dx,
\end{aligned}\tag{28}$$

where $\sigma\in[0,1]$. We define the bilinear form

$$a(u,\psi)=\Big(\nu^2+\frac{\varepsilon^2}{4}\Big)\int_0^1 u_{xx}\psi_{xx}dx+\Big(T+\frac{\nu}{\tau}\Big)\int_0^1 u_x\psi_x dx, \tag{29}$$

and the linear functional

$$\begin{aligned}
F(\psi) &= -\sigma\Big(\nu^2+\frac{\varepsilon^2}{4}\Big)\int_0^1\frac{1}{2}w_x^2\psi_{xx}dx+2J_0\sigma\nu\int_0^1 w_x(e^{-w}\psi_x)_x dx\\
&\quad+J_0^2\sigma\int_0^1 w_xe^{-2w}\psi_x dx-\frac{J_0}{\tau}\sigma\int_0^1 e^{-w}\psi_x dx-\frac{\sigma}{\lambda^2}\int_0^1\psi(e^w-C)dx.
\end{aligned}\tag{30}$$

Since the bilinear form $a(u,\psi)$ is continuous and coercive on $H_0^2(0,1)\times H_0^2(0,1)$ and the linear functional F is continuous on $H_0^2(0,1)$, we can apply the Lax-Milgram theorem to obtain the existence of a solution $u\in H_0^2(0,1)$ of (28). Thus, the operator

$$S: H_0^1(0,1)\times[0,1]\to H_0^1(0,1),\quad (w,\sigma)\mapsto u,$$

is well-defined. Moreover, it is continuous and compact since the embedding $H_0^2(0,1)\hookrightarrow H_0^1(0,1)$ is compact. Furthermore, $S(w,0)=0$. Following the steps of the proof of Lemma 3.1, we can show that $\|u\|_{H_0^2}\le$ const. for all $(u,\sigma)\in H_1^0(0,1)\times[0,1]$ satisfying $S(u,\sigma)=u$. Therefore, the existence of a fixed point u with $S(u,1)=u$ follows from the Leray-Schauder fixed-point theorem. This fixed point is a solution of (22) and, in fact, also of (21) since $|u(x)|\le M(\gamma)$. $\square$

Theorem 3.3 *Under the assumptions of Lemma 3.1, there exists a solution $(u, V) \in H^4(0,1) \times H^2(0,1)$ of (12)-(14).*

Proof. Let u be a weak solution of (21) or (12). Since $u \in H_0^2(0,1)$, it holds $u_x^2 \in H_0^1(0,1)$ and $(e^{-u}u_{xx})_x \in H^{-1}(0,1)$. Then, from (12), we infer $u_{xxxx} \in H^{-1}(0,1)$. Hence, there exists $w \in L^2(0,1)$ such that $w_x = u_{xxxx}$. This implies $u_{xxx} = w + \text{const.} \in L^2(0,1)$ and, by (12), $u_{xxxx} \in L^2(0,1)$. This allows us to conclude that $u \in H^4(0,1)$ and from the regularity of u and from (13) follows the regularity of V. □

4 Uniqueness of solutions

Theorem 4.1 *If ν, ε, and $J_0 > 0$ are sufficiently small, the problem (12)-(14) has a unique solution.*

Proof. We proceed similarly as in [14]. Let u, $v \in H_0^2(0,1)$ be weak solutions of (12). We observe that, in view of the boundary conditions for u_x,

$$u_x^2(x) = 2\int_0^x u_x(s)u_{xx}(s)ds \le 2\|u_x\|_{L^2}\|u_{xx}\|_{L^2}$$

and thus

$$\|u_x\|_{L^\infty} \le \frac{\alpha}{2}\|u_x\|_{L^2} + \frac{1}{2\alpha}\|u_{xx}\|_{L^2}$$

for all $\alpha > 0$. By Lemma 3.1 we obtain

$$\|u_x\|_{L^\infty} \le \Big(\frac{\alpha}{2\sqrt{\nu^2+\varepsilon^2/4}} + \frac{1}{2\alpha\sqrt{T+\nu/\tau}}\Big)\sqrt{K(\gamma)}.$$

Choosing $\alpha = \sqrt{(T+\nu/\tau)/K(\gamma)}$ then gives

$$\|u_x\|_{L^\infty} \le \frac{\sqrt{T+\nu/\tau}}{2\sqrt{\nu^2+\varepsilon^2/4}} + \frac{K(\gamma)}{T+\nu/\tau}.$$

Now we choose ν and ε so small that

$$\sqrt{\nu^2+\varepsilon^2/4} \le \frac{(T+\nu/\tau)^{3/2}}{2K(\gamma)}.$$

As T is positive, such a choice is possible. This implies

$$\|u_x\|_{L^\infty} \le \sqrt{\frac{T+\nu/\tau}{\nu^2+\varepsilon^2/4}}.$$

A similar estimate can be obtained for v_x. Therefore

$$\|(u+v)_x\|_{L^\infty} \le 2\sqrt{\frac{T+\nu/\tau}{\nu^2+\varepsilon^2/4}}. \tag{31}$$

Now we start to estimate the difference $u-v$. The weak formulation of the difference of the equations satisfied by u and v, with the test function $u-v$, reads as follows:

$$\begin{aligned}
\Big(\nu^2+\frac{\varepsilon^2}{4}\Big)\int_0^1 (u-v)_{xx}^2 + \Big(T+\frac{\nu}{\tau}\Big)\int_0^1 (u-v)_x^2 dx + \frac{1}{2}\Big(\nu^2+\frac{\varepsilon^2}{4}\Big)\int_0^1 (u_x^2-v_x^2)(u-v)_{xx}dx \\
= \quad -2J_0\nu\int_0^1 (u_{xx}e^{-u}-v_{xx}e^{-v})(u-v)_x dx - \frac{1}{2}J_0^2\int_0^1 (e^{-2u}-e^{-2v})_x(u-v)_x dx \\
-\frac{1}{\lambda^2}\int_0^1 (e^u-e^v)(u-v)dx + \frac{J_0}{\tau}\int_0^1 (e^{-u}-e^{-v})(u-v)_x dx \\
= \quad I_1+I_2+I_3+I_4. \qquad (32)
\end{aligned}$$

The mean value theorem and the estimate (24) with $M=M(\gamma)$ yields $|e^{-u(x)}-e^{-v(x)}|\le e^M|u(x)-v(x)|$. Therefore, using Poincaré's inequality,

$$I_4 \le \frac{J_0}{\tau}e^M\|u-v\|_{L^2}\|(u-v)_x\|_{L^2} \le \frac{J_0}{\tau}e^M\|(u-v)_x\|_{L^2}^2.$$

The monotonicity of $x\mapsto e^x$ implies $I_3\le 0$. For the estimate of the second integral we obtain similarly as above

$$I_2 = J_0^2\int_0^1 [e^{-2u}(u-v)_x^2 + (e^{-2u}-e^{-2v})v_x(u-v)_x]dx \le J_0^2K_1e^{2M}\|(u-v)_x\|_{L^2}^2,$$

where the constant $K_1>0$ depends on $\|v_x\|_{L^\infty}$. Finally, we write for the first integral

$$I_1 = -2J_0\nu\int_0^1 [e^{-u}(u-v)_{xx} + (e^{-u}-e^{-v})v_{xx}](u-v)_x dx.$$

As we do not have an L^∞ bound for v_{xx}, we integrate by parts in the second addend:

$$\begin{aligned}
I_1 \quad &= \quad -2J_0\nu\int_0^1 [e^{-u}(u-v)_{xx}(u-v)_x - e^{-u}v_x(u-v)_x^2 - (e^{-u}-e^{-v})v_x^2(u-v)_x \\
&\quad - (e^{-u}-e^{-v})v_x(u-v)_{xx}]dx \\
&\le \quad \frac{\nu^2}{2}\|(u-v)_{xx}\|_{L^2}^2 + J_0^2e^{2M}K_2\|(u-v)_x\|_{L^2}^2,
\end{aligned}$$

and $K_2>0$ depends on $\|v_x\|_{L^\infty}$. In the last step we have used again the mean value theorem and Young's and Poincaré's inequalities. We conclude from (32)

$$\begin{aligned}
I \quad &= \quad \Big(\nu^2+\frac{\varepsilon^2}{4}\Big)\int_0^1 (u-v)_{xx}^2 + \Big(T+\frac{\nu}{\tau}\Big)\int_0^1 (u-v)_x^2 dx \\
&\quad + \frac{1}{2}\Big(\nu^2+\frac{\varepsilon^2}{4}\Big)\int_0^1 (u+v)_x(u-v)_x(u-v)_{xx}dx \\
&\le \quad \frac{\nu^2}{2}\|(u-v)_{xx}\|_{L^2}^2 + J_0\Big(\frac{e^M}{\tau} + J_0K_1e^{2M} + J_0K_2e^{2M}\Big)\|(u-v)_x\|_{L^2}^2. \qquad (33)
\end{aligned}$$

The estimate of the last integral of the left-hand side of (32) is more delicate. We use the bound (31):

$$
\begin{aligned}
I \;\geq\; & \frac{1}{2}\Big(\nu^2+\frac{\varepsilon^2}{4}\Big)\int_0^1 (u-v)_{xx}^2 + \frac{1}{2}\Big(T+\frac{\nu}{\tau}\Big)\int_0^1 (u-v)_x^2 dx \\
& + \frac{1}{2}\int_0^1 \Big(\sqrt{\nu^2+\varepsilon^2/4}\,|(u-v)_{xx}| - \sqrt{T+\nu/\tau}\,|(u-v)_x|\Big)^2 dx \\
& + \sqrt{\nu^2+\varepsilon^2/4}\sqrt{T+\nu/\tau}\int_0^1 |(u-v)_x||(u-v)_{xx}|\Big(1-\frac{1}{2}\sqrt{\frac{\nu^2+\varepsilon^2/4}{T+\nu/\tau}}|(u+v)_x|\Big)dx \\
\geq\; & \frac{1}{2}\Big(\nu^2+\frac{\varepsilon^2}{4}\Big)\int_0^1 (u-v)_{xx}^2 + \frac{1}{2}\Big(T+\frac{\nu}{\tau}\Big)\int_0^1 (u-v)_x^2 dx.
\end{aligned}
$$

Thus putting together this estimate and (33), for sufficiently small $J_0 > 0$, we arrive to

$$
\frac{\varepsilon^2}{8}\int_0^1 (u-v)_{xx}^2 + \frac{1}{2}\Big(T+\frac{\nu}{\tau}\Big)\int_0^1 (u-v)_x^2 dx \leq 0.
$$

This implies $u - v = 0$ in $(0,1)$. $\square$

For the proof of Theorem 2.1 it remains to show that the solution (u, V) of (12)-(14) provides a solution (n, V) of (1)-(5). Then both formulations are equivalent and the uniqueness of solutions of (1)-(5) follows.

Let (u, V) be the unique solution of (12)-(14) and set $n = e^u$. Then we obtain (10) and (11). We differentiate (11) twice with respect to x:

$$
\begin{aligned}
V_{xx} \;=\; & -\Big(2\nu^2+\frac{\varepsilon^2}{2}\Big)\Big(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\Big)_{xx} + \Big(T+\frac{\nu}{\tau}\Big)(\log n)_{xx} + \frac{J_0^2}{2}\Big(\frac{1}{n^2}\Big)_{xx} \\
& + \frac{J_0}{\tau}\Big(\frac{1}{n}\Big)_x + 2J_0\nu\Big(\frac{n_x}{n^2}\Big)_{xx} + 2J_0\nu\Big(\frac{n_x^2}{n^3}\Big)_x,
\end{aligned}
$$

and, comparing with (10), Poisson's equation (3) follows. Furthermore, from (11) it holds:

$$
V(0) = -\Big(2\nu^2+\frac{\varepsilon^2}{2}\Big)\Big(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\Big)(0) + \frac{J_0^2}{2} = V_0,
$$

taking into account (5).

Now we differentiate (11) with respect to x and multiply the resulting equation with n:

$$
nV_x = -\Big(2\nu^2+\frac{\varepsilon^2}{2}\Big)n\Big(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\Big)_x + \Big(T+\frac{\nu}{\tau}\Big)n_x - J_0^2\frac{n_x}{n^2} + \frac{J_0}{\tau} - 2J_0\nu\Big(\frac{n_x^2}{n^2}-\frac{n_{xx}}{n}\Big).
$$

Introducing $J(x) := \nu n_x(x) + J_0$, equation (1) follows after differentiation. Finally, from

$$
-2\nu^2 n\Big(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\Big)_x + \frac{\nu}{\tau}n_x - 2J_0\nu\Big(\frac{n_x^2}{n^2}-\frac{n_{xx}}{n}\Big) + \frac{J_0}{\tau} - J_0^2\Big(\frac{n_x}{n^2}\Big) = \Big(\frac{J^2}{n}\Big)_x - \nu J_{xx} + \frac{J}{\tau}
$$

equation (2) follows.

5 Asymptotic limits

We only prove Theorem 2.4 as the proofs of Theorems 2.2 and 2.3 are very similar (and in fact, easier). The proof is a consequence of the key estimate (23) and the compact embedding $H^1(0,1) \hookrightarrow L^\infty(0,1)$. First we show:

Theorem 5.1 *Let (u_δ, V_δ) be a solution of (12)-(14) for $\delta = \nu^2 + \varepsilon^2/4 > 0$ and let (15) hold for $\nu = 0$. Set $J_\delta = \nu \exp(u_\delta) u_{\delta,x} + J_0$. Then there exists a subsequence of $(u_\delta, J_\delta, V_\delta)$ (not relabeled) such that*

$$\begin{aligned} u_\delta &\rightharpoonup u && \text{weakly in } H^1 \text{ and strongly in } L^\infty, && (34)\\ V_\delta &\rightharpoonup V && \text{weakly in } H^3, && (35)\\ J_\delta &\rightharpoonup J && \text{weakly in } H^1, && (36) \end{aligned}$$

and (u, J, V) is a solution of

$$\begin{aligned} J &= J_0, && (37)\\ Tu_{xx} &= \frac{e^u - C}{\lambda^2} + J_0^2 (u_x e^{-2u})_x - \frac{J_0}{\tau}(e^{-u})_x, && (38)\\ V(x) &= Tu(x) + \frac{1}{2} J_0^2 e^{-2u(x)} + \frac{J_0}{\tau}\int_0^x e^{-u(x)} ds,\ x \in (0,1), && (39) \end{aligned}$$

with boundary conditions

$$u(0) = u(1) = 0. \qquad (40)$$

Proof. From Lemma 3.1 and Poincaré's inequality we obtain a uniform H^1 bound for u_δ. Then there exists a subsequence of (u_δ) (not relabeled) such that (34) holds. The weak formulation of (12) reads, for any $\psi \in C_0^\infty(0,1)$, after integration by parts,

$$\begin{aligned} &-\delta \int_0^1 u_\delta \psi_{xxxx} - \frac{\delta}{2}\int_0^1 u_{\delta,x}^2 \psi_{xx} dx \\ &= \Big(T + \frac{\nu}{\tau}\Big)\int_0^1 u_{\delta,x}\psi_x dx + 2J_0\nu \int_0^1 u_{\delta,x}^2 e^{-u_\delta}\psi_x dx - 2J_0\nu\int_0^1 u_{\delta,x} e^{-u_\delta}\psi_{xx} dx \\ &\quad - J_0^2 \int_0^1 u_{\delta,x} e^{-2u_\delta}\psi_x dx + \frac{J_0}{\tau}\int_0^1 e^{-u_\delta}\psi_x dx + \frac{1}{\lambda^2}\int_0^1 (e^{u_\delta} - C)\psi dx. \end{aligned}$$

The convergences (34) allow us to pass to the limit $\delta \to 0$ in the above equation, observing that the left-hand side vanishes in the limit:

$$-T\int_0^1 u_x \psi_x dx = -J_0^2 \int_0^1 u_x e^{-2u}\psi_x dx + \frac{J_0}{\tau}\int_0^1 e^{-u}\psi_x dx + \frac{1}{\lambda^2}\int_0^1 (e^u - C)\psi dx.$$

Now we rewrite (13) as

$$\begin{aligned} V_\delta(x) &= -\delta\Big(u_{\delta,xx} + \frac{1}{2}u_{\delta,x}^2\Big) + \Big(T + \frac{\nu}{\tau}\Big)u_\delta + 2J_0\nu e^{-u_\delta}u_{\delta,x} + 2J_0\nu\int_0^x e^{-u_\delta}u_{\delta,x}^2 ds \\ &\quad + \frac{1}{2}J_0^2 e^{-2u_\delta} + \frac{J_0}{\tau}\int_0^x e^{-u_\delta} ds. \qquad (41) \end{aligned}$$

Differentiating this equation twice with respect to x and comparing to (12) yields

$$V_{\delta,xx} = \lambda^{-2}(e^{u_\delta} - C).$$

Thus, from (23) follows that V_δ is uniformly bounded in H^3 and (35) is proved.

Next we multiply (41) by $\phi \in C_0^\infty(0,1)$ and integrate over $(0,1)$. Integrating by parts and using (14), we find

$$\begin{aligned}\int_0^1 V_\delta \phi dx \;&=\; -\delta \int_0^1 u_\delta \phi_{xx} dx - \frac{\delta}{2}\int_0^1 u_{\delta,x}^2 \phi dx + \Big(T + \frac{\nu}{\tau}\Big)\int_0^1 u_\delta \phi dx \\ &\quad + 2J_0\nu \int_0^1 e^{-u_\delta} u_{\delta,x}\phi dx + 2J_0\nu \int_0^1 \phi \int_0^x e^{-u_\delta} u_{\delta,x}^2 ds dx + \frac{1}{2}J_0^2 \int_0^1 e^{-2u_\delta}\phi dx \\ &\quad + \frac{J_0}{\tau}\int_0^1 \phi \int_0^x e^{-u_\delta} ds dx. \end{aligned} \tag{42}$$

Using the uniform L^∞ and H^1 bounds of u_δ and the convergence (34), we can pass to the limit $\delta \to 0$ in (42):

$$\int_0^1 V\phi dx = \int_0^1 \Big(Tu + \frac{1}{2}J_0^2 e^{-2u} + \frac{J_0}{\tau}\int_0^x e^{-u} ds\Big)\phi dx.$$

Finally, we consider the equation

$$J_\delta = \nu e^{u_\delta} u_{\delta,x} + J_0.$$

As $\nu u_{\delta,x}$ is uniformly bounded in H^1, by Lemma 3.1, a subsequence of (J_δ) converges weakly in H^1, i.e., (36) holds. Multiplying the above equation by some $\phi \in C_0^\infty(0,1)$ and integrating over $(0,1)$ gives

$$\int_0^1 J_\delta \phi dx = -\nu \int_0^1 e^{u_\delta}\phi_x dx + J_0 \int_0^1 \phi dx,$$

and the limit $\nu \to 0$ implies (37). $\square$

Remark 5.2 For sufficiently small current densities $J_0 > 0$, the quantum hydrodynamic model (37)-(40) is uniquely solvable (see, e.g., [10, 14]). This means that the whole sequence $(u_\delta, V_\delta, J_\delta)$ converges to (u, V, J_0) in the sense of (34)-(36).

We prove Theorem 2.4. Setting $n_\delta = e^{u_\delta}$ and $n = e^u$, where u is the solution of (38), obtained as the limit of the subsequence (u_δ), the convergence results (16)-(18) hold. We rewrite (38) in the variable n:

$$T(\log n)_{xx} = \frac{n - C}{\lambda^2} + J_0^2\Big(\frac{n_x}{n^3}\Big)_x - \frac{J_0}{\tau}\Big(\frac{1}{n}\Big)_x. \tag{43}$$

Notice that n is strictly positive since $n(x) \geq \exp(-\|u\|_{L^\infty}) \geq \exp(-M(\gamma))$, $x \in (0,1)$. Differentiating (39) twice with respect to x, we obtain

$$V_{xx}(x) = T(\log n)_{xx} + \frac{J_0^2}{2}\Big(\frac{1}{n^2}\Big)_{xx} + \frac{J_0}{\tau}\Big(\frac{1}{n}\Big)_x. \tag{44}$$

Comparing (43) and (44) gives Poisson's equation (see (19)). Differentiating (39) with respect to x and multiplying by n, the resulting equation is equal to the first equation in (19). Finally, from (39) we have

$$V(0) = \frac{1}{2}J_0^2 = V_0,$$

which equals (20).

6 Appendix: Derivation and scaling of the model

The viscous quantum hydrodynamic equations are derived similarly as in [11] by applying a moment method to the following Wigner-Fokker-Planck equation:

$$\partial_t w + \frac{\hbar}{m} k \cdot \nabla_x w + \frac{q}{\hbar} \theta[V](w) = L(w), \qquad x, k \in \mathbb{R}^d, \ t > 0, \tag{45}$$

$$w(x, k, 0) = w_0(x, k), \qquad x, k \in \mathbb{R}^d. \tag{46}$$

Here, $w = w(x, k, t)$ is the Wigner distribution function depending on the space variable $x \in \mathbb{R}^d$ ($d \geq 1$), the wave vector $k \in \mathbb{R}^d$, and the time $t > 0$. The physical parameters are the reduced Planck constant $\hbar = h/2\pi$, the effective mass m of the electrons, and the elementary charge q. The operator $\theta[V]$ is defined in the sense of pseudo-differential operators [22] as

$$\begin{aligned}(\theta[V])(w)(x, k, t) \;=\; & \frac{i}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{m}{\hbar} \Big[V\Big(x + \frac{\eta}{2}, t\Big) - V\Big(x - \frac{\eta}{2}, t\Big) \Big] \\ & \times w(x, k', t) e^{-i(k-k')\cdot\eta} dk' \, d\eta,\end{aligned}$$

where $V = V(x, t)$ is the electrostatic potential, usually selfconsistently given by the Poisson equation

$$\mathrm{div}_x(\varepsilon_s \nabla_x V) = q(n - C(x)), \qquad x \in \mathbb{R}^d.$$

Here, ε_s denotes the permittivity of the semiconductor material and $C(x)$ the concentration of fixed charged background ions (doping profile). The particle density $n(x, t)$ and the current density $J(x, t)$ are related to the Wigner function by

$$n(x, t) = \int_{\mathbb{R}^d} w(x, k, t) dp, \qquad J(x, t) = \frac{q}{m} \int_{\mathbb{R}^d} w(x, k, t) p dp,$$

with the momentum $p = \hbar k$.

The quantum Fokker-Planck operator

$$L(w) = \frac{D_{pp}}{\hbar^2} \Delta_k w + \frac{1}{\tau_0} \mathrm{div}_k(kw) + \frac{D_{pq}}{\hbar} \mathrm{div}_x(\nabla_k w) + D_{qq} \Delta_x w \tag{47}$$

models the interaction of the electrons with the phonons of the crystal lattice (oscillators) with constants

$$D_{pp} = \frac{k_B T_0}{m \tau_0}, \quad D_{pq} = \frac{\Omega \hbar^2}{6\pi m^2 k_B T_0 \tau_0}, \quad D_{qq} = \frac{\hbar^2}{12 m k_B T_0 \tau_0},$$

where k_B denotes the Boltzmann constant, T_0 the lattice temperature, τ_0 the momentum relaxation time, and Ω the cut-off frequency of the reservoir oscillators. This model governs the dynamical evolution of an electron ensemble in the single-particle Hartree approximation interacting dissipatively with an idealized heat bath consisting of an ensemble of harmonic oscillators and modeling the semiconductor lattice.

The Wigner-Fokker-Planck model (45)-(47) has been derived in [7, 9] under the main assumptions that

- the thermal energy $k_B T_0$ is of the same order as the energy $\hbar\Omega$ corresponding to the cut-off frequency;

- the reservoir memory time $1/\Omega$ is much smaller than the characteristic time scale t^* of the electrons and the momentum relaxation time τ_0.

For a discussion of the model (45), we refer to [4]. The existence and uniqueness of solutions to the periodic and the whole-space problem (45)-(47) have been shown in [3, 4].

In order to derive macroscopic equations from (45) for the macroscopic variables n and J, the moment method as in [11] can be applied, i.e., equation (45) is multiplied by 1 and $p = \hbar k$, respectively, and integrated over $\mathbb{R}^d$ with respect to p. The resulting system is closed by assuming that the Wigner function w is close to the quantum thermal equilibrium density approximation by Wigner [24]. The only difference to the derivation in [11] is that the Fokker-Planck operator has to be integrated. This yields

$$
\begin{aligned}
\int_{\mathbb{R}^d} (L(w))(x,k,t)d(\hbar k) &= D_{qq}\Delta_x n(x,t), \\
\int_{\mathbb{R}^d} (L(w))(x,k,t)(\hbar k)d(\hbar k) &= -\frac{J(x,t)}{\tau_0} - \frac{q}{m}D_{pq}\nabla_x n(x,t) + D_{qq}\Delta_x J(x,t).
\end{aligned}
$$

Therefore, we obtain the *viscous quanutm hydrodynamic equations*

$$
\begin{aligned}
&\partial_t n + \frac{1}{q}\mathrm{div} J = D_{qq}\Delta n, \\
&\partial_t J + \frac{1}{q}\mathrm{div}\left(\frac{J\otimes J}{n}\right) + \frac{qk_BT_0}{m}\left(1 + \frac{D_{pq}}{k_BT_0}\right)\nabla_x n - \frac{q^2}{m}n\nabla V - \frac{q\hbar^2}{6m^2}n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right) \\
&\quad = -\frac{J}{\tau_0} + D_{qq}\Delta J,
\end{aligned}
$$

where $J\otimes J$ is a matrix with the components J_iJ_k, $i,k = 1,\ldots,d$.

In order to scale these equations, we introduce the characteristic length L and the characteristic time t^* and define the characteristic density, voltage and current density, respectively, by

$$
C^* = \sup|C|, \quad V^* = \frac{k_BT_0}{q}, \quad J^* = \frac{qk_BT_0C^*t^*}{LM}\,\frac{L}{\iota},
$$

where ι is the mean-free path defined by $\iota^2 = k_BT_0\tau_0^2/m$. After introducing the scaling

$$
x \to Lx, \quad t \to t^*t, \quad n \to C^*n, \quad C \to C^*C, \quad V \to V^*V, \quad J \to J^*J,
$$

we obtain the scaled viscous quantum hydrodynamic equations

$$
\begin{aligned}
\partial_t n + \mathrm{div} J &= \nu\Delta n, \\
\partial_t J + \mathrm{div}\left(\frac{J\otimes J}{n}\right) + T\nabla n - n\nabla V - \frac{\varepsilon^2}{2}n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right) &= -\frac{J}{\tau} + \nu\Delta J, \\
\lambda^2\Delta V &= n - C(x).
\end{aligned}
$$

The scaled parameters are

$$
\varepsilon^2 = \frac{1}{6}\left(\frac{L_b}{L}\right)^2, \quad \nu = \frac{1}{6}\left(\frac{L_b}{L}\right)^2\frac{L}{\iota}\,\frac{t^*}{\tau_0}, \quad T = 1 + \frac{1}{\sqrt{18}\pi}\,\frac{\Omega\hbar}{k_BT_0}\,\frac{L_b}{\iota},
$$
$$
\tau = \frac{\tau_0}{t^*}, \quad \lambda^2 = \frac{\varepsilon_s k_BT_0}{q^2L^2C^*},
$$

and $L_b = \hbar/\sqrt{2mk_BT_0}$ is the de Broglie length. Notice that the scaled effective temperature T is the sum of the scaled temperature (which is one) and the correction term $\Omega\hbar L_b/\sqrt{18}\pi k_BT_0\iota$. The correction is small if the mean free path is large compared to the de Broglie length, since $\Omega\hbar/k_BT_0$ is assumed to be of order one.

The parameters ε and ν can be small depending on the physical situations:

1. $T_0 = 300\,\mathrm{K}$, $\tau = 10^{-12}\,\mathrm{s}$, $L = 1\,\mu\mathrm{m}$: With these values we have $L_b/L \approx 10^{-3}$ and $\iota/L \approx 0.1$ and hence $\varepsilon^2 \ll 1$ and $\nu \ll 1$ if the characteristic time t^* is of the same order as τ_0.

2. $T_0 = 300\,\mathrm{K}$, $\tau = 10^{-12}\,\mathrm{s}$, $L = 100\,\mathrm{nm}$: We obtain $L_b/L \approx 0.01$ and $\iota/L \approx 1$. If the characteristic time is much larger than the relaxation time, $t^*/\tau_0 \gg 1$, such that $(L_b/L)^2(t^*/\tau_0)$ is of order one, this gives $\varepsilon^2 \ll 1$ but ν is of order one.

3. $T_0 = 3\,\mathrm{K}$, $\tau = 10^{-12}\,\mathrm{s}$, $L = 100\,\mathrm{nm}$: This yields $L_b/L \approx 0.1$ and $\iota/L \approx 1$. If the characteristic time is much smaller than the momentum time, $t^*/\tau_0 \ll 1$, we obtain $\nu \ll \varepsilon$.

7 Conclusions

In this paper we analyzed a macroscopic model for quantum semiconductor devices including viscous terms which model collisions of the electrons with the semiconductor lattice. The derivation of the so-called viscous quantum hydrodynamic equations from the Wigner-Fokker-Planck model is sketched. The existence and uniqueness of classical stationary solutions of the one-dimensional model is shown for "weakly supersonic" flows. This means that for sufficiently large viscosity or, equivalently, sufficiently small momentum relaxation time, the flow is allowed to be supersonic in the classical sense. Furthermore, the inviscid and semiclassical limits are investigated and it is shown that the solution of the viscous quantum hydrodynamic model converges to the quantum hydrodynamic equations in the inviscid limit and to the classical hydrodynamic equations in the combined inviscid-semiclassical limit.

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