Decay rates for solutions of degenerate parabolic systems

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Abstract

Explicit decay rates for solutions of systems of degenerate parabolic equations in the whole space or in bounded domains subject to homogeneous Dirichlet boundary conditions are proven. These systems include the scalar porous medium, fast diffusion and p-Laplace equation and strongly coupled systems of these equations. For the whole space problem, the (algebraic) decay rates turn out to be optimal. In the case of bounded domains, algebraic and exponential decay rates are shown to hold depending on the nonlinearities. The proofs of these results rely on the use of the entropy functional together with generalized Nash inequalities (for the whole space problem) or Poincaré inqualities (for the bounded domain case).

Key words. Explicit deacy rates, long-time behavior of solutions, algebraic decay, exponential decay, degenerate parabolic equations, Nash inequality.

AMS(MOS) subject classification. 35K65, 35K55, 35B40.

Acknowledgments. The authors acknowledge support from the TMR project "Asymptotic Methods in Kinetic Theory", No. ERBFRMXCT 970157, funded by the EC. The first author was supported by the Gerhard-Hess program of the Deutsche Forschungsgemeinschaft, No. JU359/3-1.

1 Introduction

In this paper we derive *explicit* and, in some situations, *optimal* decay rates for solutions of the following strongly coupled system of degenerate parabolic equations:

$$\begin{aligned} \partial_t b(u) - \operatorname{div} a(u, \nabla u) &= f(u) \quad \text{in } \Omega \times (0, \infty), && (1) \\ b(u(\cdot, 0)) &= b(u_0) \quad \text{in } \Omega, && (2) \end{aligned}$$

either in the whole space $\Omega = \mathbb{R}^d$ or in a bounded domain $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ with Lipschitzian boundary. In the second case we impose homogeneous Dirichlet boundary conditions

$$u = 0 \quad \text{on } \Omega \times (0, \infty). \tag{3}$$

Here $u : \Omega \times (0, \infty) \to \mathbb{R}^n$ is a vector-valued function, $a(\cdot, \cdot)$ is a matrix-valued function with n rows and d columns, and ∇u stands for the Jacobian of the n-dimensional vector field u, i.e. $(\nabla u)_{ij} = \frac{\partial u_i}{\partial x_j}$. The divergence of a matrix field is defined in the usual way, i.e. it is the vector whose j-th component is the scalar divergence of the j-th matrix column.

Our assumptions on the nonlinearities are such that the trivial (zero) solution is a solution of the steady-state system. The objective of this paper is to study the rate of convergence of $u(t)$ to zero in L^q and in terms of the entropy of the system (see below).

It turns out that the decay rate of $u(t)$ to zero in the whole space case is algebraic with optimal rate. In the case of bounded domains the decay rate can be better than the rate for the whole space, however, under stronger conditions on the nonlinearities. In particular situations which include the nondegenerate case, even exponential decay can be shown.

2 Main results

First we specify the assumptions on the nonlinearities.

(A1) The function $b : \mathbb{R}^n \to \mathbb{R}^n$ with $b(0) = 0$ is strictly monotone and a gradient, i.e. there exists a function $\chi \in C^1(\mathbb{R}^n)$ with $b = \nabla \chi$, $\chi(0) = 0$, and constants $\beta, B > 0$, $m > 0$ such that for all $u, v \in \mathbb{R}^n$,

$$\beta |u - v|^{1+1/m} \leq (b(u) - b(v)) \cdot (u - v) \leq B |u - v|^{1+1/m}.$$

(A2) The function $a : \mathbb{R}^n \times \mathbb{R}^{n \times d} \to \mathbb{R}^{n \times d}$ is continuous in $\mathbb{R}^n \times \mathbb{R}^{n \times d}$, satisfies $a(u, 0) = 0$ for all $u \in \mathbb{R}^n$ and is elliptic in the sense

$$(a(u, z_1) - a(u, z_2)) \cdot (z_1 - z_2) \geq \alpha |z_1 - z_2|^p$$

for all $u \in \mathbb{R}^n$, $z_1, z_2 \in \mathbb{R}^{n \times d}$, with constants $\alpha > 0$ and $p > 1$.

(A3) The function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$f(u) \cdot u \le 0, \quad |f(u)| \le Ce(b(u)),$$

for all $u \in \mathbb{R}^n$, where the function e is the Legendre transform of χ, i.e.

$$e(b(u)) = b(u) \cdot u - \chi(u), \quad u \in \mathbb{R}^n. \tag{4}$$

The "$\cdot$" product of matrices in (A2) is defined as sum over both indices of products of equally indexed matrix elements, i.e. $A \cdot B := \mathrm{trace}(AB^T)$, where "^{T}" stands for matrix transposition.

The initial datum satisfies

(A4) $e(b(u_0)) \in L^1(\Omega)$ with measurable u_0.

Systems of equations like (1)–(2) arise in a variety of physical situations. For example, they describe the evolution of a fluid in non-Newtonian filtration or the water flow through porous media (see [16] and the references therein). In this context, often *single* equations with $n = 1$ are considered (see [19]). *Systems* of equations with $n > 1$ arise, for instance, in non-equilibrium thermodynamics [8], semiconductor modeling [9, 13] and alloy solidification processes [12].

The porous medium equation ($m > 1$) or the fast diffusion equation ($0 < m < 1$)

$$\partial_t(u^{1/m}) - \Delta u = 0, \quad u \ge 0,$$

are included in (1). Furthermore, the p-Laplace equation

$$\partial_t u - \mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$$

is also included. Notice that the corresponding functions $b(u)$ and $a(u, z)$ satisfy the conditions (A1) and (A2).

We introduce our notion of weak solution of the system (1)–(2), (1)–(3) respectively (see [2]). We call $u \in L^p(0, T; W_0^{1,p}(\Omega))$ a *weak solution* of (1)–(2) ((1)–(3) respectively) on the time interval $[0, T)$, if $b(u) \in L^\infty(0, T; L^1(\Omega))$, $\partial_t b(u) \in L^{p'}(0, T;\, W_0^{-1,p'}(\Omega))$, $a(u, \nabla u) \in L^{p'}((0, T) \times \Omega)$, u satisfies (1) in the distributional sense and the initial condition (2) is satisfied in the *weak sense*, i.e.

$$\int_0^T \langle \partial_t b(u), w \rangle dt + \int_0^T \int_\Omega (b(u) - b(u_0)) \cdot \partial_t w dx dt = 0$$

for all smooth function w such that $w(x, T) = 0$ for all $x \in \Omega$. Here, $p' = p/(p-1)$. Clearly, $W_0^{1,p}(\Omega) = W^{1,p}(\mathbb{R}^d)$ if $\Omega = \mathbb{R}^d$.

Later we need an auxiliary result for integration by parts in time:

Lemma 2.1 *Let $\Omega = \mathbb{R}^d$ or let $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ be a bounded domain. Let u be a weak solution of (1)–(2), (1)–(3) respectively. Furthermore, let (A4) hold. Then $e(b(u)) \in L^\infty(0,T;\, L^1(\Omega))$ and for almost all $t \in [0,T)$ the following formula holds:*

$$\int_\Omega e(b(u(t)))dx - \int_\Omega e(b(u_0))dx = \int_0^t \langle \partial_t b(u), u\rangle dt.$$

Here $\langle \cdot, \cdot \rangle$ denotes the duality bracket between $W^{1,p}(\Omega)$ and $W_0^{-1,p'}(\Omega)$).

A proof of this result for bounded domains can be found in [2, Lemma 1.5]. For the whole space case, the proof is almost exactly the same as in the bounded domain case. Since $\langle \partial_t b(u), u\rangle \in L^1(0,T)$ the *entropy*

$$H(t) = \int_\Omega e(b(u(x,t)))dx \tag{5}$$

is actually well defined for all $t \in [0,T]$ and absolutely continuous on $[0,T]$.

The existence of (global) weak solutions of (1)–(2) in *bounded* domains subject to mixed Dirichlet-Neumann boundary conditions has been shown by Alt and Luckhaus in [2] (also see [17]). They obtained an existence result for elliptic-parabolic systems, that is, assuming the function b to be only monotone (instead of strictly monotone). This result has been extended in different directions by various authors, for instance under more general assumptions on $a(u,z)$ or $b(u_0)$ [1, 11, 18]. No existence result seems to be available for the whole space problem.

The uniqueness of weak solutions (always in bounded domains) in the case of a *single* equation has been first shown in [2] under the additional assumption $\partial_t b(u) \in L^1$. This condition could be removed by Otto in [21]. In the case of *systems* of equations, uniqueness results seem to be available only for functions $a(u,z) = Az + g(u)$ (see [2, 14]).

We now state the main theorems. The first theorem is valid in the whole space *or* for bounded domains.

Theorem 2.2 *Let $\Omega = \mathbb{R}^d$ or let $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ be a bounded domain with $\partial\Omega \in C^{0,1}$. Let the hypotheses (A1)–(A4) hold, and*

$$m > \frac{1}{2}, \qquad p > \frac{d(m+1)}{dm+1}.$$

Let u be a weak solution of the system (1)–(2), (1)–(3) respectively, for $t \in [0,\infty)$ with

$$b(u) \in L^\infty(0,\infty; L^1(\Omega)).$$

Then there exist constants C_1, C_2, $C_3 > 0$ only depending on α, β, B, β_0, d, m, n, and p with

$$\beta_0 = \|b(u)\|_{L^\infty(0,\infty;L^1(\Omega))}$$

such that for all $t > 0$,

$$\begin{aligned} H(t) &\leq (H(0)^{-\delta} + \delta C_1 t)^{-1/\delta}, && (6) \\ \|u(t)\|_{L^{1+1/m}} &\leq C_2(H(0)^{-\delta} + \delta C_1 t)^{-m/\delta(m+1)}, && (7) \end{aligned}$$

and if $m > 1$,

$$\|u(t)\|_{L^1} \leq C_3(H(0)^{-\delta} + \delta C_1 t)^{-(m-1)/\delta m}, \tag{8}$$

where

$$\delta = \frac{dm(p-1) + p - d}{dm} > 0. \tag{9}$$

We show in Remark 3.3 below that the above decay rates are optimal in the whole space case. In the following two theorems we treat the case of bounded domains; there we get (in general) better convergence rates than in Theorem 2.2 but partly under stronger conditions on m and p.

Theorem 2.3 *Let $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ be a bounded domain with $\partial\Omega \in C^{0,1}$. Let the assumptions (A1)–(A4) hold and*

$$m > \frac{1}{p-1} \quad \textit{and} \quad m \geq \frac{d-p}{d(p-1)+p}.$$

Let u be a weak solution of the system (1)–(3) for $t \in [0, \infty)$. Then there exist constants $C_1, C_2 > 0$ only depending on α, β, B, d, Ω, m, n, and p such that for all $t > 0$,

$$\begin{aligned} H(t) &\leq (H(0)^{-\gamma} + \gamma C_1 t)^{-1/\gamma}, \\ \|u(t)\|_{L^{1+1/m}} &\leq C_2(H(0)^{-\gamma} + \gamma C_1 t)^{-m/\gamma(m+1)}, \end{aligned}$$

where

$$\gamma = \frac{(p-1)m - 1}{m+1} > 0.$$

Remark 2.4 The second condition on m is equivalent to

$$p \geq \frac{d(m+1)}{dm + m + 1}.$$

Therefore, if $p \geq 3$, the conditions of Theorem 2.3 on m (for fixed p and d) are weaker than those of Theorem 2.2. In the case $p < 3$ the conditions of Theorem 2.2 can be weaker or stronger than those of Theorem 2.3, depending on the precise values of p and d. In particular, if $p = 2$, Theorem 2.2 contains the fast diffusion case $m < 1$ which is excluded in Theorem 2.3.

The decay rate $1/\delta$ of Theorem 2.2 is always smaller than or equal to the rate $1/\gamma$ of Theorem 2.3. More precisely, if $m > (d-p)/(d(p-1)+p)$ then $\delta > \gamma$, and if $m = (d-p)/(d(p-1)+p)$ then $\delta = \gamma$.

Theorem 2.5 *Let $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ be a bounded domain with $\partial\Omega \in C^{0,1}$. Let the assumptions (A1)–(A4) hold and*

$$m = \frac{1}{p-1}.$$

Let u be a weak solution of the system (1)–(3) for $t \in [0, \infty)$. Then there exist constants $C, \mu > 0$ only depending on α, β, B, d, Ω, m, n, and p such that for all $t > 0$

$$\begin{aligned} H(t) &\leq H(0)e^{-\mu t}, \\ \|u(t)\|_{L^{1+1/m}} &\leq Ce^{-m\mu/(m+1)\cdot t}. \end{aligned}$$

Remark 2.6 Notice that the decay rate $1/\gamma$ becomes arbitrarily large if $p \to 1 + 1/m$. In this sense Theorem 2.5 is a limiting case of Theorem 2.3. Indeed, if $p = 1 + 1/m$ (i.e. $m = 1/(p-1)$) than we obtain exponential decay.

Remark 2.7 Theorems 2.3 and 2.5 are valid too, if homogeneous mixed Dirichlet-Neumann boundary conditions are prescribed, i.e.

$$u_i = 0 \quad \text{on } \Gamma_D \times (0, \infty), \qquad a_i(u, \nabla u) \cdot \nu = 0 \quad \text{on } \Gamma_N \times (0, \infty),$$

where $i = 1, \ldots, n$. Here, $\partial\Omega = \Gamma_D \cup \Gamma_N$ with $\Gamma_D \cap \Gamma_N = \emptyset$, $\text{meas}_{d-1}(\Gamma_D) > 0$, Γ_N is open in $\partial\Omega$, and ν is the unit normal vector of $\partial\Omega$.

Indeed, in the proofs we use the Poincaré inequality which is valid for functions u which vanish on a part of the boundary with positive $(d-1)$-dimensional Lebesgue measure [24, Lemma 1.46].

Remark 2.8 Theorem 2.2 has been proven in [5]; Theorems 2.3 and 2.5 are new. Note that Theorem 2.5 contains the nondegenerate case $p = 2$, $m = 1$. The energy-transport equations arising in nonequilibrium thermodynamics and semiconductor theory are a special example of strongly coupled parabolic systems with $p = 2$ and $m = 1$. In [8] exponential convergence of solutions of the energy-transport system has been proven. Therefore, Theorem 2.5 is an extension of the result in [8].

3 Proof of Theorem 2.2

For the proof of Theorem 2.2 we need an inequality which relates the L^q norm of a function to the L^p norm of its gradient, for appropriate p, q. In bounded domains, this is provided by the Poincaré inequality. In the whole space case, we shall replace the Poincaré inequality by the Nash inequality.

The classical Nash inequality reads as follows [3, 4, 20]: There exists a constant $\Gamma > 0$ such that for all $w \in L^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$,

$$\|w\|_{L^2}^{1+2/d} \leq \Gamma \|w\|_{L^1}^{2/d} \|\nabla w\|_{L^2}. \tag{10}$$

For the degenerate parabolic system (1)–(2) under the assumption (A1) however, it is more natural to work in the space $L^{1+1/m}$ instead of L^2. We shall call the corresponding inequality *generalized Nash inequality*:

Lemma 3.1 *Let $\Omega = \mathbb{R}^d$ or let $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ be a bounded domain with $\partial\Omega \in C^{0,1}$. Let $m > 1/2$, $d \in \mathbb{N}$ and $p \in [1, \infty)$ such that*

$$p > \frac{d(m+1)}{dm+m+1}.$$

Then there exists a constant $\Gamma > 0$ only depending on d, m and p such that for all $w \in W_0^{1,p}(\Omega)$ with $|w|^{1/m} \in L^1(\Omega)$:

$$\|w\|_{L^{1+1/m}}^{1+\sigma} \leq \Gamma \| \, |w|^{1/m} \|_{L^1}^{\sigma m} \|\nabla w\|_{L^p}, \tag{11}$$

where

$$\sigma = \frac{dpm + (m+1)(p-d)}{dpm^2} > 0.$$

The classical Nash inequality (10) is obtained for $m = 1$ and $p = 2$.

Proof. The generalized Nash inequality is a consequence of the Gagliardo-Nirenberg *and* the Hölder inequality. This is not very surprising since there are close relations between the Sobolev, the Gagliardo-Nirenberg and the Nash inequality [3].

First, let $w \in \mathcal{D}(\Omega)$ and $r \in (1, \infty)$ with $1/m < r < 1 + 1/m$. Then there exists a constant $G > 0$ only depending on d, p and r such that the Gagliardo-Nirenberg inequality holds:

$$\|w\|_{L^{1+1/m}} \leq G \|\nabla w\|_{L^p}^{\theta} \|w\|_{L^r}^{1-\theta}, \tag{12}$$

where

$$\theta = \frac{\frac{m}{m+1} - \frac{1}{r}}{\frac{1}{p} - \frac{1}{r} - \frac{1}{d}}.$$

It is easy to check that the inequality $p > d(m+1)/(dm+m+1)$ implies $0 < \theta < 1$.

For all $v \in L^1(\Omega) \cap L^{m+1}(\Omega)$, the Hölder inequality

$$\|v\|_{L^{rm}} \leq \|v\|_{L^1}^{\alpha} \|v\|_{L^{m+1}}^{1-\alpha}$$

holds, where

$$\alpha = \frac{m+1-rm}{rm^2}.$$

The inequalities $1/m < r < 1 + 1/m$ imply $0 < \alpha < 1$. Taking $v = |w|^{1/m}$ we obtain

$$\|w\|_{L^r} \leq \| \, |w|^{1/m} \|_{L^1}^{\alpha m} \|w\|_{L^{1+1/m}}^{1-\alpha}.$$

Substituting the L^r norm of w in (12), we conclude

$$\|w\|_{L^{1+1/m}}^{1/\theta-(1-\alpha)(1-\theta)/\theta} \leq G^{1/\theta} \| \, |w|^{1/m} \|_{L^1}^{\alpha m(1-\theta)/\theta} \|\nabla w\|_{L^p}.$$

Since

$$\frac{1}{\theta} - \frac{(1-\alpha)(1-\theta)}{\theta} = 1 + \alpha \frac{1-\theta}{\theta} = 1 + \frac{dpm + (m+1)(p-d)}{dpm^2} = 1 + \sigma,$$

we obtain the Nash inequality (11) for all $w \in \mathcal{D}(\Omega)$. The assertion then follows from a density argument.

Proof of Theorem 2.2. The proof is divided into several steps.

Step 1: Entropy inequality. Using equation (1) and conditions (A2)–(A3), we obtain for $0 < s < t$ (see Lemma 2.1),

$$\begin{aligned} H(t) - H(s) &= \int_s^t \langle \partial_t b(u), u \rangle d\tau \\ &= -\int_s^t \int_\Omega a(u, \nabla u) \cdot \nabla u dx d\tau + \int_s^t \int_\Omega f(u) \cdot u dx d\tau \\ &\leq -\alpha \int_s^t \|\nabla u(\tau)\|_{L^p}^p d\tau. \end{aligned} \tag{13}$$

The condition (A1) yields $b(u) \cdot u \geq \beta |u|^{1+1/m}$ for all $u \in \mathbb{R}^n$. Therefore, for all $i = 1, \ldots, n$,

$$\| \, |u_i(t)|^{1/m} \|_{L^1} \leq \| \, |u(t)|^{1/m} \|_{L^1} \leq (1/\beta) \|b(u(t))\|_{L^1} \leq b_0/\beta,$$

where $b_0 = \sup_{t>0} \|b(u(t))\|_{L^1(\Omega)}$. Since p satisfies the hypotheses of Lemma 3.1, we can apply the generalized Nash inequality (11):

$$\|u_i(t)\|_{L^{1+1/m}}^{1+\sigma} \leq \Gamma (b_0/\beta)^{\sigma m} \|\nabla u_i(t)\|_{L^p},$$

and hence

$$\begin{aligned} \|u(t)\|_{L^{1+1/m}}^{1+\sigma} &= \Big(\sum_{i=1}^n \|u_i(t)\|_{L^{1+1/m}} \Big)^{1+\sigma} \\ &\leq \max(1, n^\sigma) \Gamma (b_0/\beta)^{\sigma m} \sum_{i=1}^n \|\nabla u_i(t)\|_{L^p} = C_0 \|\nabla u(t)\|_{L^p}, \end{aligned}$$

where

$$C_0 = \max(1, n^\sigma) \Gamma (b_0/\beta)^{\sigma m}.$$

Employing the above inequality in (13) we obtain

$$H(t) - H(s) \leq -\alpha C_0^{-p} \int_s^t \|u(\tau)\|_{L^{1+1/m}}^{p(1+\sigma)} d\tau.$$

Step 2: Relation between the entropy and $\|u\|_{L^{1+1/m}}$. In order to relate the $L^{1+1/m}$ norm of $u(\tau)$ to $H(\tau)$ we use the condition (A1). Then, for all $u \in \mathbb{R}^n$,

$$\begin{aligned} e(b(u)) &= \int_0^1 (b(u) - b(\sigma u)) \cdot u d\sigma \\ &= \int_0^1 (b(u) - b(\sigma u)) \cdot (u - \sigma u) \frac{d\sigma}{1-\sigma} \\ &\leq B \int_0^1 |u - \sigma u|^{1+1/m} \frac{d\sigma}{1-\sigma} = \frac{mB}{m+1} |u|^{1+1/m}. \end{aligned} \tag{14}$$

Therefore

$$H(\tau) \leq \frac{mB}{m+1} \|u(\tau)\|_{L^{1+1/m}}^{1+1/m}, \tag{15}$$

which yields

$$H(t) - H(s) \leq -C_1 \int_s^t H(\tau)^{mp(1+\sigma)/(1+m)} d\tau,$$

where

$$C_1 = \alpha C_0^{-p} \left(\frac{m+1}{mB} \right)^{mp(1+\sigma)/(m+1)}.$$

This implies

$$\frac{dH}{dt} \leq -C_1 H^{1+\delta}$$

for almost all $t > 0$, where $\delta > 0$ is given by (9). Notice that $\delta > 0$ if and only if $p > d(m+1)/(dm+1)$. The above differential inequality immediately implies (6). The decay (7) is obtained from condition (A1) and (14):

$$H(t) \geq \frac{m\beta}{m+1} \|u(t)\|_{L^{1+1/m}}^{1+1/m} \quad \text{for almost all } t > 0,$$

with $C_2 = ((m+1)/m\beta)^{m/(m+1)}$.

Step 3: Decay rate in L^1. In order to derive the decay rate (8), we employ the estimate (7) and the Hölder inequality

$$\|w\|_{L^m} \leq \|w\|_{L^{m+1}}^{1-1/m^2} \|w\|_{L^1}^{1/m^2},$$

applied to $w = |u_i(t)|^{1/m}$, to obtain

$$\begin{aligned}\|u(t)\|_{L^1} &= \sum_{i=1}^{n} \|u_i(t)\|_{L^1} \le (b_0/\beta)^{1/m} \sum_{i=1}^{n} \|u_i(t)\|_{L^{1+1/m}}^{1-1/m^2} \\ &\le \max(1, n^{1/m^2})(b_0/\beta)^{1/m} \|u(t)\|_{L^{1+1/m}}^{1-1/m^2} \\ &\le C_3(H(0)^{-\delta} + \delta C_1 t)^{-(m-1)/\delta m},\end{aligned}$$

where

$$C_3 = \max(1, n^{1/m^2})(b_0/\beta)^{1/m} C_2^{1-1/m^2}.$$

This proves the theorem.

Remark 3.2 The most serious restriction of Theorem 2.2 is the uniform boundedness of $b(u(t))$ in $L^1(\Omega)$. In the following two important cases sufficient assumptions can be given:

(1) Let $\Omega = \mathbb{R}^d$, let the solution $u(t) = (u_1(t), \ldots, u_n(t))$ of (1)-(2) satisfy $u_i(t) \ge 0$ for almost all $t > 0$, $i = 1, \ldots, n$, and assume

$$b_i(u) \ge 0, \quad \sum_{j=1}^{n} f_j(u) \le 0 \text{ for all } u = (u_1, \ldots, u_n) \text{ with } u_k \ge 0,\ i, k = 1, \ldots, n.$$

Also let $b(u_0) \in L^1(\mathbb{R}^d)$.

(2) $n = 1$ (scalar case) and $b(u_0) \in L^1(\Omega)$.

If (1) or (2) holds then $b(u) \in L^\infty(0, \infty; L^1(\Omega))$ for the solution $u = u(t)$ of (1)-(2). In the case (1) it is sufficient for the proof to add the rows of (1) and to integrate (formally) over $\mathbb{R}^d$:

$$\begin{aligned}\|b(u(t))\|_{L^1(\mathbb{R}^d)} &= \sum_{j=1}^{n} \int_{\mathbb{R}^d} b_j(u(t))dx = \sum_{j=1}^{d} \int_{\mathbb{R}^d} f(u(t))dx + \|b(u_0)\|_{L^1(\mathbb{R}^d)} \\ &\le \|b(u_0)\|_{L^1(\mathbb{R}^d)}.\end{aligned}$$

To be more precise, use a regularization of the characteristic function on the ball $B_R(0)$ with center 0 and radius R as test function in the weak formulation of (1). It is not difficult to see that one obtains for $R \to \infty$:

$$\|b(u(t))\|_{L^1(\mathbb{R}^d)} = \lim_{R\to\infty} \|b(u(t))\|_{L^1(B_R(0))} \le \|b(u_0)\|_{L^1(\mathbb{R}^d)}.$$

In the scalar case (2) we take an increasing regularization S^γ of the sign function (with $\gamma > 0$ the regularization parameter) such that $S^\gamma(0) = 0$ and $\text{sign} - S^\gamma \to 0$ as $\gamma \to 0$ in $L^1(\Omega)$ and multiply Eq. (1) by $S^\gamma(b(u(t)))$. Integration by parts and the limit $\gamma \to 0$ give the desired result.

Remark 3.3 We consider examples for $\Omega = \mathbb{R}^d$ and $n = 1$ (single equation) with $b(u) = |u|^{1/m-1}u$, $a(u,z) = |z|^{p-2}z$:

(1) **Heat equation** $(m = 1, p = 2)$**:** Let $u_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Then

$$\|u(t)\|_{L^2} \sim t^{-d/4} \quad \text{as } t \to \infty.$$

More precisely, we have

$$\|u(t)\|_{L^2} \leq \frac{C_2\|u_0\|_{L^2}}{(1 + 2C_1\|u_0\|_{L^2}^{4/d}t)^{d/4}},$$

which is sharper for large t than the usual estimate $\|u(t)\|_{L^2} \leq \|u_0\|_{L^2}$ (see, for instance, [23]).

(2) **Porous medium equation** $(m > 1, p = 2)$**:** Let $u_0 \in L^1_+(\mathbb{R}^d) \cap L^{1+1/m}(\mathbb{R}^d)$, where $L^1_+(\mathbb{R}^d) = \{u \in L^1(\mathbb{R}^d) : u \geq 0 \text{ in } \mathbb{R}^d\}$. Then

$$\|u(t)\|_{L^1} \sim t^{-d(m-1)/(dm+2-d)} \quad \text{as } t \to \infty. \tag{16}$$

This estimate is sharp in the sense that the Barenblatt-Prattle solution has the same decay rate. Indeed, the Barenblatt-Prattle solution

$$V(t,x) = t^{-dk}\left(\left[C - \frac{m-1}{2m}\left(\frac{|x|}{t^k}\right)^2\right]_+\right)^{1/(m-1)} \tag{17}$$

with $k = 1/(2 + d(m-1))$ and $C > 0$ solves the equation

$$\partial_t V = \Delta V^m \quad \text{in } \mathbb{R}^d, \tag{18}$$

with $V(0,x) = D\delta(x)$, where D is a constant depending on C. Thus $U = V^m$ solves the equation (1) with the special choice of the nonlinear functions a and b given above. An easy calculation shows

$$\|U(t)\|_{L^1} \sim t^{-dk(m-1)} = t^{-d(m-1)/(dm+2-d)} \quad \text{as } t \to \infty.$$

We also refer to [6, 10] for related results.

(3) **Fast diffusion equation** $(m < 1, p = 2)$**:** Let $u_0 \in L^1_+(\mathbb{R}^d) \cap L^{1+1/m}(\mathbb{R}^d)$ and assume $m > \max(1/2, 1 - 2/d)$. Then

$$\|u(t)\|_{L^{1+1/m}} \sim t^{-dm^2/(dm+2-d)(m+1)} \quad \text{as } t \to \infty. \tag{19}$$

The Barenblatt-Prattle solution V (see Eq. (17)) solves the fast diffusion equation (18) for $m > 1 - 2/d$, and the function $U = V^m$ satisfies

$$\|U(t)\|_{L^{1+1/m}} \sim t^{-mdk(m+2)/(m+1)} = t^{-dm^2/(dm+2-d)(m+1)} \quad \text{as } t \to \infty.$$

This decay rate is the same as derived above for the solution u (also see [6, 10]).

The condition $m > \max(1/2, 1-2/d)$ is weaker than the condition derived by Otto [22], i.e. $m > d/(d+2)$ and $m \geq 1 - 1/d$, if and only if $d \geq 3$. For $d = 2$, both conditions give the restriction $m > 1/2$.

(4) **p-Laplace equation ($m = 1, p \geq 2$):** Let $u_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Then

$$\|u(t)\|_{L^2} \sim t^{-d/(2d(p-2)+2p)} \quad \text{as } t \to \infty.$$

The function

$$U(t,x) = t^{-d\kappa} \left(\left[C - \frac{p-2}{p} \Big(\frac{|x|}{t^\kappa} \Big)^{p/(p-1)} \right]_+ \right)^{(p-1)/(p-2)}$$

with $\kappa = 1/(d(p-2)+p)$ and $C > 0$ solves the p-Laplace equation with $U(0,x) = D\delta(x)$ where, again, D is a constant which depends on C. This function satisfies

$$\|U(t)\|_{L^2} \sim t^{-d\kappa/2} = t^{-d/(2d(p-2)+2p)} \quad \text{as } t \to \infty,$$

which is the same decay rate as above. For related results, see, e.g., [15].

Remark 3.4 The rates of decay of the solution $u(t)$ of the equation

$$\partial_t(u^{1/m}) = \Delta u \quad \text{in } \mathbb{R}^d$$

to the Barenblatt-Prattle solution $U(t)$ (with the same mass) in $L^1(\mathbb{R}^d)$ have been recently obtained in [7, 10, 22] by spatial-temporal rescaling techniques (cf. section 3.2). For instance, from [7, Thm. 6.1] we have the estimate

$$\|u(t)^{1/m} - U(t)^{1/m}\|_{L^1} \sim t^{-1/((dm+2m-d)} \quad \text{as } t \to \infty,$$

for $m > 1$, whereas for $1 - 1/d < m < 1$ (and $d = 2,3,4$, $m \neq \frac{1}{2}$) [10, Thm. 1.2]:

$$\|u(t) - U(t)\|_{L^1} \sim t^{-(1-d(1-m))/(dm+2-d)} \quad \text{as } t \to \infty.$$

Using the triangle inequality and Remark 3.3 we can only conclude the same rate for $u(t) - U(t)$ as for $u(t)$ itself (i.e. the rate (16) in L^1 for $m > 1$ and the rate (19) in $L^{1+1/m}$ for $\max(1/2, 1-2/d) < m < 1$). Clearly, these rates are not sharp.

We do not obtain the same results on the time decay of the difference $u(t) - U(t)$ as in [7, 10, 22] since we do not control the entropy dissipation rate. However, our method is simpler and valid for a very large class of problems.

4 Proofs of Theorem 2.3 and 2.5

Proof of Theorem 2.3. As in the proof of Theorem 2.2 we have for $0 < s < t$ the inequality (see (13))

$$H(t) - H(s) \le -\alpha \int_s^t \|\nabla u(\tau)\|_{L^p}^p d\tau.$$

Instead of the generalized Nash inequality we use now the Poincaré inequality (see, e.g., [24])

$$\|u(\tau)\|_{L^{1+1/m}} \le C_0 \|\nabla u(\tau)\|_{L^p}, \tag{20}$$

since $u(\tau) \in W_0^{1,p}(\Omega)$. For this inequality we need $1 - d/p \ge -d/(1+1/m)$ which is equivalent to $m \ge (d-p)/(d(p-1)+p)$. Then, together with the relation (15), we obtain

$$H(t) - H(s) \le -C_1 \int_s^t H(\tau)^{pm/(m+1)} d\tau,$$

where

$$C_1 = \alpha C_0^{-p} \Big(\frac{m+1}{mB}\Big)^{mp/(m+1)}.$$

This implies

$$\frac{dH}{dt} \le -C_1 H^{1+\gamma}$$

for almost all $t > 0$. Notice that $\gamma = mp/(m+1) - 1 > 0$ since $m > 1/(p-1)$. Integrating this inquality gives the first assertion. The second assertion can be shown as in the proof of Theorem 2.2.

Proof of Theorem 2.5. Let $\mu > 0$ to be specified later. We use again the integration by parts formula, but now in a slightly modified form:

$$\begin{aligned} e^{\mu t} H(t) - e^{\mu s} H(s) &= \int_s^t e^{\mu\tau} (\langle \partial_t b(u), u \rangle + \mu H(\tau)) d\tau \\ &\le \int_s^t e^{\mu\tau} (-\alpha \|\nabla u(\tau)\|_{L^p}^p + \mu H(\tau)) d\tau. \end{aligned}$$

Again using the Poincaré inequality (20) and the relation (15), observing that $p = 1 + 1/m$, we obtain

$$e^{\mu t} H(t) - e^{\mu s} H(s) \le \int_s^t e^{\mu\tau} (-\alpha + \mu m B C_0^p/(m+1)) \|\nabla u(\tau)\|_{L^p}^p d\tau.$$

Choosing $0 < \mu \le \alpha m B C_0^p/(m+1)$, we see that the integral on the right-hand side is nonpositive, and therefore, for $s = 0$,

$$H(t) \le H(0) e^{-\mu t}.$$

This finishes the proof.

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