A System of Parabolic Equations in Nonequilibrium Thermodynamics including Thermal and Electrical Effects

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Abstract

The time-dependent equations for a charged gas or fluid consisting of several components, exposed to an electric field, are considered. These equations form a system of strongly coupled, quasilinear parabolic equations which in some situations can be derived from the Boltzmann equation. The model uses the duality between the thermodynamic fluxes and the thermodynamic forces. Physically motivated mixed Dirichlet-Neumann boundary conditions and initial conditions are prescribed.

The existence of weak solutions is proven. The key of the proof is *(i)* a transformation of the problem by using the entropic variables, or electro-chemical potentials, which symmetrizes the equations, and *(ii) a priori* estimates obtained by using the entropy function. Finally, the entropy inequality is employed to show the convergence of the solutions to the thermal equilibrium state as the time tends to infinity.

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1 Introduction

We consider a gas or fluid consisting of n components (species), which is influenced by thermal, electrical and diffusive forces. Each component, labeled by i and with density ρ_i, has the charge e_i per particle. The evolution of the particles is governed by the conservation laws of the mass and of the energy (see, e.g., [18, 25]) :

$$\frac{\partial \rho_i}{\partial t} + \operatorname{div} J_i = W_i, \qquad i = 1, \ldots, n, \tag{1.1}$$

$$\frac{\partial U}{\partial t} + \operatorname{div} J_{n+1} = \sum_{k=1}^{n} F_k \cdot J_k + W_{n+1} \quad \text{in } \Omega \times (0, \tau), \tag{1.2}$$

where $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) is the (bounded) domain occupied by the gas or fluid and $\tau > 0$. The carrier flux density of the i-th component is denoted by J_i, the energy flux density or heat flux is denoted by J_{n+1}, and U is the internal energy density. The functions W_i are source terms, and F_i are external forces. We assume the situation where the external forces are due to the electric potential V, i.e.

$$F_i = -e_i \nabla V, \quad i = 1, \ldots, n, \tag{1.3}$$

solving the Poisson equation

$$\Delta V = -\sum_{k=1}^{n} e_k \rho_k - C(x) \quad \text{in } \Omega \times (0, \tau). \tag{1.4}$$

The given function $C(x)$ models fixed charged background particles.

The thermodynamic fluxes J_i ($i = 1, \ldots, n+1$) are assumed to depend on the thermodynamic forces $X_i = \nabla(\mu_i/T) + e_i \nabla V/T$ ($i = 1, \ldots, n$) and $X_{n+1} = \nabla(-1/T)$ where μ_i denotes the chemical potential and T the temperature of the gas or fluid:

$$J_i = J_i(X_1, \ldots, X_{n+1}), \quad i = 1, \ldots, n+1.$$

For sufficiently small deviations from the thermal equilibrium state $J_i = 0$ ($i = 1, \ldots, n+1$), the phenomenological relations between the fluxes and the forces are given by

$$J_i = -\sum_{k=1}^{n+1} L_{ik} X_k, \quad i = 1, \ldots, n+1. \tag{1.5}$$

From Onsager's principle, the symmetry of the coefficient matrix $L = (L_{ik})$ can be concluded. The second law of thermodynamics requires that (L_{ik}) is positive definite [25]. We refer to the monographs [18, 19, 25] for a discussion of the choice of the fluxes and forces and the relation between them. Furthermore, we mention that recently, in the context of semiconductor theory, the model (1.1)-(1.5) is derived from the Boltzmann equation [1, 3]. This justifies the relation (1.5).

Setting

$$u = (u_1, \ldots, u_n, u_{n+1}) = (\mu_1/T, \ldots, \mu_n/T, -1/T) \quad \text{and} \quad \rho_{n+1} = U$$

the particle densities ρ_i and the internal energy density ρ_{n+1} are assumed to depend on u: $\rho_i = \rho_i(u)$, $i = 1, \ldots, n+1$. The main assumption on $\rho = (\rho_1, \ldots, \rho_{n+1})$ is that

$$\rho \text{ is strongly monotone} \quad \text{ and } \quad \exists \chi : \rho = \nabla \chi. \tag{1.6}$$

The thermodynamic model (1.1)-(1.5) occurs in many applications of charged particle transport, for instance in semiconductor theory [1, 23], in electro-chemistry [10, 13], and alloy solidification processes [8, 20]. We only present the energy-transport model for semiconductors as an important example for (1.1)-(1.5) and refer to [11] for the details of other examples.

Example: energy-transport model for semiconductors. This model consists of equations for the electron density ρ_1 with charge $e_1 = -1$, the electron temperature $T > 0$, and the electric potential V:

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \operatorname{div} J_n &= 0, \\ \frac{\partial U}{\partial t} + \operatorname{div} J_w &= -J_n \cdot \nabla V + W(\mu_1, T), \\ \Delta V &= \rho_1 - C(x), \end{aligned}$$

where the electron and energy current densities are given by

$$\begin{aligned} J_n &= -L_{11}\Big(\nabla(\frac{\mu_1}{T}) - \frac{\nabla V}{T}\Big) - L_{12}\nabla\Big(-\frac{1}{T}\Big), \\ J_w &= -L_{21}\Big(\nabla(\frac{\mu_1}{T}) - \frac{\nabla V}{T}\Big) - L_{22}\nabla\Big(-\frac{1}{T}\Big). \end{aligned}$$

The model describes the flow of electrons through a doped semiconductor crystal which is characterized by the doping profile $C(x)$. It can be derived rigorously from the Boltzmann equation incorporating electron-electron and elastic collisions [1, 2, 16]. The energy relaxation term $W(\mu_1, T)$ satisfies

$$W(\mu_1, T)(T - T_L) \le 0,$$

where T_L is the lattice temperature. The variables ρ_1, U and the coefficients L_{ik} depend on μ_1 and T. In the case of Boltzmann statistics and parabolic band structure, under an additional hypothesis on elastic collisions, the diffusion matrix L can be written explicitly in terms of ρ_1 and T [1]:

$$L = c_0 \rho_1 T^{1/2} \begin{pmatrix} 1 & 2T \\ 2T & 6T^2 \end{pmatrix},$$

where $c_0 > 0$, and

$$\rho_1 = T^{3/2} \exp(\mu_1/T), \quad \rho_2 = U = \frac{3}{2}\rho_1 T.$$

The energy-transport model corresponds to (1.1)-(1.5) for $n = 1$. Notice that

$$\rho(u) = \left(-\frac{1}{u_2}\right)^{3/2} \exp(u_1) \cdot \begin{pmatrix} 1 \\ -(3/2)u_2 \end{pmatrix}$$

is strongly monotone and a gradient, i.e. $\chi(u) = (-1/u_2)^{3/2} \exp(u_1)$ satisfies $\nabla\chi = \rho$.

The model (with different diffusion coefficients) is investigated numerically in many papers, see, e.g., [7, 9, 30]; in the mathematical literature, however, only the stationary model was analyzed [5, 11]. No results are available for the time-dependent equations.

This paper is concerned with the existence and long-time behavior of the solutions to (1.1)-(1.5). Introducing $e_{n+1} = 0$ we can rewrite (1.1)-(1.5) as

$$\begin{aligned} \frac{\partial}{\partial t}\rho_i(u) + \operatorname{div} J_i &= W_i(u, V), \qquad i = 1, \ldots, n, &(1.7)\\ \frac{\partial}{\partial t}\rho_{n+1}(u) + \operatorname{div} J_{n+1} &= W_{n+1}(u, V) - \sum_{k=1}^{n} e_k \nabla V \cdot J_k, &(1.8)\end{aligned}$$

$$\begin{aligned} \Delta V &= -\sum_{k=1}^{n} e_k \rho_k(u) - C(x) \qquad \text{in } \Omega \times (0, \tau), &(1.9)\\ J_i &= -\sum_{k=1}^{n+1} L_{ik}(u, V)(\nabla u_k - e_k u_{n+1} \nabla V), \quad i = 1, \ldots, n+1. &(1.10)\end{aligned}$$

The equations have to be supplemented with physically motivated initial and boundary conditions. The boundary $\partial\Omega$ of the domain Ω consists of two disjoint subsets Γ_D and Γ_N. The electric potential, the chemical potentials, and the temperature are fixed at Γ_D, whereas Γ_N models the union of insulating boundary segments (zero outflow):

$$\begin{aligned} &u = u_D, \quad V = V_D \qquad \text{on } \Gamma_D \times (0, \tau), &(1.11)\\ &J_i \cdot \nu = \nabla V \cdot \nu = 0 \qquad \text{on } \Gamma_N \times (0, \tau), \quad i = 1, \ldots, n+1, &(1.12)\end{aligned}$$

where ν denotes the exterior normal vector of $\partial\Omega$. Thanks to the relation between ρ and u, the carrier densities and the temperature instead of u can be prescribed at Γ_D. Thus the boundary condition

$$\rho = \rho_D, \quad T = T_D, \quad V = V_D \qquad \text{on } \Gamma_D \times (0, \tau),$$

is equivalent to (1.11) (note that $\rho(u)$ is invertible since it is strongly monotone). The initial condition is given by

$$\rho(u(0)) = \rho(u^0) \qquad \text{in } \Omega. \tag{1.13}$$

Mathematically, there arise two main difficulties in the analysis of (1.7)-(1.13). First, the variable of the time differential operator, i.e. ρ, and the variable of the space differential operator, i.e. u, are different and related by the algebraic relation $\rho = \rho(u)$. For a single equation with $\rho(0) = 0$, this type of differential equation is called *doubly-degenerate* [27]. For more than one equation, this type of equation with $V = 0$ is studied in, e.g., [6, 14].

Secondly, the equations form a system of strongly coupled parabolic equations and maximum principle arguments cannot be used in general to derive L^∞ bounds for the variables u_i. Now, the term $J_k \cdot \nabla V$ on the right hand side of (1.8) contains the term $L_{ik} u_{n+1} |\nabla V|^2$ which is quadratic in the gradient. Note that, due to the mixed boundary conditions (1.11)-(1.12), we cannot expect to get "regular" solutions, and generally, $|\nabla V|^2 \in L^1(\Omega)$ only holds. But then, u_{n+1} not being in $L^\infty(\Omega)$, the term $L_{ik} u_{n+1} |\nabla V|^2$ is not defined.

The key of the proof of the existence as well as the long-time behavior is *(i)* to use another set of variables which symmetrizes the problem and *(ii)* to obtain *a priori* estimates by using the entropy function (see below).

(i) Transformation of variables. Define the entropic variables, or electro-chemical potentials,

$$w_i = u_i - e_i u_{n+1} V \quad (i = 1, \ldots, n), \qquad w_{n+1} = u_{n+1}, \tag{1.14}$$

and set $w = (w_1, \ldots, w_{n+1})$. To simplify the notation, we set $e = (e_1, \ldots, e_{n+1}) = (e_1, \ldots, e_n, 0)$ and define

$$e \cdot \xi = \sum_{k=1}^{n} e_k \xi_k \quad \text{for } \xi \in \mathbb{R}^{n+1}.$$

Then the problem (1.7)-(1.13) is *equivalent* to

$$\begin{aligned} \frac{\partial}{\partial t} b_i(w, V) + \operatorname{div} I_i &= Q_i(w, V), \quad i = 1, \ldots, n, & (1.15)\\ \frac{\partial}{\partial t} b_{n+1}(w, V) + \operatorname{div} I_{n+1} &= Q_{n+1}(w, V) + V e \cdot Q(w, V) & \\ &\quad + \frac{\partial V}{\partial t} e \cdot b(w, V), & (1.16) \end{aligned}$$

$$\begin{aligned} \Delta V &= -e \cdot b(w, V) - C(x) \qquad \text{in } \Omega \times (0, \tau), & (1.17)\\ I_i &= -\sum_{k=1}^{n+1} D_{ik}(w, V) \nabla w_k, \qquad i = 1, \ldots, n+1, & (1.18) \end{aligned}$$

subject to the initial and boundary conditions

$$\begin{aligned} w = w_D, \quad V = V_D \qquad & \text{on } \Gamma_D \times (0, \tau), & (1.19)\\ I_i \cdot \nu = \nabla V \cdot \nu = 0 \qquad & \text{on } \Gamma_N \times (0, \tau), \quad i = 1, \ldots, n+1, & (1.20)\\ b(w(0), V(0)) = b(w^0, V^0) \qquad & \text{in } \Omega, & (1.21) \end{aligned}$$

where

$$b_i(w, V) = \rho_i(u), \quad i = 1, \ldots, n,$$

$$
\begin{aligned}
b_{n+1}(w,V) &= \rho_{n+1}(u) + Ve\cdot\rho(u), \\
Q_i(w,V) &= W_i(u,V), \quad i=1,\ldots,n+1, \\
w_D &= u_D - eu_{D,n+1}V_D, \\
w^0 &= u^0 - eu^0_{n+1}V^0,
\end{aligned}
$$

V^0 is the solution of $\Delta V^0 = -e\cdot\rho(u^0) - C(x)$ under the mixed boundary conditions $V^0 = V_D(0)$ on Γ_D and $\nabla V\cdot\nu = 0$ on Γ_N, and finally, the new diffusion coefficients are given by

$$
\begin{aligned}
D_{ik} &= L_{ik}, \quad i,k=1,\ldots,n, \\
D_{i,n+1} &= D_{n+1,i} = L_{i,n+1} + V\sum_{k=1}^{n} e_k L_{ik}, \quad i=1,\ldots,n, \\
D_{n+1,n+1} &= L_{n+1,n+1} + 2V\sum_{k=1}^{n} e_k L_{n+1,k} + V^2 \sum_{i,k=1}^{n} e_i e_k L_{ik}.
\end{aligned}
$$

The *equivalence* of (1.7)-(1.10) and (1.15)-(1.18) can be shown by elementary computations. Strictly speaking, the two problems are only equivalent if the corresponding solutions are regular enough. The used notion of solution as well as the needed assumptions on the nonlinear functions are stated in the following section.

The diffusion matrix $D = (D_{ik})$ is symmetric and positive definite. Indeed, it holds $D = P^T LP$, where $P = (P_{ik})$ is the regular matrix defined by

$$
P_{ik} = \begin{cases} 1 & \text{if } i = k \in \{1,\ldots,n+1\} \\ e_k V & \text{if } i = n+1,\ k\in\{1,\ldots,n\} \\ 0 & \text{else.} \end{cases}
$$

Thus, if L is symmetric, positive definite, then D is symmetric, positive definite, too.

We note that the transformation (1.14) is well known in nonequilibrium thermodynamics [18, §53] and is also used in the drift-diffusion model for semiconductors (i.e. T = const.), where w_i is called quasi-Fermi potential. Recently, this transformation appears naturally in the derivation of an energy-transport model for semiconductor heterostructures [12]. The symmetrization property of the transformation (1.14) is also observed by Albinus [4] in the case of the energy-transport model. The system (1.7)-(1.13) is mathematically investigated here for the first time.

(ii) Entropy function. To derive *a priori* estimates we use the entropy function

$$
S(t) = \int_\Omega (\rho(u)\cdot(u-u_D) - (\chi(u)-\chi(u_D))) - \frac{1}{2}u_{D,n+1}\int_\Omega |\nabla(V-V_D)|^2,
$$

where $u_{D,n+1}$ = const. < 0 and $\rho = \nabla\chi$ (see above). In Section 4 we show that $S(t)$ is non-negative and satisfies the so-called entropy inequality (if w_D = const.):

$$
S(t_2) + c\int_{t_1}^{t_2}\int_\Omega \sum_{k=1}^{n+1} |\nabla(w_k - w_{D,k})|^2 \le S(t_1) \qquad \text{for } 0\le t_1 < t_2 < \infty \tag{1.22}
$$

and for some $c > 0$. (If ∇w_D does not vanish then the right hand side of (1.22) has to be replaced by $S(t_1) + c(w_D)$.) Indeed, introducing the entropy density $s(t,x)$ by $S(t) = \int_\Omega s(t,x)dx$, it holds

$$\frac{\partial s}{\partial t} + \mathrm{div} \sum_{k=1}^{n+1} I_k(w_k - w_{D,k}) = \sum_{k=1}^{n+1} I_k \cdot \nabla(w_k - w_{D,k}) \leq 0, \tag{1.23}$$

and equation (1.23) leads (formally) to equation (1.22) since the matrix D is symmetric, positive definite. Therefore we can interpret $\sum_k I_k(w_k - w_{D,k})$ as the entropy current and $\sum_k I_k \cdot \nabla(w_k - w_{D,k})$ as the entropy production rate with the thermodynamic fluxes I_i and the generalized thermodynamic forces $\nabla(w_k - w_{D,k})$ [25]. Furthermore, if $V = 0$, the entropy density relates the extensive variables $\rho_1, \ldots, \rho_{n+1}$ and the intensive variables $u_1, \ldots, u_{n+1}$ by $\partial s/\partial \rho_i = u_i - u_{D,i}$ for $i = 1, \ldots, n+1$. Mathematically, $s(t,x)$ can be interpreted as the Legendre transform of χ.

The plan of the paper is as follows. In Section 2 we state the assumptions and the existence theorems. We get existence of weak solutions (u, V) under mixed boundary conditions in two space dimensions, and under Dirichlet boundary conditions in three space dimensions. In two dimensions, we get the regularity $V(t) \in W^{1,p}(\Omega)$ for some $p > 2$ [17, 26]. Since $u(t) \in H^1(\Omega) \hookrightarrow L^q(\Omega)$ for all $q < \infty$, this implies $J_i(t) \in L^2(\Omega)$ and hence, the product $\nabla V(t) \cdot J_i(t)$ is defined. In three dimensions, we need more regularity. Indeed, if $V(t) \in W^{1,3}(\Omega)$, then $J_i(t) \in L^2(\Omega)$ and $\nabla V(t) \cdot J_i(t) \in L^1(\Omega)$. However, for mixed boundary conditions this regularity can only be obtained under restrictive assumptions on the data [28] so that we take Dirichlet boundary conditions.

Section 3 is devoted to the proof of the existence theorems. The method of the proof is a semidiscretization of time (or method of lines, or Rothe's method) as in [15] (see also the monographs [21, 24]). The resulting elliptic problems are solved by employing Leray-Schauder's fixed point theorem. The convergence of the discrete solution to the continuous solution is obtained by compactness arguments (see, e.g., [21]).

In Section 4 the long-time behavior of the solutions is studied. Assuming that the boundary functions (u_D, V_D) are in thermal equilibrium (i.e. $J_i = 0$ for $i = 1, \ldots, n+1$ or, equivalently, $\nabla w = 0$), we show that the solution constructed in Section 3 converges to the thermal equilibrium state. More precisely,

$$u(t) \to u_D \quad \text{in } L^2(\Omega), \qquad V(t) \to V_D \quad \text{in } H^1(\Omega) \qquad (t \to \infty).$$

The proof is based on the entropy inequality (1.22). In particular, $S(t)$ is non-increasing and $S(t) \to 0$ as $t \to \infty$. Thus $S(t)$ can be interpreted as a Lyapunov functional. Similar results for $T = \text{const.}$ (or $u_{n+1} = \text{const.}$) are proved in [15, 22].

2 Assumptions, existence results, semidiscretization

We start with some notations. Denote by $W^{s,q}(\Omega)$, $W^{s,q}(\Omega;\mathbb{R}^{n+1})$ the usual Sobolev space of real-valued, vector-valued functions, respectively. The norm of $W^{s,q}(\Omega)$ and $W^{s,q}(\Omega;\mathbb{R}^{n+1})$ is denoted in both cases by $\|\cdot\|_{s,q,\Omega}$. We abreviate $W^{s,q}(X) \stackrel{\text{def}}{=} W^{s,q}(0,T;X)$ if X is a Banach space. Furthermore, set $\mathcal{V} = H_0^1(\Omega\cup\Gamma_N;\mathbb{R})$ and $\mathcal{V}^{n+1} = H_0^1(\Omega\cup\Gamma_N;\mathbb{R}^{n+1})$ (see [32]), and denote by $\mathcal{V}^*$ the dual space to $\mathcal{V}$. Then $\langle\cdot,\cdot\rangle_{\mathcal{V}^*,\mathcal{V}}$ is the duality pairing between $\mathcal{V}^*$ and $\mathcal{V}$. Finally, we set $Q_\tau = \Omega\times(0,\tau)$.

We impose the following hypotheses:

(A1) $\Omega\subset\mathbb{R}^d$ $(1\le d\le 3)$ is a bounded domain with Lipschitzian boundary $\partial\Omega = \Gamma_D\cup\Gamma_N$, $\Gamma_D\cap\Gamma_N=\emptyset$, $\text{meas}_{d-1}(\Gamma_D)>0$, and Γ_N is open in $\partial\Omega$.

(A2) $\rho = (\rho_1,\ldots,\rho_{n+1})\in W^{1,\infty}(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ is strongly monotone and a gradient, i.e.

$$(\rho(u)-\rho(v))\cdot(u-v)\ge c_0|u-v|^2 \quad \text{for } u,v\in\mathbb{R}^{n+1},$$

where $c_0>0$, and there exists a (convex) function $\chi\in C^1(\mathbb{R}^{n+1};\mathbb{R})$ such that $\rho=\nabla\chi$.

(A3) $L=(L_{ik})\in L^\infty(Q_\tau\times\mathbb{R}^{n+1}\times\mathbb{R};\mathbb{R}^{(n+1)\times(n+1)})$ is a Carathéodory function (see [32]) and a symmetric, uniformly positive definite matrix.

(A4) $W_i : Q_\tau\times\mathbb{R}^{n+1}\times\mathbb{R}\to\mathbb{R}$ $(i=1,\ldots,n+1)$ are Carathéodory functions satisfying

$$\begin{gathered}\sum_{k=1}^n (W_k(u,V)-W_k(\hat u,\hat V))(u_k-\hat u_k)\le 0,\\ e\cdot W(u,V)=0,\quad W_{n+1}(x,t,u,V)(u_{n+1}-\overline{u})\le 0,\\ |W_i(x,t,u,V)|\le c(1+|u|+|V|),\quad i=1,\ldots,n+1,\end{gathered}$$

for all u, V, $\hat u$, $\hat V$, with $\overline{u}<0$ and $e=(e_1,\ldots,e_n,0)$.

(A5) $u_D\in C^0([0,\tau];H^1(\Omega;\mathbb{R}^{n+1}))\cap H^1(0,\tau;L^2(\Omega;\mathbb{R}^{n+1}))$, $V_D\in C^0([0,\tau];L^\infty(\Omega))\cap H^1(W^{1,p_0}(\Omega))$ with $p_0>2$ if $d=2$ and $p_0=3$ if $d=3$; $u^0=(u_1^0,\ldots,u_{n+1}^0)\in L^\infty(\Omega;\mathbb{R}^{n+1})\cap H^1(\Omega;\mathbb{R}^{n+1})$; $C\in L^\infty(\Omega)$, $e_1,\ldots,e_n\in\mathbb{R}$, and $u_{D,n+1}=\overline{u}$.

In the papers [6, 14] the hypothesis (A2) is also used. It is satisfied in the energy-transport model for semiconductors (see above) if the particle density and the temperature are bounded. The assumption (A3) follows from basic physical principles, as explained in Section 1. The monotonicity condition for W_{n+1} in assumption (A4) means that W_{n+1} is a "relaxation" term. For the energy-transport model for semiconductors, this condition is verified as can be proven rigorously [1, 3]. We assume that the temperature u_{n+1} is constant at the contacts, i.e. we neglect surface thermal effects. The term W_{n+1} relaxes to the constant temperature at the boundary (assumption (A4)).

In the case of the bipolar energy-transport model for semiconductors (i.e. $n=2$ and ρ_1, ρ_2 are the electron and hole density, respectively), usually the Shockley-Read-Hall term $W_1(u) = W_2(u) = (\rho_1(u)\rho_2(u)-1)/(c_1\rho_1(u)+c_2\rho_2(u)+c_3)$ is used as source term. Since $e_1=-1$, $e_2=+1$, we get $e_1W_1+e_2W_2=0$. Furthermore, W_i is monotone in the sense of (A4). Therefore, the hypothesis (A4) is satisfied in this case.

Now we can state the existence theorems:

Theorem 2.1 *Let (A1)-(A5) hold and let $\tau>0$. If $d\le 2$ there exists $p>2$ and a solution (u,V) of (1.7)-(1.13) with*

$$u-u_D \in L^2(\mathcal{V}^{n+1}) \cap C^0([0,\tau];L^2(\Omega;\mathbb{R}^{n+1})), \tag{2.1}$$
$$\rho(u) \in H^1((\mathcal{V}^*)^{n+1}) \cap L^2(Q_\tau), \tag{2.2}$$
$$V-V_D \in L^\infty(\mathcal{V}) \cap L^\infty(W^{1,p}). \tag{2.3}$$

Theorem 2.2 *Let (A1)-(A5) hold. Let $d=3$ and $\partial\Omega=\Gamma_D\in C^{1,1}$. Then there exists a solution (u,V) of (1.7)-(1.13) such that (2.1)-(2.3) holds and, moreover, $V\in L^\infty(Q_\tau)\cap L^\infty(W^{1,3})$.*

Notice that our regularity results are sufficient to conclude that the product $\nabla V\cdot J_i$ is integrable in Q_τ.

As mentioned in Section 1, the proof of the theorems is based on a semidiscretization of time. This method is also of interest from a numerical point of view. We introduce partitions $0=t_0<t_1<\cdots<t_N=\tau$ of $[0,\tau]$ such that $h_j \stackrel{\text{def}}{=} t_j-t_{j-1}<1$ and

$$h \stackrel{\text{def}}{=} \max_{j=1,\ldots,N} h_j \to 0 \quad (N\to\infty), \qquad \sup_{N>0}\max_{j=2,\ldots,N}\left(\frac{h_j}{h_{j-1}}+\frac{h_{j-1}}{h_j}\right)<\infty. \tag{2.4}$$

For instance, uniform partitions satisfy (2.4). We denote by $C_N(0,\tau;X)$ (for a Banach space X) the space of all functions $u^{(N)}:(0,\tau]\to X$ which are constant on $(t_{j-1},t_j]$, and we set $u^{(N)}(t)=u^j$ for $t\in(t_{j-1},t_j]$, $j=1,\ldots,N$.

Let $u_D^{(N)}$, $V_D^{(N)}\in C_N(0,\tau;H^1(\Omega))$ be defined by

$$u_D^j \stackrel{\text{def}}{=} h_j^{-1}\int_{t_{j-1}}^{t_j} u_D(\tau)d\tau,$$
$$V_D^j \stackrel{\text{def}}{=} h_j^{-1}\int_{t_{j-1}}^{t_j} V_D(\tau)d\tau$$

for $j=1,\ldots,N$. Furthermore, define the shift operator $\sigma_N: C_N(0,\tau;L^2(\Omega))\to C_N(0,\tau;L^2(\Omega))$ by

$$(\sigma_N u^{(N)})^j = u^{j-1}, \qquad j=1,\ldots,N,$$

and the linear interpolation of $u^{(N)}\in C_N(0,\tau;L^2(\Omega))$ by

$$\tilde u^{(N)}(x,t) = h_j^{-1}(t_j-t)(u^j-u^{j-1})+u^j \quad \text{for } x\in\Omega,\ t\in(t_{j-1},t_j],$$

for $j = 1, \ldots, N$, where u^0 is the initial value of (A5). Then, thanks to (2.4) (see [21, Lemma 5.2.5]),

$$\begin{aligned} u_D^{(N)} &\to u_D && \text{in } L^2(H^1), \\ \tilde{u}_D^{(N)} &\to u_D && \text{in } H^1(L^2), \\ V_D^{(N)} &\to V_D && \text{in } H^1(H^1) \quad \text{as } N \to \infty. \end{aligned}$$

The time discretization of (1.7)-(1.13) is, for $j = 1, \ldots, N$

$$h_j^{-1}(\rho_i(u^j) - \rho_i(u^{j-1})) + \operatorname{div} J_i^j = W_i(u^j, V^j), \quad i = 1, \ldots, n, \tag{2.5}$$

$$h_j^{-1}(\rho_{n+1}(u^j) - \rho_{n+1}(u^{j-1})) + \operatorname{div} J_{n+1}^j = W_{n+1}(u^j, V^j) - \sum_{k=1}^{n} e_k \nabla V^j \cdot J_k^j, \tag{2.6}$$

$$\Delta V^j = -e \cdot \rho(u^j) - C(x) \qquad \text{in } \Omega, \tag{2.7}$$

$$J_i^j = -\sum_{k=1}^{n+1} L_{ik}(u^j, V^j)(\nabla u_k^j - e_k u_{n+1}^j \nabla V^j), \quad i = 1, \ldots, n+1, \tag{2.8}$$

with boundary conditions

$$u^j = u_D^j, \quad V^j = V_D^j \qquad \text{on } \Gamma_D, \tag{2.9}$$

$$J_i^j \cdot \nu = \nabla V^j \cdot \nu = 0 \qquad \text{on } \Gamma_N. \tag{2.10}$$

For given $\rho(u^0)$ and V^0 (computed from $\Delta V^0 = e \cdot \rho(u^0) - C$ with mixed boundary conditions), the equations (2.5)-(2.10) define recursively (u^j, V^j), for $j = 1, \ldots, N$. Using the discrete transformation

$$w^j = u^j - e u_{n+1}^j V^j, \qquad j \geq 0, \tag{2.11}$$

the above discrete system is (formally) equivalent, for $j = 1, \ldots, N$, to

$$h_j^{-1}(b_i(w^j, V^j) - b_i(w^{j-1}, V^{j-1})) + \operatorname{div} I_i^j = Q_i(w^j, V^j), \quad i = 1, \ldots, n, \tag{2.12}$$

$$\begin{aligned} & h_j^{-1}(b_{n+1}(w^j, V^j) - b_{n+1}(w^{j-1}, V^{j-1})) + \operatorname{div} I_{n+1}^j \\ &= Q_{n+1}(w^j, V^j) + V^j e \cdot Q(w^j, V^j) + h_j^{-1}(V^j - V^{j-1}) e \cdot b(w^{j-1}, V^{j-1}), \end{aligned} \tag{2.13}$$

$$\Delta V^j = -e \cdot b(w^j, V^j) - C(x) \qquad \text{in } \Omega, \tag{2.14}$$

$$I_i^j = -\sum_{k=1}^{n+1} D_{ik}(w^j, V^j) \nabla w_k^j, \qquad i = 1, \ldots, n+1, \tag{2.15}$$

subject to the boundary conditions

$$w^j = w_D^j, \quad V^j = V_D^j \qquad \text{on } \Gamma_D, \tag{2.16}$$

$$I_i^j \cdot \nu = \nabla V^j \cdot \nu = 0 \qquad \text{on } \Gamma_N, \tag{2.17}$$

where $w_D^j = u_D^j - e u_{D,n+1}^j V_D^j \in H^1(\Omega; \mathbb{R}^{n+1})$ thanks to (A5).

For given (w^{j-1}, V^{j-1}), the problem (2.12)-(2.17) is a system of strongly coupled elliptic equations for (w^j, V^j). If $(w^j, V^j) \in H^1(\Omega; \mathbb{R}^{n+1}) \times L^\infty(\Omega)$ is such that $V^j \in W^{1,p}(\Omega)$ with $p > 2$ if $d \leq 2$ and $p = 3$ if $d = 3$, then $u^j \in H^1(\Omega; \mathbb{R}^{n+1})$, and the problems (2.12)-(2.17) and (2.5)-(2.10) are equivalent.

3 Proof of the existence results

To prove Theorems 2.1 and 2.2 we proceed as follows. First we solve recursively (2.12)-(2.17) to get functions $w^{(N)} \in C_N(0,T;H^1)$ and $V^{(N)} \in C_N(0,T;H^1)$ where $w^{(N)} = (w^1,\dots,w^N)$ and $V^{(N)} = (V^1,\dots,V^N)$. Then, setting $u^{(N)} = w^{(N)} - ew^{(N)}_{n+1}V^{(N)} \in C_N(0,T;L^2)$, we prove that $(u^{(N)},V^{(N)})$ solves (2.5)-(2.10). Using compactness arguments we let $N \to \infty$ to get a solution to the continuous problem (1.7)-(1.13).

The proof of the existence of solutions to (2.12)-(2.17) is based on estimates on the discrete entropy function

$$\begin{aligned} S^j \stackrel{\text{def}}{=} S(u^j,V^j) &= \int_\Omega (\rho(u^j)\cdot(u^j-u^j_D) - (\chi(u^j)-\chi(u^j_D))) \\ &\quad - \frac{1}{2}u_{D,n+1}\int_\Omega |\nabla(V^j-V^j_D)|^2. \end{aligned}$$

Notice that, since $u_{D,n+1} = \overline{u} < 0$, we have $S^j \geq 0$ for all $j \geq 0$.

The following lemma provides *a priori* estimates for the existence of solutions to the discrete system (2.12)-(2.17) as well as for the proof of Theorems 2.1 and 2.2.

Lemma 3.1 *Let (A1)-(A5) hold and let $(w^j,V^j) \in H^1(\Omega;\mathbb{R}^{n+1})\times(H^1(\Omega)\cap L^\infty(\Omega))$ be a solution to (2.12)-(2.17) such that $u^j \in H^1(\Omega;\mathbb{R}^{n+1})$. Then*

$$\|u^j\|_{0,2,\Omega} + \|V^j\|_{1,2,\Omega} + h_j\|w^j\|_{1,2,\Omega} \leq c_1, \tag{3.1}$$

$$\|u^{(N)}\|_{L^\infty(L^2)} + \|V^{(N)}\|_{L^\infty(H^1)} + \|w^{(N)}\|_{L^2(H^1)} \leq c_2, \tag{3.2}$$

where $c_1 > 0$ is independent of (w^j,V^j) and $c_2 > 0$ is independent of $(w^{(N)},V^{(N)})$ and N.

Proof. Using the convexity of χ, which implies

$$\chi(u^j) - \chi(u^{j-1}) - \nabla\chi(u^{j-1})\cdot(u^j-u^{j-1}) \geq 0,$$

and the fact that (w^j,V^j) solves (2.12), (2.14), we can estimate as follows:

$$\begin{aligned} S^j - S^{j-1} &\leq \int_\Omega (\nabla\chi(u^j)-\nabla\chi(u^{j-1}))\cdot(u^j-u^j_D) \\ &\quad - \overline{u}\int_\Omega \nabla(V^j-V^{j-1})\cdot\nabla(V^j-V^j_D) \\ &\quad + \int_\Omega (-\rho(u^{j-1})\cdot(u^j_D-u^{j-1}_D) + \chi(u^j_D) - \chi(u^{j-1}_D)) \\ &\quad + \overline{u}\int_\Omega (\nabla(V^{j-1}-V^{j-1}_D)\cdot\nabla(V^j_D-V^{j-1}_D) - \frac{1}{2}|\nabla(V^j_D-V^{j-1}_D)|^2) \\ &= K_1 + \ldots + K_4. \end{aligned}$$

First we estimate the integrals K_1 and K_2. Since (w^j, V^j) solves (2.12), (2.14), we obtain

$$
\begin{aligned}
K_1 + K_2 &= -h_j \sum_{k=1}^{n} \int_\Omega \operatorname{div} I_k^j (u_k^j - u_{D,k}^j) + h_j \sum_{k=1}^{n} \int_\Omega Q_k(w^j, V^j)(u_k^j - u_{D,k}^j) \\
&\quad + \int_\Omega (\rho_{n+1}(u^j) - \rho_{n+1}(u^{j-1}))(u_{n+1}^j - u_{D,n+1}^j) \\
&\quad - \overline{u} \int_\Omega e \cdot (\rho(u^j) - \rho(u^{j-1}))(V^j - V_D^j).
\end{aligned}
$$

Taking into account (2.13), we get

$$
\begin{aligned}
K_1 + K_2 &= -h_j \sum_{k=1}^{n} \int_\Omega \operatorname{div} I_k^j (w_k^j - w_{D,k}^j) - h_j \sum_{k=1}^{n} \int_\Omega \operatorname{div} I_k^j e_k \Big(V^j (w_{n+1}^j - w_{D,n+1}) \\
&\quad + w_{D,n+1}(V^j - V_D^j) \Big) + h_j \sum_{k=1}^{n} \int_\Omega Q_k(w^j, V^j)(u_k^j - u_{D,k}^j) \\
&\quad + \int_\Omega (b_{n+1}(w^j, V^j) - b_{n+1}(w^{j-1}, V^{j-1}))(w_{n+1}^j - w_{D,n+1}) \\
&\quad - \int_\Omega \Big(e \cdot (\rho(u^j) - \rho(u^{j-1}))V^j \\
&\quad + e \cdot \rho(u^{j-1})(V^j - V^{j-1}) \Big)(w_{n+1}^j - w_{D,n+1}) \\
&\quad - \overline{u} \int_\Omega e \cdot (\rho(u^j) - \rho(u^{j-1}))(V^j - V_D^j) \\
&= -h_j \sum_{k=1}^{n+1} \int_\Omega \operatorname{div} I_k^j (w_k^j - w_{D,k}^j) - h_j \sum_{k=1}^{n} \int_\Omega \operatorname{div} I_k^j e_k \Big(V^j (w_{n+1}^j - w_{D,n+1}) \\
&\quad + w_{D,n+1}(V^j - V_D^j) \Big) + h_j \sum_{k=1}^{n+1} \int_\Omega Q_k(w^j, V^j)(u_k^j - u_{D,k}^j) \\
&\quad + h_j \int_\Omega V^j e \cdot Q(w^j, V^j)(w_{n+1}^j - w_{D,n+1}) \\
&\quad - \int_\Omega e \cdot (\rho(u^j) - \rho(u^{j-1}))V^j (w_{n+1}^j - w_{D,n+1}) \\
&\quad - \overline{u} \int_\Omega e \cdot (\rho(u^j) - \rho(u^{j-1}))(V^j - V_D^j).
\end{aligned}
$$

Using (2.12) again and the assumption (A4), we obtain

$$
\begin{aligned}
K_1 + K_2 &= -h_j \sum_{k=1}^{n+1} \int_\Omega \operatorname{div} I_k^j (w_k^j - w_{D,k}^j) - h_j \overline{u} \int_\Omega Q(w^j, V^j) \cdot e(V^j - V_D^j) \\
&\quad + h_j \sum_{k=1}^{n+1} \int_\Omega Q_k(w^j, V^j)(u_k^j - u_{D,k}^j) \\
&\le h_j \sum_{k=1}^{n+1} \int_\Omega I_k^j \cdot \nabla (w_k^j - w_{D,k}^j) - h_j \overline{u} \int_\Omega W(u^j, V^j) \cdot e(V^j - V_D^j) \\
&\quad + h_j \sum_{k=1}^{n} \int_\Omega (W_k(u^j, V^j) - W_k(u_D^j, V_D^j))(u_k^j - u_{D,k}^j)
\end{aligned}
$$

$$
\begin{aligned}
& + h_j \sum_{k=1}^{n} \int_\Omega W_k(u_D^j, V_D^j)(u_k^j - u_{D,k}^j) \\
\leq \;& -h_j \sum_{k,\ell=1}^{n+1} \int_\Omega D_{k\ell}(w^j, V^j) \nabla w_\ell^j \cdot \nabla (w_k^j - w_{D,k}^j) + \varepsilon h_j \int_\Omega |u^j - u_D^j|^2 \\
& + c(\varepsilon, u_D^j, V_D^j) h_j \sum_{k=1}^{n} \int_\Omega W_k(u_D^j, V_D^j)^2
\end{aligned}
$$

employing Young's inequality, where $\varepsilon > 0$. Taking into account the symmetry of D, we obtain

$$
\begin{aligned}
& -\sum_{k,\ell=1}^{n+1} \int_\Omega D_{k\ell}(w^j, V^j) \nabla w_\ell^j \cdot \nabla (w_k^j - w_{D,k}^j) \\
& = \; -\frac{1}{2} \sum_{k,\ell=1}^{n+1} \int_\Omega D_{k\ell}(w^j, V^j) \nabla (w_\ell^j - w_{D,\ell}^j) \cdot \nabla (w_k^j - w_{D,k}^j) \\
& \quad - \frac{1}{2} \sum_{k,\ell=1}^{n+1} \int_\Omega D_{k\ell}(w^j, V^j) \nabla w_\ell^j \cdot \nabla w_k^j + \frac{1}{2} \sum_{k,\ell=1}^{n+1} \int_\Omega D_{k\ell}(w^j, V^j) \nabla w_{D,\ell}^j \cdot \nabla w_{D,k}^j.
\end{aligned}
$$

By Stampacchia's estimate (see [31]), we get, since $d \leq 3$,

$$
\|V^j\|_{0,\infty,\Omega} \leq c(1 + \|u^j\|_{0,2,\Omega}).
$$

Since the matrix D is positive definite, there exists $\delta(u^j) > 0$ such that

$$
\sum_{i,k=1}^{n+1} D_{ik}(w^j, V^j) \xi_i \xi_k \geq \delta(u^j) |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^{n+1}. \tag{3.3}
$$

Indeed, δ depends on the $L^\infty(\Omega)$ norm of V^j, by the definition of D, and thus on the $L^2(\Omega)$ norm of u^j, by the above elliptic estimate. Taking into account $\nabla w_{D,n+1} = 0$ due to (A5), the positive definiteness of L, and $D_{ik} = L_{ik}$ for all $i, k = 1, \ldots, n$ by the definition of D, we get further

$$
\begin{aligned}
& -\sum_{\ell,k=1}^{n+1} \int_\Omega D_{k\ell}(w^j, V^j) \nabla w_\ell^j \cdot \nabla (w_k^j - w_{D,k}^j) \\
\leq \;& -\frac{1}{2} \sum_{k=1}^{n+1} \int_\Omega \delta(u^j) |\nabla w_k^j|^2 + \frac{1}{2} \sum_{k,\ell=1}^{n} \int_\Omega L_{\ell k}(u^j, V^j) \nabla w_{D,\ell}^j \cdot \nabla w_{D,k}^j \\
\leq \;& -\frac{1}{2} \sum_{k=1}^{n+1} \int_\Omega \delta(u^j) |\nabla w_k^j|^2 + c(w_D^j).
\end{aligned}
$$

This implies

$$
K_1 + K_2 \;\leq\; -\frac{1}{2} h_j \sum_{k=1}^{n+1} \int_\Omega \delta(u^j) |\nabla w_k^j|^2 + \varepsilon h_j \int_\Omega |u^j - u_D^j|^2 + c(\varepsilon) h_j.
$$

Now we estimate the remaining integrals K_3 and K_4. Recalling that $h_j < 1$, we employ (A2) and Young's inequality with $\varepsilon > 0$ to get

$$\begin{aligned} K_3 &= -\int_\Omega ((\rho(u^{j-1}) - \rho(u_D^{j-1})) \cdot (u_D^j - u_D^{j-1}) + \rho(u_D^{j-1}) \cdot (u_D^j - u_D^{j-1}) \\ &\quad - \chi(u_D^j) + \chi(u_D^{j-1})) \\ &\le \varepsilon h_{j-1} \int_\Omega |u^{j-1} - u_D^{j-1}|^2 + c(\varepsilon) h_{j-1}^{-1} \int_\Omega |u_D^j - u_D^{j-1}|^2 + c(u_D^{j-1}) \end{aligned}$$

and

$$K_4 \le \varepsilon h_{j-1} \int_\Omega |\nabla(V^{j-1} - V_D^{j-1})|^2 + c(\varepsilon) h_{j-1}^{-1} \int_\Omega |\nabla(V_D^j - V_D^{j-1})|^2$$

We proceed by estimating

$$\begin{aligned} S^m - S^0 &= \sum_{j=1}^m (S^j - S^{j-1}) \\ &\le -\frac{1}{2} \sum_{j=1}^m h_j \sum_{k=1}^{n+1} \int_\Omega \delta(u^j) |\nabla w_k^j|^2 + \varepsilon \sum_{j=0}^m h_j \int_\Omega |u^j - u_D^j|^2 \\ &\quad + \varepsilon \sum_{j=0}^m h_j \int_\Omega |\nabla(V^j - V_D^j)|^2 + c(\varepsilon) \sum_{j=1}^m h_j^{-1} \int_\Omega |u_D^j - u_D^{j-1}|^2 \\ &\quad + c(\varepsilon) \sum_{j=1}^m h_j^{-1} \int_\Omega |\nabla(V_D^j - V_D^{j-1})|^2 + c(\varepsilon) \sum_{j=1}^m h_j, \end{aligned}$$

where we have used the fact that h_j/h_{j-1} is bounded independently of j and N (see (2.4)). Since

$$\begin{aligned} \sum_{j=1}^m h_j^{-1} \int_\Omega (|u_D^j - u_D^{j-1}|^2 + |\nabla(V_D^j - V_D^{j-1})|^2) &\le \left\| \frac{\partial \tilde{u}_D^{(N)}}{\partial t} \right\|_{L^2(L^2)} + \left\| \frac{\partial \tilde{V}_D^{(N)}}{\partial t} \right\|_{L^2(H^1)} \\ &\le c \end{aligned}$$

independently of N (see Section 2), we get finally

$$\begin{aligned} S^m - S^0 &\le -\frac{1}{2} \sum_{j=1}^m h_j \sum_{k=1}^{n+1} \int_\Omega \delta(u^j) |\nabla w_k^j|^2 + \varepsilon \sum_{j=1}^m h_j \int_\Omega |u^j - u_D^j|^2 \\ &\quad + \varepsilon \sum_{j=1}^m h_j \int_\Omega |\nabla(V^j - V_D^j)|^2 + c(\varepsilon, u_D, V_D, \tau). \end{aligned}$$

On the other hand, since ρ is strongly monotone,

$$\begin{aligned} S^m &= \int_\Omega \int_0^1 (\rho(u^m) - \rho(su^m + (1-s)u_D^m)) \cdot (u^m - (su^m + (1-s)u_D^m)) \frac{\mathrm{d}s}{1-s} \\ &\quad - \frac{1}{2} u_{D,n+1} \int_\Omega |\nabla(V^m - V_D^m)|^2 \\ &\ge \frac{c_0}{2} \int_\Omega |u^m - u_D^m|^2 + \frac{1}{2} |\overline{u}| \int_\Omega |\nabla(V^m - V_D^m)|^2. \end{aligned}$$

The above two estimates imply

$$\frac{c_0}{2}\int_\Omega |u^m - u_D^m|^2 + \frac{1}{2}|\overline{u}|\int_\Omega |\nabla(V^m - V_D^m)|^2 + \frac{1}{2}\sum_{j=0}^m h_j \sum_{k=1}^{n+1}\int_\Omega \delta(u^j)|\nabla w_k^j|^2$$
$$\leq \quad S^0 + \varepsilon\sum_{j=0}^m h_j\int_\Omega |u^j - u_D^j|^2 + \varepsilon\sum_{j=0}^m h_j \int_\Omega |\nabla(V^j - V_D^j)|^2 + c(\varepsilon).$$

Applying the maximum over $m = 1, \ldots, N$ yields

$$\frac{c_0}{2}\|u^{(N)} - u_D^{(N)}\|^2_{L^\infty(L^2)} + \frac{1}{2}|\overline{u}|\|V^{(N)} - V_D^{(N)}\|^2_{L^\infty(\mathcal{V})} + \frac{1}{2}\sum_{j=0}^N h_j \sum_{k=1}^{n+1}\int_\Omega \delta(u^j)|\nabla w_k^j|^2$$
$$\leq \quad S^0 + \varepsilon\|u^{(N)} - u_D^{(N)}\|_{L^2(L^2)} + \varepsilon\|V^{(N)} - V_D^{(N)}\|_{L^2(\mathcal{V})} + c(\varepsilon). \qquad (3.4)$$

This implies, for sufficiently small $\varepsilon > 0$,

$$\|u^{(N)}\|_{L^\infty(L^2)} + \|V^{(N)}\|_{L^\infty(H^1)} \leq c,$$

where $c > 0$ only depends on the given data. In particular,

$$\|u^j\|_{0,2,\Omega} + \|V^j\|_{1,2,\Omega} \leq c, \qquad j = 1, \ldots, N.$$

It can be seen that therefore

$$\delta(u^j) \geq \delta_0 > 0 \qquad \text{for } j = 1, \ldots, N.$$

Hence we conclude from (3.4)

$$\|w^{(N)}\|^2_{L^2(H^1)} \leq c \qquad \text{and} \qquad \|w^j\|_{1,2,\Omega} \leq c$$

with $c > 0$ only depending on the data. This finishes the proof. $\square$

We are now able to prove the existence of solutions to the discrete problem (2.12)-(2.17).

Lemma 3.2 *Let (A1)-(A5) hold and let* $(w^{j-1}, V^{j-1}) \in H^1(\Omega; \mathbb{R}^{n+1}) \times (H^1(\Omega) \cap L^\infty(\Omega))$ *be given. Then there exists a solution* $(w^j, V^j) \in H^1(\Omega; \mathbb{R}^{n+1}) \times (H^1(\Omega) \cap L^\infty(\Omega))$ *to (2.12)-(2.17).*

Proof. The proof is based on the Leray-Schauder fixed point theorem. To define the fixed point operator, let $\tilde{u} \in L^2(\Omega; \mathbb{R}^{n+1})$ and let $V \in H^1(\Omega)$ be the unique solution of

$$\Delta V = -e \cdot \rho(\tilde{u}) - C(x) \text{ in } \Omega, \quad V = V_D^j \text{ on } \Gamma_D, \quad \nabla V \cdot \nu = 0 \text{ on } \Gamma_N.$$

From Stampacchia's estimate for elliptic equations [31] follows that $V \in L^\infty(\Omega)$. Thus $\tilde{w} \stackrel{\text{def}}{=} \tilde{u} - V\tilde{u}_{n+1}e \in L^2(\Omega;\mathbb{R}^{n+1})$. Now consider the linear system for w:

$$\begin{aligned}
-\mathrm{div}\sum_{k=1}^{n+1} D_{ik}(\tilde{w},V)\nabla w_k &= \sigma\big(-h_j^{-1}(b_i(\tilde{w},V)-b_i(w^{j-1},V^{j-1}))+Q_i(\tilde{w},V)\big),\\
-\mathrm{div}\sum_{k=1}^{n+1} D_{n+1,k}(\tilde{w},V)\nabla w_k &= \sigma\big(-h_j^{-1}(b_{n+1}(\tilde{w},V)-b_{n+1}(w^{j-1},V^{j-1}))\\
&\quad + Q_{n+1}(\tilde{w},V)+Ve\cdot Q(\tilde{w},V)\\
&\quad + h_j^{-1}(V-V^{j-1})e\cdot\rho(u^{j-1})\big),
\end{aligned}$$

where $i=1,\ldots,n$, subject to the boundary conditions

$$w=\sigma w_D \quad \text{on } \Gamma_D, \qquad \sum_{k=1}^{n+1} D_{ik}(\tilde{w},V)\nabla w_k\cdot\nu = 0 \quad \text{on } \Gamma_N, \qquad i=1,\ldots,n+1,$$

where $\sigma\in[0,1]$. By Lax-Milgram's theorem, there exists a unique solution $w \in H^1(\Omega;\mathbb{R}^{n+1})$, since the right-hand side of this elliptic problem lies in $L^2(\Omega;\mathbb{R}^{n+1})$. Finally, define $u = w + Vw_{n+1}e \in L^2(\Omega;\mathbb{R}^{n+1})$.

Hence the fixed point operator $S: L^2(\Omega;\mathbb{R}^{n+1})\times[0,1]\to L^2(\Omega;\mathbb{R}^{n+1}), (\tilde{u},\sigma)\mapsto u$, is well-defined. It holds $S(\tilde{u},0)=0$ for all $\tilde{u}\in L^2(\Omega;\mathbb{R}^{n+1})$. Every fixed point u of S with $\sigma=1$ solves (2.12)–(2.17). Indeed, let $S(u,1)=u$ then we only have to show that $w=\tilde{w}$. Now, $\tilde{w}_{n+1}=u_{n+1}=w_{n+1}$ and $\tilde{w}_i = u_i - Vw_{n+1}e = w_i$ for $i=1,\ldots,n$. Therefore, $w = u - Vu_{n+1}e$.

To show that S is a compact operator, let $u\in L^2(\Omega;\mathbb{R}^{n+1})$ be a fixed point of S with $\sigma\in[0,1]$. Similarly as in the proof of Lemma 3.1 we get the estimates

$$\|u\|_{0,2,\Omega}+\|w\|_{1,2,\Omega}+\|V\|_{1,2,\Omega}\le c,$$

where $c>0$ is independent of u, w, V, and σ. Thus

$$\begin{aligned}
\|\nabla u\|_{0,3/2} &\le c(\|\nabla w\|_{0,2}+\|\nabla V\|_{0,2}\|w_{n+1}\|_{0,6}+\|V\|_{0,6}\|\nabla w_{n+1}\|_{0,2})\\
&\le c(\|w\|_{1,2}+\|V\|_{1,2})\\
&\le c,
\end{aligned}$$

using $H^1(\Omega)\hookrightarrow L^6(\Omega)$. Standard arguments show that S is continuous and compact, noting that the embedding $W^{1,3/2}(\Omega)\hookrightarrow L^2(\Omega)$ is compact (for $d\le 3$). Leray-Schauder's theorem gives the desired result. $\square$

Corollary 3.3 *Let (A1)-(A5) hold and let $(u^{j-1},V^{j-1})\in H^1(\Omega;\mathbb{R}^{n+1})\times(H^1(\Omega)\cap L^\infty(\Omega))$ be given. Then there exists a solution $(u^j,V^j)\in H^1(\Omega;\mathbb{R}^{n+1})\times(H^1(\Omega)\cap L^\infty(\Omega))$ to (2.5)-(2.10).*

Taking into account the estimate (3.2) for $u^{(N)}$, the regularity result for elliptic problems of [17, 26], and Stampacchia's elliptic estimate [31]

$$\|V^j\|_{0,\infty,\Omega}\le c(1+\|u^j\|_{0,2,\Omega})$$

we conclude:

Lemma 3.4 *There exists $p > 2$ such that*

$$\|V^{(N)}\|_{L^\infty(L^\infty)} + \|V^{(N)}\|_{L^\infty(W^{1,p})} \le c, \tag{3.5}$$

where $c > 0$ is independent of N. Furthermore, $p = 3$ if $d = 3$.

We proceed now by proving further *a priori* estimates independent of N. From (3.2) and (3.5) follows:

Lemma 3.5 *It holds*

$$\begin{aligned} \|u^{(N)}\|_{L^2(H^1)} &\le c, & (3.6)\\ \|J_i^{(N)}\|_{L^2(L^2)} &\le c, \quad i = 1, \ldots, n+1, & (3.7) \end{aligned}$$

where $J_i^{(N)} \in C_N(0, T; L^2)$ is such that $J_i^{(N)} = J_i^j$ on $(t_{j-1}, t_j]$, $j = 1, \ldots, N$, and $c > 0$ is independent of N.

Proof. We have for $i = 1, \ldots, n$

$$\begin{aligned} \|\nabla u_i^{(N)}\|_{L^2(L^2)} \le\ & \|\nabla w_i^{(N)}\|_{L^2(L^2)} + c\|w_{n+1}^{(N)}\|_{L^2(L^q)}\|\nabla V^{(N)}\|_{L^\infty(L^p)} \\ & + c\|\nabla w_{n+1}^{(N)}\|_{L^2(L^2)}\|V^{(N)}\|_{L^\infty(L^\infty)}, \end{aligned}$$

where $p > 2$ and $q = 2p/(p-2)$ if $d = 2$, and $p = 3$, $q = 6$ if $d = 3$. Since the embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$ is continuous, we obtain (3.6). To prove (3.7), we use that L is bounded in $Q_\tau \times \mathbb{R}^{n+1} \times \mathbb{R}$:

$$\|J_i^{(N)}\|_{L^2(L^2)} \le c\Big(\sum_{k=1}^{n+1} \|\nabla u_k^{(N)}\|_{L^2(L^2)} + \|u_{n+1}^{(N)}\|_{L^2(L^q)}\|\nabla V^{(N)}\|_{L^\infty(L^p)}\Big) \le c, \tag{3.8}$$

where $i = 1, \ldots, n+1$ and p and q are as above. $\square$

Our final a priori estimates are necessary for the compactness argument. For this, define the linear interpolation of $\rho(u^{(N)})$ by

$$\tilde\rho^{(N)}(x,t) = h_j^{-1}(t_j - t)(\rho(u^j) - \rho(u^{j-1})) + \rho(u^j) \ \text{ for } x \in \Omega,\ t \in (t_{j-1}, t_j],$$

for all $j = 1, \ldots, N$. Recall that $h = \max_j h_j$ and that σ_N is the shift operator defined by $(\sigma_N u^{(N)})^j = u^{j-1}$ for $j = 1, \ldots, N$.

Lemma 3.6 *It holds for a positive constant c independent of N:*

$$\begin{aligned} \|u^{(N)} - \sigma_N u^{(N)}\|_{L^2(L^2)}^2 &\le ch, & (3.9)\\ \Big\|\frac{\partial \tilde\rho_i^{(N)}}{\partial t}\Big\|_{L^2(\mathcal{V}^*)} + \|\tilde\rho_i^{(N)}\|_{L^2(H^1)} &\le c, \qquad i = 1, \ldots, n+1. & (3.10) \end{aligned}$$

Proof. Use $u^j - u^{j-1} - (u_D^j - u_D^{j-1})$ as test function in the weak formulation of (2.5)-(2.6) and use the monotonicity of ρ (see (A2)) to get

$$\begin{aligned}
c_0 h_j^{-1} \int_\Omega |u^j - u^{j-1}|^2 &\le h_j^{-1} \int_\Omega (\rho(u^j) - \rho(u^{j-1})) \cdot (u^j - u^{j-1}) \\
&= h_j^{-1} \int_\Omega (\rho(u^j) - \rho(u^{j-1})) \cdot (u_D^j - u_D^{j-1}) \\
&\quad + \int_\Omega \sum_{k=1}^{n+1} J_k^j \cdot \nabla(u_k^j - u_k^{j-1} - (u_{D,k}^j - u_{D,k}^{j-1})) \\
&\quad + \int_\Omega W(u^j, V^j) \cdot (u^j - u^{j-1} - (u_D^j - u_D^{j-1})) \\
&\quad - \int_\Omega \sum_{k=1}^{n} e_k \nabla V^j \cdot J_k^j (u_{n+1}^j - u_{n+1}^{j-1} - (u_{D,n+1}^j - u_{D,n+1}^{j-1})) \\
&\le \frac{c_0}{2} h_j^{-1} \int_\Omega |u^j - u^{j-1}|^2 + c h_j^{-1} \int_\Omega |u_D^j - u_D^{j-1}|^2 \\
&\quad + \sum_{k=1}^{n+1} (\|J_k^j\|_{0,2} + \|W_k(u^j, V^j)\|_{0,2})(\|u_k^j - u_k^{j-1}\|_{1,2} + \|u_D^j - u_D^{j-1}\|_{1,2}) \\
&\quad + c \sum_{k=1}^{n} \|\nabla V^j\|_{0,p} \|J_k^j\|_{0,2} (\|u_{n+1}^j - u_{n+1}^{j-1}\|_{0,q} + \|u_{D,n+1}^j - u_{D,n+1}^{j-1}\|_{0,q})
\end{aligned}$$

with p and q as in the proof of Lemma 3.5. Summing this inequality over j from 1 to N yields:

$$\begin{aligned}
&h^{-1} \|u^{(N)} - \sigma_N u^{(N)}\|_{L^2(L^2)}^2 \\
&\le \sum_{j=1}^{N} h_j^{-1} \int_\Omega |u^j - u^{j-1}|^2 \\
&\le c\Big(1 + \|J^{(N)}\|_{L^2(L^2)} + \|u^{(N)}\|_{L^2(L^2)} + \|V^{(N)}\|_{L^2(L^2)} \\
&\qquad + \|\nabla V^{(N)}\|_{L^\infty(W^{1,p})} \|J^{(N)}\|_{L^2(L^2)}\Big) \Big(\|u^{(N)}\|_{L^2(H^1)} + \|u_D^{(N)}\|_{L^2(H^1)} \\
&\qquad + \Big\| \frac{\partial \tilde{u}_D^{(N)}}{\partial t} \Big\|_{L^2(L^2)}\Big) \\
&\le c,
\end{aligned}$$

using Lemmas 3.4 and 3.5. This proves (3.9). To prove (3.10) observe that for $i = 1, \ldots, n+1$

$$\|\nabla \tilde{\rho}_i^{(N)}\|_{L^2(L^2)}^2 \le c \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} (\|\nabla u^j\|_{0,2}^2 + \|\nabla u^{j-1}\|_{0,2}^2) \le c(1 + \|u^{(N)}\|_{L^2(H^1)}^2) \le c$$

by Lemma 3.5. Since for $i = 1, \ldots, n+1$

$$\|\mathrm{div}\, J_i^{(N)}\|_{L^2(\mathcal{V}^*)} \le c,$$

we get immediately for $i = 1, \ldots, n$

$$\begin{aligned}\Big\| \frac{\partial \tilde\rho_i^{(N)}}{\partial t} \Big\|^2_{L^2(\mathcal{V}^*)} &= \sum_{j=1}^N h_j^{-1} \|\rho(u^j) - \rho(u^{j-1})\|^2_{\mathcal{V}^*} \\ &\le \|\operatorname{div} J_i^{(N)}\|^2_{L^2(\mathcal{V}^*)} + \|W_i(u^{(N)}, V^{(N)})\|^2_{L^2(L^2)} \\ &\le c.\end{aligned}$$

For $i = n+1$, observe that $\nabla V^j \cdot J_i^j$ is bounded in $L^2(L^{2p/(p+2)})$ independently of N with p as in Lemma 3.4 and that $L^{2p/(p+2)} \hookrightarrow \mathcal{V}^*$, and therefore

$$\begin{aligned}\Big\| \frac{\partial \tilde\rho_{n+1}^{(N)}}{\partial t} \Big\|_{L^2(\mathcal{V}^*)} &\le \|\operatorname{div} J_{n+1}^{(N)}\|_{L^2(\mathcal{V}^*)} + \|W_{n+1}(u^{(N)}, V^{(N)})\|_{L^2(L^2)} \\ &\quad + \sum_{k=1}^n \|e_k \nabla V^{(N)} \cdot J_k^{(N)}\|_{L^2(L^{2p/(p+2)})} \\ &\le c.\end{aligned}$$

This proves the lemma. □

Proof of Theorems 2.1 and 2.2.

First step: strong convergence of $u^{(N)}$. We choose partitions of $[0,\tau]$ satisfying (2.4). From Lemma 3.6 follows that for all $i = 1, \ldots, n+1$ the sequences $(\tilde\rho_i{}^{(N)})_N$ are bounded in $L^2(H^1) \cap H^1(\mathcal{V}^*)$ and therefore relatively compact in $L^2(Q_\tau)$ (by Aubin's lemma, see [21, 29]). Taking into account Lemma 3.5, there exists a subsequence of N, not relabelled, such that

$$u^{(N)} \rightharpoonup u \quad \text{weakly in } L^2(H^1), \tag{3.11}$$

$$\tilde\rho^{(N)} \rightharpoonup r \quad \text{weakly in } L^2(H^1) \text{ and in } H^1((\mathcal{V}^*)^{n+1}), \tag{3.12}$$

$$\tilde\rho^{(N)} \to r \quad \text{in } L^2(Q_\tau; \mathbb{R}^{n+1}) \qquad (N \to \infty). \tag{3.13}$$

Now we identify r with $\rho(u)$ by using a monotonicity argument. From Lemma 3.6 we conclude

$$u^{(N)} - \sigma_N u^{(N)} \to 0 \qquad \text{in } L^2(Q_\tau; \mathbb{R}^{n+1}) \quad (N \to \infty),$$

thus, ρ being uniformly Lipschitz continuous,

$$\rho(u^{(N)}) - \rho(\sigma_N u^{(N)}) \to 0 \qquad \text{in } L^2(Q_\tau; \mathbb{R}^{n+1}) \quad (N \to \infty).$$

Recalling that

$$\tilde\rho^{(N)}(x,t) = h_j^{-1}(t_j - t)(\rho(u^j) - \rho(u^{j-1})) + \rho(u^j) \ \text{ for } x \in \Omega,\ t \in (t_{j-1}, t_j],$$

for all $j = 1, \ldots, N$, we conclude

$$\|\tilde\rho^{(N)} - \rho(u^{(N)})\|_{L^2(L^2)} \le \|\rho(u^{(N)}) - \rho(\sigma_N u^{(N)})\|_{L^2(L^2)} \to 0 \quad (N \to \infty),$$

which implies

$$\rho(u^{(N)}) \to r \qquad \text{in } L^2(Q_\tau; \mathbb{R}^{n+1}). \tag{3.14}$$

It holds for all $v \in L^2(Q_\tau; \mathbb{R}^{n+1})$

$$\int_{Q_\tau} (\rho(u^{(N)}) - \rho(v)) \cdot (u^{(N)} - v) \geq 0.$$

Letting $N \to \infty$ gives, taking into account (3.11) and (3.14),

$$\int_{Q_\tau} (r - \rho(v)) \cdot (u - v) \geq 0,$$

and hence $r = \rho(u)$ by the strong monotonicity of ρ. Since

$$c_0 \int_{Q_\tau} |u^{(N)} - u|^2 \leq \int_{Q_\tau} (\rho(u^{(N)}) - \rho(u)) \cdot (u^{(N)} - u) \to 0 \qquad (N \to \infty),$$

we get finally

$$u^{(N)} \to u \qquad \text{in } L^2(Q_\tau; \mathbb{R}^{n+1}). \tag{3.15}$$

Second step: convergence of $V^{(N)}$ *and* $J^{(N)}$. By Lemma 3.1 the sequence $(V^{(N)})$ is uniformly bounded in $L^2(H^1)$, i.e. there exists a subsequence (not relabelled) such that

$$V^{(N)} \rightharpoonup V \qquad \text{weakly in } L^2(H^1) \tag{3.16}$$

and therefore

$$\Delta V = -e \cdot \rho(u) - C, \qquad V - V_D \in L^2(\mathcal{V}),$$

since $V_D^{(N)} \to V_D$ in $L^2(H^1)$ (see Section 2). This result implies, together with (3.15),

$$V^{(N)} \to V \qquad \text{in } L^2(H^1). \tag{3.17}$$

The strong convergence of $u^{(N)}$ and $V^{(N)}$ in $L^2(L^2)$ implies $L_{ik}(u^{(N)}, V^{(N)}) \to L_{ik}(u, V)$ in $L^2(L^2)$. From (3.5) and (3.6) we conclude that $u_{n+1}^{(N)} \nabla V^{(N)} \to u_{n+1} \nabla V$ in $L^1(L^1)$. Hence, since $L_{ik}(u, V) \in L^\infty(Q_\tau)$, $J^{(N)} \rightharpoonup J$ weakly in $L^1(Q_\tau; \mathbb{R}^{n+1})$, where $J = (J_1, \ldots, J_{n+1})$ and

$$J_i = -\sum_{k+1}^{n+1} L_{ik}(u, V)(\nabla u_k - e_k u_{n+1} \nabla V), \quad i = 1, \ldots, n+1.$$

In view of the bound (3.7) this yields (for a subsequence)

$$J^{(N)} \rightharpoonup J \qquad \text{weakly in } L^2(L^2) \quad (N \to \infty). \tag{3.18}$$

Third step: the limit $N \to \infty$ *in the equations.* The weak formulation of (2.5) is

$$\int_0^T \left\langle \frac{\partial \tilde{\rho}_i{}^{(N)}}{\partial t}, \phi \right\rangle_{\mathcal{V}^*, \mathcal{V}} - \int_{Q_\tau} J_i^{(N)} \cdot \nabla \phi = \int_{Q_\tau} W_i(u^{(N)}, V^{(N)}) \phi,$$

for $\phi \in L^2(\mathcal{V})$, where $i = 1, \ldots, n$. Using (3.12), (3.15)-(3.18), we can let $N \to \infty$ to get the weak formulation of (1.7). In order to get the weak formulation of (1.8) we observe that for $\phi \in L^2(\mathcal{V}) \cap L^\infty(L^\infty)$

$$\sum_{k=1}^n \int_{Q_\tau} e_k J_k^{(N)} \cdot \nabla V^{(N)} \phi \to \sum_{k=1}^n \int_{Q_\tau} e_k J_k \cdot \nabla V \phi \qquad (N \to \infty),$$

using (3.17) and (3.18). Finally, $u - u_D \in L^2(\mathcal{V}^{n+1})$ and $\rho(u(0)) = \rho(u^0)$ in Ω, since the bound (3.10) implies $\rho(u) \in C^0([0,\tau]; L^2(\Omega))$, and $\tilde{\rho}^{(N)}(t = 0) = \rho(u^0)$. Furthermore, by the strong monotonicity, $u \in C^0([0,\tau]; L^2(\Omega))$. The theorems are proved. $\square$

4 Long-time behavior of the solution

In this section we show that the solution constructed in the previous section converges to the thermal equilibrium state if the boundary data are in thermal equilibrium. The thermal equilibrium state is defined by $J_i = 0$, $i = 1, \ldots, n+1$, which is equivalent to $\nabla w_i = 0$ for $i = 1, \ldots, n+1$. In particular, the temperature $T = -1/w_{n+1}$ is constant in Ω.

The following assumptions are needed:

(A6) $\sum_{k=1}^n W_k(u_D, V_D) = 0$.

(A7) $u_D \in H^1(\Omega)$, $V_D \in H^1(\Omega) \cap L^\infty(\Omega)$, $\nabla w_D = 0$, and $\Delta V_D = -e \cdot \rho(u_D) - C$.

The first assumption is needed to show that the entropy function $S(t)$ introduced in Section 1 is non-increasing. In the case of the bipolar energy-transport model for semiconductors (see Section 2), it holds $W_1(u_D, V_D) = W_2(u_D, V_D) = 0$, so the assumption (A6) is satisfied for this model. The second assumption means that the boundary data is time-independent and in thermal equilibrium. Setting $c_k = w_{D,k} = \text{const.}$, the differential equation for V_D is equivalent to

$$\Delta V_D = -e \cdot \rho(c_k + e_k \overline{u} V_D) - C.$$

Using mixed boundary conditions for V_D as in Section 2, we see that this problem is uniquely solvable in $H^1(\Omega)$ since ρ is monotone and $\overline{u} < 0$.

Theorem 4.1 *Let (A1)-(A7) hold. Then there exist c, $\mu > 0$ such that for $t > 0$*

$$\|u(t) - u_D\|_{0,2,\Omega} + \|V(t) - V_D\|_{1,2,\Omega} \le c e^{-\mu t},$$

where (u, V) is the solution constructed in Section 3. Furthermore, $S(t) \ge 0$ is non-increasing and $S(t) \to 0$ as $t \to \infty$.

Proof. In the proof we use ideas of [15]. First, introduce the entropy density

$$s(t) \stackrel{\text{def}}{=} \rho(u(t)) \cdot (u(t) - u_D) - (\chi(u(t)) - \chi(u_D)) - \frac{1}{2} u_{D,n+1} |\nabla(V(t) - V_D)|^2.$$

Thanks to (A2) and $u_{D,n+1} = \overline{u} < 0$ it holds

$$\begin{aligned} s(t) &\geq c_1(|u(t) - u_D|^2 + |\nabla(V(t) - V_D)|^2), &\quad (4.1)\\ s(t) &\leq c_2(|u(t) - u_D|^2 + |\nabla(V(t) - V_D)|^2). &\quad (4.2) \end{aligned}$$

where c_1, $c_2 > 0$. Set

$$S_0(t) = S_0(u(t)) \stackrel{\text{def}}{=} \int_\Omega (\rho(u(t)) \cdot (u(t) - u_D) - (\chi(u(t)) - \chi(u_D))).$$

Using methods of convex analysis as in [15, p. 15] and the convergence results of the previous section, it can be seen that for a.e. $0 \leq \tau_1 < \tau_2 < \infty$

$$S_0(\tau_2) - S_0(\tau_1) \leq \int_{\tau_1}^{\tau_2} \sum_{k=1}^{n+1} \Big\langle \frac{\partial}{\partial t} \rho_k(u(t)), u_k(t) - u_{D,k} \Big\rangle_{\mathcal{V}^*,\mathcal{V}} dt. \quad (4.3)$$

This result also follows directly from the convexity of χ. Indeed, let $0 = t_0 < t_1 < \cdots < t_N = \tau$ be a partition of $[0, \tau]$ and let ℓ, m be such that $\tau_1 \in (t_{\ell-1}, t_\ell]$ and $\tau_2 \in (t_{m-1}, t_m]$. Then

$$\begin{aligned} \int_{t_\ell}^{t_m} \sum_{k=1}^{n+1} \Big\langle \frac{\partial}{\partial t} \tilde{\rho}_k^{(N)}, u_k^{(N)} - u_{D,k} \Big\rangle_{\mathcal{V}^*,\mathcal{V}} dt &= \sum_{j=\ell+1}^{m} \int_\Omega (\rho(u^j) - \rho(u^{j-1})) \cdot (u^j - u_D) \\ &\geq \sum_{j=\ell+1}^{m} (S_0(u^j) - S_0(u^{j-1})) \\ &= S_0(u^m) - S_0(u^\ell) \end{aligned}$$

and thus

$$\int_{\tau_1}^{\tau_2} \sum_{k=1}^{n+1} \Big\langle \frac{\partial}{\partial t} \tilde{\rho}_k^{(N)}, u_k^{(N)} - u_{D,k}^{(N)} \Big\rangle_{\mathcal{V}^*,\mathcal{V}} dt + \mathcal{O}(h) \geq S_0(u^{(N)}(\tau_2)) - S_0(u^{(N)}(\tau_1)).$$

Letting $N \to \infty$ (thus $h \to 0$) gives (4.3).

Now we estimate similarly as in the beginning of the proof of Lemma 3.1. Employing (1.7)-(1.10), we get for all $\mu \geq 0$,

$$\begin{aligned} e^{\mu\tau} S(\tau) - S(0) &\leq \int_0^\tau e^{\mu t} \Big(\sum_{k=1}^{n+1} \Big\langle \frac{\partial}{\partial t} \rho_k(u), u_k - u_{D,k} \Big\rangle_{\mathcal{V}^*,\mathcal{V}} \\ &\qquad - u_{D,n+1} \int_\Omega \nabla \frac{\partial V}{\partial t} \cdot \nabla(V - V_D) + \mu \int_\Omega s(t) \Big) dt \\ &= \int_0^\tau e^{\mu t} \Big(- \sum_{k=1}^{n} \langle \operatorname{div} J_k, u_k - u_{D,k} \rangle_{\mathcal{V}^*,\mathcal{V}} \end{aligned}$$

$$
\begin{aligned}
& + \sum_{k=1}^{n} \langle W_k(u,V), u_k - u_{D,k} \rangle_{\mathcal{V}^*,\mathcal{V}} \\
& + \Big\langle \frac{\partial \rho_{n+1}}{\partial t}, u_{n+1} - u_{D,n+1} \Big\rangle_{\mathcal{V}^*,\mathcal{V}} \\
& - u_{D,n+1} \sum_{k=1}^{n} e_k \Big\langle \frac{\partial}{\partial t} \rho_k(u), V - V_D \Big\rangle_{\mathcal{V}^*,\mathcal{V}} + \mu \int_\Omega s(t) \Big) dt \\
= & \int_0^\tau e^{\mu t} \int_\Omega \Big(\sum_{k=1}^{n+1} I_k \cdot \nabla (w_k - w_{D,k}) \\
& + \sum_{k=1}^{n+1} W_k(u,V)(u_k - u_{D,k}) + \mu s(t) \Big) dt \\
\le & \int_0^\tau e^{\mu t} \int_\Omega \Big(-\delta(V(t)) \sum_{k=1}^{n+1} |\nabla (w_k - w_{D,k})|^2 \qquad (4.4) \\
& + c_2 \mu |u - u_D|^2 + c_2 \mu |\nabla (V - V_D)|^2 \Big) dt,
\end{aligned}
$$

where we have used (A4), (A6), (4.2), and (3.3) for the last inequality. If $\mu = 0$ we get, using (4.1),

$$
\|u(\tau) - u_D\|_{0,2,\Omega} + \|V(\tau) - V_D\|_{1,2,\Omega} \le c,
$$

with $c > 0$ independent of $\tau > 0$. Therefore, $\rho(u) \in L^\infty(0,\infty; L^2(\Omega))$ and, by Stampacchia's elliptic estimate, $V \in L^\infty(0,\infty; L^\infty(\Omega))$. Hence there exists $\delta_0 > 0$ such that $\delta(V(t)) \ge \delta_0$. For $\mu > 0$ we obtain from (4.4)

$$
\begin{aligned}
& e^{\mu\tau} c_1 \int_\Omega (|u(\tau) - u_D|^2 + |\nabla (V(\tau) - V_D)|^2) \le e^{\mu\tau} S(\tau) \\
\le & \; S(0) + \int_0^\tau e^{\mu t} \int_\Omega \Big(-\delta_0 \sum_{k=1}^{n+1} |\nabla (w_k - w_{D,k})|^2 + c_2 \mu |u - u_D|^2 \\
& + c_2 \mu |\nabla (V - V_D)|^2 \Big) dt.
\end{aligned}
$$

Now, if we can show that

$$
\int_\Omega (|u - u_D|^2 + |\nabla (V - V_D)|^2) \le c_3 \int_\Omega \sum_{k=1}^{n+1} |\nabla (w_k - w_{D,k})|^2, \qquad (4.5)
$$

then the theorem follows after choosing $\mu \le \delta_0/(c_2 c_3)$. To prove (4.5) we use (A2), (A7), and (1.9):

$$
\begin{aligned}
& \frac{c_0}{4} \int_\Omega |u - u_D|^2 + c \int_\Omega |w - w_D|^2 \ge \int_\Omega (\rho(u) - \rho(u_D)) \cdot (w - w_D) \\
\ge & \; c_0 \int_\Omega |u - u_D|^2 - \overline{u} \int_\Omega e \cdot (\rho(u) - \rho(u_D))(V - V_D) \\
& - \int_\Omega e \cdot (\rho(u) - \rho(u_D))(u_{n+1} - \overline{u}) V \\
\ge & \; \frac{c_0}{2} \int_\Omega |u - u_D|^2 - \overline{u} \int_\Omega |\nabla (V - V_D)|^2 - c(V) \int_\Omega |w_{n+1} - w_{D,n+1}|^2,
\end{aligned}
$$

where $c(V) > 0$ depends on the $L^\infty(\Omega \times (0, \infty))$ norm of V. Employing Poincare's inequality and observing $\overline{u} < 0$, we obtain (4.5). $\square$

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