Zero-Relaxation-Time Limits in the Hydrodynamic Equations for Plasmas Revisited

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Abstract

A rigorous proof of the zero-relaxation-time limits in the hydrodynamic equations for plasmas is given. The equations consist of the Euler equations for the electrons and/or ions coupled with a linear or nonlinear Poisson equation. The proof is based on high energy estimates for the Euler equations and appropriate compactness arguments. The theorem is valid for all adiabatic states for both electrons and ions and thus improves our previous results [6].

Keywords. Euler-Poisson system, zero-relaxation-time limits, high energy estimates, compensated compactness, compactness by convexity, plasmas.

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1 Introduction

The Euler-Poisson systems for plasmas have been studied intensively both analytically and numerically in the literature (see, e.g., [1, 2, 4, 8, 11]). In the zero-relaxation-time limit, the model equations can be approximated by the drift-diffusion equations, which are much simpler to treat, e.g. numerically, than the hyperbolic problem. In this paper we give a rigorous proof of the zero-relaxation-time limit for *all adiabatic states*. More precisely, consider an unmagnetized plasma consisting of electrons with mass m_e and charge $q_e = -1$ and of a single species of ions with mass m_i and charge $q_i = +1$. Denoting by n_e and j_e (respectively, n_i and j_i) the particle and current densities of the electrons (respectively, ions) and by ϕ the electric potential, these variables satisfy the following scaled Euler-Poisson system:

$$\partial_t n_\alpha + \partial_x j_\alpha = 0 \tag{1}$$

$$m_\alpha \partial_t j_\alpha + m_\alpha \partial_x (j_\alpha^2/n_\alpha) + \partial_x p_\alpha(n_\alpha) = -q_\alpha n_\alpha \partial_x \phi - m_\alpha \frac{j_\alpha}{\tau_\alpha}, \tag{2}$$

$$-\lambda^2 \partial_{xx}\phi = n_i - n_e, \tag{3}$$

where $\alpha = e, i$ and $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$. Here, $\lambda > 0$ is the scaled Debye length and $\tau_e > 0, \tau_i > 0$ are the scaled relaxation time constants for electrons and ions, respectively. The pressure functions p_α are usually of the form

$$p_\alpha(n_\alpha) = a_\alpha^2 n_\alpha^{\gamma_\alpha}, \qquad n_\alpha \geq 0,$$

where $\gamma_\alpha \geq 1$ and $a_\alpha > 0$ are constants. The plasma is called *isothermal* if $\gamma_\alpha = 1$ and *adiabatic* if $\gamma_\alpha > 1$. The system (1)–(3) is completed by the following initial and boundary conditions:

$$n_\alpha(0,x) = n_{\alpha 0}(x), \quad j_\alpha(0,x) = j_{\alpha 0}(x), \quad x \in \mathbb{R}, \tag{4}$$

$$\lim_{|x|\to\infty} \phi(t,x) = 0, \qquad \text{a.e. } t > 0. \tag{5}$$

The homogeneous boundary condition (5) means that the plasma is in equilibrium at infinity. The existence of global solutions to the problem (1)–(5) has been shown in [1, 9, 11].

In the limit $m_e \to 0$ we get formally the following equations for weakly ionized plasmas:

$$\partial_t n_i + \partial_x j_i = 0, \tag{6}$$

$$m_i\partial_t j_i + m_i\partial_x(j_i^2/n_i) + \partial_x p_i(n_i) = -n_i\partial_x\phi - m_i\frac{j_i}{\tau_i}, \tag{7}$$
$$-\lambda^2\partial_{xx}\phi = n_i - f(\phi), \tag{8}$$

where the function f is the inverse of the enthalpy h_e defined by

$$h_e'(s) = p_e'(s)/s \quad \text{for } s > 0.$$

The existence of global weak entropy solutions to (6)– (8) is proved in [2] when $\gamma_e = \gamma_i = 1$, and in [10] when $\gamma_i > 1$. In the second case, the following assumptions are needed :

(A1) $0 \le n_{i0},\ j_{i0}/n_{i0} \in L^\infty(\mathbb{R}),\ n_{i0}(x) = 0$ for $|x| \ge L$ for some $L > 0,\ \alpha = e, i$.

(A2) $f \in C^1([0,\infty)),\ f(0) = 0,\ f(+\infty) = +\infty,\ f'(s) > 0$ for $s > 0$.

It is easy to check that the assumption (A2) includes the case $\gamma_e > 1$. The limit $m_e \to 0$ is studied in [3]. We also refer to [7] for a related zero-electron-mass limit.

For the zero-relaxation-time limit in the models (1)– (3) and (6)–(8) we introduce a scaling of time $s = \tau t$ and define

$$N_\alpha(s,x) = n_\alpha(\frac{s}{\tau}, x), \quad J_\alpha(s,x) = \frac{1}{\tau} j_\alpha(\frac{s}{\tau}, x), \quad \Phi(s,x) = \phi(\frac{s}{\tau}, x),$$

where, for simplicity, we have taken $\tau = \tau_e = \tau_i$. We also set $m_e = m_i = \lambda = 1$. Setting again $t = s$, the problem (1)–(3) become, for $\alpha = e, i$,

$$\partial_t N_\alpha + \partial_x J_\alpha = 0, \tag{9}$$
$$\tau^2\partial_t J_\alpha + \tau^2\partial_x(J_\alpha^2/N_\alpha) + \partial_x p_\alpha(N_\alpha) = -q_\alpha N_\alpha \partial_x\Phi - J_\alpha, \tag{10}$$
$$-\partial_{xx}\Phi = N_i - N_e. \tag{11}$$

Letting formally $\tau \to 0$, we get the drift-diffusion equations

$$\partial_t N_\alpha - \partial_x(\partial_x p_\alpha(N_\alpha) + q_\alpha N_\alpha \partial_x\Phi) = 0, \quad \alpha = e, i \tag{12}$$
$$-\partial_{xx}\Phi = N_i - N_e, \tag{13}$$

with initial and boundary conditions ($\alpha = e, i$)

$$N_\alpha(0,x) = n_{0\alpha}(x),\ x \in \mathbb{R},\ \lim_{|x|\to\infty} \Phi(t,x) = 0,\ \text{a.e. } t > 0. \tag{14}$$

Furthermore, using the above relaxation-time scaling in equations (6)–(8) we obtain

$$\partial_t N_i + \partial_x J_i = 0, \tag{15}$$

$$\tau^2 \partial_t J_i + \tau^2 \partial_x (J_i^2/N_i) + \partial_x p_i(N_i) = -N_i \partial_x \Phi - J_i, \tag{16}$$

$$-\partial_{xx}\Phi = N_i - f(\Phi), \tag{17}$$

which yields, after letting formally $\tau \to 0$,

$$\partial_t N_i - \partial_x(\partial_x p_i(N_i) + N_i \partial_x \Phi) = 0, \tag{18}$$

$$-\partial_{xx}\Phi = N_i - f(\Phi). \tag{19}$$

The zero-relaxation-time limit in the hydrodynamic equations (9)–(11) for isothermal plasmas has been proved in [5]. For adiabatic plasmas, this limit has been obtained in [8] under the assumption of given uniform L^∞ estimates. The authors proved in [6] rigorously the zero-relaxation-time limit for adiabatic plasmas under the two (main) assumptions:

(H1) $f \in C^1([0,\infty)),\ f(0) = 0,\ f'(s) > f_0 > 0$ for all $s \geq 0$;

(H2) $p_i(s) = a_i^2 s^\gamma$ for $s \geq 0$ with $\gamma = 1 + 2/m,\ m \in \mathbb{N}^*$,

where $f_0 > 0$ is a constant. The condition (H1) includes only the pressure function

$$p_e(s) = a_e^2 s^{\gamma_e}, \quad s \geq 0,$$

for $\gamma_e \geq 2$. In this paper the zero-relaxation-time limit is shown for *all* adiabatic states

$$p_i(s) = a_i^2 s^{\gamma_i},\ p_e(s) = a_e^2 s^{\gamma_e}, \quad s \geq 0,\ \text{with } \gamma_i, \gamma_e > 1. \tag{20}$$

More precisely, we suppose that

(H1') $f(s) = f_0 s^\delta, \quad s \geq 0$, with $f_0 > 0,\ \delta > 0$;

(H2') $p_i(s) = a_i^2 s^\gamma, \quad s \geq 0$, with $\gamma > 1$.

Clearly, these assumptions include the functions (20) with $\delta = 1/(\gamma_e - 1)$.

In the next section we state the main results of this paper, Theorems 2.1 and 2.2. In Section 3 we derive the necessary uniform a priori estimates for N_α, J_α, and Φ. Finally, the proofs of the main theorems are given in Section 4.

2 Statement of the main results

We rewrite the Euler-Poisson system (15)–(17) as follows:

$$\partial_t N_\tau + \partial_x J_\tau = 0, \tag{21}$$

$$\tau^2 \partial_t J_\tau + \tau^2 \partial_x (J_\tau^2 / N_\tau) + \partial_x p(N_\tau) = -N_\tau \partial_x \Phi_\tau - J_\tau \tag{22}$$

$$-\partial_{xx} \Phi_\tau = N_\tau - f(\Phi_\tau), \tag{23}$$

for $x \in \mathbb{R},\ t > 0$, with the initial and boundary conditions

$$N_\tau(0,x) = n_{i0}(x),\ J_\tau(0,x) = \tau^{-1} j_{i0}(x),\ x \in \mathbb{R}, \tag{24}$$

$$\lim_{|x|\to\infty} \Phi_\tau(t,x) = 0, \quad \text{a.e. } t > 0. \tag{25}$$

In [10] it is shown under the assumptions (A1)–(A2), (H2') that the problem (21)–(25) has a global weak entropy solution $(N_\tau, J_\tau, \Phi_\tau)$ satisfying

$$0 \le N_\tau,\ J_\tau/N_\tau \in L^\infty(Q_T),\ 0 \le \Phi_\tau \in L^\infty(0,T; W^{2,\infty}(\mathbb{R})),$$

where $Q_T = (0,T) \times \mathbb{R}$ and $T > 0$. Our first main result is as follows:

Theorem 2.1 *Let (A1), (H1') and (H2') hold and let $T > 0$. Let $(N_\tau, J_\tau, \Phi_\tau)$ be a weak entropy solution of (21)–(25). Then, as $\tau \to 0$, passing if necessary to subsequences, $(N_\tau, J_\tau, \Phi_\tau)$ converges to (N, J, Φ), a solution of (18)–(19), (14), in the following sense:*

$$\begin{aligned} N_\tau \to N &\quad \text{in} \quad L^p_{loc}(Q_T) \text{ strongly, for any } p \in (1,\infty), \\ J_\tau \rightharpoonup J &\quad \text{in} \quad L^2(Q_T) \text{ weakly}, \\ \tau^2 J_\tau^2 / N_\tau \to 0 &\quad \text{in} \quad L^2(Q_T) \text{ strongly}, \\ \Phi_\tau \to \Phi &\quad \text{in} \quad L^p_{loc}(Q_T) \text{ strongly, for any } p \in [1+\delta, \infty), \end{aligned}$$

where (N, J, Φ) satisfies

$$\begin{aligned} N &\in L^\infty(0,T; L^p(\mathbb{R})) \text{ for any } p \in (1,\infty), \\ J &\in L^2(Q_T), \\ \Phi &\in L^\infty(0,T; W^{2,p}(\mathbb{R})) \text{ for any } p \in [\alpha_0, \infty),\ \text{with } \alpha_0 = \max(2, 1+\delta). \end{aligned}$$

The proof of the zero-relaxation-time limit in the equations (9)–(11) is completely contained in the proof of Theorem 2.1, since the Poisson equation (11) is linear (cf. Remark 1 in [6]). For the sake of completeness, we state the result as follows:

Theorem 2.2 *Let (A1) hold and let the pressure functions p_e and p_i be given by (20). Let $(N_{\alpha\tau}, J_{\alpha\tau}, \Phi_\tau)$ be a weak entropy solution of (9)–(11), with*

$$N_{\alpha\tau}(0,x) = n_{\alpha 0}(x), \; J_{\alpha\tau}(0,x) = \tau^{-1} j_{\alpha 0}(x), \; x \in \mathbb{R},$$
$$\lim_{|x|\to\infty} \Phi_\tau(t,x) = 0, \quad a.e. \; t > 0, \; \alpha = e, i.$$

Then, as $\tau \to 0$, passing eventually to subsequences, $(N_{\alpha\tau}, J_{\alpha\tau}, \Phi_\tau)$ converges to $(N_\alpha, J_\alpha, \Phi)$, a solution of (12)–(14), in the following sense $(\alpha = e, i)$

$$\begin{aligned}
N_{\alpha\tau} \to N_\alpha \quad & \text{in} \quad L^p_{loc}(Q_T) \text{ strongly, for any } p \in (1,\infty), \\
J_{\alpha\tau} \rightharpoonup J_\alpha \quad & \text{in} \quad L^2(Q_T) \text{ weakly}, \\
\tau^2 J_{\alpha\tau}^2 / N_{\alpha\tau} \to 0 \quad & \text{in} \quad L^2(Q_T) \text{ strongly}, \\
\Phi_\tau \to \Phi \quad & \text{in} \quad L^p_{loc}(Q_T) \text{ strongly, for any } p \in [1+\delta,\infty).
\end{aligned}$$

The functions N_α, J_α, Φ satisfy the regularity properties

$$\begin{aligned}
N_\alpha &\in L^\infty(0,T; L^p(\mathbb{R})) \text{ for any } p \in (1,\infty), \\
J_\alpha &\in L^2(Q_T), \\
\Phi &\in L^\infty(0,T; W^{2,p}(\mathbb{R})) \text{ for any } p \in [\alpha_0,\infty), \text{ with } \alpha_0 = \max(2, 1+\delta).
\end{aligned}$$

3 A priori estimates

The following result is proved in [6]:

Lemma 3.1 *Let (A1) and (A2) hold. Then:*

(i) The sequence $(f(\Phi_\tau))_{\tau>0}$ is bounded in $L^\infty(0,T; L^1(\mathbb{R}))$ and $(\partial_x \Phi_\tau)_{\tau>0}$ is bounded in $L^\infty(Q_T)$.

(ii) The sequence $(N_\tau)_{\tau>0}$ is bounded in $L^\infty(0,T; L^p(\mathbb{R}))$ for any $p \in [1,\infty)$.

(iii) The sequence $(N_\tau^{\alpha_1^{(k)}} J_\tau^2)_{\tau>0}$ is bounded in $L^1(Q_T)$, $k \geq 1$.

(iv) The sequence $(\tau^2 N_\tau^{\alpha_2^{(k)}} J_\tau^4)_{\tau>0}$ is bounded in $L^1(Q_T)$, $k \geq 2$.

For (iii) and (iv) we have introduced the numbers

$$\alpha_i^{(k)} = (k-i)(\gamma-1) - 2i + 1, \quad 0 \leq i \leq k.$$

Part (i) of the above lemma is a consequence of Lemma 3 and (4.3) of [6]. The parts (ii)–(iv) are proved in Lemmas 4 and 5 of [6] by using high energy estimates for the Euler equation.

In the following, we assume that the hypotheses (H1')–(H2') hold.

Lemma 3.2 *It holds:*

(i) The sequence $(J_\tau)_{\tau>0}$ *is bounded in* $L^2(Q_T)$.

(ii) The sequence $(\tau J_\tau^2/N_\tau)_{\tau>0}$ *is bounded in* $L^2(Q_T)$.

Proof. : We use the following simple inequality to prove Lemma 3.2. If $a_1 \le a_2 \le a_3$, then

$$x^{a_2} \le x^{a_1} + x^{a_3},\ \forall\, x \ge 0, \tag{26}$$

where we make the convention that $x^{a_1} = \infty$ if $x = 0$ and $a_1 < 0$.

Take $k_0 \ge 1$ large enough such that $\alpha_1^{(k_0)} \ge 0$. Since $\alpha_1^{(1)} = -1$, we obtain

$$J_\tau^2 \le N_\tau^{\alpha_1^{(1)}} J_\tau^2 + N_\tau^{\alpha_1^{(k_0)}} J_\tau^2. \tag{27}$$

Therefore, (i) follows from Lemme 3.1(iii). Similarly, take k_1 large enough such that $\alpha_2^{(k_1)} \ge -2$. Since $\alpha_2^{(2)} = -3$, we have

$$\tau N_\tau^{-1} J_\tau^2 \le \tau N_\tau^{\alpha_2^{(2)}/2} J_\tau^2 + \tau N_\tau^{\alpha_2^{(k_1)}/2} J_\tau^2, \tag{28}$$

which implies (ii) from Lemma 3.1(iv).

Lemma 3.3 *It holds:*

(i) $(\Phi_\tau)_{\tau>0}$ *is bounded in* $L^\infty(0,T;L^p(\mathbb{R}))$ *for any* $p \in [1+\delta,\infty]$.

(ii) $(\partial_x\Phi_\tau)_{\tau>0}$ *is bounded in* $L^\infty(0,T;L^p(\mathbb{R}))$ *for any* $p \in [2,\infty]$.

(iii) $(\partial_{xx}\Phi_\tau)_{\tau>0}$ *is bounded in* $L^\infty(0,T;L^p(\mathbb{R}))$ *for any* $p \in [1,\infty)$.

Proof. : Multiply equation (23) by Φ_τ, integrate over $\mathbb{R}$ and use (H1') to get

$$\begin{aligned}
&\int_{\mathbb{R}} |\partial_x \Phi_\tau(t,x)|^2 dx + f_0 \int_{\mathbb{R}} \Phi_\tau(t,x)^{1+\delta} dx \\
\le\ & \int_{\mathbb{R}} N_\tau(t,x)\Phi_\tau(t,x)dx \\
\le\ & \|N_\tau(t,\cdot)\|_{L^{1+1/\delta}(\mathbb{R})} \|\Phi_\tau(t,\cdot)\|_{L^{1+\delta}(\mathbb{R})} \\
\le\ & \frac{1}{2f_0}\|N_\tau(t,\cdot)\|^{1+1/\delta}_{L^{1+1/\delta}(\mathbb{R})} + \frac{f_0}{2}\|\Phi_\tau(t,\cdot)\|^{1+\delta}_{L^{1+\delta}(\mathbb{R})}.
\end{aligned}$$

Therefore, by Lemma 3.1(ii), the sequence $(\Phi_\tau)_{\tau>0}$ is bounded in $L^\infty(0,T;L^{1+\delta}(\mathbb{R}))$ and the sequence $(\partial_x\Phi_\tau)_{\tau>0}$ is bounded in $L^\infty(0,T;L^2(\mathbb{R}))$. Using Lemma 3.1(i) and an interpolation argument, we have proved (ii).

Furthermore, if $\delta\geq 1$ then $(\Phi_\tau)_{\tau>0}$ is bounded in $L^\infty(0,T;W^{1,1+\delta}(\mathbb{R}))\hookrightarrow L^\infty(Q_T)$. Otherwise, if $\delta<1$, we obtain

$$\begin{aligned}\Phi_\tau(t,x)^2 &= \int_{-\infty}^x \partial_y(\Phi_\tau(t,y)^2)dy\\ &\leq 2\int_{\mathbb{R}}|\Phi_\tau(t,y)|\,|\partial_y\Phi_\tau(t,y)|dy\\ &\leq 2\|\Phi_\tau(t,\cdot)\|_{L^{1+\delta}(\mathbb{R})}\|\partial_x\Phi_\tau(t,\cdot)\|_{L^{1+1/\delta}(\mathbb{R})}.\end{aligned}$$

Since $1+1/\delta>2$, we deduce that $(\Phi_\tau)_{\tau>0}$ is bounded in $L^\infty(Q_T)$. On the other hand, $(\Phi_\tau)_{\tau>0}$ is bounded in $L^\infty(0,T;L^{1+\delta}(\mathbb{R}))$. Thus, (i) follows from interpolation.

Finally, (i) and Lemma 3.1(i) imply that $(f(\Phi_\tau))_{\tau>0}$ is bounded in $L^\infty(0,T;L^p(\mathbb{R}))$ for any $p\in[1,\infty]$. Hence, (iii) follows from Lemma 3.1(ii) and the Poisson equation.

Lemma 3.4 *The sequence $(\partial_t\Phi_\tau^d)_{\tau>0}$ is bounded in $L^2(0,T;H^1(\mathbb{R}))$, where $d=\max(1,(\delta+1)/2)$.*

Proof. : Define the quantities

$$\begin{aligned}\Phi_\tau^h(t,x) &= \frac{1}{h}(\Phi_\tau(t+h,x)-\Phi_\tau(t,x)),\\ F_\tau^h(t,x) &= \frac{f(\Phi_\tau(t+h,x))-f(\Phi_\tau(t,x))}{\Phi_\tau(t+h,x)-\Phi_\tau(t,x)},\end{aligned}$$

if $\Phi_\tau(t+h,x)\neq\Phi_\tau(t,x)$ and $F_\tau^h(x,t)=0$ else, for $x\in\mathbb{R}$, $0<t<T-h,\ h>0$. We want to show that for any $T_0>0$ such that $T_0+h<T$, the sequence

$$\left(\frac{1}{h}(\Phi_\tau(t+h,\cdot)^d-\Phi_\tau(t,\cdot)^d)\right)_{\tau>0}$$

is bounded in $L^2(0,T_0;H^1(\mathbb{R}))$. This implies the conclusion of the lemma.

First, let $\delta<1$. The following equations hold

$$\begin{aligned}-\partial_{xx}\Phi_\tau^h+F_\tau^h\Phi_\tau^h &= N_\tau^h \quad \text{in } \mathbb{R},\\ \lim_{|x|\to\infty}\Phi_\tau^h(t,x) &= 0 \qquad \text{a.e. } t\in(0,T_0),\end{aligned} \tag{29}$$

where

$$N_\tau^h(t+h,x) = \frac{1}{h}(N_\tau(t+h,x) - N_\tau(t,x)).$$

Multiply equation (29) by Φ_τ^h and integrate over $\mathbb{R}$ to get

$$\int_{\mathbb{R}} |\partial_x \Phi_\tau^h(t,x)|^2 dx + \int_{\mathbb{R}} F_\tau^h (\Phi_\tau^h)^2(t,x)dx \le \|N_\tau^h(t,\cdot)\|_{H^{-1}(\mathbb{R})} \|\Phi_\tau^h(t,\cdot)\|_{H^1(\mathbb{R})}.$$

Since $\delta < 1$ and $(\Phi_\tau)_{\tau>0}$ is bounded in $L^\infty(Q_T)$, by Lemma 3.3(i), we have $F_\tau^h(t,x) \ge f_1$ for some constant $f_1 > 0$. Therefore, Young's inequality yields

$$\|\Phi_\tau^h\|_{L^2(0,T_0;H^1(\mathbb{R}))} \le \max(1, 1/f_1)\|N_\tau^h\|_{L^2(0,T_0;H^{-1}(\mathbb{R}))}.$$

By Lemma 3.2(i), the sequence $(\partial_x J_\tau)_{\tau>0}$ is bounded in $L^2(0,T;H^{-1}(\mathbb{R}))$, i.e. the sequence $(\partial_t N_\tau)_{\tau>0} = -(\partial_x J_\tau)_{\tau>0}$ is bounded in the same space. This implies the boundedness of $(N_\tau^h)_{\tau>0}$ in $L^2(0,T_0;\ H^{-1}(\mathbb{R}))$. Hence, $(\Phi_\tau^h)_{\tau>0}$ is bounded in $L^2(0,T_0;H^1(\mathbb{R}))$.

Now, let $\delta \ge 1$. Then $(N_\tau)_{\tau>0}$ is bounded in $H^1(0,T;H^{-1}(\mathbb{R})) \hookrightarrow C^{0,1/2}([0,T];\ H^{-1}(\mathbb{R}))$ which implies

$$N_\tau(t,\cdot) - N_\tau(0,\cdot) = \int_0^t \partial_s N_\tau(s,\cdot)ds = -\int_0^t \partial_x J_\tau(s,\cdot)ds.$$

Therefore

$$N_\tau^h(t,x) = -\frac{1}{h}\int_t^{t+h} \partial_x J_\tau(s,x)ds. \tag{30}$$

Multiplying equation (29) by Φ_τ^h and integrating over $\mathbb{R}$, we have, by using equation (30),

$$\begin{aligned}
&\int_{\mathbb{R}} |\partial_x \Phi_\tau^h(t,x)|^2 dx + \int_{\mathbb{R}} F_\tau^h (\Phi_\tau^h)^2(t,x)dx \\
&= \Big\langle -\frac{1}{h}\int_t^{t+h} \partial_x J_\tau(s,\cdot)ds, \Phi_\tau^h(t,\cdot)\Big\rangle_{H^{-1}(\mathbb{R}),H^1(\mathbb{R})} \\
&= \frac{1}{h}\int_t^{t+h} \langle J_\tau(s,\cdot), \partial_x \Phi_\tau^h(t,\cdot)\rangle_{H^{-1}(\mathbb{R}),H^1(\mathbb{R})} ds \\
&= \int_{\mathbb{R}} \frac{1}{h}\int_t^{t+h} J_\tau(s,x)ds\, \partial_x \Phi_\tau^h(t,x)dx \\
&\le \frac{1}{2}\|\partial_x \Phi_\tau^h(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\|\frac{1}{h}\int_t^{t+h} J_\tau(s,\cdot)ds\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Since $(J_\tau)_{\tau>0}$ is bounded in $L^2(Q_T)$, by Lemma 3.2(i), there exists a constant $C_0 > 0$, independent of $\tau > 0$, such that

$$\frac{1}{2}\int_{\mathbb{R}} |\partial_x \Phi_\tau^h(t,x)|^2 dx + \int_{\mathbb{R}} F_\tau^h (\Phi_\tau^h)^2(t,x)dx \le C_0. \tag{31}$$

Hence,

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \frac{f_0}{h^2}\left(\Phi_\tau(t+h,x)^\delta - \Phi_\tau(t,x)^\delta\right)(\Phi_\tau(t+h,x) - \Phi_\tau(t,x))dxdt \\
&= \int_0^T \int_{\mathbb{R}} F_\tau^h (\Phi_\tau^h)^2(t,x)dxdt \\
&\le C_0.
\end{aligned}$$

Suppose that there exists a constant $C(\delta) > 0$ such that for all $x, y \ge 0$

$$(x^\delta - y^\delta)(x-y) \ge C(\delta)\left(x^{(\delta+1)/2} - y^{(\delta+1)/2}\right)^2 \tag{32}$$

holds. Then we conclude that

$$\int_0^T \int_{\mathbb{R}} \left[\frac{1}{h}(\Phi_\tau(t+h,x)^{(\delta+1)/2} - \Phi_\tau(t,x)^{(\delta+1)/2})\right]^2 dxdt \le C_0/f_0 C(\delta),$$

and the assertion follows.

It remains to prove (32). Let $x, y \ge 0$ with $x \ne y$. Then, by Jensen's inequality,

$$\begin{aligned}
\left(\frac{x^{(\delta+1)/2} - y^{(\delta+1)/2}}{x-y}\right)^2 &= \left(\int_0^1 \frac{\delta+1}{2}(\lambda x + (1-\lambda)y)^{(\delta-1)/2} d\lambda\right)^2 \\
&\le \frac{(\delta+1)^2}{4}\int_0^1 (\lambda x + (1-\lambda)y)^{\delta-1} d\lambda \\
&= \frac{(\delta+1)^2}{4\delta}\frac{x^\delta - y^\delta}{x-y},
\end{aligned}$$

and the inequality (32) is proved with $C(\delta) = 4\delta/(\delta+1)^2$.

4 Proof of Theorem 2.1

The proof is essentially a consequence of the results of [6] and the preceding lemmas. Indeed, from Lemmas 3.1, 3.2 and 3.3, there exist functions $N \in L^\infty(0,T;L^q(\mathbb{R}))$,

$\Phi \in L^\infty(0,T;L^{q+\delta}(\mathbb{R}))$, $J \in L^2(Q_T)$, $\overline{p} \in L^\infty(0,T;L^q(\mathbb{R}))$, for any $q \in (1,\infty)$, and a subsequence (not relabeled) such that as $\tau \to 0$,

$$
\begin{aligned}
N_\tau &\rightharpoonup N && \text{in } L^q(Q_T) \text{ weakly},\\
J_\tau &\rightharpoonup J && \text{in } L^2(Q_T) \text{ weakly},\\
p_i(N_\tau) &\rightharpoonup \overline{p} && \text{in } L^q(Q_T) \text{ weakly},\\
\Phi_\tau &\rightharpoonup \Phi && \text{in } L^{q+\delta}(Q_T) \text{ weakly.}
\end{aligned}
$$

Furthermore, by Lemma 3.2(ii),

$$\tau^2 J_\tau^2/N_\tau \to 0 \quad \text{in } L^2(Q_T) \text{ strongly,}$$

and therefore,

$$\tau^2 J_\tau \to 0 \quad \text{in } L^2(Q_T) \text{ strongly.}$$

As in Lemma 9 of [6], we can prove the existence of a subsequence, still denoted by $(N_\tau, \Phi_\tau)_{\tau>0}$, such that, as $\tau \to 0$,

$$N_\tau \partial_x \Phi_\tau \rightharpoonup N \partial_x \Phi \quad \text{in } \mathcal{D}'(Q_T).$$

This convergence follows from the div-curl lemma and only uses the boundedness of $(\partial_{xx}\Phi_\tau)_{\tau>0}$ in $L^2(Q_T)$, which is a consequence of Lemma 3.3(iii). In [6, Lemma 10] it is shown, by means of the div-curl lemma and the monotonicity of $s \mapsto p(s)$ that it holds for a subsequence (not relabeled), as $\tau \to 0$,

$$N_\tau \to N \quad \text{in } L^p_{loc}(Q_T) \text{ strongly,}$$

for any $p \in (1,\infty)$. For this result we need the boundedness of the sequences $(\tau^2 J_\tau^2/N_\tau + p(N_\tau))_{\tau>0}$, $(\tau^2 J_\tau)_{\tau>0}$ and $(N_\tau \partial_x \Phi_\tau + J_\tau)_{\tau>0}$ in $L^2(Q_T)$, which follows from Lemmas 3.1, 3.2 and 3.3. These results allow to pass to the limit in the Euler equation. It remains to perform the limit $\tau \to 0$ in the Poisson equation.

If $\delta < 1$ then $(\Phi_\tau)_{\tau>0}$ is bounded in $L^\infty(0,T;H^1(\mathbb{R}))$ and $(\partial_t\Phi_\tau)_{\tau>0}$ is bounded in $L^2(0,T;H^1(\mathbb{R}))$, by Lemmas 3.3 and 3.4. Thus $(\Phi_\tau)_{\tau>0}$ is relatively compact in $L^p_{loc}(Q_T)$ for any $p \in [2,\infty)$, in view of Aubin's lemma. Therefore, there exists a subsequence, still denoted by $(\Phi_\tau)_{\tau>0}$ such that as $\tau \to 0$

$$\Phi_\tau \to \Phi \quad \text{in } L^p_{loc}(Q_T) \text{ strongly.} \tag{33}$$

This allows to pass to the limit in the Poisson equation.

Now, let $\delta \geq 1$. Then, by Lemma 3.3, $(\partial_x \Phi_\tau^{(1+\delta)/2})_{\tau>0} = ((1+\delta)/2 \cdot \Phi_\tau^{(\delta-1)/2} \partial_x \Phi_\tau)_{\tau>0}$ is bounded in $L^\infty(0,T;L^p(\mathbb{R}))$ for any $p \in [2,\infty)$. Set $\Psi_\tau = \Phi_\tau^{(\delta+1)/2}$. The sequence $(\Psi_\tau)_{\tau>0}$ is bounded in $L^\infty(0,T;H^1(\mathbb{R}))$ and $(\partial_t \Psi_\tau)_{\tau>0}$ is bounded in $L^2(0,T;H^1(\mathbb{R}))$, by Lemma 3.4. Employing Aubin's lemma, we get the existence of a subsequence (not relabeled) such that as $\tau \to 0$

$$\Psi_\tau \to \Psi \quad \text{in } L^p_{loc}(Q_T) \text{ strongly,}$$

where $0 \leq \Psi \in L^2(Q_T)$ and $p \in [2,\infty)$. Let $\widetilde{\Phi} = \Psi^{2/(1+\delta)}$. We have to show that for $q \in [1+\delta,\infty)$,

$$\Phi_\tau \to \widetilde{\Phi} \quad \text{in } L^q_{loc}(Q_T) \text{ strongly, as } \tau \to 0, \tag{34}$$

since this implies $\widetilde{\Phi} = \Phi$.

Suppose that for all $\alpha > 1$ there exists a constant $C(\alpha) > 0$ such that for all $x, y \geq 0$ it holds

$$|x-y|^\alpha \leq C(\alpha)|x^\alpha - y^\alpha|. \tag{35}$$

Then, for $p \in [2,\infty)$,

$$\begin{aligned} |\Phi_\tau - \widetilde{\Phi}|^{p(1+\delta)/2} &\leq C(\delta)|\Phi_\tau^{(1+\delta)/2} - \widetilde{\Phi}^{(1+\delta)/2}|^p \\ &= C(\delta)|\Psi_\tau - \Psi|^p, \end{aligned}$$

since $\alpha = (1+\delta)/2 \geq 1$. In view of the above convergence result for $(\Psi_\tau)_{\tau>0}$, this implies (34).

It remains to show (35). Let $x, y \geq 0$. Then

$$\begin{aligned} |x^\alpha - y^\alpha| &= |x-y| \int_0^1 \alpha(\lambda x + (1-\lambda)y)^{\alpha-1} d\lambda \\ &\geq \alpha|x-y| \int_0^1 (\lambda(1-\lambda)|x-y|)^{\alpha-1} d\lambda \\ &= \alpha \int_0^1 (\lambda(1-\lambda))^{\alpha-1} d\lambda \cdot |x-y|^\alpha, \end{aligned}$$

and the inequality (35) follows with $C(\alpha) = \left(\alpha \int_0^1 (\lambda(1-\lambda))^{\alpha-1} d\lambda\right)^{-1}$.

Thus we have proved Theorem 2.1.

References

[1] S. Cordier. Global solutions to the isothermal Euler-Poisson plasma model. *Appl. Math. Lett.* 8 (1995), 19–24.

[2] S. Cordier and Y.-J. Peng. Système Euler-Poisson non linéaire – existence globale de solutions faibles entropiques. *Mod. Math. Anal. Num.* 32 (1998), 1–23.

[3] T. Goudon, A. Jüngel, and Y.-J. Peng. Zero-electron-mass limits in hydrodynamic models for plasmas. To appear in *Appl. Math. Lett.*, 1998.

[4] X. Jiang. *Numerical Simulations of the Semiconductor Devices by Streamline-Diffusion Methods.* PhD thesis, University of British Columbia, Vancouver, Canada, 1995.

[5] S. Junca and M. Rascle. Relaxation of the isothermal Euler-Poisson system to the drift-diffusion equations. To appear in *Quart. Appl. Math.*, 1998.

[6] A. Jüngel and Y.-J. Peng. A hierarchy of hydrodynamic models for plasmas. Zero-relaxation-time limits. To appear in *Comm. P. D. E.*, 1998.

[7] A. Jüngel and Y.-J. Peng. A hierarchy of hydrodynamic models for plasmas. Zero-electron-mass limits in the drift-diffusion equations. To appear in *Annales de l'I. H. P., Anal. Non Lin.*, 1999.

[8] P. Marcati and R. Natalini. Weak solutions to a hydrodynamic model for semiconductors and relaxation to the drift diffusion equations. *Arch. Rat. Mech. Anal.* 129 (1995), 129–145.

[9] P. Marcati and R. Natalini. Weak solutions to a hydrodynamic model for semiconductors: The Cauchy problem. *Proc. Roy. Soc. Edinburgh* A125 (1995), 115–131.

[10] Y.-J. Peng. Convergence of the fractional step Lax-Friedrichs scheme and Godunov scheme for a nonlinear Euler-Poisson system. To appear in *Nonlin. Anal.*, 1998.

[11] F. Poupaud, M. Rascle, and J. Vila. Global solutions to the isothermal Euler-Poisson system with arbitrarily large data. *J. Diff. Eqs.* 123 (1995), 93–121.