Zero-mass-electrons limits in hydrodynamic models for plasmas

Thierry Goudon

Laboratoire J.A.Dieudonné, CNRS UMR 6621,
Université de Nice, F-06108 Nice cédex 2, France
goudon@math.unice.fr

Ansgar Jüngel

Fachbereich Mathematik, Technische Universität Berlin,
Straβe des 17. Juni 136, D-10623 Berlin, Germany
jungel@math.tu-berlin.de

Yue-Jun Peng

Laboratoire de Mathématiques Appliquées, CNRS UMR 6620,
Université Blaise Pascal, F-63177 Aubière cédex, France
peng@ucfma.univ-bpclermont.fr

Abstract : We study the zero-mass-electrons limits in the hydrodynamic models for plasmas consisting of electrons and ions. We prove, under suitable assumptions, that the evolution of the ions is governed by the Euler equations coupled with a nonlinear Poisson equation, as the mass of electrons tends to zero.

Keywords : Plasmas, Euler-Poisson system, Zero-mass-electrons limits.

I. Introduction

In previous works, we introduced a hierarchy of hydrodynamic models for plasmas and proved rigorously the zero-mass-electrons limits in the drift-diffusion equations [4] and the zero-relaxation-time limits in the hydrodynamic equations [5]. In this Note, we study the zero-mass-electrons limits in the hydrodynamic equations.

Let us consider an unmagnetized plasma consisting of electrons with mass m_e and charge $q_e = -1$, and of a single species of ions with mass m_i and charge $q_i = 1$. We denote by $n_e(t,x)$, $j_e(t,x)$ (respectively, $n_i(t,x)$, $j_i(t,x)$) the scaled density and electric current of electrons (respectively, ions), and by $E(t,x)$ the scaled electric field at time $t > 0$ and position $x \in I\!R$. For simplicity, set the scaled relaxation time τ_α and the scaled Debye length λ equal to 1, then this model is described by the following Euler-Poisson system **(HD-EI)** :

$$m_\alpha \partial_t n_\alpha + m_\alpha \partial_x j_\alpha = 0, \ \ t > 0, \ x \in I\!R, \tag{1}$$

$$m_\alpha \partial_t j_\alpha + \partial_x \left(\frac{m_\alpha j_\alpha^2}{n_\alpha} + p_\alpha(n_\alpha) \right) = q_\alpha n_\alpha E - m_\alpha j_\alpha, \ \ t > 0, \ x \in I\!R, \tag{2}$$

$$\partial_x E = n_i - n_e, \ \ t > 0, \ x \in I\!R, \tag{3}$$

subject to the initial and boundary conditions :

$$n_\alpha(0,x) = n_{\alpha 0}(x), \ j_\alpha(0,x) = j_{\alpha 0}(x), \ x \in I\!R, \ \ \lim_{x \to -\infty} E(t,x) = E^-(t), \ t > 0, \tag{4}$$

where the subscripts $\alpha = e$ and $\alpha = i$ stand for respectively electrons and ions, and E^- is a given function. The pressure functions are of the form :

$$p_\alpha(n_\alpha) = a_\alpha^2 n_\alpha^{\gamma_\alpha}, \quad n_\alpha \geq 0, \tag{5}$$

where $\gamma_\alpha > 1$ and $a_\alpha > 0$ are constants.

Usually, the ions are heavy compared to the electrons, i.e. $m_i \gg m_e$. Therefore, the problem of the zero-mass-electrons limit of (1)-(3) is to study the asymptotic limit as $m_e \to 0$ and $m_i = 1$ fixed. The formal limit of this problem is the model **(HD-I)** :

$$\partial_t n + \partial_x j = 0, \quad t > 0, \ x \in I\!R, \tag{6}$$

$$\partial_t j + \partial_x \left(\frac{j^2}{n} + p(n) \right) = nE - j, \quad p(n) = a^2 n^\gamma, \quad t > 0, \ x \in I\!R, \tag{7}$$

$$\partial_x p_e(n_e) = -n_e E, \quad \partial_x E = n - n_e, \quad t > 0, \ x \in I\!R, \tag{8}$$

where we have used a simplified notations $n = n_i$, $j = j_i$ and $p = p_i$ etc. If we define the electric potential φ by $-\partial_x \varphi = E$, then for $n_e > 0$, we can eliminate the variable n_e in (8) and obtain a nonlinear Poisson equation :

$$-\partial_{xx}\varphi = n - f(\varphi), \quad t > 0, \ x \in I\!R, \tag{9}$$

where the function f is defined by

$$q^{-1}(\varphi) := f(\varphi) \geq 0, \quad q'(n_e) = p_e'(n_e)/n_e, \quad n_e > 0,$$

and q^{-1} is the inverse function of q.

The existence of weak solutions to the system (1)-(3) and (6), (7), (9) has been proved in [10, 11]. The goal of this Note is to give a justification of the asymptotic limit (HD-EI) $\longrightarrow$ (HD-I) under suitable assumptions. We observe that the system satisfied by (n_e, j_e) is similar to that treated in [5] for which an a priori L^p estimate is available. That is why we suppose below a L^p type estimate for (n_e, j_e) uniform in m_e. The proof is based on the kinetic formulation, the monotonicity and compactness arguments.

II. The main results

For simplicity, we denote by $\varepsilon = m_e > 0$. Then the system (1)-(3) can be written as :

$$\partial_t n^\varepsilon + \partial_x j^\varepsilon = 0, \quad t > 0, \ x \in I\!R, \tag{10}$$

$$\partial_t j^\varepsilon + \partial_x \left(\frac{(j^\varepsilon)^2}{n^\varepsilon} + p(n^\varepsilon) \right) = n^\varepsilon E^\varepsilon - j^\varepsilon, \quad t > 0, \ x \in I\!R, \tag{11}$$

$$\varepsilon \partial_t n_e^\varepsilon + \varepsilon \partial_x j_e^\varepsilon = 0, \quad t > 0, \ x \in I\!R, \tag{12}$$

$$\varepsilon \partial_t j_e^\varepsilon + \partial_x \left(\frac{\varepsilon (j_e^\varepsilon)^2}{n_e^\varepsilon} + p_e(n_e^\varepsilon) \right) = -\varepsilon j_e^\varepsilon - n_e^\varepsilon E^\varepsilon, \quad t > 0, \ x \in I\!R, \tag{13}$$

$$\partial_x E^\varepsilon = n^\varepsilon - n_e^\varepsilon, \quad t > 0, \ x \in I\!R. \tag{14}$$

Let $T > 0$ and $Q_T = [0, T[\times I\!R$. We assume that

(H1) $0 \le n_{\alpha 0}$, $n_{\alpha 0}$, $j_{\alpha 0}/n_{\alpha 0} \in L^\infty(I\!R)$, $E^- \in L^\infty_{loc}(I\!R^+)$, supp $n_{\alpha 0} \subset I\!R$, $\alpha = e, i$,

(H2) the sequence $(n_e^\varepsilon)_{\varepsilon>0}$ is bounded in $L^{2\gamma_e}_{loc}(Q_T)$ with $\gamma_e > 1$,

(H3) the sequence $((j_e^\varepsilon)^2/n_e^\varepsilon)_{\varepsilon>0}$ is bounded in $L^2_{loc}(Q_T)$.

THEOREM. - Under the assumptions (H1)-(H3), there is a subsequence, still denoted by $(n^\varepsilon, n_e^\varepsilon, j^\varepsilon, j_e^\varepsilon, E^\varepsilon)$, such that as $\varepsilon \to 0$, $(n^\varepsilon, n_e^\varepsilon, j^\varepsilon, j_e^\varepsilon, E^\varepsilon)$ converges to (n, n_e, j, j_e, E) in the following sense :

$$n^\varepsilon \longrightarrow n, \;\; j^\varepsilon \longrightarrow j, \;\; \text{in} \;\; L^p_{loc}(Q_T) \;\; \text{strongly for any} \;\; p \in [1, +\infty[, \tag{15}$$

$$n_e^\varepsilon \longrightarrow n, \;\; \text{in} \;\; L^p_{loc}(Q_T) \;\; \text{strongly for any} \;\; p \in [1, \gamma_e + 1], \tag{16}$$

$$j_e^\varepsilon \rightharpoonup j_e, \;\; \text{in} \;\; L^2_{loc}(Q_T) \;\; \text{weakly}, \tag{17}$$

$$E^\varepsilon \rightharpoonup E, \;\; \text{in} \;\; L^\infty(Q_T) \;\; \text{weak} - *, \tag{18}$$

where $(n, n_e, j, , E)$ is a weak solution of the problem (6)-(8).

REMARK. - A direct consequence of (H2)-(H3) is that $(j_e^\varepsilon)_{\varepsilon>0}$ is bounded in $L^2_{loc}(Q_T)$, which implies (17). Hypothesis (H3) can be replaced by a weaker one :

(H3') the sequences $(j_e^\varepsilon)_{\varepsilon>0}$ and $(\sqrt{\varepsilon}(j_e^\varepsilon)^2/n_e^\varepsilon)_{\varepsilon>0}$ are bounded in $L^2_{loc}(Q_T)$.

III. The proofs of the Theorem

Under (15)-(18), the passage to the limit (HD-EI) $\longrightarrow$ (HD-I) as $\varepsilon \to 0$ is evident. According to the previous remark, it suffices to prove (15)-(16) and (18).

We first show that the sequence $(E^\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty(Q_T)$ which yields (18). Indeed, from (4) and (14), we have

$$E^\varepsilon(t, x) = E^-(t) + \int_{-\infty}^x (n^\varepsilon(t, y) - n_e^\varepsilon(t, y))dy.$$

Since j^ε and j_e^ε have compact support in $I\!R_x$ and $n_e^\varepsilon \ge 0$, $n_e^\varepsilon \ge 0$ (see [10, 11]), integrating (10) and (12) over $I\!R$, we obtain

$$\| n^\varepsilon(t, .) \|_{L^1(I\!R)} = \| n_{i0} \|_{L^1(I\!R)}, \;\; \| n_e^\varepsilon(t, .) \|_{L^1(I\!R)} = \| n_{e0} \|_{L^1(I\!R)} .$$

Hence

$$\| E^\varepsilon \|_{L^\infty(Q_T)} \le \| E^- \|_{L^\infty(0,T)} + \| n_{i0} \|_{L^1(I\!R)} + \| n_{e0} \|_{L^1(I\!R)} .$$

Next, we show the strong compactness of the sequence $(n_e^\varepsilon)_{\varepsilon>0}$. By (H2), there is $n_e \in L^{2\gamma_e}_{loc}(Q_T)$ such that for any $p \in]1, 2\gamma_e]$, up to a subsequence,

$$n_e^\varepsilon \rightharpoonup n_e, \;\; \text{in} \;\; L^p_{loc}(Q_T) \;\; \text{weakly}, \;\; \text{as} \;\; \varepsilon \to 0.$$

We define now two sequences $U^\varepsilon = (n_e^\varepsilon, j_e^\varepsilon)^t$ and $V^\varepsilon = (\varepsilon(j_e^\varepsilon)^2/n_e^\varepsilon + p_e(n_e^\varepsilon), -\varepsilon j_e^\varepsilon)^t$. It is easy to see that $(U^\varepsilon)_{\varepsilon>0}$ and $(V^\varepsilon)_{\varepsilon>0}$ are all bounded in $(L^2_{loc}(Q_T))^2$. Since $(\varepsilon j_e^\varepsilon + n_e^\varepsilon E^\varepsilon)_{\varepsilon>0}$ is bounded in $L^2_{loc}(Q_T)$, by (12)-(13), $(div_{t,x}U^\varepsilon)_{\varepsilon>0}$ is bounded in $L^2_{loc}(Q_T)$ and $(rot_{t,x}V^\varepsilon)_{\varepsilon>0}$ is bounded in $(L^2_{loc}(Q_T))^4$. Thus, the div-curl lemma [12] can be applied to $(U^\varepsilon, V^\varepsilon)$ to obtain

$$\lim_{\varepsilon\to 0} n_e^\varepsilon p_e(n_e^\varepsilon) = n_e \overline{p}, \quad \text{in} \quad \mathcal{D}'(Q_T), \tag{19}$$

where

$$\overline{p} = \lim_{\varepsilon\to 0} p_e(n_e^\varepsilon) \quad \text{in} \quad L^2_{loc}(Q_T) \quad \text{weakly}.$$

Since $s \to p_e(s)$ is monotone increasing on $I\!R^+$, by monotonicity argument, we deduce [5],

$$\overline{p} = p_e(n_e). \tag{20}$$

Hence, (5) and (19)-(20) imply that

$$\lim_{\varepsilon\to 0} \| n_e^\varepsilon \|_{L^{\gamma_e+1}(\Omega)} = \| n_e \|_{L^{\gamma_e+1}(\Omega)}, \quad \forall\, \Omega \subset Q_T \quad \text{bounded}.$$

This proves (16).

As a consequence of (16) and (18), we have, passing if necessary to subsequences :

$$n_e^\varepsilon E^\varepsilon \rightharpoonup n_e E, \quad \text{in} \quad \mathcal{D}'(Q_T), \quad \text{as} \quad \varepsilon \to 0.$$

Now we prove (15). Since $(E^\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty(Q_T)$, from [8], the sequences $(n^\varepsilon)_{\varepsilon>0}$ and $(j^\varepsilon/n^\varepsilon)_{\varepsilon>0}$ are bounded in $L^\infty(Q_T)$, as limits of the strong convergence of the approximate solutions $(n_h^\varepsilon)_{h>0}$ and $(j_h^\varepsilon/n_h^\varepsilon)_{h>0}$, constructed by the fractional step Lax-Friedrichs scheme or Godunov scheme with the space mesh size $h > 0$. Following [6], it remains to prove that for any pair of weak entropy-flux (η, q) of (10)-(11), the sequence $(\Lambda^\varepsilon)_{\varepsilon>0} := (\partial_t \eta(n^\varepsilon, j^\varepsilon) + \partial_x q(n^\varepsilon, j^\varepsilon))_{\varepsilon>0}$ lies in a compact set of $H^{-1}_{loc}(Q_T)$.

To this end, we apply the kinetic formulation to system (10)-(11). Let us denote by

$$\chi^\varepsilon(t,x,\xi) = [(w_2^\varepsilon - \xi)(\xi - w_1^\varepsilon)]_+^\lambda, \quad \zeta^\varepsilon(t,x,\xi) = (\theta\xi + (1-\theta)u^\varepsilon)\chi^\varepsilon(t,x,\xi),$$

with $u^\varepsilon = j^\varepsilon/n^\varepsilon$ and

$$w_1^\varepsilon = u^\varepsilon - \frac{2a\sqrt{\gamma}}{\gamma-1}(n^\varepsilon)^\theta, \quad w_2^\varepsilon = u^\varepsilon + \frac{2a\sqrt{\gamma}}{\gamma-1}(n^\varepsilon)^\theta, \quad \lambda = \frac{3-\gamma}{2(\gamma-1)}, \quad \theta = \frac{\gamma-1}{2}\,.$$

Then $(n^\varepsilon, j^\varepsilon)$ is a solution of (10)-(11) if and only if ([6, 7])

$$\partial_t \chi^\varepsilon + \partial_x \zeta^\varepsilon = (n^\varepsilon E^\varepsilon - j^\varepsilon)\frac{\partial \chi^\varepsilon}{\partial j^\varepsilon} + \frac{\partial^2 m^\varepsilon}{\partial \xi^2}, \quad \text{in} \quad \mathcal{D}'(Q_T \times I\!R_\xi),$$

where $(m^\varepsilon)_{\varepsilon>0}$ is a bounded sequence of non-positive measures defined on $Q_T \times I\!R_\xi$. Due to (H1), $(n^\varepsilon, j^\varepsilon)$ has compact support in x uniform in ε. Therefore, m^ε has compact support in x and ξ uniform in ε.

Let $g : I\!R \longrightarrow I\!R$ be an arbitrary smooth function. Then any pair of weak entropy-flux $(\eta^\varepsilon, q^\varepsilon) = (\eta(n^\varepsilon, j^\varepsilon), q(n^\varepsilon, j^\varepsilon))$ of (10)-(11) can be written as :

$$\eta^\varepsilon = \int_{I\!R} g(\xi)\chi^\varepsilon(t,x,\xi)d\xi, \quad q^\varepsilon = \int_{I\!R} g(\xi)\zeta^\varepsilon(t,x,\xi)d\xi.$$

By using $\frac{\partial\chi^\varepsilon}{\partial j^\varepsilon} = -\frac{1}{n^\varepsilon}\frac{\partial\chi^\varepsilon}{\partial\xi}$, we obtain

$$\partial_t\eta^\varepsilon + \partial_x q^\varepsilon = (E^\varepsilon - u^\varepsilon)\int_{I\!R} g'(\xi)\chi^\varepsilon(t,x,\xi)d\xi + < g'', m^\varepsilon > . \tag{21}$$

The first term in the right hand side of (21) is bounded in $L^\infty(Q_T) \subset$ compact set of $H^{-1}_{loc}(Q_T)$, the second term is a bounded measure on Q_T. Since $(\eta^\varepsilon, q^\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty(Q_T)$, we conclude from Murat's lemma [9] that $(\Lambda^\varepsilon)_{\varepsilon>0}$ is compact in $H^{-1}_{loc}(Q_T)$. The proof of the Theorem is finished.

Acknowledgments

The second author acknowledges partial support from the DAAD-PROCOPE Program, from the Deutsche Forschungsgemeinschaft, grant numbers MA 1662/ -1 and -2, and from the TMR Project "Asymptotic Methods in Kinetic Theory", grant number ERB FMRX CT97 0157. The research of the third author was partially supported by the TMR network of the Project "Hyperbolic Systems of Conservation Laws" (HCL), grant number ERB FMRX CT96 0033.

References

[1] S. Cordier and Y.J. Peng, "Système Euler-Poisson non linéaire - Existence globale de solutions faibles entropiques", Mod. Math. Anal. Num., 1998, to appear.

[2] A. Jüngel, "On the existence and uniqueness of transient solutions of a degenerate nonlinear drift-diffusion model for semiconductors", Math. Mod. Meth. Appl. Sci., 1994, 4, No. 5, p. 677-703.

[3] A. Jüngel, "A nonlinear drift-diffusion system with electric convection arising in electrophoretic and semiconductor modeling", Math. Nachr., 1997, 185, p. 85-110.

[4] A. Jüngel and Y.J. Peng, "A hierarchy of hydrodynamics models for plasmas. Part I : Zero-mass-electron limits", preprint, 1998, LMA, Univ. Blaise Pascal, no. 98-1.

[5] A. Jüngel and Y.J. Peng, "A hierarchy of hydrodynamics models for plasmas. Part II : Zero-relaxation-time limits", preprint, 1998, LMA, Univ. Blaise Pascal, no. 98-3.

[6] P.L. Lions, B. Perthame and P.E. Souganidis, "Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates", Comm. Pure Appl. Math., 1996, 49, p. 599-638.

[7] P.L. Lions, B. Perthame and E. Tadmor, "Kinetic formulation for the isentropic gas dynamics and p-system", Comm. Math. Phys., 1994, 163, p. 415-431.

[8] P. Marcati and R. Natalini, "Weak solutions to a hydrodynamic model for semiconductors : the Cauchy problem", Proc. Roy. Soc. Edinburgh Sect. A, 1995, 125, p. 115-131.

[9] F. Murat, "L'injection du cône positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout $q < 2$", J. Math. Pures et Appl., 60, 1981, p. 309-322.

[10] R. Natalini, "The bipolar hydrodynamic model for semiconductors and the drift-diffusion equation", J. Math. Analysis and Appl., 1996, 198, p. 262-281.

[11] Y.J. Peng, "Convergence of the fractional step Lax-Friedrichs scheme and Godunov scheme for a nonlinear Euler-Poisson system", preprint, 1997, LMA, Univ. Blaise Pascal, no. 97-12.

[12] L. Tartar, "Compensated compactness and applications to partial differential equations", Research Note in mathematics, nonlinear analysis and mechanics, Heriot-Watt symposium, 1979, 4, R.J. Knops ed., New york, Pitman Press, p. 136-212.