Exponential time decay of solutions to a nonlinear fourth-order parabolic equation

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Abstract

In this paper we investigate the large-time behavior of weak solutions to the nonlinear fourth-order parabolic equation $n_t = -(n(\log n)_{xx})_{xx}$ modeling interface fluctuations in spin systems. We study here the case $x \in \Omega = (0,1)$, with $n = 1$, $n_x = 0$ on $\partial\Omega$. In particular, we prove exponential decay of $u(x,t)$ towards the constant steady state $n_\infty = 1$ in the L^1 norm for long times and we give the explicit rate of decay. The result is based on classical entropy estimates, and on detailed lower bounds for the entropy production.

Keywords: asymptotic behavior, entropy dissipation, higher-order parabolic equation, diffusion equation.

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1 Introduction

In the last years, so called entropy dissipation methods have been successfully used to prove (sometimes optimal) decay rates of weak solutions to nonlinear second-order parabolic equations and their systems (see, e.g., [1, 4, 6]). The aim of this paper is to show that these methods can also be applied to the nonlinear fourth-order equation

$$n_t = -(n(\log n)_{xx})_{xx}, \tag{1.1}$$

which arises in the study of interface fluctuations in spin systems [3] and in quantum semiconductor modeling [9]. More specifically, we study the asymptotic behavior of the solution to the problem

$$\begin{aligned} n_t = -(n(\log n)_{xx})_{xx} &\qquad \text{in } \Omega = (0,1), && (1.2)\\ n = 1,\ n_x = 0 &\qquad \text{on } \partial\Omega, && (1.3)\\ n(\cdot, 0) = n_0 &\qquad \text{in } \Omega. && (1.4) \end{aligned}$$

The existence of global-in-time non-negative weak solutions has been shown in [8] for initial data which are only measurable and satisfy

$$\int_\Omega (n_0 - \log n_0) dx < \infty.$$

We refer to Section 2 for the precise definition of a weak solution to (1.2)-(1.4). In the paper [8] it has been also shown that $\log n(t)$ converges to 0 in the $L^1(\Omega)$-norm as $t \to \infty$; however, no decay rate has been derived.

As far as the asymptotic behavior of higher-order diffusion equations is concerned, only few results are presently available. Entropy techniques have been employed only recently in [7] to analyze the asymptotic behavior of a one-dimensional nonlinear degenerate diffusion equation modeling the surface tension dominated motion of thin viscous films and spreading droplets [11]:

$$\frac{\partial n}{\partial t} = -(n n_{xxx})_x, \qquad x \in \mathbb{R},\ t > 0, \tag{1.5}$$

with

$$n(x, t = 0) = n_0(x) \geq 0, \qquad x \in \mathbb{R}. \tag{1.6}$$

The result of [7] refers to diffusion in the whole line and follows from a suitable generalization of methods borrowed from [6]. On a bounded set, with periodic boundary conditions, various results on the asymptotic behaviour of the solution have been obtained for the thin film equation

$$\frac{\partial n}{\partial t} = -(n^k n_{xxx})_x, \qquad x \in \mathbb{R},\ t > 0. \tag{1.7}$$

We refer mainly to the papers of Bertozzi and Pugh [5], which analyzed the long-time behavior of weak solutions to (1.7) by introducing a family of regularized convex entropies. This technique allows them to handle the large-time problem when $0 < k < 3$. In addition, for all $k > 0$ they also show exponential decay of the distributional solutions (corresponding to strictly positive initial data) in L^∞ norm. Later on, exponential in time decay in the L^1 norm, with an explicit dependence of the exponential rate on k has been obtained in [10]. The result is based on classical entropy estimates, and on detailed lower bounds for the entropy production.

The idea of the entropy dissipation method we use here is generally as follows. Assume that a partial differential equation for the variable $n = n(x,t)$ (n may be a scalar function or vector valued) possesses a functional $\eta(n)$ such that the following *entropy inequality* holds:

$$\eta(n(t)) + \int_0^t P(n(\tau), n_x(\tau), \ldots) d\tau \le E(n(0)),$$

where we call $\eta(n(t))$ an *entropy* and P the corresponding *entropy production*. The second assumption is that the entropy production has the following lower bound:

$$P(n, n_x, \ldots) \ge \gamma E(n)^\alpha,$$

where n is an appropriate function and $\gamma > 0$ and $\alpha \ge 1$ are some constants. The above two inequalities imply a differential inequality for $\eta(t) = \eta(n(t))$:

$$\frac{d\eta}{dt} \le -\gamma \eta^\alpha, \quad t > 0, \qquad \eta(0) = \eta(n(0)).$$

If $\alpha > 1$ we obtain the *algebraic* decay rate $\eta(n(t)) = O(t^{1/(\alpha-1)})$ as $t \to \infty$, whereas for $\alpha = 1$, *exponential* decay can be concluded: $\eta(n(t)) \le \eta(0)\exp(-\gamma t)$, $t > 0$. Using Csiszar-Kullback type inequalities (see, e.g., [2, 4]), the decay rates in the 'entropy norm' give explicit decay rates in the L^1 norm.

In this paper we will show that the entropy production induced by the (semiclassical) relative entropy

$$\eta(n) = \int_\Omega (n(\log n - 1) + 1)\ dx, \tag{1.8}$$

is an upper bound for the entropy (1.8). This implies *exponential* convergence towards $n_\infty = 1$ with an explicit rate. Then, L^1-convergence follows by a Csiszar-Kullback type inequality. With respect to the studies of [5, 10], we face here with a problem which is not mass-conserving. This introduces additional difficulties.

Our main result is as follows:

Theorem 1.1 *Assume that*

$$\eta_0 \stackrel{\text{def}}{=} \int_\Omega (n_0(\log n_0 - 1) + 1)\ dx < \infty, \tag{1.9}$$

$$q_0 \stackrel{\text{def}}{=} \int_\Omega (n_0 - \log n_0)\ dx < \infty \tag{1.10}$$

hold and let n *be the weak solution to (1.2)-(1.4) constructed in [8] (also see Section 2). Then the entropy*

$$\eta(t) \stackrel{\text{def}}{=} \int_\Omega (n(t)(\log n(t) - 1) + 1)\ dx$$

satisfies

$$\eta(t) \le \eta_0 e^{-4t/(q_0+1)} \quad \textit{for } t \ge 0.$$

Moreover,

$$\|n(t) - 1\|_{L^1(\Omega)} \le \sqrt{\eta_0}(1 + \sqrt{2q_0})e^{-2t/(q_0+1)} \quad \textit{for } t \ge 0.$$

As explained above, we use the functional (1.8) in order to derive the entropy inequality. However, since the solution n is only non-negative, the treatment of terms like $\log n$ is not obvious. Therefore, we construct as in [8] solutions to a *semi-discrete* problem, which are strictly positive. This allows us to perform all computations, containing terms like $\log n$, rigorously. In Section 2 we recall the definition and the properties of the semi-discrete solutions. The proof of Theorem 1.1 is based on several lemmas shown in Section 3. Finally, Section 4 contains the proof of Theorem 1.1.

2 The semi-discrete problem

The starting point of our analysis will be the existence results obtained in [8]. For the sake of clarity, we will briefly collect them in a series of theorems. The main tool to deal with equation (1.2) is to perform a semi-discretization in time, considering, for any fixed $\tau > 0$ the problem

$$\frac{1}{\tau}(n_k - n_{k-1}) = -(n_k(\log n_k)_{xx})_{xx} \qquad \text{in } \Omega, \tag{2.11}$$

$$n_k = 1, \quad n_{k,x} = 0 \qquad \text{on } \partial\Omega, \tag{2.12}$$

We recall the following results from [8].

Theorem 2.1 *Let $\tau > 0$ and (1.10) hold. Then, for any $k \in \mathbb{N}$, there exists a weak solution $n_k \in H^2(\Omega)$ of (2.11), (2.12) satisfying $\log n_k \in H_0^2(\Omega)$ and $n_k > 0$ in Ω.*

Introduce the piecewise constant (in time) functions

$$\begin{aligned} n^{(\tau)}(\cdot, t) &= n_k, \\ \tilde{n}^{(\tau)}(\cdot, t) &= \frac{t - \tau(k-1)}{\tau}(n_k - n_{k-1}) + n_{k-1}, \end{aligned}$$

if $\tau(k-1) < t \le \tau k,\ k \in \mathbb{N}$. Then it holds:

Theorem 2.2 *Under the assumptions of Theorem 2.1 there exists a subsequence of $(n^{(\tau)})$ (not relabeled) such that*

$$\begin{aligned} \log n^{(\tau)} \rightharpoonup \log n \qquad & \textit{weakly in } L^2(0,T; H^2(\Omega)), \\ n^{(\tau)} \to n \qquad & \textit{strongly in } L^2(0,T; L^\infty(\Omega)), \\ \partial_t \tilde{n}^{(\tau)} \rightharpoonup \partial_t n \qquad & \textit{weakly in } L^1(0,T; H^{-2}(\Omega)), \end{aligned}$$

for any $T > 0,\ n \ge 0$ in $\Omega \times (0,T)$, and n solves

$$\int_0^T \langle \partial_t n, \phi \rangle \ dt = -\int_0^T \int_\Omega n(\log n)_{xx} \ \phi_{xx} \ dx \ dt$$

for all $\phi \in L^\infty(0,T; H_0^2(\Omega))$, where $\langle \cdot, \cdot \rangle$ is the duality product of $H^{-2}(\Omega)$ and $H_0^2(\Omega)$.

3 Auxiliary results

Let $n_k \in H^2(\Omega)$ be a weak solution to (1.2)-(1.4) and introduce the discrete entropies

$$\begin{aligned} q_k &= \int_\Omega (n_k - \log n_k)\ dx, \\ \eta_k &= \int_\Omega (n_k(\log n_k - 1) + 1)\ dx, \end{aligned}$$

for $k \in \mathbb{N}$.

Lemma 3.1 *1. Assume $\eta_{k-1} < \infty,\ k \geq 1$. Then*

$$\frac{1}{\tau}(\eta_k - \eta_{k-1}) + \int_\Omega n_k (\log n_k)^2_{xx}\ dx \leq 0. \tag{3.13}$$

2. Assume $q_{k-1} < \infty,\ k \geq 1$. Then

$$\frac{1}{\tau}(q_k - q_{k-1}) + \int_\Omega (\log n_k)^2_{xx}\ dx \leq 0. \tag{3.14}$$

3. Assume $q_0 < \infty$. Then, for any $k \in \mathbb{N}$,

$$0 < e^{-q_0} \leq N_k \stackrel{\text{def}}{=} \int_\Omega n_k\ dx \leq 2q_0 < \infty. \tag{3.15}$$

Proof. The first two parts of Lemma 3.1 have already been proved in [8]. For the sake of completeness we repeat the proof.

In order to prove the first part of the lemma, we use $\log n_k \in H_0^2(\Omega)$ as test function in the weak formulation of (2.11)-(2.12):

$$\frac{1}{\tau}\int_\Omega (n_k - n_{k-1}) \log n_k\ dx = -\int_\Omega n_k (\log n_k)^2_{xx}\ dx.$$

Using the inequality $-\log x + x - 1 \geq 0$ for all $x > 0$, we obtain from

$$\begin{aligned} &\frac{1}{\tau}\int_\Omega (n_k - n_{k-1}) \log n_k\ dx \\ &= \frac{1}{\tau}(\eta_k - \eta_{k-1}) + \frac{1}{\tau}\int_\Omega n_{k-1}\Big(- \log\frac{n_k}{n_{k-1}} + \frac{n_k}{n_{k-1}} - 1\Big)\ dx \\ &\geq \frac{1}{\tau}(\eta_k - \eta_{k-1}) \end{aligned}$$

the inequality (3.13).

For the proof of the second part, we use $1 - n_k^{-1} \in H_0^2(\Omega)$ as test function in the weak formulation of (2.11)-(2.12):

$$\begin{aligned} &\frac{1}{\tau}\int_\Omega (n_k - n_{k-1} - 1 + n_{k-1}/n_k)\ dx + \int_\Omega (\log n_k)_{xx}^2\ dx \\ &= \int_\Omega \left(\frac{n_{k,xx}n_{k,x}^2}{n_k^3} - \frac{n_{k,x}^4}{n_k^4}\right) dx. \end{aligned}$$

The boundary condition $n_{k,x} = 0$ on $\partial\Omega$ implies

$$\int_\Omega \frac{n_{k,xx}n_{k,x}^2}{n_k^3}\ dx = \frac{1}{3}\int_\Omega \frac{1}{n_k^3}(n_{k,x}^3)_x\ dx = \int_\Omega \frac{n_{k,x}^4}{n_k^4}\ dx.$$

Furthermore, the inequality $x - 1 \geq \log x$ for $x \geq 0$ gives

$$\begin{aligned} &\int_\Omega (n_k - n_{k-1} - 1 + n_{k-1}/n_k)\ dx \\ &= q_k - q_{k-1} + \int_\Omega (n_{k-1}/n_k - 1 - \log(n_{k-1}/n_k))\ dx \\ &\geq q_k - q_{k-1}. \end{aligned}$$

Therefore,

$$\frac{1}{\tau}(q_k - q_{k-1}) + \int_\Omega (\log n_k)_{xx}^2\ dx \leq 0,$$

which proves the second part.

In order to prove the third part of the lemma, we estimate as follows. From $x - \log x \geq \frac{1}{2}x$ for $x > 0$ follows that

$$q_k \geq \frac{1}{2}\int_\Omega n_k\ dx$$

and hence

$$\int_\Omega n_k\ dx \leq 2q_k \leq 2q_0.$$

Since $x \mapsto -\log x$ is convex, we can apply Jensen's inequality to get

$$-\log \int_\Omega n_k\ dx \leq -\int_\Omega \log n_k\ dx.$$

Therefore

$$-\log \int_\Omega n_k \, dx \le \int_\Omega n_k \, dx - \log \int_\Omega n_k \, dx \le q_k \le q_0.$$

The assertion of the lemma follows.

Lemma 3.2 *Let $n \in H^2(\Omega)$ be such that $n > 0$ in Ω and $n = 1$, $n_x = 0$ on $\partial\Omega$. Then*

$$\int_\Omega n(\log n)_{xx}^2 \, dx \ge 8\|\sqrt{n} - 1\|_{L^\infty(\Omega)}^2.$$

Proof. By integration by parts, using $n_x = 0$ on $\partial\Omega$, we have

$$\int_\Omega \frac{n_{xx}n_x^2}{n^2} \, dx = \frac{1}{3}\int_\Omega \frac{1}{n^2}(n_x^3)_x \, dx = \frac{2}{3}\int_\Omega \frac{n_x^4}{n^3} \, dx$$

and therefore,

$$\begin{aligned} \int_\Omega n(\log n)_{xx}^2 \, dx &= \int_\Omega \left(\frac{n_{xx}^2}{n} - \frac{2n_{xx}n_x^2}{n^2} + \frac{n_x^4}{n^3} \right) dx \\ &= \int_\Omega \left(\frac{n_{xx}^2}{n} - \frac{1}{3}\frac{n_x^4}{n^3} \right) dx. \end{aligned}$$

This gives

$$\begin{aligned} \int_\Omega (\sqrt{n})_{xx}^2 \, dx &= \frac{1}{4}\int_\Omega \left(\frac{n_{xx}^2}{n} - \frac{n_{xx}n_x^2}{n^2} + \frac{n_x^4}{4n^3} \right) dx \\ &= \frac{1}{4}\int_\Omega \left(\frac{n_{xx}^2}{n} - \frac{5}{12}\frac{n_x^4}{n^3} \right) dx \\ &\le \frac{1}{4}\int_\Omega \left(\frac{n_{xx}^2}{n} - \frac{1}{3}\frac{n_x^4}{n^3} \right) dx \\ &= \frac{1}{4}\int_\Omega n(\log n)_{xx}^2 \, dx. \end{aligned}$$

From the Poincaré's inequalities

$$\begin{aligned} \|\sqrt{n} - 1\|_{L^\infty(\Omega)} &\le \|(\sqrt{n})_x\|_{L^2(\Omega)}, \\ \|(\sqrt{n})_x\|_{L^2(\Omega)} &\le \frac{1}{\sqrt{2}}\|(\sqrt{n})_{xx}\|_{L^2(\Omega)} \end{aligned}$$

we conclude

$$\int_\Omega n(\log n)_{xx}^2 \ dx \geq 4\|(\sqrt{n})_{xx}\|_{L^2(\Omega)}^2 \geq 8\|\sqrt{n}-1\|_{L^\infty(\Omega)}^2,$$

which proves the lemma.

Lemma 3.3 *Let $n \in L^\infty(\Omega)$ be such that $n \geq 0$ in Ω. Then*

$$\int_\Omega (n(\log n - 1) + 1) \ dx \leq (N+2)\|\sqrt{n}-1\|_{L^\infty(\Omega)}^2,$$

where $N = \int_\Omega n \ dx$.

Proof. Introduce the function $f(x) = x^2(\log x^2 - 1) + 1, \ x \geq 0$. By Taylor's expansion around $x = 1$ we find

$$n(\log n - 1) + 1 = f(\sqrt{n}) = \frac{1}{2} f''(\sqrt{\lambda})(\sqrt{n}-1)^2 = (\log\lambda + 2)(\sqrt{n}-1)^2,$$

where λ lies between 1 and n. The inequality $\log\lambda \leq \lambda - 1$ gives

$$\begin{aligned} n(\log n - 1) + 1 &\leq (\lambda+1)(\sqrt{n}-1)^2 \leq (\max(1,n)+1)(\sqrt{n}-1)^2 \\ &\leq (n+2)(\sqrt{n}-1)^2, \end{aligned}$$

and we conclude

$$\begin{aligned} \int_\Omega (n(\log n - 1) + 1) \ dx &\leq \|n+2\|_{L^1(\Omega)}\|\sqrt{n}-1\|_{L^\infty(\Omega)}^2 \\ &= (N+2) \ \|\sqrt{n}-1\|_{L^\infty(\Omega)}^2. \end{aligned}$$

The final lemma is a Csiszar-Kullback-type inequality:

Lemma 3.4 *Let $0 \leq n \in L^1(\Omega)$ be such that*

$$q = \int_\Omega (n - \log n) \, dx < \infty. \tag{3.16}$$

Then, the following inequality holds

$$\int_\Omega [n(\log n - 1) + 1] \ dx \geq \frac{1}{(1+\sqrt{2q})^2}\left[\int_\Omega |n-1| \, dx\right]^2. \tag{3.17}$$

Proof. By Taylor's expansion around $n = 1$ we get

$$n(\log n - 1) + 1 = \frac{1}{2[1 + \theta(n)(n-1)]}(n-1)^2, \tag{3.18}$$

where $0 < \theta(n) < 1$. On the other hand

$$\int_{n<1} |n-1|\, dx \le \int_{n<1} \frac{|n-1|}{[1+\theta(n-1)]^{1/2}}\, dx \le \left[\int_{n<1} \frac{(n-1)^2}{1+\theta(n-1)}\, dx\right]^{1/2}$$

while, by the Cauchy-Schwarz inequality and Lemma 3.1.3 (recall (3.16)),

$$\begin{aligned}
\int_{n>1} |n-1|\, dx &= \int_{n>1} \frac{|n-1|}{[1+\theta(n-1)]^{1/2}}[1+\theta(n-1)]^{1/2}\, dx \\
&\le \left[\int_{n>1} \frac{(n-1)^2}{1+\theta(n-1)}\, dx\right]^{1/2} \left[\int_{n>1} (1+\theta(n-1))\, dx\right]^{1/2} \\
&\le \sqrt{2q}\left[\int_{n>1} \frac{(n-1)^2}{1+\theta(n-1)}\, dx\right]^{1/2}.
\end{aligned}$$

Finally,

$$\int_{\Omega} |n-1|\, dx \le \left[\int_{n<1} \frac{(n-1)^2}{1+\theta(n-1)}\, dx\right]^{1/2} + \sqrt{2q}\left[\int_{n>1} \frac{(n-1)^2}{1+\theta(n-1)}\, dx\right]^{1/2},$$

which implies

$$\left[\int_{\Omega} |n-1|\, dx\right]^2 \le (1+\sqrt{2q})^2 \int_{\Omega} \frac{(n-1)^2}{1+\theta(n-1)}\, dx.$$

Now use (3.18) to conclude the proof.

4 Proof of Theorem 1.1

Let $n_k \in H^2(\Omega)$ be a weak solution to the semi-discrete problem (2.11)-(2.12). From Lemma 3.1.1 and Lemmas 3.2 and 3.3 we obtain

$$\begin{aligned}
\eta_k - \eta_{k-1} &\le -\tau \int_{\Omega} n_k (\log n_k)_{xx}^2\, dx \le -8\tau \|\sqrt{n_k} - 1\|_{L^\infty(\Omega)}^2 \\
&\le -\frac{8\tau}{N_k + 2}\eta_k \le -\frac{4\tau}{q_0 + 1}\eta_k.
\end{aligned}$$

This yields

$$\eta_k \leq \left(1+\frac{4\tau}{q_0+1}\right)^{-1}\eta_{k-1} \leq \left(1+\frac{4\tau}{q_0+1}\right)^{-k}\eta_0$$

and, for $(k-1)\tau < t \leq k\tau$,

$$\eta_k \leq \left(1+\frac{4\tau}{q_0+1}\right)^{-t/\tau}\eta_0.$$

Introducing the function

$$\eta^{(\tau)}(t) = \int_\Omega \left(n^{(\tau)}(t)(\log n^{(\tau)}(t)-1)+1\right)\,dx,$$

we can rewrite this inequality as

$$\eta^{(\tau)}(t) \leq \eta_0\left(1+\frac{4\tau}{q_0+1}\right)^{-t/\tau}. \tag{4.19}$$

The convergence results of Theorem 2.2 imply, as $\tau \to 0$,

$$n^{(\tau)}(x,t) \to n(x,t) \qquad \text{a.e. in } (x,t),$$

which gives

$$n^{(\tau)}(x,t)\log n^{(\tau)}(x,t) \to n(x,t)\log n(x,t) \qquad \text{a.e. in } (x,t).$$

Due to inequality (4.19) the $L^1(\Omega)$ norm of $n^{(\tau)}(x,t)(\log n^{(\tau)}(x,t)-1)+1$ is bounded uniformly in τ, for fixed $t \in (0,T)$, and we can apply Lebesgue's convergence theorem to conclude

$$\eta^{(\tau)}(t) \to \eta(t) \qquad \text{for } t \in (0,T).$$

Hence

$$\eta(t) \leq \lim_{\tau\to 0}\eta_0\left(1+\frac{4\tau}{q_0+1}\right)^{-t/\tau} = \eta_0 e^{-4t/(q_0+1)},$$

and the theorem is proved.

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