Exponential decay in time of solutions of the viscous quantum hydrodynamic equations

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Abstract. The long-time asymptotics of solutions of the viscous quantum hydrodynamic model in the one space dimension is studied. This model consists of continuity equations for the particle density and the current density, coupled to the Poisson equation for the electrostatic potential. The equations are a dispersive and viscous regularization of the Euler equations. It is shown that the solutions converge exponentially fast to the (unique) thermal equilibrium state as the time tends to infinity. For the proof we employ the entropy dissipation method, applied for the first time to a third-order differential equation.

Keywords. Quantum hydrodynamics, Wigner-Fokker-Planck equation, long-time behavior of solutions, entropy dissipation method.

1 Introduction

It is well known that quantum models, like the Schrödinger equation or the Wigner equation, allow for a fluiddynamical description in terms of macroscopic quantities, like the particle density or the current density, satisfying the so-called Madelung or quantum hydrodynamic equations [9]. These equations do not include a model of collisions. Collisions of the particles with an oscillator bath can be modeled by the Wigner-Fokker-Plank equation [1, 2]. From this equation, the macroscopic viscous quantum hydrodynamic model can be derived, using a moment method as in [4]. This model consists of the continuity equations for the particle density $n(x,t)$ and the current density $J(x,t)$, coupled to the Poisson equation for the electrostatic potential $V(x,t)$. The scaled equations in one space dimension read as follows:

$$\begin{aligned} n_t + J_x &= \nu n_{xx}, && (1)\\ J_t + \left(\frac{J^2}{n} + Tn\right)_x - nV_x - \frac{\varepsilon^2}{2} n\left(\frac{\sqrt{n}_{xx}}{\sqrt{n}}\right)_x &= \nu J_{xx} - \frac{J}{\tau}, && (2)\\ \lambda^2 V_{xx} &= n-1, \qquad x\in\Omega,\ t>0, && (3) \end{aligned}$$

with initial conditions

$$n(\cdot,0) = n_I, \quad J(\cdot,0) = J_I \qquad \text{in } \Omega. \tag{4}$$

The (scaled) physical parameters are the temperature constant $T>0$, the Planck constant $\varepsilon>0$, the momentum relaxation time constant $\tau>0$, and the Debye length $\lambda>0$. The viscosity $\nu>0$ models the strength of interaction of the particles with the oscillators. Equations (1)-(2) can be derived exactly as in [4], where the (linear) Fokker-Planck term gives the viscous contributions nun_{xx} and νJ_{xx} to the right-hand sides of (1)-(2). In the Poisson equation, we have prescribed a constant concentration of fixed background charges. For the choice of boundary conditions, see below.

For $\varepsilon = 0$ and $\nu = 0$, the system (1)-(3) is the hydrodynamic model for the ensemble of charged particles, for instance, electrons moving in a semiconductor crystal [6]. For $\varepsilon > 0$ and $\nu = 0$ (and $T = 0$, $1/\tau = 0$), equations (1)-(3) are the Madelung equations used in the modeling of superfluids [8]. With this choice of parameters, (1)-(3) are formally equivalent to the Schrödinger-Poisson system. Finally, the equations (1)-(3) with $\nu = 0$ are known as the quantum hydrodynamic equations which have been used to model quantum semiconductor devices [4, 6]. No results are available in the mathematical literature for (1)-(3) with $\varepsilon > 0$ and $\nu > 0$.

In this paper we study the long-time asymptotics of the solutions of (1)-(3) to the so-called thermal equilibrium state where no bias is applied. We choose the boundary conditions:

$$n = 1, \quad n_x = 0, \quad V = 0 \qquad \text{on } \partial\Omega \times (0, \infty), \tag{5}$$

$$\int_{\partial\Omega} J\left[J_x\left(\frac{\varepsilon^2}{4\nu} + \nu\right) - \frac{1}{2}J^2\right](\cdot, t)ds = 0, \qquad t > 0. \tag{6}$$

The boundary data is in thermal equilibrium. Indeed, thermal equilibrium means $J = 0$. Thus, by (1) and (5), $n = 1$ in Ω and, by (3) and (5), $V = 0$ in Ω. The boundary condition (6) can be interpreted as a generalized thermal equilibrium condition for the current density (see Remark 1). We prove that any strong solution of (1)-(6) converges exponentially fast to the (unique) thermal equilibrium state $(n, J, V) = (1, 0, 0)$. The rate of convergence for $n(\cdot, t)$ and $V(\cdot, t)$ depends on the viscosity constant $\nu > 0$. If $\nu = 0$, no convergence rate can be obtained. The proof is based on the entropy dissipation method. This method has been used to derive explicit convergence rates of *second-* and *fourth*-order equations [3, 7]. Here we apply the method for the first time to a *third*-order equation.

The entropy dissipation method is based on a priori estimates for the entropy (or, more precisely, free energy) functional:

$$E(t) = \int_0^1 \left[\frac{\varepsilon^2}{2}(\sqrt{n})_x^2 + T(n(\log n - 1) + 1) + \frac{\lambda^2}{2}V_x^2 + \frac{J^2}{2n}\right](x, t)dx \geq 0 \tag{7}$$

consisting of the quantum energy, thermodynamic entropy, electric energy, and kinetic energy of the system. The idea is to derive an inequality of the form:

$$E(t) + \int_0^t \int_0^1 P(x, t)dxds \leq E(0),$$

where the entropy dissipation rate $P(x, t) \geq 0$ depends on the variables and his derivatives. We show that

$$\int_0^1 P(x, t)dx \geq \gamma E(t)$$

for some $\gamma > 0$ and thus Gronwall's lemma implies

$$E(t) \leq E(0)e^{-\gamma t}, \qquad t \geq 0.$$

More precisely, our main result reads as follows.

Theorem 1 *Let $n \in H^1(0, T^*; L^2(\Omega)) \cap L^2(0, T^*; H^3(\Omega))$, $J \in H^1(0, T^*; L^2(\Omega)) \cap L^2(0, T^*; H^2(\Omega))$, $V \in L^2(0, T^*; H^2(\Omega))$ be a solution to (1)-(3) for any $T^* > 0$ such that $n > 0$ in*

$\Omega \times (0,T^*)$ and let $n_I \in H^1(\Omega)$, $J_I \in L^2(\Omega)$ such that $n_I > 0$ in Ω. Then

$$\begin{aligned} \varepsilon^2 \|n(\cdot,t)-1\|^2_{L^\infty(\Omega)} &\le 2E(0)e^{-4\nu(1+2T/\varepsilon^2)t}, \\ \lambda^2 \|V(\cdot,t)\|^2_{L^\infty(\Omega)} &\le 2E(0)e^{-4\nu t}, \\ \left\| \frac{J(\cdot,t)}{\sqrt{n(\cdot,t)}} \right\|^2_{L^2(\Omega)} &\le 2E(0)e^{-2t/\tau}, \qquad t>0. \end{aligned}$$

Remark 1 (i) The global-in-time existence for (1)-(3) will be studied in [5].

(ii) The boundary condition (6) is needed for technical reasons. It is a weaker condition than the (physically reasonable, but mathematically over-determining) boundary conditions $J(0,t) = J(1,t) = 0$, $t > 0$.

(iii) As expected, no convergence rate for n and V can be expected if $\nu = 0$. However, the kinetic energy $J^2(\cdot,t)/n(\cdot,t)$ converges to zero exponentially in the $L^1(\Omega)$ norm with a decay rate $1/\tau$. This is physically reasonable since τ models the momentum relaxation time.

2 Proof of Theorem 1

Let $T^* > 0$ and fix $t \in (0,T^*)$. Multiply (2) by J/n, integrate over $\Omega = (0,1)$ and integrate by parts:

$$\begin{aligned} \int_0^1 J_t \frac{J}{n} dx &= -\int_0^1 \left(\frac{J^2}{n}\right)_x \frac{J}{n} dx - T\int_0^1 J\frac{n_x}{n}dx + \int_0^1 V_x J dx - \frac{\varepsilon^2}{2}\int_0^1 J_x\left(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\right)dx \\ &\quad + \frac{\varepsilon^2}{2}\int_{\partial\Omega} J\left(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\right)ds - \nu\int_0^1 J_x\left(\frac{J}{n}\right)_x dx + \nu\int_{\partial\Omega} J_x\left(\frac{J}{n}\right)ds \\ &\quad - \frac{1}{\tau}\int_0^1 \frac{J^2}{n}dx \\ &= A_1 + \cdots + A_8. \end{aligned} \tag{8}$$

Multiply (2) by the function $T\log n - J^2/2n^2 - V - \varepsilon^2(\sqrt{n})_{xx}/2\sqrt{n}$, integrate over Ω and integrate by parts:

$$\begin{aligned} &\int_0^1 n_t\left(T\log n - \frac{J^2}{2n^2} - V - \varepsilon^2\frac{(\sqrt{n})_{xx}}{2\sqrt{n}}\right)dx \\ &= -\nu T\int_0^1 \frac{n_x^2}{n}dx + \nu\int_0^1 \frac{1}{2}\frac{J^2}{2n^2}n_x dx + \nu\int_0^1 V_x n_x dx - \nu\frac{\varepsilon^2}{2}\int_0^1 n_{xx}\left(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\right)dx \\ &\quad + T\int_0^1 J(\log n)_x dx + \int_0^1 \frac{J^2 J_x}{2n^2}dx - \int_0^1 V_x J dx + \frac{\varepsilon^2}{2}\int_0^1 J_x\left(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\right)dx = \\ &= B_1 + \cdots + B_8. \end{aligned} \tag{9}$$

Here, we have used that $n_x = 0$ and $V = 0$ on $\partial\Omega$.

First we consider the terms on the left-hand side of (8) and (9). Since $n_x = 0$ and $V = 0$ on $\partial\Omega$, it holds:

$$\begin{aligned} \int_0^1 \left(\frac{JJ_t}{n} - \frac{1}{2}\frac{J^2 n_t}{n^2}\right)dx &= \partial_t \int_0^1 \frac{1}{2}\frac{J^2}{n}dx, \\ \int_0^1 n_t \log n dx &= \partial_t \int_0^1 (n(\log n - 1) + 1)dx, \end{aligned}$$

$$\begin{aligned}
-\int_0^1 V n_t dx &= -\lambda^2 \int_0^1 V_{xxt} V dx = \lambda^2 \int_0^1 V_{xt} V_x dx = \frac{\lambda^2}{2}\partial_t \int_0^1 V_x^2 dx, \\
-\int_0^1 \frac{\varepsilon^2}{2} n_t \Big(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\Big) dx &= -\varepsilon^2 \int_0^1 \frac{n_t}{2\sqrt{n}} (\sqrt{n})_{xx} dx = -\varepsilon^2 \int_0^1 (\sqrt{n})_t (\sqrt{n})_{xx} dx \\
&= \frac{\varepsilon^2}{2} \partial_t \int_0^1 (\sqrt{n})_x^2 dx.
\end{aligned}$$

Hence, the sum of the left-hand sides of (8) and (9) is equal to $\partial_t E(t)$ where $E(t)$ is defined in (7).

Now we compute the right-hand sides of (8) and (9). Notice that $A_2 + B_5 = 0$, $A_4 + B_8 = 0$, $A_3 + B_7 = 0$. Using $n = 1$ on $\partial\Omega$, we obtain:

$$A_1 + B_6 = \int_0^1 \left(\frac{n_x J^3}{n^3} - \frac{3}{2} \frac{J^2 J_x}{n^2} \right) dx = -\frac{1}{2} \int_0^1 \left(\frac{J^3}{n^2} \right)_x dx = \frac{1}{2} J^3(0) - \frac{1}{2} J^3(1).$$

A computation gives

$$A_6 + B_2 = \nu \int_0^1 \left(-\frac{J_x^2}{n} + \frac{2 J J_x n_x}{n^2} - \frac{J^2 n_x^2}{n^3} \right) dx = -\nu \int_0^1 \left(\frac{J_x}{\sqrt{n}} - \frac{n_x J}{\sqrt{n}^3} \right)^2 dx.$$

By integration by parts and $n_x = 0$ on $\partial\Omega$ we obtain

$$\int_0^1 \frac{n_x^2 n_{xx}}{n^2} dx = \frac{2}{3} \int_0^1 \frac{n_x^4}{n^3} dx,$$

and therefore,

$$\begin{aligned}
B_4 &= -\nu \frac{\varepsilon^2}{2} \int_0^1 n_{xx} \Big(\frac{(\sqrt{n})_{xx}}{\sqrt{n}} \Big) dx = \varepsilon^2 \nu \int_0^1 \left(\frac{1}{8} \frac{n_x^2 n_{xx}}{n^2} - \frac{1}{4} \frac{n_{xx}^2}{n} \right) dx \\
&= -\varepsilon^2 \nu \int_0^1 \left((\sqrt{n})_{xx}^2 + \frac{1}{48} \frac{n_x^4}{n^3} \right) dx.
\end{aligned}$$

Finally, the boundary condition $n = 1$ on $\partial\Omega$ implies

$$B_3 = \nu \int_0^1 V_x (n-1)_x dx = -\frac{\nu}{\lambda^2} \int_0^1 (n-1)^2 dx \tag{10}$$

Here we need that $n = \text{const.}$ on $\partial\Omega$.

The above computations show that the sum of (8) and (9), integrated over $(0, t)$, can be written as:

$$\begin{aligned}
E(t) - E(0) &= -\int_0^t \int_0^1 \Big[\varepsilon^2 \nu (\sqrt{n})_{xx}^2 + \varepsilon^2 \nu \frac{1}{48} \frac{n_x^4}{n^3} + 4\nu T (\sqrt{n})_x^2 + \frac{\nu}{\lambda^2} (n-1)^2 \\
&\quad + \nu \Big(\frac{J_x}{\sqrt{n}} - \frac{n_x J}{n^{3/4}} \Big)^2 + \frac{1}{\tau} \frac{J^2}{n} \Big] dx dt \\
&\quad + \int_0^t \int_{\partial\Omega} \Big[\frac{\varepsilon^2}{2} \Big(\frac{(\sqrt{n})_{xx}}{\sqrt{n}} J \Big) + \nu J J_x - \frac{1}{2} J^3 \Big] ds dt.
\end{aligned}$$

Since $n_x = 0$ and $n = 1$ on $\partial\Omega$, we can write the boundary integral as

$$\int_0^t \int_{\partial\Omega} \left[\frac{\varepsilon^2}{4} n_{xx} J + \nu J J_x - \frac{1}{2} J^3 \right] ds dt$$

and from (1), because of $n \in H^1(0, T^*, L^2(\Omega))$ and $J \in L^2(0, T^*, H^2(\Omega))$, yields $\nu n_{xx}(\cdot, t) = J_x(\cdot, t)$ on $\partial\Omega$. Then, by (6), the boundary integral vanishes:

$$\int_0^t \int_{\partial\Omega} \left[\frac{\varepsilon^2}{4} n_{xx} J + \nu J J_x - \frac{1}{2} J^3 \right] ds dt = \int_0^t \int_{\partial\Omega} \left[\frac{\varepsilon^2}{4\nu} J J_x + \nu J J_x - \frac{1}{2} J^3 \right] ds dt = 0.$$

The remaining integral on the right-hand side is the entropy dissipation rate. We apply the Poincaré inequality

$$\|u\|_{L^2(\Omega)} \leq \frac{1}{\sqrt{2}} \|u_x\|_{L^2(\Omega)} \qquad \forall u \in H_0^1(\Omega)$$

to $u = (\sqrt{n})_x$ to obtain

$$\frac{\varepsilon^2}{2} \int_0^1 (\sqrt{n})_x^2(x,t) dx \leq E(t) \leq E(0) - (2\nu\varepsilon^2 + 4\nu T) \int_0^1 \int_0^t (\sqrt{n})_x^2(x,t) dx dt$$

Gronwall's lemma implies

$$\varepsilon^2 \|(\sqrt{n})_x(\cdot, t)\|_{L^2(\Omega)}^2 \leq 2E(0) e^{-4\nu(1+2T/\varepsilon^2)t}, \qquad t \geq 0.$$

The Sobolev-Poincaré inequality

$$\|u - 1\|_{L^\infty(\Omega)} \leq \|u_x\|_{L^2(\Omega)}, \qquad \forall u - 1 \in H_0^1(\Omega)$$

then gives

$$\varepsilon^2 \|\sqrt{n}(\cdot, t) - 1\|_{L^\infty(\Omega)}^2 \leq 2E(0) e^{-4\nu(1+2T/\varepsilon^2)t}, \qquad t \geq 0.$$

Furthermore, we conclude from

$$\frac{1}{2} \int_0^1 \frac{J^2}{n} dx \leq E(t) \leq E(0) - \frac{1}{\tau} \int_0^1 \int_0^t \frac{J^2}{n} dx dt$$

the convergence rate

$$\left\| \frac{J(\cdot, t)}{\sqrt{n(\cdot, t)}} \right\|_{L^2(\Omega)}^2 \leq 2E(0) e^{-2t/\tau}, \qquad t \geq 0.$$

Finally, from the elliptic estimate

$$\sqrt{2}\lambda^2 \|V\|_{L^2(\Omega)} \leq \|n - 1\|_{L^2(\Omega)}$$

we infer

$$\frac{\lambda^2}{2} \int_0^1 \int_0^t V_x^2 dx dt \leq E(t) \leq E(0) - 2\nu\lambda^2 \int_0^1 \int_0^t V_x^2 dx dt,$$

and hence

$$\lambda^2 \|V(\cdot, t)\|_{L^\infty(\Omega)}^2 \leq \lambda^2 \|V_x(\cdot, t)\|_{L^2(\Omega)}^2 \leq 2E(0) e^{-4\nu t}, \qquad t \geq 0.$$

This proves the theorem.

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