Recent Progress on Quantum Hydrodynamic Models for Semiconductors

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1 Introduction

1.1 General Remarks on the QHD Model and Its Derivation

For ultra-small semiconductor devices in which quantum effects are present, the corresponding mathematical models have to incorporate the quantum mechanical phenomena. Roughly speaking, these models can be divided into two classes: microscopic and macroscopic quantum models.

Microscopic models include *quantum kinetic equations*, like the Wigner equation and the Wigner-(Poisson-)Fokker-Plank model, and Schrödinger-Poisson systems [4, 30]. It has been known since the early days of quantum mechanics [26, 36] that these microscopic models can be written in hydrodynamic form using macroscopic quantities like the electron density or current density. The hydrodynamic form of quantum kinetic equations is termed *quantum hydrodynamic models*.

The advantage of quantum hydrodynamic models is that in semiconductor applications, the numerical simulation of microscopic models is quite expensive [32] while the numerical approximation of macroscopic models seems to be easier than for the microscopic ones.

Quantum hydrodynamic models (QHD models for short) can be derived from the quantum Wigner-Boltzmann equation using a velocity-moment method [10] or from the Schrödinger equation [9, 25]. It governs the evolution of the electron density ρ, the electron current density J and the electrostatic potential V. Assuming that the typical time scale is given by the momentum relaxation time constant τ (see [13]), we obtain the following unipolar scaled QHD equations:

$$\begin{aligned} \partial_t \rho + \mathrm{div} J &= 0, &(1)\\ \partial_t J + \mathrm{div}\left(\frac{J\otimes J}{\rho}\right) + \nabla p(\rho) - \rho\nabla V - \frac{\varepsilon^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) &= -\frac{J}{\tau}, &(2)\\ \lambda^2\Delta V &= \rho - C(x), &(3) \end{aligned}$$

for $x\in\mathbb{R}^d$ and $t>0$. Here, $\frac{\varepsilon^2}{2}\rho\nabla\left(\Delta\sqrt{\rho}/\sqrt{\rho}\right)$ is the quantum Bohm correction and $p(\rho)$ is the pressure term. The scaled parameters are the Planck constant ε (the ratio of the de Broglie length and the device length) and the Debye length λ. The doping profile $C(x)$ models fixed background charges. Also small mean-free-path asymptotics have been used to derive QHD models in the case of high electric fields. The obtained models have linear quantum corrections [27].

1.2 Connection of the QHD Model with the Schrödinger-Poisson and the Wigner Equation

We start with a single state nonlinear Schrödinger-Poisson system in $\mathbb{R}^d$:

$$
\begin{aligned}
i\epsilon\partial_t\Psi &= -\frac{\epsilon^2}{2}\Delta\Psi + \big(V + Th(|\Psi|^2)\big)\,\Psi, \qquad x\in\mathbb{R}^d,\ t>0, &(4)\\
\lambda^2\Delta V &= |\Psi|^2 - C(x), \qquad x\in\mathbb{R}^d,\ t>0, &(5)\\
\Psi(0,\cdot) &= \Psi_I \qquad \text{in } \mathbb{R}^d. &(6)
\end{aligned}
$$

The variables are the wave function $\Psi = \Psi(t,x)$, and the electrostatic potential $V = V(t,x)$. The function $h(s)$ can be interpreted as a self-interacting potential, and $T > 0$ is a constant (see below for the physical meaning of these quantities).

Now introduce the Wigner transform [2, 25, 28]

$$
w(t,x,v) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\overline{\Psi}\Big(t,x+\frac{\eta}{2}\Big)\Psi\Big(t,x-\frac{\eta}{2}\Big)e^{i\eta\cdot v}d\eta.
$$

A simple calculation shows that w satisfies the so-called Wigner equation

$$
\begin{aligned}
\partial_t w + v\cdot\nabla w - \Theta[V+h(\rho)]w &= 0, \qquad x,v\in\mathbb{R}^d,\ t>0,\\
w(0,x,v) &= w_I(x,v), \qquad x,v\in\mathbb{R}^d,
\end{aligned} \tag{7}
$$

where $\rho = \int w dv$, $\Theta[V+h(\rho)]$ is the pseudo-differential operator [33]

$$
\begin{aligned}
&(\Theta[V+h(\rho)]w)(t,x,v)\\
&= \frac{i}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{\varepsilon}\Big\{(V+h(n))\Big(t,x+\varepsilon\frac{\eta}{2}\Big) - (V+h(n))\Big(t,x-\varepsilon\frac{\eta}{2}\Big)\Big\}\\
&\quad\times w_j(t,x,v')e^{i(v-v')\cdot\eta}d\eta dv',
\end{aligned}
$$

and the initial datum w_I is the Wigner transform of the initial state Ψ_I given by (6). Easy calculations show that the lowest-order moments, i.e. the particle density $\rho = |\Psi|^2$, the current density $J = \varepsilon\mathrm{Im}\,(\overline{\Psi}\nabla\Psi)$ and the energy density $W = \varepsilon^2|\nabla\Psi|^2$, are given by [8, 25]

$$
\begin{aligned}
\rho(t,x) &= \int_{\mathbb{R}^d} w(t,x,v)dv, &(8)\\
J(t,x) &= \int_{\mathbb{R}^d} vw(t,x,v)dv, &(9)\\
W(t,x) &= \int_{\mathbb{R}^d} |v|^2w(t,x,v)dv. &(10)
\end{aligned}
$$

Therefore, multiplying the Wigner equation (7) by 1 and v and integrating over $\mathbb{R}^d$, we obtain the continuity equation (1) and the momentum equation (2), respectively, where the pressure $p(s)$ is related to $h(s)$ by $p'(s) = Tsh'(s)$, and T can be interpreted as a temperature (constant). For details, we refer to [8, 9, 25].

We also mention that the momentum variable v of the Wigner equation can be restrained to live in the Brillouin zone for the Schrödinger equation with an oscillating periodic potential $V(x/\varepsilon)$ [29] or in the discrete integer set $\mathbb{Z}^d$ for the Schrödinger equation on the torus [35].

Clearly, a *physical* derivation of the QHD model has to start with the *linear* Schrödinger equation (4) without the self-interacting potential $h(\rho)$. Performing the same calculations as above, we arrive to the QHD equations (1)-(2) without the pressure term $p(\rho)$. A non-zero pressure can be obtained, for instance, by a mixed-state approach; see [8].

1.3 Relaxation-time Limit from the QHD Model to the QDD Model

Rescaling the current density via $J \to \tau J$ and the time via $t \to t/\tau$ and performing the formal limit $\tau \to 0$, we obtain the scaled *quantum drift-diffusion equations* (see, for instance, [15]):

$$\partial_t \rho + \mathrm{div} J = 0, \quad J = \frac{\varepsilon^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} - Th(\rho) + V\right), \tag{11}$$

$$\lambda^2 \Delta V = \rho - C(x), \qquad x \in \Omega,\ t > 0. \tag{12}$$

This paper is organized as follows. In Section 2 we present some results on the existence and large-time behavior of solutions to the QHD and QDD models, and Section 3 is concerned with the corresponding semi-classical limits.

2 Existence and Large-Time Behavior of Solutions

2.1 The Stationary QHD Equations

The third-order term $\rho\nabla(\Delta\sqrt{\rho}/\sqrt{\rho})$ in (2) is the main difficulty in the analytical treatment of the QHD equations. Two strategies have been considered in the mathematical literature: First a potential-flow formulation has been assumed and the stationary momentum equation (2) is integrated once, yielding a *second-order* elliptic equation. Second, the stationary equation (2) is differentiated with respect to the spatial variable which gives a *fourth-order* elliptic equation. We review these strategies now in more detail.

Assume that the current density can be written as $J = \rho\nabla S$ with the quantum quasi-Fermi (or velocity) potential S. This means that we assume a potential flow with irrotational velocity $J/\rho = \nabla S$.

After division by ρ and integration of (2) we obtain the stationary equations:

$$\tfrac{\varepsilon^2}{2}\Delta\sqrt{\rho} = \sqrt{\rho}\left(\tfrac{1}{2}|\nabla S|^2 + Th(\rho) - V + \tfrac{1}{\tau}S\right), \tag{13}$$

$$\mathrm{div}(\rho\nabla S) = 0, \quad \lambda^2\Delta V = \rho - C(x) \quad \text{in } \Omega, \tag{14}$$

subject to Dirichlet boundary conditions:

$$\rho = \rho_D,\ S = S_D,\ V = V_D \quad \text{on } \partial\Omega. \tag{15}$$

For a different choice of boundary conditions, see below. In the above formulation, the function h, defined by $\rho h'(\rho) = p'(\rho)$ and $h(1) = 0$, is called *enthalpy*. We assume that $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ is a bounded domain with $\partial\Omega \in C^{1,1}$. Using a Stampacchia truncation method and elliptic estimates, the following result has been proved in [14] (also see [16]).

Theorem 2.1 *Let $h(\rho) = \log\rho$ (to simplify) and assume that $\sqrt{\rho_D} \in W^{2,p}(\Omega)$ with $p > d/2$ and $\inf_\Omega \rho_D > 0$, $S_D \in C^{1,1}(\overline{\Omega})$, $V_D \in H^1(\Omega) \cap L^\infty(\Omega)$, and $C \in L^2(\Omega)$. Then there exists $\eta > 0$ such that if*

$$T^{-1}\|S_D\|^2_{C^{1,1}(\overline{\Omega})} \leq \eta, \tag{16}$$

there exists a solution (ρ, S, V) of (13)-(15) satisfying

$$\sqrt{\rho} \in W^{2,p}(\Omega),\ S \in C^1(\overline{\Omega}),\ V \in H^1(\Omega) \cap L^\infty(\Omega),\ \text{and } \inf_\Omega \rho > 0.$$

The assumption (16) can be interpreted as a subsonic-type condition. Indeed, (16) and elliptic estimates imply that

$$|J/\rho| = |\nabla S| \leq \|S\|_{C^1(\overline{\Omega})} \leq c\|S_D\|_{C^1(\overline{\Omega})} \leq c\sqrt{\eta} \cdot \sqrt{T},$$

and thus, for sufficiently small $\eta > 0$, $|J/\rho| < \sqrt{T}$.

It is well known that uniqueness of solutions to semiconductor equations cannot hold in general as there are devices which are based on multiple steady states [34]. Generally, uniqueness of solutions is to be expected close to the *thermal equilibrium state* (defined by $J = 0$) or, equivalently, for sufficiently small applied voltage. The uniqueness of solutions of (13)-(15) is shown in [14] under the condition (16) and for sufficiently large $\varepsilon > 0$. Notice that S_D is related to the applied voltage such that the first condition is natural; the second condition is needed for technical reasons.

An existence and *non-existence* result based on the same strategy but with different boundary conditions has been presented in [5]. In this paper, the one-dimensional model in $\Omega = (0, 1)$ has been analyzed using the set of boundary conditions

$$\rho(0) = \rho_0 > 0, \ \rho(1) = \rho_1 > 0, \ V(0) = V_0, \ V_x(0) = -E_0, \ S(0) = K, \qquad (17)$$

where $K < 0$ is a specific (given) constant depending on the data [5]. Notice that in one space dimension, $J = \text{const}$. The following result has been proved in [5].

Theorem 2.2 *Let $\Omega = (0, 1)$ and $C \in L^2(\Omega)$. Then there exist $J_0, J_1 > 0$ such that*

(i) if $h(\rho) = \log \rho$ (to simplify) and $|J| \leq J_0$ then there exists a classical solution (ρ, V) of the stationary equations (1)-(2), (17) satisfying $\min_\Omega \rho > 0$, and

(ii) if $h(\rho) = \rho^\gamma - 1$ with $\gamma > 2$ and $J \geq J_1$ then the stationary QHD equations cannot have a weak solution for any $K \in \mathbb{R}$.

It is unknown if the QHD equations (13)-(14) subject to the boundary conditions (15) possess a (weak) solution for sufficiently large S_D (in an appropriate norm). For a simplified QHD model, it is shown in [14] by a simple example that the electron density vanishes for sufficiently large S_D and consequently, no weak solution can be expected in this case.

The second strategy to analyze the one-dimensional QHD model is to differentiate (2) with respect to x and to use the Poisson equation (3). This leads to an equation for the variable $u = \log \rho$:

$$\frac{\varepsilon^2}{2}\left(u_{xx} + \frac{u_x^2}{2}\right)_{xx} + J^2(e^{-2u}u_x)_x - Tu_{xx} + \lambda^{-2}(e^{-u} - C(x)) = \frac{J}{\tau}(e^{-u})_x \text{ in } \Omega = (0, 1), \qquad (18)$$

where we have taken $h(\rho) = \log \rho$. The boundary conditions are

$$u(0) = u_0, \ u(1) = u_1, \ u_x(0) = u_x(1) = 0. \qquad (19)$$

In [12] the following result is shown.

Theorem 2.3 *Let $C \in L^2(\Omega)$. Then there exists $J_0 > 0$ such that if $|J| \leq J_0$ there exists a unique solution $u \in H^4(\Omega)$ of (18)-(19). Moreover, (ρ, V) with $\rho = e^u$ (and V satisfying (3)) is a solution of the stationary problem (1)-(3) such that $\min_\Omega \rho > 0$.*

The positivity of the particle density cannot be proven using the maximum principle since the equation (18) is of *fourth* order. Instead, Sobolev estimates are derived for the solution u (using the free energy of the problem) such that $u \in L^\infty(\Omega)$. This implies the positivity of $n = e^u$.

The assumption $h(\rho) = \log \rho$ in the above theorem can be substantially weakened. In fact, the pressure p does not need to be a monotone or convex function; for details we refer to [17].

2.2 The Time-Dependent QHD Equations

There are only few results in the mathematical literature concerning the transient QHD equations. Here we review two results. Local-in-time solutions have been studied in [19]. More precisely, existence of strong solutions of the potential-flow formulation

$$\partial_t \rho + \operatorname{div}(\rho \nabla S) = 0, \quad \lambda^2 \Delta V = \rho - C(x), \tag{20}$$

$$\partial_t S + \tfrac{1}{2}|\nabla S|^2 + Th(\rho) - V - \tfrac{\varepsilon^2}{2}\tfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}} = -\tfrac{S}{\tau} \quad \text{in } \Omega \times (0,\infty) \tag{21}$$

subject to the initial-boundary conditions

$$\rho = \rho_D,\ S = S_D,\ V = V_D \qquad \text{on } \partial\Omega \times (0,\infty), \tag{22}$$

$$\rho(\cdot,0) = \rho_I,\ S(\cdot,0) = S_I \qquad \text{in } \Omega, \tag{23}$$

has been proved. Here, $\Omega \subset \mathbb{R}^d$ ($d \le 3$) is supposed to be a bounded domain with $\partial\Omega \in C^2$.

Theorem 2.4 *Assume that* $p \in C^{2,1}(0,\infty)$, $C \in L^2(\Omega)$, ρ_D, S_D, V_D, ρ_I, $S_I \in H^2(\Omega)$ *with* $\rho_I = \rho_D$, $S_I = S_D$ *on* $\partial\Omega$ *and* $\min_\Omega \rho_I > 0$. *Then there exist* $T^* > 0$ *and a unique solution* (ρ, S, V) *of (20)-(23) satisfying*

$$\rho, S, V \in C^0([0,T^*]; H^2(\Omega)) \quad \text{and} \quad \min_\Omega \rho(\cdot,t) > 0 \text{ for } 0 \le t \le T^*.$$

The proof of this result is based on a reformulation of (20)-(21) as the nonlinear Schrödinger equation

$$i\varepsilon\partial_t\psi = -\frac{\varepsilon^2}{2}\Delta\psi - V\psi + h(|\psi|^2)\psi + \frac{1}{\tau}S\psi$$

and the use of semigroup theory.

To our knowledge, the existence of *global-in-time* solutions has been shown only in the case of small perturbations from the steady state in one space dimension [18]. The initial and boundary conditions are as follows:

$$\rho(0,t) = \rho_0,\ \rho(1,t) = \rho_1,\ \rho_x(0,t) = \rho_x(1,t) = 0, \tag{24}$$

$$V(0,t) = V_0,\ V(1,t) = V_1 \qquad \text{for } t > 0, \tag{25}$$

$$\rho(x,0) = \rho_I(x),\ J(x,0) = J_I(x) \qquad \text{for } x \in \Omega. \tag{26}$$

Theorem 2.5 *Assume that* $\Omega = (0,1)$, $p \in C^4(0,\infty)$ *is non-decreasing (to simplify),* $C \in H^2(\Omega)$, $\sqrt{\rho_I} \in H^6(\Omega)$ *with* $\min_\Omega \rho_I > 0$, *and* $J_I \in H^5(\Omega)$. *Then there exist a classical solution* (ρ_0, J_0, V_0) *of the stationary QHD equations and a number* $\eta > 0$ *such that if*

$$\|J_I/\rho_I\|_{C^1(\overline{\Omega})} + \|C_x\|_{H^1(\Omega)} + \|\rho_I - \rho_0\|_{H^6(\Omega)} + \|J_I - J_0\|_{H^5(\Omega)} \le \eta,$$

there exists a classical solution (ρ, J, V) *of (1)-(3), (24)-(26) for all time and, moreover,*

$$\|\rho(\cdot,t) - \rho_0\|_{H^6(\Omega)} + \|J(\cdot,t) - J_0\|_{H^5(\Omega)} + \|V(\cdot,t) - V_0\|_{H^4(\Omega)} \le ce^{-\alpha t},$$

where $c, \alpha > 0$ *are some constants.*

Clearly, the initial and boundary data have to satisfy appropriate compatibility conditions ensuring the needed regularity of the solutions. The idea of the proof is to write (1)-(2) as a fourth-order wave equation and to obtain a-priori estimates which allow to extend the local-in-time solution. The existence of local-in-time solutions is based on a construction of a sequence of approximate solutions converging to the local solution of the transient problem.

2.3 The QDD Equations

The QDD model is quite well understood both analytically and numerically. The existence of solutions to the stationary and the time-dependent equations has been proved.

First we review the well-posedness of the stationary model [1]. We assume that $\Omega \subset \mathbb{R}^d$ $(d \leq 3)$ is a bounded domain with $\partial\Omega = \Gamma_D \cup \Gamma_N \in C^{1,1}$, $\Gamma_D \cap \Gamma_N = \emptyset$, Γ_N is relatively open, and $\mathrm{meas}(\Gamma_D) > 0$.

We impose the following boundary conditions:

$$\rho = \rho_D,\ V = V_D,\ S := \frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} - Th(\rho) + V = S_D \qquad \text{on } \Gamma_D, \tag{27}$$

$$\nabla S \cdot \nu = \nabla V \cdot \nu = 0 \qquad \text{on } \Gamma_N, \tag{28}$$

where ν is the exterior unit normal to $\partial\Omega$. The function S can be interpreted as a quantum quasi-Fermi potential, similarly to the QHD model (see section 2.1). However, observe that here, the function S is given explicitly in terms of ρ and V (and its derivatives) whereas in the QHD model, S is a solution of a differential equation.

For the function h, we can choose (for simplicity) $h(\rho) = \rho^{\gamma-1} - 1$, with $\gamma > 1$, or $h(\rho) = \log\rho$.

Theorem 2.6 *In addition to the above assumptions, let ρ_D, F_D $V_D \in L^\infty(\Omega) \cap H^1(\Omega)$ such that $\inf_\Omega \rho_D \geq 0$. (On the parts of the boundary where $\rho_D = 0$ we do not impose boundary conditions for S.) Then there exists a weak solution $(\rho, V) \in L^\infty(\Omega) \cap H^1(\Omega)$ of the stationary equations (11)-(12), (27)-(28) satisfying*

$$\rho, F, V \in C^0(\Omega) \quad \textit{and} \quad \inf_\Omega \rho > 0.$$

Notice that we allow for *vanishing* electron density on some part of the boundary. Surprisingly, it is possible to prove that the particle density is *positive* in the domain Ω. For the proof of the above theorem, show first that there exists a unique maximizer of an appropriately truncated energy functional corresponding to the problem (11)-(12), (27)-(28) and then, using the uniform estimates coming from the energy inequality, let the truncation parameter tend to zero. The limiting function is the solution of the problem under consideration. The positivity of the electron density is shown by using the Harnack inequality (if $h(\rho) = \rho^{\gamma-1} - 1$) or the classical maximum principle (if $h(\rho) = \log\rho$) and the Hopf principle. We refer to [1] for details and generalizations of the above result. To our knowledge, there does not exist a uniqueness result for the stationary QDD equations.

The time-dependent equations in the one-dimensional interval $\Omega = (0,1)$ are considered with the following initial-boundary conditions:

$$\rho(\cdot,t) = 1,\ \rho_x(\cdot,t) = 0,\ V(\cdot,t) = V_D \qquad \text{on } \partial\Omega \times (0,\infty), \tag{29}$$

$$\rho(\cdot,0) = \rho_I \qquad \text{in } \Omega. \tag{30}$$

The following result can be derived from the results of [22] and [21].

Theorem 2.7 *Let $\Omega = (0,1)$ and $\rho_I \in H^2(\Omega)$. Then there exists a solution (ρ, V) of (11)-(12), (29)-(30) for all time satisfying*

$$\rho \in L^2_{\mathrm{loc}}(0,\infty; H^2(\Omega)) \cap C^0(\overline{\Omega} \times [0,\infty)),\ V \in C^0([0,\infty); C^2(\overline{\Omega})),\ \min_\Omega \rho \geq 0.$$

The boundary condition $\rho_{xx} = 0$ at $x = 0, 1$ is satisfied in a generalized sense; see [22]. The main idea of the proof is a semi-discretization in time such that elliptic problems have to be solved.

The non-negativity property is proved by using $1-\log\rho$ as a test function in (11) which yields $H^1(\Omega)$ estimates for $\log\rho$. In one space dimension this implies $\log\rho \in L^\infty(\Omega)$ and hence $\rho > 0$. In the limit of vanishing time discretization parameter the strict inequality relaxes to $\rho \geq 0$ in $\Omega\times(0,\infty)$.

An existence result for a simplified equation ($T=0$ and $V=0$ in (11)) with the boundary conditions $\rho=1$, $\rho_x=0$ and $V=V_D$ has been presented in [21].

3 Semi-classical Limits

In this section we review two results on the semi-classical limit in the quantum models leading to classical fluiddynamical equations, in particular the *combined* semi-classical and quasi-neutral limit in the Schrödinger-Poisson system and the semi-classical limit in the QHD equations. In the literature, semi-classical limits have also been performed for other quantum models, for instance for the stationary QDD equations [1].

3.1 From the Schrödinger-Poisson to the Compressible Euler Equation

The combined semi-classical quasi-neutral limit in the bipolar Schrödinger-Poisson system gives the compressible Euler equations. This limit has been made rigorous in [23]. More precisely, consider the Schrödinger-Poisson system

$$i\varepsilon\partial_t\psi_j = -\frac{\varepsilon^2}{2}\Delta\psi_j + (q_jV + h(|\psi_j|^2))\psi_j, \quad \psi_j(\cdot,0)=\psi_{I,j}, \quad j=1,2, \tag{31}$$

$$\lambda^2\Delta V = |\psi_1|^2 - |\psi_2|^2 \quad \text{in } \mathbb{R}^d\times(0,\infty), \tag{32}$$

where $d\geq 1$, $q_1=-1$ and $q_2=+1$. Let $(\psi_1^\delta,\psi_2^\delta,V^\delta)$ be a solution of (31)-(32) with $\delta := (\varepsilon,\lambda)$. In the limit $\delta\to 0$, it is possible to prove that the particle densities ρ_j^δ and the current densities J_j^ε, defined by

$$\rho_j^\delta = |\psi_j^\delta|^2, \quad J_j^\delta = \varepsilon\,\mathrm{Im}(\overline{\psi_j^\delta}\nabla\psi_j^\delta),$$

converge to ρ and ρu, respectively, where (ρ,u) is a (smooth) solution of

$$\partial_t\rho + \mathrm{div}(\rho u) = 0, \quad \partial_t u + u\cdot\nabla u + \nabla h(\rho) = 0 \quad \text{in } \mathbb{R}^d\times(0,T^*), \tag{33}$$

$$\rho(\cdot,0)=\rho_I, \quad u(\cdot,0)=u_0 \quad \text{in } \mathbb{R}^d, \tag{34}$$

and $(0,T^*)$ is the maximal time interval in which there exists a smooth solution (ρ,u) of (33)-(34).

Theorem 3.1 *Let $h(\rho)=\rho^{\gamma-1}-1$ with $\gamma\geq 2$, $\rho_0 := |\psi_{I,1}^\delta|^2 = |\psi_{I,2}^\delta|^2 \in L^\gamma(\mathbb{R}^d)$, $u_0\in H^s(\mathbb{R}^d)$, where $s > d/2+2$. Furthermore, assume that $\|\psi_j^\delta\|_{H^s(\mathbb{R}^d)}$ is sufficiently small and that*

$$\|u_0\psi_{I,j}^\delta + i\varepsilon\nabla\psi_{I,j}^\delta\|_{L^2(\mathbb{R}^d)} \to 0 \tag{35}$$

as $\delta\to 0$ ($j=1,2$). Then, as $\delta=(\varepsilon,\lambda)\to 0$,

$$\rho_j^\delta \to \rho \qquad \textit{weakly in } L^\infty(0,T^*;L^\gamma(\mathbb{R}^d)),$$
$$J_j^\delta \to \rho u \qquad \textit{in the sense of } L^\infty(0,\infty;\mathcal{M}(\mathbb{R}^d)),$$

where $\mathcal{M}(\mathbb{R}^d)$ is the d-vector-valued space of Radon measures and (ρ,u) is the unique $H^s(\mathbb{R}^d)$ solution of (33)-(34).

If ψ_j^δ can be decomposed as $\psi_j^\delta = \sqrt{\rho_j^\delta}\exp(iS_j^\delta/\varepsilon)$ the assumption (35) can be written in the form

$$\varepsilon\|\nabla\sqrt{\rho_j^\delta}\|^2_{L^2(\Omega)}+\|\sqrt{\rho_j^\delta}(u_0-\nabla S_j^\delta)\|^2_{L^2(\mathbb{R}^d)} = \|u_0\psi_{I,j}+i\varepsilon\nabla\psi_{I,j}^\delta\|^2_{L^2(\mathbb{R}^d)} \to 0 \quad \text{as } \delta\to 0.$$

In particular, if $\rho_0 = \rho_{I,j}^\delta$ and $u_0 = \nabla S_j^\delta$ $(j = 1,2)$, then (35) is trivially satisfied since

$$\|u_0\psi_{I,j}^\delta + i\varepsilon\nabla\psi_{I,j}^\delta\|^2_{L^2(\mathbb{R}^d)} = \varepsilon\|\nabla\sqrt{\rho_0}\|^2_{L^2(\Omega)} \to 0 \quad \text{as } \delta\to 0.$$

For the proof of Theorem 3.1, the Schrödinger-Poisson system (31)-(32) is formulated as a quantum hydrodynamic model in which the convection term is replaced by an expression which contains the Wigner function. Using estimates from a modulated energy functional, it is possible to prove that $\rho_j^\delta \to \rho$ weakly* in $L^\infty(0,T^*;L^\gamma(\mathbb{R}^d))$. Moreover, the Wigner function converges weakly to some Radon measure w as shown in [9, 25]. Then the estimates for the modulated energy functional imply that $w(x,v,t) = \rho(x,t)\delta(v-u(x,t))$, where δ is the Dirac δ-distribution. This concludes the proof (see [23] for details).

The combined semi-classical quasi-neutral limit in the *linear* Schrödinger equation, coupled to the Poisson equation for one carrier type, to the *incompressible* Euler equations has been shown in [31].

We notice that there are only a few results on the *quasi-neutral* limit $\lambda\to 0$. Brenier [3] and Grenier [11] performed the limit $\lambda\to 0$ in Vlasov-Poisson systems. The quasi-neutral limit in drift-diffusion equations has been made rigorous in [7, 20].

Another way to perform the semi-classical limit rigorously is to make a WKB ansatz and to reformulate the (defocusing) Schrödinger-Poisson system as a quantum hydrodynamic model. This model can be seen as a dispersive perturbation of the Euler-Poisson problem written in the form of a quasi-linear symmetric hyperbolic system to which the Lax-Friedrich-Kato theory can be applied. In [24] it is proved that the solution of the (unipolar) Schrödinger-Poisson system converges to a (local-in-time smooth) solution of the compressible Euler-Poisson system as $\varepsilon\to 0$.

3.2 From the Quantum Hydrodynamic to the Subsonic Hydrodynamic Equations

Let $(\rho_\varepsilon, V_\varepsilon) \in H^4(\Omega)\times H^2(\Omega)$ be the solution of the one-dimensional stationary QHD model (1)-(3) with $\Omega = (0,1)$ and boundary conditions (19) where $\rho = e^u$. Then it is shown in [12] that $(\rho_\varepsilon, V_\varepsilon)$ converges to the solution (n, V) of the hydrodynamic equations

$$\left(\frac{J^2}{\rho} + Tp(\rho)\right)_x - \rho V_x = -\frac{J}{\tau}, \quad \lambda^2 V_{xx} = \rho - C(x), \tag{36}$$

$$\rho(0) = \rho_0 > 0,\ \rho(1) = \rho_1 > 0,\ V(0) = V_0, \tag{37}$$

if the (constant) current density is small enough and if $p(\rho) = \rho$.

Theorem 3.2 *Let $C\in L^2(\Omega)$ and $p(\rho)=\rho$. Then, for sufficiently small $J>0$, as $\varepsilon\to 0$,*

$$\begin{aligned} \rho_\varepsilon &\to \rho && \text{weakly in } H^1(\Omega) \text{ and strongly in } C^0(\overline{\Omega}), \\ V_\varepsilon &\to V && \text{weakly in } L^2(\Omega), \end{aligned}$$

and $(\rho, V)\in H^1(\Omega)\times H^1(\Omega)$ is the unique solution of (36)-(37) satisfying $\min_\Omega \rho > 0$.

The idea of the proof is to use $\log \rho_\varepsilon - \log \tilde{\rho}$ with $\tilde{\rho}(0) = \rho_0$ and $\tilde{\rho}(1) = \rho_1$ as a test function in the stationary equation (2). The pressure term provides an $H^1(\Omega)$ estimate for $\log \rho_\varepsilon$ independent of ε from which we can conclude the result.

A similar result using different boundary conditions has been presented in [6].

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