BLOW-UP IN TWO-COMPONENT NONLINEAR SCHRÖDINGER SYSTEMS WITH AN EXTERNAL DRIVEN FIELD

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A system of two nonlinear Schrödinger equations in up to three space dimensions is analyzed. The equations are coupled through cubic mean-field terms and a linear term which models an external driven field described by the Rabi frequency. The intraspecific mean-field expressions may be non-cubic. The system models, for instance, two components of a Bose-Einstein condensate in a harmonic trap. Sufficient conditions on the various model parameters for global-in-time existence of strong solutions are given. Furthermore, the finite-time blow-up of solutions is proved under suitable conditions on the parameters and in the presence of at least one focusing nonlinearity. Numerical simulations in one and two space dimensional equations verify and complement the theoretical results. It turns out that the Rabi frequency of the driven field may be used to control the mass transport and hence to influence the blow-up behavior of the system.

Keywords: Nonlinear Schrödinger equations, global existence, blow-up of solutions, Rabi frequency, time-splitting sine-spectral method, blow-up time.

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1. Introduction

We consider the following system of two nonlinear Schrödinger equations:

$$i\partial_t\psi_1 = -\frac{1}{2}\Delta\psi_1 + \frac{\gamma^2}{2}|x|^2\psi_1 + \beta_{11}|\psi_1|^\alpha\psi_1 + \beta_{12}|\psi_2|^2\psi_1 + \lambda\psi_2, \tag{1.1}$$

$$i\partial_t\psi_2 = -\frac{1}{2}\Delta\psi_2 + \frac{\gamma^2}{2}|x|^2\psi_2 + \beta_{12}|\psi_1|^2\psi_2 + \beta_{22}|\psi_2|^\alpha\psi_2 + \lambda\psi_1, \tag{1.2}$$

where $x \in \mathbb{R}^N$ $(N \leq 3)$ and $t > 0$, with the initial conditions

$$\psi_1(\cdot, 0) = \psi_1^0, \quad \psi_2(\cdot, 0) = \psi_2^0 \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

The parameters are the mean-field exponent $\alpha > 0$, the magnetic trapping strength $\gamma > 0$, the external driven field constant $\lambda \in \mathbb{R}$, the intraspecific scattering lengths

β_{jj}, the interspecific scattering length β_{12}, and the space dimension $N \leq 3$. The wave functions ψ_1 and ψ_2 describe two components of a quantum system. They are coupled through the mean-field expressions $|\psi_1|^2\psi_2$, $|\psi_2|^2\psi_1$ and through the linear coupling terms $\lambda\psi_2$, $\lambda\psi_1$. If β_{jj} is negative (positive), the self-interaction is attractive (repulsive). Similarly, if $\beta_{12} < 0$ ($\beta_{12} > 0$), the interspecies interaction is attractive (repulsive).

When $\alpha = 2$ and $\lambda = 0$, the above system models, for instance, a two-component beam in Kerr-like nonlinear optical media (see Formulas (1)-(2) in Ref. 1) or a mixture of Bose-Einstein condensates consisting of two different hyperfine spin states of Rubidium atoms below the critical temperature (see Formula (1) in Ref. 6). In the latter example, the two components of the condensate are placed in the same harmonic trap. By applying a weak magnetic (driven) field with the Rabi frequency λ, the two components are coupled in the overlap region. This coupling realizes a Josephson-type junction and gives rise to nonlinear oscillations in the relative populations.[27]

Non-cubic nonlinearities $\alpha \neq 2$ occur, for instance, in lower dimensional mean-field models describing cigar-shaped Bose-Einstein condensates; see Section 2.4 in Ref. 13. Cubic-quintic nonlinear terms arise in the modeling of Bose-Einstein condensates trapped in an optimal lattice, where the quintic term ($\alpha = 4$) represents three-body collisions.[15] We allow for general nonlinearities also for mathematical reasons. Indeed, we wish to study the possible blow-up behavior of strong solutions in the presence of at least one focusing nonlinearity.

More precisely, we analyze under which conditions on the model parameters the global existence of solutions or the finite-time blow-up of the above system can be proved. Furthermore, the influence of the Rabi frequency λ on the blow-up behavior is studied. Some of our results may be summarized as follows:

- Let $\lambda = 0$. If the interspecies interaction is attractive, blow-up may occur even when the self-interaction is repulsive (see Theorem 3.1). This result is known in the literature (see, e.g., Refs. 18, 29) but our conditions are more general in certain cases (see Remark 3.1).
- The Rabi frequency may trigger or delay blow-up in the following sense. We observe numerically that blow-up in the case of one focusing component seems to occur for appropriately chosen λ even when the corresponding initial mass is subcritical (the total initial mass has to be supercritical). On the other hand, numerical experiments indicate that certain values of λ may avoid or delay blow-up (see Section 5).
- If $\alpha = 2$, we derive a semi-explicit formula for the time evolution of the masses of the components. The mass exchange is quasi-periodic, perturbed by the interaction coefficients. If all β_{jk} are equal, the evolution of the masses is exactly periodic with frequency 2λ (see Section 4).

It is well known that the strong solution of the nonlinear Schrödinger equation

$$i\partial_t \psi = -\frac{1}{2}\Delta\psi - |\psi|^\alpha \psi \quad \text{in } \mathbb{R}^N$$

blows up in finite time (i.e. the L^2 norm of $|\nabla\psi|$ becomes infinite) if $\alpha \geq 4/N$ (L^2-critical and supercritical case) and the initial data is sufficiently large in the H^1 norm. We refer to the monograph Ref. 9 for more details. When an external potential is included in the L^2-critical nonlinear Schrödinger equation ($\alpha = 4/N$), Carles showed that the blow-up rate does not change, and only the blow-up time is affected.[7,8]

Let us comment on related results in the literature for coupled nonlinear Schrödinger systems. Most of the papers are concerned with system (1.1)-(1.2) with $\alpha = 2$ and $\lambda = 0$. For focusing nonlinearities, $\beta_{jk} < 0$, Chen and Wei[11] proved the global-in-time existence of solutions in the energy space $\Sigma = \{\psi \in H^1(\mathbb{R}^N) : |x\psi| \in L^2(\mathbb{R}^N)\}$ if the initial datum in Σ is smaller than the L^2 norm of a certain ground-state solution. Simultaneous blow-up of the two components (without confinement, $\gamma = 0$) was shown by Lin and Wei[18] if β_{11}, $\beta_{22} < 0$, and $\beta_{12} < \sqrt{\beta_{11}\beta_{22}}$. A similar result, but including harmonic confinement, was proved by Zhongxue and Zuhan.[28] The two-dimensional case $N = 2$ of a special system without confinement, assuming $\beta_{11} = \beta_{22} = 0$ and $\beta_{12} < 0$, was treated by Ma and Schulze.[19] Finally, we mention the work of Pyrtula et al.[24] in which a coupled system of $M \geq 2$ nonlinear Schrödinger equations was studied and conditions on the initial data and the parameters β_{jk} for finite-time blow-up were found.

Schrödinger systems with general parameter α have been examined in the mathematical literature as well. These systems, arising in nonlinear optics, have typically the structure

$$\begin{aligned} i\partial_t\psi_1 &= -\frac{1}{2}\Delta\psi_1 + \beta_{11}|\psi_1|^{2p}\psi_1 + \beta_{12}|\psi_2|^{p+1}\psi_1, \\ i\partial_t\psi_2 &= -\frac{1}{2}\Delta\psi_2 + \beta_{22}|\psi_2|^{2p}\psi_2 + \beta_{12}|\psi_1|^{p+1}\psi_2 \quad \text{in } \mathbb{R}^N, \end{aligned}$$

where $p > 0$ and β_{11}, β_{22}, $\beta_{12} < 0$, i.e., all the nonlinearities are focusing. When $p = 1$, we recover (1.1)-(1.2) with $\alpha = 2$. Fanelli and Montefusco[12] proved that there exists a global strong solution if $p < 2/N$. In the critical case $p = 2/N$, there exist initial data such that the solutions blow up in finite time. These results were improved by Li, Wu, and Lai[17] who derived sharp blow-up thresholds. Chen and Guo[10] assumed that only the interspecific term is attractive, $\beta_{12} < -\sqrt{\beta_{11}\beta_{22}}$. They show that there exist global solutions if $p < 2/N$, whereas the solutions blow up in finite time if $p > 2/N$ and either the initial energy is negative or the time derivative of the variance, $I'(t) = \partial_t \int_{\mathbb{R}^N} |x|^2(|\psi_1|^2 + |\psi_2|^2)dx$, is sufficiently negative initially. In the critical case $p = 2/N$, the solutions blow up like the Dirac δ distribution. Systems of more than two equations were examined by Ma and Zhao[21] and Song.[25]

Only a few papers are concerned with Schrödinger systems involving the Rabi frequency λ (and always $\alpha = 2$). One of the first papers seems to be due to Williams et al.[27] examining the nonlinear oscillations in the relative populations of the two components by numerical experiments. Oscillations were also predicted by Park and Eberly.[23] Kim and Liu[16] have shown that a strong external driven field causes vortex filaments. The creation and stability of dark solitons was investigated by Öhberg and Santos.[22] From a more mathematical point of view, ground-state solutions were numerically computed by Bao[2] and Bao and Cai.[4] Finally, Zhongxue and Zuhan[29] proved finite-time blow-up of solutions for "large" initial data when all coefficients β_{jk} are negative. The results seem to be valid in the two-dimensional case only.

The originality of the present paper consists in the fact that we consider (1.1)-(1.2) for general $\alpha > 0$ and $\lambda \in \mathbb{R}$ and that we present a thorough global existence and blow-up analysis allowing for various combinations of the signs of the coefficients β_{jk}. Moreover, we present numerical simulations in one and two space dimensions using a time-splitting sine-spectral method which underlines and complements the analytical findings.

The paper is organized as follows. In Section 2 we prove the local-in-time and global-in-time existence of strong solutions. Sufficient conditions for the finite-time blow-up of solutions are given in Section 3. The role of the Rabi frequency λ is examined in Section 4. Section 5 is devoted to numerical examples. We conclude in Section 6.

2. Local and global existence of solutions

First we introduce some definitions. For functions $\Psi = (\psi_1, \psi_2) : \mathbb{R}^N \to \mathbb{C}^2$, we define the norms

$$\|\Psi\|_p = \left(\sum_{j=1}^{2} \int_{\mathbb{R}^N} |\psi_j(x)|^p dx \right)^{1/p}, \quad \|\nabla\Psi\|_p = \left(\sum_{j=1}^{2} \int_{\mathbb{R}^N} |\nabla\psi_j(x)|^p dx \right)^{1/p}$$

for $1 \le p < \infty$ with the corresponding Banach spaces $L^p(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N) = \{\Psi \in L^2(\mathbb{R}^N) : |\nabla\Psi| \in L^2(\mathbb{R}^N)\}$. We write the coupled Schrödinger equations as the system

$$i\partial_t \Psi = H\Psi + g(\Psi) + B\Psi \quad \text{in } \mathbb{R}^N,\ t > 0, \quad \Psi(\cdot, 0) = \Psi^0, \tag{2.1}$$

where $\Psi^0 = (\psi_1^0, \psi_2^0)$,

$$H = \begin{pmatrix} -\frac{1}{2}\Delta + \frac{1}{2}\gamma^2|x|^2 & 0 \\ 0 & -\frac{1}{2}\Delta + \frac{1}{2}\gamma^2|x|^2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix},$$

and the nonlinearity is given by the matrix

$$g(\Psi) = \begin{pmatrix} \beta_{11}|\psi_1|^\alpha \psi_1 + \beta_{12}|\psi_2|^2 \psi_1 \\ \beta_{12}|\psi_1|^2 \psi_2 + \beta_{22}|\psi_2|^\alpha \psi_2 \end{pmatrix}.$$

It is not difficult to show that g and B are locally Lipschitz continuous. In particular, the assumptions of Theorem 3.3.9 in Ref. 9 are satisfied, showing the local well-posedness of (2.1). More precisely, we introduce the energy-type space

$$\Sigma = \{\Psi \in H^1(\mathbb{R}^n) : |x\Psi| \in L^2(\mathbb{R}^N)\}. \tag{2.2}$$

Furthermore, let $N \le 3$, $0 \le \alpha < 4/(N-2)$ ($\alpha < \infty$ if $N \le 2$), and $\Psi^0 \in \Sigma$. Then there exists a unique maximal solution $\Psi = (\psi_1, \psi_2) \in C^0((-T_{\min}, T_{\max}); \Sigma) \cap C^1((-T_{\min}, T_{\max}); \Sigma^*)$ to (1.1)-(1.2). Moreover, the total mass

$$M(t) = M_1(t) + M_2(t) = \int_{\mathbb{R}^N} |\psi_1(x,t)|^2 dx + \int_{\mathbb{R}^N} |\psi_2(x,t)|^2 dx$$

and the total energy $E(t) = E_1(t) + E_2(t) + E_{12}(t)$, consisting of the energies of the components,

$$E_j(t) = \int_{\mathbb{R}^N} \Big(\frac{1}{2}|\nabla\psi_j|^2 + \frac{\gamma^2}{2}|x|^2|\psi_j|^2 + \frac{2\beta_{jj}}{\alpha+2}|\psi_j|^{\alpha+2} + \frac{\beta_{12}}{2}|\psi_1|^2|\psi_2|^2 \Big)(x,t)dx, \quad (2.3)$$

and the interaction energy,

$$E_{12}(t) = 2\lambda \int_{\mathbb{R}^N} \Re(\psi_1^* \psi_2)(x,t)dx, \quad (2.4)$$

are conserved quantities, $M(t) = M(0)$ and $E(t) = E(0)$ for all $t \ge 0$. Here, $\Re(\psi)$ denotes the real part of ψ and ψ^* its complex conjugate.

Next, we wish to specify conditions on the data under which global existence holds. First, we prove the following lemma which generalizes an idea used in Section 2.4 of Ref. 24.

Lemma 2.1. *Let $\delta > 2$ and a_0, a_2, $a_\delta > 0$. Furthermore, let $F(x) = a_\delta x^\delta - a_2 x^2 + a_0$, $x \ge 0$, and let $v : [0,\infty) \to \mathbb{R}$ be a continuous function satisfying $F(v(t)) \ge 0$ for all $t \ge 0$. If*

$$v(0) \le \sqrt{\frac{a_0}{a_2}}, \quad a_0 < (\delta-2)\Big(\frac{4a_2^\delta}{\delta^\delta a_\delta^2}\Big)^{1/(\delta-2)}, \quad (2.5)$$

then there exists $C > 0$ such that $v(t) \le C$ for all $t \ge 0$.

Proof. The function F has a local maximum at $x_1 = 0$ (since $F'(x) < 0$ for small $x \ge 0$) and a global minimum at $x_2 = (2a_2/\delta a_\delta)^{1/(\delta-2)}$ (since $F''(x_2) = 2(\delta-2)a_2 > 0$). Condition (2.5) on a_0 implies that $F(x_2) < 0$. Thus, if $v(0) \le x_3$ for some $x_3 > 0$ which is smaller than the first root of F, $v(t)$ is bounded for $t \ge 0$. We choose $x_3 = \sqrt{a_0/a_2}$. This choice gives $F(x_3) = a_\delta(a_0/a_2)^{\delta/2} > 0$, and condition (2.5) yields

$$a_0^{(\delta-2)/2} < \frac{2(\delta-2)^{(\delta-2)/2} a_2^{\delta/2}}{\delta^{\delta/2} a_\delta} < \frac{2\delta^{(\delta-2)/2} a_2^{\delta/2}}{\delta^{\delta/2} a_\delta} = \frac{2a_2}{\delta a_\delta} a_2^{(\delta-2)/2}$$

which implies that $x_3 < x_2$. We infer that $v(t) \le x_2$ for $t \ge 0$. □

We also need the Gagliardo-Nirenberg inequality

$$\|\psi\|_{\alpha+2}^{\alpha+2} \le C_{\alpha,N} \|\nabla\psi\|_2^{\alpha N/2} \|\psi\|_2^{\alpha+2-\alpha N/2}, \quad \psi \in H^1(\mathbb{R}^N), \quad (2.6)$$

where $C_{\alpha,N} > 0$. Furthermore, for any solution (ψ_1, ψ_2) to (1.1)-(1.2), we set

$$G(t) = \|\nabla\psi_1(\cdot,t)\|_2^2 + \|\nabla\psi_2(\cdot,t)\|_2^2$$

and $x^+ = \max\{0, x\} \geq 0$ for $x \in \mathbb{R}$. Our main result on the global existence of solutions reads as follows.

Theorem 2.1. *Let* $N \leq 3$, $0 \leq \alpha < 4/(N-2)$ ($\alpha < \infty$ *if* $N \leq 2$), *and set* $\beta = \max\{(-\beta_{11})^+, (-\beta_{22})^+\}$. *Then there exists a global-in-time strong solution to* (1.1)-(1.2) *in the following cases:*

(1) $\min\{\beta_{11}, \beta_{22}\} \geq 0$, $\beta_{12} \geq 0$;
(2) $\min\{\beta_{11}, \beta_{22}\} \geq 0$, $\beta_{12} < 0$:

(a) $N = 1$,
(b) $N = 2$ *and* $M(0) < 2/(C_{2,2}|\beta_{12}|)$,
(c) $N = 3$, $G(0) \leq 2(E(0) + |\lambda| M(0))$, *and*

$$M(0)(E(0) + |\lambda| M(0)) < \frac{8}{27 C_{2,3}^2 \beta_{12}^2};$$

(3) $\min\{\beta_{11}, \beta_{22}\} < 0$, $\beta_{12} \geq 0$:

(a) $\alpha < 4/N$,
(b) $\alpha = 4/N$ *and* $M(0) < ((N+2)/(2N C_{\alpha,N} \beta))^{N/2}$,
(c) $\alpha > 4/N$, $G(0) \leq 2(E(0) + |\lambda| M(0))$, *and*

$$M(0)^{\alpha+2-\alpha N/2}(E(0) + |\lambda| M(0))^{(\alpha N-4)/2} < \frac{4(\alpha N - 4)^{(\alpha N-4)/2}(\alpha+2)^2}{(2\alpha N)^{\alpha N/2} C_{\alpha,N}^2 \beta^2};$$

(4) $\min\{\beta_{11}, \beta_{22}\} < 0$, $\beta_{12} < 0$:

(a) $N = 1$: $\alpha < 4$ *or* ($\alpha = 4$ *and* $M(0) < (3/(2C_{4,1}\beta))^{1/2}$),
(b) $N = 2$:

$$\begin{aligned}
&\alpha < 2 \quad \textit{and} \quad M(0) < 2/(C_{2,2}|\beta_{12}|), \\
&\alpha = 2 \quad \textit{and} \quad M(0) < 4/(C_{2,2}(2\beta + |\beta_{12}|)), \\
&\alpha > 2, \quad a_2 := \frac{1}{2} - \frac{1}{4} C_{2,2}|\beta_{12}| M(0) > 0, \quad \textit{and} \\
&\qquad (E(0) + |\lambda| M(0))^{\alpha-2} M(0)^2 < \frac{(\alpha-2)^{\alpha-2}(\alpha+2)^2 a_2^\alpha}{\alpha^\alpha C_{\alpha,2}^2 \beta^2};
\end{aligned}$$

(c) $N = 3$: $G(0) \leq 2(E(0) + |\lambda| M(0))$ *and*

$$\begin{aligned}
&\alpha = 2, \quad M(0)(E(0) + |\lambda| M(0)) < \frac{8}{27 C_{2,3}^2 (2\beta + |\beta_{12}|)^2}, \\
&\alpha = \frac{4}{3}, \quad a_2 := \frac{1}{2} - \frac{3}{5} C_{4/3,3} \beta M(0)^{2/3} > 0, \\
&\qquad M(0)(E(0) + |\lambda| M(0)) < \frac{64 a_2^3}{27 C_{2,3}^2 \beta_{12}^2}, \\
&\alpha \neq \frac{4}{3}, 2, \quad M(0) \textit{ sufficiently small.}
\end{aligned}$$

Proof. We can write the energy conservation as

$$\begin{aligned}\frac{1}{2}G(t) &= E(0) - \frac{\gamma^2}{2}\int_{\mathbb{R}^N}|x|^2(|\psi_1|^2+|\psi_2|^2)dx - \sum_{j=1}^{2}\frac{2\beta_{jj}}{\alpha+2}\|\psi_j\|_{\alpha+2}^{\alpha+2} \\ &\quad - \beta_{12}\int_{\mathbb{R}^N}|\psi_1|^2|\psi_2|^2dx - 2\lambda\int_{\mathbb{R}^N}\Re(\psi_1^*\psi_2)dx \\ &\le E(0) + \frac{2}{\alpha+2}\sum_{j=1}^{2}(-\beta_{jj})^+\|\psi_j\|_{\alpha+2}^{\alpha+2} + (-\beta_{12})^+\|\psi_1\|_4^2\|\psi_2\|_4^2 \\ &\quad + 2|\lambda|\,\|\psi_1\|_2\|\psi_2\|_2.\end{aligned}$$

Applying the Gagliardo-Nirenberg inequality (2.6), this becomes

$$\begin{aligned}\frac{1}{2}G(t) &\le E(0) + \frac{2C_{\alpha,N}}{\alpha+2}\sum_{j=1}^{2}(-\beta_{jj})^+\|\nabla\psi_j\|_2^{\alpha N/2}\|\psi_j\|_2^{\alpha+2-\alpha N/2} \\ &\quad + C_{2,N}(-\beta_{12})^+\|\nabla\psi_1\|_2^{N/2}\|\psi_1\|_2^{2-N/2}\|\nabla\psi_2\|_2^{N/2}\|\psi_2\|_2^{2-N/2} \\ &\quad + |\lambda|\big(\|\psi_1\|_2^2+\|\psi_2\|_2^2\big) \\ &= E(0) + I_2 + I_3 + I_4. \end{aligned} \tag{2.7}$$

We estimate the right-hand side term by term.

If $\beta_{jj} \ge 0$ for $j = 1, 2$, the second term I_2 vanishes. Otherwise, we employ the Hölder inequality:

$$\begin{aligned}I_2 &\le \frac{2C_{\alpha,N}\beta}{\alpha+2}\left(\sum_{j=1}^{2}\|\nabla\psi_j\|_2^{\alpha pN/2}\right)^{1/p}\left(\sum_{j=1}^{2}\|\psi_j\|_2^{q(2\alpha+4-\alpha N)/2}\right)^{1/q} \\ &= \frac{2C_{\alpha,N}\beta}{\alpha+2}\left(\sum_{j=1}^{2}(\|\nabla\psi_j\|_2^2)^{\alpha pN/4}\right)^{1/p}\left(\sum_{j=1}^{2}(\|\psi_j\|_2^2)^{q(2\alpha+4-\alpha N)/4}\right)^{1/q},\end{aligned}$$

where $1/p + 1/q = 1$. We distinguish the cases $\alpha N/4 < 1$ and $\alpha N/4 \ge 1$. If $\alpha N/4 < 1$, we choose $p = 4/(\alpha N) > 1$ and $q = 4/(4 - \alpha N) > 1$. Then, with $r = q(2\alpha + 4 - \alpha N)/4 = 1 + 2\alpha/(4 - \alpha N) > 1$,

$$\begin{aligned}&\left(\sum_{j=1}^{2}(\|\nabla\psi_j\|_2^2)^{\alpha pN/4}\right)^{1/p}\left(\sum_{j=1}^{2}(\|\psi_j\|^2)_2^{q(2\alpha+4-\alpha N)/4}\right)^{1/q} \\ &\le \left(\sum_{j=1}^{2}\|\nabla\psi_j\|_2^2\right)^{1/p}\left(\sum_{j=1}^{2}\|\psi_j\|_2^{2r}\right)^{1/q} \le \left(\sum_{j=1}^{2}\|\nabla\psi_j\|_2^2\right)^{\alpha N/4}\left(\sum_{j=1}^{2}\|\psi_j\|_2^2\right)^{r/q},\end{aligned}$$

using $x^r + y^r \le (x + y)^r$ for all x, $y \ge 0$, since $r > 1$.

Next, let $\alpha N/4 \ge 1$. If $N \le 2$, we choose any $1 < p, q < \infty$ satisfying $1/p+1/q = 1$. If $N = 3$, we take $p = 4/\alpha$ and $q = 4/(4 - \alpha)$. Since $\alpha < 4/(N - 2) = 4$ by assumption, it holds that $p > 1$. Furthermore, $1/p + 1/q = 1$. Therefore, applying the inequality $x^r + y^r \leq (x+y)^r$ to $r = \alpha pN/4$ and $r = q(2\alpha + 4 - \alpha N)/4 \geq 1$, respectively, we obtain

$$\left(\sum_{j=1}^{2} (\|\nabla\psi_j\|_2^2)^{\alpha pN/4}\right)^{1/p} \leq \left(\sum_{j=1}^{2} \|\nabla\psi_j\|_2^2\right)^{\alpha N/4},$$

$$\left(\sum_{j=1}^{2} (\|\psi_j\|^2)_2^{q(2\alpha+4-\alpha N)/4}\right)^{1/q} \leq \left(\sum_{j=1}^{2} \|\psi_j\|_2^2\right)^{(2\alpha+4-\alpha N)/4}.$$

Summarizing, we infer that

$$\begin{aligned} I_2 &\leq \frac{2C_{\alpha,N}\beta}{\alpha+2}\left(\sum_{j=1}^{2} \|\nabla\psi_j\|_2^2\right)^{\alpha N/4} \left(\sum_{j=1}^{2} \|\psi_j\|_2^2\right)^{(2\alpha+4-\alpha N)/4} \\ &= \frac{2C_{\alpha,N}\beta}{\alpha+2} G(t)^{\alpha N/4} M(0)^{(2\alpha+4-\alpha N)/4}. \end{aligned}$$

The term I_3 is estimated by using Young's inequality, $xy \leq \frac{1}{2}(x^2+y^2)$ for x, $y \geq 0$, and mass conservation, $M(t) = M(0)$:

$$\begin{aligned} I_3 &= C_{2,N}(-\beta_{12})^+ \big(\|\nabla\psi_1\|_2\|\nabla\psi_2\|_2\big)^{N/2} \big(\|\psi_1\|_2\|\psi_2\|_2\big)^{2-N/2} \\ &\leq \frac{C_{2,N}}{4}(-\beta_{12})^+ \big(\|\nabla\psi_1\|_2^2 + \|\nabla\psi_2\|_2^2\big)^{N/2} \big(\|\psi_1\|_2^2 + \|\psi_2\|_2^2\big)^{2-N/2} \\ &= \frac{C_{2,N}}{4}(-\beta_{12})^+ G(t)^{N/2} M(0)^{2-N/2}. \end{aligned}$$

Finally, we write $I_4 = |\lambda| M(0)$. Hence, (2.7) becomes

$$\begin{aligned} \frac{1}{2}G(t) \leq E(0) + |\lambda|M(0) &+ \frac{2C_{\alpha,N}\beta}{\alpha+2} G(t)^{\alpha N/4} M(0)^{(2\alpha+4-\alpha N)/4} \\ &+ \frac{C_{2,N}}{4}(-\beta_{12})^+ G(t)^{N/2} M(0)^{2-N/2}. \end{aligned} \tag{2.8}$$

Now, we consider the various cases of the signs of $\min\{\beta_{11}, \beta_{22}\}$ and β_{12}.

1. When $\min\{\beta_{11}, \beta_{22}\} \geq 0$ and $\beta_{12} \geq 0$, it holds that $\beta = 0$ and $(-\beta_{12})^+ = 0$. Hence, (2.8) shows that $G(t)$ is uniformly bounded for all $t \geq 0$. This implies the global existence of strong solutions.[9]

2. Let $\min\{\beta_{11}, \beta_{22}\} \geq 0$ and $\beta_{12} < 0$. Then (2.8) reduces to

$$\frac{1}{2}G(t) \leq E(0) + |\lambda|M(0) + \frac{C_{2,N}}{4}|\beta_{12}|G(t)^{N/2}M(0)^{2-N/2},$$

and we have to distinguish the space dimension. If $N = 1$, $G(t)$ remains bounded uniformly in $t \geq 0$. If $N = 2$, the coefficient of $G(t)$ has to be positive,

$$\frac{1}{2} - \frac{C_{2,2}}{4}|\beta_{12}|M(0) > 0,$$

which leads to the condition in case (2b). Finally, if $N = 3$, we apply Lemma 2.1 to $v(t) = \sqrt{G(t)}$, $\delta = 3$, $a_0 = E(0) + |\lambda|M(0)$, $a_2 = 1/2$, and $a_\delta = C_{2,3}|\beta_{12}|M(0)^{1/2}/4$, which gives the conditions in case (2c).

3. Let $\min\{\beta_{11},\beta_{22}\}<0$ and $\beta_{12}\geq 0$. Inequality (2.8) writes as

$$\frac{1}{2}G(t)\leq E(0)+|\lambda|M(0)+\frac{2C_{\alpha,N}\beta}{\alpha+2}G(t)^{\alpha N/4}M(0)^{(2\alpha+4-\alpha N)/4}.$$

If $\alpha N/4<1$, it is clear that $G(t)$ is uniformly bounded. If $\alpha N/4=1$, the coefficient of $G(t)$ has to be positive,

$$\frac{1}{2}-\frac{2C_{\alpha,N}\beta}{\alpha+2}M(0)^{2/N}>0,$$

which equals the condition in case (3b). If $\alpha N>4$, we apply again Lemma 2.1 to $v(t)=\sqrt{G(t)}$, now with $\delta=\alpha N/2>2$, $a_0=E(0)+|\lambda|M(0)$, $a_2=1/2$, and $a_\delta=2C_{\alpha,N}\beta\times M(0)^{(2\alpha+4-\alpha N)/4}/(\alpha+2)$.

4. Finally, let $\min\{\beta_{11},\beta_{22}\}<0$ and $\beta_{12}<0$. First, let $N=1$ and $\alpha<4$. Then the exponents of $G(t)$ on the right-hand side of (2.8) are smaller than one, and thus, $G(t)$ is uniformly bounded. If $N=1$ and $\alpha=4$, we find from (2.8) that

$$\Big(\frac{1}{2}-\frac{C_{4,1}}{3}\beta M(0)^2\Big)G(t)\leq E(0)+|\lambda|M(0)+\frac{C_{2,1}}{4}|\beta_{12}|M(0)^{3/2}G(t)^{1/2},$$

and $G(t)$ is uniformly bounded if $M(0)^2<3/(2C_{4,1}\beta)$, which is the condition in case (4a).

When $N=2$, $\alpha<2$, or $N=2$, $\alpha=2$, the coefficient of $G(t)$ in (2.8) has to be positive, which leads to the conditions in case (4b). When $N=2$, $\alpha>2$, we write (2.8) as

$$\Big(\frac{1}{2}-\frac{C_{2,2}}{4}|\beta_{12}|M(0)\Big)G(t)\leq E(0)+|\lambda|M(0)+\frac{2C_{\alpha,2}\beta}{\alpha+2}G(t)^{\alpha/2}M(0)$$

and apply Lemma 2.1 to $v(t)=\sqrt{G(t)}$ with $\delta=\alpha>2$, $a_0=E(0)+|\lambda|M(0)$, $a_2=1/2-C_{2,2}|\beta_{12}|M(0)/4$, and $a_\delta=2C_{\alpha,2}\beta M(0)/(\alpha+2)$. The assumptions of the lemma and $a_2>0$ correspond to the conditions in case (4b).

In the case $N=3$, (2.8) rewrites as

$$\frac{1}{2}G(t)\leq E(0)+|\lambda|M(0)+\frac{2C_{\alpha,3}\beta}{\alpha+2}G(t)^{3\alpha/4}M(0)^{(4-\alpha)/4}+\frac{C_{2,3}}{4}|\beta_{12}|G(t)^{3/2}M(0)^{1/2}.$$

This inequality simplifies in the cases $\alpha=2$ and $\alpha=4/3$. If $\alpha=2$, both exponents of $G(t)$ on the right-hand side equal 3/2, and we can apply Lemma 2.1 to $v(t)=\sqrt{G(t)}$ with $\delta=3$, $a_0=E(0)+|\lambda|M(0)$, $a_2=1/2$, and $a_\delta=C_{2,3}(2\beta+|\beta_{12}|)M(0)^{1/2}/4$. If $\alpha=4/3$, (2.8) becomes

$$\Big(\frac{1}{2}-\frac{3C_{4/3,3}\beta}{5}M(0)^{2/3}\Big)G(t)\leq E(0)+|\lambda|M(0)+\frac{C_{2,3}}{4}|\beta_{12}|M(0)^{1/2}G(t)^{3/2}.$$

If the coefficient of $G(t)$ is positive, we can apply Lemma 2.1 to $v(t)=\sqrt{G(t)}$, which leads to the conditions in case (4c). In the general case, (2.8) is an inequality of the type

$$F(v(t))=a_0-v(t)^2+a_\alpha v(t)^{3\alpha/2}+a_3v(t)^3\geq 0,\quad t\geq 0,$$

where $v(t) = \sqrt{G(t)}$ and a_0, a_α, a_3 are some positive parameters satisfying $a_\alpha \to 0$ and $a_3 \to 0$ as $M(0) \to 0$, since $\alpha < 4$. For sufficiently small a_α and a_3, the function $F(x)$ has a global minimum at some point $x_1 > 0$ and $F(x_1) < 0$. The proof of Lemma 2.1 can be generalized to the above function, showing that if additionally $v(0)$ is sufficiently small, $F(v(t)) \geq 0$ implies that $v(t)$ is uniformly bounded. In fact, it is sufficient to require that $M(0)$ is small enough to conclude this result. □

Remark 2.1. The conditions in Theorem 2.1 roughly mean that $\psi_1(0)$ and $\psi_2(0)$ have to be sufficiently small in certain L^p norms to achieve global existence of solutions. In cases (4b) and (4c), we have to assume additionally that $E(0)$ is sufficiently small since the exponent of $M(0)$ on both sides of the respective inequality is the same.

When only one of the coefficients β_{11} or β_{22} is negative, say $\beta_{11} < 0$, case (3) of Theorem 2.1 can be improved. Indeed, it is sufficient to impose constraints on $\|\nabla\psi_1\|_2$ only instead on $G(t)$. More precisely, the condition $G(0) \leq 2(E(0) + |\lambda|M(0))$ in (3c), $\alpha > 4/N$ can be replaced by the weaker condition $\|\nabla\psi_1(0)\|_2^2 \leq 2(E(0) + |\lambda|M(0))$.

Moreover, for $\lambda = 0$ we can modify the condition in cases (3b) and (3c) to

(3b') $\alpha = 4/N$ and

$$\|\psi_1(0)\|_2^2 < \Big(\frac{N+2}{2NC_{\alpha,N}\beta}\Big)^{N/2},$$

(3c') $\alpha > 4/N$, $\|\nabla\psi_1(0)\|_2^2 \leq 2E(0)$, and

$$\|\psi_1(0)\|_2^{2\alpha+4-\alpha N} E(0)^{(\alpha N-4)/2} < \frac{4(\alpha N-4)^{(\alpha N-4)/2}(\alpha+2)^2}{(2\alpha N)^{\alpha N/2} C_{\alpha,N}^2 \beta^2}.$$

□

Remark 2.2. For critical power nonlinearities $\alpha = 4/N$, the best constant in the Gagliardo-Nirenberg inequality can be computed by solving the equation $\Delta u - u + u^{\alpha+1} = 0$ in $\mathbb{R}^N$ and setting $C_{\alpha,N} = (\alpha+2)/(2\|u\|_2^\alpha)$.[26]

Slightly sharper constraints can be derived by using a vector-valued Gagliardo-Nirenberg inequality, see, e.g., Corollary 6 in Ref. 20. This result applies to our situation for $\alpha = 2$, leading to an improved constant $C_{2,2}/4$ instead of $C_{2,2}$. □

3. Blow-up of solutions

In this section, we give sufficient conditions yielding finite-time blow-up of the solutions. We set

$$I(t) = \int_{\mathbb{R}^N} |x|^2\big(|\psi_1|^2 + |\psi_2|^2\big)(x,t)dx. \tag{3.1}$$

Theorem 3.1. *Let $\Psi^0 \in \Sigma$ (see (2.2)) and $0 < \alpha < 4/(N-2)$ ($\alpha < \infty$ if $N \leq 2$). If one of the two sets of conditions*

$$E(0) + |\lambda| M(0) < \frac{\gamma^2}{2} I(0), \quad or \tag{3.2}$$

$$I'(0) < 0, \quad E(0) + |\lambda| M(0) < -\frac{\gamma}{2} I'(0) \tag{3.3}$$

is satisfied, the solution $\Psi = (\psi_1, \psi_2)$ to (1.1)-(1.2) blows up at time $t^* \leq \pi/(2\gamma)$ or $t^* \leq \pi/(4\gamma)$, respectively, i.e.

$$\lim_{t \to t^*} \|\nabla \Psi\|_2 = +\infty,$$

if additionally one of the following conditions is fulfilled:

(1) $\beta_{11} \geq 0$, $\beta_{22} \geq 0$:
- *(a) $N = 2$: $\alpha \leq 2$,*
- *(b) $N = 3$: $\alpha \leq 4/3$ and $\beta_{12} \leq 0$;*

(2) $\beta_{11} \geq 0$, $\beta_{22} < 0$, or $\beta_{11} < 0$, $\beta_{22} \geq 0$:
- *(a) $N = 1$: $\alpha = 4$ and $\beta_{12} \geq 0$,*
- *(b) $N = 2$: $\alpha = 2$,*
- *(c) $N = 3$: $\alpha = 4/3$ and $\beta_{12} \leq 0$;*

(3) $\beta_{11} < 0$, $\beta_{22} < 0$:
- *(a) $N = 1$: $\alpha \geq 4$ and either $\beta_{12} \geq 0$, or $\beta_{12} < 0$ and both $\min\{|\beta_{11}|, |\beta_{22}|\}$ and $1/|\beta_{12}|$ are sufficiently large,*
- *(b) $N = 2$: $\alpha \geq 2$,*
- *(c) $N = 3$: either $\alpha \geq 4/3$, $\beta_{12} \leq 0$, or $\alpha \geq 2$, $\beta_{12} > 0$, and both $\min\{|\beta_{11}|, |\beta_{22}|\}$ and $1/|\beta_{12}|$ are sufficiently large.*

Notice that the first condition, $I'(0) = 0$, is satisfied if the initial data are real.

Proof. The idea of the proof is to employ the classical method of Glassey[14] which consists in calculating the time derivatives of the variance $I(t)$, defined in (3.1), and to prove that $I(t_0) < 0$ for some $t_0 > 0$, contradicting $I(t) \geq 0$. The unboundedness of the gradient than follows from the inequality (see p. 573 in Ref. 26)

$$\|\psi_j^0\|_2^2 = \|\psi_j(\cdot, t)\|_2^2 \leq \frac{2}{N} \|\nabla \psi_j(\cdot, t)\|_2 \|x \psi_j(\cdot, t)\|_2.$$

A tedious but straightforward calculation shows that

$$I'(t) = 2\Im \int_{\mathbb{R}^N} \psi_1^*(x,t) x \cdot \nabla \psi_1(x,t) dx + 2\Im \int_{\mathbb{R}^n} \psi_2^*(x,t) x \cdot \nabla \psi_2(x,t) dx,$$

$$I''(t) = 2\|\nabla \psi_1(\cdot,t)\|_2^2 + 2\|\nabla \psi_2(\cdot,t)\|_2^2 - 2\gamma^2 I(t) + \frac{2\alpha N \beta_{11}}{\alpha+2} \|\psi_1(\cdot,t)\|_{\alpha+2}^{\alpha+2}$$
$$+ \frac{2\alpha N \beta_{22}}{\alpha+2} \|\psi_2(\cdot,t)\|_{\alpha+2}^{\alpha+2} + 2N\beta_{12} \int_{\mathbb{R}^N} |\psi_1(\cdot,t)|^2 |\psi_2(\cdot,t)|^2 dx.$$

These computations can be made rigorous using the techniques of Ref. 9; see Proposition 6.5.1 and Remark 9.2.10. Since mass and energy are conserved, the second derivative can be formulated as

$$\begin{aligned} I''(t) &= 4E(0) - 4\gamma^2 I(t) + \frac{2(\alpha N-4)\beta_{11}}{\alpha+2}\|\psi_1(\cdot,t)\|_{\alpha+2}^{\alpha+2} \\ &\quad + \frac{2(\alpha N-4)\beta_{22}}{\alpha+2}\|\psi_2(\cdot,t)\|_{\alpha+2}^{\alpha+2} + 2(N-2)\beta_{12}\int_{\mathbb{R}^N}|\psi_1(x,t)|^2|\psi_2(x,t)|^2dx \\ &\quad - 8\lambda\Re\int_{\mathbb{R}^N}\psi_1^*(x,t)\psi_2(x,t)dx. \end{aligned}$$

Using $|\psi_1+\psi_2|^2 + |\psi_1-\psi_2|^2 = 2(|\psi_1|^2+|\psi_2|^2)$, we obtain for the last integral:

$$\begin{aligned} -8\lambda\Re\int_{\mathbb{R}^N}(\psi_1^*\psi_2)(x,t)dx &= -2\lambda\int_{\mathbb{R}^N}\big(|\psi_1+\psi_2|^2 - |\psi_1-\psi_2|^2\big)(x,t)dx \\ &= -4|\lambda|\int_{\mathbb{R}^N}\big(|\psi_1+\mathrm{sign}(\lambda)\psi_2|^2 - (|\psi_1|^2+|\psi_2|^2)\big)(x,t)dx \\ &= -4|\lambda|\,\|\psi_1+\mathrm{sign}(\lambda)\psi_2\|_2^2 + 4|\lambda|M(0). \end{aligned}$$

Therefore, I'' solves

$$I''(t) = 4(E(0)+|\lambda|M(0)) - 4\gamma^2 I(t) + R(t), \quad t>0,$$

where

$$\begin{aligned} R(t) &= \frac{2(\alpha N-4)\beta_{11}}{\alpha+2}\|\psi_1(\cdot,t)\|_{\alpha+2}^{\alpha+2} + \frac{2(\alpha N-4)\beta_{22}}{\alpha+2}\|\psi_2(\cdot,t)\|_{\alpha+2}^{\alpha+2} \\ &\quad + 2(N-2)\beta_{12}\int_{\mathbb{R}^N}|\psi_1(x,t)|^2|\psi_2(x,t)|^2dx - 4|\lambda|\,\|\psi_1+\mathrm{sign}(\lambda)\psi_2\|_2^2. \end{aligned}$$

The solution of the above differential equation is given by

$$\begin{aligned} I(t) &= \cos(2\gamma t)I(0) + \frac{1}{2\gamma}\sin(2\gamma t)I'(0) + \frac{1}{\gamma^2}(E(0)+|\lambda|M(0))(1-\cos(2\gamma t)) \\ &\quad + \frac{1}{2\gamma}\int_0^t\sin(2\gamma(t-\tau))R(\tau)d\tau, \quad t\ge 0. \end{aligned}$$

We claim that, if either (3.2) or (3.3) holds and if $R(\tau)\le 0$ for all $0\le\tau\le t$, $I(t)$ becomes negative. Indeed, if (3.2) and $R(\tau)\le 0$ hold,

$$I\Big(\frac{\pi}{2\gamma}\Big) \le -I(0) + \frac{2}{\gamma^2}(E(0)+|\lambda|M(0)) < 0,$$

and if (3.3) and $R(\tau)\le 0$ hold,

$$I\Big(\frac{\pi}{4\gamma}\Big) \le \frac{I'(0)}{2\gamma} + \frac{1}{\gamma^2}(E(0)+|\lambda|M(0)) < 0.$$

Thus, it remains to determine conditions under which $R(\tau)\le 0$ holds for all $\tau>0$. This is the case if $(N\alpha-4)\beta_{11}\le 0$, $(N\alpha-4)\beta_{22}\le 0$, and $(N-2)\beta_{12}\le 0$, which leads to the conditions stated in the theorem except the restrictions on $|\beta_{12}|$ in the case $\beta_{11}<0$, $\beta_{22}<0$.

Thus, let $\beta_{11} < 0$, $\beta_{22} < 0$, and $\alpha N - 4 \geq 0$. The idea is to control the integral involving β_{12} by the $L^{\alpha+2}$ norms,

$$2|(N-2)\beta_{12}| \int_{\mathbb{R}^N} (|\psi_1|^2|\psi_2|^2)(x,t)dx \leq \frac{2(\alpha N-4)}{\alpha+2} \sum_{j=1}^{2} |\beta_{jj}| \, \|\psi_j\|_{\alpha+2}^{\alpha+2}, \tag{3.4}$$

which implies that $R(t) \leq 0$. For this, we assume that $\alpha \geq 2$ and employ the Cauchy-Schwarz and interpolation inequalities:

$$\begin{aligned} 2|(N-2)\beta_{12}| \int_{\mathbb{R}^N} |\psi_1|^2|\psi_2|^2 dx &\leq 2|(N-2)\beta_{12}| \, \|\psi_1\|_4^2 \|\psi_2\|_4^2 \\ &\leq 2|(N-2)\beta_{12}| \, \|\psi_1\|_2^{(\alpha-2)/2} \|\psi_1\|_{\alpha+2}^{(\alpha+2)/2} \|\psi_2\|_2^{(\alpha-2)/2} \|\psi_2\|_{\alpha+2}^{(\alpha+2)/2}. \end{aligned}$$

Under the condition

$$2|(N-2)\beta_{12}| \leq \frac{2|\alpha N-4|}{\alpha+2} |\beta_{11}\beta_{22}|^{1/\alpha} M(0)^{(2-\alpha)/\alpha} \tag{3.5}$$

and employing the Young inequality as well as the Hölder inequality for sums, we find that

$$\begin{aligned} &2|(N-2)\beta_{12}| \int_{\mathbb{R}^N} (|\psi_1|^2|\psi_2|^2)(x,t)dx \\ &\quad \leq \frac{|\alpha N-4|}{\alpha+2} M(0)^{(2-\alpha)/\alpha} \|\psi_1\|_2^{(\alpha-2)/\alpha} \big(|\beta_{11}| \, \|\psi_1\|_{\alpha+2}^{\alpha+2}\big)^{1/\alpha} \\ &\qquad \times \|\psi_2\|_2^{(\alpha-2)/\alpha} \big(|\beta_{22}| \, \|\psi_2\|_{\alpha+2}^{\alpha+2}\big)^{1/\alpha} \\ &\quad \leq \frac{2|\alpha N-4|}{\alpha+2} M(0)^{(2-\alpha)/\alpha} \sum_{j=1}^{2} \|\psi_j\|_2^{2(\alpha-2)/\alpha} \big(|\beta_{jj}| \, \|\psi_j\|_{\alpha+2}^{\alpha+2}\big)^{2/\alpha} \\ &\quad \leq \frac{2|\alpha N-4|}{\alpha+2} M(0)^{(2-\alpha)/\alpha} \left(\sum_{j=1}^{2} \|\psi_j\|_2^{2p(\alpha-2)/\alpha} \right)^{1/p} \\ &\qquad \times \left(\sum_{j=1}^{2} \big(|\beta_{jj}| \, \|\psi_j\|_{\alpha+2}^{\alpha+2}\big)^{2q/\alpha} \right)^{1/q}, \end{aligned}$$

where $1/p + 1/q = 1$. If $\alpha = 2$, (3.4) follows immediately. For $\alpha > 2$, we choose $p = \alpha/(\alpha-2)$ and $q = \alpha/2$. Then, by mass conservation,

$$\begin{aligned} &2|(N-2)\beta_{12}| \int_{\mathbb{R}^N} (|\psi_1|^2|\psi_2|^2)(x,t)dx \\ &\quad \leq \frac{2|\alpha N-4|}{\alpha+2} M(0)^{(2-\alpha)/\alpha} \left(\sum_{j=1}^{2} \|\psi_j\|_2^2 \right)^{(\alpha-2)/\alpha} \left(\sum_{j=1}^{2} |\beta_{jj}| \, \|\psi_j\|_{\alpha+2}^{\alpha+2} \right)^{2/\alpha} \\ &\quad = \frac{2|\alpha N-4|}{\alpha+2} \left(\sum_{j=1}^{2} |\beta_{jj}| \, \|\psi_j\|_{\alpha+2}^{\alpha+2} \right)^{2/\alpha}. \end{aligned}$$

Now, if $\alpha N - 4 \geq 0$ and

$$\sum_{j=1}^{2} |\beta_{jj}| \, \|\psi_j\|_{\alpha+2}^{\alpha+2} \geq 1, \tag{3.6}$$

our claim (3.4) follows. In order to prove (3.6), we argue that there exists a bounded set $B \subset \mathbb{R}^N$ such that

$$\int_B \big(|\psi(x,t)|^2 + |\psi_2(x,t)|^2\big) dx \geq \frac{M(0)}{2} \quad \text{for } 0 \leq t \leq t^*.$$

In order to prove this assertion, we observe that

$$\int_{\mathbb{R}^N} (|\psi_1(x,t)|^2 + |\psi_2(x,t)|^2) dx = M(0),$$

and we let $0 \leq t \leq t^*$. Because of $\Psi(t) \in \Sigma$, we have

$$\|x\psi_1(t)\|_2^2 + \|x\psi_2(t)\|_2^2 \leq C_{t^*} < \infty \quad \text{for all } t \in [0, t^*],$$

where $C_{t^*} > 0$ does not depend on t. Let k be a positive constant. Then

$$\begin{aligned}
\frac{C_{t^*}}{k^2} &\geq \frac{1}{k^2} \int_{\mathbb{R}^N} \big(|x\psi_1(x,t)|^2 + |x\psi_2(x,t)|^2\big) \, dx \\
&\geq \frac{1}{k^2} \left(\int_{|x|>k} \big(|x\psi_1(x,t)|^2 + |x\psi_2(x,t)|^2\big) \, dx \right) \\
&\geq \frac{1}{k^2} \left(\int_{|x|>k} k^2 \, \big(|\psi_1(x,t)|^2 + |\psi_2(x,t)|^2\big) \, dx \right) \\
&= \int_{|x|>k} \big(|\psi_1(x,t)|^2 + |\psi_2(x,t)|^2\big) \, dx.
\end{aligned}$$

Thus, choosing $k > 0$ such that $C_{t^*}/k^2 \leq M_0/2$ and $B = \{|x| \leq k\}$, the result follows since

$$\begin{aligned}
\int_B \big(|\psi_1(x,t)|^2 + |\psi_2(x,t)|^2\big) \, dx &= M(0) - \int_{|x|>k} \big(|\psi_1(x,t)|^2 + |\psi_2(x,t)|^2\big) \, dx \\
&\geq \frac{M(0)}{2} \quad \text{for } 0 \leq t \leq t^*.
\end{aligned}$$

We continue with the proof of (3.6):

$$\begin{aligned}
\sum_{j=1}^{2} |\beta_{jj}| \, \|\psi_j\|_{\alpha+2}^{\alpha+2} &= \sum_{j=1}^{2} |\beta_{jj}| \int_B |\psi_j(x,t)|_{\alpha+2}^{\alpha+2} dx \\
&\geq \min\{|\beta_{11}|, |\beta_{22}|\} \left(\int_B |\psi_1(x,t)|^{\alpha+2} dx + \int_B |\psi_2(x,t)|^{\alpha+2} dx \right).
\end{aligned}$$

Since B is bounded, the embedding $L^{\alpha+2}(B) \hookrightarrow L^2(B)$ is continuous and $\|f\|_{L^2(B)} \leq \text{meas}(B)^{\alpha/(\alpha+2)} \|f\|_{L^{\alpha+2}(B)}$. We infer that

$$\sum_{j=1}^{2} |\beta_{jj}| \, \|\psi_j\|_{\alpha+2}^{\alpha+2} \geq \min\{|\beta_{11}|, |\beta_{22}|\} \text{meas}(B)^{-\alpha/2} \big(\|\psi_1\|_2^{\alpha+2} + \|\psi_2\|_2^{\alpha+2}\big).$$

With the inequality $(x+y)^{(\alpha+2)/2} \leq 2^{\alpha/2}(x^{(\alpha+2)/2}+y^{(\alpha+2)/2})$ for x, $y \geq 0$, it follows that

$$\sum_{j=1}^{2} |\beta_{jj}|\, \|\psi_j\|_{\alpha+2}^{\alpha+2} \geq \min\{|\beta_{11}|, |\beta_{22}|\}(2\text{meas}(B))^{-\alpha/2}\big(\|\psi_1\|_2^2 + \|\psi_2\|_2^2\big)^{(\alpha+2)/2}$$
$$\geq \min\{|\beta_{11}|, |\beta_{22}|\}(2\text{meas}(B))^{-\alpha/2}\Big(\frac{M(0)}{2}\Big)^{(\alpha+2)/2}$$
$$= \frac{\min\{|\beta_{11}|, |\beta_{22}|\}}{2^{\alpha+1}\text{meas}(B)^{\alpha/2}} M(0)^{(\alpha+2)/2}.$$

Therefore, (3.6) holds if

$$M(0)^{(\alpha+2)/2} \geq \frac{2^{\alpha+1}\text{meas}(B)^{\alpha/2}}{\min\{|\beta_{11}|, |\beta_{22}|\}}. \tag{3.7}$$

This shows that $R(t) \leq 0$ for all $t > 0$. The initial mass has to satisfy conditions (3.5) and (3.7):

$$\frac{2^{2(\alpha+1)/(\alpha+2)}\text{meas}(B)^{\alpha/(\alpha+2)}}{\min\{|\beta_{11}|, |\beta_{22}|\}^{2/(\alpha+2)}} \leq M(0) \leq \frac{(\alpha N-4)^{\alpha/(\alpha-2)}|\beta_{11}\beta_{22}|^{1/(\alpha-2)}}{(\alpha+2)^{\alpha/(\alpha-2)}|(N-2)\beta_{12}|^{\alpha/(\alpha-2)}},$$

and we recall that we have assumed that $\alpha \geq 2$, $\alpha N - 4 \geq 0$, and $(N-2)\beta_{12} \geq 0$. (If $(N-2)\beta_{12} = 0$, the right-hand side of the chain of inequalities becomes infinite.) The above inequalities are satisfied if $\min\{|\beta|_{11}, |\beta_{22}|\}$ is sufficiently large and either $N = 2$ or $|\beta_{12}|$ is sufficiently large. This concludes the proof. □

Remark 3.1. Similar conditions as in Theorem 3.1 were derived by Lin and Wei (see Theorem 1.1 in Ref. 18) without an external driven field, assuming that $\alpha = 2$, all β_{jk} are negative, and $\beta_{12} < \sqrt{\beta_{11}\beta_{22}}$ if $N = 3$. We need only the condition $\beta_{12} \leq 0$ and can allow for nonnegative values for the coefficients β_{jk}, thus generalizing the results of [18] in this situation.

4. The Rabi frequency of the external driven field

In this section we examine the role of the Rabi frequency λ. We introduce the mass of each component:

$$M_1(t) = \int_{\mathbb{R}^N} |\psi_1(x,t)|^2 dx, \quad M_2(t) = \int_{\mathbb{R}^N} |\psi_2(x,t)|^2 dx.$$

The total mass equals $M = M_1 + M_2$ and, by mass conservation, it is constant for all time. We also define

$$M_{12}(t) = \Im \int_{\mathbb{R}^N} \psi_1(x,t)\psi_2^*(x,t) dx,$$

recalling that $\Im(z)$ denotes the imaginary part of $z \in \mathbb{C}$.

Lemma 4.1. *The quantities M_2 and M_{12} satisfy the following differential equations:*

$$\partial_t M_2(t) = 2\lambda M_{12}(t), \quad \partial_t M_{12}(t) = \lambda M(0) - 2\lambda M_2(t) - Q(t), \quad t > 0, \tag{4.1}$$

where

$$Q(t) = \Re \int_{\mathbb{R}^N} \psi_1\psi_2^* \big(\beta_{11}|\psi_1|^\alpha - \beta_{22}|\psi_2|^\alpha - \beta_{12}(|\psi_1|^2 - |\psi_2|^2)\big)(x,t)dx.$$

Proof. Multiplying (1.2) by ψ_2^*, integrating over $\mathbb{R}^N$, and taking the imaginary part leads to

$$\begin{aligned}\partial_t M_2 &= \partial_t \int_{\mathbb{R}^N} |\psi_2|^2(x,t)dx = 2\Re \int_{\mathbb{R}^N} (\partial_t\psi_2\psi_2^*)(x,t)dx = 2\Im \int_{\mathbb{R}^N} (i\partial_t\psi_2\psi_2^*)(x,t)dx \\ &= 2\lambda\Im \int_{\mathbb{R}^N} (\psi_1\psi_2^*)(x,t)dx = 2\lambda M_{12}(t).\end{aligned}$$

For the derivative of M_{12}, we multiply (1.1) and (1.2) by ψ_2^* and ψ_1^*, respectively, integrate with respect to $x \in \mathbb{R}^N$, and take the real part:

$$\begin{aligned}-\Im \int_{\mathbb{R}^N} (\partial_t\psi_1\psi_2^*)(x,t)dx &= \Re \int_{\mathbb{R}^N} \Big(\frac{1}{2}\nabla\psi_1\cdot\nabla\psi_2^* + \frac{\gamma^2}{2}|x|^2\psi_1\psi_2^* + \beta_{11}|\psi_1|^\alpha\psi_1\psi_2^* \\ &\qquad + \beta_{12}|\psi_2|^2\psi_1\psi_2^* + \lambda|\psi_2|^2\Big)(x,t)dx, \\ -\Im \int_{\mathbb{R}^N} (\partial_t\psi_2\psi_1^*)(x,t)dx &= \Re \int_{\mathbb{R}^N} \Big(\frac{1}{2}\nabla\psi_2\cdot\nabla\psi_1^* + \frac{\gamma^2}{2}|x|^2\psi_2\psi_1^* + \beta_{22}|\psi_2|^\alpha\psi_2\psi_1^* \\ &\qquad + \beta_{12}|\psi_1|^2\psi_2\psi_1^* + \lambda|\psi_1|^2\Big)(x,t)dx.\end{aligned}$$

Since $\Im(z_1^*z_2) = -\Im(z_1z_2^*)$ and $\Re(z_1^*z_2) = \Re(z_1z_2^*)$ for z_1, $z_2 \in \mathbb{C}$, the difference of the above equations becomes

$$\begin{aligned}-\partial_t M_{12}(t) &= -\partial_t\Im \int_{\mathbb{R}^N} (\psi_1\psi_2^*)(x,t)dx = Q(t) + \lambda \int_{\mathbb{R}^N} (|\psi_2|^2 - |\psi_1|^2)(x,t)dx \\ &= Q(t) - \lambda M(t) + 2\lambda M_2(t).\end{aligned}$$

Then mass conservation $M(t) = M(0)$ gives the differential equation for M_{12}. □

The functions $M_2(t)$ and $M_{12}(t)$ can be computed explicitly from the linear system (4.1). Then $M_1(t) = -M_2(t) + M(0)$. The solution reads as

$$\begin{aligned}M_1(t) &= -\sin(2\lambda t)M_{12}(0) + \cos(2\lambda t)M_1(0) + \frac{1}{2}(1-\cos(2\lambda t))M(0) \\ &\quad + \int_0^t \sin(2\lambda(t-s))Q(s)ds, \qquad (4.2) \\ M_2(t) &= \sin(2\lambda t)M_{12}(0) + \cos(2\lambda t)M_2(0) + \frac{1}{2}(1-\cos(2\lambda t))M(0) \\ &\quad - \int_0^t \sin(2\lambda(t-s))Q(s)ds. \qquad (4.3)\end{aligned}$$

These functions show the role of the Rabi frequency λ. Indeed, the components exchange their mass periodically. In the special case $\alpha = 2$ and $\beta_{11} = \beta_{22} = \beta_{12}$, this exchange occurs actually with the frequency 2λ. In the general case, the periodic structure is perturbed by the inhomogeneity $Q(t)$. Notice that for $\lambda = 0$, the mass of each component is constant in time, i.e., mass exchange is only possible for nonvanishing Rabi frequencies.

One might ask if a similar relation as above also holds for the energies of each component. Unfortunately, this is not the case since the time derivative $\partial_t E_j$ (see (2.3) for the definition of E_j), cannot be easily expressed in terms of E_1, E_2, and the interaction energy E_{12} (see (2.4)).

The case $\alpha = 2$ and $\beta := \beta_{11} = \beta_{22} = \beta_{12}$ can also be understood by observing that system (1.1)-(1.2) can be transformed into a system without λ-term. Indeed, if (ψ_1, ψ_2) solves (1.1)-(1.2), the transformed functions

$$\phi_1 = \frac{1}{\sqrt{2}} e^{i\lambda t}(\psi_1 + \psi_2), \quad \phi_2 = \frac{1}{\sqrt{2}} e^{-i\lambda t}(\psi_1 - \psi_2)$$

solve in $\mathbb{R}^N$ the system

$$\begin{aligned} i\partial_t \phi_1 &= -\frac{1}{2}\Delta\phi_1 + \frac{\gamma^2}{2}|x|^2\phi_1 + \beta(|\phi_1|^2 + |\phi_2|^2)\phi_1, \\ i\partial_t \phi_2 &= -\frac{1}{2}\Delta\phi_2 + \frac{\gamma^2}{2}|x|^2\phi_2 + \beta(|\phi_1|^2 + |\phi_2|^2)\phi_2. \end{aligned}$$

Thus, since the masses of ϕ_j are conserved, both components will blow up simultaneuously if there is blow up. There is a mass exchange between the two components, and the individual masses $M_1(t)$ and $M_2(t)$ oscillate with frequency 2λ according to (4.2)-(4.3) with $Q(t) = 0$ for all $t > 0$. The finite-time blow-up of solutions to this system was studied in, e.g., Ref. 11, 18, 24. By Theorem 3.1, the solution blows up in finite time if $\beta < 0$.

5. Numerical simulations

We solve system (1.1)-(1.2) in one and two space dimensions by employing the time-splitting sine-spectral method.[5] We choose this explicit method since it is unconditionally stable, time reversible, of spectral-order accuracy in space and second-order accuracy in time, and it conserves the discrete total mass.[2] For simplicity of notation, we introduce the method in one space dimension only. The extension to two space dimensions is straightforward.

5.1. *Numerical scheme*

The equations are solved on a bounded interval $I = (a, b)$, where $a < b$. We use a uniform spatial grid with mesh size $h > 0$ and grid points $x_j = x_0 + jh$, $j = 0, \ldots, K$, where $K + 1 \in \mathbb{N}$ is the (odd) number of grid points. Then $h = (b - a)/K$. The time grid is given by $t_n = n\tau$, $n \in \mathbb{N}_0$, where $\tau > 0$ is the time step size. We set $(\psi_k)_j^n := \psi_k(x_j, t_n)$, where $k = 1, 2$, $j = 0, \ldots, K$, and $n \in \mathbb{N}_0$. We split system

(1.1)-(1.2) into the three subsystems

$$i\partial_t\psi_1 = \frac{\gamma^2}{2}|x|^2\psi_1 + \beta_{11}|\psi_1|^\alpha\psi_1 + \beta_{12}|\psi_2|^2\psi_1, \tag{5.1}$$

$$i\partial_t\psi_2 = \frac{\gamma^2}{2}|x|^2\psi_2 + \beta_{22}|\psi_2|^\alpha\psi_2 + \beta_{12}|\psi_1|^2\psi_2, \tag{5.2}$$

$$i\partial_t\psi_k = -\frac{1}{2}\Delta\psi_k, \quad k = 1,2, \tag{5.3}$$

$$i\partial_t\psi_1 = \lambda\psi_2, \quad i\partial_t\psi_2 = \lambda\psi_1, \tag{5.4}$$

considered on $[t_n, t_{n+1}]$ and subject to some initial data. These subsystems are solved as follows:

Step 1 The quantity $|\psi_k|^2$, computed from the evolution of (5.1)-(5.2), remains unchanged. Therefore, we "freeze" these values at time t_n and solve the resulting linear ODEs exactly in the interval $[t_n, t_n+\tau/2]$, giving at time $t_n+\tau/2$:

$$(\psi_1)_j^* = \exp\left(-i\frac{\tau}{2}\Big(\frac{\gamma^2}{2}x_j^2 + \beta_{11}|(\psi_1)_j^n|^\alpha + \beta_{12}|(\psi_2)_j^n|^2\Big)\right)(\psi_1)_j^n, \tag{5.5}$$

and analogously for $(\psi_2)_j^*$.

Step 2 We solve (5.3) for $k=1,2$ in the interval $[t_n, t_n+\tau/2]$, discretized in space by the Fourier spectral method and solved exactly in time:

$$(\psi_k)_j^{**} = \frac{1}{K}\sum_{m=-K/2}^{K/2}\exp\Big(-i\frac{\tau}{2}\frac{\mu_m}{2}\Big)(\widehat{\psi}_k)_m^*\exp\big(i\mu_j(x_j-x_0)\big), \quad k=1,2, \tag{5.6}$$

where $\mu_m = 2\pi m/(x_K - x_0)$ and

$$(\widehat{\psi}_k)_m^* = \sum_{j=0}^{K-1}(\psi_k)_j^*\exp\big(-i\mu_m(x_m-x_0)\big), \quad m = -\frac{K}{2},\ldots,\frac{K}{2}-1.$$

Step 3 Equations (5.4) are solved on $[t_n, t_{n+1}]$ exactly, yielding

$$(\psi_1)_j^{***} = \cos(\lambda\tau)(\psi_1)_j^{**} + i\sin(\lambda\tau)(\psi_2)_j^{**},$$
$$(\psi_2)_j^{***} = i\sin(\lambda\tau)(\psi_1)_j^{**} + \cos(\lambda\tau)(\psi_2)_j^{**}.$$

Step 4 We solve (5.3) on $[t_n+\tau/2, t_{n+1}]$ using the discretization of Step 2 with $(\psi_k)_j^{***}$ instead of $(\psi_k)_j^*$ and obtain $(\psi_k)_j^{****}$.

Step 5 We solve (5.1)-(5.2) on $[t_n+\tau/2, t_{n+1}]$ using the discretization of Step 1 with $(\psi_k)_j^{****}$ instead of $(\psi_k)_j^n$ and obtain $(\psi_k)_j^{n+1}$.

In the numerical tests below, we have solved system (1.1)-(1.2) in the interval $I=[-4,4]$ with periodic boundary conditions and initial conditions corresponding to the ground state of the harmonic oscillator,

$$\psi_k(x,0) = \kappa\Big(\frac{1}{\gamma\pi}\Big)^{1/4}\exp\Big(-\frac{\gamma}{2}|x|^2\Big), \quad k=1,2,$$

τ	$\|\Psi - \Psi^\tau\|$	p
0.0025	$1.450 \cdot 10^{-4}$	–
0.005	$5.802 \cdot 10^{-4}$	2.001
0.01	$2.321 \cdot 10^{-3}$	2.005

Table 1. Temporal accuracy of the scheme.

K	$\|\Psi(\cdot, t^0) - \Psi^K(\cdot, t^0)\|_2$	p
32	$4.494 \cdot 10^{-4}$	–
64	$1.276 \cdot 10^{-8}$	15.167
128	$9.429 \cdot 10^{-13}$	13.725

Table 2. Spatial accuracy of the scheme.

where $\kappa > 0$ is a normalization constant. The initial functions are normalized to one if $\kappa = 1$. The numerical parameters are $K = 1024$ and $\tau = 10^{-5}$ if not stated otherwise.

5.2. *Temporal and spatial accuracy*

First, we report the temporal accuracy of the numerical solutions in the $L^\infty(0, T; L^2(I))$ norm $\| \cdot \|$ with $T = 20$. The model parameters are $N = 1$, $\alpha = 4$, $\beta_{11} = \beta_{12} = \beta_{22} = 1$, $\gamma = 4$, and $\lambda = -5$. The numerical solution $\Psi^\tau = (\psi_1^\tau, \psi_2^\tau)$ with time step size τ will be compared to the reference solution $\Psi = (\psi_1, \psi_2)$, computed with the parameters $K = 1024$ and $\tau = 10^{-5}$. For the errors, we expect an expansion of the type

$$\|\Psi(\cdot, t_j) - \Psi^\tau(\cdot, t_j)\|_2 = \tau^*(t_j)\tau^p,$$

where $\tau^*(t_j)$ depends on the time step t_j. Then

$$p = \log_2 \frac{\|\Psi - \Psi^\tau\|}{\|\Psi - \Psi^{\tau/2}\|}.$$

The numerical results are reported in Table 1, showing the expected second-order time accuracy.

Next, we test the spatial accuracy by reporting the error $\|\Psi(\cdot, t^0) - \Psi^K(\cdot, t^0)\|_2$, where $\Psi^K(\cdot, t^0)$ is the numerical solution at time $t^0 = 0.1$ with grid number K and $\Psi(\cdot, t^0)$ is the reference solution with $2^{14} = 16\,384$ grid points and $\tau = 10^{-5}$. We consider the spatial error at the beginning of the evolution since at later times, the temporal error dominates. The model parameters are as above. Table 2 illustrates the spatial accuracy. For grid points $K \geq 256$, the absolute L^2 error is of the order 10^{-12}, which is close to machine precision.

Before we present the numerical experiments, we define the numerical blow-up

K	256	512	1024	2048	4096
t^*	0.103	0.103	0.103	0.104	0.104

Table 3. Blow-up time t^* as a function of the grid number K.

time t^*. We set

$$t^* = \min_{T>0} \max_{t\in[0,T]} \log(\|\nabla\Psi(\cdot,t)\|_2^2) > 9.21 = \log(10\,000), \tag{5.7}$$

and we say that blow-up occurs if $t^* \le T = 200$. Table 3 shows that the numerical blow-up time t^* varies only slightly with the grid number K (the model parameters are as in Experiment 5a with $\lambda = -5$ below).

5.3. *Experiment 1: One focusing component, blow up*

The intention of this experiment is to verify the blow-up condition (2a) in Theorem 3.1. The model parameters are chosen as follows: $N = 1$, $\alpha = 4$, $\beta_{11} = -1$, $\beta_{12} = \beta_{22} = 1$, $\gamma = 4$, and $\lambda = 0$, i.e., we have one focusing nonlinearity and mass conservation in every component. The initial masses are normalized to $\|\psi_1(\cdot,0)\|_2^2 = 3$ and $\|\psi_2(\cdot,0)\|_2^2 = 0.5$. Since the initial data are real, we have $I'(0) = 0$. Our choice guarantees that the sufficient condition (3.3) is satisfied:

$$E(0) - \frac{\gamma^2}{2} I(0) = -1.8885 < 0.$$

Figure 1 illustrates the values of $\|\nabla\psi_1(\cdot,t)\|_2^2$ and $\|\nabla\psi_2(\cdot,t)\|_2^2$ versus time in logarithmic scale. As expected, the gradient of the first component becomes very large, and its position density seems to approach the Dirac δ distribution (Figure 2).

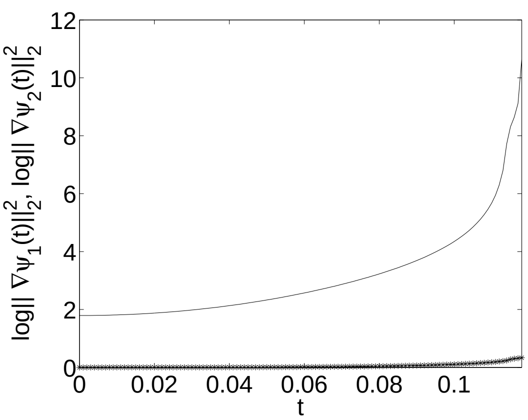

Fig. 1. Experiment 1: $\|\nabla\psi_1(\cdot,t)\|_2^2$ (thin line) and $\|\nabla\psi_2(\cdot,t)\|_2^2$ (bold line) versus time.

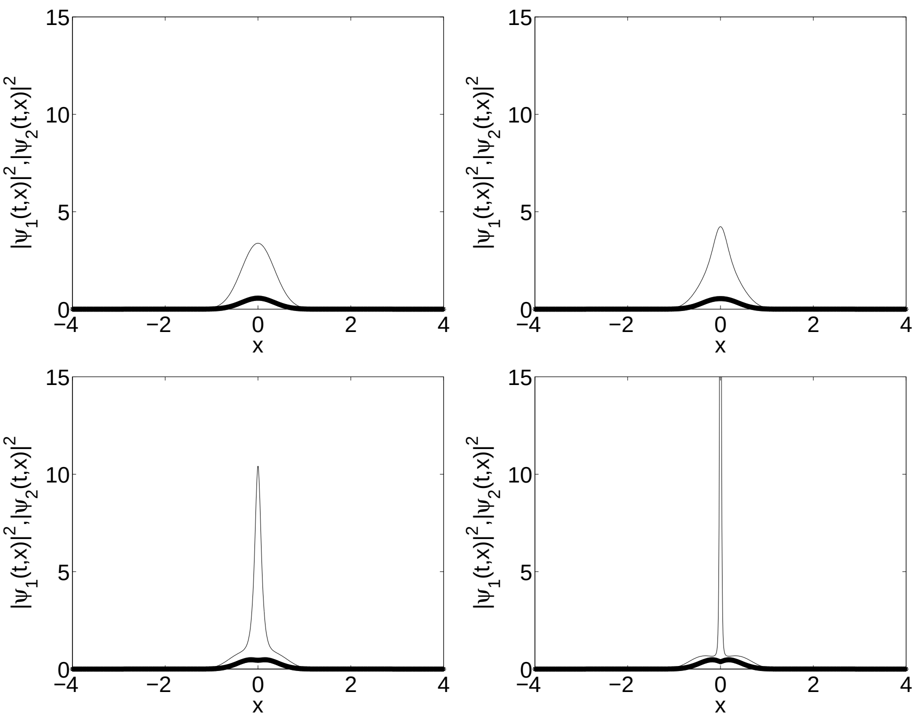

Fig. 2. Experiment 1: Position densities $|\psi_1(x,t)|^2$ (thin line) and $|\psi_2(x,t)|^2$ (bold line) at times $t = 0.001$, 0.05 (upper row) and 0.1, 0.115 (lower row).

5.4. *Experiment 2: Two focusing components, blow up*

In this experiment, we consider two focusing nonlinearities in two space dimensions, choosing $N = 2$, $\alpha = 2$, $\beta_{11} = \beta_{12} = \beta_{22} = -1$, $\gamma = 4$, and $\lambda = -3$. We solve the system on $[-4, 4] \times [-4, 4]$ using 512×512 grid points. The initial masses are normalized to $\|\psi_1(\cdot, 0)\|_2^2 = 4.5$ and $\|\psi_2(\cdot, 0)\|_2^2 = 3.5$. Again, we have $I'(0) = 0$. The sufficient condition of Theorem 3.1, case (3b), is satisfied since

$$E(0) + |\lambda| M(0) - \frac{\gamma^2}{2} I(0) = -4.1941 < 0.$$

As expected, the position density $|\psi_1(x,t)|^2$ seems to approach a Dirac δ distribution at time $t^* \approx 0.21$ (Figure 3). The second component behaves in a similar way.

5.5. *Experiment 3: One focusing component, global existence*

We wish to check the conditions for global existence of solutions with small initial data.

(a) First, we choose the same parameters as in Experiment 1 but different initial masses: $M_1(0) = \|\psi_1(\cdot, 0)\|_2^2 = 1.5$ and $M_2(0) = \|\psi_2(\cdot, 0)\|_2^2 = 1.5$. This corresponds

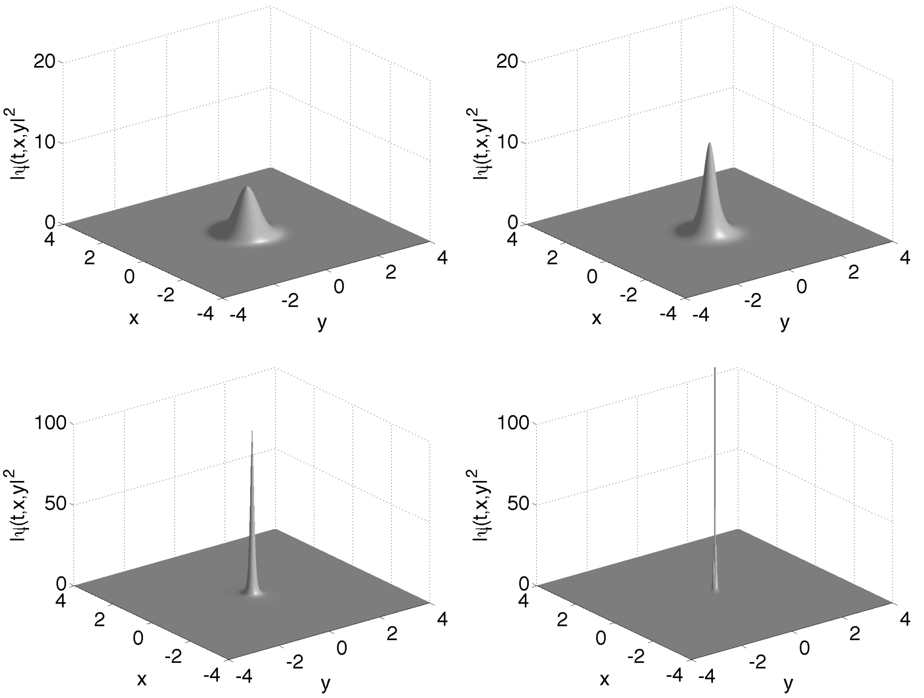

Fig. 3. Experiment 2: Surfaces of the position density $|\psi_1(x,t)|^2$ at times $t = 0,\ 0.1$ (upper row) and 0.2, 0.21 (lower row).

to case (3b) in Theorem 2.1 (observing Remark 2.1) since

$$M_1(0) < M_{\text{crit}} := \Big(\frac{N+1}{2|\beta_{11}|C_{4,1}}\Big)^{1/2} = \Big(\frac{3}{2\cdot 0.40529921\ldots}\Big)^{1/2} = 1.9238\ldots$$

The optimal Gagliardo-Nirenberg constant $C_{4,1}$ was computed by minimizing the energy functional corresponding to the elliptic problem in Remark 2.2. For this, we employed the numerical method of Ref. 3. The theory predicts global existence. Indeed, the numerical results show that our blow-up condition (5.7) is never satisfied until times $T = 200$ (Figure 4).

(b) A similar result is obtained for the α-supercritical case (3c) in Theorem 2.1. The parameters are as above except $\alpha = 6$ (in order to have supercriticality) and $\lambda = 1$. The initial masses are normalized such that $M_1(0) = 0.5$ and $M_2(0) = 0.25$. Then the sufficient conditions

$$G(0) \le 2(E(0) + M(0)), \quad M(0)^5(E(0) + M(0)) < \frac{8}{27C_{6,1}},$$

where $C_{6,1} = 0.5134\ldots$, are satisfied. Figure 5 illustrates the gradient of the wave functions in the L^2 norm, confirming the global existence result of Theorem 2.1, case (3c).

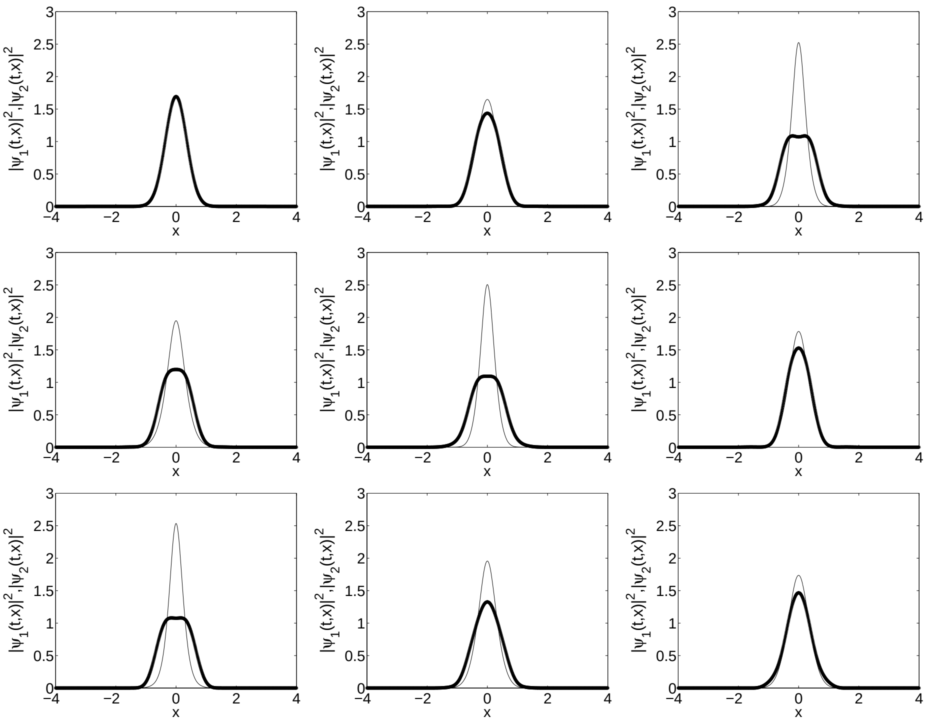

Fig. 4. Experiment 3a: Position densities $|\psi_1(x,t)|^2$ (thin line) and $|\psi_2(x,t)|^2$ (bold line) at times $t = 0$, 10, 25 (upper row), 50, 100, 125 (middle row), and 150, 175, 200 (lower row).

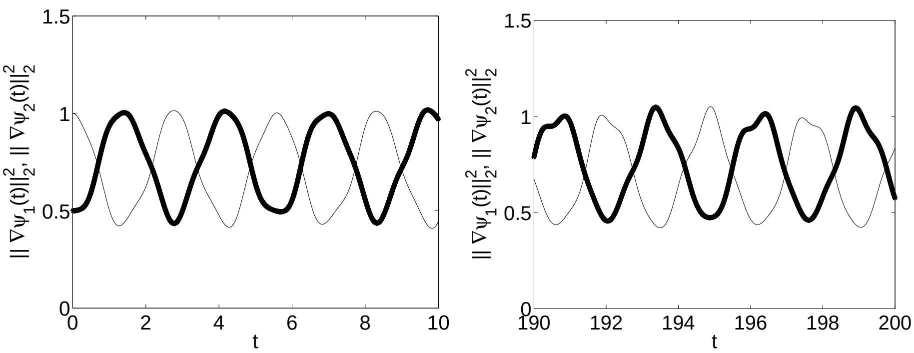

Fig. 5. Experiment 3b: $\|\nabla\psi_1(\cdot,t)\|_2^2$ (thin line) and $\|\nabla\psi_2(\cdot,t)\|_2^2$ (bold line) at times $t = 0,\ldots,10$ (left) and $t = 190,\ldots,200$ (right).

5.6. *Experiment 4: Two defocusing components, global existence*

In Section 4, we have derived explicit formulas for the masses $M_1(t)$ and $M_2(t)$. We check the observations numerically by choosing $N = 2$, $\alpha = 2$, $\beta_{11} = \beta_{12} =$

$\beta_{22} = 1$, $\gamma = 4$, and $\lambda = -3$. The exchange of the mass components occurs with a frequency depending on the Rabi frequency λ, and there are no "perturbations" in the amplitude (Figure 6).

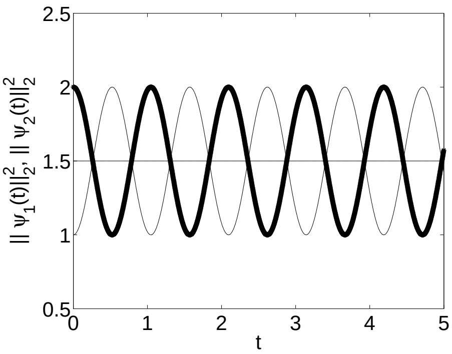

Fig. 6. Experiment 4: Masses $M_1(t)$ (thin line) and $M_2(t)$ (bold line) versus time.

5.7. *Experiment 5: Rabi frequency and blow-up*

Finally, we want to examine the effect of the Rabi frequency λ on the blow-up behavior. We choose the set-up of Experiment 1, i.e. the case of one focusing nonlinearity. In the following, we vary the initial masses and the Rabi frequency. We observe that, if the initial mass of the first (focusing) component is smaller than the critical mass $M_{\text{crit}} = 1.9238\ldots$ (see Experiment 3a), there is no blow-up for $\lambda = 0$, independently of the initial mass of the second component. However, for $\lambda \neq 0$, the total initial mass is crucial (see Theorem 2.1, case (3b)).

The numerical blow-up times t^* are presented in Figure 4. The grid number equals $K = 2048$ except for the values marked with * for which we have chosen $K = 8192$. A hyphen "$-$" means that we did not observe blow-up (in the sense defined above) up to $T = 200$. If $\lambda = 0$, blow-up occurs only if the initial mass $M_1(0)$ of the focusing component is larger than the critical mass M_{crit} (see Remark 2.1). This is the case for cases (a) and (d). In cases (b), (c), and (e) there is no blow-up when $\lambda = 0$. If $\lambda \neq 0$, blow-up depends on the total initial mass. We discuss these cases now.

In case (a), we have blow-up for all λ since, as explained above, the initial mass of the focusing component is larger than M_{crit}. Blow-up also occurs in case (b) for $\lambda \neq 0$ although the initial mass $M_1(0)$ is smaller than M_{crit}. The reason is that the Rabi coupling causes an exchange of mass in such a way that the mass of the first component becomes supercritical and blow-up occurs. The blow-up times are larger than in case (a). Both initial masses are smaller than the critical mass in case (c). Still, we observe blow-up until $T = 200$ (except for $\lambda = 5$). Possibly, blow-up

λ	Case (a)	Case (b)	Case (c)	Case (d)	Case (e)
	$M_1(0) = 3.0$ $M_2(0) = 1.0$	$M_1(0) = 1.0$ $M_2(0) = 3.0$	$M_1(0) = 1.5$ $M_2(0) = 1.5$	$M_1(0) = 2.0$ $M_2(0) = 0.1$	$M_1(0) = 0.1$ $M_2(0) = 2.0$
−5	0.104	0.262	0.319	–	29.666
−4	0.105	0.295	0.327	–	–
−3	0.107	0.341	0.354	5.948	1.925*
−2	0.109	0.414	0.384	0.387*	0.783
−1	0.111	0.576	0.479	0.306	31.409
0	0.117	–	–	0.319	–
1	0.126	73.576*	5.350*	0.957	84.724
2	0.142	2.271	15.410*	132.773*	–
3	0.393	1.485	1.912	1.054	–
4	0.980	2.133	25.220	–	–
5	1.037	37.597	–	–	–

Table 4. Experiment 5: Blow-up times t^*. The values marked with * have been computed with grid number $K = 8192$, otherwise with $K = 2048$.

λ	1.0	1.5	1.9	2.0	2.1	2.5	3.0
t^*	0.957	7.002	9.131	132.773	–	2.735	1.054

Table 5. Experiment 5d: Blow-up times t^*.

λ	-2	-1.5	-1	0	0.5	1	1.5
t^*	0.783	2.476	31.409	–	–	84.724	12.764

Table 6. Experiment 5e: Blow-up times t^*.

also occurs for $\lambda = 5$; in this sense, the Rabi coupling "delays" blow-up. In case (d), the total mass is slightly larger than the critical mass. If $\lambda = -4$, the mass $M_1(t)$ exceeds the critical value at certain times. However, it seems that the first component cannot concentrate since, after some time, the mass exchange leads to a decrease of $M_1(t)$ (Figure 7, upper row). Therefore, no blow-up occurs although the critical mass is exceeded locally in time. In case (e) for $\lambda = 2$, we observe numerically that the mass $M_1(t)$ of the first component is always smaller than the critical mass, and we do not expect blow-up (Figure 7, lower row).

The results of Table 4 may lead to the conjecture that the blow-up time depends continuously on λ. This seems to be evident for cases (a)-(c) but it is less clear in case (d), in particular around $\lambda = 2$. Table 5 shows the blow-up times for some intermediate values of λ, which seem to support our conjecture. Similar results are obtained also in case (e) (Table 6).

Summarizing, it seems that we can distinguish three cases. First, if the total

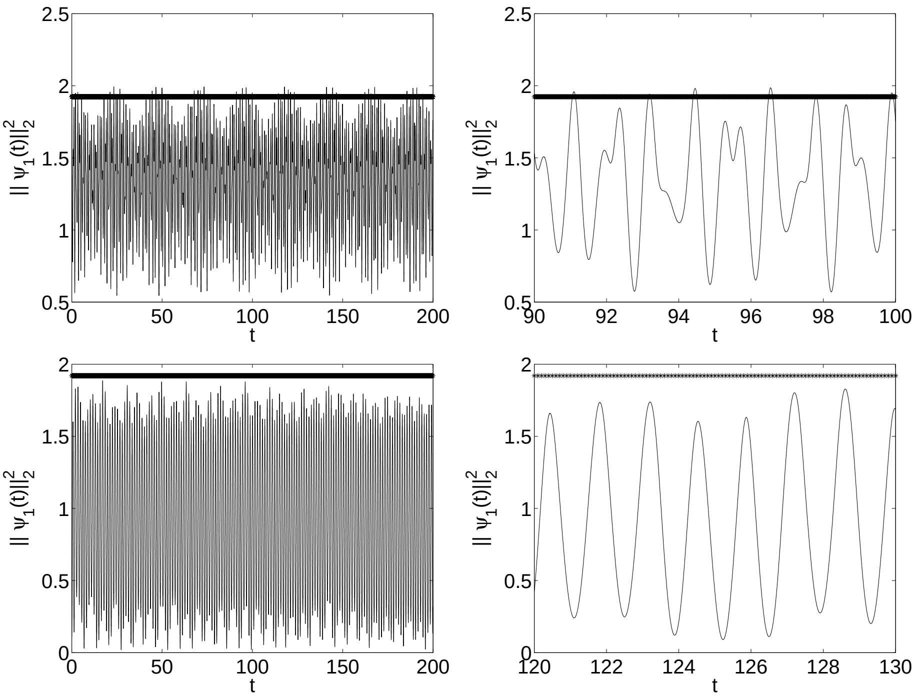

Fig. 7. Experiment 5: Mass $M_1(t)$ of the first component versus time for case (d), $\lambda = -4$ (upper row) and case (e), $\lambda = 2$ (lower row). The bold line indicates the critical mass. The right figure shows details of the time evolution.

mass is much larger than the critical mass (cases (a) and (b)), we observe blow-up for all values of $\lambda \in [-5, 5]$, independently of the values of the respective initial masses $M_1(0)$ and $M_2(0)$. Second, if the total initial mass is larger than the critical mass but the initial mass of the focusing component is smaller than the critical mass (cases (c) and (e)), the Rabi coupling may "induce" blow-up. Third, it may happen that there is no blow-up (in the sense defined above) even if the total initial mass and the initial mass of the focusing component exceed the critical mass.

6. Conclusions

We have proved, under some assumptions on the data, the global existence of strong solutions to the nonlinear Schrödinger system with external driven field by standard energy estimates. As expected, the solutions are global if both components are defocusing, or if at least one component is focusing and the initial mass and energy are sufficiently small. Furthermore, sufficient conditions of finite-time blow-up of solutions are given using the method of Glassey. We derived a semi-explicit formula describing the mass evolution, indicating the role of the Rabi frequency λ.

Although the Rabi coupling is linear, the mass evolution is nonlinear, which complicates the proof of sharp conditions for the blow-up behavior. We have shown analytically and confirmed numerically that in the mass critical case, it is sufficient to choose a total initial mass smaller than a critical value in order to have global existence of solutions. When the Schrödinger equations decouple (i.e. $\lambda = 0$), blow-up is determined by the value of the critical mass. The Rabi coupling (i.e. $\lambda \neq 0$) induces a mass transport between the two components of the system, and new effects occur. For instance, in the case of one focusing component and a total initial mass larger than the critical mass, we cannot prove global existence of solutions analytically although this seems to be true numerically. The Rabi coupling mixes the mass components in such a way that mass cannot be accumulated in the focusing component, thus possibly avoiding blow-up.

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