ANALYSIS AND MEAN-FIELD DERIVATION OF A POROUS-MEDIUM EQUATION WITH FRACTIONAL DIFFUSION

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Abstract. A mean-field-type limit from stochastic moderately interacting many-particle systems with singular Riesz potential is performed, leading to nonlocal porous-medium equations in the whole space. The nonlocality is given by the inverse of a fractional Laplacian, and the limit equation can be interpreted as a transport equation with a fractional pressure. The proof is based on Oelschläger's approach and a priori estimates for the associated diffusion equations, coming from energy-type and entropy inequalities as well as parabolic regularity. An existence analysis of the fractional porous-medium equation is also provided, based on a careful regularization procedure, new variants of fractional Gagliardo–Nirenberg inequalities, and the div-curl lemma. A consequence of the mean-field limit estimates is the propagation of chaos property.

1. Introduction

The aim of this paper is to derive and analyze the following nonlocal porous-medium equation:

$$\partial_t\rho = \operatorname{div}(\rho\nabla P), \quad P = (-\Delta)^{-s}f(\rho), \quad \rho(0) = \rho^0 \quad \text{in } \mathbb{R}^d, \tag{1}$$

where $0 < s < 1$, $d \geq 2$, and $f \in C^1([0,\infty))$ is a nondecreasing function satisfying $f(0) = 0$. This model describes a particle system that evolves according to a continuity equation for the density $\rho(x,t)$ with velocity $v = -\nabla P$. The velocity is assumed to be the gradient of a potential, which expresses Darcy's law. The pressure P is related to the density in a nonlinear and nonlocal way through $P = (-\Delta)^{-s}f(\rho)$. The nonlocal operator $(-\Delta)^{-s}$ can be written as a convolution operator with a singular kernel,

$$(-\Delta)^{-s}u = \mathcal{K} * u, \quad \mathcal{K}(x) = c_{d,-s}|x|^{2s-d}, \quad x \in \mathbb{R}^d, \tag{2}$$

where $c_{d,-s} = \Gamma(d/2-s)/(4^s\pi^{d/2}\Gamma(s))$ and Γ denotes the Gamma function [46, Theorem 5].

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If $s = 0$, we recover the porous-medium equation (for nonnegative solutions), while the case $s = 1$ was investigated in [10, 19] with $f(u) = u$ for the evolution of the vortex density in a superconductor. Other applications include particle systems with long-range interactions and dislocation dynamics as a continuum [48, Sec. 6.2].

Equation (1) was first analyzed in [8] with $f(u) = u$ for nonnegative solutions and in [2] with $f(u) = |u|^{m-2}u$ $(m > 1)$ for sign-changing solutions. The nonnegative solutions have the interesting property that they propagate with finite speed, which is not common in other fractional diffusion models [8, 44]. Equation (1) was probabilistically interpreted in [38], and it was shown that the probability density of a so-called random flight process is given by a Barenblatt-type profile. Previous mean-field limits leading to (1) were concerned with the linear case $f(u) = u$ only; see [18] (using the technique of [40]) and [36] (including additional diffusion as in (7) below). In [13], equation (1) (with $f(u) = u$) was derived in the high-force regime from the Euler–Riesz equations, which can be derived in the mean-field limit from interacting particle systems [41]. A direct derivation from particle systems with Lévy noise was proved in [17] for cross-diffusion systems, but still with $f(u) = u$. Up to our knowledge, a rigorous derivation of (1) from stochastic interacting particle systems for general nonlinearities $f(u)$ like power functions is missing in the literature. In this paper, we fill this gap.

1.1. **Problem setting.** Equation (1) is derived from an interacting particle system with N particles, moving in the whole space $\mathbb{R}^d$. Because of the singularity of the integral kernel and the degeneracy of the nonlinearity, we approximate (1) using three levels. First, we introduce a parabolic regularization adding a Brownian motion to the particle system with diffusivity $\sigma \in (0, 1)$ and replacing f by a smooth approximation f_σ. Second, we replace the interaction kernel $\mathcal{K}$ by a smooth kernel $\mathcal{K}_\zeta$ with compact support, where $\zeta > 0$. Third, we consider interaction functions W_β with $\beta \in (0, 1)$, which approximate the delta distribution. (We refer to Subsection 1.3 for the precise definitions.)

The particle positions are represented on the *microscopic level* by the stochastic processes $X_i^N(t)$ evolving according to

$$\begin{aligned} dX_i^N(t) &= -\nabla\mathcal{K}_\zeta * f_\sigma\bigg(\frac{1}{N}\sum_{j=1,\, j\neq i}^N W_\beta(X_j^N(t) - X_i^N(t))\bigg)dt + \sqrt{2\sigma}dB_i^N(t), \\ X_i^N(0) &= \xi_i, \quad i = 1, \ldots, N, \end{aligned} \tag{3}$$

where the convolution has to be understood with respect to x_i, $(B_i^N(t))_{t\geq 0}$ are independent d-dimensional Brownian motions defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, and ξ_i are independent identically distributed random variables in $\mathbb{R}^d$ with the same probability density function ρ_σ^0 (defined in (12) below).

The mean-field-type limit is performed in three steps. First, for fixed (σ, β, ζ), system (3) is approximated for $N \to \infty$ on the *intermediate level* by

$$\begin{aligned} d\bar X_i^N(t) &= -\nabla\mathcal{K}_\zeta * f_\sigma\big(W_\beta * \rho_{\sigma,\beta,\zeta}(\bar X_i^N(t), t)\big)dt + \sqrt{2\sigma}dB_i^N(t), \\ \bar X_i^N(0) &= \xi_i, \quad i = 1, \ldots, N, \end{aligned} \tag{4}$$

where $\rho_{\sigma,\beta,\zeta}$ is the probability density function of $\bar{X}_i^N$ and a strong solution to

$$\partial_t\rho_{\sigma,\beta,\zeta} - \sigma\Delta\rho_{\sigma,\beta,\zeta} = \operatorname{div}\big(\rho_{\sigma,\beta,\zeta}\nabla\mathcal{K}_\zeta * f_\sigma(W_\beta * \rho_{\sigma,\beta,\zeta})\big), \quad \rho_{\sigma,\beta,\zeta}(0) = \rho_\sigma^0 \quad \text{in } \mathbb{R}^d. \tag{5}$$

System (4) is uncoupled, since $\bar{X}_i^N$ depends on N only through the initial datum.

Second, passing to the limit $(\beta,\zeta)\to 0$ in the intermediate system leads on the *macroscopic level* to

$$\begin{aligned} d\widehat{X}_i^N(t) &= -\nabla\mathcal{K} * f_\sigma(\rho_\sigma(\widehat{X}_i^N(t),t))dt + \sqrt{2\sigma}dB_i^N(t), \\ \widehat{X}_i^N(0) &= \xi_i, \quad i = 1,\ldots,N, \end{aligned} \tag{6}$$

where ρ_σ is the density function of $\widehat{X}_i^N$ and a weak solution to

$$\partial_t\rho_\sigma = \sigma\Delta\rho_\sigma + \operatorname{div}(\rho_\sigma\nabla(-\Delta)^{-s}f_\sigma(\rho_\sigma)), \quad \rho_\sigma(0) = \rho_\sigma^0 \quad \text{in } \mathbb{R}^d. \tag{7}$$

We perform the limits $N\to\infty$ and $(\beta,\zeta)\to 0$ simultaneously. The logarithmic scaling $\beta\sim(\log N)^{-\mu}$ for some $\mu>0$ corresponds to the moderately interacting particle regime, according to the notation of Oelschläger [34], while the smoothing parameter ζ is allowed to depend algebraically on N, i.e. $\zeta\sim N^{-\nu}$ for some $\nu>0$; see Theorem 2 for details. Our approach also implies the two-step limit but leading to weak convergence only, compared to the convergence in expectation obtained in Theorem 2.

Third, the limit $\sigma\to 0$ is performed on the level of the diffusion equation, based on a priori estimates uniform in σ and the div-curl lemma.

1.2. **State of the art.** We already mentioned that the existence of weak solutions to (1) with $f(u)=u$ was proved first in [8]. The convergence of the weak solution to a self-similar profile was shown by the same authors in [7]. The convergence becomes exponential, at least in one space dimension, when adding a confinement potential [9]. Equation (1) with $f(u)=u$ was identified as the Wasserstein gradient flow of a square fractional Sobolev norm [30], implying time decay as well as energy and entropy estimates. The Hölder regularity of solutions to (1) was proved in [6] for $f(u)=u$ and in [25] for $f(u)=u^{m-1}$ and $m\geq 2$.

In the literature, related equations have been analyzed too. Equation (1) for $f(u)=u$ and the limit case $s=1$ was shown in [1] to be the Wasserstein gradient flow on the space of probability measures, leading to the well-posedness of the equation and energy-dissipation inequalities. The existence of local smooth solutions to the regularized equation (7) are proved in [14]. The solutions $\partial_t\rho=\operatorname{div}(\rho^{m-1}\nabla P)$ with $P=(-\Delta)^{-s}\rho$ in $\mathbb{R}^d$ propagate with finite speed if and only if $m\geq 2$ [44]. The existence of weak solutions to this equation with $P=(-\Delta)^{-s}(\rho^n)$ and $n>0$ is proved in [31] (in bounded domains). While (1) has a parabolic-elliptic structure, parabolic-parabolic systems have been also investigated. For instance, the global existence of weak solutions to $\partial_t\rho=\operatorname{div}(\rho\nabla P)$ and $\partial_t P+(-\Delta)^s P=\rho^\beta$, where $\beta>1$, was shown in [5]. In [16], the algebraic decay towards the steady state was proved in the case $\beta=2$. We also mention that fractional porous-medium equations of the type $\partial_t\rho+(-\Delta)^{s/2}f(\rho)=0$ in $\mathbb{R}^d$ have been studied in the literature; see, e.g., [37]. Compared to (1), this problem has infinite speed of propagation. For a review and comparison of this model and (1), we refer to [47].

There is a huge literature concerning mean-field limits leading to diffusion equations, and the research started already in the 1980s; we refer to the reviews [23, 26] and the classical works of Sznitman [42, 43]. Oelschläger proved the mean-field limit in weakly interacting particle systems [33], leading to deterministic nonlinear processes, and moderately interacting particle systems [35], giving porous-medium-type equations with quadratic diffusion. First investigations of moderate interactions in stochastic particle systems with nonlinear diffusion coefficients were performed in [27]. The approach of moderate interactions was extended in [11, 12] to multi-species systems, deriving population cross-diffusion systems. Reaction-diffusion equations with nonlocal terms were derived in the mean-field limit in [24]. The large population limit of point measure-valued Markov processes leads to nonlocal Lotka–Volterra systems with cross diffusion [22]. Further references can be found in [36, Sec. 1.3].

Compared to previous works, we consider a singular kernel $\mathcal{K}$ and derive a partial differential equation without Laplace diffusion by taking the limit $\sigma \to 0$. The authors of [21] derived the viscous porous-medium equation by starting from a stochastic particle system with a double convolution structure in the drift term, similar to (4). The main difference to our work is that (besides different techniques for the existence and regularity of solutions to the parabolic problems) we consider a singular kernel in one part of the convolution and a different scaling for the approximating regularized kernel $\mathcal{K}_\zeta = \mathcal{K}\omega_\zeta * W_\zeta$, where ω_ζ is a $W^{1,\infty}(\mathbb{R}^d)$ cutoff function (see Section 1.3 for the exact definition), in comparison to the interaction scaling $W_\beta * \rho_{\sigma,\beta,\zeta}$. The two different scalings β and ζ allow us to establish a result, for which the kernel regularization on the particle level does not need to be of logarithmic type but of power-law type only.

1.3. **Main results and key ideas.** We impose the following hypotheses:

- (H1) Data: Let $0 < s < 1$, $d \geq 2$.
- (H2) $\rho^0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ satisfies $\rho^0 \geq 0$ in $\mathbb{R}^d$ and $\int_{\mathbb{R}^d} \rho^0(x)|x|^{2d/(d-2s)}dx < \infty$.
- (H3) Nonlinearity: $f \in C^1([0,\infty))$ is nondecreasing, $f(0) = 0$, and $u \mapsto uf(u)$ for $u > 0$ is strictly convex.

Let us discuss these assumptions. We assume that $d \geq 2$; the case $d = 1$ can be treated if $s < 1/2$; see [8]. Extending the range of s to $s < 0$ leads to the fractional (higher-order) thin-film equation, which is studied in [29]. The case $1 < s < d/2$ may be considered too, since it yields better regularity results; we leave the details to the reader. On the other hand, the case $s \geq d/2$ is more delicate since the multiplier in the definition of $(-\Delta)^{-s}$ using Fourier transforms does not define a tempered distribution. The case $s = d/2$ for $d \leq 2$ (with a logarithmic Riesz kernel) was analyzed in [18]. We need the moment bound for the initial datum ρ^0 to prove the same moment bound for ρ_σ, which in turn is used several times, for instance to show the entropy balance and the convergence $\rho_\sigma \to \rho$ as $\sigma \to 0$ in the sense of $C^0_{\rm weak}([0,T]; L^1(\mathbb{R}^d))$. The monotonicity of f and the strict convexity of $u \mapsto uf(u)$ are needed to prove the strong convergence of (ρ_σ), which then allows us to identify the limit of $(f_\sigma(\rho_\sigma))$. An example of a function satisfying Hypothesis (H3) is $f(u) = u^\beta$ with $\beta \geq 1$.

Our first main result is concerned with the existence analysis of (1). We write $\|\cdot\|_p$ for the $L^p(\mathbb{R}^d)$ norm.

Theorem 1 (Existence of weak solutions to (1))**.** *Let Hypotheses (H1)–(H3) hold. Then there exists a weak solution* $\rho\geq 0$ *to* (1) *satisfying* (i) *the regularity*

$$\rho\in L^\infty(0,\infty;L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)),\quad \nabla(-\Delta)^{-s/2}f(\rho)\in L^2(0,\infty;L^2(\mathbb{R}^d)),$$
$$\partial_t\rho\in L^2(0,\infty;H^{-1}(\mathbb{R}^d)),$$

(ii) *the weak formulation*

$$\int_0^T\langle\partial_t\rho,\phi\rangle dt+\int_0^T\int_{\mathbb{R}^d}\rho\nabla(-\Delta)^{-s}f(\rho)\cdot\nabla\phi dxdt=0 \tag{8}$$

for all $\phi\in L^2(0,T;H^1(\mathbb{R}^d))$ *and* $T>0$, (iii) *the initial datum* $\rho(0)=\rho^0$ *in the sense of* $H^{-1}(\mathbb{R}^d)$, *and* (iv) *the following properties for* $t>0$*:*

- *Mass conservation:* $\|\rho(t)\|_1=\|\rho^0\|_1$,
- *Dissipation of the* L^∞ *norm:* $\|\rho(t)\|_\infty\leq\|\rho^0\|_\infty$,
- *Moment estimate:* $\sup_{0<t<T}\int_{\mathbb{R}^d}\rho(x,t)|x|^{2d/(d-2s)}dx\leq C(T)$,
- *Entropy inequality:*

$$\int_{\mathbb{R}^d}h(\rho(t))dx+\int_0^t\int_{\mathbb{R}^d}|\nabla(-\Delta)^{-s/2}f(\rho)|^2dxds\leq\int_{\mathbb{R}^d}h(\rho^0)dx.$$

Note that the Hardy–Littlewood–Sobolev-type inequality (68) (see Appendix B) implies that

$$\|\rho\nabla(-\Delta)^{-s}f(\rho)\|_2=\|\rho(-\Delta)^{-s/2}[\nabla(-\Delta)^{-s/2}f(\rho)]\|_2\leq C\|\rho\|_{d/(2s)}\|\nabla(-\Delta)^{-s/2}f(\rho)\|_2,$$

such that $\rho\nabla(-\Delta)^{-s}f(\rho)\in L^2(\mathbb{R}^d)$, and the weak formulation (8) is defined.

The key ideas of the proof of Theorem 1 are as follows. A priori estimates for strong solutions ρ_σ to the regularized equation (7) are derived from mass conservation, the entropy inequality, and energy-type bounds. The energy-type bound allows us to show, for sufficiently small $\sigma>0$, that the L^∞ norm of ρ_σ is bounded by the L^∞ norm of ρ^0, up to some factor depending on the moment bound for ρ^0. The existence of a strong solution ρ_σ is proved by regularizing (7) in a careful way to deal with the singular kernel. The regularized equation is solved locally in time by Banach's fixed-point theorem. Entropy estimates allow us to extend this solution globally in time and to pass to the de-regularization limit. The second step is the limit $\sigma\to 0$ in (7). Since the bounds only provide weak convergence of (a subsequence of) ρ_σ, the main difficulty is the identification of the nonlinear limit $f_\sigma(\rho_\sigma)$. This is done by applying the div-curl lemma and exploiting the monotonicity of f and the strict convexity of $u\mapsto uf(u)$ [20].

We already mentioned that the existence of local smooth solutions ρ_σ to (7) has been proven in [13]. However, we provide an independent proof that allows for global strong solutions and that yields a priori estimates needed in the mean-field limit.

Our second main result is concerned with the mean-field-type limit. For this, we need some definitions. Define

$$f_\sigma(u) = \int_0^u (\Gamma_\sigma * (f' 1_{[0,\infty)}))(w)\widetilde{\Xi}(\sigma w)dw \quad u \in \mathbb{R}, \tag{9}$$

where the mollifier Γ_σ for $\sigma > 0$ is given by $\Gamma_\sigma(x) = \sigma^{-1}\Gamma_1(x/\sigma)$, and $\Gamma_1 \in C_0^\infty(\mathbb{R})$ satisfies $\Gamma_1 \geq 0$, $\|\Gamma_1\|_1 = 1$, while the cutoff function $\widetilde{\Xi} \in C_0^\infty(\mathbb{R})$ satisfies $0 \leq \widetilde{\Xi} \leq 1$ in $\mathbb{R}$ and $\widetilde{\Xi}(x) = 1$ for $|x| \leq 1$. Then $f_\sigma \in C^\infty(\mathbb{R})$, $f'_\sigma \geq 0$, $f_\sigma(0) = 0$, and the derivatives $\mathrm{D}^k f_\sigma$ are bounded and compactly supported for all $k \geq 1$. In a similar way, we introduce the mollifier function W_β for $\beta > 0$ and $x \in \mathbb{R}^d$ by

$$W_\beta(x) = \beta^{-d}W_1(x/\beta), \quad W_1 \in C_0^\infty(\mathbb{R}^d) \text{ is symmetric, } W_1 \geq 0, \ \|W_1\|_1 = 1. \tag{10}$$

Let us define the cutoff version of the singular kernel $\mathcal{K}$ by

$$\begin{aligned} &\widetilde{\mathcal{K}}_\zeta := \mathcal{K}\omega_\zeta, \text{ where the cut-off function } \omega_\zeta \in W^{1,\infty}(\mathbb{R}^d) \text{ is such that} \\ &0 \leq \omega_\zeta(x) \leq 1 \text{ for } x \in \mathbb{R}^d, \quad \|\nabla\omega_\zeta\|_\infty \leq 2\zeta, \\ &\omega_\zeta(x) = 1 \text{ for all } |x| \leq \zeta^{-1}, \quad \omega_\zeta(x) = 0 \text{ for all } |x| \geq 2\zeta^{-1}. \end{aligned} \tag{11}$$

Then the regularized kernel $\mathcal{K}_\zeta$ is given by

$$\mathcal{K}_\zeta(x) := \widetilde{\mathcal{K}}_\zeta * W_\zeta(x) \quad \text{for all } x \in \mathbb{R}^d,$$

where $\zeta > 0$. Let the cutoff function $\Xi \in C_0^\infty(\mathbb{R}^d)$ satisfy $0 \leq \Xi \leq 1$ in $\mathbb{R}^d$ and $\Xi(x) = 1$ for $|x| \leq 1$. Then we define the regularized initial datum for $x \in \mathbb{R}^d$ by

$$\rho_\sigma^0(x) = \kappa_\sigma(W_\sigma * \rho^0)(x)\Xi(\sigma x), \quad \text{where } \kappa_\sigma = \frac{\int_{\mathbb{R}^d} \rho^0(y)dy}{\int_{\mathbb{R}^d}(W_\sigma * \rho^0)(y)\Xi(\sigma y)dy}. \tag{12}$$

This definition guarantees the mass conservation since $\|\rho_\sigma^0\|_1 = \|\rho^0\|_1$; see Section 2.1.

Theorem 2 (Error estimate for the stochastic system)**.** *Let X_i^N and $\widehat{X}_i^N$ be the solutions to* (3) *and* (6)*, respectively. We assume that $\zeta^{-2s-1} \leq C_1 N^{1/4}$ for some constant $C_1 > 0$. Let $\delta \in (0, 1/4)$ and $a := \min\{1, d-2s\} > 0$. Then there exist constants $\varepsilon > 0$, depending on σ and δ, and $C_2 > 0$, depending on σ and T, such that if $\beta^{-3d-7} \leq \varepsilon \log N$ then*

$$\mathbb{E}\bigg(\sup_{0<s<T} \max_{i=1,\ldots,N} |(X_i^N - \widehat{X}_i^N)(s)|\bigg) \leq C_2(\beta + \zeta^a) \to 0 \ \textit{as} \ (N, \zeta, \beta) \to (\infty, 0, 0).$$

The theorem is proved by estimating the differences

$$\begin{aligned} E_1(t) &:= \mathbb{E}\bigg(\sup_{0<s<t} \max_{i=1,\ldots,N} |(X_i^N - \bar{X}_i^N)(s)|\bigg), \\ E_2(t) &:= \mathbb{E}\bigg(\sup_{0<s<t} \max_{i=1,\ldots,N} |(\bar{X}_i^N - \widehat{X}_i^N)(s)|\bigg), \end{aligned}$$

and applying the triangle inequality. For the first difference, we estimate expressions like $\|\mathrm{D}^k\mathcal{K}_\zeta * u\|_\infty$ for appropriate functions u and $\|\mathrm{D}^k W_\beta\|_\infty$ for $k \in \mathbb{N}$ in terms of negative powers of β (here, D^k denotes the kth-order partial derivatives). Using properties of Riesz potentials, in particular Hardy–Littlewood–Sobolev-type inequalites (see Lemmas 21 and 22), we show that for some $\mu_i > 0$ ($i = 1, 2, 3$),

$$E_1(t) \le C(\sigma)\beta^{-\mu_1} \int_0^t E_1(s)ds + C(\sigma)\beta^{-\mu_2}\zeta^{-\mu_3}N^{-1/2}.$$

By applying the Gronwall lemma and choosing a logarithmic scaling for β and an algebraic scaling for ζ with respect to N, we infer that $E_1(t) \le C(\sigma)N^{-\mu_4}$ for some $\mu_4 \in (0, 1/4)$. For the second difference E_2, we need the estimates $\|W_\beta * u - u\|_\infty \le C(\sigma)\beta$ (Lemma 20), and $\|(\mathcal{K}_\zeta - \mathcal{K}) * \rho_\sigma\| \le C(\sigma)\zeta^a$, $\|\rho_{\sigma,\beta,\zeta} - \rho_\sigma\|_\infty \le C(\sigma)(\beta + \zeta^a)$ (Proposition 13), recalling that $a = \min\{1, d - 2s\}$. The proof of these estimates is very technical. The idea is to apply several times fractional Gagliardo–Nirenberg inequalities that are proved in Appendix B and Hardy–Littlewood–Sobolev inequalities that are recalled in Lemmas 21–22. Then, after suitable computations,

$$E_2(t) \le C(\sigma)(\beta + \zeta^a) + C(\sigma) \int_0^t E_2(s)ds,$$

and we conclude with Gronwall's lemma that $E_2(t) \le C(\sigma)(\beta + \zeta^a)$.

Theorem 2 and calculations for $\sigma \to 0$ yield the following propagation of chaos result.

Theorem 3 (Propagation of chaos for (1)). *Let the assumptions of Theorem 2 hold and let* $\mathrm{P}^k_{N,\sigma,\beta,\zeta}(t)$ *be the joint distribution of* $(X_1^N(t), \ldots, X_k^N(t))$ *for a fixed* $t \in (0, T)$. *Then there exists a subsequence in* σ *such that*

$$\lim_{\sigma\to 0} \lim_{N\to\infty,\,(\beta,\zeta)\to 0} \mathrm{P}^k_{N,\sigma,\beta,\zeta}(t) = \mathrm{P}^{\otimes k}(t),$$

where the limit is locally uniform in t, *the limit* $N \to \infty$, $(\beta, \zeta) \to 0$ *has to be understood in the sense of Theorem 2, and the measure* $\mathrm{P}(t)$ *is absolutely continuous with respect to the Lebesgue measure with the probability density function* $\rho(t)$ *that is a weak solution to* (1).

If equation (1) was uniquely solvable, we would obtain the convergence of the whole sequence in σ. Unfortunately, the regularity of the solution ρ to (1) is too weak to conclude the uniqueness of weak solutions. Up to our knowledge, none of the known methods, such as [3, 15], seem to be applicable to equation (1).

The paper is organized as follows. The existence of global nonnegative weak solutions to (1) is proved in Section 2 by establishing an existence analysis for (7) and performing the limit $\sigma \to 0$. Some uniform estimates for the solution $\rho_{\rho,\beta,\zeta}$ to (5) and for the difference $\rho_{\sigma,\beta,\zeta} - \rho_\sigma$ are shown in Section 3. Section 4 is devoted to the proof of the error estimate in Theorem 2 and the propagation of chaos in Theorem 3. In Appendices A–C we recall some auxiliary results and Hardy–Littlewood–Sobolev-type inequalities, prove new variants of fractional Gagliardo–Nirenberg inequalities, and formulate a result on parabolic regularity.

Notation. We write $\|\cdot\|_p$ for the $L^p(\mathbb{R}^d)$ or $L^p(\mathbb{R})$ norm with $1 \le p \le \infty$. The ball around the origin with radius $R > 0$ is denoted by B_R. The partial derivative $\partial/\partial x_i$ is abbreviated as ∂_i for $i = 1, \ldots, d$, and D^α denotes a partial derivative of order $|\alpha|$, where $\alpha \in \mathbb{N}_0^d$ is a multiindex. The notation D^k refers to the kth-order tensor of partial derivatives of order $k \in \mathbb{N}$. In this situation, the norm $\|\mathrm{D}^k u\|_p$ is the sum of all L^p norms of partial derivatives of u of order k. Finally, $C > 0$, $C_1 > 0$, etc. denote generic constants with values changing from line to line.

2. Analysis of equation (1)

In this section, we prove the existence of global nonnegative weak solutions to (1) and an estimate for the difference $\rho_{\sigma,\beta,\zeta} - \rho_\sigma$ of the solutions to (5) and (7), respectively, needed in the mean-field limit. We first prove the existence of a solution ρ_σ to (7) by a fixed-point argument and then perform the limit $\sigma \to 0$. Recall definition (12) of the number κ_σ, which is stated in (iv) below.

Theorem 4. *Let Hypotheses (H1)–(H3) hold. Then for all $\sigma > 0$, there exists a unique weak solution $\rho_\sigma \geq 0$ to* (7) *satisfying* (i) *the regularity*

$$\rho_\sigma \in L^\infty(0,\infty; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)) \cap C^0([0,\infty); L^2(\mathbb{R}^d)),$$
$$\nabla\rho_\sigma \in L^2(0,\infty; L^2(\mathbb{R}^d)), \quad \partial_t\rho_\sigma \in L^2(0,\infty; H^{-1}(\mathbb{R}^d)),$$

(ii) *the weak formulation of* (7) *with test functions $\phi \in L^2(0,T; H^1(\mathbb{R}^d))$,* (iii) *the inital datum $\rho_\sigma(0) = \rho_\sigma^0$ in $L^2(\mathbb{R}^d)$, and* (vi) *the following properties for $t > 0$, which are uniform in σ for sufficiently small $\sigma > 0$:*

- *Mass conservation: $\|\rho_\sigma(t)\|_1 = \|\rho^0\|_1$.*
- *Dissipation of the L^∞ norm: $\|\rho_\sigma\|_{L^\infty(0,\infty;L^\infty(\mathbb{R}^d))} \leq \kappa_\sigma\|\rho^0\|_{L^\infty(\mathbb{R}^d)} \leq C\|\rho^0\|_{L^\infty(\mathbb{R}^d)}$.*
- *Moment estimate: $\sup_{t\in[0,\infty)} \int_{\mathbb{R}^d} \rho_\sigma(x,t)|x|^{\frac{2d}{d-2s}} dx \leq C_T$.*
- *Entropy inequality:*

$$\int_{\mathbb{R}^d} h(\rho_\sigma(T))dx + 4\sigma \int_0^T \int_{\mathbb{R}^d} f_\sigma'(\rho_\sigma)|\nabla\sqrt{\rho_\sigma}|^2 dxdt$$
$$+ \int_0^T \int_{\mathbb{R}^d} |\nabla(-\Delta)^{-s/2} f_\sigma(\rho_\sigma)|^2 dxdt \leq \int_{\mathbb{R}^d} h(\rho_\sigma^0)dx \quad \text{for all } T > 0.$$

Additionally, for any $T > 0$, $1 < p < \infty$, and $2 \leq q < \infty$, there exists $C > 0$, depending on T, σ, p, and q, such that

$$\|\rho_\sigma\|_{L^p(0,T;W^{3,p}(\mathbb{R}^d))} + \|\partial_t\rho_\sigma\|_{L^p(0,T;W^{1,p}(\mathbb{R}^d))} + \|\rho_\sigma\|_{C^0([0,T];W^{2,1}(\mathbb{R}^d)\cap W^{3,q}(\mathbb{R}^d))} \leq C,$$

i.e., ρ_σ is even a strong solution to (7) *and $\rho_\sigma \in C^0([0,T]; W^{2,1}(\mathbb{R}^d) \cap W^{3,q}(\mathbb{R}^d))$ for $q \geq 2$.*

2.1. **Basic estimates for ρ_σ.** We prove a priori estimates in L^p spaces and an energy-type estimate. Let $\sigma \in (0,1)$ and let ρ_σ be a nonnegative strong solution to (7). Integration of (7) in $\mathbb{R}^d$ and the definition of ρ_σ^0 immediately yield the mass conservation

$$\|\rho_\sigma(t)\|_1 = \|\rho_\sigma^0\|_1 = \|\rho^0\|_1 \quad \text{for } t > 0. \tag{13}$$

Lemma 5 (Energy-type estimate)**.** *Let $F\in C^2([0,\infty))$ be convex and let $F(\rho_\sigma^0)\in L^1(\mathbb{R}^d)$. Then*

$$\begin{aligned}(14)\qquad \frac{d}{dt}\int_{\mathbb{R}^d}F(\rho_\sigma)dx &= -\sigma\int_{\mathbb{R}^d}F''(\rho_\sigma)|\nabla\rho_\sigma|^2dx \\ &\quad -\frac{c_{d,1-s}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(G(\rho_\sigma(x))-G(\rho_\sigma(y)))(f_\sigma(\rho_\sigma(x))-f_\sigma(\rho_\sigma(y)))}{|x-y|^{d+2(1-s)}}dxdy\le 0,\end{aligned}$$

where $G(u):=\int_0^u vF''(v)dv$ for $u\ge 0$ and $c_{d,1-s}$ is defined after (2)*.*

Proof. First, we assume that F'' is additionally bounded. Then $F'(\rho_\sigma)-F'(0)$ is an admissible test function in the weak formulation of (7), since $|F'(\rho_\sigma)-F'(0)|\le\|F''\|_\infty|\rho_\sigma|$. It follows from definition (66) of the fractional Laplacian and integration by parts that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}F(\rho_\sigma)dx+\sigma\int_{\mathbb{R}^d}F''(\rho_\sigma)|\nabla\rho_\sigma|^2dx &= -\int_{\mathbb{R}^d}F''(\rho_\sigma)\rho_\sigma\nabla\rho_\sigma\cdot\nabla(-\Delta)^{-s}f_\sigma(\rho_\sigma)dx \\ &= -\int_{\mathbb{R}^d}\nabla G(\rho_\sigma)\cdot\nabla(-\Delta)^{-s}f_\sigma(\rho_\sigma)dx = -\int_{\mathbb{R}^d}G(\rho_\sigma)(-\Delta)^{1-s}f_\sigma(\rho_\sigma)dx \\ &= -c_{d,1-s}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}G(\rho_\sigma(x))\frac{f_\sigma(\rho_\sigma(x))-f_\sigma(\rho_\sigma(y))}{|x-y|^{d+2(1-s)}}dxdy.\end{aligned}$$

A symmetrization of the last integral yields (14).

In the general case, we introduce $F_k(u)=F(0)+F'(0)u+\int_0^u\int_0^v\min\{F''(w),k\}dwdv$ for $k>0$. Then $F_k''(u)$ is bounded and (14) follows for F replaced by F_k. The result follows after taking the limit $k\to\infty$ using monotone convergence. □

We need a bound on κ_σ, defined in (12), to derive uniform $L^\infty(\mathbb{R}^d)$ bounds for ρ_σ.

Lemma 6 (Bound for κ_σ)**.** *There exists $C>0$ such that, for sufficiently small $\sigma>0$,*

$$1\le\kappa_\sigma\le\frac{1}{1-C\sigma E},\quad \textit{where } E:=\frac{1}{\|\rho^0\|_1}\int_{\mathbb{R}^d}(1+|x|^{2d/(d-2s)})\rho^0(x)dx.$$

Proof. By Young's convolution inequality (Lemma 18), we have

$$\int_{\mathbb{R}^d}(W_\sigma*\rho^0)(x)\Xi(\sigma x)dx\le\|W_\sigma*\rho^0\|_1\le\|W_\sigma\|_1\|\rho^0\|_1=\|\rho^0\|_1,$$

which shows that $\kappa_\sigma\ge 1$. To prove the upper bound, we use the triangle inequality $|x|\le|x-y|+|y|$:

$$\begin{aligned}\int_{\mathbb{R}^d}(W_\sigma*\rho^0)(x)\Xi(\sigma x)dx &\ge \int_{\{|x|\le 1/\sigma\}}\int_{\mathbb{R}^d}W_\sigma(x-y)\rho^0(y)dydx \\ &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}W_\sigma(x-y)dx\right)\rho^0(y)dy-\int_{\{|x|>1/\sigma\}}\int_{\mathbb{R}^d}W_\sigma(x-y)\rho^0(y)dydx \\ &\ge \int_{\mathbb{R}^d}\rho^0(y)dy-\sigma^{2d/(d-2s)}\int_{\{|x|>1/\sigma\}}\int_{\mathbb{R}^d}|x|^{2d/(d-2s)}W_\sigma(x-y)\rho^0(y)dydx\end{aligned}$$

$$\geq \int_{\mathbb{R}^d} \rho^0(y)dy - \sigma^{2d/(d-2s)} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |x-y|^{2d/(d-2s)} W_\sigma(x-y)\rho^0(y)dydx$$
$$- \sigma^{2d/(d-2s)} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |y|^{2d/(d-2s)} W_\sigma(x-y)\rho^0(y)dydx.$$

Using the property $\int_{\mathbb{R}^d} |z|^{2d/(d-2s)} W_\sigma(z)dz \leq C\sigma^{2d/(d-2s)}$ for the second term on the right-hand side and $\|W_\beta\|_{L^1(\mathbb{R}^d)} = 1$ for the third term, we find that

$$\int_{\mathbb{R}^d} (W_\sigma * \rho^0)(x)\Xi(\sigma x)dx \geq \int_{\mathbb{R}^d} \rho^0(y)dy - C\sigma^{4d/(d-2s)} \int_{\mathbb{R}^d} \rho^0(y)dy$$
$$- \sigma^{2d/(d-2s)} \int_{\mathbb{R}^d} |y|^{2d/(d-2s)} \rho^0(y)dy.$$

Because of $\sigma^{2d/(d-2s)} \leq \sigma$ for $\sigma \leq 1$, we obtain

$$\frac{\|\rho^0\|_1}{\kappa_\sigma} = \int_{\mathbb{R}^d} (W_\sigma * \rho^0)(x)\Xi(\sigma x)dx \geq \int_{\mathbb{R}^d} \rho^0(y)dy - C\sigma \int_{\mathbb{R}^d} (1 + |y|^{2d/(d-2s)})\rho^0(y)dy$$
$$\geq \int_{\mathbb{R}^d} \rho^0(y)dy - C\sigma \int_{\mathbb{R}^d} \rho^0(y)dy \cdot E = \|\rho^0\|_1(1 - C\sigma E),$$

which proves the lemma. ☐

Lemma 7 (Bounds for ρ_σ)**.** *The following bounds hold:*

$$\|\rho_\sigma(t)\|_\infty \leq \kappa_\sigma \|\rho^0\|_\infty \leq C\|\rho^0\|_\infty, \quad t > 0, \tag{15}$$
$$\sqrt{\sigma}\|\rho_\sigma\|_{L^2(0,T;H^1(\mathbb{R}^d))} \leq C(T, \|\rho^0\|_1, \|\rho^0\|_2), \tag{16}$$

where (15) *holds for sufficiently small* $\sigma > 0$.

Lemma 7 and mass conservation imply that $\|\rho_\sigma(t)\|_p$ is bounded for all $t > 0$ and $1 \leq p \leq \infty$. Observe that $\kappa_\sigma \to 1$ as $\sigma \to 0$. So, if $\rho_\sigma(t) \to \rho(t)$ a.e., the dissipation of the L^∞ norm follows, as stated in Theorem 1 (iv).

Proof. The convexity of F shows that G, defined in Lemma 5, is nondecreasing. Therefore, $(d/dt)\int_{\mathbb{R}^d} F(\rho_\sigma)dx \leq 0$ and

$$\sup_{t>0} \int_{\mathbb{R}^d} F(\rho_\sigma(t))dx \leq \int_{\mathbb{R}^d} F(\rho_\sigma^0)dx.$$

We choose a convex function $F \in C^2([0,\infty))$ such that $F(u) = 0$ for $u \leq \|\rho_\sigma^0\|_\infty$, $F(u) > 0$ for $u > \|\rho_\sigma^0\|_\infty$ and satisfying $F(u) \leq Cu$ for $u \to \infty$. Then

$$0 \leq \int_{\mathbb{R}^d} F(\rho_\sigma(t))dx \leq \int_{\mathbb{R}^d} F(\rho_\sigma^0)dx = 0 \quad \text{for } t > 0.$$

Consequently, $\rho_\sigma(x,t) \leq \|\rho_\sigma^0\|_\infty \leq \kappa_\sigma\|\rho^0\|_\infty$ for $t > 0$, showing the $L^\infty(\mathbb{R}^d)$ bound. Finally, choosing $F(u) = u^2$ in Lemma 5, the $L^2(0,T;H^1(\mathbb{R}^d))$ estimate follows. ☐

2.2. **Entropy and moment estimates.** We need a fractional derivative estimate for $f_\sigma(\rho_\sigma)$, which is not an immediate consequence of Lemma 5. To this end, we define the entropy density

$$h_\sigma(u) = \int_0^u \int_1^v \frac{f_\sigma'(w)}{w} dwdv, \quad u \geq 0.$$

Lemma 8 (Entropy balance)**.** *It holds for all $t > 0$ that*

$$\frac{d}{dt}\int_{\mathbb{R}^d} h_\sigma(\rho_\sigma)dx + 4\sigma \int_{\mathbb{R}^d} f_\sigma'(\rho_\sigma)|\nabla \rho_\sigma^{1/2}|^2 dx + \int_{\mathbb{R}^d} |\nabla(-\Delta)^{-s/2} f_\sigma(\rho_\sigma)|^2 dx = 0.$$

In particular, for all $T > 0$, there exists $C > 0$ such that

$$\|f_\sigma(\rho_\sigma)\|_{L^2(0,T;H^{1-s}(\mathbb{R}^d))} \leq C. \tag{17}$$

Proof. The idea is to apply Lemma 5. Since $h_\sigma \notin C^2([0,\infty))$, we cannot use the lemma directly. Instead, we apply it to the regularized function

$$h_\sigma^\delta(u) = \int_0^u \int_1^v \frac{f_\sigma'(w)}{w+\delta} dwdv, \quad u \geq 0,$$

where $\delta > 0$. Choosing $F = h_\sigma^\delta$ in Lemma 5 gives

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^d} h_\sigma^\delta(\rho_\sigma)dx &+ 4\sigma \int_{\mathbb{R}^d} f_\sigma'(\rho_\sigma)\frac{\rho_\sigma}{\rho_\sigma+\delta}|\nabla \rho_\sigma^{1/2}|^2 dx \\ &= -\frac{c_{d,1-s}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(f_\sigma^\delta(\rho_\sigma(x)) - f_\sigma^\delta(\rho_\sigma(y)))(f_\sigma(\rho_\sigma(x)) - f_\sigma(\rho_\sigma(y))}{|x-y|^{d+2(1-s)}} dxdy, \end{aligned} \tag{18}$$

where $f_\sigma^\delta(u) := \int_0^u (v/(v+\delta))f_\sigma'(v)dv$ for $u \geq 0$.

Step 1: Estimate of h_σ^δ. The pointwise limit $h_\sigma^\delta(\rho_\sigma) \to h_\sigma(\rho_\sigma)$ holds a.e. in $\mathbb{R}^d \times (0,T)$ as $\delta \to 0$. We observe that for all $0 < u \leq 1$,

$$|h_\sigma^\delta(u)| \leq \sup_{0<v<1} f'(v) \int_0^u \int_v^1 \frac{dw}{w} dv \leq Cu(|\log u| + 1),$$

while for all $u > 1$, since $f_\sigma' \geq 0$ in $[0,\infty)$,

$$\begin{aligned} |h_\sigma^\delta(u)| &\leq \int_0^1 \int_v^1 \frac{f_\sigma'(w)}{w+\delta} dwdv + \int_1^u \int_1^v \frac{f_\sigma'(w)}{w+\delta} dwdv \\ &\leq C + \int_1^u \int_1^v f_\sigma'(w)dwdv \leq C + \int_0^u f_\sigma(v)dv \leq C + uf_\sigma(u). \end{aligned}$$

The last inequality follows after integration of $f_\sigma(v) \leq f_\sigma(v) + vf_\sigma'(v) = (vf_\sigma(v))'$ in $(0,u)$. Therefore, since $\rho_\sigma \leq \|\rho_\sigma^0\|_\infty$ a.e. in $\mathbb{R}^d \times (0,\infty)$, we find that

$$|h_\sigma^\delta(\rho_\sigma)| \leq C\rho_\sigma(|\log \rho_\sigma| + 1)1_{\{\rho_\sigma \leq 1\}} + C1_{\{\rho_\sigma > 1\}} \leq C(\rho_\sigma^\theta + \rho_\sigma),$$

where $\theta \in (0,1)$ is arbitrary, and consequently, because of mass conservation,

$$\int_{\mathbb{R}^d} |h_\sigma^\delta(\rho_\sigma)|dx \leq C + C\int_{\mathbb{R}^d} \rho_\sigma^\theta dx. \tag{19}$$

Step 2: Estimate of $\int_{\mathbb{R}^d}\rho_\sigma^\theta dx$. Let $0<\alpha<1$ and $d/(d+\alpha)<\theta<1$. Then, by Young's inequality,

$$\begin{aligned}\int_{\mathbb{R}^d}\rho_\sigma^\theta dx &= \int_{\mathbb{R}^d}(1+|x|^2)^{\alpha\theta/2}\rho_\sigma^\theta(1+|x|^2)^{-\alpha\theta/2}dx \\ &\le \int_{\mathbb{R}^d}(1+|x|^2)^{\alpha/2}\rho_\sigma dx + C\int_{\mathbb{R}^d}(1+|x|^2)^{-\alpha\theta/(2(1-\theta))}dx \\ &\le \int_{\mathbb{R}^d}(1+|x|^2)^{\alpha/2}\rho_\sigma dx + C,\end{aligned}$$

since the choice of θ guarantees that $-\alpha\theta/(2(1-\theta))<-d/2$, so $\int_{\mathbb{R}^d}(1+|x|^2)^{-\alpha\theta/(2(1-\theta))}dx<\infty$. To control the right-hand side, we need to bound a suitable moment of ρ_σ. For this, we use the test function $(1+|x|^2)^{\alpha/2}$ in the weak formulation of (7). (Actually, we need to use a cutoff to guarantee integrability, but we leave the technical details to the reader.) We find that

$$\begin{aligned}\int_{\mathbb{R}^d}(1+|x|^2)^{\alpha/2}\rho_\sigma(x,t)dx &= \int_{\mathbb{R}^d}(1+|x|^2)^{\alpha/2}\rho_\sigma^0 dx + \sigma\int_0^t\int_{\mathbb{R}^d}\rho_\sigma\Delta(1+|x|^2)^{\alpha/2}dxds \\ &\quad - \alpha\int_0^t\int_{\mathbb{R}^d}\rho_\sigma(1+|x|^2)^{\alpha/2-1}x\cdot\nabla(-\Delta)^{-s}f_\sigma(\rho_\sigma)dxds.\end{aligned}$$

Since $\alpha<1$, the terms $\Delta(1+|x|^2)^{\alpha/2}$ and $x(1+|x|^2)^{\alpha/2-1}$ are bounded in $\mathbb{R}^d$. Thus, taking into account the assumption on ρ^0 and mass conservation,

$$\sup_{0<t<T}\int_{\mathbb{R}^d}(1+|x|^2)^{\alpha/2}\rho_\sigma(x,t)dx \le C + C\int_0^T\int_{\mathbb{R}^d}\rho_\sigma(-\Delta)^{-s/2}|\nabla(-\Delta)^{-s/2}f_\sigma(\rho_\sigma)|dxdt.$$

Next, we apply the Hardy–Littlewood–Sobolev inequality (see Appendix B) and the Young inequality (see Lemma 21) and use the fact that $\rho_\sigma(t)$ is bounded in any $L^p(\mathbb{R}^d)$:

$$\begin{aligned}\sup_{0<t<T}&\int_{\mathbb{R}^d}(1+|x|^2)^{\alpha/2}\rho_\sigma(x,t)dx \\ &\le C + \int_0^T\|\rho_\sigma\|_{2d/(d+2s)}\|(-\Delta)^{-s/2}[\nabla(-\Delta)^{-s/2}f_\sigma(\rho_\sigma)]\|_{2d/(d-2s)}dt \\ &\le C + \int_0^T\|\rho_\sigma\|_{2d/(d+2s)}\|\nabla(-\Delta)^{-s/2}f_\sigma(\rho_\sigma)\|_2 dt \\ &\le C(\eta) + \eta\int_0^T\|\nabla(-\Delta)^{-s/2}f_\sigma(\rho_\sigma)\|_2^2 dt\end{aligned}$$

for all $\eta>0$. This proves that

$$\int_{\mathbb{R}^d}\rho_\sigma^\theta dx \le C(\eta) + \eta\int_0^T\|\nabla(-\Delta)^{-s/2}f_\sigma(\rho_\sigma)\|_2^2 dt.$$

Step 3: A priori estimate. Inserting the previous estimate into (19) leads to

$$\sup_{0<t<T}\int_{\mathbb{R}^d}|h_\sigma^\delta(\rho_\sigma(x,t))|dx \le C(\eta)+\eta\int_0^T\|\nabla(-\Delta)^{-s/2}f_\sigma(\rho_\sigma)\|_2^2dt.$$

We integrate (18) in time and use the previous estimate:

$$\begin{aligned}4\sigma&\int_0^T\int_{\mathbb{R}^d}f_\sigma'(\rho_\sigma)\frac{\rho_\sigma}{\rho_\sigma+\delta}|\nabla\rho_\sigma^{1/2}|^2dxdt\\&+\frac{c_{d,1-s}}{2}\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(f_\sigma^\delta(\rho_\sigma(x))-f_\sigma^\delta(\rho_\sigma(y)))(f_\sigma(\rho_\sigma(x))-f_\sigma(\rho_\sigma(y))}{|x-y|^{d+2(1-s)}}dxdydt\\&\le\int_{\mathbb{R}^d}|h_\sigma^\delta(\rho_\sigma(T))|dx+\int_{\mathbb{R}^d}|h_\sigma^\delta(\rho_\sigma^0)|dx\le C(\eta)+\eta\int_0^T\|\nabla(-\Delta)^{-s/2}f_\sigma(\rho_\sigma)\|_2^2dt.\end{aligned}$$

We wish to pass to the limit $\delta\to 0$ in the previous inequality. We deduce from dominated convergence that $f_\sigma^\delta(\rho_\sigma)\to f_\sigma(\rho_\sigma)$ a.e. in $\mathbb{R}^d\times[0,\infty)$. The integrand of the second term on the left-hand side is nonnegative, and we obtain from Fatou's lemma that

$$\begin{aligned}4\sigma\int_0^T\int_{\mathbb{R}^d}f_\sigma'(\rho_\sigma)|\nabla\rho_\sigma^{1/2}|^2dxdt+\frac{c_{d,1-s}}{2}\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(f_\sigma(\rho_\sigma(x))-f_\sigma(\rho_\sigma(y)))^2}{|x-y|^{d+2(1-s)}}dxdydt\\\le C(\eta)+\eta\int_0^T\|\nabla(-\Delta)^{-s/2}f_\sigma(\rho_\sigma)\|_2^2dt.\end{aligned}\tag{20}$$

By the integral representation of the fractional Laplacian,

$$\frac{c_{d,1-s}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(f_\sigma(\rho_\sigma(x))-f_\sigma(\rho_\sigma(y)))^2}{|x-y|^{d+2(1-s)}}dxdy=\|\nabla(-\Delta)^{-s/2}f_\sigma(\rho_\sigma)\|_2^2,$$

the last term in (20) can be absorbed for sufficiently small $\eta>0$ by the second term on the left-hand side. This leads to the estimate

$$4\sigma\int_0^T\int_{\mathbb{R}^d}f_\sigma'(\rho_\sigma)|\nabla\rho_\sigma^{1/2}|^2dxdt+\int_0^T\int_{\mathbb{R}^d}|\nabla(-\Delta)^{-s/2}f_\sigma(\rho_\sigma)|^2dxdt\le C.$$

Thus, we can pass to the limit $\delta\to 0$ in (18) giving the desired entropy balance. Finally, bound (17) follows from the definition of the $H^{1-s}(\mathbb{R}^d)$ norm and the facts that $f_\sigma(\rho_\sigma)\in L^2(\mathbb{R}^d)$ since f_σ is locally Lipschitz continuous, $f_\sigma(0)=0$, and ρ_σ is bounded both in $L^\infty(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$ independently of σ. □

Lemma 9 (Moment estimate)**.** *It holds that*

$$\sup_{0<t<T}\int_{\mathbb{R}^d}\rho_\sigma(x,t)|x|^{2d/(d-2s)}dx\le C,$$

where $C>0$ depends on T and the $L^1(\mathbb{R}^d)$ norms of ρ^0 and $|\cdot|^{2d/(d-2s)}\rho^0$.

Proof. For the following computations, we would need to use cut-off functions to make the calculations rigorous. We leave the details to the reader, as we wish to simplify the presentation. Let $m = 2d/(d-2s)$. Since $|\cdot|^m\rho^0 \in L^1(\mathbb{R}^d)$ by assumption, we can compute

$$\frac{d}{dt}\int_{\mathbb{R}^d}\rho_\sigma(t)\frac{|x|^m}{m}dx = \sigma(m-2+d)\int_{\mathbb{R}^d}|x|^{m-2}\rho_\sigma dx - \int_{\mathbb{R}^d}\rho_\sigma|x|^{m-2}x\cdot\nabla(-\Delta)^{-s}f_\sigma(\rho_\sigma)dx$$

$$\le C\||\cdot|^{m-2}\rho_\sigma\|_1 + \||\cdot|^{m-1}\rho_\sigma\|_{2d/(d+2s)}\|\nabla(-\Delta)^{-s}f_\sigma(\rho_\sigma)\|_{2d/(d-2s)}. \tag{21}$$

By Young's inequality and mass conservation, we have

$$\||\cdot|^{m-2}\rho_\sigma\|_1 \le C\int_{\mathbb{R}^d}(1+|x|^m)\rho_\sigma dx \le C + C\int_{\mathbb{R}^d}|x|^m\rho_\sigma dx.$$

It follows from (17) that $\nabla(-\Delta)^{-s}f_\sigma(\rho_\sigma)$ is bounded in $L^2(0,T;H^s(\mathbb{R}^d))$. In particular, because of the Sobolev embedding $H^s(\mathbb{R}^d)\hookrightarrow L^m(\mathbb{R}^d)$,

$$\|\nabla(-\Delta)^{-s}f_\sigma(\rho_\sigma)\|_{L^2(0,T;L^m(\mathbb{R}^d))} \le C.$$

Furthermore, using $\rho_\sigma \in L^\infty(0,\infty;L^\infty(\mathbb{R}^d))$, Young's inequality, and the property $2d/(d+2s)\ge 1$ (recall that $d\ge 2$)

$$\||\cdot|^{m-1}\rho_\sigma\|_{2d/(d+2s)}^{2d/(d+2s)} = \int_{\mathbb{R}^d}\rho_\sigma^{2d/(d+2s)}|x|^{2d(m-1)/(d+2s)}dx$$

$$\le C + C\int_{\mathbb{R}^d}\rho_\sigma|x|^{2d(m-1)/(d+2s)}dx.$$

Thus, we infer from (21) and the identity $2d(m-1)/(d+2s) = m$ that

$$\frac{d}{dt}\int_{\mathbb{R}^d}\rho_\sigma(t)\frac{|x|^m}{m}dx \le C + C\int_{\mathbb{R}^d}\rho_\sigma(t)|x|^m dx,$$

and Gronwall's lemma concludes the proof. □

2.3. **Higher-order estimate.**

We need some estimates in higher-order Sobolev spaces.

Proposition 10 (Higher-order regularity)**.** *Let $T>0$, $1<p<\infty$ and $2\le q<\infty$. Then there exists $C>0$, depending on T, σ, p, and q, such that*

$$\|\rho_\sigma\|_{L^p(0,T;W^{3,p}(\mathbb{R}^d))} + \|\partial_t\rho_\sigma\|_{L^p(0,T;W^{1,p}(\mathbb{R}^d))} + \|\rho_\sigma\|_{C^0([0,T];W^{2,q}(\mathbb{R}^d))} \le C.$$

Proof. Step 1: Case $s>1/2$. If $s>1/2$ then $w := \rho_\sigma\nabla(-\Delta)^{-s}f_\sigma(\rho_\sigma)$ does not involve any derivative of ρ_σ. Thus $w\in L^p(0,T;L^p(\mathbb{R}^d))$ for $p<\infty$ and Lemma 25 in Appendix C implies that $\rho_\sigma \in L^p(0,T;W^{1,p}(\mathbb{R}^d))$. Iterating the argument leads to the conclusion. Thus, in the following, we can assume that $0<s\le 1/2$.

Step 2: Estimate of $\operatorname{div} w$ *in* $L^p(0,T;W^{-1,p}(\mathbb{R}^d))$. We claim that w can be estimated in $L^p(0,T;L^p(\mathbb{R}^d))$ for any $p<\infty$. Then, by Lemma 25, $\nabla\rho_\sigma\in L^p(0,T;L^p(\mathbb{R}^d))$. We use the L^∞ bound for ρ_σ, the fractional Gagliardo–Nirenberg inequality (Lemma 23), and Young's inequality to find that

$$\|w\|_p \le C\|\nabla(-\Delta)^{-s}f_\sigma(\rho_\sigma)\|_p \le C\|f_\sigma(\rho_\sigma)\|_p^{2s}\|\nabla f_\sigma(\rho_\sigma)\|_p^{1-2s} \le C(\eta) + \eta\|\nabla\rho_\sigma\|_p,$$

where $\eta>0$ is arbitrary. By estimate (72) in Lemma 25,

$$\|\rho_\sigma\nabla(-\Delta)^{-s}f_\sigma(\rho_\sigma)\|_p = \|w\|_p \le C(\eta) + \eta\big(\|\rho_\sigma\nabla(-\Delta)^{-s}f_\sigma(\rho_\sigma)\|_p + T^{1/p}\|\nabla\rho^0\|_p\big).$$

Choosing $\eta > 0$ sufficiently small shows the claim.

Step 3: Estimate of $\operatorname{div} w$ *in* $L^p(0,T;L^p(\mathbb{R}^d))$. We use Hölder's inequality with $1/p = 2s/(d+p) + 1/q$ to obtain

$$\begin{aligned}\|\operatorname{div} w\|_p &\le \|\nabla\rho_\sigma\cdot\nabla(-\Delta)^{-s}f_\sigma(\rho_\sigma)\|_p + \|\rho_\sigma(-\Delta)^{1-s}f_\sigma(\rho_\sigma)\|_p \\ &\le \|\nabla\rho_\sigma\|_{(d+p)/(2s)}\|\nabla(-\Delta)^{-s}f_\sigma(\rho_\sigma)\|_q + C\|(-\Delta)^{1-s}f_\sigma(\rho_\sigma)\|_p.\end{aligned}$$

By the fractional Gagliardo–Nirenberg inequality (Lemma 24 with $\theta = 1 + d/p - d/q - 2s$ and Lemma 23 with s replaced by $1-s$) and Young's inequality, it follows that

$$\begin{aligned}\|\operatorname{div} w\|_p &\le C\|\nabla\rho_\sigma\|_{(d+p)/(2s)}\|f_\sigma(\rho_\sigma)\|_p^{1-\theta}\|\nabla f_\sigma(\rho_\sigma)\|_p^\theta + C\|f_\sigma(\rho_\sigma)\|_p^s\|\mathrm{D}^2 f_\sigma(\rho_\sigma)\|_p^{1-s} \\ &\le C\|\nabla\rho_\sigma\|_{(d+p)/(2s)}\|\nabla\rho_\sigma\|_p^\theta + C\|f_\sigma'(\rho_\sigma)\mathrm{D}^2\rho_\sigma + f_\sigma''(\rho_\sigma)\nabla\rho_\sigma\otimes\nabla\rho_\sigma\|_p^{1-s} \\ &\le C(\eta) + C\|\nabla\rho_\sigma\|_{(d+p)/(2s)}^{1/(1-\theta)} + C\|\nabla\rho_\sigma\|_p + C\|\nabla\rho_\sigma\|_{2p}^2 + \eta\|\mathrm{D}^2\rho_\sigma\|_p,\end{aligned}$$

where $\eta > 0$ is arbitrary. Taking the $L^p(0,T)$ norm of the previous inequality and observing that $p/(1-\theta) = (d+p)/(2s)$ (because of $\theta = d(1/p-1/q)+1-2s$), it follows that

$$\begin{aligned}\|\operatorname{div} w\|_{L^p(0,T;L^p(\mathbb{R}^d))} &\le C + C\|\nabla\rho_\sigma\|_{L^{(d+p)/(2s)}(0,T;L^{(d+p)/(2s)}(\mathbb{R}^d))}^{1/(1-\theta)} + C\|\nabla\rho_\sigma\|_{L^p(0,T;L^p(\mathbb{R}^d))} \\ &\quad + C\|\nabla\rho_\sigma\|_{L^{2p}(0,T;L^{2p}(\mathbb{R}^d))}^2 + \eta\|\mathrm{D}^2\rho_\sigma\|_{L^p(0,T;L^p(\mathbb{R}^d))}.\end{aligned}$$

Lemma 25 and Step 2 ($\nabla\rho_\sigma \in L^p(0,T;L^p(\mathbb{R}^d))$) show that

$$\|\partial_t\rho_\sigma\|_{L^p(0,T;L^p(\mathbb{R}^d))} + (1-C\eta)\|\mathrm{D}^2\rho_\sigma\|_{L^p(0,T;L^p(\mathbb{R}^d))} \le C.$$

Choosing $\eta > 0$ sufficiently small, this yields $\partial_t\rho_\sigma \in L^p(0,T;L^p(\mathbb{R}^d))$ and $\rho_\sigma \in L^p(0,T;W^{2,p}(\mathbb{R}^d))$. We deduce from Lemma 19, applied to $\nabla\rho_\sigma$, that $\nabla\rho_\sigma \in L^\infty(0,T;L^q(\mathbb{R}^d))$ for any $2 \le q < \infty$. (At this point, we need the restriction $q \ge 2$.)

Step 4: Higher-order regularity. To improve the regularity of ρ_σ, we differentiate (7) in space. Recall that $\partial_i = \partial/\partial x_i$, $i = 1,\ldots,d$. Then

$$\begin{aligned}\partial_t\partial_i\rho_\sigma - \sigma\Delta\partial_i\rho_\sigma &= \sum_{j=1}^d \partial_i\partial_j\big(\rho_\sigma\partial_j(-\Delta)^{-s}f_\sigma(\rho_\sigma)\big) = \sum_{j=1}^d \big(\partial_{ij}^2\rho_\sigma\partial_j(-\Delta)^s f_\sigma(\rho_\sigma) \\ &\quad + \partial_i\rho_\sigma\partial_{jj}^2(-\Delta)^{-s}f_\sigma(\rho_\sigma) + \partial_j\rho_\sigma\partial_{ij}^2(-\Delta)^{-s}f_\sigma(\rho_\sigma) + \rho_\sigma\partial_{ijj}^3(-\Delta)^{-s}f_\sigma(\rho_\sigma)\big).\end{aligned} \tag{22}$$

We estimate the right-hand side term by term. Let $0 < s \le 1/2$. First, by Hölder's inequality with $1/p = 1/q + 1/r$, $1 < p < q < \infty$, $\max\{2,p\} < r < \infty$ and the fractional Gagliardo–Nirenberg inequality (Lemma 23),

$$\begin{aligned}\|\partial_{ij}^2\rho_\sigma\partial_j(-\Delta)^s f_\sigma(\rho_\sigma)\|_{L^p(0,T;L^p(\mathbb{R}^d))}^p &\le \int_0^T \|\partial_{ij}^2\rho_\sigma\|_q^p\|\partial_j(-\Delta)^s f_\sigma(\rho_\sigma)\|_r^p dt \\ &\le C\int_0^T \|\partial_{ij}^2\rho_\sigma\|_q^p\|f_\sigma(\rho_\sigma)\|_r^{(1-2s)p}\|\nabla f_\sigma(\rho_\sigma)\|_r^{2sp}dt \\ &\le C\|f_\sigma(\rho_\sigma)\|_{L^\infty(0,T;L^r(\mathbb{R}^d))}^{(1-2s)p}\|\nabla f_\sigma(\rho_\sigma)\|_{L^\infty(0,L^r(\mathbb{R}^d))}^{2sp}\int_0^T\|\partial_{ij}^2\rho_\sigma\|_q^p dt \le C.\end{aligned}$$

The second and third term on the right-hand side of (22) can be treated in a similar way, observing that $\partial_{ij}^2(-\Delta)^{-s} = \partial_j(-\Delta)^{-s}\partial_i$. The last term is estimated according to

$$\begin{aligned}\|\rho_\sigma\partial_{ijj}^3(-\Delta)^{-s}f_\sigma(\rho_\sigma)\|_p &\le C\|\partial_{ijj}^3(-\Delta)^{-s}f_\sigma(\rho_\sigma)\|_p \le C\|\partial_{jj}^2 f_\sigma(\rho_\sigma)\|_p^{2s}\|\nabla\partial_{jj}^2 f_\sigma(\rho_\sigma)\|_p^{1-2s} \\ &\le C(\eta)\|\partial_{jj}^2 f_\sigma(\rho_\sigma)\|_p + \eta\|\nabla\partial_{jj}^2 f_\sigma(\rho_\sigma)\|_p,\end{aligned}$$

and the last expression can be absorbed by the corresponding estimate of $\Delta\partial_i\rho_\sigma$ from the left-hand side of (22). Then we deduce from Lemma 25 that $\partial_t\partial_i\rho_\sigma$, $\partial_{ijj}^3\rho_\sigma \in L^p(0,T;L^p(\mathbb{R}^d))$ for all $p>1$ and Lemma 19, applied to $\partial_{ij}^2\rho_\sigma$, yields $\partial_{ij}^2\rho_\sigma \in C^0([0,T];L^q(\mathbb{R}^d))$ for all $q\ge 2$.

Next, if $1/2 < s < 1$, we use the second inequality in Lemma 23 and argue similarly as before. This finishes the proof. □

Lemma 11. *Under the assumptions of Proposition 10, for every $q \ge 2$, there exists a constant $C = C(q) > 0$, depending on σ, such that*

$$\|\rho_\sigma\|_{C^0([0,T];W^{2,1}(\mathbb{R}^d)\cap W^{3,q}(\mathbb{R}^d))} \le C.$$

The embedding $W^{3,q}(\mathbb{R}^d) \hookrightarrow W^{2,\infty}(\mathbb{R}^d)$ for $q > d$ yields a bound for ρ_σ in $C^0([0,T];W^{2,\infty}(\mathbb{R}^d))$.

Proof. We first prove the bound in $C^0([0,T];W^{3,q}(\mathbb{R}^d))$. By differentiating (7) twice in space, estimating similarly as in Step 4 of the previous proof, and using the regularity results of Proposition 10, we can show that ρ_σ is bounded in $L^\infty(0,T;W^{3,q}(\mathbb{R}^d))$ for any $q\ge 2$.

It remains to show the $C^0([0,T];W^{2,1}(\mathbb{R}^d))$ bound for ρ_σ. In view of mass conservation and Gagliardo–Nirenberg–Sobolev's inequality, it suffices to show a bound for $\mathrm{D}^2\rho_\sigma$ in $L^\infty(0,T;L^1(\mathbb{R}^d))$. To this end, we define the weights $\gamma_n = (1+|x|^2)^{n/2}$ for $n\ge 0$ and test equation (7) for ρ_σ with $v_n := \gamma_n\rho_\sigma$. Then

$$\begin{aligned}&\partial_t v_n - \sigma\Delta v_n = \operatorname{div}\big(v_n\nabla\mathcal{K} * f_\sigma(\rho_\sigma)\big) + I_n, \quad v_n(0) = \gamma_n\rho_\sigma^0 \quad \text{in } \mathbb{R}^d, \\ &\text{where } I_n = -2\sigma\nabla\gamma_n\cdot\nabla\rho_\sigma - \sigma\rho_\sigma\Delta\gamma_n - \rho_\sigma\nabla\gamma_n\cdot\nabla\mathcal{K} * f_\sigma(\rho_\sigma).\end{aligned}$$

Arguing as in Step 4 of the previous proof, we can find a bound in $L^\infty(0,T;W^{2,p}(\mathbb{R}^d))$ for v_n. Indeed, we can proceed by induction over n, since the additional terms in I_n can be controlled by Sobolev norms of $v_0,\ldots,v_{n-1}$. The definition of ρ_σ^0 implies that $\gamma_n\rho_\sigma^0$, $\gamma_n\nabla\rho_\sigma^0 \in L^\infty(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$ for every $n\ge 0$. Then choosing $n > d$ yields, for $0\le t\le T$, that

$$\|\gamma_n\mathrm{D}^2\rho_\sigma\|_p \le \|\mathrm{D}^2(\gamma_n\rho_\sigma)\|_p + 2\|\nabla\gamma_n\cdot\nabla\rho_\sigma\|_p + \|\rho_\sigma\mathrm{D}^2\gamma_n\|_p \le C(T).$$

We conclude from $\gamma_n^{-1} \in L^\infty(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$ that

$$\|\mathrm{D}^2\rho_\sigma\|_1 \le \|\gamma_n^{-1}\|_{p/(p-1)}\|\gamma_n\mathrm{D}^2\rho_\sigma\|_p \le C(T).$$

This proves the desired bound. □

2.4. **Existence of solutions to** (7)**.** We show that the regularized equation (7) possesses a unique strong solution ρ_σ.

Step 1: Existence for an approximated system. Let $T > 0$ arbitrary, define the spaces

$$X_T := L^2(0,T;H^1(\mathbb{R}^d)) \cap H^1(0,T;H^{-1}(\mathbb{R}^d)) \hookrightarrow Y_T := C^0([0,T];L^2(\mathbb{R}^d)),$$
$$Y_{T,R} := \{u \in Y_T : \|u - \rho_\sigma^0\|_{L^\infty(0,T;L^2(\mathbb{R}^d))} \le R\},$$

and consider the mapping $S : v \in Y_T \mapsto u \in Y_T$,

$$\begin{aligned} &\partial_t u - \sigma\Delta u = \operatorname{div}(u\nabla\mathcal{K}_s^{(\delta)} * f_\sigma^{(\eta)}(v)) \quad \text{in } \mathbb{R}^d \times (0,T), \\ &u(0) = \rho_\sigma^0 \quad \text{in } \mathbb{R}^d, \end{aligned} \tag{23}$$

where $\mathcal{K}_s^{(\delta)} : \mathbb{R}^d \to \mathbb{R}_+$ is a regularized version of $\mathcal{K}_s$, defined by

$$\mathcal{K}_s^{(\delta)} = \widetilde{\mathcal{K}}_{s/2}^{(\delta)} * \widetilde{\mathcal{K}}_{s/2}^{(\delta)},$$
$$\widetilde{\mathcal{K}}_{s/2}^{(\delta)}(x) = c_{d,-s/2}\begin{cases} \delta^{s-d} + (s-d)\delta^{s-d-1}(|x|-\delta) & \text{for } |x| < \delta, \\ |x|^{s-d} & \text{for } \delta \le |x| \le \delta^{-1}, \\ [\delta^{d-s} + (s-d)\delta^{d+1-s}(|x|-\delta^{-1})]_+ & \text{for } |x| > \delta^{-1}, \end{cases}$$

and $f_\sigma^{(\eta)}$ is given by

$$f_\sigma^{(\eta)}(\rho) = \int_0^{|\rho|} f_\sigma'(u)\min(1,u\eta^{-1})du + \frac{\eta}{2}\rho^2, \quad \rho \in \mathbb{R}.$$

The regularization with parameter η is needed for the entropy estimates.

We derive some estimates for $f_\sigma^{(\eta)}$. First, we have $0 \le f_\sigma^{(\eta)}(\rho) \le C_\eta\rho^2$ for $\rho \in \mathbb{R}$, since

$$f_\sigma^{(\eta)}(\rho) \le \left(\eta + \eta^{-1}\max_{[0,\eta]} f_\sigma'\right)\frac{\rho^2}{2} \quad \text{for } |\rho| \le \eta,$$
$$f_\sigma^{(\eta)}(\rho) \le f_\sigma(|\rho|) + \frac{\eta}{2}\rho^2 \le \left(\|f_\sigma\|_\infty\eta^{-2} + \frac{\eta}{2}\right)\rho^2 \quad \text{for } |\rho| > \eta.$$

Furthermore,

$$|\mathrm{D}f_\sigma^{(\eta)}(\rho)| = \left|\frac{\rho}{|\rho|}f_\sigma'(|\rho|)\min(1,|\rho|\eta^{-1}) + \eta\rho\right| \le (\eta + \eta^{-1}\|f_\sigma'\|_\infty)|\rho|,$$

which implies that $|\mathrm{D}f_\sigma^{(\eta)}(\rho)| \le C_\eta|\rho|$ for $\rho \in \mathbb{R}$. This shows that there exists $C(\eta) > 0$ such that for any ρ_1, $\rho_2 \in \mathbb{R}$,

$$|f_\sigma^{(\eta)}(\rho_1) - f_\sigma^{(\eta)}(\rho_2)| \le C(\eta)(|\rho_1| + |\rho_2|)|\rho_1 - \rho_2|.$$

It follows that $f_\sigma^{(\eta)}(v) \in L^\infty(0,T;L^1(\mathbb{R}^d))$ for $v \in Y_T$.

Since $\nabla\mathcal{K}_s^{(\delta)} \in L^\infty(\mathbb{R}^d)$, a standard argument shows that (23) has a unique solution $u \in X_T \hookrightarrow Y_T$. Therefore, the mapping S is well-defined. Additionally, the nonnegativity of u follows immediately after by testing (23) with $\min(0,u)$.

We show now that S is a contraction on $Y_{T,R}$ for sufficiently small $T>0$. We start with a preparation. By testing (23) with u and taking into account the L^∞ bound for $\nabla\mathcal{K}_s^{(\delta)}$, we deduce from Young's inequality for products and convolutions that

$$\int_{\mathbb{R}^d} u(t)^2dx + \frac{\sigma}{2}\int_0^t\int_{\mathbb{R}^d}|\nabla u|^2dxd\tau \le \int_{\mathbb{R}^d}|\rho_\sigma^0|^2dx + C(\delta,\eta,\sigma)\int_0^t\|u\|_2^2\|v\|_2^4d\tau,$$

since $\|f_\sigma^{(\eta)}(v)\|_1\le C_\eta\|v\|_2^2$ for $v\in Y_T$. Then, if $v\in Y_{T,R}$, we infer from Gronwall's lemma that

$$\int_{\mathbb{R}^d} u(t)^2dx + \sigma\int_0^t\int_{\mathbb{R}^d}|\nabla u|^2dxd\tau \le e^{C(\sigma,\delta,\eta)R^4t}\int_{\mathbb{R}^d}|\rho_\sigma^0|^2dx \quad \text{for } 0\le t\le T. \tag{24}$$

Let $v_i\in Y_{T,R}$ and set $u_i=S(v_i)$, $i=1,2$. We compute

$$\begin{aligned}
&\|u_1\nabla\mathcal{K}_s^{(\delta)}*f_\sigma^{(\eta)}(v_1) - u_2\nabla\mathcal{K}_s^{(\delta)}*f_\sigma^{(\eta)}(v_2)\|_2 \\
&\quad\le \|(u_1-u_2)\nabla\mathcal{K}_s^{(\delta)}*f_\sigma^{(\eta)}(v_1)\|_2 + \|u_2\nabla\mathcal{K}_s^{(\delta)}*(f_\sigma^{(\eta)}(v_1)-f_\sigma^{(\eta)}(v_2))\|_2 \\
&\quad\le \|u_1-u_2\|_2\|\nabla\mathcal{K}_s^{(\delta)}*f_\sigma^{(\eta)}(v_1)\|_\infty + \|u_2\|_2\|\nabla\mathcal{K}_s^{(\delta)}*(f_\sigma^{(\eta)}(v_1)-f_\sigma^{(\eta)}(v_2))\|_\infty \\
&\quad\le \|u_1-u_2\|_2\|\nabla\mathcal{K}_s^{(\delta)}\|_\infty\|f_\sigma^{(\eta)}(v_1)\|_1 + \|u_2\|_2\|\nabla\mathcal{K}_s^{(\delta)}\|_\infty\|f_\sigma^{(\eta)}(v_1)-f_\sigma^{(\eta)}(v_2)\|_1 \\
&\quad\le C(\delta,\eta)\big(\|u_1-u_2\|_2\|v_1\|_2^2 + \|u_2\|_2(\|v_1\|_2+\|v_2\|_2)\|v_1-v_2\|_2\big).
\end{aligned}$$

Therefore, using (24), for $v_1,v_2\in Y_{T,R}$,

$$\|u_1\nabla\mathcal{K}_s^{(\delta)}*f_\sigma^{(\eta)}(v_1) - u_2\nabla\mathcal{K}_s^{(\delta)}*f_\sigma^{(\eta)}(v_2)\|_2 \le C(\delta,\eta,R,T)(\|u_1-u_2\|_2+\|v_1-v_2\|_2). \tag{25}$$

Next, we write (23) for (u_i,v_i) in place of (u,v), $i=1,2$, take the difference between the two equations, and test the resulting equation with u_1-u_2:

$$\begin{aligned}
&\frac12\|u_1-u_2\|_2^2(t) + \sigma\int_0^t\int_{\mathbb{R}^d}|\nabla(u_1-u_2)|^2dxd\tau \\
&\quad= -\int_0^t\int_{\mathbb{R}^d}\nabla(u_1-u_2)\cdot(u_1\nabla\mathcal{K}_s^{(\delta)}*f_\sigma^{(\eta)}(v_1) - u_2\nabla\mathcal{K}_s^{(\delta)}*f_\sigma^{(\eta)}(v_2))dxd\tau \\
&\quad\le \frac{\sigma}{2}\int_0^t\int_{\mathbb{R}^d}|\nabla(u_1-u_2)|^2dxd\tau + \frac{1}{2\sigma}\int_0^t\|u_1\nabla\mathcal{K}_s^{(\delta)}*f_\sigma^{(\eta)}(v_1) - u_2\nabla\mathcal{K}_s^{(\delta)}*f_\sigma^{(\eta)}(v_2)\|_2^2d\tau.
\end{aligned}$$

It follows from (25) that

$$\|u_1-u_2\|_2^2(t) + \sigma\int_0^t\int_{\mathbb{R}^d}|\nabla(u_1-u_2)|^2dxd\tau \le C(\delta,\eta,R,T,\sigma)\int_0^t(\|u_1-u_2\|_2^2+\|v_1-v_2\|_2^2)d\tau,$$

and we conclude from Gronwall's lemma that

$$\|u_1-u_2\|_2^2(t) \le e^{C(\delta,\eta,R,T,\sigma)t}\int_0^T\|v_1-v_2\|_2^2d\tau \quad \text{for } 0\le t\le T.$$

This inequality implies that S is a contraction in $Y_{T,R}$, provided that T is sufficiently small. Therefore, by Banach's theorem, S admits a unique fixed point $u\in Y_{T,R}\subset Y_T$ for $T>0$ sufficiently small.

It remains to show that the local solution can be extended to a global one. To this end, we note that the function $u \in X_T$ satisfies (23) with $v = u$:

$$\begin{aligned} &\partial_t u - \sigma\Delta u = \operatorname{div}(u\nabla\mathcal{K}_s^{(\delta)} * f_\sigma^{(\eta)}(u)) \quad \text{in } \mathbb{R}^d \times (0,T), \\ &u(\cdot,0) = \rho_\sigma^0 \quad \text{in } \mathbb{R}^d. \end{aligned} \tag{26}$$

Then, defining the truncated entropy density

$$h^{(\eta)}(\rho) = \int_0^\rho \int_0^u \mathrm{D}f_\sigma^{(\eta)}(v)v^{-1}dvdu, \quad \rho \geq 0,$$

and testing (26) with $\mathrm{D}h^{(\eta)}(u)$ yields, in view of the definition of $\mathcal{K}_s^{(\delta)}$, that

$$\begin{aligned} \int_{\mathbb{R}^d} h^{(\eta)}(u(t))dx &+ \sigma\int_0^t\int_{\mathbb{R}^d} \mathrm{D}f_\sigma^{(\eta)}(u)u^{-1}|\nabla u|^2 dxd\tau \\ &+ \int_0^t\int_{\mathbb{R}^d} |\nabla\widetilde{\mathcal{K}}_{s/2}^{(\delta)} * f_\sigma^{(\eta)}(u)|^2 dxd\tau = \int_{\mathbb{R}^d} h^{(\eta)}(\rho_\sigma^0)dx \end{aligned} \tag{27}$$

for $0 \leq t \leq T$. This inequality and the definitions of $f_\sigma^{(\eta)}$ and $h^{(\eta)}$ yield a (δ,T)-uniform bound for u in $L^2(0,T;H^1(\mathbb{R}^d))$, which in turn (together with (26)) implies a (δ,T)-uniform bound for u in X_T, and a fortiori in Y_T. This means that the solution u can be prolonged to the whole time interval $[0,\infty)$ and exists for all times.

Finally, we point out that, since $\nabla\mathcal{K}_s^{(\delta)} \in L^2(\mathbb{R}^d)$, then $\nabla\mathcal{K}_s^{(\delta)} * f_\sigma^{(\eta)}(u) \in L^\infty(0,T;L^2(\mathbb{R}^d))$ and so $u\nabla\mathcal{K}_s^{(\delta)} * f_\sigma^{(\eta)}(u) \in L^\infty(0,T;L^1(\mathbb{R}^d))$. This fact yields the conservation of mass for u, i.e. $\int_{\mathbb{R}^d} u(t)dx = \int_{\mathbb{R}^d} \rho_\sigma^0 dx$ for $t > 0$. Indeed, it is sufficient to test (26) with a cutoff $\psi_R \in C_0^1(\mathbb{R}^d)$ satisfying $\psi_R(x) = 1$ for $|x| < R$, $\psi_R(x) = 0$ for $|x| > 2R$, $|\nabla\psi_R(x)| \leq CR^{-1}$ for $x \in \mathbb{R}^d$, and then to take the limit $R \to \infty$.

Step 2: Limit $\delta \to 0$. Let $u^{(\delta)}$ be the solution to (26). An adaption of the proof of [5, Lemma 1] shows that the embedding $H^1(\mathbb{R}^d) \cap L^1(\mathbb{R}^d;(1+|x|^2)^{\kappa/2}) \hookrightarrow L^2(\mathbb{R}^d)$ is compact. Thus, because of the δ-uniform bounds for $u^{(\delta)}$, the Aubin–Lions Lemma implies that (up to a subsequence) $u^{(\delta)} \to u$ strongly in $L^2(0,T;L^2(\mathbb{R}^d))$ for every $T > 0$. We wish now to study the convergence of the nonlinear and nonlocal terms in (26)–(27) as $\delta \to 0$.

It follows from (27) that (up to a subsequence)

$$\nabla\widetilde{\mathcal{K}}_{s/2}^{(\delta)} * f_\sigma^{(\eta)}(u^{(\delta)}) \rightharpoonup U \quad \text{weakly in } L^2(\mathbb{R}^d \times (0,T)) \quad \text{as } \delta \to 0. \tag{28}$$

In order to identify the limit U, we first notice that, by construction, $0 \leq \widetilde{\mathcal{K}}_{s/2}^{(\delta)} \nearrow \mathcal{K}_{s/2}$ a.e. in $\mathbb{R}^d$. Furthermore, the Hardy–Littlewood–Sobolev inequality, the bound for $f_\sigma^{(\eta)}$, and then the Gagliardo-Nirenberg-Sobolev inequality yield that

$$\begin{aligned} \|\mathcal{K}_{s/2} * f_\sigma^{(\eta)}(u)\|_{(d+2)/(d-s)} &\leq C\|f_\sigma^{(\eta)}(u)\|_{(d+2)/(d+2s/d)} \leq C(\eta)\|u\|_{(2d+4)/(d+2s/d)}^2 \\ &\leq C(\eta)\|u\|_2^{2(s+2)/(d+2)}\|\nabla u\|_2^{2(d-s)/(d+2)}. \end{aligned}$$

Therefore, since $u \in L^\infty(0,T;L^2(\mathbb{R}^d)) \cap L^2(0,T;H^1(\mathbb{R}^d))$,

$$\int_0^T \|\mathcal{K}_{s/2} * f_\sigma^{(\eta)}(u)\|_{(d+2)/(d-s)}^{(d+2)/(d-s)} dt \le C(\eta)\|u\|_{L^\infty(0,T;L^2(\mathbb{R}^d))}^{2(s+2)/(d-s)} \int_0^T \|\nabla u\|_2^2 dt \le C(\eta,T),$$

meaning that $\mathcal{K}_{s/2} * f_\sigma^{(\eta)}(u) \in L^{(d+2)/(d-s)}(\mathbb{R}^d \times (0,T))$. Taking into account that $f_\sigma^{(\eta)}(u) \ge 0$, we deduce from monotone convergence that

$$\widetilde{\mathcal{K}}_{s/2}^{(\delta)} * f_\sigma^{(\eta)}(u) \to \mathcal{K}_{s/2} * f_\sigma^{(\eta)}(u) \quad \text{strongly in } L^{(d+2)/(d-s)}(\mathbb{R}^d \times (0,T)). \tag{29}$$

Furthermore, arguing as before and using the estimates for $\mathrm{D} f_\sigma^{(\eta)}$ leads to

$$\begin{aligned} \|\widetilde{\mathcal{K}}_{s/2}^{(\delta)} * (f_\sigma^{(\eta)}(u^{(\delta)}) - f_\sigma^{(\eta)}(u))\|_{(d+2)/(d-s)} &\le \|\widetilde{\mathcal{K}}_{s/2} * |f_\sigma^{(\eta)}(u^{(\delta)}) - f_\sigma^{(\eta)}(u)|\|_{(d+2)/(d-s)} \\ &\le C\|f_\sigma^{(\eta)}(u^{(\delta)}) - f_\sigma^{(\eta)}(u)\|_{(d+2)/(d+2s/d)} \\ &\le C(\eta)\big\||u| + |u^{(\delta)}|\big\|_{(2d+4)/(d+2s/d)} \|u - u^{(\delta)}\|_{(2d+4)/(d+2s/d)} \\ &\le C(\eta)\big(\|u\|_2^{(s+2)/(d+2)}\|\nabla u\|_2^{(d-s)/(d+2)} + \|u^{(\delta)}\|_2^{(s+2)/(d+2)}\|\nabla u^{(\delta)}\|_2^{(d-s)/(d+2)}\big) \\ &\quad \times \|u - u^{(\delta)}\|_2^{(s+2)/(d+2)}\|\nabla(u - u^{(\delta)})\|_2^{(d-s)/(d+2)}. \end{aligned}$$

Since $u^{(\delta)}$ is bounded in $L^\infty(0,T;L^2(\mathbb{R}^d)) \cap L^2(0,T;H^1(\mathbb{R}^d))$, it follows that (up to a subsequence) $\widetilde{\mathcal{K}}_{s/2}^{(\delta)} * (f_\sigma^{(\eta)}(u^{(\delta)}) - f_\sigma^{(\eta)}(u))$ converges weakly to some limit in $L^{(d+2)/(d-s)}(\mathbb{R}^d \times (0,T))$. However, Hölder's inequality and the fact that $u^{(\delta)} \to u$ strongly in $L^p(0,T;L^2(\mathbb{R}^d))$ for every $2 \le p < \infty$, which follows from

$$\int_0^T \|u^{(\delta)} - u\|_2^p dt \le \sup_{0<t<T} \|(u^{(\delta)} - u)(t)\|_2^{p-2} \int_0^T \|u^{(\delta)} - u\|_2^2 dt \to 0 \quad \text{as } \delta \to 0,$$

imply that

$$\widetilde{\mathcal{K}}_{s/2}^{(\delta)} * (f_\sigma^{(\eta)}(u^{(\delta)}) - f_\sigma^{(\eta)}(u)) \to 0 \quad \text{strongly in } L^p(0,T;L^{(d+2)/(d-s)}(\mathbb{R}^d)),\ p < \frac{d+2}{d-s}.$$

We conclude that

$$\widetilde{\mathcal{K}}_{s/2}^{(\delta)} * (f_\sigma^{(\eta)}(u^{(\delta)}) - f_\sigma^{(\eta)}(u)) \rightharpoonup 0 \quad \text{weakly in } L^{(d+2)/(d-s)}(\mathbb{R}^d \times (0,T)). \tag{30}$$

We deduce from (29)–(30) that

$$\begin{aligned} &\widetilde{\mathcal{K}}_{s/2}^{(\delta)} * f_\sigma^{(\eta)}(u^{(\delta)}) - \mathcal{K}_{s/2} * f_\sigma^{(\eta)}(u) \\ &\qquad = \big(\widetilde{\mathcal{K}}_{s/2}^{(\delta)} * f_\sigma^{(\eta)}(u) - \mathcal{K}_{s/2} * f_\sigma^{(\eta)}(u)\big) + \widetilde{\mathcal{K}}_{s/2}^{(\delta)} * (f_\sigma^{(\eta)}(u^{(\delta)}) - f_\sigma^{(\eta)}(u)) \\ &\qquad \rightharpoonup 0 \quad \text{weakly in } L^{(d+2)/(d-s)}(\mathbb{R}^d \times (0,T)), \end{aligned}$$

which, together with (28), implies that $U = \nabla \mathcal{K}_{s/2} * f_\sigma^{(\eta)}(u)$, that is,

$$\nabla \widetilde{\mathcal{K}}_{s/2}^{(\delta)} * f_\sigma^{(\eta)}(u^{(\delta)}) \rightharpoonup \nabla \mathcal{K}_{s/2} * f_\sigma^{(\eta)}(u) \quad \text{weakly in } L^2(\mathbb{R}^d \times (0,T)). \tag{31}$$

Let $\psi \in C_0^\infty(\mathbb{R}^d \times (0,T))$. Because of

$$\nabla \mathcal{K}_s^{(\delta)} * f_\sigma^{(\eta)}(u^{(\delta)}) = \widetilde{\mathcal{K}}_{s/2}^{(\delta)} * \big(\nabla \widetilde{\mathcal{K}}_{s/2}^{(\delta)} * f_\sigma^{(\eta)}(u^{(\delta)})\big),$$

we find that

$$\int_0^T\int_{\mathbb{R}^d}\psi\cdot\nabla\mathcal{K}_s^{(\delta)}*f_\sigma^{(\eta)}(u^{(\delta)})dxdt=\int_0^T\int_{\mathbb{R}^d}\big(\nabla\widetilde{\mathcal{K}}_{s/2}^{(\delta)}*f_\sigma^{(\eta)}(u^{(\delta)})\big)\cdot\big(\widetilde{\mathcal{K}}_{s/2}^{(\delta)}*\psi\big)dxdt.$$

Our goal is to show that $\widetilde{\mathcal{K}}_{s/2}^{(\delta)}*\psi\to\mathcal{K}_{s/2}*\psi$ strongly in $L^2(\mathbb{R}^d\times(0,T))$ as $\delta\to 0$. We can assume without loss of generality that $\psi\geq 0$ a.e. in $\mathbb{R}^d\times(0,T)$. Indeed, for general functions ψ, we may write $\psi=\psi_++\psi_-$, where $\psi_+=\max\{0,\psi\}$ and $\psi_-=\min\{0,\psi\}$, and we have $\widetilde{\mathcal{K}}_{s/2}^{(\delta)}*\psi=\widetilde{\mathcal{K}}_{s/2}^{(\delta)}*\psi_+-\widetilde{\mathcal{K}}_{s/2}^{(\delta)}*(-\psi_-)$. Once again, since $\widetilde{\mathcal{K}}_{s/2}^{(\delta)}\nearrow\widetilde{\mathcal{K}}_{s/2}$ a.e. in $\mathbb{R}^d$, it is sufficient to show that $\mathcal{K}_{s/2}*\psi\in L^2(\mathbb{R}^d\times(0,T))$. The Hardy–Littlewood–Sobolev inequality (see Appendix B) yields

$$\int_0^T\|\mathcal{K}_{s/2}*\psi\|_2^2dt\leq C\int_0^T\|\psi\|_{2d/(d+2s)}^2dt.$$

It follows from (31), the previous argument, and the fact that $\mathcal{K}_s*u=(-\Delta)^{-s}u=\mathcal{K}_{s/2}*\mathcal{K}_{s/2}*u$ that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d}\psi\cdot\nabla\mathcal{K}_s^{(\delta)}*f_\sigma^{(\eta)}(u^{(\delta)})dxdt&\to\int_0^T\int_{\mathbb{R}^d}\big(\nabla\mathcal{K}_{s/2}*f_\sigma^{(\eta)}(u)\big)\cdot(\mathcal{K}_{s/2}*\psi)dxdt\\&=\int_0^T\int_{\mathbb{R}^d}\psi\cdot\nabla\mathcal{K}_s*f_\sigma^{(\eta)}(u)dxdt\end{aligned}$$

for every $\psi\in L^2(0,T;L^{2d/(d+2s)}(\mathbb{R}^d))$, which means that

$$\nabla\mathcal{K}_s^{(\delta)}*f_\sigma^{(\eta)}(u^{(\delta)})\rightharpoonup\nabla\mathcal{K}_s*f_\sigma^{(\eta)}(u)\quad\text{weakly in }L^2(0,T;L^{2d/(d-2s)}(\mathbb{R}^d)).\tag{32}$$

Since $u^{(\delta)}\to u$ strongly in $L^2(0,T;L^2(\mathbb{R}^d))$ and $(u^{(\delta)})$ is bounded in $L^\infty(0,T;L^1(\mathbb{R}^d))$ (via mass conservation), it also holds that $u^{(\delta)}\to u$ strongly in $L^2(0,T;L^{2d/(d+2s)}(\mathbb{R}^d))$. Therefore, the convergence (32) is sufficient to pass to the limit $\delta\to 0$ in (26).

Step 3: Limit $\eta\to 0$ and conclusion. The limit $\delta\to 0$ in (26) shows that the limit u solves

$$\begin{aligned}&\partial_t u-\sigma\Delta u=\operatorname{div}(u\nabla\mathcal{K}_s*f_\sigma^{(\eta)}(u))\quad\text{in }\mathbb{R}^d\times(0,T),\\&u(\cdot,0)=\rho_\sigma^0\quad\text{in }\mathbb{R}^d.\end{aligned}\tag{33}$$

Fatou's Lemma and the weakly lower semicontinuity of the L^2 norm allow us to infer from (27) that for $t>0$,

$$\begin{aligned}\int_{\mathbb{R}^d}h^{(\eta)}(u(t))dx&+\sigma\int_0^t\int_{\mathbb{R}^d}\mathrm{D}f_\sigma^{(\eta)}(u)u^{-1}|\nabla u|^2dxd\tau\\&+\int_0^t\int_{\mathbb{R}^d}|\nabla\mathcal{K}_{s/2}*f_\sigma^{(\eta)}(u)|^2dxd\tau\leq\int_{\mathbb{R}^d}h^{(\eta)}(\rho_\sigma^0)dx.\end{aligned}\tag{34}$$

At this point, all the bounds for u, derived in the previous subsections, and the moment estimate, contained in Lemma 9, can be proved like in Sections 2.1–2.2. All these estimates are uniform in η. It is rather straightforward to perform the limit $\eta\to 0$ in (33)–(34) to obtain a weak solution to (7). However, the higher regularity bounds obtained in Section

2.3 imply that u is actually a strong solution to (7), which in turn yields the uniqueness of u as a weak solution to (7). This finishes the proof of Theorem 4.

2.5. **Limit $\sigma \to 0$.** We prove that there exists a subsequence of (ρ_σ) that converges strongly in $L^1(\mathbb{R}^d \times (0,T))$ to a weak solution ρ to (1).

The uniform $L^\infty(\mathbb{R}^d \times (0,T))$ bound for ρ_σ in Lemma 7 implies that, up to a subsequence, $\rho_\sigma \rightharpoonup^* \rho$ weakly* in $L^\infty(\mathbb{R}^d \times (0,T))$ as $\sigma \to 0$. We deduce from the uniform $L^\infty(0,T;L^1(\mathbb{R}^d))$ bound (13) and the moment bound for ρ_σ in Lemma 9 that (ρ_σ) is equi-integrable. Thus, by the Dunford–Pettis theorem, again up to a subsequence, $\rho_\sigma \rightharpoonup \rho$ weakly in $L^1(\mathbb{R}^d \times (0,T))$. It follows from the $L^2(0,T;H^1(\mathbb{R}^d))$ estimate (16) that $\sigma\Delta\rho_\sigma \to 0$ strongly in $L^2(0,T;H^{-1}(\mathbb{R}^d))$. The estimates in (17) and Lemma 7 show that $(\partial_t\rho_\sigma)$ is bounded in $L^2(0,T;H^{-1}(\mathbb{R}^d))$ and consequently, up to a subsequence, $\partial_t\rho_\sigma \rightharpoonup \partial_t\rho$ weakly in $L^2(0,T;H^{-1}(\mathbb{R}^d))$. Therefore, the limit $\sigma \to 0$ in (7) leads to

$$\partial_t\rho = \operatorname{div}\overline{(\rho_\sigma\nabla(-\Delta)^{-s}f_\sigma(\rho_\sigma))} \quad \text{in } L^2(0,T;H^{-1}(\mathbb{R}^d)), \tag{35}$$

where the overline denotes the weak limit of the corresponding sequence.

We need to identify the weak limit on the right-hand side. The idea is to use the div-curl lemma [20, Theorem 10.21]. For this, we define the vector fields with $d+1$ components

$$U_\sigma := \big(\rho_\sigma, -\rho_\sigma\nabla(-\Delta)^{-s}f_\sigma(\rho_\sigma)\big), \quad V_\sigma := \big(f_\sigma(\rho_\sigma), 0, \ldots, 0\big).$$

Let $R > 0$ be arbitrary and write B_R for the ball around the origin with radius R. The $L^\infty(\mathbb{R}^d)$ bound (15) for ρ_σ and the $L^2(0,T;H^{1-s}(\mathbb{R}^d))$ bound (17) for $f_\sigma(\rho_\sigma)$ show that (U_σ) is bounded in $L^p(B_R \times (0,T))$ for some $p > 1$, while (V_σ) is bounded in $L^\infty(B_R \times (0,T))$. Furthermore, by (17),

$$\begin{aligned}
&\operatorname{div}_{(t,x)} U_\sigma = \sigma\Delta\rho_\sigma \to 0 \quad \text{strongly in } L^2(0,T;H^{-1}(B_R)) \hookrightarrow H^{-1}(B_R \times (0,T)), \\
&\|\operatorname{curl}_{(t,x)} V_\sigma\|_{L^2(0,T;H^{-s}(B_R))} \le C\|\nabla f_\sigma(\rho_\sigma)\|_{L^2(0,T;H^{-s}(B_R))} \le C,
\end{aligned}$$

where $\operatorname{curl}_{(t,x)} V_\sigma$ is the antisymmetric part of the Jacobian matrix of V_σ. Hence, by the compact embedding $H^{-s}(B_R \times (0,T)) \hookrightarrow W^{-1,r}(B_R \times (0,T))$ (since $L^2(0,T;H^{-s}(B_R)) \subset H^{-s}(B_R \times (0,T))$), the sequence $(\operatorname{curl}_{(t,x)} V_\sigma)$ is relatively compact in $W^{-1,r}(B_R \times (0,T))$ for some $r > 1$. Therefore, we can apply the div-curl lemma giving $\overline{U_\sigma \cdot V_\sigma} = \overline{U_\sigma} \cdot \overline{V_\sigma}$ or

$$\overline{\rho_\sigma f_\sigma(\rho_\sigma)} = \rho\overline{f_\sigma(\rho_\sigma)} \quad \text{a.e. in } B_R \times (0,T).$$

By definition (9) of $f_\sigma(\rho_\sigma)$, it follows for arbitrary $\rho_\sigma \in [0,L]$ and sufficiently large $L > 0$, that

$$\begin{aligned}
f_\sigma(\rho_\sigma) &= \int_0^{\rho_\sigma} (\Gamma_\sigma * (f'1_{[0,\infty)}))(u)\widetilde{\Xi}(\sigma u)du = \int_0^{\rho_\sigma}\int_0^\infty \Gamma_\sigma(u-w)f'(w)dw\widetilde{\Xi}(\sigma u)du \\
&= \int_0^{\rho_\sigma}\int_0^\infty \Gamma_\sigma'(u-w)f(w)dw\widetilde{\Xi}(\sigma u)du = \int_0^\infty \bigg(\int_0^{\rho_\sigma} \Gamma_\sigma'(u-w)\widetilde{\Xi}(\sigma u)du\bigg)f(w)dw.
\end{aligned}$$

We use the properties that (ρ_σ) is uniformly bounded and $\widetilde{\Xi} = 1$ in $[-1,1]$. Then, choosing $\sigma > 0$ sufficiently small,

$$\begin{aligned} f_\sigma(\rho_\sigma) &= \int_0^\infty \bigg(\int_0^{\rho_\sigma} \Gamma'_\sigma(u-w)du \bigg) f(w)dw \\ &= \int_0^\infty \Gamma_\sigma(\rho_\sigma - w) f(w)dw - \int_0^\infty \Gamma_\sigma(-w)f(w)dw \\ &= \int_{\mathbb{R}} \Gamma_\sigma(\rho_\sigma - w)\tilde{f}(w)dw - \int_{\mathbb{R}} \Gamma_\sigma(-w)\tilde{f}(w)dw, \end{aligned}$$

setting $\tilde{f} := f 1_{[0,\infty)}$. Hence, using $f(0) = 0$, we find that

$$f_\sigma(\rho_\sigma) - f(\rho_\sigma) = \int_{\mathbb{R}} \Gamma_\sigma(u)(\tilde{f}(u+\rho_\sigma) - \tilde{f}(\rho_\sigma))du - \int_{\mathbb{R}} \Gamma_\sigma(-w)(\tilde{f}(w) - \tilde{f}(0))dw.$$

Taking into account the fundamental theorem of calculus for the function $\tilde{f} \in C^0 \cap W^{1,1}(\mathbb{R})$, we can estimate as follows:

$$\begin{aligned} |f_\sigma(\rho_\sigma) - f(\rho_\sigma)| &\le \operatorname*{ess\,sup}_{u \in \operatorname{supp}(\Gamma_\sigma)\setminus\{0\}} \bigg(\frac{|\tilde{f}(u+\rho_\sigma) - \tilde{f}(\rho_\sigma)|}{|u|} + \frac{|\tilde{f}(u) - \tilde{f}(0)|}{|u|} \bigg) \int_{\mathbb{R}} \Gamma_\sigma(w)|w|dw \\ &\le \bigg(\max_{\xi \in \operatorname{supp}(\Gamma_\sigma)\cap[0,\infty)} (f'(\xi+\rho_\sigma) + f'(\xi)) \bigg) \int_{\mathbb{R}} \Gamma_\sigma(w)|w|dw. \end{aligned}$$

Then, since $\Gamma_\sigma(u) = \sigma^{-1}\Gamma_1(\sigma^{-1}u)$, $\operatorname{supp}(\Gamma_\sigma) \subset B_\sigma(0)$ is compact, $f \in C^1([0,\infty))$, and (ρ_σ) is uniformly bounded, we conclude that

$$|f_\sigma(\rho_\sigma) - f(\rho_\sigma)| \le C\sigma.$$

This means that $f_\sigma(\rho_\sigma) - f(\rho_\sigma) \to 0$ strongly in $L^\infty(B_R \times (0,T))$, and it shows that $\overline{\rho_\sigma f(\rho_\sigma)} = \rho\overline{f(\rho_\sigma)}$ a.e. in $B_R \times (0,T)$. As f is nondecreasing, we can apply [20, Theorem 10.19] to infer that $\overline{f(\rho_\sigma)} = f(\rho)$ a.e. in $B_R \times (0,T)$. Consequently, $\overline{\rho_\sigma f(\rho_\sigma)} = \rho f(\rho)$. As $u \mapsto uf(u)$ is assumed to be strictly convex, we conclude from [20, Theorem 10.20] that (ρ_σ) converges a.e. in $B_R \times (0,T)$. Since (ρ_σ) is bounded in $L^\infty(\mathbb{R}^d \times (0,T))$, it follows that $\rho_\sigma \to \rho$ strongly in $L^p(B_R \times (0,T))$ for all $p < \infty$. Using the moment estimate from Lemma 9, we infer from

$$\begin{aligned} \limsup_{\sigma\to 0} \int_0^T \int_{\mathbb{R}^d} |\rho_\sigma - \rho| dxdt &= \limsup_{\sigma\to 0} \int_0^T \int_{\mathbb{R}^d\setminus B_R} |\rho_\sigma - \rho| dxdt \\ &\le R^{-2d/(d-2s)} \limsup_{\sigma\to 0} \int_0^T \int_{\mathbb{R}^d\setminus B_R} \rho_\sigma(t,x)|x|^{2d/(d-2s)} dx \\ &\le R^{-2d/(d-2s)} C \to 0 \quad \text{as } R \to \infty \end{aligned}$$

that $\rho_\sigma \to \rho$ strongly in $L^p(\mathbb{R}^d \times (0,T))$ for all $p < \infty$. The strong convergences of ρ_σ and $f_\sigma(\rho_\sigma)$ in $L^p(\mathbb{R}^d \times (0,T))$ for all $p < \infty$ allow us to identify the weak limit in (35), proving the weak formulation (8).

Finally, we deduce from the uniform $L^2(0,T;H^{-1}(\mathbb{R}^d))$ bound for $\partial_t\rho_\sigma$ and the fact that $\rho_\sigma\to\rho$ strongly in $L^p(\mathbb{R}^d)$ for any $p<\infty$ that $\rho(0)=\rho^0$ in the sense of $H^{-1}(\mathbb{R}^d)$. Properties (iv) of Theorem 1 follow from the corresponding expressions satisfied by ρ_σ in the limit $\sigma\to 0$.

The following lemma is needed in the proof of Theorem 3.

Corollary 12. *Under the assumptions of Theorem 1, it holds for all $\phi\in L^\infty(\mathbb{R}^d)$ that, possibly for a subsequence,*

$$\int_{\mathbb{R}^d}\rho_\sigma\phi dx\to\int_{\mathbb{R}^d}\rho\phi dx \quad \textit{uniformly in } [0,T].$$

Proof. Let $\phi\in C_0^1(\mathbb{R}^d)$ and $0\le t_1<t_2\le T$. The uniform $L^2(0,T;H^{-1}(\mathbb{R}^d))$ bound of $\partial_t\rho_\sigma$ implies that

$$\begin{aligned}\bigg|\int_{\mathbb{R}^d}\rho_\sigma(t_2)\phi dx-\int_{\mathbb{R}^d}\rho_\sigma(t_1)\phi dx\bigg| &= \bigg|\int_{t_1}^{t_2}\langle\partial_t\rho_\sigma,\phi\rangle dt\bigg| \\ &\le |t_2-t_1|^{1/2}\|\partial_t\rho_\sigma\|_{L^2(0,T;H^{-1}(\mathbb{R}^d))}\|\phi\|_{H^1(\mathbb{R}^d)}\le C|t_2-t_1|^{1/2}\|\phi\|_{H^1(\mathbb{R}^d)}.\end{aligned}$$

Hence, the sequence of functions $t\mapsto\int_{\mathbb{R}^d}\rho_\sigma(t)\phi ds$ is bounded and equicontinuous in $[0,T]$. By the Ascoli–Arzelá theorem, up to a ϕ-depending subsequence, $\int_{\mathbb{R}^d}\rho_\sigma\phi dx\to\xi_\phi$ strongly in $C^0([0,T])$ as $\sigma\to 0$. Since $\rho_\sigma\rightharpoonup^*\rho$ weakly* in $L^\infty(0,T;L^\infty(\mathbb{R}^d))$, we can identify the limit, $\xi_\phi=\int_{\mathbb{R}^d}\rho\phi dx$. Since $H^1(\mathbb{R}^d)$ is separable, a Cantor diagonal argument together with a density argument allows us to find a subsequence (which is not relabeled) such that for all $\phi\in H^1(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}\rho_\sigma\phi dx\to\int_{\mathbb{R}^d}\rho\phi dx \quad \text{strongly in } C^0([0,T]). \tag{36}$$

Since (ρ_σ) is bounded in $L^\infty(0,T;L^2(\mathbb{R}^d))$, another density argument shows that this limit also holds for all $\phi\in L^2(\mathbb{R}^d)$.

Now, let $\phi\in L^\infty(\mathbb{R}^d)$. Using $\phi 1_{\{|x|<R\}}\in L^2(\mathbb{R}^d)$, it follows from (36) and the moment estimate for ρ_σ that

$$\begin{aligned}\limsup_{\sigma\to 0}\sup_{0<t<T}&\bigg|\int_{\mathbb{R}^d}\rho_\sigma(t)\phi dx-\int_{\mathbb{R}^d}\rho(t)\phi dx\bigg| \\ &\le\limsup_{\sigma\to 0}\sup_{0<t<T}\bigg|\int_{\mathbb{R}^d}\rho_\sigma(t)\phi 1_{\{|x|>R\}}dx-\int_{\mathbb{R}^d}\rho(t)\phi 1_{\{|x|>R\}}dx\bigg| \\ &\le R^{-2d/(d-2s)}\|\phi\|_\infty\limsup_{\sigma\to 0}\sup_{0<t<T}\int_{\mathbb{R}^d}(\rho_\sigma(x,t)+\rho(x,t))|x|^{2d/(d-2s)}dx \\ &\le C(T)R^{-2d/(d-2s)}\|\phi\|_\infty\to 0 \quad \text{as } R\to\infty.\end{aligned}$$

This shows that

$$\lim_{\sigma\to 0}\sup_{0<t<T}\bigg|\int_{\mathbb{R}^d}\rho_\sigma(t)\phi dx-\int_{\mathbb{R}^d}\rho(t)\phi dx\bigg|=0,$$

concluding the proof. □

3. Analysis of equation (5)

This section is devoted to the analysis of equation (5),

$$\begin{aligned}&\partial_t\rho_{\sigma,\beta,\zeta} - \sigma\Delta\rho_{\sigma,\beta,\zeta} = \operatorname{div}\big(\rho_{\sigma,\beta,\zeta}\nabla\mathcal{K}_\zeta * f_\sigma(W_\beta * \rho_{\sigma,\beta,\zeta})\big), \quad t>0,\\ &\rho_{\sigma,\beta,\zeta}(0) = \rho_\sigma^0 \quad \text{in } \mathbb{R}^d,\end{aligned} \tag{37}$$

where $\mathcal{K}_\zeta = \widetilde{\mathcal{K}}_\zeta * W_\zeta$ and W_β is defined in (10), as well as to an estimate for the difference $\rho_{\sigma,\beta,\zeta} - \rho_\sigma$, which is needed in the mean-field analysis. The existence and uniqueness of a strong solution to (37) follows from standard parabolic theory, since we regularized the singular kernel and smoothed the nonlinearity.

Proposition 13 (Uniform estimates)**.** *Let Hypotheses (H1)–(H3) hold and let $T>0$, $p>d$. Set $a := \min\{1, d-2s\}$, let ρ_σ be the strong solution to* (7)*, and let $\rho_{\sigma,\beta,\zeta}$ be the strong solution to* (5)*. Then there exist constants $C_1>0$, and $\varepsilon_0>0$, both depending on σ, p, and T, such that if $\beta+\zeta^a<\varepsilon_0$ then*

$$\|\rho_{\sigma,\beta,\zeta} - \rho_\sigma\|_{L^\infty(0,T;W^{2,p}(\mathbb{R}^d))} \le C_1(\beta+\zeta^a), \tag{38}$$

$$\|\rho_{\sigma,\beta,\zeta}\|_{L^\infty(0,T;W^{2,p}(\mathbb{R}^d))} \le C_1. \tag{39}$$

Furthermore, for every $q\ge 2$, there exists $C_2 = C_2(q)>0$, depending on σ and T, such that

$$\|(\mathcal{K}_\zeta - \mathcal{K}) * \rho_\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \le C_2\zeta^a, \tag{40}$$

$$\|\rho_{\sigma,\beta,\zeta}\|_{L^\infty(0,T;W^{2,1}(\mathbb{R}^d)\cap W^{3,q}(\mathbb{R}^d))} \le C_2. \tag{41}$$

The proof is presented in the following subsections.

3.1. **Proof of** (38)**.** We introduce the difference $u := \rho_{\sigma,\beta,\zeta} - \rho_\sigma$, which satisfies

$$\begin{aligned}\partial_t u - \sigma\Delta u &= \operatorname{div}\big[(u+\rho_\sigma)\nabla\mathcal{K}_\zeta * f_\sigma(W_\beta * (u+\rho_\sigma)) - \rho_\sigma\nabla\mathcal{K} * f_\sigma(\rho_\sigma)\big]\\ &= D[u] + R[\rho_\sigma,u] + S[\rho_\sigma,u] \quad \text{in } \mathbb{R}^d,\ t>0,\end{aligned} \tag{42}$$

and the initial datum $u(0)=0$ in $\mathbb{R}^d$, where

$$\begin{aligned}D[u] &= \operatorname{div}\big[u\nabla\mathcal{K} * f_\sigma(W_\beta * u)\big],\\ R[\rho_\sigma,u] &= \operatorname{div}\big[u\nabla\mathcal{K} * \big(f_\sigma(W_\beta*(u+\rho_\sigma)) - f_\sigma(W_\beta * u)\big)\\ &\quad + \rho_\sigma\nabla\mathcal{K} * \big(f_\sigma(W_\beta*(u+\rho_\sigma)) - f_\sigma(W_\beta*\rho_\sigma)\big) + \rho_\sigma\nabla\mathcal{K} * \big(f_\sigma(W_\beta*\rho_\sigma) - f_\sigma(\rho_\sigma)\big)\big],\\ S[\rho_\sigma,u] &= \operatorname{div}\big[(u+\rho_\sigma)\nabla(\mathcal{K}_\zeta - \mathcal{K}) * f_\sigma(W_\beta*(u+\rho_\sigma))\big].\end{aligned}$$

We show first an estimate for D^2u that depends on a lower-order estimate for u.

Lemma 14 (Conditional estimate for u)**.** *For any $p>d$, there exists a number $\Gamma_p\in(0,1)$ such that, if $\sup_{0<t<T}\|u(t)\|_{W^{1,p}(\mathbb{R}^d)} \le \Gamma_p$ then*

$$\|\mathrm{D}^2u\|_{L^p(0,T;L^p(\mathbb{R}^d))} \le C\big(\|u\|_{L^p(0,T;W^{1,p}(\mathbb{R}^d))} + \beta + \zeta^a\big),$$

recalling that $a=\min\{1,d-2s\}$, and where $C>0$ is independent of u, β, and ζ, but may depend on σ.

Proof. Let $\Gamma_p \in (0,1)$ be such that $\sup_{0<t<T}\|u(t)\|_{W^{1,p}(\mathbb{R}^d)} \le \Gamma_p$. We will find a constraint for Γ_p at the end of the proof. The aim is to derive an estimate for the right-hand side of (42) in $L^p(0,T;L^p(\mathbb{R}^d))$. We observe that $\|u(t)\|_1 \le \|\rho_{\sigma,\beta,\zeta}\|_1 + \|\rho_\sigma\|_1 \le 2\|\rho^0\|_1$ for $t\in[0,T]$. In the following, we denote by $C>0$ a generic constant that may depend on σ, without making this explicit. Furthermore, we denote by μ a generic exponent in $(0,1)$, whose value may vary from line to line.

Step 1: Estimate of $D[u]$. Let $1/2 < s < 1$. Then, by the Hardy–Littlewood–Sobolev-type inequality (68),

$$\begin{aligned}\|D[u]\|_p &\le \|\nabla u\cdot\nabla\mathcal{K}*f_\sigma(W_\beta*u)\|_p + \|u\nabla\mathcal{K}*[f_\sigma'(W_\beta*u)W_\beta*\nabla u]\|_p \\ &\le C\|\nabla u\|_p\|f_\sigma(W_\beta*u)\|_{d/(2s-1)} + C\|u\|_{d/(2s-1)}\|f_\sigma'(W_\beta*u)\|_\infty\|\nabla u\|_p.\end{aligned}$$

We use the Young convolution inequality, the Gagliardo–Nirenberg inequality, the smoothness of f_σ, the property $f_\sigma(0)=0$, and the fact $\|W_\beta\|_{L^1(\mathbb{R}^d)}=1$ to estimate the terms on the right-hand side:

$$\begin{aligned}\|W_\beta*u\|_\infty &\le \|u\|_\infty \le \|u\|_1^{1-\lambda}\|\nabla u\|_p^\lambda \le C\Gamma_p^\lambda \le C, \\ \|f_\sigma(W_\beta*u)\|_\infty &\le \max_U|f_\sigma'|\,\|W_\beta*u\|_\infty \le C, \\ \|f_\sigma'(W_\beta*u)\|_\infty &\le |f_\sigma'(0)| + \max_U|f_\sigma''|\|W_\beta*u\|_\infty \le C, \\ \|u\|_{d/(2s-1)} &\le \|u\|_1^{1-\mu}\|u\|_\infty^\mu \le C\|u\|_{W^{1,p}(\mathbb{R}^d)}^\mu \le C\Gamma_p^\mu \le C,\end{aligned}$$

where $U := [-\|W_\beta*u\|_\infty, \|W_\beta*u\|_\infty]$ and $\lambda>0$, $\mu>0$. Therefore, $\|D[u]\|_p \le C\|\nabla u\|_p$ and

$$\|D[u]\|_{L^p(0,T;L^p(\mathbb{R}^d))} \le C\|u\|_{L^p(0,T;W^{1,p}(\mathbb{R}^d))}. \tag{43}$$

Next, let $0 < s \le 1/2$. Then we write

$$\begin{aligned}D[u] &= \nabla u\cdot\mathcal{K}*[f_\sigma'(W_\beta*u)W_\beta*\nabla u] + u\mathcal{K}*[f_\sigma''(W_\beta*u)|W_\beta*\nabla u|^2] \\ &\quad + u\mathcal{K}*[f_\sigma'(W_\beta*u)W_\beta*\Delta u] =: D_1 + D_2 + D_3.\end{aligned}$$

By the Hardy–Littlewood–Sobolev-type inequality (Lemma 21),

$$\|D_1\|_p \le C\|\nabla u\|_{d/(2s)}\|f_\sigma'(W_\beta*u)W_\beta*\nabla u\|_p \le C\|\nabla u\|_{d/(2s)}\|\nabla u\|_p.$$

Next, we apply the Gagliardo–Nirenberg inequality with $\lambda = (1+1/d-2s/d)/(1+2/d-1/p)$:

$$\|\nabla u\|_{d/(2s)} \le C\|u\|_1^{1-\lambda}\|\mathrm{D}^2u\|_p^\lambda \le C\|\mathrm{D}^2u\|_p^\lambda,$$

which is possible as long as $\lambda \ge 1/2$ or equivalently $d \ge 2s$, which is true. Consequently, using $\Gamma_p \le 1$,

$$\|D_1\|_p \le C\|\nabla u\|_p\|\mathrm{D}^2u\|_p^\lambda \le C\Gamma_p^\lambda\|\nabla u\|_p^{1-\lambda}\|\mathrm{D}^2u\|_p^\lambda \le C(\delta)\|\nabla u\|_p + \delta\|\mathrm{D}^2u\|_p,$$

where $\delta>0$ is arbitrary. It follows from the Hardy-Littlewood-Sobolev-type inequality and the Gagliardo–Nirenberg inequality

$$\|\nabla u\|_{2p}^2 \le C\|\mathrm{D}^2u\|_p^{d/p}\|\nabla u\|_p^{2-d/p} \le C\Gamma_p\|\mathrm{D}^2u\|_p^{d/p}\|\nabla u\|_p^{1-d/p}$$

that

$$\|D_2\|_p \le C\|u\|_{d/2s}\Gamma_p\|D^2u\|_p^{d/p}\|\nabla u\|_p^{1-d/p} \le C(\delta)\|\nabla u\|_p + \delta\|\mathrm{D}^2u\|_p.$$

Finally, using similar ideas, we obtain

$$\|D_3\|_p \le C\|u\|_{d/(2s)}\|\Delta u\|_p \le C\Gamma_p^\mu\|\mathrm{D}^2u\|_p.$$

Summarizing the estimates for D_1, D_2, and D_3 and integrating in time leads to

$$\|D[u]\|_{L^p(0,T;L^p(\mathbb{R}^d))} \le C\|u\|_{L^p(0,T;W^{1,p}(\mathbb{R}^d))} + C\Gamma_p^\mu\|\mathrm{D}^2u\|_{L^p(0,T;L^p(\mathbb{R}^d))}. \tag{44}$$

Step 2: Estimate of $R[\rho_\sigma,u]$. We write $R[\rho_\sigma,u]=R_1+R_2+R_3$ for the three terms in the definition of $R[\rho_\sigma,u]$ below (42).

Step 2a: Estimate of R_1. If $s>1/2$, we can argue similarly as in the derivation of (43), which gives

$$\|R_1\|_{L^p(0,T;L^p(\mathbb{R}^d))} \le C\|u\|_{L^p(0,T;W^{1,p}(\mathbb{R}^d))}.$$

If $0<s\le 1/2$, we write $R_1=R_{11}+\cdots+R_{16}$, where

$$\begin{aligned}
R_{11} &= \nabla u\cdot\mathcal{K}*\big[f_\sigma'(W_\beta*(u+\rho_\sigma))W_\beta*\nabla\rho_\sigma\big],\\
R_{12} &= u\mathcal{K}*\big[f_\sigma''(W_\beta*(u+\rho_\sigma))(W_\beta*\nabla\rho_\sigma)\cdot(W_\beta*\nabla(u+\rho_\sigma))\big],\\
R_{13} &= u\mathcal{K}*\big[f_\sigma'(W_\beta*(u+\rho_\sigma))W_\beta*\Delta\rho_\sigma\big],\\
R_{14} &= \nabla u\cdot\mathcal{K}*\big[\big(f_\sigma'(W_\beta*(u+\rho_\sigma))-f_\sigma'(W_\beta*u)\big)W_\beta*\nabla u\big],\\
R_{15} &= u\mathcal{K}*\big[\big(f_\sigma''(W_\beta*(u+\rho_\sigma))W_\beta*\nabla(u+\rho_\sigma)\\
&\qquad - f_\sigma''(W_\beta*u)(W_\beta*\nabla u)\big)\cdot(W_\beta*\nabla u)\big],\\
R_{16} &= u\mathcal{K}*\big[\big(f_\sigma'(W_\beta*(u+\rho_\sigma))-f_\sigma'(W_\beta*u)\big)W_\beta*\Delta u\big].
\end{aligned}$$

All terms except the last one can be treated by the Hardy–Littlewood–Sobolev and Gagliardo–Nirenberg inequalities as before. For the last term, we use these inequalities and the $L^\infty(\mathbb{R}^d)$ bound for ρ_σ:

$$\begin{aligned}
\|R_{16}\|_p &\le C\|u\|_{d/(2s)}\big\|\big(f_\sigma'(W_\beta*(u+\rho_\sigma))-f_\sigma'(W_\beta*u)\big)W_\beta*\Delta u\big\|_p\\
&\le C\|u\|_{d/(2s)}\|f_\sigma''\|_\infty\|W_\beta*\rho_\sigma\|_\infty\|W_\beta*\Delta u\|_p \le C\|u\|_{d/(2s)}\|\Delta u\|_p \le C\Gamma_p^\mu\|\mathrm{D}^2u\|_p.
\end{aligned}$$

We infer that (possibly with a different $\mu>0$ than before)

$$\|R_1\|_{L^p(0,T;L^p(\mathbb{R}^d))} \le C\|u\|_{L^p(0,T;W^{1,p}(\mathbb{R}^d))} + C\Gamma_p^\mu\|\mathrm{D}^2u\|_{L^p(0,T;L^p(\mathbb{R}^d))}.$$

Step 2b: Estimate of R_2. Since $|f_\sigma'|$ is bounded on the interval $[-\|u\|_\infty-\|\rho_\sigma\|_\infty,\|u\|_\infty+\|\rho_\sigma\|_\infty]$, we obtain for $s>1/2$,

$$\|R_2\|_{L^p(0,T;L^p(\mathbb{R}^d))} \le C\|u\|_{L^p(0,T;W^{1,p}(\mathbb{R}^d))}.$$

For $0<s\le 1/2$, we write $R_2=R_{21}+\cdots+R_{27}$, where

$$\begin{aligned}
R_{21} &= \nabla\rho_\sigma\cdot\mathcal{K}*\big[f_\sigma'(W_\beta*(u+\rho_\sigma))W_\beta*\nabla u\big],\\
R_{22} &= \rho_\sigma\mathcal{K}*\big[f_\sigma''(W_\beta*(u+\rho_\sigma))W_\beta*\nabla(u+\rho_\sigma)\cdot(W_\beta*\nabla u)\big],\\
R_{23} &= \rho_\sigma\mathcal{K}*\big[f_\sigma'(W_\beta*(u+\rho_\sigma))W_\beta*\Delta u\big],
\end{aligned}$$

$$
\begin{aligned}
R_{24} &= \nabla\rho_\sigma\cdot\mathcal{K}*\big[\big(f_\sigma'(W_\beta*(u+\rho_\sigma))-f_\sigma'(W_\beta*\rho_\sigma)\big)W_\beta*\nabla\rho_\sigma\big], \\
R_{25} &= \rho_\sigma\mathcal{K}*\big[f_\sigma''(W_\beta*(u+\rho_\sigma))(W_\beta*\nabla u)\cdot(W_\beta*\nabla\rho_\sigma)\big], \\
R_{26} &= \rho_\sigma\mathcal{K}*\big[\big(f_\sigma''(W_\beta*(u+\rho_\sigma))-f_\sigma''(W_\beta*\rho_\sigma)\big)|W_\beta*\nabla\rho_\sigma|^2\big], \\
R_{27} &= \rho_\sigma\mathcal{K}*\big[\big(f_\sigma'(W_\beta*(u+\rho_\sigma))-f_\sigma'(W_\beta*\rho_\sigma)\big)W_\beta*\Delta\rho_\sigma\big].
\end{aligned}
$$

Similar estimations as before allow us to treat all terms except the third one:

$$
\begin{aligned}
\|R_{23}\|_p &\le \big\|\rho_\sigma\mathcal{K}*\big[\big(f_\sigma'(W_\beta*(u+\rho_\sigma))-f_\sigma'(W_\beta*\rho_\sigma)\big)W_\beta*\Delta u\big]\big\|_p \\
&\quad + \|\rho_\sigma\mathcal{K}*[f_\sigma'(W_\beta*\rho_\sigma)W_\beta*\Delta u]\|_p =: Q_{231}+Q_{232}.
\end{aligned}
$$

The first term can be estimated similarly as above by $Q_{231}\le C\Gamma_p^\mu\|\mathrm{D}^2u\|_p$, while

$$
\begin{aligned}
Q_{232} &\le \|\rho_\sigma\Delta\mathcal{K}*[f_\sigma'(W_\beta*\rho_\sigma)W_\beta*u]\|_p + \|\rho_\sigma\mathcal{K}*[\Delta f_\sigma'(W_\beta*\rho_\sigma)W_\beta*u]\|_p \\
&\quad + 2\|\rho_\sigma\mathcal{K}*[\nabla f_\sigma'(W_\beta*\rho_\sigma)\cdot(W_\beta*\nabla u)]\|_p.
\end{aligned}
$$

It follows from $-\Delta\mathcal{K}*v=(-\Delta)^{1-s}v$ and the fractional Gagliardo–Nirenberg inequality (Lemma 23) that

$$
\begin{aligned}
Q_{232} &\le C\|u\|_{W^{1,p}(\mathbb{R}^d)} + \|\rho_\sigma(-\Delta)^{1-s}[f_\sigma'(W_\beta*\rho_\sigma)W_\beta*u]\|_p \\
&\le C\|u\|_{W^{1,p}(\mathbb{R}^d)} + C\|\rho_\sigma\|_\infty\|f_\sigma'(W_\beta*\rho_\sigma)W_\beta*u\|_p^s\|\mathrm{D}^2[f_\sigma'(W_\beta*\rho_\sigma)W_\beta*u]\|_p^{1-s} \\
&\le C\|u\|_{W^{1,p}(\mathbb{R}^d)} + C\|u\|_p^s\big(\|u\|_p^{1-s}+\|\nabla u\|_p^{1-s}+\|\mathrm{D}^2u\|_p^{1-s}\big) \\
&\le C\|u\|_{W^{1,p}(\mathbb{R}^d)} + C\Gamma_p\|\mathrm{D}^2u\|_p.
\end{aligned}
$$

This shows that $\|R_{23}\|_p\le C\|u\|_{W^{1,p}(\mathbb{R}^d)}+C\Gamma_p^\mu\|\mathrm{D}^2u\|_p$, and we conclude that

$$
\|R_2\|_{L^p(0,T;L^p(\mathbb{R}^d))} \le C\|u\|_{L^p(0,T;W^{1,p}(\mathbb{R}^d))} + C\Gamma_p^\mu\|\mathrm{D}^2u\|_{L^p(0,T;L^p(\mathbb{R}^d))}.
$$

Step 2c: Estimate of R_3. We write $R_3=R_{31}+\cdots+R_{37}$, where

$$
\begin{aligned}
R_{31} &= \nabla\rho_\sigma\cdot\mathcal{K}*\big[\big(f_\sigma'(W_\beta*\rho_\sigma)-f_\sigma'(\rho_\sigma)\big)W_\beta*\nabla\rho_\sigma\big], \\
R_{32} &= \rho_\sigma\mathcal{K}*\big[\big(f_\sigma''(W_\beta*\rho_\sigma)-f_\sigma''(\rho_\sigma)\big)|W_\beta*\nabla\rho_\sigma|^2\big], \\
R_{33} &= \rho_\sigma\mathcal{K}*\big[f_\sigma''(\rho_\sigma)(W_\beta*\nabla\rho_\sigma-\nabla\rho_\sigma)\cdot(W_\beta*\nabla\rho_\sigma)\big], \\
R_{34} &= \nabla\rho_\sigma\cdot\mathcal{K}*\big[f_\sigma'(\rho_\sigma)(W_\beta*\nabla\rho_\sigma-\nabla\rho_\sigma)\big], \\
R_{35} &= \rho_\sigma\mathcal{K}*\big[f_\sigma''(\rho_\sigma)\nabla\rho_\sigma\cdot(W_\beta*\nabla\rho_\sigma-\nabla\rho_\sigma)\big], \\
R_{36} &= \rho_\sigma\mathcal{K}*\big[f_\sigma'(\rho_\sigma)(W_\beta*\Delta\rho_\sigma-\Delta\rho_\sigma)\big] \\
R_{37} &= \rho_\sigma\mathcal{K}*\big[(f_\sigma'(W_\beta*\rho_\sigma)-f_\sigma'(\rho_\sigma))W_\beta*\Delta\rho_\sigma\big]
\end{aligned}
$$

We start with the estimate of R_{31}. We use the Hardy–Littlewood–Sobolev inequality (Lemma 21) and Lemma 20 to estimate $W_\beta*\rho_\sigma-\rho_\sigma$:

$$
\begin{aligned}
R_{31} &\le C\|\nabla\rho_\sigma\|_{d/s}\|f_\sigma'(W_\beta*\rho_\sigma)-f_\sigma'(\rho_\sigma)\|_p\|W_\beta*\nabla\rho_\sigma\|_{d/s} \\
&\le C\|\nabla\rho_\sigma\|_{d/s}^2\max_{[0,2\|\rho_\sigma\|_\infty]}|f_\sigma''|\,\|W_\beta*\rho_\sigma-\rho_\sigma\|_p \le C(\sigma)\beta,
\end{aligned}
$$

also taking into account the $L^\infty(0,T;L^q(\mathbb{R}^d))$ bound for $\nabla\rho_\sigma$; see Proposition 10. With this regularity, we can estimate all other terms except R_{34} and R_{36}. Since they have similar structures, we only treat R_{34}. This term is delicate since the factor $f_\sigma'(\rho_\sigma)$ cannot be bounded in $L^q(\mathbb{R}^d)$ for any $q<\infty$. Therefore, one might obtain via Hardy–Littlewood–Sobolev's inequality factors like $\|\nabla\rho_\sigma\|_{q_1}$ and $\|\mathrm{D}^2\rho_\sigma\|_{q_2}$ with either $q_1<2$ or $q_2<2$. However, for such factors, an L^∞ bound in time is currently lacking (Proposition 10 provides such a bound only for $q\geq 2$). Our idea is to add and subtract the term $f_\sigma'(0)$ since

$$|f_\sigma'(\rho_\sigma)-f_\sigma'(0)|\leq \rho_\sigma \max_{[0,\|\rho_\sigma\|_\infty]}|f_\sigma''|\leq C\rho_\sigma$$

can be controlled. This leads to

$$\begin{aligned}\|R_{34}\|_p &\leq \|\nabla\rho_\sigma\cdot\mathcal{K}*[(f_\sigma'(\rho_\sigma)-f_\sigma'(0))(W_\beta*\nabla\rho_\beta-\nabla\rho_\beta)\|_p \\ &\quad + \|f_\sigma'(0)\nabla\rho_\sigma\cdot\mathcal{K}*(W_\beta*\nabla\rho_\sigma-\nabla\rho_\sigma)\|_p \\ &\leq C\beta+|f_\rho'(0)|\|\nabla\rho_\sigma\cdot\mathcal{K}*(W_\beta*\nabla\rho_\sigma-\nabla\rho_\sigma)\|_p =: C\beta+Q_{341},\end{aligned}$$

as the first term can be estimated in a standard way. For the estimate of Q_{341}, we need to distinguish two cases.

If $1/2<s\leq 1$, we infer from the Hardy–Littlewood–Sobolev-type inequality (68) that

$$Q_{341}\leq C\|\nabla\rho_\sigma\|_{d/(2s-1)}\|W_\beta*\rho_\sigma-\rho_\sigma\|_p\leq C\|\nabla\rho_\sigma\|_{d/(2s-1)}\|\nabla\rho_\sigma\|_p\beta\leq C\beta.$$

Next, let $0<s\leq 1/2$. Then we apply the Hardy–Littlewood–Sobolev-type inequality (67), the standard Gagliardo–Nirenberg inequality for some $\lambda>0$, and Lemma 20:

$$Q_{341}\leq C\|\nabla\rho_\sigma\|_{d/(2s)}\|W_\beta*\nabla\rho_\sigma-\nabla\rho_\sigma\|_p\leq C\|\rho_\sigma\|_1^{1-\lambda}\|\mathrm{D}^2\rho_\sigma\|_p^\lambda(\beta\|\mathrm{D}^2\rho_\sigma\|_p)\leq C\beta.$$

We conclude that $\|R_{34}\|_p\leq C\beta$ and eventually

$$\|R_3\|_{L^p(0,T;L^p(\mathbb{R}^d))}\leq C\beta.$$

Summarizing the estimates for R_1, R_2, and R_3 finishes this step:

$$\|R[\rho_\sigma,u]\|_{L^p(0,T;L^p(\mathbb{R}^d))}\leq C\|u\|_{L^p(0,T;W^{1,p}(\mathbb{R}^d))}+C\beta+C\Gamma_p^\mu\|\mathrm{D}^2u\|_{L^p(0,T;W^{1,p}(\mathbb{R}^d))}. \tag{45}$$

Step 3: Estimate of $S[\rho_\sigma,u]$. We formulate this term as $S[\rho_\sigma,u]=S_1+\cdots+S_4$, where

$$\begin{aligned}S_1&=\operatorname{div}\big[u\nabla(\mathcal{K}_\zeta-\mathcal{K})*\big(f_\sigma(W_\beta*(u+\rho_\sigma))-f_\sigma(W_\beta*\rho_\sigma)\big)\big],\\ S_2&=\operatorname{div}\big(u\nabla(\mathcal{K}_\zeta-\mathcal{K})*f_\sigma(W_\beta*\rho_\sigma)\big),\\ S_3&=\operatorname{div}\big[\rho_\sigma\nabla(\mathcal{K}_\zeta-\mathcal{K})*\big(f_\sigma(W_\beta*(u+\rho_\sigma))-f_\sigma(W_\beta*\rho_\sigma)\big)\big],\\ S_4&=\operatorname{div}\big(\rho_\sigma\nabla(\mathcal{K}_\zeta-\mathcal{K})*f_\sigma(W_\beta*\rho_\sigma)\big).\end{aligned}$$

The terms S_1, S_2, and S_3 can be treated as the terms in $R[\rho_\sigma,u]$, since they have the same structure and the techniques used to estimate integrals involving $\mathcal{K}$ can be applied to those involving $\mathcal{K}_\zeta$. This leads to (for some $\mu>0$)

$$\|S_1+S_2+S_3\|_{L^p(0,T;L^p(\mathbb{R}^d))}\leq C\|u\|_{L^p(0,T;W^{1,p}(\mathbb{R}^d))}+C\Gamma_p^\mu\|\mathrm{D}^2u\|_{L^p(0,T;L^p(\mathbb{R}^d))}. \tag{46}$$

It remains to estimate S_4. We write $S_4=S_{41}+S_{42}+S_{43}$, where

$$S_{41}=\nabla\rho_\sigma\cdot(\mathcal{K}_\zeta-\mathcal{K})*\big[f_\sigma'(W_\beta*\rho_\sigma)W_\beta*\nabla\rho_\sigma\big],$$

$$S_{42} = \rho_\sigma(\mathcal{K}_\zeta - \mathcal{K}) * \big[f''_\sigma(W_\beta * \rho_\sigma)|W_\beta * \nabla\rho_\sigma|^2\big],$$
$$S_{43} = \rho_\sigma(\mathcal{K}_\zeta - \mathcal{K}) * \big[f'_\sigma(W_\beta * \rho_\sigma)W_\beta * \Delta\rho_\sigma\big].$$

Observe that, because of the definition of $\mathcal{K}_\zeta = \widetilde{\mathcal{K}}_\zeta * W_\zeta$ with $\widetilde{\mathcal{K}}_\zeta = \mathcal{K}\omega_\zeta$ (defined in (11)), we have $(\mathcal{K}_\zeta - \mathcal{K}) * v = \mathcal{K} * (W_\zeta * v - v) - (\mathcal{K}(1-\omega_\zeta)) * W_\zeta * v$ for every function v for which the convolution is defined, and therefore, by the Hardy–Littlewood–Sobolev-type inequality (67), Young's convolution inequality, and Lemma 20,

$$\begin{aligned}\|\rho_\sigma(\mathcal{K}_\zeta - \mathcal{K}) * v\|_p &\le C\|\rho_\sigma\|_{d/(2s)}\|W_\zeta * v - v\|_p + C\|\rho_\sigma\|_p\|(\mathcal{K}(1-\omega_\zeta)) * v\|_\infty \\ &\le C\|\rho_\sigma\|_{d/(2s)}\|\nabla v\|_p\zeta + C\|\rho_\sigma\|_p\|\mathcal{K}1_{\mathbb{R}^d\setminus B(0,\zeta^{-1})}\|_\infty\|v\|_1 \\ &\le C\|\rho_\sigma\|_{d/(2s)}\|\nabla v\|_p\zeta + C\zeta^{d-2s}\|\rho_\sigma\|_p\|v\|_1,\end{aligned}$$

Given the regularity properties of ρ_σ (see Lemma 11) and the assumptions on f_σ, it follows that

$$\|S_4\|_{L^p(0,T;L^p(\mathbb{R}^d))} \le C\zeta^{\min\{1,d-2s\}}. \tag{47}$$

We conclude from (46) and (47) that

$$\|S[\rho_\sigma, u]\|_{L^p(0,T;L^p(\mathbb{R}^d))} \le C\|u\|_{L^p(0,T;W^{1,p}(\mathbb{R}^d))} + C\zeta^a + C\Gamma_p^\mu\|\mathrm{D}^2u\|_{L^p(0,T;L^p(\mathbb{R}^d))}, \tag{48}$$

where $a := \min\{1, d-2s\}$.

Step 4: End of the proof. Summarizing (44), (45), and (48), we infer that the right-hand side of (42) can be bounded (for some $\mu > 0$) by

$$\begin{aligned}&\|D[u] + R[\rho_\sigma, u] + S[\rho_\sigma, u]\|_{L^p(0,T;L^p(\mathbb{R}^d))} \\ &\qquad \le C\|u\|_{L^p(0,T;W^{1,p}(\mathbb{R}^d))} + C(\beta + \zeta^a) + C\Gamma_p^\mu\|\mathrm{D}^2u\|_{L^p(0,T;L^p(\mathbb{R}^d))}.\end{aligned}$$

By parabolic regularity (71),

$$\|\mathrm{D}^2u\|_{L^p(0,T;L^p(\mathbb{R}^d))} \le C\|u\|_{L^p(0,T;W^{1,p}(\mathbb{R}^d))} + C(\beta + \zeta^a) + C\Gamma_p^\mu\|\mathrm{D}^2u\|_{L^p(0,T;L^p(\mathbb{R}^d))}.$$

Choosing $\Gamma_p > 0$ sufficiently small finishes the proof. ☐

It remains to estimate the $L^p(0,T;W^{1,p}(\mathbb{R}^d))$ norm of u. This is done in the following lemma.

Lemma 15 (Unconditional estimate for u)**.** *For any $p > d$, there exist constants $C > 0$, and $\varepsilon_0 > 0$, both depending on σ, p, and T, such that for $\beta + \zeta^a < \varepsilon_0$,*

$$\|u\|_{L^\infty(0,T;W^{1,p}(\mathbb{R}^d))} \le C(\beta + \zeta^a).$$

recalling that $a := \min\{1, d-2s\}$.

Proof. The idea is to test (42) with $p|u|^{p-2}u - p\operatorname{div}(|\nabla u|^{p-2}\nabla u)$. Integration by parts and some elementary computations lead to

$$\int_{\mathbb{R}^d} p\operatorname{div}(|\nabla u|^{p-2}\nabla u)\Delta u dx = -p\sum_{i,j}\int_{\mathbb{R}^d}|\nabla u|^{p-2}\partial_i u\partial_i\partial_{jj}^2 u dx$$

$$= p\sum_{i,j}\int_{\mathbb{R}^d}\partial_j(|\nabla u|^{p-2}\partial_i u)\partial_{ij}^2 u dx$$
$$= p\int_{\mathbb{R}^d}|\nabla u|^{p-2}|D^2u|^2dx + \frac{p}{2}\sum_j\int_{\mathbb{R}^d}\partial_j(|\nabla u|^{p-2})\partial_j(|\nabla u|^2)dx$$
$$= p\int_{\mathbb{R}^d}|\nabla u|^{p-2}|D^2u|^2dx + \sum_j\int_{\mathbb{R}^d}\frac{4}{p}(p-2)\big(\partial_j(|\nabla u|^{p/2})\big)^2dx.$$

Consequently, we have

$$\begin{aligned}
(49)\qquad & p\|u(t)\|^p_{W^{1,p}(\mathbb{R}^d)} + \sigma p(p-1)\int_0^t\int_{\mathbb{R}^d}|u|^{p-2}|\nabla u|^2dxds \\
&\quad + \sigma\int_0^t\int_{\mathbb{R}^d}\big(p|\nabla u|^{p-2}|D^2u|^2 + 4(p-2)p^{-1}\big|\nabla(|\nabla u|^{p/2})\big|^2\big)dxds \\
&= p\int_0^t\int_{\mathbb{R}^d}\big(|u|^{p-2}u - \operatorname{div}(|\nabla u|^{p-2}\nabla u)\big)\big(D[u] + R[\rho_\sigma,u] + S[\rho_\sigma,u]\big)dxds \\
&=: Q[u].
\end{aligned}$$

We infer from Lemmas 19 and 25 that $u\in C^0([0,T];W^{1,p}(\mathbb{R}^d))$. Therefore, since $u(0)=0$, it holds that $\|u(t)\|_{W^{1,p}(\mathbb{R}^d)}\le\Gamma_p$ for all $t\in[0,T^*]$ and $T^* := \sup\{t_0\in(0,T) : \|u(t)\|_{W^{1,p}(\mathbb{R}^d)}\le\Gamma_p \text{ for } 0\le t\le t_0\}$. Let $t\in[0,T^*]$. We have shown in the proof of the previous lemma that

$$\|D[u] + R[\rho_\sigma,u] + S[\rho_\sigma,u]\|_{L^p(0,t;L^p(\mathbb{R}^d))}\le C\|u\|_{L^p(0,t;W^{1,p}(\mathbb{R}^d))} + C(\beta+\zeta^a).$$

Hence, we can estimate the right-hand side $Q[u]$ of (49) as follows:

$$\begin{aligned}
Q[u] &\le C\int_0^t\int_{\mathbb{R}^d}\big(|u|^{p-1} + |\nabla u|^{p-2}|\mathrm{D}^2u|\big)\big|D[u] + R[\rho_\sigma,u] + S[\rho_\sigma,u]\big|dx \\
&\le C\big(\|u\|^{p-1}_{L^p(0,t;L^p(\mathbb{R}^d))} + \|\nabla u\|^{p/2-1}_{L^p(0,t;L^p(\mathbb{R}^d))}\||\nabla u|^{p/2-1}|\mathrm{D}^2u|\|_{L^2(0,t;L^2(\mathbb{R}^d))}\big) \\
&\quad\times\big(\|u\|_{L^p(0,t;W^{1,p}(\mathbb{R}^d))} + \beta + \zeta^a\big) \\
&\le C(\delta,p,t)\big(\|u\|^p_{L^p(0,t;W^{1,p}(\mathbb{R}^d))} + (\beta+\zeta^a)^p\big) + \delta\||\nabla u|^{p/2-1}|\mathrm{D}^2u|\|^2_{L^2(0,t;L^2(\mathbb{R}^d))},
\end{aligned}$$

where $\delta>0$. Choosing δ sufficiently small, the last term is absorbed by the corresponding expression on the left-hand side of (49), and we infer from (49) that for $0\le t\le T^*$,

$$\|u(t)\|^p_{W^{1,p}(\mathbb{R}^d)}\le C(p,t)\int_0^t\|u\|^p_{W^{1,p}(\mathbb{R}^d)}ds + C(p,t)(\beta+\zeta^a)^p.$$

We assume without loss of generality that $C(p,t)$ is nondecreasing in t. Then Gronwall's lemma implies that for $0\le t\le T^*$,

$$\|u(t)\|^p_{W^{1,p}(\mathbb{R}^d)}\le C(p,T)(\beta+\zeta^a)^p\int_0^t e^{C(p,T)(t-s)}ds\le(\beta+\zeta^a)e^{C(p,T)t}.$$

Choosing $\varepsilon_0 = \frac{1}{2}\Gamma_p \exp(-C(p,T)T/p) < 1$, we find that $\|u(t)\|_{W^{1,p}(\mathbb{R}^d)} \le \Gamma_p/2$ for $\beta + \zeta^a < \varepsilon_0$ and $0 \le t \le T^*$. By definition of T^*, it follows that $T^* = T$. In particular, $\|u(t)\|_{W^{1,p}(\mathbb{R}^d)} \le C(\beta+\zeta^a)$ for $0 < t < T$, which finishes the proof. □

3.2. **Proof of** (38) **and** (39)**.** Combining Lemmas 14 and 15 leads to

$$\|u\|_{L^p(0,T;W^{2,p}(\mathbb{R}^d))} \le C(\sigma,p,T)(\beta+\zeta^a), \quad \text{where } a = \min\{1, d-2s\}, \tag{50}$$

as long as $\beta + \zeta^a < \varepsilon_0$ and $p > d$. Next, we differentiate (42) with respect to x_i (writing ∂_i for $\partial/\partial x_i$):

$$\partial_t(\partial_i u) - \sigma\Delta(\partial_i u) = \partial_i\big(D[u] + R[\rho_\sigma, u] + S[\rho_\sigma, u]\big), \quad \partial_i u(0) = 0 \quad \text{in } \mathbb{R}^d.$$

Taking into account estimate (50) and arguing as in the proof of Lemma 14, we can show that for $\delta > 0$,

$$\|\partial_i(D[u] + R[\rho_\sigma,u] + S[\rho_\sigma,u])\|_{L^p(0,T;L^p(\mathbb{R}^d))} \le C(p,\sigma,\delta)(\beta+\zeta^a) + \delta\|\mathrm{D}^3 u\|_{L^p(0,T;L^p(\mathbb{R}^d))}.$$

We infer from parabolic regularity (Lemma 25) for sufficiently small $\delta > 0$ that

$$\|\partial_t \mathrm{D}u\|_{L^p(0,T;L^p(\mathbb{R}^d))} + \|\mathrm{D}^3 u\|_{L^p(0,T;L^p(\mathbb{R}^d))} \le C(p,\sigma)(\beta+\zeta^a).$$

Then Lemma 19, applied to $\mathrm{D}u$, leads to (38), which with Proposition 10 implies (39).

3.3. **Proof of** (40)**.** Let $x \in \mathbb{R}^d$. We use the definitions of $\mathcal{K}_\zeta$ and W_ζ to find that

$$\begin{aligned} |(\mathcal{K}_\zeta - \mathcal{K}) * \rho_\sigma(x)| &= \left|\int_{\mathbb{R}^d} W_\zeta(x-y)\big((\mathcal{K}*\rho_\sigma)(x) - ((\mathcal{K}\omega_\zeta)*\rho_\sigma)(y)\big)dy\right| \\ &\le \int_{\mathbb{R}^d} W_\zeta(x-y)|x-y|\frac{|(\mathcal{K}*\rho_\sigma)(x) - (\mathcal{K}*\rho_\sigma)(y)|}{|x-y|}dy + \|(\mathcal{K}(1-\omega_\zeta))*\rho_\sigma\|_\infty \\ &\le \|\nabla\mathcal{K}*\rho_\sigma\|_\infty \int_{\mathbb{R}^d} W_\zeta(z)|z|dz + \|\mathcal{K}1_{\mathbb{R}^d\setminus B(0,\zeta^{-1})}\|_\infty \|\rho_\sigma\|_1 \\ &\le \zeta\|\nabla\mathcal{K}*\rho_\sigma\|_\infty \int_{\mathbb{R}^d} W_1(y)|y|dy + \zeta^{d-2s}\|\rho_\sigma\|_1. \end{aligned}$$

Let $\phi \in C_0^\infty(\mathbb{R}^d)$ be such that $\mathrm{supp}(\phi) \subset B_2$ and $\phi = 1$ in B_1. Then (since we can assume without loss of generality that $\zeta < 1$), by arguing like in the derivation of (47), we obtain

$$|(\mathcal{K}_\zeta - \mathcal{K}) * \rho_\sigma(x)| \le C\zeta^{\min\{1,d-2s\}}\big(\|\nabla(\mathcal{K}\phi)*\rho_\sigma\|_\infty + \|\nabla(\mathcal{K}(1-\phi))*\rho_\sigma\|_\infty + \|\rho_\sigma\|_1\big),$$

A computation shows that for $p > \max\{d/(2s), 2\}$,

$$\begin{aligned} \|\nabla(\mathcal{K}\phi)*\rho_\sigma\|_\infty &= \|(\mathcal{K}\phi)*\nabla\rho_\sigma\|_\infty \le \|\mathcal{K}\phi\|_{p/(p-1)}\|\nabla\rho_\sigma\|_p \le C\|\nabla\rho_\sigma\|_p, \\ \|\nabla(\mathcal{K}(1-\phi))*\rho_\sigma\|_\infty &\le \|\nabla(\mathcal{K}(1-\phi))\|_\infty\|\rho_\sigma\|_1 \le C\|\rho_\sigma\|_1, \end{aligned}$$

where we note that $\mathcal{K}1_{B_2} \in L^{p/(p-1)}$ if $p > d/(2s)$. Then, in view of the regularity of ρ_σ in Lemma 11, we find that

$$\|(\mathcal{K}_\zeta - \mathcal{K}) * \rho_\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \le C\zeta^a.$$

3.4. **Proof of** (41)**.** The $L^\infty(0,T;W^{2,1}(\mathbb{R}^d) \cap W^{3,q}(\mathbb{R}^d))$ bound for $\rho_{\sigma,\beta,\zeta}$ is shown in a similar way as the corresponding bound for ρ_σ in Lemma 11.

4. Mean-field analysis

This section is devoted to the proof of Theorems 2 and 3.

4.1. **Existence of density functions.** We claim that the solution $\widehat{X}^N$ to (6) is absolutely continuous with respect to the Lebesgue measure, which implies that this process possesses a probability density $\widehat{u} \in L^\infty(0,T;L^1(\mathbb{R}^d))$. The claim follows from [32, Theorem 2.3.1] if the coefficients of the stochastic differential equation (6), satisfied by $\widehat{X}^N$, are globally Lipschitz continuous and of at most linear growth. The latter condition follows from

$$\begin{aligned}|\nabla\mathcal{K} * f_\sigma(\rho_\sigma(x,t))| &\le \|\mathcal{K} * \nabla f_\sigma(\rho_\sigma)\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \\ &\le C\|\mathcal{K} * \nabla f_\sigma(\rho_\sigma)\|_{L^\infty(0,T;W^{1,p}(\mathbb{R}^d))} \le C\|\nabla f_\sigma(\rho_\sigma)\|_{L^\infty(0,T;W^{1,r}(\mathbb{R}^d))} \le C(\sigma),\end{aligned}$$

where $p > d$ and $r = dp/(d+2s)$ according to the Hardy–Littlewood–Sobolev inequality, and we used the regularity bounds for ρ_σ from Lemma 25. The global Lipschitz continuity is a consequence of the mean-value theorem, the Hardy–Littlewood–Sobolev inequality, and the $W^{2,\infty}(\mathbb{R}^d)$ regularity of ρ_σ from Lemma 11:

$$\begin{aligned}\sup_{0<t<T}\big|\nabla\mathcal{K} * f_\sigma(\rho_\sigma(x,t)) - \nabla\mathcal{K} * f_\sigma(\rho_\sigma(y,t))\big| &\le \sup_{0<t<T}\|\mathrm{D}^2\mathcal{K} * f_\sigma(\rho_\sigma(\cdot,t))\|_\infty |x-y| \\ = \sup_{0<t<T}\big\|\mathcal{K} * \big(f_\sigma''(\rho_\sigma)\nabla\rho_\sigma \otimes \nabla\rho_\sigma + f_\sigma'(\rho_\sigma)\mathrm{D}^2\rho_\sigma\big)(\cdot,t)\big\|_\infty &|x-y| \le C(\sigma)|x-y|.\end{aligned}$$

Similar arguments show that $\bar{X}_i^N(t)$ has a density function $\bar{u} \in L^\infty(0,T;L^1(\mathbb{R}^d))$.

Next, we show that $\widehat{u}$ and $\bar{u}$ can be identified with the weak solutions ρ_σ and $\rho_{\sigma,\beta,\zeta}$, respectively, using Itô's lemma. Indeed, let $\phi \in C_0^\infty(\mathbb{R}^d \times [0,T])$. We infer from Itô's formula that

$$\begin{aligned}\phi(\widehat{X}_i^N(t),t) &= \phi(\widehat{X}_i^N(0),0) + \int_0^t \partial_s\phi(\widehat{X}_i^N(s),s)ds + \sigma\int_0^t \Delta\phi(\widehat{X}_i^N(s),s)ds \\ &- \int_0^t \nabla\mathcal{K} * f_\sigma(\rho_\sigma(\widehat{X}_i^N(s),s)) \cdot \nabla\phi(\widehat{X}_i^N(s),s)ds + \sqrt{2\sigma}\int_0^t \nabla\phi(\widehat{X}_i^N(s),s)\cdot dB_i^N(s).\end{aligned}$$

Taking the expectation, the Itô integral vanishes, and we end up with

$$\begin{aligned}\int_{\mathbb{R}^d}\phi(x,t)\widehat{u}(x,t)dx &= \int_{\mathbb{R}^d}\phi(x,0)\rho_\sigma^0(x)dx + \int_0^t\int_{\mathbb{R}^d}\partial_s\phi(x,s)\widehat{u}(x,s)dxds \\ &+ \sigma\int_0^t\int_{\mathbb{R}^d}\Delta\phi(x,s)\widehat{u}(x,s)dxds - \int_0^t\int_{\mathbb{R}^d}\nabla\mathcal{K} * f_\sigma(\rho_\sigma(x,s))\cdot\nabla\phi(x,s)\widehat{u}(x,s)dxds.\end{aligned} \tag{51}$$

Hence, $\widehat{u}$ is a very weak solution in the space $L^\infty(0,T;L^1(\mathbb{R}^d))$ to the linear equation

$$\partial_t\widehat{u} = \sigma\Delta\widehat{u} + \operatorname{div}(\widehat{u}\nabla\mathcal{K} * f_\sigma(\rho_\sigma)), \quad \widehat{u}(0) = \rho_\sigma^0 \quad \text{in } \mathbb{R}^d, \tag{52}$$

where ρ_σ is the unique solution to (7).

It can be shown that (52) is uniquely solvable in the class of functions in $L^\infty(0,T;L^1(\mathbb{R}^d))$. This implies that $\widehat{u} = \rho_\sigma$ in $\mathbb{R}^d \times (0,T)$ (and similarly $\bar{u} = \rho_{\sigma,\beta,\zeta}$). The proof is technical but standard; see, e.g., [11, Theorem 7] for a sketch of a proof.

Another approach is as follows. Because of the linearity of (51), it is sufficient to prove that $\widehat{u} \equiv 0$ in $\mathbb{R}^d \times (0,T)$ if $\rho_\sigma^0 = 0$. First, we verify that $v := \nabla\mathcal{K} * f_\sigma(\rho_\sigma) \in L^\infty(0,T;W^{1,\infty}(\mathbb{R}^d))$ and $\widehat{u} \in L^p(0,T;L^p(\mathbb{R}^d))$ for $p < d/(d-1)$. Then, by density, (51) holds for all $\phi \in W^{1,q}(0,T;L^q(\mathbb{R}^d)) \cap L^q(0,T;W^{2,q}(\mathbb{R}^d))$ with $q > d$ and $\phi(T) = 0$. Choosing ψ to be the unique strong solution to the dual problem

$$\partial_t\psi + \sigma\Delta\psi = v\cdot\nabla\psi + g, \quad \psi(T) = 0 \quad \text{in } \mathbb{R}^d$$

in the very weak formulation of (51), we find that $\int_0^T\int_{\mathbb{R}^d}\widehat{u}g\,dxdt = 0$ for all $g \in C_0^\infty(\mathbb{R}^d \times (0,T))$, which implies that $\widehat{u} = 0$.

4.2. Estimate of $X_i^N - \overline{X}_i^N$.

Lemma 16. *Let X_i^N and $\bar{X}_i^N$ be the solutions to* (3) *and* (4)*, respectively, and let $\delta \in (0,1/4)$. Under the assumptions of Theorem 2 on β and ζ, it holds that*

$$\mathbb{E}\bigg(\sup_{0<s<T}\max_{i=1,\ldots,N}|(X_i^N - \bar{X}_i^N)(s)|\bigg) \le CN^{-1/4+\delta}.$$

Proof. To simplify the presentation, we set

$$\Psi(x,t) := f_\sigma\bigg(\frac{1}{N}\sum_{j=1,\,j\neq i}^N W_\beta(X_j^N(t)-x)\bigg),\ \bar{\Psi}(x,t) := f_\sigma\bigg(\frac{1}{N}\sum_{j=1,\,j\neq i}^N W_\beta(\bar{X}_j^N(t)-x)\bigg),$$

and we write $\rho := \rho_{\sigma,\beta,\zeta}$. Taking the difference of equations (3) and (4) in the integral formulation leads to

$$\begin{aligned}\sup_{0<s<t}|(X_i^N - \bar{X}_i^N)(s)| &\le \int_0^t\big|\nabla\mathcal{K}_\zeta * \big(\Psi(X_i^N(s),s) - f_\sigma(W_\beta*\rho(\bar{X}_i^N(s),s))\big)\big|ds \\ &\le \int_0^t\big|\nabla\mathcal{K}_\zeta * \big(\Psi(X_i^N(s),s) - \bar{\Psi}(\bar{X}_i^N(s),s)\big)\big|ds \\ &\quad + \int_0^t\big|\nabla\mathcal{K}_\zeta * \big(\bar{\Psi}(\bar{X}_i^N(s),s) - f_\sigma(W_\beta*\rho(\bar{X}_i^N(s),s))\big)\big|ds =: I_1 + I_2. \end{aligned} \tag{53}$$

Step 1: Estimate of I_1. To estimate I_1, we formulate $I_1 = I_{11} + I_{12} + I_{13}$, where

$$\begin{aligned} I_{11} &= \int_0^t\big|\nabla\mathcal{K}_\zeta * \big(\Psi(X_i^N(s),s) - \Psi(\bar{X}_i^N(s),s)\big)\big|ds, \\ I_{12} &= \int_0^t\big|\nabla\mathcal{K}_\zeta * \big(\Psi(\bar{X}_i^N(s),s) - \bar{\Psi}(X_i^N(s),s)\big)\big|ds, \\ I_{13} &= \int_0^t\big|\nabla\mathcal{K}_\zeta * \big(\bar{\Psi}(X_i^N(s),s) - \bar{\Psi}(\bar{X}_i^N(s),s)\big)\big|ds. \end{aligned}$$

We start with the first integral:

$$I_{11} \le \int_0^t\|\mathrm{D}^2\mathcal{K}_\zeta * \Psi(\cdot,s)\|_\infty \sup_{0<r<s}\max_{i=1,\ldots,N}|(X_i^N - \bar{X}_i^N)(r)|ds.$$

We claim that

$$\|\mathrm{D}^k\mathcal{K}_\zeta * \Psi(\cdot,s)\|_\infty \le C(\sigma)\beta^{-(k+1)(d+k)-1}, \quad k\in\mathbb{N}. \tag{54}$$

For the proof, we introduce

$$\Phi(x,y) := f_\sigma\bigg(\frac{1}{N}\sum_{j=1}^{N-1} W_\beta(y_j-x)\bigg) \quad \text{for } x\in\mathbb{R}^d,\ y=(y_1,\ldots,y_{N-1})\in\mathbb{R}^{(N-1)d}.$$

Then, by definition of $\mathcal{K}_\zeta$,

$$\|\mathrm{D}^k\mathcal{K}_\zeta * \Psi(\cdot,t)\|_\infty \le \sup_{y\in\mathbb{R}^{N-1}} \|W_\zeta * \mathcal{K}\omega_\zeta * \mathrm{D}^k\Phi(\cdot,y)\|_\infty.$$

We estimate the right-hand side:

$$\begin{aligned}\|W_\zeta * (\mathcal{K}\omega_\zeta * \mathrm{D}^k\Phi(\cdot,y))\|_\infty &\le \|W_\zeta\|_1\|\mathcal{K}\omega_\zeta * \mathrm{D}^k\Phi(\cdot,y)\|_\infty \le C\|\mathcal{K}\omega_\zeta * \mathrm{D}^k\Phi(\cdot,y)\|_{W^{1,p}(\mathbb{R}^d)} \\ &\le C\|\mathcal{K} * |\mathrm{D}^k\Phi(\cdot,y)|\|_p + C\|\mathcal{K} * |\mathrm{D}^{k+1}\Phi(\cdot,y)|\|_p \\ &\le C\|\mathrm{D}^k\Phi(\cdot,y)\|_r + C\|\mathrm{D}^{k+1}\Phi(\cdot,y)\|_r,\end{aligned}$$

where we used the Hardy–Littlewood–Sobolev inequality for $r = dp/(d+2ps)$ in the last step. It follows from the Faà di Bruno formula, after an elementary computation, that the last term is estimated according to

$$\begin{aligned}\|\mathrm{D}^{k+1}\Phi(\cdot,y)\|_r^r &= \int_{\mathbb{R}^d}\bigg|\mathrm{D}^{k+1}\bigg(f_\sigma\bigg(\frac{1}{N}\sum_{j=1}^{N-1}W_\beta(y_j-x)\bigg)\bigg)\bigg|^r dx \\ &\le C(k,N)\max_{\ell=1,\ldots,k+1}\|f_\sigma^{(\ell)}\|_\infty^r\|\mathrm{D}^kW_\beta\|_\infty^{kr}\max_{0\le j\le k}\int_{\mathbb{R}^d}|\mathrm{D}^{j+1}W_\beta(x)|^r dx \\ &\le C(k,N)\max_{\ell=1,\ldots,k+1}\|f_\sigma^{(\ell)}\|_\infty^r\beta^{-(d+k)kr}\beta^{-(d+k+1)r+d} \le C(k,N,\sigma)\beta^{-(d+k)(k+1)r-r},\end{aligned}$$

since $\|\mathrm{D}^kW_\beta\|_\infty \le C\beta^{-(d+k)}$ and $\|\mathrm{D}^{j+1}W_\beta\|_r \le C\beta^{-(d+j+1)+d/r}$. This verifies (54). We infer from (54) with $k=2$ that

$$I_{11} \le C\beta^{-3d-7}\int_0^t \sup_{0<r<s}\max_{i=1,\ldots,N}|(X_i^N-\bar{X}_i^N)(r)|ds.$$

The term I_{13} is estimated in a similar way, with Ψ replaced by $\bar\Psi$:

$$I_{13} \le C\beta^{-3d-7}\int_0^t \sup_{0<r<s}\max_{i=1,\ldots,N}|(X_i^N-\bar{X}_i^N)(r)|ds.$$

The estimate of the remaining term I_{12} is more involved. Since W_β is assumed to be symmetric, we find that

$$I_{12} = \bigg|\int_0^t\int_{\mathbb{R}^d}\mathcal{K}_\zeta(y)\nabla\bigg\{f_\sigma\bigg(\frac{1}{N}\sum_{j=1,\,j\neq i}^{N}W_\beta(X_j^N(s)-\bar{X}_i^N(s)+y)\bigg)$$

$$
\begin{aligned}
&\quad - f_\sigma\bigg(\frac{1}{N}\sum_{j=1,\,j\neq i}^N W_\beta(\bar X_j^N(s)-X_i^N(s)+y)\bigg)\bigg\}dyds\bigg| \\
&\le C\int_0^t\int_{\mathbb{R}^d}\mathcal{K}_\zeta(y)\bigg|f_\sigma'\bigg(\frac{1}{N}\sum_{j\neq i}W_\beta(X_j^N(s)-\bar X_i^N(s)+y)\bigg) \\
&\quad \times\frac{1}{N}\sum_{j\neq i}\nabla\big(W_\beta(X_j^N(s)-\bar X_i^N(s)+y)-W_\beta(\bar X_j^N(s)-X_i^N(s)+y)\big) \\
&\quad +\bigg\{f_\sigma'\bigg(\frac{1}{N}\sum_{j\neq i}W_\beta(X_j^N(s)-\bar X_i^N(s)+y)\bigg)-f_\sigma'\bigg(\frac{1}{N}\sum_{j\neq i}W_\beta(\bar X_j^N(s)-X_i^N(s)+y)\bigg)\bigg\} \\
&\quad \times\frac{1}{N}\sum_{j\neq i}\nabla W_\beta(\bar X_j^N(s)-X_i^N(s)+y)\bigg|dyds \\
&\le C\|f_\sigma'\|_\infty\int_0^t\sup_{0<s<t}\max_{i=1,\ldots,N}|(X_i^N-\bar X_i^N)(s)|\frac{1}{N}\sum_{j\neq i}\int_{\mathbb{R}^d}\mathcal{K}_\zeta(y)|\mathrm{D}^2W_\beta(y+\xi_{ij}(s))|dyds \\
&\quad + C\|f_\sigma''\|_\infty\int_0^t\sup_{0<s<t}\max_{i=1,\ldots,N}|(X_i^N-\bar X_i^N)(s)|\,\|\mathcal{K}_\zeta*\nabla W_\beta\|_\infty ds,
\end{aligned}
$$

where $\xi_{ij}(s)$ is a random value. We write $\mathcal{K}^1=\mathcal{K}|_{B_1}$, $\mathcal{K}^2=\mathcal{K}|_{\mathbb{R}^d\setminus B_1}$ and note that $\widetilde{\mathcal{K}}_\zeta\le\mathcal{K}$ for all $\zeta>0$. Then

$$
\begin{aligned}
\int_{\mathbb{R}^d}\mathcal{K}_\zeta(y)|\mathrm{D}^2W_\beta(y+\xi_{ij}(s))|dy &\le \int_{B_{1+\zeta}}(\mathcal{K}^1*W_\zeta)(y)|\mathrm{D}^2W_\beta(y+\xi_{ij}(s))|dy \\
&\quad + \int_{\mathbb{R}^d\setminus B_{1-\zeta}}(\mathcal{K}^2*W_\zeta)(y)|\mathrm{D}^2W_\beta(y+\xi_{ij}(s))|dy \\
&\le \|\mathcal{K}^1*W_\zeta\|_{L^{\theta/(\theta-1)}(B_{1+\zeta})}\|\mathrm{D}^2W_\beta(\cdot+\xi_{ij}(s))\|_{L^\theta(B_{1+\zeta})} \\
&\quad + \|\mathcal{K}^2*W_\zeta\|_\infty\|\mathrm{D}^2W_\beta(\cdot+\xi_{ij}(s))\|_{L^1(\mathbb{R}^d\setminus B_{1-\zeta})} \\
&\le \|\mathcal{K}^1\|_{L^{\theta/(\theta-1)}(B_1)}\|\mathrm{D}^2W_\beta(\cdot+\xi_{ij}(s))\|_{L^\theta(B_{1+\zeta})}+\|\mathcal{K}^2\|_\infty\|\mathrm{D}^2W_\beta(\cdot+\xi_{ij}(s))\|_{L^1(\mathbb{R}^d)} \\
&\le C\big(\|\mathrm{D}^2W_\beta\|_\infty+\|\mathrm{D}^2W_\beta\|_1\big)\le C\beta^{-d-2}.
\end{aligned}
$$

Observe that we did not use the compact support for $\widetilde{\mathcal{K}}_\zeta$ (which depends on ζ), because a negative exponent of ζ at this point would lead to a logarithmic connection between ζ and N in the end, which we wish to avoid.

Furthermore, by the convolution, Sobolev, and Hardy–Littlewood–Sobolev inequalities as well as the fact that $|\widetilde{\mathcal{K}}_\zeta*\nabla W_\beta|=|(\mathcal{K}w_\zeta)*W_\zeta*\nabla W_\beta|\le\mathcal{K}*|W_\zeta|*|\nabla W_\beta|$,

$$
\begin{aligned}
\|\mathcal{K}_\zeta*\nabla W_\beta\|_\infty &= \|W_\zeta*\widetilde{\mathcal{K}}_\zeta*\nabla W_\beta\|_\infty\le\|\widetilde{\mathcal{K}}_\zeta*\nabla W_\beta\|_\infty\le\|\widetilde{\mathcal{K}}_\zeta*\nabla W_\beta\|_\infty \\
&\le C\|\widetilde{\mathcal{K}}_\zeta*\nabla W_\beta\|_{W^{1,p}(\mathbb{R}^d)}\le C(\|\mathcal{K}*|\nabla W_\beta|\|_p^p+\|\mathcal{K}*|D^2W_\beta|\|_p^p)^{1/p} \\
&\le C\|\nabla W_\beta\|_{W^{1,r}(\mathbb{R}^d)}\le C\beta^{-d-2+d/r},
\end{aligned}
$$

where we recall that $r > d/(2s)$ and we choose $p > d$ satisfying $1/p = 2s/d - 1/r$. The previous two estimates lead to

$$I_{12} \leq C(\sigma)\beta^{-d-2} \int_0^t \sup_{0<r<s} \max_{i=1,\ldots,N} |(X_i^N - \bar{X}_i^N)(r)| ds.$$

We summarize:

$$I_1 \leq C(\sigma)\beta^{-3d-7} \int_0^t \sup_{0<r<s} \max_{i=1,\ldots,N} |(X_i^N - \bar{X}_i^N)(r)| ds. \tag{55}$$

Step 2: Estimate of I_2. We take the expectation of I_2 and use the mean-value theorem:

$$\begin{aligned} \mathbb{E}(I_2) &= \int_0^t \mathbb{E}\bigg| \int_{\mathbb{R}^d} \nabla \mathcal{K}_\zeta(y) \bigg\{ f_\sigma\bigg(\frac{1}{N} \sum_{j \neq i} W_\beta(\bar{X}_j^N(s) - \bar{X}_i^N(s) + y) \bigg) \\ &\qquad - f_\sigma\big(W_\beta * \rho(\bar{X}_i^N(s) - y, s) \big) \bigg\} dy \bigg| ds \\ &\leq N^{-1} \|f_\sigma'\|_\infty \|\widetilde{\mathcal{K}}_\zeta * \nabla W_\zeta\|_1 \int_0^t \sup_{y \in \mathbb{R}^d} \mathbb{E}\bigg(\sum_{j \neq i} |b_{ij}(y,s)| \bigg) ds, \end{aligned} \tag{56}$$

where

$$b_{ij}(y,s) = W_\beta(\bar{X}_j^N(s) - \bar{X}_i^N(s) + y) - \frac{N}{N-1} W_\beta * \rho(\bar{X}_i^N(s) - y, s).$$

We deduce from $\|\nabla W_\zeta\|_{L^1(\mathbb{R}^d)} \leq C\zeta^{-1}$ that

$$\|\widetilde{\mathcal{K}}_\zeta * \nabla W_\zeta\|_1 \leq C\zeta^{-1} \|\widetilde{\mathcal{K}}_\zeta\|_1 \leq C\zeta^{-2s-1},$$

due to the compact support of $\widetilde{\mathcal{K}}_\zeta(x) = |x|^{2s-d}\omega_\zeta(x) \leq C|x|^{2s-d} 1_{|x| \leq 2\zeta^{-1}}$ and

$$\int_{\{|x|<2/\zeta\}} |x|^{2s-d} dx = \int_{\{|y|<2\}} \zeta^{-d} |y/\zeta|^{2s-d} dy = C\zeta^{-2s}.$$

We claim that $\mathbb{E}(\sum_{j \neq i} |b_{ij}(y,s)|) \leq C(\sigma)\beta^{-d/2} N^{1/2}$ for all $y \in \mathbb{R}^d$. To show the claim, we compute the expectation $\mathbb{E}[(\sum_{j \neq i} b_{ij}(y,s))^2]$. We estimate first the terms with $k \neq j$ (omitting the argument (y,s) to simplify the notation). Then an elementary but tedious computation leads to

$$\begin{aligned} \mathbb{E}(b_{ji}b_{ki}) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \bigg(W_\beta(x_j - x_i + y) - \frac{N}{N-1} W_\beta * \rho(x_i - y) \bigg) \\ &\quad \times \bigg(W_\beta(x_k - x_i + y) - \frac{N}{N-1} W_\beta * \rho(x_i - y) \bigg) \rho(x_i)\rho(x_j)\rho(x_k) dx_i dx_j dx_k \\ &= \int_{\mathbb{R}^d} \bigg(W_\beta * \rho(x_i - y) - \frac{N}{N-1} W_\beta * \rho(x_i - y) \bigg)^2 \rho(x_i) dx_i \\ &\leq N^{-2} \|\rho\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \|W_\beta * \rho\|^2_{L^\infty(0,T;L^2(\mathbb{R}^d))} \\ &\leq C(\sigma) N^{-2} \|W_\beta\|_1^2 \leq C(\sigma) N^{-2}. \end{aligned}$$

The diagonal terms contribute in the following way:

$$\begin{aligned}\mathbb{E}(b_{ji}^2) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\bigg(W_\beta(x_j-x_i+y) - \frac{N}{N-1}W_\beta*\rho(x_i-y)\bigg)^2\rho(x_i)\rho(x_j)dx_idx_j \\ &= \int_{\mathbb{R}^d}\bigg((W_\beta^2*\rho)(x_i-y) - \frac{2N}{N-1}(W_\beta*\rho)(x_i-y)^2 \\ &\quad + \frac{N^2}{(N-1)^2}(W_\beta*\rho)(x_i-y)^2\bigg)\rho(x_i)dx_i \\ &\le C(\sigma)\big(\|W_\beta^2*\rho\|_{L^\infty(0,T;L^1(\mathbb{R}^d))} + \|W_\beta*\rho\|^2_{L^\infty(0,T;L^2(\mathbb{R}^d))}\big) \le C(\sigma)\beta^{-d},\end{aligned}$$

since $\|W_\beta^2*\rho\|_2 \le \|W_\beta^2\|_1\|\rho\|_2 \le C\|W_\beta\|_2^2 \le \beta^{-d}C$. This shows that

$$\mathbb{E}\bigg(\sum_{j\ne i}|b_{ji}(y,s)|\bigg) \le \bigg(\mathbb{E}\bigg[\sum_{j\ne i}b_{ji}(y,s)\bigg]^2\bigg)^{1/2} \le C(\sigma)\beta^{-d/2}N^{1/2}.$$

We infer that (56) becomes

$$I_2 \le C(\sigma)\zeta^{-2s-1}\beta^{-d/2}N^{-1/2}. \tag{57}$$

Step 3: End of the proof. We insert (55) and (57) into (53) to infer that

$$\begin{aligned}E_1(t) &:= \mathbb{E}\bigg(\sup_{0<s<t}\max_{i=1,\ldots,N}|(X_i^N-\bar{X}_i^N)(s)|\bigg) \\ &\le C(\sigma)\beta^{-3d-7}\int_0^t E_1(s)ds + C(\sigma)\zeta^{-2s-1}\beta^{-d/2}N^{-1/2}.\end{aligned}$$

By Gronwall's lemma,

$$E_1(t) \le C(\sigma)\zeta^{-2s-1}\beta^{-d/2}N^{-1/2}\exp\big(C(\sigma)\beta^{-3d-7}T\big), \quad 0\le t\le T.$$

We choose $\varepsilon = \widetilde{\delta}/(C(\sigma)T)$ for some arbitrary $\widetilde{\delta}\in(0,1/4)$. Then, since by assumption, $\beta^{-d/2}\le\beta^{-3d-7}\le\varepsilon\log N$ and $\zeta^{-2s-1}\le C_1N^{1/4}$, we find that

$$E_1(t) \le C(\sigma)C_1\varepsilon\log(N)N^{-1/4}\exp\big(C(\sigma)T\varepsilon\log N\big) = \frac{C_1\widetilde{\delta}}{T}\log(N)N^{-1/4+\widetilde{\delta}},$$

proving the result. □

4.3. Estimate of $\bar{X}_i^N - \widehat{X}_i^N$.

Lemma 17. *Let $\bar{X}_i^N$ and $\widehat{X}_i^N$ be the solutions to* (4) *and* (6)*, respectively. Then there exists a constant $C>0$, depending on σ, such that*

$$\mathbb{E}\bigg(\sup_{0<t<T}\max_{i=1,\ldots,N}|(\bar{X}_i^N-\widehat{X}_i^N)(t)|\bigg) \le C(\beta+\zeta^a),$$

where $a := \min\{1, d-2s\}$.

Proof. We compute the difference

$$\begin{aligned}|(\bar X_i^N-\widehat X_i^N)(t)| &= \bigg|\int_0^t\big(\nabla\mathcal{K}_\zeta * f_\sigma(W_\beta*\rho(\bar X_i^N(s),s)) - \nabla\mathcal{K}*f_\sigma(\rho_\sigma(\widehat X_i^N(s),s))\big)ds\bigg| \\ &\le J_1+J_2+J_3,\end{aligned}$$

where $\rho:=\rho_{\sigma,\beta,\zeta}$, the convolution is taken with respect to x_i, and

$$\begin{aligned}J_1 &= \bigg|\int_0^t \nabla\mathcal{K}_\zeta*\big(f_\sigma(W_\beta*\rho(\bar X_i^N(s),s)) - f_\sigma(W_\beta*\rho(\widehat X_i^N(s),s))\big)ds\bigg|, \\ J_2 &= \bigg|\int_0^t \nabla\mathcal{K}_\zeta*\big(f_\sigma(W_\beta*\rho(\widehat X_i^N(s),s)) - f_\sigma(\rho_\sigma(\widehat X_i^N(s),s))\big)ds\bigg|, \\ J_3 &= \bigg|\int_0^t \nabla(\mathcal{K}_\zeta-\mathcal{K})*f_\sigma(\rho_\sigma(\widehat X_i^N(s),s))ds\bigg|.\end{aligned}$$

Step 1: Estimate of J_1. We write $\nabla\mathcal{K}_\zeta * f_\sigma(\cdots)=\mathcal{K}_\zeta*\nabla f_\sigma$ and add and subtract the expression $f_\sigma'(W_\beta*\rho(\bar X_i^N-y))\nabla W_\beta*\rho(\widehat X_i^N-y)$:

$$\begin{aligned}J_1 &= \int_0^t\int_{\mathbb{R}^d}\mathcal{K}_\zeta(y)\Big(f_\sigma'(W_\beta*\rho(\bar X_i^N(s)-y))\nabla W_\beta*\big[\rho(\bar X_i^N(s)-y)-\rho(\widehat X_i^N(s)-y)\big] \\ &\quad - \big[f_\sigma'(W_\beta*\rho(\widehat X_i^N(s)-y)) - f_\sigma'(W_\beta*\rho(\bar X_i^N(s)-y))\big]\nabla W_\beta*\rho(\widehat X_i^N(s)-y)\Big)dyds \\ &\le \|f_\sigma'\|_\infty\int_0^t\int_{\mathbb{R}^d}\Big|\mathcal{K}_\zeta(y)\nabla W_\beta*\big(\rho(\bar X_i^N(s)-y)-\rho(\widehat X_i^N(s)-y)\big)\Big|dyds \\ &\quad + \|f_\sigma''\|_\infty\|\nabla W_\beta*\rho\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \\ &\quad \times\int_0^t\int_{\mathbb{R}^d}\big|\mathcal{K}_\zeta(y)W_\beta*\big(\rho(\widehat X_i^N(s)-y)-\rho(\bar X_i^N(s)-y)\big)\big|dyds.\end{aligned}$$

By the mean-value theorem and using $\|W_\beta\|_1=1$, we obtain for some random variable $\xi_{ij}(s)$,

$$\begin{aligned}J_1 &\le \|f_\sigma\|_{W^{2,\infty}(\mathbb{R})}\|\nabla\rho\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\int_0^t\sup_{0<r<s}\sup_{i=1,\ldots,N}|(\bar X_i^N-\widehat X_i^N)(r)| \\ &\quad\times\int_{\mathbb{R}^d}\sum_{k=1}^2\big|\mathcal{K}_\zeta(y)\mathrm{D}^kW_\beta*\rho(y+\xi_{ij}(s),s)\big|dyds. \qquad (58)\end{aligned}$$

We need to estimate the last integral. For this, we write for $k=1,2$

$$\begin{aligned}&\int_{\mathbb{R}^d}\big|\mathcal{K}_\zeta(y)\mathrm{D}^kW_\beta*\rho(y+\xi_{ij}(s),s)\big|dy \le K_1^k+K_2^k,\quad\text{where} \\ &K_1^k := \int_{B_{1+\zeta}}\big|\mathcal{K}^1*W_\zeta(y)\mathrm{D}^kW_\beta*\rho(y+\xi_{ij}(s),s)\big|dy, \\ &K_2^k := \int_{\mathbb{R}^d\setminus B_{1-\zeta}}\big|\mathcal{K}^2*W_\zeta(y)\mathrm{D}^kW_\beta*\rho(y+\xi_{ij}(s),s)\big|dy,\end{aligned}$$

where $\mathcal{K}^1 = \mathcal{K}|_{B_1}$ and $\mathcal{K}^2 = \mathcal{K}|_{\mathbb{R}^d\setminus B_1}$. Note that $\widetilde{\mathcal{K}}_\zeta \le \mathcal{K}$. A similar argument as for the estimate of I_{12} in the proof of Lemma 16 shows that for $\theta > \max\{d/(2s), d\}$,

$$\begin{aligned} K_1^k + K_2^k &\le C\big(\|\mathrm{D}^k W_\beta * \rho\|_{L^\infty(0,T;L^\theta(\mathbb{R}^d))} + \|\mathrm{D}^k W_\beta * \rho\|_{L^\infty(0,T;L^1(\mathbb{R}^d))}\big) \\ &\le C\big(\|\mathrm{D}^k \rho\|_{L^\infty(0,T;L^\theta(\mathbb{R}^d))} + \|\mathrm{D}^k \rho\|_{L^\infty(0,T;L^1(\mathbb{R}^d))}\big) \le C(\sigma), \end{aligned}$$

where we used Proposition 13 ((39) and (41)) with $p = \theta$ in the last inequality. We conclude from (58) that

$$J_1 \le C(\sigma) \int_0^t \sup_{0<r<s} \max_{i=1,\ldots,N} |(\bar{X}_i^N - \widehat{X}_i^N)(r)| ds. \tag{59}$$

Step 2: Estimate of J_2. We treat the two cases $s < 1/2$ and $s \ge 1/2$ separately. Let first $s \ge 1/2$. Then

$$\begin{aligned} J_2 &= \left| \int_0^t \nabla\widetilde{\mathcal{K}}_\zeta * W_\zeta * \big(f_\sigma(W_\beta * \rho(\widehat{X}_i^N(s), s)) - f_\sigma(\rho_\sigma(\widehat{X}_i^N(s), s))\big) ds \right| \\ &\le T\|\nabla\widetilde{\mathcal{K}}_\zeta * (f_\sigma(W_\beta * \rho) - f_\sigma(\rho_\sigma))\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}. \end{aligned}$$

Recalling the definition of $\widetilde{\mathcal{K}}_\zeta = \mathcal{K}\omega_\zeta$ in (11) and writing $\nabla\widetilde{\mathcal{K}}_\zeta * u = \nabla\mathcal{K} * u - [(1-\omega_\zeta)\nabla\mathcal{K}] * u + [\mathcal{K}\nabla\omega_\zeta] * u$ for $u = f_\sigma(W_\beta * \rho) - f_\sigma(\rho_\sigma)$, we find that

$$\begin{aligned} J_2 \le C(T)\big(&\|\nabla\mathcal{K} * u\|_{L^\infty([0,T];L^\infty(\mathbb{R}^d))} + \|[(1-\omega_\zeta)\nabla\mathcal{K}] * u\|_{L^\infty([0,T];L^\infty(\mathbb{R}^d))} \\ &+ \|[\mathcal{K}\nabla\omega_\zeta] * u\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\big). \end{aligned} \tag{60}$$

We estimate the right-hand side term by term. Because of

$$\nabla\mathcal{K} * v = \begin{cases} \nabla(-\Delta)^{-1/2} v & \text{for } s = 1/2 \\ (\nabla\mathcal{K}) * v & \text{for } s > 1/2, \end{cases}$$

we use Sobolev's embedding $W^{1,p}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ for any $p > d$ and then the boundedness of the Riesz operator $\nabla(-\Delta)^{-1/2} : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ [45, Chapter IV, §3.1] in case $s = 1/2$ or the Hardy–Littlewood–Sobolev inequality for $\widetilde{\alpha} = \alpha - 1/2 > 0$ (see Lemma 21) in case $s > 1/2$ to control the first norm in (60) by

$$\begin{aligned} \|\nabla\mathcal{K} * u\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} &\le C\Bigg(\|\nabla\mathcal{K} * u\|_{L^\infty(0,T;L^p(\mathbb{R}^d))} + \sum_{j=1}^d \|\nabla\mathcal{K} * D^j u\|_{L^\infty(0,T;L^p(\mathbb{R}^d))} \Bigg) \\ &\le C\|u\|_{L^\infty(0,T;W^{1,r}(\mathbb{R}^d))} = C\|f_\sigma(W_\beta * \rho) - f_\sigma(\rho_\sigma)\|_{L^\infty(0,T;W^{1,r}(\mathbb{R}^d))}, \end{aligned}$$

where $r = p$ in case $s = 1/2$ and $r = pd/(d + 2s - 1)$ in case $s > 1/2$. Choosing $p > d + (2s-1)$ guarantees that $r > d$ always holds.

For the second norm in (60), Hölder's inequality yields for $q > d$ and $1/q + 1/q' = 1$, for every $t > 0$,

$$\begin{aligned} \|[(1-\omega_\zeta)\nabla\mathcal{K}] * u(t)\|_{L^\infty(\mathbb{R}^d)} &\le \|1-\omega_\zeta\|_{L^\infty(\mathbb{R}^d)} \|\nabla\mathcal{K}\|_{L^{q'}(\{|x|>2\zeta^{-1}\})} \|u(t)\|_{L^q(\mathbb{R}^d)} \\ &\le \|\nabla\mathcal{K}\|_{L^{q'}(\{|x|>2\zeta^{-1}\})} \|u(t)\|_{L^q(\mathbb{R}^d)}, \end{aligned}$$

which can be bounded by $C\zeta^{1-2s+d/q}\|u(t)\|_{L^q(\mathbb{R}^d)}$, since

$$\begin{aligned}\|\nabla\mathcal{K}\|^{q'}_{L^{q'}(\{|x|>2\zeta^{-1}\})} &\le C\int_{\{|x|>2\zeta^{-1}\}}|x|^{(2s-d-1)q'}dx = C\zeta^{-d}\int_{\{|y|>2\}}|y/\zeta|^{(2s-d-1)q'}dy \\ &\le C\zeta^{-d+(1+d-2s)q'}.\end{aligned}$$

By similar arguments and the fact that $\|\nabla\omega_\zeta\|_{L^\infty}\le C\zeta$, we find that

$$\|\mathcal{K}\nabla\omega_\zeta\|_{L^{q'}(\{|x|<2\zeta^{-1}\})}\le C\zeta^{1+d-2s-d/q'},$$

and hence, using $q'=q/(q-1)$, we conclude for the second and third term in (60) that

$$\|[(1-\omega_\zeta)\nabla\mathcal{K}]*u(t)\|_{L^\infty(\mathbb{R}^d)}+\|[\mathcal{K}\nabla\omega_\zeta]*u(t)\|_{L^\infty(\mathbb{R}^d)}\le C\zeta^{1-2s+d/q}\|u(t)\|_{L^q(\mathbb{R}^d)}.$$

The choice $d<q\le d/(2s-1)$ guarantees on the one hand that $q>d$ and on the other hand that the exponent $1-2s+d/q$ is strictly positive (which allows us to use the property $\zeta^{1-2s+d/q}<1$).

Using these estimates in (60), we arrive (for $s\ge 1/2$) at

$$J_2\le C(T)\big(\|f_\sigma(W_\beta*\rho)-f_\sigma(\rho_\sigma)\|_{L^\infty(0,T;W^{1,r}(\mathbb{R}^d))}+\|f_\sigma(W_\beta*\rho)-f_\sigma(\rho_\sigma)\|_{L^\infty(0,T;L^q(\mathbb{R}^d))}\big),$$

where we recall that $r,q>d$. These norms can be estimated by $\|f_\sigma(W_\beta*\rho(t))-f_\sigma(\rho_\sigma(t))\|_{L^q(\mathbb{R}^d)}\le\|f'_\sigma\|_\infty\|W_\beta*\rho(t)-\rho_\sigma(t)\|_{L^q(\mathbb{R}^d)}$ and

$$\begin{aligned}\|\nabla(f_\sigma(W_\beta*\rho)-f_\sigma(\rho_\sigma))(t)\|_{L^r(\mathbb{R}^d)} &\le \|f'_\sigma\|_\infty\|(W_\beta*\nabla\rho-\nabla\rho_\sigma)(t)\|_{L^r(\mathbb{R}^d)} \\ &\quad+\|f''_\sigma\|_\infty\|(W_\beta*\rho-\rho_\sigma)(t)\|_{L^r(\mathbb{R}^d)}\|\nabla\rho_\sigma(t)\|_{L^\infty(\mathbb{R}^d)}.\end{aligned}$$

The $L^\infty(\mathbb{R}^d\times(0,T))$ bound for $\nabla\rho_\sigma$ from Lemma 11 and the definition of f_σ finally show for $s\ge 1/2$ and $r,q>d$ that

$$J_2\le C(\sigma,T)\big(\|W_\beta*\rho-\rho_\sigma\|_{L^\infty(0,T;W^{1,r}(\mathbb{R}^d))}+\|W_\beta*\rho-\rho_\sigma\|_{L^\infty(0,T;L^q(\mathbb{R}^d))}\big). \tag{61}$$

Now, let $s<1/2$. In this case, we cannot estimate $\nabla\mathcal{K}$ and put the gradient to the second factor of the convolution. Adding and subtracting an appropriate expression as in Step 1, using the embedding $W^{1,p}(\mathbb{R}^d)\hookrightarrow L^\infty(\mathbb{R}^d)$ for $p>d$, the estimate $\mathcal{K}_\zeta\le\mathcal{K}$, and the Hardy–Littlewood–Sobolev inequality, we find that

$$\begin{aligned}J_2 &= \bigg|\int_0^t\int_{\mathbb{R}^d}\mathcal{K}_\zeta(y)\Big(\big(f'_\sigma(W_\beta*\rho(\widehat{X}_i^N(s)-y))-f'_\sigma(\rho_\sigma(\widehat{X}_i^N(s)-y))\big)\nabla W_\beta*\rho(\widehat{X}_i^N(s)-y) \\ &\qquad -f'_\sigma(\rho_\sigma(\widehat{X}_i^N(s)-y))\big(\nabla\rho_\sigma(\widehat{X}_i^N(s)-y)-\nabla W_\beta*\rho(\widehat{X}_i^N(s)-y)\big)\Big)dyds\bigg| \\ &\le \|f''_\sigma\|_\infty\|W_\beta*\nabla\rho\|_\infty\int_0^t\int_{\mathbb{R}^d}\mathcal{K}_\zeta(y)\big|\rho_\sigma(\widehat{X}_i^N(s)-y)-W_\beta*\rho(\widehat{X}_i^N(s)-y)\big|dyds \\ &\qquad +\|f'_\sigma\|_\infty\int_0^t\int_{\mathbb{R}^d}\mathcal{K}_\zeta(y)\big|\nabla\rho_\sigma(\widehat{X}_i^N(s)-y)-W_\beta*\nabla\rho(\widehat{X}_i^N(s)-y)\big|dyds \\ &\le \max\{\|\nabla\rho\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))},1\}\|f'_\sigma\|_{W^{1,\infty}}T\big(\|\mathcal{K}*|(W_\beta*\rho-\rho_\sigma)|\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \\ &\qquad +\|\mathcal{K}*|(W_\beta*\nabla\rho-\nabla\rho_\sigma)|\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\big)\end{aligned}$$

$$\le C(\sigma,T)\big(\|\nabla\rho\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}+1\big)\sum_{|\alpha|\le 2}\|W_\beta*\mathrm{D}^\alpha\rho-\mathrm{D}^\alpha\rho_\sigma\|_{L^\infty(0,T;L^r(\mathbb{R}^d))},$$

where $r>d$ is such that $1/r=2s/d+1/p$ (this is needed for the Hardy–Littlewood–Sobolev inequality) and $p>d$ (because of Sobolev's emebdding). Note that $r>d$ can be only guaranteed if $s<1/2$. Together with the fact that $\|\nabla\rho\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\le C(\sigma)$ (choose $q>d$ in (41) and use Sobolev's embedding), this shows that for $s<1/2$,

$$J_2\le C(\sigma,T)\sum_{|\alpha|\le 2}\|W_\beta*\mathrm{D}^\alpha\rho-\mathrm{D}^\alpha\rho_\sigma\|_{L^\infty(0,T;L^r(\mathbb{R}^d))}. \tag{62}$$

It follows from estimate (38) and Lemma 20 in Appendix A for $p>d$ that

$$\|(W_\beta*D^\alpha\rho-D^\alpha\rho_\sigma)(t)\|_{L^p(\mathbb{R}^d)}\le C\big(\|D^\alpha\nabla\rho\|_{L^p(\mathbb{R}^d)}\beta+\beta+\zeta^a\big)\le C(\sigma,T)(\beta+\zeta^a),$$

where we used the $L^\infty(0,T;W^{3,p}(\mathbb{R}^d))$ estimate for $\rho=\rho_{\sigma,\beta,\zeta}$ in (41). Then we deduce from estimates (61) and (62) that for all $0<s<1$,

$$J_2\le C(\sigma,T)(\beta+\zeta^a),$$

where we recall that $a=\min\{1,d-2s\}$.

Step 3: Estimate of J_3 and end of the proof. Arguing similarly as in Section 3.3, we have

$$\|(\mathcal{K}_\zeta-\mathcal{K})*\nabla\rho_\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\le C\zeta^a\big(\|\mathrm{D}^2\rho_\sigma\|_{L^\infty(0,T;L^p(\mathbb{R}^d))}+\|\nabla\rho_\sigma\|_{L^\infty(0,T;L^1(\mathbb{R}^d))}\big).$$

This implies that

$$J_3\le\|f'_\sigma\|_\infty\|(\mathcal{K}_\zeta-\mathcal{K})*\nabla\rho_\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\le C(\sigma)\zeta^a. \tag{63}$$

Taking the expectation, we infer from (59)–(63) that

$$E_2(t):=\mathbb{E}\bigg(\sup_{0<s<t}\max_{i=1,\ldots,N}|(\bar X_i^N-\widehat X_i^N)(s)|\bigg)\le C(\sigma)(\beta+\zeta^a)+C(\sigma)\int_0^t E_2(s)ds,$$

An application of Gronwall's lemma gives the result. □

4.4. **Proof of Theorems 2 and 3.** Lemmas 16 and 17 show that

$$\mathbb{E}\bigg(\sup_{0<s<T}\max_{i=1,\ldots,N}|(X_i^N-\widehat X_i^N)(s)|\bigg)\le C(N^{-1/4+\delta}+\beta+\zeta^{\min\{1,d-2s\}}),$$

and this expression converges to zero as $N\to\infty$ and $(\beta,\zeta)\to 0$ under the conditions stated in Theorem 2. This result implies the convergence in probability of the k-tuple $(X_1^N,\ldots,X_k^N)$ to $(\widehat X_1^N,\ldots,\widehat X_k^N)$. Since convergence in probability implies convergence in distribution, we obtain

$$\lim_{N\to\infty,\,(\beta,\zeta)\to 0}\mathrm{P}^k_{N,\beta,\sigma}(t)=\mathrm{P}_\sigma^{\otimes k}(t)\quad\text{locally uniform in time},$$

where $\mathrm{P}^k_{N,\beta,\sigma}(t)$ and $\mathrm{P}_\sigma^{\otimes k}(t)$ denote the joint distributions of $(X_1^N,\ldots,X_k^N)(t)$ and $(\widehat X_1^N,\ldots,\widehat X_k^N)(t)$, respectively. By Section 4.1, $\mathrm{P}_\sigma(t)$ is absolutely continuous with the density function $\rho_\sigma(t)$. Using the test function $\phi = 1_{(-\infty,x]^d}$ in Corollary 12, we have, up to a subsequence,

$$\mathrm{P}_\sigma(t,(-\infty,x]^d) = \int_{(-\infty,x]^d} \rho_\sigma(y,t)dy \to \int_{(-\infty,x]^d} \rho(y,t)dy =: \mathrm{P}(t,(-\infty,x]^d)$$

locally uniformly for $t > 0$. Since the convergence also holds for the initial condition, the result is shown.

Appendix A. Auxiliary results

We recall some known results. The following result is proved in [4, Theorem 4.33].

Lemma 18 (Young's convolution inequality)**.** *Let $1 \le p,\ r \le \infty$, $u \in L^p(\mathbb{R}^d)$, $v \in L^q(\mathbb{R}^d)$, and $1/p + 1/q = 1 + 1/r$. Then $u * v \in L^r(\mathbb{R}^d)$ and*

$$\|u * v\|_r \le \|u\|_p \|v\|_q.$$

The following lemma slightly extends [39, Lemma 7.3] from the L^2 to the L^p setting.

Lemma 19. *Let $p \ge 2$ and $T > 0$. Then the following embedding is continuous:*

$$L^p(0,T;W^{1,p}(\mathbb{R}^d)) \cap W^{1,p}(0,T;W^{-1,p}(\mathbb{R}^d)) \hookrightarrow C^0([0,T];L^p(\mathbb{R}^d)).$$

Proof. Let $u \in L^p(0,T;W^{1,p}(\mathbb{R}^d)) \cap W^{1,p}(0,T;W^{-1,p}(\mathbb{R}^d))$ and $0 \le t_1 \le t_2 \le T$. Then

$$\begin{aligned}(64)\quad \left| \int_{\mathbb{R}^d} |u(t_2)|^p dx - \int_{\mathbb{R}^d} |u(t_1)|^p dx \right| &= \left| \int_{t_1}^{t_2} \langle \partial_t u, p|u|^{p-2}u \rangle dt \right| \\ &\le p\|\partial_t u\|_{L^p(t_1,t_2;W^{-1,p}(\mathbb{R}^d))} \| |u|^{p-2}u\|_{L^{p'}(t_1,t_2;W^{1,p'}(\mathbb{R}^d))},\end{aligned}$$

where $p' = p/(p-1)$. Direct computations using Young's inequality lead to

$$\begin{aligned}\| |u|^{p-2}u\|^{p'}_{L^{p'}(t_1,t_2;W^{1,p'}(\mathbb{R}^d))} &= C\int_{t_1}^{t_2}\int_{\mathbb{R}^d} \big(|u|^p + |u|^{p'(p-2)}|\nabla u|^{p'}\big)dxdt \\ &\le C\int_{t_1}^{t_2} \|u(t)\|^p_{W^{1,p}(\mathbb{R}^d)}dt.\end{aligned}$$

We infer from (64) and the continuity of the integrals with respect to the time integration boundaries that $t \mapsto \|u(t)\|_p$ is continuous and

$$(65)\quad \sup_{0<t<T} \|u(t)\|_p \le \|u(0)\|_p + C\|\partial_t u\|_{L^p(t_1,t_2;W^{-1,p}(\mathbb{R}^d))} + C\|u\|_{L^p(0,T;W^{1,p}(\mathbb{R}^d))}.$$

Next, let $t \in [0,T]$ be arbitrary and let $\tau_n \to 0$ as $n \to \infty$ such that $t + \tau_n \in [0,T]$. Estimate (65) implies that $(u(t+\tau_n))_{n\in\mathbb{N}}$ is bounded in $L^p(\mathbb{R}^d)$. Thus, there exists a subsequence $(\tau_{n'})$ of (τ_n) such that $u(t+\tau_{n'}) \rightharpoonup v(t)$ weakly in $L^p(\mathbb{R}^d)$ as $n' \to \infty$ for some $v(t) \in L^p(\mathbb{R}^d)$. We can show, using estimate (65) and dominated convergence for the integral

$$\int_0^T \int_{\mathbb{R}^d} (u(t+\tau_{n'},x) - v(t,x))\phi(t,x)dx \quad \text{for } \phi \in C_0^\infty(\mathbb{R}^d \times (0,T))$$

that in the limit $n' \to \infty$

$$\int_0^T \int_{\mathbb{R}^d} (u(t,x) - v(t,x))\phi(t,x)dx = 0,$$

which yields $v(t) = u(t)$.

Moreover, since $t \mapsto \|u(t)\|_p$ is continuous, we have $\|u(t+\tau_{n'})\|_p \to \|u(t)\|_p$. Since $L^p(\mathbb{R}^d)$ is uniformly convex, we deduce from [4, Prop. 3.32] that $u(t+\tau_{n'}) \to u(t)$ strongly in $L^p(\mathbb{R}^d)$. Since the limit is unique, the whole sequence converges. Together with (65), this concludes the proof. □

Let $W_1 \in C_0^\infty(\mathbb{R}^d)$ be nonnegative with $\int_{\mathbb{R}^d} W_1(x)dx = 1$ and define $W_\beta(x) = \beta^{-d} W_1(x/\beta)$ for $x \in \mathbb{R}^d$ and $\beta > 0$.

Lemma 20. *Let $1 \le p < \infty$ and $u \in W^{1,p}(\mathbb{R}^d)$. Then*

$$\|W_\beta * u - u\|_p \le C\beta \|\nabla u\|_p.$$

Proof. We use Hölder's inequality and the fact that $\|W_\beta\|_{L^1(\mathbb{R}^d)} = 1$ to find that

$$\begin{aligned}
\|W_\beta * u - u\|_p^p &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} W_\beta(x-y)(u(x)-u(y))dy \right|^p dx \\
&\le \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} W_\beta(x-y)dy \right)^{p-1} \left(\int_{\mathbb{R}^d} W_\beta(x-y)|u(x)-u(y)|^p dy \right) dx \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W_\beta(z)|z|^p \frac{|u(y+z)-u(y)|^p}{|z|^p} dydz \\
&\le \|\nabla u\|_p^p \int_{\mathbb{R}^d} W_\beta(z)|z|^p dz \le C\beta^p \|\nabla u\|_p^p,
\end{aligned}$$

which shows the lemma. □

Appendix B. Fractional Laplacian

We recall that the fractional Laplacian $(-\Delta)^s$ for $0 < s < 1$ can be written as the pointwise formula

$$(-\Delta)^s u(x) = c_{d,s} \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+2s}} dy, \quad \text{where } c_{d,s} = \frac{4^s \Gamma(d/2+s)}{\pi^{d/2}|\Gamma(-s)|}, \tag{66}$$

$u \in H^s(\mathbb{R}^d)$, and the integral is understood as principal value if $1/2 \le s < 1$ [46, Theorem 2]. The inverse fractional Laplacian $(-\Delta)^{-s}$ is defined in (2). The following lemma can be found in [45, Chapter V, Section 1.2].

Lemma 21 (Hardy–Littlewood–Sobolev inequality)**.** *Let $0 < s < 1$ and $1 < p < \infty$. Then there exists a constant $C > 0$ such that for all $u \in L^p(\mathbb{R}^d)$,*

$$\|(-\Delta)^{-s} u\|_q \le C\|u\|_p, \quad \text{where } \frac{1}{p} = \frac{1}{q} + \frac{2s}{d}.$$

Applying Hölder's and then Hardy–Littlewood–Sobolev's inequality gives the following result.

Lemma 22. *Let $0 < s < 1$ and $1 \le p < q < \infty$. Then there exists $C > 0$ such that for all $u \in L^q(\mathbb{R}^d)$, $v \in L^r(\mathbb{R}^d)$,*

$$\|u(-\Delta)^{-s}v\|_p \le C\|u\|_q\|v\|_r, \quad \frac{1}{q}+\frac{1}{r}=\frac{1}{p}+\frac{2s}{d}, \tag{67}$$

$$\|u\nabla(-\Delta)^{-s}v\|_p \le C\|u\|_q\|v\|_r, \quad \frac{1}{q}+\frac{1}{r}=\frac{1}{p}+\frac{2s-1}{d},\ s>\frac{1}{2}. \tag{68}$$

Lemma 23 (Fractional Gagliardo–Nirenberg inequality I)**.** *Let $d \ge 2$ and $1 < p < \infty$. Then there exists $C > 0$ such that for all $u \in W^{1,p}(\mathbb{R}^d)$ or $u \in W^{2,p}(\mathbb{R}^d)$, respectively,*

$$\|(-\Delta)^s u\|_p \le C\|u\|_p^{1-2s}\|\nabla u\|_p^{2s} \quad \text{if } 0 < s \le 1/2,$$
$$\|(-\Delta)^s u\|_p \le C\|u\|_p^{1-s}\|\mathrm{D}^2 u\|_p^{s} \quad \text{if } 1/2 < s \le 1.$$

Proof. It follows from the properties of the Riesz and Bessel potentials [45, Theorem 3, page 96] that the operator $(-\Delta)^s : W^{1,p}(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is bounded for $0 < s \le 1/2$, while the operator $(-\Delta)^s : W^{2,p}(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is bounded for $1/2 < s \le 1$. Thus, if $0 < s \le 1/2$,

$$\|(-\Delta)^s u\|_p \le C(\|u\|_p + \|\nabla u\|_p) \quad \text{for } u \in W^{1,p}(\mathbb{R}^d).$$

Replacing u by $u_\lambda(x) = \lambda^{d/p-2s}u(\lambda x)$ with $\lambda > 0$ yields

$$\|(-\Delta)^s u\|_p = \|(-\Delta)^s u_\lambda\|_p \le C(\|u_\lambda\|_p + \|\nabla u_\lambda\|_p) = C\lambda^{-2s}(\|u\|_p + \lambda\|\nabla u\|_p).$$

We minimize the right-hand side with respect to λ giving the value $\lambda_0 = 2s(1-2s)^{-1}\|u\|_p$ $\|\nabla u\|_p^{-1}$ and therefore,

$$\|(-\Delta)^s u\|_p \le C\|u\|^{1-2s}\|\nabla u\|_p^{2s}.$$

The case $1/2 < s \le 1$ is proved in a similar way. ☐

Lemma 24 (Fractional Gagliardo–Nirenberg inequality II)**.** *Let $d \ge 2$, $0 < s \le 1/2$, $p \in (1,\infty)$, and $q \in [p,\infty)$. If $p < d/(2s)$, we assume additionally that $q \le dp/(d-2sp)$. Then there exists $C > 0$ such that for all $u \in W^{1,p}(\mathbb{R}^d)$,*

$$\|(-\Delta)^{-s}\nabla u\|_q \le C\|u\|_p^{1-\theta}\|\nabla u\|_p^{\theta},$$

where $\theta = 1 + d/p - d/q - 2s \in [0,1]$.

Proof. The statement is true for $s = 1/2$ since the operator $(-\Delta)^{-1/2}\nabla : L^q(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is bounded for any $q \in (1,\infty)$ [45, Theorem 3, page 96]. Then the inequality follows from the standard Gagliardo–Nirenberg inequality.

Thus, let $0 < s < 1/2$. We claim that it is sufficient to prove that $(-\Delta)^{-s}\nabla : W^{1,p}(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is bounded. Indeed, assume that

$$\|(-\Delta)^{-s}\nabla u\|_q \le C(\|u\|_p + \|\nabla u\|_p) \quad \text{for } u \in W^{1,p}(\mathbb{R}^d). \tag{69}$$

Replacing, as in the proof of Lemma 23, u by $u_\lambda(x) = \lambda^{d/q-1+2s}u(\lambda x)$ with $\lambda > 0$ yields

$$\|(-\Delta)^{-s}\nabla u\|_q \le C\lambda^{-\theta}(\|u\|_p + \lambda\|\nabla u\|_p),$$

where θ is defined in the statement of the theorem. Minimizing the right-hand side with respect to λ gives the value $\lambda_0 = \theta(1-\theta)^{-1}\|u\|_p\|\nabla u\|_p^{-1}$ and therefore,

$$\|(-\Delta)^{-s}\nabla u\|_q \le C\|u\|_p^{1-\theta}\|\nabla u\|_p^\theta.$$

It remains to show (69). To this end, we distinguish two cases. First, let $p < d/(2s)$. By assumption, $p \le q \le r(1) := dp/(d-2sp)$. We apply the Hardy–Littlewood–Sobolev inequality (Lemma 21) to find that

$$\|(-\Delta)^{-s}\nabla u\|_{r(1)} \le C\|\nabla u\|_p \le C(\|u\|_p + \|\nabla u\|_p).$$

Furthermore, by using (in this order) the boundedness of $(-\Delta)^{-1/2}\nabla : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$, Lemma 2 in [45, page 133], equation (40) in [45, page 135], and Theorem 3 in [45, page 135f],

$$\begin{aligned}
\|(-\Delta)^{-s}\nabla u\|_p &= \|\nabla(-\Delta)^{-1/2}(-\Delta)^{1/2-s}u\|_p \le C\|(-\Delta)^{1/2-s}u\|_p \\
&\le C\|(I-\Delta)^{1/2-s}u\|_p \le C\|(I-\Delta)^{1/2}u\|_p \le C(\|u\|_p + \|\nabla u\|_p).
\end{aligned} \tag{70}$$

These inequalities hold for any $p \in (1,\infty)$. Now, it is sufficient to interpolate with $1/q = \mu/p + (1-\mu)/r(1)$:

$$\|(-\Delta)^{-s}\nabla u\|_q \le \|(-\Delta)^{-s}\nabla u\|_p^\mu \|(-\Delta)^{-s}\nabla u\|_{r(1)}^{1-\mu} \le C(\|u\|_p + \|\nabla u\|_p).$$

Second, let $p \ge d/(2s)$. We choose $\lambda \in (0, d/(2sp)) \subset (0,1)$ and apply the Hardy–Littlewood–Sobolev inequality:

$$\|(-\Delta)^{-s}\nabla u\|_{r(\lambda)} = \|(-\Delta)^{-\lambda s}(-\Delta)^{-(1-\lambda)s}\nabla u\|_{r(\lambda)} \le C\|(-\Delta)^{-(1-\lambda)s}\nabla u\|_p,$$

where $r(\lambda) = dp/(d-2s\lambda p)$. Since $(1-\lambda)s < 1/2$, we deduce from (70) that

$$\|(-\Delta)^{-s}\nabla u\|_{r(\lambda)} \le C(\|u\|_p + \|\nabla u\|_p).$$

Since $r(\lambda) \to \infty$ as $\lambda \to d/(2sp)$, the result follows. □

Appendix C. Parabolic regularity

Lemma 25 (Parabolic regularity). *Let $1 < p < \infty$, $T > 0$ and let u be the (weak) solution to the heat equation*

$$\partial_t u - \Delta u = v, \quad u(0) = u^0 \quad \text{in } \mathbb{R}^d,$$

where $v \in L^p(0,T;L^p(\mathbb{R}^d))$ and $u^0 \in W^{2,p}(\mathbb{R}^d)$. Then there exists $C > 0$, depending on T and p, such that

$$\|\partial_t u\|_{L^p(0,T;L^p(\mathbb{R}^d))} + \|\mathrm{D}^2 u\|_{L^p(0,T;L^p(\mathbb{R}^d))} \le C\big(\|v\|_{L^p(0,T;L^p(\mathbb{R}^d))} + \|\mathrm{D}^2 u^0\|_{L^p(\mathbb{R}^d)}\big). \tag{71}$$

Furthermore, if $v = \operatorname{div} w$ for some $w \in L^p(0,T;L^p(\mathbb{R}^d;\mathbb{R}^d))$ then

$$\|\nabla u\|_{L^p(0,T;L^p(\mathbb{R}^d))} \le C\big(\|w\|_{L^p(0,T;L^p(\mathbb{R}^d))} + T^{1/p}\|\nabla u^0\|_{L^p(\mathbb{R}^d)}\big). \tag{72}$$

Proof. We use a known result on the parabolic regularity for the equation

$$\partial_t\widehat{u} - \Delta\widehat{u} = v, \quad \widehat{u}(0) = 0 \quad \text{in } \mathbb{R}^d. \tag{73}$$

It holds that [28]

$$\|\partial_t\widehat{u}\|_{L^p(0,T;L^p(\mathbb{R}^d))} + \|\mathrm{D}^2\widehat{u}\|_{L^p(0,T;L^p(\mathbb{R}^d))} \le C\|v\|_{L^p(0,T;L^p(\mathbb{R}^d))}. \tag{74}$$

We apply this result to $\widehat{u} = u - e^{t\Delta}u^0$, where $e^{t\Delta}u^0$ is the solution to the homogeneous heat equation in $\mathbb{R}^d$ with initial datum u^0. Then $\widehat{u}$ solves (73) and satisfies estimate (74). Inserting the definition of $\widehat{u}$ and observing that $\|\mathrm{D}^2(e^{t\Delta}u^0)\|_p \le C\|\mathrm{D}^2u^0\|_p$, we obtain (71).

If $v = \operatorname{div} w$ for some $w \in L^p(0,T;L^p(\mathbb{R}^d;\mathbb{R}^d))$, the uniqueness of solutions to the heat equation yields $u = e^{t\Delta}u^0 + \operatorname{div} U$, where U solves

$$\partial_t U - \Delta U = w, \quad U(0) = 0 \quad \text{in } \mathbb{R}^d.$$

Then we deduce from the regularity result of [28] with $\widehat{u} = U$ and $v = w$ that

$$\|\mathrm{D}^2U\|_{L^p(0,T;L^p(\mathbb{R}^d))} \le C\|w\|_{L^p(0,T;L^p(\mathbb{R}^d))}.$$

Since $\nabla u = e^{t\Delta}\nabla u^0 + \nabla \operatorname{div} U$, inequality (72) follows. □

References

[1] L. Ambrosio and S. Serfaty. A gradient flow approach to an evolution problem arising in superconductivity. *Commun. Pure Appl. Math.* 61 (2008), 1495–1539.

[2] P. Biler, C. Imbert, and G. Karch. Barenblatt profiles for a non local porous medium equation. *C. R. Acad. Sci. Paris, Ser. I* 349 (2011), 641–645.

[3] M. Bonforte, Y. Sire, and J. L. Vazquez. Existence, uniqueness and asymptotic behaviour for fractional porous medium equations on bounded domains. *Discrete Contin. Dyn. Sys.* 35 (2015), 5725–5767.

[4] H. Brézis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Springer, New York, 2011.

[5] L. Caffarelli, M. Gualdani, and N. Zamponi. Existence of weak solutions to a continuity equation with space time nonlocal Darcy law. *Commun. Partial Differ. Eqs.* 45 (2020), 1799–1819.

[6] L. Caffarelli, F. Soria, F., and J. L. Vázquez. Regularity of solutions of the fractional porous medium flow. *J. Eur. Math. Soc.* 15 (2013), 1701–1746.

[7] L. Caffarelli and J. L. Vázquez. Asymptotic behaviour of a porous medium equation with fractional diffusion. *Discrete Contin. Dyn. Sys.* 29 (2011), 1393–1404.

[8] L. Caffarelli and J. L. Vazquez. Nonlinear porous medium flow with fractional potential pressure. *Arch. Ration. Mech. Anal.* 202 (2011), 537–565.

[9] J. A. Carrillo, Y. Huang, M. C. Santos, and J. L. Vazquez. Exponential convergence towards stationary states for the 1D porous medium equation with fractional pressure. *J. Differ. Eqs.* 258 (2015), 736–763.

[10] J. Chapman, J. Rubinstein, and M. Schatzman. A mean-field model for superconducting vortices. *Eur. J. Appl. Math.* 7 (1996), 97–111.

[11] L. Chen, E. S. Daus, A. Holzinger, and A. Jüngel. Rigorous derivation of population cross-diffusion systems from moderately interacting particle systems. Submitted for publication, 2020. arXiv:2010.12389.

[12] L. Chen, E. S. Daus, and A. Jüngel. Rigorous mean-field limit and cross diffusion. *Z. Angew. Math. Phys.* 70 (2019), no. 122, 21 pages.

[13] Y.-P. Choi and I.-J. Jeong. Relaxation to fractional porous medium equation from Euler–Riesz system. Submitted for publication, 2021. arXiv:2102.01817.

[14] Y.-P. Choi and I.-J. Jeong. Classical solutions for fractional porous medium flow. Submitted for publication, 2021. arXiv:2102.01816.

[15] S. Cifani and E. Jakobsen. Entropy solution theory for fractional degenerate convection-diffusion equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 28 (2011), 413–441.
[16] E. S. Daus, M. Gualdani, and N. Zamponi. Long time behavior and weak-strong uniqueness for a nonlocal porous media equation. *J. Differ. Eqs.* 268 (2020), 1820–1839.
[17] E. S. Daus, M. Ptashnyk, and C. Raithel. Derivation of a fractional cross-diffusion system as the limit of a stochastic many-particle system driven by Lévy noise. Submitted for publication, 2020. arXiv:2006.00277.
[18] M. Duerinckx. Mean-field limits for some Riesz interaction gradient flows. *SIAM J. Math. Anal.* 48 (2016), 2269–2300.
[19] W. E. Dynamics of vortex liquids in Ginzburg–Landau theories with applications in superconductivity. *Phys. Rev. B* 50 (1994), 1126–1135.
[20] E. Feireisl A. Novotný. *Singular Limits in Thermodynamics of Viscous Fluids*. Birkhäuser, Basel 2009.
[21] A. Figalli and R. Philipowski. Convergence to the viscous porous medium equation and propagation of chaos. *Alea* 4 (2008), 185–203.
[22] J. Fontbona and S. Méléard. Non local Lotka–Volterra system with cross-diffusion in an heterogeneous medium. *J. Math. Biol.* 70 (2015), 829–854.
[23] F. Golse. The mean-field limit for the dynamics of large particle systems. *Journées Equations aux dérivées partielles* (2003), 1–47.
[24] K. Ichikawa, M. Rouzimaimaiti, and T. Suzuki. Reaction diffusion equation with non-local term arises as a mean field limit of the master equation. *Discrete Cont. Dyn. Sys. S* 5 (2012), 115–126.
[25] C. Imbert, R. Tarhini, and F. Vigneron. Regularity of solutions of a fractional porous medium equation. *Interfaces Free Bound.* 22 (2020), 401–442.
[26] P.-E. Jabin and Z. Wang. Mean field limit for stochastic particle systems. In: N. Bellomo, P. Degond, and E. Tadmor, *Active Particles*, vol. 1, pp. 379–402. Springer, Boston, 2017.
[27] B. Jourdain and S. Méléard. Propagation of chaos and fluctuations for a moderate model with smooth initial data. *Ann. Inst. H. Poincaré Probab. Stat.* 34 (1998), 727–766.
[28] G. M. Lieberman. *Second Order Parabolic Differential Equations*. World Scientific, Singapore, 1996.
[29] S. Lisini. Fractional high order thin film equation: gradient flow approach. Submitted for publication, 2020. arXiv:2007.00459.
[30] S. Lisini, E. Mainini, and A. Segatti. A gradient flow approach to the porous medium equation with fractional pressure. *Arch. Ration. Mech. Anal.* 277 (2018), 567–606.
[31] Q.-H. Nguyen and J. L. Vazquez. Porous medium equation with nonlocal pressure in a bounded domain. *Commun. Partial Differ. Eqs.* 43 (2018), 1502–1539.
[32] D. Nualart. *The Malliavin Calculus and Related Topics*. Springer, Berlin, 2006.
[33] K. Oelschläger. A martingale approach to the law of large numbers for weakly interacting stochastic processes. *Ann. Probab.* 12 (1984), 458–479.
[34] K. Oelschläger. On the derivation of reaction-diffusion equations as limit dynamics of systems of moderately interacting stochastic processes. *Probab. Th. Rel. Fields* 82 (1989), 565–586.
[35] K. Oelschläger. Large systems of interacting particles and the porous medium equation. *J. Differ. Eqs.* 88 (1990), 294–346.
[36] C. Olivera, A. Richard, and M. Tomašević. Quantitative particle approximation of nonlinear Fokker–Planck equations with singular kernel. Submitted for publication, 2020. arXiv:2011.00537.
[37] A. de Pablo, F. Quiros, A. Rodriguez, and J. L. Vázquez. A general fractional porous medium equation. *Commun. Pure Appl. Math.* 65 (2012), 1242–1284.
[38] L. Płociniczak. Derivation of the nonlocal pressure form of the fractional porous medium equation in the hydrological setting. *Commun. Nonlin. Sci. Numer. Simul.* 76 (2019), 66–70.
[39] T. Roubíček. *Nonlinear Partial Differential Equations with Applications*. Birkhäuser, Basel, 2013.
[40] S. Serfaty. Mean-field limits of the Gross–Pitaevskii and parabolic Ginzburg–Landau equations. *J. Amer. Math. Soc.* 30 (2017), 713–768.
[41] S. Serfaty. Mean field limit for Coulomb-type flows. *Duke Math. J.* 169 (2020), 2887–2935.

[42] A. S. Sznitman. Nonlinear reflecting diffusion process, and the propagation of chaos and fluctuations associated. *J. Funct. Anal.* 56 (1984), 311–336.

[43] A. S. Sznitman. Topics in propagation of chaos. In: P. L. Hennequin (ed.), *École d'Été de Probabilités de Saint-Flour* XIX-1989, Lect. Notes Math. 1464, pp. 165–251. Berlin, Springer, 1991.

[44] D. Stan, F. del Teso, and J. L. Vázquez. Finite and infinite speed of propagation for porous medium equations with fractional pressure. *C. R. Acad. Sci. Paris, Ser. I* 352 (2014), 123–128.

[45] E. Stein. *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press, Princeton, 1970.

[46] P. R. Stinga. User's guide to the fractional Laplacian and the method of semigroups. In: A. Kochubei and Y. Luchko (eds.), *Handbook of Fractional Calculus with Applications*, vol. 2, pp. 235–266. De Gruyter, Berlin, 2019.

[47] J. L. Vázquez. Nonlinear diffusion with fractional Laplacian operators. In: H. Holden and K. Karlsen (eds.). *Nonlinear Partial Differential Equations*, Abel Symposia, vol. 7, pp. 1–27. Springer, Berlin, 2012.

[48] J. L. Vázquez. The mathematical theories of diffusion. Nonlinear and fractional diffusion. In: M. Bonforte and G. Grillo (eds.), *Nonlocal and Nonlinear Diffusions and Interactions: New Methods and Directions.* Lect. Notes Math. 2186, pp. 205–278. Springer, Cham, 2017.

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