EXISTENCE ANALYSIS OF A STATIONARY COMPRESSIBLE FLUID MODEL FOR HEAT-CONDUCTING AND CHEMICALLY REACTING MIXTURES

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Abstract. The existence of large-data weak solutions to a steady compressible Navier–Stokes–Fourier system for chemically reacting fluid mixtures is proved. General free energies are considered satisfying some structural assumptions which include ideal gas mixtures. The model is thermodynamically consistent and contains the Maxwell–Stefan cross-diffusion equations as a special case. Compared to previous works, a very general model class is analyzed, including cross-diffusion effects, temperature gradients, compressible fluids, and different molar masses. A priori estimates are derived from the entropy balance and the total energy balance. The compactness for the total mass density follows from an improved estimate for the density in L^γ with $\gamma > 3/2$, the effective viscous flux identity, and uniform bounds related to Feireisl's oscillations defect measure. These bounds rely heavily on the convexity of the free energy and the strong convergence of the relative chemical potentials.

Contents

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1. Introduction

This paper is concerned with the mathematical analysis of a stationary model for fluid mixtures, coupling Maxwell–Stefan diffusion with a Navier–Stokes–Fourier model. A key feature of the present paper is that the fluid mixture is modeled in a thermodynamically consistent way. Compared to [6], we include temperature effects, and compared to [31, 32], our constitutive equations are different and we allow for temperature gradients inside the diffusive fluxes, which yields a cross-diffusion coupling between the equations for the partial mass densities and the equation for the energy. The most important issue which allows for better results than in previous papers for steady compressible models of chemically reacting mixtures is the convexity of the Helmholtz free energy, similarly as in [6]–[8], where an evolutionary model was studied, however, under the assumption that the temperature is constant. On the other hand, we neglect chemical reactions on the boundary which is an important issue in the aforementioned papers.

1.1. **Balance equations.** The stationary balance equations for the partial mass densities ρ_i, the momentum $\rho\boldsymbol{v}$ and the total energy ρE are

$$\operatorname{div}(\rho_i \boldsymbol{v} + \boldsymbol{J}_i) = r_i, \quad i = 1, \ldots, N, \tag{1.1}$$

$$\operatorname{div}(\rho \boldsymbol{v} \otimes \boldsymbol{v} - \mathbb{S}) + \nabla p = \rho \boldsymbol{b}, \tag{1.2}$$

$$\operatorname{div}\left(\rho E \boldsymbol{v} + \boldsymbol{Q} - \mathbb{S}\boldsymbol{v} + p\boldsymbol{v}\right) = \rho \boldsymbol{b} \cdot \boldsymbol{v}, \tag{1.3}$$

where the physical meaning of the variables is explained in Table 1.

Variable	Physical meaning
ρ_i	partial mass density of the ith species
$\rho = \sum_{i=1}^N \rho_i$	total mass density
m_i	molar mass of the ith species
n_i	number density, $n_i = \rho_i/m_i$
$\overline{n}$	total number density, $\overline{n} = \sum_{i=1}^N n_i$
$\boldsymbol{J}_i$	partial flux of the ith species
$\boldsymbol{v}$	barycentric velocity
r_i	reaction term for the ith species
p	pressure
$\mathbb{S}$	viscous stress tensor
$\boldsymbol{b}$	momentum force term
$\boldsymbol{\mathcal{J}}$	entropy flux
Ξ	entropy production
$\boldsymbol{Q}$	internal heat flux
μ_i	chemical potential of the ith species
θ	thermodynamic temperature
s	specific entropy
e	specific internal energy
$E = e + \|\boldsymbol{v}\|^2/2$	specific total energy
ψ	specific Helmholtz free energy

TABLE 1. Physical meaning of the variables.

Equations (1.1)–(1.3) are solved in a bounded domain $\Omega \subset \mathbb{R}^3$ and are supplemented with the following boundary conditions on $\partial\Omega$:

$$\boldsymbol{J}_i \cdot \boldsymbol{\nu} = 0, \quad i = 1, \ldots, N, \tag{1.4}$$

$$\boldsymbol{v} \cdot \boldsymbol{\nu} = 0, \quad (\mathbb{I} - \boldsymbol{\nu} \otimes \boldsymbol{\nu})(\mathbb{S}\boldsymbol{\nu} + \alpha_1 \boldsymbol{v}) = \mathbf{0}, \tag{1.5}$$

$$\boldsymbol{Q} \cdot \boldsymbol{\nu} + \alpha_2(\theta_0 - \theta) = 0, \tag{1.6}$$

where α_2 and θ_0 are positive constants, $\alpha_1 \geq 0$, and $\boldsymbol{\nu}$ is the exterior unit normal vector to $\partial\Omega$. The matrix $\mathbb{I} - \boldsymbol{\nu} \otimes \boldsymbol{\nu}$ projects onto the orthogonal complement of span$\{\boldsymbol{\nu}\}$. Equations (1.4) are no-flux boundary conditions, (1.5) are the Navier slip boundary conditions (partial if $\alpha_1 > 0$, complete for $\alpha_1 = 0$), and (1.6) means that the normal component of the heat flux $\boldsymbol{Q}$ is proportional to the temperature difference $\theta - \theta_0$, where θ_0 has the physical meaning of the outer temperature and can be generally nonconstant (which we do not consider here in order to simplify the presentation).

We also prescribe the total mass

$$\frac{1}{|\Omega|} \int_\Omega \rho \, dx = \overline{\rho}, \tag{1.7}$$

where $|\Omega|$ denotes the measure of Ω, and $\overline{\rho}$ is assumed to be positive. Note that the total mass is in fact $|\Omega|\overline{\rho}$ and the quantity $\overline{\rho}$ has the meaning of the average density.

In some situations, especially when the integrability of the density is low, we replace the total energy balance by the entropy inequality

$$-\operatorname{div}\boldsymbol{\mathcal{J}} + \Xi \le 0. \tag{1.8}$$

Note that we replaced one equality by one inequality. Hence, we may obtain too many solutions to our problem which are non-physical. However, due to mathematical reasons (and physically, it is not surprizing either), as explained below, we cannot expect equality for the balance of entropy. To avoid this problem, similarly as for the single-component steady Navier–Stokes–Fourier system (see, e.g., [18]), we also add the total energy balance (1.3) integrated over Ω,

$$\int_{\partial\Omega} \alpha_1 |\boldsymbol{v}|^2\,ds = \int_{\partial\Omega} \alpha_2(\theta_0 - \theta)\,dx + \int_\Omega \rho\boldsymbol{b}\cdot\boldsymbol{v}\,dx. \tag{1.9}$$

We will discuss this issue later; at this point, we just note that (1.8) together with (1.9) possesses the property of weak-strong compatibility, i.e., any smooth solution to (1.1), (1.2), (1.8), and (1.9) is in fact a smooth solution to (1.1)–(1.3).

1.2. **Notation.** The unit matrix in $\mathbb{R}^{m\times m}$ is denoted by $\mathbb{I}$, where $m \in \mathbb{N}$. Given two matrices A, $B \in \mathbb{R}^{m\times n}$, we define $A : B \equiv \sum_{i=1}^m \sum_{j=1}^n A_{ij}B_{ij}$. We denote by a bold letter a vector $\boldsymbol{a} \in \mathbb{R}^3$ with the components (a_1, a_2, a_3), by a blackboard bold letter a matrix $\mathbb{A} \in \mathbb{R}^{3\times 3}$ with the coefficients (A_{ij}) or $(\mathbb{A}_{ij})$, and by $\vec{a}$ the vector $(a_1, \ldots, a_N)$ in $\mathbb{R}^N$. Recall that N denotes the number of species in equation (1.1). We also set $\mathbb{R}_+ = [0, \infty)$.

1.3. **Constitutive equations.** We specify the entropy flux $\boldsymbol{\mathcal{J}}$, entropy production Ξ, viscous stress tensor $\mathbb{S}$, heat flux $\boldsymbol{Q}$, diffusion fluxes $\boldsymbol{J}_i$, and reaction terms r_i.

The **entropy flux** $\boldsymbol{\mathcal{J}}$ is the sum of the contributions of free transport, diffusion fluxes, and heat flux,

$$\boldsymbol{\mathcal{J}} = \rho s\boldsymbol{v} - \sum_{i=1}^N \frac{\mu_i}{\theta}\boldsymbol{J}_i + \frac{1}{\theta}\boldsymbol{Q}.$$

The **entropy production** Ξ keeps into account the contributions from the diffusion fluxes, the heat flux, the viscous stress and the reaction terms,

$$\Xi = -\sum_{i=1}^N \boldsymbol{J}_i\cdot\nabla\frac{\mu_i}{\theta} + \boldsymbol{Q}\cdot\nabla\frac{1}{\theta} + \frac{\mathbb{S}:\nabla\boldsymbol{v}}{\theta} - \sum_{i=1}^N r_i\frac{\mu_i}{\theta}.$$

According to the second law of thermodynamics [21], the entropy production Ξ must be nonnegative. This is achieved by requiring that

$$-\sum_{i=1}^N \boldsymbol{J}_i\cdot\nabla\frac{\mu_i}{\theta} + \boldsymbol{Q}\cdot\nabla\frac{1}{\theta} \ge 0, \quad \frac{\mathbb{S}:\nabla\boldsymbol{v}}{\theta} \ge 0, \quad -\sum_{i=1}^N r_i\frac{\mu_i}{\theta} \ge 0. \tag{1.10}$$

The following definitions of $\mathbb{S}$, $\boldsymbol{Q}$, $\boldsymbol{J}_i$, and r_i are chosen in such a way that these requirements are satisfied.

The **viscous stress tensor** is assumed to be a linear function of the symmetric velocity gradient,

$$\mathbb{S} = 2\lambda_1(\theta)\bigg(\mathbb{D}(\boldsymbol{v}) - \frac{1}{3}\operatorname{div}(\boldsymbol{v})\mathbb{I}\bigg) + \lambda_2(\theta)\operatorname{div}(\boldsymbol{v})\mathbb{I}, \tag{1.11}$$

where $\lambda_1(\theta)$ and $\lambda_2(\theta)$ are the temperature-dependent shear and bulk viscosity coefficients, respectively.

The **heat flux** consists of the Fourier law and the molecular diffusion term,

$$\boldsymbol{Q} = -\kappa(\theta)\nabla\theta - \sum_{i=1}^N M_i\nabla\frac{\mu_i}{\theta}, \tag{1.12}$$

where $\kappa(\theta)$ is the thermal conductivity and the coefficients M_i depend on $\vec{\rho}$ and θ. The molecular diffusion term plays an important role in the energy identity.

The **diffusion flux** is a linear combination of the thermodynamic forces $\nabla(\vec{\mu}/\theta)$ and $\nabla(1/\theta)$,

$$\boldsymbol{J}_i = -\sum_{j=1}^N M_{ij}\nabla\frac{\mu_j}{\theta} - M_i\nabla\frac{1}{\theta}, \quad i = 1,\ldots,N, \tag{1.13}$$

where $M_{ij} = M_{ij}(\vec{\rho},\theta)$ are diffusion coefficients. By Onsager's principle, the coefficient matrix (M_{ij}) is symmetric [21].

To fulfill the mass conservation equation $\operatorname{div}(\rho\boldsymbol{v}) = 0$, the sum of the diffusion fluxes and the sum of the reaction terms should vanish, i.e. $\sum_{i=1}^N \boldsymbol{J}_i = \boldsymbol{0}$, $\sum_{i=1}^N r_i = 0$. Hence, the diffusion matrix has a nontrivial kernel, and we assume that

$$\sum_{i=1}^N M_{ij} = \sum_{i=1}^N M_i = 0, \quad j = 1,\ldots,N. \tag{1.14}$$

To be consistent with the first inequality in (1.10) and relations (1.12) and (1.13), we assume that $\kappa(\theta) > 0$ and that the mobility matrix (M_{ij}) is nonnegative. More precisely,

$$\exists C_M > 0: \quad \sum_{i,j=1}^N M_{ij}z_iz_j \geq C_M|\Pi\vec{z}|^2 \quad \text{for all } \vec{z}\in\mathbb{R}^N, \tag{1.15}$$

where $\Pi = \mathbb{I} - \vec{1}\otimes\vec{1}/N$ is the orthogonal projector on $\operatorname{span}\{\vec{1}\}^\perp$. For the structure of the mobility matrix and its connection to the Maxwell–Stefan theory, we refer to [1].

According to the third inequality in (1.10), the entropy production due to reactions, $-\sum_{i=1}^N r_i\mu_i/\theta$, must be nonnegative. Therefore, we suppose for the **reaction terms** that

$$\begin{cases} r_i = r_i(\Pi(\vec{\mu}/\theta),\theta), \quad i = 1,\ldots,N, \\ \exists C_r > 0,\, \zeta > 0,\, \beta > 0: \quad -\sum_{i=1}^N r_i(\Pi(\vec{q}),\theta)q_i \geq C_r|\Pi\vec{q}|^2, \\ |r_i(\Pi(\vec{q}),\theta)| \leq C_r(|\Pi\vec{q}|^{5(6-\zeta)/6} + \theta^{5(3\beta-\zeta)/6}) \quad \text{for all } \vec{q}\in\mathbb{R}^N, \theta > 0, \end{cases} \tag{1.16}$$

for some $\zeta > 0$ possibly very small, $\beta > 0$, and all $i = \{1, 2, \ldots, N\}$. This condition is satisfied for the reaction terms used in [6],

$$r_i = -\sum_{j=1}^{l} \frac{\partial \Phi}{\partial D_j}(D^R)\gamma_i^j, \quad \text{where } D_j^R = \sum_{k=1}^{N} \gamma_k^j \frac{\mu_k}{\theta},\ i = 1, \ldots, N,\ j = 1, \ldots, l, \tag{1.17}$$

where Φ: $\mathbb{R}^l \to \mathbb{R}$ is a convex potential with suitable growth and $\vec{\gamma}^j \in \mathbb{R}^N$ is a vectorial coefficient associated to the jth reaction; see Remark 1.3.

The remaining variables—**chemical potentials** μ_i, **pressure** p, **total internal energy** e, and **entropy** s—are determined from the Helmholtz free energy density $\rho\psi$ which is assumed to be a function of $\vec{\rho}$ and θ:

$$\begin{aligned} \mu_i &= \frac{\partial(\rho\psi)}{\partial \rho_i}, \quad p = -\rho\psi + \sum_{i=1}^{N} \rho_i \mu_i, \\ \rho e &= \rho\psi - \theta \frac{\partial(\rho\psi)}{\partial\theta}, \quad \rho s = -\frac{\partial(\rho\psi)}{\partial\theta}. \end{aligned} \tag{1.18}$$

The definition of the pressure is known as the Gibbs–Duhem relation. In this paper, we allow for general free energies satisfying Hypothesis (H6) below. A specific example is given in Remark 1.2.

For the mathematical analysis, we rename the **free energy density** by writing $h_\theta(\vec{\rho}) = (\rho\psi)(\vec{\rho}, \theta)$ and denote by

$$h_\theta^*(\vec{\mu}) = \sup_{\vec{\rho} \in \mathbb{R}_+^N} (\vec{\rho} \cdot \vec{\mu} - h_\theta(\vec{\rho}))$$

the Legendre transform of h_θ, which in fact equals the pressure p. It is well defined on $D_\theta^* = \{\vec{\mu} \in \mathbb{R}^N : \exists \vec{\rho} \in \mathbb{R}_+^N : \vec{\mu} = \nabla h_\theta(\vec{\rho})\}$ [35, §26]. If $h_\theta \in C^2(\mathbb{R}_+^N)$ depends smoothly on θ, then $h_\theta^* \in C^2(D_\theta^*)$ also depends smoothly on θ, where D_θ^* denotes the domain of definition of h_θ^*. Moreover, for any $\vec{\mu} \in D_\theta^*$ and $\vec{\rho} \in \mathbb{R}_+^N$ [35, Theorem 26.5],

$$\vec{\mu} = \nabla h_\theta(\vec{\rho}) \quad \text{if and only if} \quad \vec{\rho} = \nabla h_\theta^*(\vec{\mu}).$$

Note that $\nabla h_\theta(\vec{\rho})$ means $\nabla_{\vec{\rho}} H_\theta(\vec{\rho})$, i.e., the derivatives are taken with respect to ρ_i, $i = 1, 2, \ldots, N$.

1.4. **Hypotheses.** We impose the following mathematical hypotheses:

(H1) *Domain:* $\Omega \subset \mathbb{R}^3$ is a bounded domain with a C^2 boundary that is not axially symmetric.

(H2) *Data:* $\alpha_1 \geq 0$, $\alpha_2 > 0$, $\theta_0 \in L^\infty(\Omega)$, $\operatorname{ess\,inf}_\Omega \theta_0 > 0$, $\boldsymbol{b} \in L^\infty(\Omega; \mathbb{R}^N)$.

(H3) *Viscosity and heat conductivity:* λ_1, λ_2, $\kappa \in C^0(\mathbb{R}_+)$ and there exist constants c_1, c_2, κ_1, κ_2, $\beta > 0$ such that for all $\theta > 0$,

$$\begin{aligned} c_1(1+\theta) &\leq \lambda_1(\theta) \leq c_2(1+\theta), \\ 0 &\leq \lambda_2(\theta) \leq c_2(1+\theta), \\ \kappa_1(1+\theta)^\beta &\leq \kappa(\theta) \leq \kappa_2(1+\theta)^\beta. \end{aligned}$$

(H4) *Diffusion coefficients:* For all $i,j=1,\ldots,N$, the coefficients M_{ij}, $M_i\in C^0(\mathbb{R}^N_+\times\mathbb{R}_+)$ satisfy (1.14), (1.15), and

$$|M_{ij}(\vec\rho,\theta)|+\frac{|M_i(\vec\rho,\theta)|}{\theta}\le \widetilde C_M(\rho^{(\gamma+\nu-\zeta)/3}+\theta^{(3\beta-\zeta)/3})$$

for all $(\vec\rho,\theta)\in\mathbb{R}^N_+\times\mathbb{R}_+$ and some constants $\widetilde C_M$, γ, $\nu>0$. The constant ν is the improvement of the estimate of pressure by the Bogovskii estimate, see Lemma 2.3, and ζ is a fixed small positive number.

(H5) *Reaction terms:* $\vec r=(r_1,\ldots,r_N)\in C^0(\mathbb{R}^N\times\mathbb{R}_+;\mathbb{R}^N)$ satisfies (1.16) and $\sum_{i=1}^N r_i=0$.

(H6) *Free energy density:* $h_\theta\in C^2(\mathbb{R}^N_+)$ is strictly convex and depends smoothly on $\theta>0$.

- For all $R>0$, there exist K_1, $K_2>0$ and a continuous function ω fulfilling $\omega(0)=0$ such that for all $\theta_1,\theta>0$ and $\mu\in\mathbb{R}^N$, if

$$\theta+\theta_1+\theta_1^{-1}+\theta^{-1}+\sum_{i=1}^N\frac{\partial h^*_\theta}{\partial\mu_i}(\vec\mu)\le R\quad\text{then}\quad\left|\frac{\partial^2h^*_\theta}{\partial\mu_i\partial\mu_j}(\vec\mu)\right|\le K_1,$$
$$\text{and}\quad|\nabla h^*_\theta(\vec\mu)-\nabla h^*_{\theta_1}(\vec\mu)|+|h^*_\theta(\vec\mu)-h^*_{\theta_1}(\vec\mu)|\le K_1\omega(\theta_1-\theta). \tag{1.19}$$

- For all $\theta>0$ and $\vec\mu\in\mathbb{R}^N$,

$$\theta\sum_{i,j=1}^N\frac{\partial^2h^*_\theta}{\partial\mu_i\partial\mu_j}(\vec\mu)\le K_2\sum_{i=1}^N\frac{\partial h^*_\theta}{\partial\mu_i}(\vec\mu), \tag{1.20}$$

- For every $\kappa\in(0,1)$, there exists $C>0$ such that for every $\theta\in(\kappa,\kappa^{-1})$ and for all $\vec\rho\in\mathbb{R}^N_+$,

$$\rho_i\in(\kappa,\kappa^{-1})\text{ implies that }\mu_i:=\frac{h_\theta}{\partial\rho_i}(\vec\rho)\ge -C. \tag{1.21}$$

- There exists $\frac12<\alpha_0<1$, $\beta_0\ge0$ such that for every $\kappa\in(0,1)$, there exists $k>0$ such that,

$$\theta\in(\kappa,\kappa^{-1})\text{ implies that }\sum_{i=1}^N\rho_i^{\alpha_0}\left|\frac{\partial^2h_\theta}{\partial\theta\partial\rho_i}(\vec\mu)\right|\le k(1+|\vec\rho|^{\beta_0}). \tag{1.22}$$

- We require the following growth conditions. For arbitrary $\gamma>\frac32$ and $\beta>\frac23$, we assume that for all $\vec\rho\in\mathbb{R}^N_+$, $\theta>0$,

$$|h_\theta(\vec\rho)|\le C_h(1+\rho^{5(\gamma+\nu-\zeta)/6}+\theta^{5(3\beta-\zeta)/6}),$$
$$\left|\frac{\partial h_\theta}{\partial\theta}(\vec\rho)\right|\le C_h(1+\rho^{5(\gamma+\nu-\zeta)/6}+\theta^{5(3\beta-\zeta)/6}+|\ln\theta|^a). \tag{1.23}$$

If, furthermore, both $\gamma>5/3$ and $\beta>1$, we replace $(1.23)_2$ by

$$\left|\frac{\partial h_\theta}{\partial\theta}(\vec\rho)\right|\le C_h(1+\rho^{(5\beta-2)(\gamma+\nu-\zeta)/(6\beta)}+\theta^{5(3\beta-\zeta)/6-1}+|\ln\theta|^a), \tag{1.24}$$

for some constants C_h, ζ, $\nu>0$, $a<6$, and ν and ζ are as above.

(H7) *Pressure:* p is defined by (1.18) and satisfies

$$c_p(\rho\theta + \rho^\gamma) \le p(\vec{\rho}, \theta) \le C_p(1 + \rho\theta + \rho^\gamma)$$

for all $(\vec{\rho}, \theta) \in \mathbb{R}_+^N \times \mathbb{R}_+$, where $\gamma > 3/2$, $c_p > 0$, and $C_p > 0$.

Remark 1.1 (Discussion of the hypotheses)**.** (H1) If we approximate Ω by smooth domains, it is possible to extend our existence result to less regular domains as, e.g., Lipschitz ones. As this would technically complicate the paper, we skip this and only point out the paper [27], where such a technique is used in the case of compressible Navier–Stokes equations.

(H2) The force term $\boldsymbol{b}$ is assumed to be dependent only on x just for simplicity. It may also depend on $(\vec{\rho}, \theta)$; then $\boldsymbol{b}(x, \vec{\rho}, \theta)$ needs to be measurable in the first variable, continuous in the last $N + 1$ variables, and bounded. We may assume that $\alpha_1 = 0$ (but not $\alpha_2 = 0$); then we need that Ω is not axially symmetric to apply the Korn inequality [28, (4.17.19)]. In case $\alpha_1 > 0$, we have the Korn inequality also for axially symmetric domains, but the estimate of the velocity gradient depends additionally on the tangential trace of the velocity. However, as shown below in Lemma 2.2, the estimate of the trace of $\boldsymbol{v}$ depends also on the density and therefore, the problem becomes slightly more complex and leads to additional restrictions on the exponents β and γ (cf. [18]). To avoid such problems, we prefer to assume that the domain is not axially symmetric.

(H3) The linear growth of the viscosities leads to optimal a priori estimates. The lower bound avoids degeneracies in the coefficients.

(H4) The growth assumption on M_{ij}, M_i is necessary to have strong relative compactness $M_{ij}(\vec{\rho}_\delta, \theta_\delta)$, $M_i(\vec{\rho}_\delta, \theta_\delta)$ in $L^3(\Omega)$ (for a suitable subsequence of $(\vec{\rho}_\delta, \theta_\delta)$).

(H5) We show in Remark 1.3 that the reaction terms (1.17) satisfy condition (1.16). This condition excludes vanishing reaction terms. In fact, it is needed to derive an $L^2(\Omega)$ bound for $\Pi\vec{q}$ and, together with (1.15), an $H^1(\Omega)$ bound for $\Pi\vec{q}$ (see Lemma 2.2). Condition (1.16) can be replaced by a Robin-type boundary condition for the diffusion fluxes involving the chemical potentials, as done in [5, Hypothesis (H8)].

(H6) The fact that $\rho \mapsto h_\theta(\vec{\rho})$ is strictly convex implies that $\nabla_{\vec{\rho}} h_\theta(\vec{\rho})$ is a strictly monotone (and therefore invertible) operator from $\mathbb{R}_+^N$ to $\mathbb{R}^N$. We do not impose any assumptions directly on the growth of $\nabla_{\vec{\rho}} h_\theta(\vec{\rho})$, since we require some growth conditions on the pressure stated in Hypothesis (H7). Condition (1.22), fulfilled by our example of a free energy from Remark 1.2 presented below, is rather technical and is used only in the construction of a solution, not in the a priori estimates or compactness part. Thus, it can be viewed as a technical assumption, and we expect that it can be avoided by using a better approximation scheme.

(H7) We present below an example of the free energy for which $p(\vec{\rho}, \theta) = n\theta + (\gamma - 1)n^\gamma$. This pressure satisfies Hypothesis (H7). The condition $\gamma > 3/2$ is needed to derive a bound of ρ in $L^{\gamma+\nu}(\Omega)$; see Lemma 2.3. This allows us to control the pressure in a better space than $L^1(\Omega)$; see Lemma 2.4.

Remark 1.2 (Example of a free energy)**.** An example of the free energy density fulfilling Hypothesis (H6) is given by

$$\rho\psi = \theta\sum_{i=1}^N n_i\log n_i + \overline{n}^\gamma - c_W\rho\theta\log\theta, \tag{1.25}$$

where $\gamma > 1$ is some exponent and $c_W > 0$ is the heat capacity; see Appendix B for the proof. The first term is related to the mixing of the components, the second one is needed for the mathematical analysis (to obtain an estimate for the total mass density; a certain physical justification of this term can be found in [12, Chapter 1]), and the third one describes the thermal energy. We have assumed that the specific volumes of all components are the same. In the paper [6, Formula (23)], the first term is defined in a different way:

$$\theta\overline{n}\sum_{i=1}^N \frac{n_i}{\overline{n}}\log\frac{n_i}{\overline{n}} = \theta\sum_{i=1}^N n_i\log n_i - \theta\overline{n}\log\overline{n}.$$

This expression does not contribute to the pressure, while our mixing entropy term gives the pressure contribution $n\theta$ of an ideal gas. With the free energy density (1.25), we obtain

$$\begin{aligned}
\mu_i &= \frac{\theta}{m_i}(\log n_i + 1) + \frac{\gamma}{m_i}\overline{n}^{\gamma-1} - c_W\theta\log\theta, \\
p &= \overline{n}\theta + (\gamma-1)\overline{n}^\gamma, \\
\rho e &= \overline{n}^\gamma + c_W\rho\theta, \\
\rho s &= -\sum_{i=1}^N n_i\log n_i + c_W\rho(\log\theta + 1).
\end{aligned}$$

The first term of the pressure p represents the ideal gas law, while the second term is usually called the cold pressure.

Remark 1.3 (Example of reaction terms)**.** We claim that (1.17) satisfies (1.16). Here, $\Phi \in C^1(\mathbb{R}_+)$ is a uniformly convex potential such that $\Phi(0) = 0$ and $\nabla\Phi(0) = 0$. The vectors $\vec\gamma^1, \ldots, \vec\gamma^l \in \mathbb{R}$ satisfy $\sum_{i=1}^N \gamma_i^j = 0$ for $j = 1, \ldots, l$, and $\operatorname{span}\{\vec\gamma^1, \ldots, \vec\gamma^l\} = \operatorname{span}\{\vec 1\}^\perp$. Indeed, set $\vec q = \vec\mu/\theta$. Note that due to $\sum_{i=1}^N \gamma_i^j = 0$ the functions r_i in fact depend only on $\Pi(\vec q)$. Then a Taylor expansion around $D^R = 0$ yields

$$-\sum_{i=1}^N r_i q_i = \sum_{j=1}^l \frac{\partial\Phi}{\partial D_j^R}(D^R)D_j^R \geq \Phi(D^R) \geq c|D^R|^2 = \frac{C}{\theta^2}\sum_{k=1}^l |\vec\mu\cdot\vec\gamma^k|^2,$$

where $c > 0$ is the convexity constant for Φ. Since $(\vec\gamma^k)_{k=1}^N$ spans $\operatorname{span}\{\vec 1\}^\perp$ and $\Pi(\vec q)$ lies in $\operatorname{span}\{\vec 1\}^\perp$, there exists another constant $C > 0$ such that

$$\frac{1}{\theta^2}\sum_{k=1}^N |\vec\mu\cdot\vec\gamma^k|^2 = \sum_{k=1}^N |\vec q\cdot\vec\gamma^k|^2 \geq C|\Pi(\vec q)|^2.$$

We infer that $-\sum_{i=1}^N r_i\mu_i/\theta \geq c|\Pi(\vec q)|^2$, proving the claim.

The definition of r_i differs slightly from that one in [6] because of the role played by the temperature. In fact, the entropy inequality provides us with an estimate for $\vec{\gamma}^j \cdot \vec{\mu}/\theta$ which in turn yields an $L^2(\Omega)$ bound for $\Pi(\vec{\mu}/\theta)$ (see Lemma 2.2). We require that $\sum_{i=1}^N \gamma_i^j = 0$ for all $j = 1, \ldots, l$ in order to achieve $\sum_{i=1}^N r_i = 0$, needed for mass conservation. As a consequence, $\vec{\gamma}^j \in \operatorname{span}\{\vec{1}\}^\perp$. We assume that the linear hull of all $\vec{\gamma}^j$ is in fact equal to $\operatorname{span}\{\vec{1}\}^\perp$, which implies that $l \geq N-1$. This condition is necessary since Theorem 8.3 in [7], which gives an $L^2(\Omega)$ estimate for $\Pi(\vec{\mu}/\theta)$, cannot be generalized in a straightforward way to the non-isothermal case.

For all $p \in [1, \infty]$, we introduce the following spaces:

$$W^{1,p}_{\boldsymbol{\nu}}(\Omega;\mathbb{R}^3) = \{\boldsymbol{u} \in W^{1,p}(\Omega;\mathbb{R}^3) : \boldsymbol{u}\cdot\boldsymbol{\nu} = 0 \text{ on } \partial\Omega\}, \quad H^1_{\boldsymbol{\nu}}(\Omega;\mathbb{R}^3) = W^{1,2}_{\boldsymbol{\nu}}(\Omega;\mathbb{R}^3).$$

1.5. **Solution concept and main result.** Before we formulate our main result, we explain two types of solutions. We consider so-called weak solutions, i.e. solutions which fulfill equations (1.1)–(1.3) with boundary conditions (1.4)–(1.6) in the weak sense, and so-called variational entropy solutions (we use the terminology from [29]), i.e. solutions which fulfill (1.1), (1.2) weakly, the integrated form of the total energy balance (1.9), and the entropy inequality (1.8). For a certain choice of parameters, we always obtain the latter while the former will be satisfied only if some terms from the total energy balance are integrable, hence for a smaller set of the parameters.

Let us explain the definition of our solutions more precisely. Before doing so, we detail the weak formulation of our equations. We consider

- *the weak formulation of the mass balance*,

$$\sum_{i=1}^N \int_\Omega \Big(-\rho_i \boldsymbol{v} + \sum_{j=1}^N M_{ij} \nabla \frac{\mu_j}{\theta} + M_i \nabla \frac{1}{\theta} \Big) \cdot \nabla \phi_i \, dx = \sum_{i=1}^N \int_\Omega r_i \phi_i \, dx, \tag{1.26}$$

  for all $\phi_1, \ldots, \phi_N \in W^{1,\infty}(\Omega)$;
- *the weak formulation of the momentum balance*,

$$\int_\Omega (-\rho \boldsymbol{v} \otimes \boldsymbol{v} + \mathbb{S}) : \nabla \boldsymbol{u} \, dx + \int_{\partial\Omega} \alpha_1 \boldsymbol{v} \cdot \boldsymbol{u} \, ds = \int_\Omega (p \operatorname{div} \boldsymbol{u} + \rho \boldsymbol{b} \cdot \boldsymbol{u}) \, dx, \tag{1.27}$$

  for all $\boldsymbol{u} \in W^{1,\infty}_{\boldsymbol{\nu}}(\Omega)$;
- *the weak formulation of the total energy balance*,

$$\int_\Omega (-\rho E \boldsymbol{v} - \boldsymbol{Q} + \mathbb{S}\boldsymbol{v} - p\boldsymbol{v}) \cdot \nabla \varphi \, dx + \int_{\partial\Omega} (\alpha_1 |\boldsymbol{v}|^2 + \alpha_2(\theta - \theta_0)) \varphi \, ds = \int_\Omega \rho \boldsymbol{b} \cdot \boldsymbol{v} \varphi \, dx, \tag{1.28}$$

  for all $\varphi \in W^{1,\infty}(\Omega)$; and

- *the weak formulation of the entropy inequality,*

$$\begin{aligned}&\int_\Omega\bigg(\rho s\boldsymbol{v}+\sum_{i=1}^N\frac{\mu_i}{\theta}\bigg(\sum_{j=1}^N M_{ij}\nabla\frac{\mu_j}{\theta}+M_i\nabla\frac{1}{\theta}\bigg)-\frac{1}{\theta}\bigg(\kappa(\theta)\nabla\theta+\sum_{i=1}^N M_i\nabla\frac{\mu_i}{\theta}\bigg)\bigg)\cdot\nabla\Phi\,dx\\ &\quad+\int_\Omega\bigg(\sum_{i,j=1}^N M_{ij}\nabla\frac{\mu_i}{\theta}\cdot\nabla\frac{\mu_j}{\theta}+\kappa(\theta)|\nabla\log\theta|^2+\frac{\mathbb{S}:\nabla\boldsymbol{v}}{\theta}-\sum_{i=1}^N r_i\frac{\mu_i}{\theta}\bigg)\Phi\,dx\\ &\leq\alpha_2\int_{\partial\Omega}\frac{\theta-\theta_0}{\theta}\Phi\,ds,\end{aligned}\tag{1.29}$$

for every $\Phi\in W^{1,\infty}(\Omega)$ with $\Phi\geq 0$ a.e. in Ω.

We also use the term *the global total energy equality*, which is nothing but equality (1.28) with $\psi\equiv 1$, i.e.

$$\int_{\partial\Omega}(\alpha_1|\boldsymbol{v}|^2+\alpha_2(\theta-\theta_0))\,ds=\int_\Omega\rho\boldsymbol{b}\cdot\boldsymbol{v}\,dx.\tag{1.30}$$

Finally, recall that if we choose $\phi_i\equiv 1$ in (1.26) and add the weak formulation for all constituents, we obtain in the sum *the weak formulation of the continuity equation*,

$$\int_\Omega\rho\boldsymbol{v}\cdot\nabla\Phi\,dx=0,\tag{1.31}$$

for all $\Phi\in W^{1,\infty}(\Omega)$. However, we shall work with a slightly stronger definition of a solution to this equation, with *the renormalized solution to the continuity equation*, which is an important tool in the theory of weak solutions to the compressible Navier–Stokes equations to show compactness of the sequence of densities. Hence, we consider only renormalized solutions in what follows, i.e., solutions satisfying in addition to the weak formulation (1.31) for $u\in W^{1,\infty}(\Omega)$ and $b\in C^{0,1}(\mathbb{R})$ with b' having compact support,

$$\int_\Omega\big(b(\rho)\boldsymbol{v}\cdot\nabla u-u(\rho b'(\rho)-b(\rho))\operatorname{div}\boldsymbol{v}\big)\,dx=0.\tag{1.32}$$

Definition 1.4 (Weak and variational entropy solutions)**.** *We call the functions*

$$\rho_1,\ldots,\rho_N\in L^\gamma(\Omega),\quad\boldsymbol{v}\in H^1_{\boldsymbol{\nu}}(\Omega),\quad\log\theta,\theta^{\beta/2}\in H^1(\Omega)$$

such that

$$\rho|\boldsymbol{v}|^2\boldsymbol{v},\ \mathbb{S}(\theta,\nabla\boldsymbol{v})\boldsymbol{v},\ p(\vec{\rho},\theta)\boldsymbol{v}\in L^1(\Omega;\mathbb{R}^3)$$

a renormalized weak solution *to problem* (1.1)–(1.13), (1.16)–(1.18) *if there holds the weak formulations of the species equation* (1.26)*, momentum equation* (1.27)*, total energy balance* (1.28)*, and the total density* $\rho:=\sum_{i=1}^N\rho_i$ *is a renormalized solution to* (1.32)*.*

We call the functions

$$\rho_1,\ldots,\rho_N\in L^\gamma(\Omega),\quad\boldsymbol{v}\in H^1_{\boldsymbol{\nu}}(\Omega),\quad\log\theta,\theta^{\beta/2}\in H^1(\Omega)$$

such that

$$\rho|\boldsymbol{v}|^2\in L^1(\Omega)$$

a renormalized variational entropy solution *to problem* (1.1)–(1.13)*,* (1.16)–(1.18) *if there holds the weak formulations of the species equation* (1.26)*, momentum equation* (1.27)*, entropy inequality* (1.29)*, and global total energy balance* (1.30)*, and the total density* $\rho := \sum_{i=1}^N \rho_i$ *is a renormalized solution to* (1.32)*.*

The main result of this paper is the following theorem.

Theorem 1.5 (Large-data existence of solutions)**.** *Let Hypotheses (H1)–(H7) hold. Let* $\beta > 2/3$ *and* $\gamma > 3/2$*. Then there exists a renormalized variational entropy solution to* (1.1)–(1.13)*,* (1.16)–(1.18)*. Moreover, if* $\beta > 1$ *and* $\gamma > 5/3$*, then the solution is also a renormalized weak solution.*

Remark 1.6. Replacing the estimates of the total density computed from the Bogovskii operator estimates by a different technique used, e.g., in [30] (see also [18, 26, 31, 32]), we could treat the problem with arbitrary $\gamma > 1$ and certain bounds on β depending on γ. However, it would significantly complicate and extend this quite technical and long paper. Therefore we prefer not to do it here.

Remark 1.7. Note that in (1.26) we may use test functions $\phi_i = 0$ and ϕ_j being a non-zero constant for all $j \neq i$. This leads to a sort of compatibility condition,

$$\int_\Omega r_j \, dx = 0, \qquad j = 1, 2, \ldots, N.$$

We do not require this condition directly, but due to our assumptions on r_j and due to the construction, we are able to fulfil these conditions; see Remark 5.3.

1.6. **Key ideas of the proof.** We first prove the existence of solutions to an approximate Navier–Stokes–Fourier system, then derive a priori estimates from the entropy and total energy balances, and finally pass to the limit of vanishing approximation parameters. The approximate system is obtained from the mass densities and momentum balance equations (1.1), (1.2), as well as from the following *internal energy balance equation:*

$$\operatorname{div}(\rho e \boldsymbol{v} + \boldsymbol{Q}) - (\mathbb{S} - p\,\mathbb{I}) : \nabla \boldsymbol{v} = 0,$$

which we consider in place of the average total energy balance (1.3) and entropy inequality (1.8). It is obtained by computing the difference of $\boldsymbol{v} \times (1.2)$ and (1.3).

We use several levels of approximation. They are described in detail in Section 4 dealing with existence of solutions. In the construction of a solution for the approximate problem, we use for the velocity a Galerkin approximation with dimension $n \in \mathbb{N}$; we add lower-order regularizations with parameter $\varepsilon > 0$ and higher-order regularizations with parameters (χ, δ, ξ) to the other equations, leading to $H^2(\Omega)$ regularity; and we regularize the free energy with parameter $\eta > 0$ to obtain higher integrability of the total mass density. Moreover, we construct not the temperature, but rather its logarithm which allows us to deduce the a.e. positivity of the temperature.

We cannot use the artificial viscosity regularization in the total mass continuity equation as in [12, Section 3.3.1], yielding a linear elliptic equation with drift, because of the cross-diffusion terms, which significantly complicates our analysis. In fact, we need two approximation levels, distinguishing between higher-order and lower-order regularizations.

The existence of a solution to the approximate scheme is proved in Section 4. The positivity of the temperature is obtained from the bound $\log\theta \in W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$. The uniform estimates from the entropy and total energy balance allow us to pass to the limits $\delta \to 0$ and $n \to \infty$ (in this order). For the limits $\chi \to 0$, $\varepsilon \to 0$, and $\xi \to 0$ (in this order), we need an estimate for the total mass density in $L^{\gamma+\nu}(\Omega)$ for some $\nu > 0$. It yields a uniform bound for the pressure in a space better than $L^1(\Omega)$. This is shown by using a test function involving the Bogovskii operator in the weak formulation of the momentum balance equation (see [28, Section 7.3.3]).

The most difficult part of the proof is the strong convergence of the approximate mass densities (ρ_ε). It is based on an effective viscous flux identity or weak compactness identity for the pressure [12, Section 3.7.4] (Lemma 3.2) and some properties related to Feireisl's oscillations defect measure. More precisely, we shall prove that

$$\overline{p(\vec\rho_\varepsilon,\theta_\varepsilon)T_k(\rho_\varepsilon)} - \overline{p(\vec\rho_\varepsilon,\theta_\varepsilon)}\;\overline{T_k(\rho_\varepsilon)} \tag{1.33}$$

converges strongly in $L^1(\Omega)$ to some function, where the "overline" signifies the weak limit and T_k is some truncation operator (Lemma 3.3), and that

$$\overline{(p(\vec\rho_\varepsilon,\theta) - p(\vec\rho_\delta))(T_k(\rho_\varepsilon) - \rho_\varepsilon T_k'(\rho_\varepsilon) - T_k(\rho_\delta) + \rho_\delta T_k'(\rho_\delta))} \tag{1.34}$$

converges strongly in $L^1(\Omega)$ to zero (Lemma 3.5). The proofs of the strong limits of (1.33) and (1.34) rely heavily on the convexity of the free energy density and the strong convergence of the relative chemical potentials. These limits are necessary to deal with the cross-diffusion terms, and the proofs seem to be new. Another ingredient is the proof that the weak limit ρ of (ρ_ε) is a renormalized solution to the mass continuity equation. Using a special test function and renormalization function, we are able to control the oscillations defect measure and to prove the strong convergence of (ρ_ε) to ρ.

We finally underline that the method used in this paper is essentially new in the aspect how we treat the strong convergence of the total density. Note that this allows us to prove the result under minimal conditions and in fact also to improve the known results for the steady compressible Navier–Stokes–Fourier equations.

1.7. **State of the art and originality of the paper.** The literature on compressible Navier–Stokes and Navier–Stokes–Fourier systems is very extensive. First results on the existence of solutions to the stationary compressible Navier–Stokes equations (for a single species) without assumptions on the size of the data goes back to P.-L. Lions [22]. He assumed the pressure relation $p(\rho) = \rho^\gamma$ with $\gamma > 5/3$ for the stationary flow. The most difficult part of the proof, the strong convergence of the sequence of approximate densities, is based on a weak continuity property of the effective viscous flux $p(\rho) + (2\lambda_1 + \lambda_2)\operatorname{div}\boldsymbol{v}$ and the theory of renormalized solutions applied to the continuity equation. A first improvement on the pressure exponent γ is due to Březina and Novotný [3], who assumed that $\gamma > (1+\sqrt{13})/3$. Their proof is based on some ideas due to Plotnikov and Sokolowski [33] and on Feireisl's idea of the oscillations defect measure estimate (see

[9] in the evolutionary case), described in the steady case in [28] for a special class of non-volume forces. Further improvements of the lower bound for the exponent γ are due to Frehse, Steinhauer, and Weigant [15]. The existence of weak solutions to the steady compressible Navier–Stokes equations for any $\gamma > 1$ was shown in [20] (for space periodic boundary conditions), in [19] (for slip boundary conditions), and in [34] (for Dirichlet boundary conditions). In the case of evolutionary Navier–Stokes equations, the existence of a solution was proved in [22] for $\gamma \geq 9/5$ and the lower bound was decreased to $\gamma > 3/2$ in [13].

The existence theory for the Navier–Stokes–Fourier system employs the techniques of both Lions and Feireisl. The first result for the stationary compressible Navier–Stokes–Fourier system was proved by P.-L. Lions [22] under the assumption that the density is bounded in some L^q space for sufficiently large values of q. Without this assumption, when the density is a priori controlled only in $L^1(\Omega)$, the existence of weak solutions was shown in [24] for $\gamma > 3$ and in [25] for $\gamma > 7/3$. These results were improved in a series of papers, see [29, 30] for Dirichlet boundary conditions and [18] for the Navier boundary conditions, showing the existence of a variational entropy solution (satisfying the entropy inequality and the global total energy balance) for any $\gamma > 1$ and the existence of a weak solution (satisfying the total energy balance) for any $\gamma > 5/4$ (Navier boundary conditions) or $\gamma > 4/3$ (Dirichlet boundary conditions). We refer to [26] for further information.

For results on the time-dependent compressible Navier–Stokes and Navier–Stokes–Fourier equations, we refer to the monographs [11, 12, 28]. One difficulty is the proof of the strong convergence of the sequence of approximate temperatures which makes necessary the application of the div-curl lemma and the effective viscous flux relation by using a cancellation property different from the isothermal model. The transient Navier–Stokes equations with density-dependent viscosities satisfying a certain structure condition allow for new a priori estimates thanks to the Bresch–Desjardin entropy [2]. However, these estimates are not available for the steady problem. The evolutionary problem for a heat conducting fluid with basically the same restriction on the adiabatic coefficient γ was considered in [10, 12] for different formulations of the energy balance (internal energy inequality and entropy inequality, respectively).

All these results concern the single-species case. The theory of fluid mixtures requires some careful modeling to maintain thermodynamic consistency; we refer to [1, 4, 16] for the thermodynamic theory of fluid mixtures. One of the first results was proved in [36], namely the existence of weak solutions to the stationary Navier–Stokes equations assuming Fick's law and $\gamma > 7/3$. This result was improved in [17] for Maxwell–Stefan-type fluxes and $\gamma > 5/3$. Another improvement was achieved in [31, 32] for variational entropy solutions with $\gamma > 1$ and for weak solutions with $\gamma > 4/3$. These results are based on the assumption of same molar masses for each constituent. Concerning the evolutionary problem, the first global in time result for arbitrarily large data is due to [14] for Fick's law. An existence result for a general thermodynamically consistent transient Navier–Stokes model with $\gamma > 3/2$ was recently presented in [6]. Electrically charged dynamical incompressible mixtures were analyzed in [5]. We also mention the work [23] in which a multicomponent viscous compressible fluid model with separate velocities of the components was studied.

Our existence result generalizes previous works on fluid mixtures. Indeed, the mobility matrix in [17, 31, 32] has no contributions to the temperature gradients, the pressure and internal energy are not defined through the free energy, and the molar masses are assumed to be equal. These restrictions were removed in [5], but the authors consider incompressible fluids. In the works [6]–[8], a very general compressible fluid model is analyzed but no temperature effects have been taken into account.

In this paper, we combine all the features studied in the above-mentioned works, namely we allow for temperature gradients, a thermodynamically consistent modeling starting from the Helmholtz free energy, compressible fluids, and different molar masses. Moreover, we obtain a new proof for the strong convergence of the sequence of approximate densities by exploiting the convexity of the free energy.

The paper is organized as follows. A priori estimates for smooth solutions are derived in Section 2. We prove in Section 3 the compactness of the sequence of total mass densities satisfying the Navier–Stokes–Fourier mixture model. This step highlights the key features and novelties of the proof without obstructing it by the numerous approximating terms. The construction of smooth approximate solutions and the deregularization limits is presented in Sections 4 and 5, respectively. In the Appendix, we recall some auxiliary results needed in this paper and we show that the free energy density in Remark 1.2 satisfies Hypothesis (H6).

2. A priori estimates for smooth solutions

This section is devoted to the derivation of suitable a priori estimates for smooth solutions. Although we consider later weak solutions only, the computations help us to identify the key estimates of the existence proof. In fact, we need several regularizations for the full model which may abstruse the main arguments.

We assume the existence of a smooth solution to the following Navier–Stokes–Fourier system with chemically reacting species:

$$\operatorname{div}\left(\rho_i \boldsymbol{v} - \sum_{j=1}^N M_{ij}\nabla\frac{\mu_j}{\theta} - M_i\nabla\frac{1}{\theta}\right) = r_i, \quad i=1,\ldots,N \qquad \text{in } \Omega, \tag{2.1}$$

$$\operatorname{div}(\rho\boldsymbol{v}\otimes\boldsymbol{v} - \mathbb{S}) + \nabla p = \rho\boldsymbol{b} \qquad \text{in } \Omega, \tag{2.2}$$

$$\operatorname{div}\left(\rho e\boldsymbol{v} - \kappa(\theta)\nabla\theta - \sum_{i=1}^N M_i\nabla\frac{\mu_i}{\theta}\right) - (\mathbb{S} - p\,\mathbb{I}) : \nabla\boldsymbol{v} = 0 \qquad \text{in } \Omega, \tag{2.3}$$

$$\left(\sum_{j=1}^N M_{ij}\nabla\frac{\mu_j}{\theta} + M_i\nabla\frac{1}{\theta}\right)\cdot\boldsymbol{\nu} = 0, \quad i=1,\ldots,N \qquad \text{on } \partial\Omega, \tag{2.4}$$

$$\boldsymbol{v}\cdot\boldsymbol{\nu} = 0, \quad (\mathbb{I} - \boldsymbol{\nu}\otimes\boldsymbol{\nu})(\mathbb{S}(\boldsymbol{v})\boldsymbol{\nu} + \alpha_1\boldsymbol{v}) = 0 \qquad \text{on } \partial\Omega, \tag{2.5}$$

$$\left(\kappa(\theta)\nabla\theta + \sum_{i=1}^N M_i\nabla\frac{\mu_i}{\theta}\right)\cdot\boldsymbol{\nu} - \alpha_2(\theta_0 - \theta) = 0 \qquad \text{on } \partial\Omega. \tag{2.6}$$

Note that on this level, we can freely switch from the internal energy balance (2.3) to the total energy balance or the entropy equality; see Lemma 2.1 below. In order to use the procedure also during the construction of the solution, we immediately obtain the integrated entropy and total energy balances. To obtain the weak formulation of the entropy and total energy balances, we can proceed as in the proof of the lemma below, we just multiply the corresponding strong formulation additionally by a smooth test function ψ, which gives several additional terms containing the derivatives of ψ. In case of the entropy (in)equality, later on, we also require that the test function is nonnegative.

Lemma 2.1 (Entropy and total energy balances)**.** *Let $(\vec{\rho}, \boldsymbol{v}, \theta)$ be a smooth solution to* (2.1)–(2.6) *such that $\theta > 0$. Then the entropy balance*

$$\begin{aligned} &\int_\Omega \bigg(\sum_{i,j=1}^N M_{ij} \nabla \frac{\mu_i}{\theta} \cdot \nabla \frac{\mu_j}{\theta} + \kappa(\theta) |\nabla \log \theta|^2 + \frac{1}{\theta} \mathbb{S} : \nabla \boldsymbol{v} - \sum_{i=1}^N r_i \frac{\mu_i}{\theta} \bigg) dx \\ &\qquad = \alpha_2 \int_{\partial\Omega} \frac{\theta - \theta_0}{\theta} \, ds, \end{aligned} \tag{2.7}$$

and the total energy balance

$$\alpha_1 \int_{\partial\Omega} |\boldsymbol{v}|^2 \, ds + \alpha_2 \int_{\partial\Omega} (\theta - \theta_0) \, ds = \int_\Omega \rho \boldsymbol{b} \cdot \boldsymbol{v} \, dx \tag{2.8}$$

hold.

Proof. We multiply (2.1) by μ_i/θ and (2.3) by $-1/\theta$, sum both equations, sum over $i = 1, \ldots, N$, integrate over Ω, and integrate by parts (up to one term). The terms involving the coefficients M_i cancel, and, taking into account the boundary conditions (2.4)–(2.6), it follows that

$$\begin{aligned} &\int_\Omega \bigg(\rho e \boldsymbol{v} \cdot \nabla \frac{1}{\theta} \bigg) + \sum_{i,j=1}^N M_{ij} \nabla \frac{\mu_i}{\theta} \cdot \nabla \frac{\mu_j}{\theta} + \kappa(\theta) |\nabla \log \theta|^2 \bigg) dx \\ &\quad + \int_\Omega \bigg(\sum_{i=1}^N \operatorname{div}(\rho_i \boldsymbol{v}) \frac{\mu_i}{\theta} - \frac{p}{\theta} \operatorname{div} \boldsymbol{v} + \frac{1}{\theta} \mathbb{S} : \nabla \boldsymbol{v} \bigg) dx = \alpha_2 \int_{\partial\Omega} \frac{\theta - \theta_0}{\theta} \, ds + \int_\Omega \sum_{i=1}^N r_i \frac{\mu_i}{\theta} \, dx. \end{aligned} \tag{2.9}$$

We claim that some of the terms cancel, namely

$$\int_\Omega \bigg(\rho e \boldsymbol{v} \cdot \nabla \frac{1}{\theta} + \sum_{i=1}^N \operatorname{div}(\rho_i \boldsymbol{v}) \frac{\mu_i}{\theta} - \frac{p}{\theta} \operatorname{div} \boldsymbol{v} \bigg) dx = 0. \tag{2.10}$$

To prove this, we use the thermodynamic relations (1.18) to deduce that (we denote $H(\vec{\rho}, \theta) = \rho \psi(\vec{\rho}, \theta)$ for a moment)

$$-\boldsymbol{v} \cdot \sum_{i=1}^N \nabla \rho_i \frac{\mu_i}{\theta} - \rho e \boldsymbol{v} \cdot \nabla \frac{1}{\theta} + \frac{p \operatorname{div} \boldsymbol{v}}{\theta} - \sum_{i=1}^N \rho_i \mu_i \frac{\operatorname{div} \boldsymbol{v}}{\theta}$$

$$= \frac{\boldsymbol{v}}{\theta} \cdot \bigg(\sum_{i=1}^N -\frac{\partial H}{\partial \rho_i} \nabla \rho_i + \Big(H - \theta \frac{\partial H}{\partial \theta} \Big) \frac{\nabla \theta}{\theta} \bigg) - \frac{H}{\theta} \operatorname{div} \boldsymbol{v}$$

$$= -\frac{\boldsymbol{v}}{\theta} \cdot \nabla H - \frac{H}{\theta} \operatorname{div} \boldsymbol{v} - H \boldsymbol{v} \cdot \nabla \frac{1}{\theta} = \operatorname{div} \Big(\frac{\boldsymbol{v} H}{\theta} \Big).$$

Hence, we deduce (2.10) after an integration by parts, and (2.9) simplifies to (2.7).

Next, we multiply (2.2) by $\boldsymbol{v}$ and add to the resulting equation the energy balance (2.3), integrate over Ω, and integrate by parts. The integrals involving $\mathbb{S} : \nabla \boldsymbol{v}$ and p cancel and we end up with

$$-\int_\Omega \rho(\boldsymbol{v} \otimes \boldsymbol{v}) : \nabla \boldsymbol{v} \, dx - \int_{\partial\Omega} (\mathbb{S}\boldsymbol{v}) \cdot \boldsymbol{\nu} \, ds = \alpha_2 \int_{\partial\Omega} (\theta_0 - \theta) \, ds + \int_\Omega \rho \boldsymbol{b} \cdot \boldsymbol{v} \, dx.$$

The first term vanishes. Indeed, the sum of (2.1) from $i = 1, \ldots, N$ yields $\operatorname{div}(\rho \boldsymbol{v}) = 0$. Multiplying this equation by $|\boldsymbol{v}|^2/2$ and integrating over Ω, an integration by parts gives

$$0 = \frac{1}{2} \int_\Omega \rho \boldsymbol{v} \cdot \nabla |\boldsymbol{v}|^2 \, dx = \int_\Omega \rho(\boldsymbol{v} \otimes \boldsymbol{v}) : \nabla \boldsymbol{v} \, dx.$$

By (2.5), it yields the second identity (2.8), finishing the proof of the lemma. □

The entropy and total energy balances yield some a priori estimates. We define

$$q_i = \frac{\mu_i}{\theta}, \quad \Pi(\vec{q})_i = q_i - \frac{1}{N} \sum_{j=1}^N q_j, \quad i = 1, \ldots, N,$$

recalling that $\Pi = \mathbb{I} - \vec{1} \otimes \vec{1}/N$ projects onto $\operatorname{span}\{\vec{1}\}^\perp$.

Lemma 2.2 (Estimates from the entropy balance)**.** *The following a priori estimates hold:*

$$\|\boldsymbol{v}\|_{H^1(\Omega)} + \|\Pi(\vec{q})\|_{H^1(\Omega)} \le C, \tag{2.11}$$

$$\|\nabla \log \theta\|_{L^2(\Omega)} + \|\nabla \theta^{\beta/2}\|_{L^2(\Omega)} + \|1/\theta\|_{L^1(\partial\Omega)} \le C, \tag{2.12}$$

$$\|\theta\|_{L^1(\partial\Omega)} + \|\log \theta\|_{H^1(\Omega)} + \|\theta^{\beta/2}\|_{H^1(\Omega)}^{2/\beta} + \|\theta\|_{L^{3\beta}(\Omega)} \le C\big(1 + \|\rho\|_{L^{6/5}(\Omega)}\big), \tag{2.13}$$

where here and in the following, $C > 0$ denotes a generic constant dependent only on the given data.

Proof. We claim that every term on the left-hand side of (2.7) is nonnegative. In view of (1.15), we need to consider only the last two terms. We deduce from Hypothesis (H3) and the Korn inequality (Lemma A.3 in Appendix A), taking into account that Ω is not axially symmetric thanks to Hypothesis (H1), that for all $\boldsymbol{v} \in H^1_{\boldsymbol{\nu}}(\Omega; \mathbb{R}^3)$,

$$\int_\Omega \frac{1}{\theta} \mathbb{S} : \mathbb{D}(\boldsymbol{v}) \, dx \ge C \|\boldsymbol{v}\|_{H^1}^2.$$

The $L^2(\Omega)$ bound for $\nabla \Pi(\vec{q})$ is a consequence of (1.15), and (1.16) gives an $L^2(\Omega)$ bound for $\Pi(\vec{q})_i$, $i = 1, \ldots, N$. At this point, we need the nonvanishing reaction terms. Thus, $\Pi(\vec{q})$ is bounded in $H^1(\Omega)$.

By Hypothesis (H3),

$$\int_\Omega \kappa(\theta)|\nabla \log\theta|^2\,dx \geq \kappa_1\int_\Omega\left(|\nabla\log\theta|^2 + \frac{4}{\beta^2}|\nabla\theta^{\beta/2}|^2\right)dx,$$

which gives the $L^2(\Omega)$ bounds for $\nabla\log\theta$ and $\nabla\theta^{\beta/2}$. The entropy balance (2.7) implies that $1/\theta$ is bounded in $L^1(\partial\Omega)$. The total energy balance (2.8) and the $H^1(\Omega)$ bound for $\boldsymbol{v}$ together with the continuous embedding $H^1(\Omega)\hookrightarrow L^6(\Omega)$ show that

$$\int_{\partial\Omega}\alpha_2\theta\,ds \leq \int_{\partial\Omega}\alpha_2\theta_0\,ds + C\|\boldsymbol{b}\|_{L^\infty(\Omega)}\|\rho\|_{L^{6/5}(\Omega)}\|\boldsymbol{v}\|_{H^1(\Omega)} \leq C(1+\|\rho\|_{L^{6/5}(\Omega)}).$$

If $\beta<2$, we conclude that

$$\|\log\theta\|_{L^1(\partial\Omega)} + \|\theta^{\beta/2}\|_{L^1(\partial\Omega)} \leq C\big(1+\|\theta\|_{L^1(\partial\Omega)}\big) \leq C\big(1+\|\rho\|_{L^{6/5}(\Omega)}\big),$$

and the Poincaré inequality yields the remaining estimates. If $\beta>2$, we find that

$$\begin{aligned}\int_\Omega|\nabla\theta|^2dx &= \int_{\{\theta\leq K\}}\theta^2|\nabla\log\theta|^2\,dx + \frac{4}{\beta^2}\int_{\{\theta>K\}}\theta^{2-\beta}|\nabla\theta^{\beta/2}|^2\,dx \\ &\leq K^2\int_\Omega|\nabla\log\theta|^2\,dx + \frac{4}{\beta^2}K^{2-\beta}\int_\Omega|\nabla\theta^{\beta/2}|^2\,dx \leq C.\end{aligned}$$

By the Poincaré inequality, $\|\theta\|_{H^1(\Omega)}\leq C(1+\|\rho\|_{L^{6/5}(\Omega)})$, and this allows us to control also the $L^6(\Omega)$ norm of θ. A bootstrapping argument then yields a control of the $L^1(\Omega)$ norm of $\theta^{\beta/2}$. Applying the Poincaré inequality to $\theta^{\beta/2}$ finishes the proof. □

Exploiting Hypothesis (H7) on the pressure, we are able to derive an $L^s(\Omega)$ bound for ρ with $s>\gamma$, provided $\gamma>3/2$ and $\beta>2/3$.

Lemma 2.3 (Estimate for the total mass density)**.** *Let*

$$\nu := \gamma\min\left\{\frac{2\gamma-3}{\gamma},\frac{3\beta-2}{3\beta+2}\right\}.$$

Then there exists a constant $C>0$ depending only on the given data such that

$$\|\rho\|_{L^{\gamma+\nu}(\Omega)}\leq C.$$

Proof. The proof is based on estimates from the momentum balance (2.2) using the Bogovskii operator $\mathcal{B}$. We refer to Theorem A.1 in Appendix A for some properties of this operator. We multiply (2.2) by $\boldsymbol{\phi}=\mathcal{B}(\rho^\nu-\langle\rho^\nu\rangle)$, where $\langle\rho^\nu\rangle = |\Omega|^{-1}\int_\Omega\rho^\nu\,dx$. Then

$$\int_\Omega p\rho^\nu\,dx = \int_\Omega\big(p\langle\rho^\nu\rangle - \rho(\boldsymbol{v}\otimes\boldsymbol{v}):\nabla\boldsymbol{\phi} + \mathbb{S}:\nabla\boldsymbol{\phi} - \rho\boldsymbol{b}\cdot\boldsymbol{\phi}\big)\,dx. \tag{2.14}$$

Recall that, by Hypothesis (H7), $\int_\Omega p\rho^\nu\,dx \geq c_p\int_\Omega(\rho^{\gamma+\nu}+\rho^{1+\nu}\theta)\,dx$. We estimate the right-hand side of (2.14) term by term. We start with the two delicate terms which lead to the restrictions on the exponent ν. We have for $\alpha>3/2$,

$$\left|\int_\Omega\rho(\boldsymbol{v}\otimes\boldsymbol{v}):\nabla\boldsymbol{\phi}\,dx\right| \leq \|\boldsymbol{v}\|^2_{L^6(\Omega)}\|\rho\|_{L^\alpha(\Omega)}\|\nabla\boldsymbol{\phi}\|_{L^{3\alpha/(2\alpha-3)}(\Omega)}$$

$$\le C\|\boldsymbol{v}\|_{L^6(\Omega)}^2\|\rho\|_{L^\alpha(\Omega)}\|\rho^\nu-\langle\rho^\nu\rangle\|_{L^{3\alpha/(2\alpha-3)}(\Omega)}$$
$$\le C\|\boldsymbol{v}\|_{L^6(\Omega)}^2\|\rho\|_{L^\alpha(\Omega)}\big(\|\rho^\nu\|_{L^{3\alpha/(2\alpha-3)}(\Omega)}+\|\rho^\nu\|_{L^1(\Omega)}\big),$$

and choosing $\alpha=\gamma+\nu_1$ and $3\alpha/(2\alpha-3)=(\gamma+\nu_1)/\nu_1$, we end up with $\nu_1=2\gamma-3$.

Next, in view of Hypothesis (H3),

$$\begin{aligned}\left|\int_\Omega \mathbb{S}:\nabla\boldsymbol{\phi}\,dx\right| &\le C(1+\|\theta\|_{L^{3\beta}(\Omega)})\|\nabla\boldsymbol{v}\|_{L^2(\Omega)}\|\nabla\boldsymbol{\phi}\|_{L^{6\beta/(3\beta-2)}(\Omega)}\\ &\le C(1+\|\rho\|_{L^{6/5}(\Omega)})\big(\|\rho^{\nu_2}\|_{L^{6\beta/(3\beta-2)}(\Omega)}+\|\rho^{\nu_2}\|_{L^1(\Omega)}\big)\\ &\le C\big(\|\rho\|_{L^{\gamma+\nu_2}(\Omega)}^{1+\nu_2}+\|\rho\|_{L^1(\Omega)}\big),\end{aligned}$$

provided $6\beta/(3\beta-2)=(\gamma+\nu_2)/\nu_2$, i.e. $\nu_2=\gamma(3\beta-2)/(3\beta+2)$. Furthermore, because of $3/2<\gamma+\nu$,

$$\begin{aligned}\left|\int_\Omega \rho\boldsymbol{b}\cdot\boldsymbol{\phi}\,dx\right| &\le \|\boldsymbol{b}\|_{L^\infty(\Omega)}\|\rho\|_{L^{3/2}(\Omega)}\|\boldsymbol{\phi}\|_{L^3(\Omega)} \le C\|\boldsymbol{b}\|_{L^\infty(\Omega)}\|\rho\|_{L^{3/2}(\Omega)}\|\boldsymbol{\phi}\|_{W^{1,3/2}(\Omega)}\\ &\le C\|\boldsymbol{b}\|_{L^\infty(\Omega)}\|\rho\|_{L^{\gamma+\nu}(\Omega)}\big(\|\rho^\nu\|_{L^{3/2}(\Omega)}+\|\rho^\nu\|_{L^1(\Omega)}\big)\le C(\|\rho\|_{L^{\gamma+\nu}(\Omega)}^{1+\nu}+1),\end{aligned}$$

since the restriction $\nu\le 2\gamma-3$ yields $\frac{3}{2}\nu<\gamma+\nu$. Finally, by Hypothesis (H7),

$$\left|\int_\Omega p\langle\rho^\nu\rangle\,dx\right|\le C_p\int_\Omega(1+\rho\theta+\rho^\gamma)\,dx\int_\Omega\rho^\nu\,dx.$$

As we control the $L^1(\Omega)$ norm of the density (see (1.7)), we can control $\int_\Omega\rho^\gamma\,dx\int_\Omega\rho^\nu\,dx$ by $C\|\rho\|_{L^{\gamma+\nu}(\Omega)}^\lambda$ for some $\lambda<\gamma+\nu$, by interpolating between the L^1 and $L^{\gamma+\nu}$ norms. Hence, we only need to deal with the part of the first integral containing the temperature.

Let us first consider the case $\nu\le 1$. Then $\int_\Omega\rho^\nu\,dx$ is bounded by a constant and

$$\int_\Omega\rho\theta\,dx=\int_{\{\rho\le K\}}\rho\theta\,dx+\int_{\{\rho>K\}}\rho\theta\,dx\le K\int_\Omega\theta\,dx+K^{-\nu}\int_\Omega\theta\rho^{1+\nu}\,dx.$$

The first term on the right-hand side is bounded, and the last term can be absorbed by the left-hand side of (2.14) for sufficiently large K. Next, note that for $\nu>1$ we have $2\gamma-3>1$, i.e. $\gamma>2$. Then, by Hölder's inequality,

$$\int_\Omega\rho\theta\,dx\int_\Omega\rho^\nu\,dx\le C\|\rho\|_{L^\gamma(\Omega)}\|\theta\|_{L^{\gamma/(\gamma-1)}(\Omega)}\|\rho\|_{L^\nu(\Omega)}.$$

It follows from $\gamma>2$ and $\beta>2/3$ that $\gamma/(\gamma-1)<3\beta$. Hence, using once more $\gamma>2$,

$$\int_\Omega\rho\theta\,dx\int_\Omega\rho^\nu\,dx\le C\|\rho\|_{L^{\gamma+\nu}(\Omega)}^{1+\nu}\|\rho\|_{L^{6/5}(\Omega)}\le C\|\rho\|_{L^{\gamma+\nu}(\Omega)}^{2+\nu}.$$

Collecting all estimates, we deduce from $2+\nu<\gamma+\nu$ that

$$c_p\int_\Omega\rho^{\gamma+\nu}\,dx\le\int_\Omega p\langle\rho^\nu\rangle\,dx\le C(\|\rho\|_{L^{\gamma+\nu}(\Omega)}^\lambda+1),$$

where $\lambda<\gamma+\nu$. This leads to the desired estimate of ρ. □

Lemma 2.4 (Estimate for the pressure)**.** *For*

$$\alpha = \min\Big\{1+\frac{\nu}{\gamma}, \frac{(1+\nu)3\beta}{\nu+3\beta}\Big\} > 1,$$

there exists a constant $C>0$ such that

$$\|p(\vec\rho,\theta)\|_{L^\alpha(\Omega)} \le C.$$

Proof. Because of Hypothesis (H7), we have $p(\vec\rho,\theta) \le C_p(1+\rho^\gamma+\rho\theta)$. Taking into account Lemma 2.3, it is sufficient to verify that $\rho\theta \in L^\alpha(\Omega)$:

$$\begin{aligned}\int_\Omega (\rho\theta)^\alpha\,dx &= \int_\Omega (\rho\theta^{1/(1+\nu)})^\alpha \theta^{\alpha\nu/(1+\nu)}\,dx \\ &\le \Big(\int_\Omega \rho^{1+\nu}\theta\,dx\Big)^{\alpha/(1+\nu)}\Big(\int_\Omega \theta^{\alpha\nu/(1+\nu-\alpha)}\,dx\Big)^{(1+\nu-\alpha)/(1+\nu)} \le C,\end{aligned}$$

provided that $\alpha\nu/(1+\nu-\alpha) \le 3\beta$. This is true if $\alpha = 3\beta(1+\nu)/(\nu+3\beta)$. □

3. Weak sequential compactness for smooth solutions

In this section, we focus on the weak sequential stability of a weak solution and formulate it as an independent result. Then, in Section 4, we will just adapt the method introduced here and use it to prove the existence of a weak solution. The main result of this part is the following theorem.

Theorem 3.1. *Let Hypotheses (H1)–(H7) be satisfied. Let the sequence $(\boldsymbol{b}_\delta, \overline{\rho}_\delta, (\theta_0)_\delta)$ fulfil*

$$\begin{aligned} \boldsymbol{b}_\delta &\to \boldsymbol{b} && \text{strongly in } L^p(\Omega;\mathbb{R}^3) \text{ for all } p<\infty, \\ \boldsymbol{b}_\delta &\rightharpoonup^* \boldsymbol{b} && \text{weakly}^* \text{ in } L^\infty(\Omega;\mathbb{R}^3), \\ \overline{\rho}_\delta &\to \overline{\rho} > 0 && \text{in } \mathbb{R}, \\ (\theta_0)_\delta &\to \theta_0 && \text{strongly in } L^1(\partial\Omega). \end{aligned} \tag{3.1}$$

Let $(\vec\rho_\delta, \boldsymbol{v}_\delta, \theta_\delta)$ be a sequence of weak solutions to (2.1)–(2.6)*, corresponding to $\boldsymbol{b}_\delta$, $\overline{\rho}_\delta$ and $(\theta_0)_\delta$. Let $\gamma > 3/2$ and $\beta > 2/3$. Then $(\vec\rho_\delta, \boldsymbol{v}_\delta, \theta_\delta)$ satisfies the uniform bounds stated in Lemmata 2.2–2.4, and there exists a subsequence (not relabeld) such that for $s = \min\{3\beta/(\beta+1), 2\} \in (1,2]$,*

$$\begin{aligned} \vec\rho_\delta &\rightharpoonup \vec\rho && \text{weakly in } L^{\gamma+\nu}(\Omega;\mathbb{R}^N), \quad \nu=\nu(\beta,\gamma) \text{ is from Lemma 2.3}, \\ \boldsymbol{v}_\delta &\rightharpoonup \boldsymbol{v} && \text{weakly in } H^1(\Omega;\mathbb{R}^3) \text{ and strongly in } L^q(\Omega;\mathbb{R}^3), \quad q<6, \\ \theta_\delta &\rightharpoonup \theta && \text{weakly in } W^{1,s}(\Omega) \text{ and strongly in } L^q(\Omega), \quad q<3\beta, \end{aligned} \tag{3.2}$$

where the triple $(\vec\rho, \boldsymbol{v}, \theta)$ is a variational entropy solution corresponding to $(\boldsymbol{b}, \overline{\rho}, \theta_0)$. In addition, if $\gamma > 5/3$ and $\beta > 1$, then it is also a weak solution. Moreover, $\vec\rho_\delta \to \vec\rho$ strongly in $L^1(\Omega;\mathbb{R}^N)$.

We shall prove the theorem in several steps and each step is described in one of the following subsections.

3.1. **Convergence results based on a priori estimates.** Based on Lemmata 2.2–2.4, we collect all weak convergence results that allow us to pass to the limit in the weak formulation. It remains to show that the densities $\vec{\rho}_\delta$ converge pointwise to identify the pressure and the chemical potentials as functions of the densities and the temperature.

Limit in the mass balance. First, using the facts that (up to subsequences) $\vec{\rho}_\delta \to \vec{\rho}$ weakly in $L^{\gamma+\nu}(\Omega;\mathbb{R}^N)$, $\theta_\delta \to \theta$ strongly in $L^r(\Omega)$, $1 \le r < 3\beta$, together with Hypothesis (H4), we see that

$$M_{ij}(\vec{\rho}_\delta, \theta_\delta) \rightharpoonup \overline{M_{ij}(\vec{\rho}_\delta, \theta_\delta)}, \quad M_i(\vec{\rho}_\delta, \theta_\delta)/\theta_\delta \rightharpoonup \overline{M_i(\vec{\rho}_\delta, \theta_\delta)/\theta_\delta} \quad \text{weakly in } L^1(\Omega),$$

where i, $j = 1, 2, \ldots, N$ as $\delta \to 0$ and a bar over a quantity denotes its weak limit. Since the partial densities converge only weakly, we cannot generally identify the weak limits with $M_{ij}(\vec{\rho}, \theta)$ and $M_i(\vec{\rho}, \theta)$, respectively. Furthermore, recalling that

$$\Pi\Big(\frac{\vec{\mu}_\delta}{\theta_\delta}\Big) \to \Pi(\vec{q}) \quad \text{strongly in } L^r(\Omega;\mathbb{R}^N),\ r < 6,$$

we deduce from Hypothesis (H5) that

$$\vec{r}\Big(\Pi\Big(\frac{\mu_\delta}{\theta_\delta}\Big), \theta_\delta\Big) \to \vec{r}(\Pi(\vec{q}), \theta) \quad \text{strongly in } L^1(\Omega;\mathbb{R}^N).$$

We do not know at this moment whether $\vec{q} = \vec{\mu}/\theta$, where $\vec{\mu}$ is given by (1.18). Therefore, letting $\delta \to 0$ in the weak formulation (1.26) of the mass balance, we infer from Hypotheses (H4) and (H5) that, for all $\phi_1, \ldots, \phi_N \in W^{1,\infty}(\Omega)$,

$$\sum_{i=1}^N \int_\Omega \Big(-\rho_i \boldsymbol{v} + \sum_{j=1}^N \overline{M_{ij}(\vec{\rho}_\delta, \theta_\delta)} \nabla q_j - \overline{\frac{M_i(\vec{\rho}_\delta, \theta_\delta)}{\theta_\delta}} \frac{\nabla\theta}{\theta} \Big) \cdot \nabla\phi_i \, dx = \sum_{i=1}^n \int_\Omega r_i(\vec{q}, \theta)\phi_i \, dx.$$

Limit in the momentum balance. By Lemma 2.4,

$$p_\delta := p(\vec{\rho}_\delta, \theta_\delta) \rightharpoonup \overline{p(\vec{\rho}_\delta, \theta_\delta)} =: p \quad \text{weakly in } L^\alpha(\Omega) \text{ as } \delta \to 0. \tag{3.3}$$

At this point, it is not clear whether $\overline{p(\vec{\rho}_\delta, \theta_\delta)} =: p = p(\rho, \theta)$ and this will be proved later. The weak convergence of (a subsequence of) $\boldsymbol{v}_\delta$ in $H^1(\Omega)$ and the strong convergence of (θ_δ) in $L^r(\Omega)$, $r \ge 2$, implies that

$$\mathbb{S}_\delta \rightharpoonup \mathbb{S} \quad \text{weakly in } L^q(\Omega) \quad \text{for some } q \in [1, 2), \tag{3.4}$$

where $\mathbb{S}_\delta$ and $\mathbb{S}$ are the stress tensors (1.11) associated to $\theta_\delta, \boldsymbol{v}_\delta$ and $\theta, \boldsymbol{v}$, respectively. We use Hypothesis (H3) to control the viscosities. Therefore, we can perform the limit $\delta \to 0$ in the weak formulation (1.27) of the momentum balance; hence for all $\boldsymbol{u} \in W^{1,\infty}_{\boldsymbol{\nu}}(\Omega)$,

$$\int_\Omega (-\rho\boldsymbol{v} \otimes \boldsymbol{v} + \mathbb{S}(\theta, \nabla\boldsymbol{v})) : \nabla\boldsymbol{u} \, dx + \int_{\partial\Omega} \alpha_1 \boldsymbol{v} \cdot \boldsymbol{u} \, ds = \int_\Omega (\overline{p(\vec{\rho}_\delta, \theta_\delta)} \operatorname{div} \boldsymbol{u} + \rho\boldsymbol{b} \cdot \boldsymbol{u}) \, dx.$$

Limit in the entropy inequality. In view of (1.18), Hypothesis (H6) (in particular (1.23) and (1.24)), and the bounds on the temperature and densities, we obtain

$$\partial_\theta h_{\theta_\delta}(\vec{\rho_\delta}) = \rho_\delta s(\vec{\rho_\delta}, \theta_\delta) \rightharpoonup \overline{\rho_\delta s(\vec{\rho_\delta}, \theta_\delta)} \quad \text{weakly in } L^q(\Omega),\ q > 6/5.$$

Using the weak lower semicontinuity in several terms, the previous weak limits, Hypotheses (H3)–(H6), and Lemma A.4, we conclude from (1.29) in the limit $\delta \to 0$ that

$$\begin{aligned}
&\int_\Omega \Bigg[\overline{\rho_\delta s(\vec{\rho_\delta}, \theta_\delta)}\boldsymbol{v} + \sum_{i=1}^N q_i \Bigg(\sum_{j=1}^N \frac{\overline{M_{ij}(\vec{\rho_\delta}, \theta_\delta)}}{\theta_\delta} \nabla q_j - \frac{\overline{M_i(\vec{\rho_\delta}, \theta_\delta)}}{\theta_\delta} \frac{\nabla\theta}{\theta}\Bigg) \\
&\qquad - \bigg(\frac{\kappa(\theta)}{\theta}\nabla\theta + \sum_{i=1}^N \frac{\overline{M_i(\vec{\rho_\delta}, \theta_\delta)}}{\theta_\delta}\nabla q_i \bigg)\Bigg] \cdot \nabla\Phi \, dx \\
&\qquad + \int_\Omega \bigg(\sum_{i,j=1}^N \overline{M_{ij}(\vec{\rho_\delta}, \theta_\delta)} \nabla\frac{\mu_i}{\theta} \cdot \nabla\frac{\mu_j}{\theta} + \kappa(\theta)|\nabla \log\theta|^2 + \frac{\mathbb{S} : \nabla\boldsymbol{v}}{\theta} - \sum_{i=1}^N r_i \frac{\mu_i}{\theta} \bigg) \Phi \, dx \\
&\quad \le \alpha_2 \int_{\partial\Omega} \frac{\theta - \theta_0}{\theta} \Phi \, ds,
\end{aligned}$$

for every $\Phi \in W^{1,\infty}(\Omega)$, $\Phi \ge 0$ a.e. in Ω.

Limit in the total energy balance. The problem with the total energy balance is more complex. We can easily pass to the limit if the test function is constant, yielding the global total energy equality (1.30). To obtain a suitable limit in the weak formulation (1.28) of the total energy balance, we have to assume that $\gamma > 5/3$ and $\beta > 1$. This ensures that $\gamma + \nu > 2$ and

$$\begin{aligned}
\rho_\delta|\boldsymbol{v}_\delta|^2\boldsymbol{v}_\delta &\rightharpoonup \rho|\boldsymbol{v}|^2\boldsymbol{v} \quad \text{weakly in } L^r(\Omega;\mathbb{R}^3), \\
\mathbb{S}(\theta_\delta, \nabla\boldsymbol{v}_\delta)\boldsymbol{v}_\delta &\rightharpoonup \mathbb{S}(\theta, \nabla\boldsymbol{v})\boldsymbol{v} \quad \text{weakly in } L^r(\Omega;\mathbb{R}^3)
\end{aligned}$$

for some $r > 1$. Moreover, in view of Hypothesis (H6),

$$\rho_\delta e(\vec{\rho_\delta}, \theta_\delta)\boldsymbol{v}_\delta \rightharpoonup \overline{\rho_\delta e(\vec{\rho_\delta}, \theta_\delta)}\boldsymbol{v} \quad \text{weakly in } L^r(\Omega;\mathbb{R}^3)$$

for some $r > 1$. Therefore, letting $\delta \to 0$ in (1.28), it follows for all $\varphi \in W^{1,\infty}(\Omega)$ that

$$\begin{aligned}
&\int_\Omega \bigg(-\frac{1}{2}\rho|\boldsymbol{v}|^2\boldsymbol{v} - \overline{\rho_\delta e(\vec{\rho_\delta}, \theta_\delta)}\boldsymbol{v} + \kappa(\theta)\nabla\theta + \sum_{i=1}^N \frac{\overline{M_i(\vec{\rho_\delta}, \theta_\delta)}}{\theta_\delta}\nabla q_i \\
&\qquad + \mathbb{S}\boldsymbol{v} - \overline{p(\vec{\rho_\delta}, \theta_\delta)}\boldsymbol{v} \bigg) \cdot \nabla\varphi \, dx + \int_{\partial\Omega} (\alpha_1|\boldsymbol{v}|^2 + \alpha_2(\theta - \theta_0))\varphi \, ds = \int_\Omega \rho\boldsymbol{b} \cdot \boldsymbol{v}\varphi \, dx.
\end{aligned}$$

It remains to verify that

$$\vec{\rho_\delta} \to \vec{\rho} \quad \text{a.e. in } \Omega \tag{3.5}$$

as well as to identify $\Pi(\vec{q})$ with $\Pi(\vec{\mu}/\theta)$, where $\vec{\mu}$ is given by (1.18). The rest of this section is devoted to the proof of (3.5). For the sake of notational simplicity, we introduce the following notation for the double limit $(\delta,\varepsilon)\to 0$, i.e., $\overline{f(\vec{\rho}_\varepsilon,\vec{\rho}_\delta)}\in L^1(\Omega)$ is defined by

$$\int_\Omega \overline{f(\vec{\rho}_\varepsilon,\vec{\rho}_\delta)}\chi\,dx := \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_\Omega f(\vec{\rho}_\varepsilon,\vec{\rho}_\delta)\chi\,dx \quad \text{for all } \chi\in L^\infty(\Omega).$$

We also do not mention explicitly that we are working with subsequences and therefore, we do not relabel any sequence. Since we use only countably many relabelings, such procedure can be made rigorous by the standard diagonal procedure. Note that in all cases considered below, the order of the limit passages is not important; for the sake of clarity, we will assume that we let first $\delta\to 0$ and afterwards $\varepsilon\to 0$.

To end this first part, we also introduce the truncation function $T_k:\mathbb{R}_+\to\mathbb{R}_+$ of class $C^1(\mathbb{R}_+)$, which will be needed later. For arbitrary $k\in\mathbb{N}$ and $z\ge 0$, we set

$$\begin{aligned} T_1(z) &:= \begin{cases} z & \text{for } 0\le z\le 1,\\ \text{concave, increasing, } C^1\text{-function} & \text{for } 1<z<3,\\ 2 & \text{for } z\ge 3.\end{cases}\\ T_k(z) &:= kT_1(z/k). \end{aligned} \tag{3.6}$$

3.2. **Effective viscous flux.** We first focus on an effective viscous flux identity. We follow the procedure developed in [9, 22] very closely, and the proof is presented here for the sake of completeness.

Lemma 3.2 (Effective viscous flux identity)**.** *Let* $(\boldsymbol{b}_\delta,\rho_\delta,\mathbb{S}_\delta,\theta_\delta,\boldsymbol{v}_\delta)$ *satisfy* (1.11), (3.1), (3.2), (3.3), *and* (3.4). *Then it holds for every* $k\in\mathbb{N}$ *and* T_k, *defined in* (3.6), *that*

$$\overline{p_\delta T_k(\rho_\delta)} - p\overline{T_k(\rho_\delta)} = \left(\lambda_2(\theta)+\frac{4}{3}\lambda_1(\theta)\right)\left(\overline{T_k(\rho_\delta)\operatorname{div}\boldsymbol{v}_\delta} - \overline{T_k(\rho_\delta)}\operatorname{div}\boldsymbol{v}\right). \tag{3.7}$$

Proof. Thanks to our assumptions and since we consider the proper, not relabeled subsequence, all terms in (3.7) are well defined. We introduce an auxiliary function ϕ_δ as the solution to

$$\Delta\phi_\delta = T_k(\rho_\delta) \quad \text{in } \Omega, \qquad \phi_\delta = 0 \quad \text{on } \partial\Omega. \tag{3.8}$$

As $\partial\Omega$ is of class C^2, we have $\phi_\delta\in W^{2,q}(\Omega)$ for all $q<\infty$. (The proof would also work for open bounded domains Ω by arguing locally, since the regularity holds true away from the boundary.) Since $(T_k(\rho_\delta))$ is bounded in $L^\infty(\Omega)$, the sequence (ϕ_δ) is bounded in $W^{2,q}(\Omega)$ for all $q<\infty$, implying, up to a subsequence, that $\phi_\delta\to\phi$ weakly in $W^{2,q}(\Omega)$ and strongly in $W^{1,q}(\Omega)$ for all $q<\infty$, where ϕ solves

$$\Delta\phi = \overline{T_k(\rho_\delta)} \quad \text{in } \Omega, \qquad \phi = 0 \quad \text{on } \partial\Omega. \tag{3.9}$$

We set $\boldsymbol{\psi}_\delta := \nabla\phi_\delta$ and $\boldsymbol{\psi} := \nabla\phi$. The convergence properties of ϕ_δ yield the strong convergence $\boldsymbol{\psi}_\delta\to\boldsymbol{\psi}$ in $L^q(\Omega;\mathbb{R}^3)$ and the weak convergence $\nabla\boldsymbol{\psi}_\delta\rightharpoonup\nabla\boldsymbol{\psi}$ in $L^q(\Omega;\mathbb{R}^{3\times 3})$ for all $q<\infty$.

Relation (1.27) is a weak formulation of

$$\operatorname{div}(-\mathbb{T}_\delta+\rho_\delta\boldsymbol{v}_\delta\otimes\boldsymbol{v}_\delta) = \rho_\delta\boldsymbol{b}_\delta, \quad \text{where } \mathbb{T}_\delta = -p_\delta\mathbb{I}+\mathbb{S}_\delta,$$

and therefore, we can apply the div-curl lemma (see Lemma A.2) to the matrix-valued functions $\mathbb{T}_\delta - \rho_\delta \boldsymbol{v}_\delta \otimes \boldsymbol{v}_\delta$ and $\nabla \boldsymbol{\psi}_\delta$. Since the divergence of the former sequence is bounded in $L^r(\Omega;\mathbb{R}^3)$ for some $r>1$ and the curl of $\nabla\boldsymbol{\psi}_\delta$ vanishes, we infer that

$$(\mathbb{T}_\delta - \rho_\delta \boldsymbol{v}_\delta \otimes \boldsymbol{v}_\delta) : \nabla\boldsymbol{\psi}_\delta \rightharpoonup \overline{\mathbb{T}_\delta - \rho_\delta \boldsymbol{v}_\delta \otimes \boldsymbol{v}_\delta} : \nabla\boldsymbol{\psi} \quad \text{weakly in } L^1(\Omega).$$

Since $\boldsymbol{v}_\delta \otimes \boldsymbol{v}_\delta \to \boldsymbol{v}\otimes\boldsymbol{v}$ strongly in $L^{q/2}(\Omega)$ for $q<6$ and $\rho_\delta \rightharpoonup \rho$ weakly in $L^{\gamma+\nu}(\Omega)$ for $\gamma > 3/2$, the product converges weakly to the product of the limits, i.e.

$$\overline{\rho_\delta \boldsymbol{v}_\delta \otimes \boldsymbol{v}_\delta} = \rho \boldsymbol{v}\otimes\boldsymbol{v} \quad \text{in } L^1(\Omega).$$

Hence,

$$\overline{(\mathbb{T}_\delta - \rho_\delta \boldsymbol{v}_\delta \otimes \boldsymbol{v}_\delta) : \nabla\boldsymbol{\psi}_\delta} = \overline{\mathbb{T}_\delta} : \nabla\boldsymbol{\psi} - (\rho\boldsymbol{v}\otimes\boldsymbol{v}) : \nabla\boldsymbol{\psi} \quad \text{a.e. in } \Omega, \tag{3.10}$$

which is the starting point of further investigations.

First, we focus on the term involving the tensorial product of velocities. Note that the sum of (1.26) gives $\operatorname{div}(\rho_\delta \boldsymbol{v}_\delta) = 0$ in the weak sense and that

$$\operatorname{curl}(\nabla\boldsymbol{\psi}_\delta \boldsymbol{v}_\delta) = \nabla\left(\nabla\boldsymbol{\psi}_\delta \boldsymbol{v}_\delta\right) - \left(\nabla\left(\nabla\boldsymbol{\psi}_\delta \boldsymbol{v}_\delta\right)\right)^T = \nabla\boldsymbol{\psi}_\delta \left(\nabla\boldsymbol{v}_\delta\right)^T - \nabla\boldsymbol{v}_\delta \left(\nabla\boldsymbol{\psi}_\delta\right)^T$$

is bounded in $L^q(\Omega;\mathbb{R}^{3\times 3})$ for $q<2$ due to the properties of $\boldsymbol{\psi}_\delta$ and (3.2). Second, (3.2) implies that $(\rho_\delta \boldsymbol{v}_\delta)$ is bounded in $L^s(\Omega;\mathbb{R}^3)$ for some $s > 6/5$ and $(\nabla\boldsymbol{\psi}_\delta \boldsymbol{v}_\delta)$ is bounded in $L^q(\Omega;\mathbb{R}^3)$ for all $q<6$. Therefore, using the div-curl lemma again,

$$\overline{(\rho_\delta \boldsymbol{v}_\delta \otimes \boldsymbol{v}_\delta) : \nabla\boldsymbol{\psi}_\delta} = \overline{\rho_\delta \boldsymbol{v}_\delta} \cdot \overline{\nabla\boldsymbol{\psi}_\delta \boldsymbol{v}_\delta} = (\rho\boldsymbol{v}) \cdot (\nabla\boldsymbol{\psi}\boldsymbol{v}) = (\rho\boldsymbol{v}\otimes\boldsymbol{v}) : \nabla\boldsymbol{\psi} \quad \text{in } L^1(\Omega)$$

(and thus a.e. in Ω), where the second equality follows from the a.e. convergence of $(\boldsymbol{v}_\delta)$. Hence, we deduce from (3.10) that

$$\overline{\mathbb{T}_\delta : \nabla\boldsymbol{\psi}_\delta} = \overline{\mathbb{T}_\delta} : \nabla\boldsymbol{\psi}.$$

Recall that $\overline{\mathbb{T}_\delta} = \overline{\mathbb{S}_\delta} - \overline{p_\delta}\mathbb{I} = \overline{\mathbb{S}_\delta} - p\mathbb{I}$. In view of the definitions of $\boldsymbol{\psi}_\delta$ and $\boldsymbol{\psi}$, this shows that

$$\overline{\mathbb{S}_\delta : \nabla^2\phi_\delta - p_\delta \Delta\phi_\delta} = \overline{\mathbb{S}_\delta} : \nabla^2\phi - p\Delta\phi.$$

Finally, by the definitions of ϕ_δ and ϕ (see (3.8) and (3.9), respectively), we obtain

$$\overline{p_\delta T_k(\rho_\delta)} - p\overline{T_k(\rho_\delta)} = \overline{\mathbb{S}_\delta : \nabla^2\phi_\delta} - \overline{\mathbb{S}_\delta} : \nabla^2\phi \quad \text{a.e. in } \Omega. \tag{3.11}$$

The left-hand side corresponds to that one of (3.7). It remains to identify the terms on the right-hand side.

The right-hand side is uniquely defined, so we just need to identify it almost everywhere in Ω. Using convergences (3.2) and the Egorov theorem, we can find for any $\varepsilon > 0$ a measurable set $\Omega_\varepsilon \subset \Omega$ such that $|\Omega\backslash\Omega_\varepsilon| \le \varepsilon$ and $\theta_\delta \to \theta$ strongly in $L^\infty(\Omega_\varepsilon)$. Consequently, using definition (1.11) of the viscous stress tensor, the previous convergence result, and convergences (3.2) again, we can identify the weak limits in Ω_ε and conclude (with the help of (3.8) and (3.9)) that

$$\overline{\mathbb{S}_\delta : \nabla^2\phi_\delta} = 2\lambda_1(\theta)\overline{\left(\mathbb{D}(\boldsymbol{v}_\delta) - \frac{1}{3}\operatorname{div}\boldsymbol{v}_\delta\mathbb{I}\right) : \nabla^2\phi_\delta} + \lambda_2(\theta)\overline{\operatorname{div}\boldsymbol{v}_\delta\mathbb{I} : \nabla^2\phi_\delta}$$

$$= 2\lambda_1(\theta)\overline{\mathbb{D}(\boldsymbol{v}_\delta) : \nabla^2\phi_\delta} + \left(\lambda_2(\theta) - \frac{2}{3}\lambda_1(\theta)\right)\overline{\operatorname{div}\boldsymbol{v}_\delta\Delta\phi_\delta}$$

$$\overline{\mathbb{S}_\delta} : \nabla^2\phi = 2\lambda_1(\theta)\mathbb{D}(\boldsymbol{v}) : \nabla^2\phi + \left(\lambda_2(\theta) - \frac{2}{3}\lambda_1(\theta)\right)\operatorname{div}\boldsymbol{v}\Delta\phi \quad \text{a.e. in } \Omega_\varepsilon.$$

Then, substituting these expressions into (3.11),

$$\begin{aligned} \overline{p_\delta T_k(\rho_\delta)} - p\overline{T_k(\rho_\delta)} &= 2\lambda_1(\theta)\left(\overline{\mathbb{D}(\boldsymbol{v}_\delta) : \nabla^2\phi_\delta} - \mathbb{D}(\boldsymbol{v}) : \nabla^2\phi\right) \\ &+ \left(\lambda_2(\theta) - \frac{2}{3}\lambda_1(\theta)\right)\left(\overline{\operatorname{div}\boldsymbol{v}_\delta\Delta\phi_\delta} - \operatorname{div}\boldsymbol{v}\Delta\phi\right) \quad \text{a.e. in } \Omega_\varepsilon. \end{aligned} \tag{3.12}$$

But since $|\Omega \setminus \Omega_\varepsilon| \leq \varepsilon$ for any $\varepsilon > 0$, relation (3.12) holds a.e. in Ω.

Thus, it remains to identify the first term, i.e., we wish to relate the difference involving $\nabla^2\phi_\delta$ and $\nabla^2\phi$ with a difference involving $\Delta\phi_\delta$ and $\Delta\phi$. For this, let $\chi \in C_0^\infty(\Omega)$ be a test function. In the following, we use only formal manipulations which can, however, be justified by approximation by smooth functions. It follows that

$$\begin{aligned} &\int_\Omega \left(\overline{\mathbb{D}(\boldsymbol{v}_\delta) : \nabla^2\phi_\delta} - \mathbb{D}(\boldsymbol{v}) : \nabla^2\phi\right)\chi\, dx = \int_\Omega \left(\overline{\nabla\boldsymbol{v}_\delta : \nabla^2\phi_\delta} - \nabla\boldsymbol{v} : \nabla^2\phi\right)\chi\, dx \\ &= \int_\Omega \left(\overline{\nabla(\boldsymbol{v}_\delta\chi) : \nabla^2\phi_\delta} - \nabla(\boldsymbol{v}\chi) : \nabla^2\phi\right) dx - \int_\Omega \left(\overline{(\boldsymbol{v}_\delta \otimes \nabla\chi) : \nabla^2\phi_\delta} - (\boldsymbol{v} \otimes \nabla\chi) : \nabla^2\phi\right) dx. \end{aligned}$$

The second term vanishes because of the strong convergence of $\boldsymbol{v}_\delta$. Thus, integrating by parts twice,

$$\begin{aligned} \int_\Omega \left(\overline{\nabla(\boldsymbol{v}_\delta\chi) : \nabla^2\phi_\delta} - \nabla(\boldsymbol{v}\chi) : \nabla^2\phi\right) dx &= \int_\Omega \left(\overline{\operatorname{div}(\boldsymbol{v}_\delta\chi)\Delta\phi_\delta} - \operatorname{div}(\boldsymbol{v}\chi)\Delta\phi\right) dx \\ = \int_\Omega \left(\overline{\operatorname{div}\boldsymbol{v}_\delta\Delta\phi_\delta} - \operatorname{div}\boldsymbol{v}\Delta\phi\right)\chi\, dx &+ \int_\Omega \left(\overline{(\boldsymbol{v}_\delta \cdot \nabla\chi)\Delta\phi_\delta} - (\boldsymbol{v} \cdot \nabla\chi)\Delta\phi\right) dx, \end{aligned}$$

and the second term again vanishes because of the strong convergence of $\boldsymbol{v}_\delta$. As the test function χ is arbitrary, we infer that

$$\overline{\mathbb{D}(\boldsymbol{v}_\delta) : \nabla^2\phi_\delta} - \mathbb{D}(\boldsymbol{v}) : \nabla^2\phi = \overline{\operatorname{div}\boldsymbol{v}_\delta\Delta\phi_\delta} - \operatorname{div}\boldsymbol{v}\Delta\phi \quad \text{a.e. in } \Omega.$$

Thus, inserting this expression into (3.12) and using the properties of ϕ_δ and ϕ, i.e., (3.8) and (3.9), respectively, we deduce (3.7) a.e. in Ω. □

3.3. **Estimates based on the convexity of the free energy.** In this part, we show that the left-hand side of the effective viscous flux identity (3.7) gives us the important description of the possible oscillations of the total density ρ_δ, provided we assume the convexity of the free energy with respect to the partial densities. This is summarized in the following lemma.

Lemma 3.3. *Let the free energy satisfy the hypothesis (H6) and the sequence $(\vec{\rho}_\delta, \theta_\delta, \vec{\mu}_\delta, p_\delta)$ with $\theta_\delta > 0$ a.e. in Ω fulfil*

$$\vec{\rho}_\delta \rightharpoonup \vec{\rho} \quad \textit{weakly in } L^1(\Omega; \mathbb{R}^N), \tag{3.13}$$

$$p_\delta \rightharpoonup p \quad \textit{weakly in } L^1(\Omega), \tag{3.14}$$

$$\theta_\delta \to \theta \quad \textit{strongly in } L^1(\Omega), \tag{3.15}$$

$$\ln\theta_\delta \to \ln\theta \quad \textit{strongly in } L^1(\Omega),$$

$$\Pi(\vec\mu_\delta/\theta_\delta) \to \overline{\Pi(\vec\mu_\delta/\theta_\delta)} \quad \textit{strongly in } L^1(\Omega;\mathbb{R}^N), \tag{3.16}$$

$$\vec\mu_\delta = \nabla_{\vec\rho} h_{\theta_\delta}(\vec\rho_\delta) \quad \textit{a.e. in } \Omega,$$

$$p_\delta = p(\vec\rho_\delta,\theta_\delta) \quad \textit{a.e. in } \Omega,$$

where $p(\vec\rho_\delta,\theta_\delta) := -h_{\theta_\delta}(\vec\rho_\delta) + \vec\mu_\delta\cdot\vec\rho_\delta = h^*_{\theta_\delta}(\vec\mu_\delta)$ *and* $\partial_i h^*_\theta(\vec z_t) = \rho_i$, *where* $\partial_i = \partial/\partial\mu_i$. *For* $k\in\mathbb{N}$, *define (for a proper subsequence)*

$$W_k := \overline{p_\delta T_k(\rho_\delta)} - p\overline{T_k(\rho_\delta)}, \tag{3.17}$$

where $\rho_\delta := \sum_{i=1}^d \rho_{\delta,i}$ *and* $\rho := \sum_{i=1}^d \rho_i$. *Then for all* $k\in\mathbb{N}$,

$$0 \le W_k \le W_{k+1} \quad \textit{a.e. in } \Omega, \tag{3.18}$$

$$0 \le \theta\big(\overline{\rho_\delta T_k(\rho_\delta)} - \rho\overline{T_k(\rho_\delta)}\big) \le K_2 W_k \quad \textit{a.e. in } \Omega, \tag{3.19}$$

where $K_2>0$ *is given in* (1.20).

Proof. Step 1: Introduction of a proper set. We start the proof by introducing the proper subsets of Ω. Indeed, since we know that all weak limits exist, we need to identify them on sufficiently large subsets of Ω. Hence, using (3.15)–(3.16) and the Egorov theorem, we know that for arbitrary $\eta>0$, there exists a measurable set Ω_η such that $|\Omega\setminus\Omega_\eta|\le\eta$ and

$$\theta_\delta \to \theta \quad \text{strongly in } L^\infty(\Omega_\eta),$$

$$\ln\theta_\delta \to \ln\theta \quad \text{strongly in } L^\infty(\Omega_\eta),$$

$$\Pi(\vec\mu_\delta/\theta_\delta) \to \overline{\Pi(\vec\mu_\delta/\theta_\delta)} \quad \text{strongly in } L^\infty(\Omega_\eta;\mathbb{R}^N),$$

and consequently also

$$\Pi\vec\mu_\delta \to \overline{\Pi\vec\mu_\delta} \quad \text{strongly in } L^\infty(\Omega_\eta;\mathbb{R}^N). \tag{3.20}$$

Furthermore, introducing the linear mapping $\mathrm{P}:\mathbb{R}^N\to\mathbb{R}^N$ by $\mathrm{P}(\vec u)_i := u_i - u_N$ for $i=1,\ldots,N$, we know that $|\mathrm{P}(\vec u)|\le C|\Pi\vec u|$ (see Lemma A.5 and Remark A.6), and due to the linearity of P, we deduce from (3.20) that

$$\mathrm{P}\vec\mu_\delta \to \overline{\mathrm{P}\vec\mu_\delta} \quad \text{strongly in } L^\infty(\Omega_\eta;\mathbb{R}^N).$$

Therefore, we just need to show that (3.18)–(3.19) holds true a.e. in Ω_η, and since $|\Omega\setminus\Omega_\eta|\le\eta$, the equalities will necessarily hold also a.e. in Ω.

Step 2: Using the conjugate h^*_θ. We characterize W_k as

$$W_k = \frac12\overline{(p(\vec\rho_\varepsilon,\theta_\varepsilon) - p(\vec\rho_\delta,\theta_\delta))(T_k(\rho_\varepsilon) - T_k(\rho_\delta))} \quad \text{a.e. in } \Omega.$$

Thus, we need to compute the differences $p(\vec{\rho}_\varepsilon,\theta_\varepsilon)-p(\vec{\rho}_\delta,\theta_\delta)$ and $T_k(\rho_\varepsilon)-T_k(\rho_\delta)$ for $\varepsilon>0$, $\delta>0$. As the pressure is equal to the Legendre transform of h_θ, we can write

$$\begin{aligned} p(\vec{\rho}_\varepsilon,\theta_\varepsilon)-p(\vec{\rho}_\delta,\theta_\delta) &= h^*_{\theta_\varepsilon}(\vec{\mu}_\varepsilon)-h^*_{\theta_\delta}(\vec{\mu}_\delta) \\ &= (h^*_\theta(\vec{\mu}_\varepsilon)-h^*_\theta(\vec{\mu}_\delta))+(h^*_{\theta_\varepsilon}(\vec{\mu}_\varepsilon)-h^*_\theta(\vec{\mu}_\varepsilon))+(h^*_\theta(\vec{\mu}_\delta)-h^*_{\theta_\delta}(\vec{\mu}_\delta)) \\ &=: Y^1_{\varepsilon,\delta}+Y^2_\varepsilon+Y^3_\delta. \end{aligned} \tag{3.21}$$

The first term $Y^1_{\varepsilon,\delta}$ is formulated as

$$Y^1_{\varepsilon,\delta}=\int_0^1\frac{d}{dt}h^*_\theta(\vec{z}_t)\,dt=\int_0^1\sum_{i=1}^N\partial_i h^*_\theta(\vec{z}_t)(\mu_{\varepsilon,i}-\mu_{\delta,i})\,dt,$$

where $\vec{z}_t:=t\vec{\mu}_\varepsilon+(1-t)\vec{\mu}_\delta$. Using the decomposition

$$\mu_{\varepsilon,i}-\mu_{\delta,i}=\mathrm{P}(\vec{\mu}_\varepsilon-\vec{\mu}_\delta)_i+(\mu_{\varepsilon,N}-\mu_{\delta,N}),\quad i=1,\ldots,N, \tag{3.22}$$

it follows that

$$Y^1_{\varepsilon,\delta}=\int_0^1\bigg(\sum_{i=1}^{N-1}\partial_i h^*_\theta(\vec{z}_t)\mathrm{P}(\vec{\mu}_\varepsilon-\vec{\mu}_\delta)_i+\sum_{i=1}^N\partial_i h^*_\theta(\vec{z}_t)(\mu_{\varepsilon,N}-\mu_{\delta,N})\bigg)\,dt. \tag{3.23}$$

In a very similar way, using the fact that $\vec{\rho}_\varepsilon=\nabla h^*_{\theta_\varepsilon}(\vec{\mu}_\varepsilon)$, we find that

$$\begin{aligned} T_k(\rho_\varepsilon)-T_k(\rho_\delta) &= T_k\bigg(\sum_{i=1}^N\partial_i h^*_{\theta_\varepsilon}(\vec{\mu}_\varepsilon)\bigg)-T_k\bigg(\sum_{i=1}^N\partial_i h^*_{\theta_\delta}(\vec{\mu}_\delta)\bigg) \\ &= \bigg[T_k\bigg(\sum_{i=1}^N\partial_i h^*_\theta(\vec{\mu}_\varepsilon)\bigg)-T_k\bigg(\sum_{i=1}^N\partial_i h^*_\theta(\vec{\mu}_\delta)\bigg)\bigg] \\ &\quad+\bigg[T_k\bigg(\sum_{i=1}^N\partial_i h^*_{\theta_\varepsilon}(\vec{\mu}_\varepsilon)\bigg)-T_k\bigg(\sum_{i=1}^N\partial_i h^*_\theta(\vec{\mu}_\varepsilon)\bigg)\bigg] \\ &\quad+\bigg[T_k\bigg(\sum_{i=1}^N\partial_i h^*_\theta(\vec{\mu}_\delta)\bigg)-T_k\bigg(\sum_{i=1}^N\partial_i h^*_{\theta_\delta}(\vec{\mu}_\delta)\bigg)\bigg]=: Z^1_{\varepsilon,\delta}+Z^2_\varepsilon+Z^3_\delta, \end{aligned} \tag{3.24}$$

and rewrite $Z^1_{\varepsilon,\delta}$ as

$$Z^1_{\varepsilon,\delta}=\int_0^1\frac{d}{ds}T_k\bigg(\sum_{i=1}^N\partial_i h^*_\theta(\vec{z}_s)\bigg)\,ds=\int_0^1\Lambda_k(s)\sum_{i,j=1}^N\partial^2_{ij}h^*_\theta(\vec{z}_s)(\mu_{\varepsilon,j}-\mu_{\delta,j})\,ds, \tag{3.25}$$

where we defined for arbitrary $0\le s\le 1$ and $k\in\mathbb{N}$,

$$\Lambda_k(s):=T_k'\bigg(\sum_{i=1}^N\partial_i h^*_\theta(\vec{z}_s)\bigg)=T_1'\bigg(\frac{1}{k}\sum_{i=1}^N\partial_i h^*_\theta(\vec{z}_s)\bigg).$$

The sum over $i,j=1,\ldots,N$ in (3.25) can be rewritten, using (3.22), as

$$\sum_{i,j=1}^N \partial_{ij}^2 h_\theta^*(\vec{z}_s)(\mu_{\varepsilon,i}-\mu_{\delta,i}) = \sum_{i,j=1}^N \partial_{ij}^2 h_\theta^*(\vec{z}_s)\big(\mathrm{P}(\vec{\mu}_\varepsilon-\vec{\mu}_\delta)_i + (\mu_{\varepsilon,N}-\mu_{\delta,N})\big)$$

such that

$$Z_{\varepsilon,\delta}^1 = \int_0^1 \Lambda_k(s)\left(\sum_{i,j=1}^N \partial_{ij}^2 h_\theta^*(\vec{z}_s)\mathrm{P}(\vec{\mu}_\varepsilon-\vec{\mu}_\delta)_i + (\mu_{\varepsilon,N}-\mu_{\delta,N})\sum_{i,j=1}^N \partial_{ij}^2 h_\theta^*(\vec{z}_s)\right) ds. \tag{3.26}$$

Step 3: Proof of (3.18). We restrict our analysis to Ω_η, since for every fixed $k\in\mathbb{N}$,

$$\sup_{\varepsilon,\delta>0}\int_{\Omega\setminus\Omega_\eta}\big|(p(\vec{\rho}_\varepsilon,\theta_\varepsilon)-p(\vec{\rho}_\delta,\theta_\delta))(T_k(\rho_\varepsilon)-T_k(\rho_\delta))\big|\,dx \to 0 \quad \text{as } \eta\to 0, \tag{3.27}$$

which follows from assumption (3.14). Furthermore, we define the sets

$$\Omega_{\eta,R}^{\varepsilon,\delta} := \{x\in\Omega_\eta:\ \rho_\varepsilon(x)+\rho_\delta(x)\le R\}.$$

Similarly as before, we again have for every fixed $k\in\mathbb{N}$,

$$\sup_{\varepsilon,\delta>0}\int_{\Omega_\eta\setminus\Omega_{\eta,R}^{\varepsilon,\delta}}\big|(p(\vec{\rho}_\varepsilon,\theta_\varepsilon)-p(\vec{\rho}_\delta,\theta_\delta))(T_k(\rho_\varepsilon)-T_k(\rho_\delta))\big|\,dx \to 0 \quad \text{as } R\to\infty. \tag{3.28}$$

Therefore, we focus on the behavior of the sequence on the sets $\Omega_{\eta,R}^{\varepsilon,\delta}$. We use Hypothesis (H6) to show that some terms $Y_{\varepsilon,\delta}^j$ and $Z_{\varepsilon,\delta}^j$ vanish in the limit. Since we assume that $\rho_\varepsilon(x)+\rho_\delta(x)\le R$ for $x\in\Omega_{\eta,R}^{\varepsilon,\delta}$, it follows on this set that

$$\sum_{i=1}^N \partial_i h_{\theta_\varepsilon}^*(\vec{\mu}_\varepsilon)+\partial_i h_{\theta_\delta}^*(\vec{\mu}_\delta)\le R. \tag{3.29}$$

Consequently, since we know that θ_δ, θ_ε, and θ are bounded from below and above in Ω_η, Hypothesis (H6) implies that

$$\sum_{i=1}^N \partial_i h_\theta^*(\vec{\mu}_\varepsilon)+\partial_i h_\theta^*(\vec{\mu}_\delta)\le C(R,\omega,\eta) \quad \text{a.e. in } \Omega_{\eta,R}^{\varepsilon,\delta},$$

where ω refers to the modulus of continuity of h_θ^*, introduced in (1.19). Finally, thanks to the continuity, for any $t\in(0,1)$ and $\vec{z}_t = t\vec{\mu}_\varepsilon+(1-t)\vec{\mu}_\delta$,

$$\sum_{i=1}^N \partial_i h_\theta^*(\vec{z}_t)\le C(R,\omega,\eta,h_\theta^*) \quad \text{a.e. in } \Omega_{\eta,R}^{\varepsilon,\delta}, \tag{3.30}$$

and Hypothesis (H6) again implies that

$$|\partial_{ij}h_\theta^*(\vec{z}_t)|\le C(R,\omega,\eta,h_\theta^*) \quad \text{a.e. in } \Omega_{\eta,R}^{\varepsilon,\delta}. \tag{3.31}$$

With these auxiliary results, we can now focus on the limiting process. Let the function $\chi \in L^\infty(\Omega)$ be arbitrary and nonnegative. It follows from definition (3.21), the uniform convergence of (θ_ε) in Ω_η, estimate (3.29), and Hypothesis (H6) that

$$\limsup_{\varepsilon\to 0}\limsup_{\delta\to 0}\int_{\Omega^{\varepsilon,\delta}_{\eta,R}}\big(|Y^2_\varepsilon|+|Y^3_\delta|\big)\,dx=0.$$

Similarly, again thanks to Hypothesis (H6) and the proper definition of the set $\Omega^{\varepsilon,\delta}_{\eta,R}$,

$$\limsup_{\varepsilon\to 0}\limsup_{\delta\to 0}\|Z^2_\varepsilon+Z^3_\delta\|_{L^\infty(\Omega^{\varepsilon,\delta}_{\eta,R})}=0.$$

Hence, using the weak convergence (3.14) and definitions (3.21) and (3.24),

$$\begin{aligned}\lim_{\varepsilon\to 0}\lim_{\delta\to 0}&\int_{\Omega^{\varepsilon,\delta}_{\eta,R}}\big(p(\vec\rho_\varepsilon,\theta_\varepsilon)-p(\vec\rho_\delta,\theta_\delta)\big)\big(T_k(\rho_\varepsilon)-T_k(\rho_\delta)\big)\chi\,dx\\&=\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{\Omega^{\varepsilon,\delta}_{\eta,R}}Y^1_{\varepsilon,\delta}Z^1_{\varepsilon,\delta}\chi\,dx=\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{\Omega^{\varepsilon,\delta}_{\eta,R}}(I^1_{\varepsilon,\delta}+I^2_{\varepsilon,\delta}+I^3_{\varepsilon,\delta})\chi\,dx,\end{aligned}$$

where we identify the terms $I^j_{\varepsilon,\delta}$ for $j=1,2,3$ with the help of (3.23) and (3.26) as

$$\begin{aligned}I^1_{\varepsilon,\delta}&=(\mu_{\varepsilon,N}-\mu_{\delta,N})^2\int_0^1\int_0^1\sum_{i,j,\ell=1}^N\partial_i h^*_\theta(\vec z_t)\partial^2_{j\ell}h^*_\theta(\vec z_s)\Lambda_k(s)\,ds\,dt,\\I^2_{\varepsilon,\delta}&=(\mu_{\varepsilon,N}-\mu_{\delta,N})\int_0^1\int_0^1\sum_{i,j,\ell=1}^N\big(\partial_i h^*_\theta(\vec z_t)\partial^2_{j\ell}h^*_\theta(\vec z_s)+\partial_j h^*_\theta(\vec z_t)\partial^2_{i\ell}h^*_\theta(\vec z_s)\big)\\&\qquad\times\mathrm{P}(\vec\mu_\varepsilon-\vec\mu_\delta)_i\Lambda_k(s)\,ds\,dt,\\I^3_{\varepsilon,\delta}&=\int_0^1\int_0^1\sum_{i,j,\ell=1}^N\partial_i h^*_\theta(\vec z_t)\partial^2_{j\ell}h^*_\theta(\vec z_s)\mathrm{P}(\vec\mu_\varepsilon-\vec\mu_\delta)_i\mathrm{P}(\vec\mu_\varepsilon-\vec\mu_\delta)_j\Lambda_k(s)\,ds\,dt.\end{aligned}\tag{3.32}$$

We start with the easiest term, which is $I^3_{\varepsilon,\delta}$. We deduce from (3.30) and (3.31) and the uniform convergence of $\Pi\vec\mu_\varepsilon$ in Ω_η (and consequently also of $\mathrm{P}(\vec\mu_\varepsilon)$) that

$$\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{\Omega^{\varepsilon,\delta}_{\eta,R}}|I^3_{\varepsilon,\delta}|\chi\,dx=0.\tag{3.33}$$

Next, since h^*_θ is convex, the Hessian of h^*_θ is positive semidefinite, i.e. $\sum_{j,\ell=1}^N\partial^2_{j\ell}h^*_\theta(\vec z_s)\ge 0$. Furthermore, we know that the function $\partial_i h^*_\theta(\vec z_t)$ is nonnegative (recall that $\partial_i h^*_\theta(\vec z_t)=\rho_i$) and then it obviously follows that $I^1_{\varepsilon,\delta}\ge 0$.

It remains to analyze the integral $I^2_{\varepsilon,\delta}$. For this, we use the Cauchy–Schwarz and Young inequalities for some $\kappa>0$, and the positive semi-definiteness of $(\partial^2_{j\ell}h^*_\theta)$:

$$|I^2_{\varepsilon,\delta}|\le|\mu_{\varepsilon,N}-\mu_{\delta,N}||\mathrm{P}(\vec\mu_\varepsilon-\vec\mu_\delta)|\int_0^1\int_0^1\sum_{i,j,\ell=1}^N\partial_i h^*_\theta(\vec z_t)\partial^2_{j\ell}h^*_\theta(\vec z_s)\Lambda_k(s)\,ds\,dt$$

$$
\begin{aligned}
&\quad + \int_0^1 \bigg| \sum_{i,\ell=1}^N \partial^2_{i\ell} h^*_\theta(\vec{z}_s) \bigg(|\mu_{\varepsilon,N} - \mu_{\delta,N}|^2 \Lambda_k(s) \int_0^1 \sum_{j=1}^N \partial_j h^*_\theta(\vec{z}_t)\,dt \bigg)^{1/2} \\
&\quad \times \mathrm{P}(\vec{\mu}_\varepsilon - \vec{\mu}_\delta)_i \bigg(\Lambda_k(s) \int_0^1 \sum_{j=1}^N \partial_j h^*_\theta(\vec{z}_t)\,dt \bigg)^{1/2} \bigg|\,ds \\
&\le 2\big(I^1_{\varepsilon,\delta}\big)^{1/2} \bigg(|\mathrm{P}(\vec{\mu}_\varepsilon - \vec{\mu}_\delta)|^2 \int_0^1 \int_0^1 \sum_{i,j,\ell=1}^N \partial_i h^*_\theta(\vec{z}_t) \partial^2_{j\ell} h^*_\theta(\vec{z}_s) \Lambda_k(s)\,ds\,dt \bigg)^{1/2} \\
&\le C(R,\eta)\big(I^1_{\varepsilon,\delta}\big)^{1/2} |\mathrm{P}(\vec{\mu}_\varepsilon - \vec{\mu}_\delta)| \le \kappa I^1_{\varepsilon,\delta} + C(R,\eta,\kappa)|\mathrm{P}(\vec{\mu}_\varepsilon - \vec{\mu}_\delta)|^2,
\end{aligned}
$$

where in the last line we used (3.30)–(3.31). As $\kappa > 0$ can be taken arbitrarily small, we use the uniform convergence of $(\mathrm{P}(\vec{\mu}_\varepsilon))$ and (3.33) to deduce that for any $\chi \in L^\infty(\Omega)$,

$$
\lim_{\varepsilon\to 0}\lim_{\delta\to 0} \int_{\Omega^{\varepsilon,\delta}_{\eta,R}} (p(\vec{\rho}_\varepsilon,\theta_\varepsilon) - p(\vec{\rho}_\delta,\theta_\delta))\big(T_k(\rho_\varepsilon) - T_k(\rho_\delta)\big)\chi\,dx = \lim_{\varepsilon\to 0}\lim_{\delta\to 0} \int_{\Omega^{\varepsilon,\delta}_{\eta,R}} I^1_{\varepsilon,\delta}\chi\,dx \ge 0. \tag{3.34}
$$

This inequality, together with (3.27) and (3.28), shows that $W_k \ge 0$. Moreover, since for all k we know that $\Lambda_k(s) \le \Lambda_{k+1}(s)$ (it follows from $T_k' \le T_{k+1}'$), we see from the definition of $I^1_{\varepsilon,\delta}$ that $W_k \le W_{k+1}$.

Step 4: Proof of (3.19). Here, we can repeat the arguments from Step 3 almost step by step. Indeed, we have

$$
2\theta\big(\overline{\rho_\delta T_k(\rho_\delta)} - \rho\overline{T_k(\rho_\delta)}\big) = \theta\overline{(\rho_\delta - \rho_\varepsilon)(T_k(\rho_\delta) - T_k(\rho_\varepsilon))}.
$$

Again, we just need to identify the inequality on the set $\Omega^{\varepsilon,\delta}_{\eta,R}$, since the remaining parts vanish due to (3.13). Proceeding in the same way as in (3.24), we write

$$
\begin{aligned}
\rho_\varepsilon - \rho_\delta &= \sum_{i=1}^N \big(\partial_i h^*_\theta(\vec{\mu}_\varepsilon) - \partial_i h^*_\theta(\vec{\mu}_\delta)\big) + \sum_{i=1}^N \big(\partial_i h^*_{\theta_\varepsilon}(\vec{\mu}_\varepsilon) - \partial_i h^*_\theta(\vec{\mu}_\varepsilon)\big) \\
&\quad + \sum_{i=1}^N \big(\partial_i h^*_\theta(\vec{\mu}_\delta) - \partial_i h^*_{\theta_\delta}(\vec{\mu}_\delta)\big) =: \widehat{Z}^1_{\varepsilon,\delta} + \widehat{Z}^2_\varepsilon + \widehat{Z}^3_\delta
\end{aligned}
$$

and rewrite $\widehat{Z}^1_{\varepsilon,\delta}$ as (compare with (3.26))

$$
\widehat{Z}^1_{\varepsilon,\delta} = \int_0^1 \bigg(\sum_{i,j=1}^N \partial^2_{ij} h^*_\theta(\vec{z}_s)\mathrm{P}(\vec{\mu}_\varepsilon - \vec{\mu}_\delta)_i + (\mu_{\varepsilon,N} - \mu_{\delta,N}) \sum_{i,j=1}^N \partial^2_{ij} h^*_\theta(\vec{z}_s) \bigg)\,ds.
$$

Hence, repeating the procedure from the previous step, we deduce that for any bounded nonnegative function χ,

$$
\lim_{\varepsilon\to 0}\lim_{\delta\to 0} \int_{\Omega^{\varepsilon,\delta}_{\eta,R}} \theta(\rho_\varepsilon - \rho_\delta)\big(T_k(\rho_\varepsilon) - T_k(\rho_\delta)\big)\chi\,dx = \lim_{\varepsilon\to 0}\lim_{\delta\to 0} \int_{\Omega^{\varepsilon,\delta}_{\eta,R}} \widehat{I}^1_{\varepsilon,\delta}\chi\,dx, \tag{3.35}
$$

where

$$\widehat{I}^1_{\varepsilon,\delta} = \theta(\mu_{\varepsilon,N} - \mu_{\delta,N})^2 \int_0^1 \int_0^1 \sum_{i,j,\ell,m=1}^N \partial_{\ell m} h^*_\theta(\vec{z}_t)\partial^2_{ij} h^*_\theta(\vec{z}_s)\Lambda_k(s)\, ds\, dt. \tag{3.36}$$

Hypothesis (H6), namely (1.20), definitions (3.32) and (3.36) then show that $\widehat{I}^1_{\varepsilon,\delta} \le K_2 I^1_{\varepsilon,\delta}$ and comparing (3.34) and (3.35) then leads to (3.19). ☐

The next lemma combines the results coming from the convexity of h_θ and the effective viscous flux identity, which finally lead to a uniform bound on (W_k), defined in (3.17).

Lemma 3.4. *Let all assumptions of Lemma 3.3 and* (3.2) *be satisfied. Then there exists a constant $C > 0$ such that for all $k \in \mathbb{N}$,*

$$\int_\Omega \frac{W_k}{\lambda_1(\theta) + \lambda_2(\theta)}\, dx \le C \sup_{\delta>0} \int_\Omega \frac{\lambda_1(\theta_\delta) + \lambda_2(\theta_\delta)}{\theta_\delta} (\operatorname{div} \boldsymbol{v}_\delta)^2\, dx. \tag{3.37}$$

Consequently, if the right-hand side in (3.37) is finite, we infer from the monotonicity of (W_k) (see (3.18)) and monotone convergence, that the sequence $(W_k/(\lambda_1(\theta)+\lambda_2(\theta))_{k\in\mathbb{N}}$ is strongly converging in $L^1(\Omega)$ to a nonnegative integrable function.

Proof. We start the proof with a simple inequality, using (3.19), (3.17), and (3.7):

$$\begin{aligned}
\overline{\frac{\theta_\delta}{\lambda_1(\theta_\delta) + \lambda_2(\theta_\delta)}(T_k(\rho_\varepsilon) - T_k(\rho_\delta))^2} &= \frac{\theta}{\lambda_1(\theta) + \lambda_2(\theta)} \overline{(T_k(\rho_\varepsilon) - T_k(\rho_\delta))^2} \\
&\le \frac{\theta}{\lambda_1(\theta) + \lambda_2(\theta)} \overline{(\rho_\varepsilon - \rho_\delta)(T_k(\rho_\varepsilon) - T_k(\rho_\delta))} \le \frac{CW_k}{\lambda_1(\theta) + \lambda_2(\theta)} \\
&= C\frac{\lambda_2(\theta) + \frac{4}{3}\lambda_1(\theta)}{\lambda_1(\theta) + \lambda_2(\theta)}\big(\overline{T_k(\rho_\delta)\operatorname{div} \boldsymbol{v}_\delta} - \overline{T_k(\rho_\delta)}\operatorname{div} \boldsymbol{v}\big).
\end{aligned}$$

Using (3.7) and (3.19) again, we observe that $\overline{T_k(\rho_\delta)\operatorname{div} \boldsymbol{v}_\delta} - \overline{T_k(\rho_\delta)}\operatorname{div} \boldsymbol{v} \ge 0$, leading to

$$\begin{aligned}
\overline{\frac{\theta_\delta}{\lambda_1(\theta_\delta) + \lambda_2(\theta_\delta)}(T_k(\rho_\varepsilon) - T_k(\rho_\delta))^2} &\le C\big(\overline{T_k(\rho_\delta)\operatorname{div} \boldsymbol{v}_\delta} - \overline{T_k(\rho_\delta)}\operatorname{div} \boldsymbol{v}\big) \\
&= C\overline{\operatorname{div} \boldsymbol{v}_\delta(T_k(\rho_\delta) - T_k(\rho_\varepsilon))}.
\end{aligned}$$

It follows from the Cauchy–Schwarz inequality that

$$\begin{aligned}
&\int_\Omega \overline{\frac{\theta_\delta}{\lambda_1(\theta_\delta) + \lambda_2(\theta_\delta)}(T_k(\rho_\varepsilon) - T_k(\rho_\delta))^2}\, dx \le C\int_\Omega \overline{\operatorname{div} \boldsymbol{v}_\delta(T_k(\rho_\delta) - T_k(\rho_\varepsilon))}\, dx \\
&\quad \le \bigg(\int_\Omega \frac{\lambda_1(\theta_\delta) + \lambda_2(\theta_\delta)}{\theta_\delta}(\operatorname{div} \boldsymbol{v}_\delta)^2\, dx\bigg)^{1/2} \bigg(\int_\Omega \overline{\frac{\theta_\delta(T_k(\rho_\varepsilon) - T_k(\rho_\delta))^2}{\lambda_1(\theta_\delta) + \lambda_2(\theta_\delta)}}\, dx,\bigg)^{1/2}
\end{aligned}$$

which yields

$$\int_\Omega \overline{\frac{\theta_\delta(T_k(\rho_\varepsilon) - T_k(\rho_\delta))^2}{\lambda_1(\theta_\delta) + \lambda_2(\theta_\delta)}}\, dx \le \sup_{\delta>0} \int_\Omega \frac{\lambda_1(\theta_\delta) + \lambda_2(\theta_\delta)}{\theta_\delta}(\operatorname{div} \boldsymbol{v}_\delta)^2\, dx,$$

$$\int_\Omega \overline{\operatorname{div} \boldsymbol{v}_\delta (T_k(\rho_\delta) - T_k(\rho_\varepsilon))}\,dx \le \sup_{\delta>0} \int_\Omega \frac{\lambda_1(\theta_\delta)+\lambda_2(\theta_\delta)}{\theta_\delta} (\operatorname{div} \boldsymbol{v}_\delta)^2\,dx.$$

We infer that

$$\begin{aligned}\int_\Omega \frac{W_k}{\lambda_1(\theta)+\lambda_2(\theta)}\,dx &= \int_\Omega \frac{\lambda_2(\theta)+\frac{4}{3}\lambda_1(\theta)}{\lambda_1(\theta)+\lambda_2(\theta)} \big(\overline{T_k(\rho_\delta)\operatorname{div} \boldsymbol{v}_\delta} - \overline{T_k(\rho_\delta)}\operatorname{div} \boldsymbol{v}\big)\,dx \\ &= \int_\Omega \frac{\lambda_2(\theta)+\frac{4}{3}\lambda_1(\theta)}{\lambda_1(\theta)+\lambda_2(\theta)} \overline{\operatorname{div} \boldsymbol{v}_\delta (T_k(\rho_\delta)-T_k(\rho_\varepsilon))}\,dx \\ &\le C \sup_{\delta>0} \int_\Omega \frac{\lambda_1(\theta_\delta)+\lambda_2(\theta_\delta)}{\theta_\delta} (\operatorname{div} \boldsymbol{v}_\delta)^2\,dx,\end{aligned}$$

finishing the proof. □

The following lemma is the key step in the proof. It shows that we can use the truncation function $T_k(\rho)$ instead of ρ in all estimates, and this change creates only a small error, which can be neglected in a suitable topology.

Lemma 3.5. *Let all assumptions of Lemmata 3.3 and 3.4 be satisfied and let*

$$\sup_{\delta>0} \int_\Omega \frac{\lambda_1(\theta_\delta)+\lambda_2(\theta_\delta)}{\theta_\delta} (\operatorname{div} \boldsymbol{v}_\delta)^2\,dx < \infty.$$

The quantities

$$Q_k := \overline{p_\varepsilon(T_k(\rho_\varepsilon) - \rho_\varepsilon T_k'(\rho_\varepsilon))} - p\overline{(T_k(\rho_\varepsilon) - \rho_\varepsilon T_k'(\rho_\varepsilon))}, \tag{3.38}$$

$$O_k := T_k(\rho) - \overline{T_k(\rho_\varepsilon)} \tag{3.39}$$

are nonnegative and satisfy for all $k \in \mathbb{N}$,

$$\theta O_k^2 \le C W_k \quad \textit{and} \quad \lim_{k\to\infty} \int_\Omega \frac{Q_k + \theta O_k^2}{\lambda_1(\theta)+\lambda_2(\theta)}\,dx = 0. \tag{3.40}$$

Proof. For the identification of Q_k, we can repeat the proof of Lemma 3.3 step by step, where instead of (3.24)–(3.25), we use a similar computation for $(T_k(\rho_\varepsilon) - \rho_\varepsilon T_k'(\rho_\varepsilon)) - (T_k(\rho_\delta) - \rho_\delta T_k'(\rho_\delta))$. Heuristically, this means that we replace Λ_k, which corresponds to the derivative of $T_k(s)$, by $\widetilde{\Lambda}_k$, which corresponds to the derivative of $T_k(s) - sT_k'(s)$. This leads to the identification of the limit (compare with (3.34)):

$$\begin{aligned}&\lim_{\varepsilon\to 0}\lim_{\delta\to 0} \int_{\Omega^{\varepsilon,\delta}_{\eta,R}} \big(p(\vec\rho_\varepsilon,\theta_\varepsilon) - p(\vec\rho_\delta,\theta_\delta)\big)\big(T_k(\rho_\varepsilon) - \rho_\varepsilon T_k'(\rho_\varepsilon) - T_k(\rho_\delta) + \rho_\delta T_k'(\rho_\delta)\big)\chi\,dx \\ &\quad = \lim_{\varepsilon\to 0}\lim_{\delta\to 0} \int_{\Omega^{\varepsilon,\delta}_{\eta,R}} \widetilde{I}^1_{\varepsilon,\delta}\chi\,dx,\end{aligned}$$

where the term

$$\widetilde{I}^1_{\varepsilon,\delta} := \theta(\mu_{\varepsilon,N} - \mu_{\delta,N})^2 \int_0^1\int_0^1 \sum_{i=1}^N \partial_i h^*_\theta(\vec z_t) \sum_{j,\ell=1}^N \partial^2_{j\ell} h^*_\theta(\vec z_s) \widetilde{\Lambda}_k(s)\,ds\,dt$$

corresponds to (3.32), just replacing the function Λ_k by

$$\widetilde{\Lambda}_k(s) := -\sum_{i=1}^{N} \partial_i h_\theta^*(\vec{z}_s)\, T_k''\bigg(\sum_{i=1}^{N} \partial_i h_\theta^*(\vec{z}_s)\bigg) = -\frac{1}{k}\sum_{i=1}^{N} \partial_i h_\theta^*(\vec{z}_s)\, T_1''\bigg(\frac{1}{k}\sum_{i=1}^{N} \partial_i h_\theta^*(\vec{z}_s)\bigg).$$

Since T_1 is concave, we have $\widetilde{\Lambda}_k \geq 0$. An elementary computation shows that there exists $C > 0$ such that for all $s \geq 0$,

$$-sT_1''(s) \leq C(T_2'(s) - T_{1/2}'(s)) = C(T_1'(s/2) - T_1'(2s)),$$

which implies that $\widetilde{\Lambda}_k(s) \leq C(\Lambda_{2k}(s) - \Lambda_{k/2}(s))$. Therefore,

$$0 \leq Q_k \leq C(W_{2k} - W_{k/2}). \tag{3.41}$$

Finally, by Lemma 3.3, (W_k) is monotone and, by Lemma 3.4, $(W_k/(\lambda_1(\theta) + \lambda_2(\theta)))$ is bounded in $L^1(\Omega)$ and monotone as well. We deduce from the monotone convergence theorem that $(W_k/(\lambda_1(\theta) + \lambda_2(\theta)))$ is strongly converging in $L^1(\Omega)$ which implies that

$$\lim_{k\to\infty} \int_\Omega \frac{W_{2k} - W_{k/2}}{\lambda_1(\theta) + \lambda_2(\theta)}\, dx = 0.$$

Inequality (3.41) then directly leads to

$$\lim_{k\to\infty} \int_\Omega \frac{Q_k}{\lambda_1(\theta) + \lambda_2(\theta)}\, dx = 0.$$

Now, we focus on O_k, defined in (3.39). Since T_k is concave, O_k is nonnegative. First, we show that $O_k \to 0$ a.e. in Ω. As $\vec{\rho}_\varepsilon \rightharpoonup \vec{\rho}$ weakly in $L^1(\Omega;\mathbb{R}^N)$, the sequence (ρ_ε) is uniformly equiintegrable. Therefore, by the weak lower semicontinuity of the $L^1(\Omega)$ norm,

$$\begin{aligned}
\int_\Omega |O_k|\, dx &\leq \int_\Omega |\rho - T_k(\rho)|\, dx + \liminf_{\varepsilon\to 0} \int_\Omega |\rho_\varepsilon - T_k(\rho_\varepsilon)|\, dx \\
&\leq \int_{\{\rho>k\}} |\rho - k|\, dx + \liminf_{\varepsilon\to 0} \int_{\{\rho_\varepsilon>k\}} |\rho_\varepsilon - k|\, dx \\
&\leq \int_{\{\rho>k\}} \rho\, dx + \liminf_{\varepsilon\to 0} \int_{\{\rho_\varepsilon>k\}} \rho_\varepsilon\, dx.
\end{aligned}$$

Because of the weak convergence of $(\vec{\rho}_\varepsilon)$, it holds that $|\{\rho > k\}| + |\{\rho_\varepsilon > k\}| \leq C/k$ and consequently, the uniform integrability of (ρ_ε) shows that

$$\lim_{k\to\infty} \int_\Omega |O_k|\, dx = 0,$$

which implies for a subsequence that $O_k \to 0$ a.e. in Ω. Second, we show the proper bound on O_k, which will enable us to use the dominated convergence theorem. We deduce from the weak lower semicontinuity of the $L^2(\Omega)$ norm that

$$|O_k|^2 = |\overline{T_k(\rho_\varepsilon)} - T_k(\rho)|^2 \leq \overline{|T_k(\rho_\varepsilon) - T_k(\rho)|^2}.$$

Substituting the algebraic inequality

$$|T_k(s_1) - T_k(s_2)|^2 \leq s_1 T_k(s_1) + s_2 T_k(s_2) - s_1 T_k(s_2) - s_2 T_k(s_1)$$

into the relation for O_k, we obtain

$$|O_k|^2 \leq \overline{\rho_\varepsilon T_k(\rho_\varepsilon) + \rho T_k(\rho) - \rho_\varepsilon T_k(\rho) - \rho T_k(\rho_\varepsilon)} = \overline{\rho_\varepsilon T_k(\rho_\varepsilon)} - \rho \overline{T_k(\rho_\varepsilon)}.$$

Consequently, (3.19) implies that

$$\frac{\theta |O_k|^2}{\lambda_1(\theta)+\lambda_2(\theta)} \leq \frac{K_2 W_k}{\lambda_1(\theta)+\lambda_2(\theta)} \quad \text{a.e. in } \Omega.$$

Arguing as before, the right-hand side is convergent in $L^1(\Omega)$, and we can apply the dominated convergence theorem to conclude the second part of (3.40). □

3.4. **Renormalized continuity equation.** We prove that the weak limit ρ of (ρ_δ) is a renormalized solution to the mass continuity equation. Note that the proof is different to the standard ones. Indeed, for our purpose, we just need the pressure having at least linear growth with respect to the density, which is not the case in other works.

Lemma 3.6. *Let all assumptions of Lemmata 3.3 and 3.4 be satisfied and let the sequence $(\rho_\delta, \boldsymbol{v}_\delta)$ satisfy (1.32), i.e., it solves for every $b \in C^{0,1}(\mathbb{R})$ with compactly supported derivative and all $\psi \in W^{1,\infty}(\Omega)$ the renormalized continuity equation*

$$-\int_\Omega b(\rho_\delta)\boldsymbol{v}_\delta \cdot \nabla\psi dx + \int_\Omega \big(b'(\rho_\delta)\rho_\delta - b(\rho_\delta)\big)\operatorname{div}(\boldsymbol{v}_\delta)\psi\, dx = 0. \tag{3.42}$$

Then the weak limit $(\rho, \boldsymbol{v})$ satisfies it as well.

Proof. We use (3.42) with $b = T_k$:

$$-\int_\Omega T_k(\rho_\delta)\boldsymbol{v}_\delta \cdot \nabla\psi\, dx + \int_\Omega \big(\rho_\delta T_k'(\rho_\delta) - T_k(\rho_\delta)\big)\operatorname{div}(\boldsymbol{v}_\delta)\psi\, dx = 0,$$

pass to the limit $\delta \to 0$, and use the fact that $\boldsymbol{v}_\delta \to \boldsymbol{v}$ strongly in $L^2(\Omega)$ to infer that

$$-\int_\Omega \overline{T_k(\rho_\delta)}\boldsymbol{v} \cdot \nabla\psi\, dx + \int_\Omega \overline{(\rho_\delta T_k'(\rho_\delta) - T_k(\rho_\delta))\operatorname{div}\boldsymbol{v}_\delta}\psi\, dx = 0,$$

which is the weak formulation of

$$\operatorname{div}(\overline{T_k(\rho_\delta)}\boldsymbol{v}) + \overline{(\rho_\delta T_k'(\rho_\delta) - T_k(\rho_\delta))\operatorname{div}\boldsymbol{v}_\delta} = 0 \quad \text{in the sense of distributions.}$$

Since $\overline{T_k(\rho_\delta)} \in L^\infty(\Omega)$ and $\boldsymbol{v} \in W^{1,2}(\Omega)$, this equation can be renormalized; see, e.g., [28, Theorem 10.29]. It follows for any Lipschitz continuous function β with compactly supported derivative and any function $\psi \in W^{1,\infty}(\Omega)$ that

$$\begin{aligned} -\int_\Omega b\big(\overline{T_k(\rho_\delta)}\big)\boldsymbol{v}\cdot\nabla\psi\, dx &+ \int_\Omega \Big(\overline{T_k(\rho_\delta)}b'\big(\overline{T_k(\rho_\delta)}\big) - b\big(\overline{T_k(\rho_\delta)}\big)\Big)\operatorname{div}(\boldsymbol{v})\psi\, dx \\ &= \int_\Omega b'\big(\overline{T_k(\rho_\delta)}\big)\overline{(T_k(\rho_\delta) - \rho_\delta T_k'(\rho_\delta))\operatorname{div}\boldsymbol{v}_\delta}\psi\, dx. \end{aligned} \tag{3.43}$$

To identify and estimate the term on the right-hand side, we use definition (3.38) of Q_k, the effective viscous flux identity (3.7), and a similar identity with $\rho_\delta T_k'(\rho_\delta)$ instead of $T_k(\rho_\delta)$:

$$Q_k = \left(\lambda_2(\theta) + \frac{4}{3}\lambda_1(\theta)\right)\Big(\overline{(T_k(\rho_\delta) - \rho_\delta T_k'(\rho_\delta))\operatorname{div}\boldsymbol{v}_\delta} - \overline{(T_k(\rho_\delta) - \rho_\delta T_k'(\rho_\delta))}\operatorname{div}\boldsymbol{v}\Big).$$

We insert this relation into (3.43) to obtain

$$\begin{aligned}&-\int_\Omega b\big(\overline{T_k(\rho_\delta)}\big)\boldsymbol{v}\cdot\nabla\psi\,dx+\int_\Omega\Big(\overline{T_k(\rho_\delta)}b'\big(\overline{T_k(\rho_\delta)}\big)-b\big(\overline{T_k(\rho_\delta)}\big)\Big)\operatorname{div}(\boldsymbol{v})\psi\,dx\\&\quad=\int_\Omega b'\big(\overline{T_k(\rho_\delta)}\big)\overline{\big(T_k(\rho_\delta)-\rho_\delta T_k'(\rho_\delta)\big)}\operatorname{div}(\boldsymbol{v})\psi\,dx+\int_\Omega\frac{b'(\overline{T_k(\rho_\delta)})Q_k}{\lambda_2(\theta)+\frac{4}{3}\lambda_1(\theta)}\psi\,dx.\end{aligned}\tag{3.44}$$

Our goal is to perform the limit $k\to\infty$ in (3.44) to recover (3.42) with $(\rho_\delta,\boldsymbol{v}_\delta)$ replaced by $(\rho,\boldsymbol{v})$. We first show that $\overline{T_k(\rho_\delta)}\to\rho$ as $k\to\infty$ a.e. in Ω. To this end, we recall the weak lower semicontinuity of the $L^1(\Omega)$ norm leading to

$$\begin{aligned}\int_\Omega\big|\overline{T_k(\rho_\delta)}-\rho\big|\,dx&\le\liminf_{\delta\to0}\int_\Omega|T_k(\rho_\delta)-\rho_\delta|\,dx\le\liminf_{\delta\to0}\int_{\{\rho_\delta\ge k\}}(\rho_\delta-T_k(\rho_\delta))\,dx\\&\le\sup_{\delta>0}\int_{\{\rho_\delta\ge k\}}\rho_\delta\,dx.\end{aligned}$$

Since (ρ_δ) is weakly convergent in $L^1(\Omega)$ (assumption (3.13)), we have $|\{\rho_\delta\ge k\}|\le C/k$, and since the sequence is uniformly equiintegrable, we deduce that

$$\limsup_{k\to\infty}\int_\Omega\big|\overline{T_k(\rho_\delta)}-\rho\big|\,dx\le\limsup_{k\to\infty}\left(\sup_{\delta>0}\int_{\{\rho_\delta\ge k\}}\rho_\delta\,dx\right)=0\tag{3.45}$$

which shows the pointwise convergence of $\overline{T_k(\rho_\delta)}$ (at least for a subsequence). Similarly,

$$\lim_{k\to\infty}\int_\Omega\big|\overline{T_k(\rho_\delta)-\rho_\delta T_k'(\rho_\delta)}\big|\,dx=0.\tag{3.46}$$

The concavity of T_k implies that $|T_k(s)-sT_k'(s)|\le T_k(s)$ and consequently,

$$\big|\overline{T_k(\rho_\delta)-\rho_\delta T_k'(\rho_\delta)}\big|\le\overline{T_k(\rho_\delta)}\quad\text{a.e. in }\Omega.$$

Thus, since b is bounded and its derivative is compactly supported, there exists $C>0$ only depending on b (but not on k) such that

$$\big|b\big(\overline{T_k(\rho_\delta)}\big)\big|+\big|\overline{T_k(\rho_\delta)}b'\big(\overline{T_k(\rho_\delta)}\big)-b\big(\overline{T_k(\rho_\delta)}\big)\big|+\big|b'\big(\overline{T_k(\rho_\delta)}\big)\overline{\big(T_k(\rho_\delta)-\rho_\delta T_k'(\rho_\delta)\big)}\big|\le C.$$

This estimate and (3.45)–(3.46) lead to

$$\begin{aligned}b\big(\overline{T_k(\rho_\delta)}\big)&\to b(\rho)&&\text{strongly in }L^q(\Omega),\\b'\big(\overline{T_k(\rho_\delta)}\big)\overline{T_k(\rho_\delta)-\rho_\delta T_k'(\rho_\delta)}&\to0&&\text{strongly in }L^q(\Omega),\\b'\big(\overline{T_k(\rho_\delta)}\big)\overline{T_k(\rho_\delta)}-b\big(\overline{T_k(\rho_\delta)}\big)&\to b'(\rho)\rho-b(\rho)&&\text{strongly in }L^q(\Omega)\end{aligned}$$

for any $q<\infty$. These convergence results, estimate (3.40) for Q_k, and the boundedness of b' allow us to pass to the limit $k\to\infty$ in (3.44) which gives (3.42) for $(\rho,\boldsymbol{v})$. ☐

3.5. **Strong convergence of the densities.** We finally show that $\vec\rho_\delta\to\vec\rho$ a.e. in Ω. First, we prove that the total mass density ρ_δ converges pointwise and then, exploiting the compactness of $\Pi\vec\mu_\delta$ and the convexity of the free energy, we deduce the compactness of the sequence $(\vec\rho_\delta)$. We start with the convergence of the total mass densities.

Lemma 3.7 (Strong convergence of the total mass densities)**.** *Let the assumptions of Lemmata 3.2–3.4 be satisfied and let $(\rho_\delta,\boldsymbol{v}_\delta)$ solve the renormalized mass continuity equation* (3.42)*. Then*

$$\rho_\delta\to\rho \qquad \text{strongly in } L^1(\Omega) \text{ as } \delta\to 0. \tag{3.47}$$

Proof. The idea of the proof is to use the effective flux identity (3.7) and combine it with the properties of W_k, defined in (3.17). We prove the key property

$$\lim_{k\to\infty}\int_\Omega\big(\overline{T_k(\rho_\delta)\operatorname{div}\boldsymbol{v}_\delta}-\overline{T_k(\rho_\delta)}\operatorname{div}\boldsymbol{v}\big)\,dx=0. \tag{3.48}$$

We claim that if $(\rho,\boldsymbol{v})$ or $(\rho_\delta,\boldsymbol{v}_\delta)$, respectively, are renormalized solutions, then

$$0=\int_\Omega T_k(\rho)\operatorname{div}\boldsymbol{v}\,dx=\int_\Omega T_k(\rho_\delta)\operatorname{div}\boldsymbol{v}_\delta\,dx. \tag{3.49}$$

For this purpose, we choose $\psi=1$ in (3.42) such that for any bounded b with compactly supported derivative, we have

$$\int_\Omega\big(b'(\rho)\rho-b(\rho)\big)\operatorname{div}\boldsymbol{v}\,dx=0,$$

and a similar identity holds for ρ_δ. For any continuous compactly supported function f in $(0,\infty)$, we define

$$b_f(s):=-s\int_s^\infty\frac{f(t)}{t^2}\,dt,\quad s\ge 0.$$

Then b_f is bounded, has a compactly supported derivative, and it holds that $b_f'(s)s-b_f(s)=f(s)$ for $s\ge 0$. Hence,

$$0=\int_\Omega\big(b_f'(\rho)\rho-b_f(\rho)\big)\operatorname{div}\boldsymbol{v}\,dx=\int_\Omega f(\rho)\operatorname{div}\boldsymbol{v}\,dx.$$

Let (f^m) be a sequence of compactly supported functions f^m satisfying $f^m\nearrow T_k$ pointwise as $m\to\infty$. Choosing $f=f^m$ in the previous identity and passing to the limit $m\to\infty$ leads to (3.49). Of course, the same argument applies to $(\rho_\delta,\boldsymbol{v}_\delta)$. Thus,

$$\int_\Omega\overline{T_k(\rho_\delta)\operatorname{div}\boldsymbol{v}_\delta}\,dx=\lim_{\delta\to 0}\int_\Omega T_k(\rho_\delta)\operatorname{div}\boldsymbol{v}_\delta\,dx=0,$$

and the first part in (3.48) vanishes. It remains to estimate the second part. It follows from (3.49), (3.39), (3.40), and the Cauchy–Schwarz inequality that

$$\lim_{k\to\infty}\left|\int_\Omega\overline{T_k(\rho_\delta)}\operatorname{div}\boldsymbol{v}\,dx\right|=\lim_{k\to\infty}\left|\int_\Omega(\overline{T_k(\rho_\delta)}-T_k(\rho))\operatorname{div}\boldsymbol{v}\,dx\right|\le\lim_{k\to\infty}\int_\Omega|O_k|\,|\operatorname{div}\boldsymbol{v}|\,dx$$

$$\leq \left(\int_\Omega \frac{\lambda_1(\theta)+\lambda_2(\theta)}{\theta}|\operatorname{div}\boldsymbol{v}|^2\,dx\right)^{1/2}\lim_{k\to\infty}\left(\int_\Omega \frac{\theta O_k^2}{\lambda_1(\theta)+\lambda_2(\theta)}\,dx\right)^{1/2} = 0,$$

and (3.48) holds true. Thus, using (3.7) and (3.17) in (3.48), it follows that

$$\begin{aligned} 0 &= \lim_{k\to\infty}\int_\Omega \left(\overline{T_k(\rho_\delta)\operatorname{div}\boldsymbol{v}_\delta} - \overline{T_k(\rho_\delta)}\operatorname{div}\boldsymbol{v}\right)dx \\ &= \lim_{k\to\infty}\int_\Omega \frac{\overline{p_\delta T_k(\rho_\delta)} - p\overline{T_k(\rho_\delta)}}{\lambda_2(\theta)+\frac{4}{3}\lambda_1(\theta)}\,dx = \lim_{k\to\infty}\int_\Omega \frac{W_k}{\lambda_2(\theta)+\frac{4}{3}\lambda_1(\theta)}\,dx. \end{aligned}$$

By Hypothesis (H3), this yields $\lim_{k\to\infty}\int_\Omega W_k\,dx = 0$. It follows from $W_k \geq 0$ that $\lim_{k\to\infty} W_k = 0$, and the monotonicity $W_k \leq W_{k+1}$ implies that

$$W_k \equiv 0 \quad \text{a.e. in } \Omega,\ k\in\mathbb{N}.$$

Hence, (3.19) shows that $\theta(\overline{\rho_\delta T_k(\rho_\delta)} - \rho\overline{T_k(\rho_\delta)}) = 0$ in Ω such that, since $\theta > 0$ a.e. in Ω,

$$0 = \int_\Omega \left(\overline{\rho_\delta T_k(\rho_\delta)} - \rho\overline{T_k(\rho_\delta)}\right)dx = \lim_{\varepsilon\to0}\lim_{\delta\to0}\int_\Omega (\rho_\delta-\rho_\varepsilon)(T_k(\rho_\delta)-T_k(\rho_\varepsilon))\,dx. \tag{3.50}$$

Finally, using the weak lower semicontinuity of the $L^1(\Omega)$ norm, the Cauchy–Schwarz inequality, and (3.50),

$$\begin{aligned} &\lim_{\varepsilon\to0}\int_\Omega|\rho_\varepsilon-\rho|\,dx \leq \lim_{\varepsilon\to0}\lim_{\delta\to0}\int_\Omega|\rho_\varepsilon-\rho_\delta|\,dx \\ &\leq \lim_{\varepsilon\to0}\lim_{\delta\to0}\int_{\{\rho_\delta>k\}\cup\{\rho_\varepsilon>k\}}|\rho_\varepsilon-\rho_\delta|\,dx + \lim_{\varepsilon\to0}\lim_{\delta\to0}\int_\Omega|T_k(\rho_\varepsilon)-T_k(\rho_\delta)|\,dx \\ &\leq \sup_{\varepsilon,\delta>0}\int_{\{\rho_\delta>k\}\cup\{\rho_\varepsilon>k\}}|\rho_\varepsilon-\rho_\delta|\,dx + |\Omega|^{1/2}\lim_{\varepsilon\to0}\lim_{\delta\to0}\left(\int_\Omega|T_k(\rho_\varepsilon)-T_k(\rho_\delta)|^2\,dx\right)^{1/2} \\ &\leq \sup_{\varepsilon,\delta>0}\int_{\{\rho_\delta>k\}\cup\{\rho_\varepsilon>k\}}|\rho_\varepsilon-\rho_\delta|\,dx + |\Omega|^{1/2}\lim_{\varepsilon\to0}\lim_{\delta\to0}\left(\int_\Omega(\rho_\varepsilon-\rho_\delta)(T_k(\rho_\varepsilon)-T_k(\rho_\delta))\,dx\right)^{1/2} \\ &= \sup_{\varepsilon,\delta>0}\int_{\{\rho_\delta>k\}\cup\{\rho_\delta>k\}}|\rho_\varepsilon-\rho_\delta|\,dx. \end{aligned}$$

The left-hand side is independent of k and therefore, we may perform the limit $k\to\infty$ also on the right-hand side. Then, thanks to uniform equiintegrability of (ρ_δ), we deduce (3.47), which finishes the proof. □

We end this subsection, and also the proof of Theorem 3.1, by the following lemma, which shows the compactness of the vector $\vec{\rho}$ of mass densities. In addition, we show that either the total mass density vanishes, i.e., all partial mass densities are zero, or all partial mass densities are strictly positive a.e. in Ω.

Lemma 3.8 (Strong convergence of the vector of mass densities)**.** *Let all assumptions of Lemmata 3.2–3.4 be satisfied and let $(\rho_\delta, \boldsymbol{v}_\delta)$ solve the renormalized mass continuity* *equation* (3.42). *Then*

$$\vec\rho_\delta \to \vec\rho \textit{ strongly in } L^1(\Omega;\mathbb{R}^N) \quad \textit{as } \delta\to 0. \tag{3.51}$$

Moreover, for a.e. $x\in\Omega$ *and all* $i=1,\dots,N$,

$$\rho(x)>0 \implies \rho_i(x)>0. \tag{3.52}$$

Proof. Taking into account the strong convergence results, derived in this section, Egorov's theorem implies that for any $\eta>0$, there exists a set Ω_η such that $|\Omega\setminus\Omega_\eta|\le\eta$ and

$$\begin{aligned} \rho_\delta \to \rho &\quad \text{strongly in } L^\infty(\Omega_\eta), \\ \Pi\vec\mu_\delta \to \overline{\Pi\vec\mu_\delta} &\quad \text{strongly in } L^\infty(\Omega_\eta;\mathbb{R}^N), \\ \theta_\delta \to \theta &\quad \text{strongly in } L^\infty(\Omega_\eta), \\ \ln\theta_\delta \to \ln\theta &\quad \text{strongly in } L^\infty(\Omega_\eta). \end{aligned}$$

We collect some facts about the free energy density h_θ. Since it is strictly convex with respect to $\vec\rho$ and $\nabla_{\vec\rho} h_\theta$ is a strictly monotone invertible mapping from $\mathbb{R}^N_+$ to $\mathbb{R}$, for every $\kappa>0$, there exists $C(\kappa)>0$ such that for all $\theta\in(\kappa,\kappa^{-1})$,

$$\begin{aligned} \rho_i>\kappa &\implies \mu_i = \frac{\partial h_\theta}{\partial\rho_i}(\vec\rho) \ge -C(\kappa), \\ |\vec\rho|\le\kappa^{-1} &\implies \mu_i = \frac{\partial h_\theta}{\partial\rho_i}(\vec\rho) \le C(\kappa) \text{ for all } i=1,\dots,N. \end{aligned} \tag{3.53}$$

Finally, we focus on the strong convergence of the densities. We consider two cases: the total density vanishes or the total density remains positive. The first case is easy, so we assume that the total density is positive. We define the set

$$\Omega_{\eta,\kappa} := \{x\in\Omega_\eta;\ \rho(x)\ge 2N\kappa\}.$$

Thanks to the uniform convergence of (ρ_δ), there exists $\delta_0>0$ such that for all $0<\delta<\delta_0$, it holds that $\rho_\delta(x)\ge N\kappa$ for $x\in\Omega_{\eta,\kappa}$. Consequently, for $x\in\Omega_{\eta,\kappa}$ and $0<\delta<\delta_0$, there exists $i\in\{1,\dots,N\}$ such that

$$\kappa \le \rho_{\delta,i}(x) \le C(\eta),$$

since $(\rho_{\delta,i})$ converges uniformly in $\Omega_{\eta,\kappa}\subset\Omega_\eta$. Therefore, using (3.53), we infer that $|\mu_{\delta,i}(x)|\le C(\kappa,\eta)$. In addition, thanks to the uniform convergence of (ρ_δ) and (3.53), $\mu_{\delta,j}(x)\le C_1(\eta,\kappa)$ for $j=1,\dots,N$. By the uniform convergence of $\Pi\vec\mu_\delta$, we obtain

$$\sum_{j=1}^N \mu_{\delta,j}(x) = N\vec\mu_{\delta,i}(x) - N(\Pi\vec\mu_\delta)_i(x) \ge -C_2(\eta,\kappa,N) \quad \text{for } x\in\Omega_{\eta,\kappa}.$$

The bounds $\mu_j\le C(\eta,\kappa)$ and $\sum_{j=1}^N\mu_j\ge -C_2(\eta,\kappa,N)$ in $\Omega_{\eta,\kappa}$ imply that

$$\mu_j \ge -C_2(\eta,\kappa,N) - \sum_{\ell\ne j}\mu_\ell \ge C_2(\eta,\kappa,N) - (N-1)C(\eta,\kappa) \quad \text{in } \Omega_{\eta,\kappa},\ j=1,\dots,N.$$

We conclude that

$$C_3(\kappa, N, \eta) \le \mu_{\delta,j}(x) \le C_4(\kappa, \eta, N) \quad \text{for } x \in \Omega_{\eta,\kappa},\ j = 1, \dots, N. \tag{3.54}$$

Consequently, since $\vec{\mu}_\delta = \nabla_{\vec{\rho}} h_{\theta_\delta}(\vec{\rho}_\delta)$ and $\theta_\delta \to \theta$ a.e. in Ω, we have

$$\lim_{\delta\to 0} |\vec{\mu}_\delta(x) - \nabla_{\vec{\rho}} h_{\theta(x)}(\vec{\rho}_\delta(x))| = 0. \tag{3.55}$$

Using the convexity of h_θ, (3.55), and the compactness of (ρ_δ), $(\Pi\vec{\mu}_\delta)$, we have in $\Omega_{\eta,\kappa}$,

$$\begin{aligned} 0 &\le \lim_{\varepsilon\to 0}\lim_{\delta\to 0}(\vec{\rho}_\varepsilon - \vec{\rho}_\delta)\cdot(\nabla_{\vec{\rho}} h_\theta(\vec{\rho}_\varepsilon) - \nabla_{\vec{\rho}} h_\theta(\vec{\rho}_\delta)) = \lim_{\varepsilon\to 0}\lim_{\delta\to 0}(\vec{\rho}_\varepsilon - \vec{\rho}_\delta)\cdot(\vec{\mu}_\varepsilon - \vec{\mu}_\delta) \\ &= \lim_{\varepsilon\to 0}\lim_{\delta\to 0}(\vec{\rho}_\varepsilon - \vec{\rho}_\delta)\cdot((\vec{\mu}_\varepsilon - \vec{\mu}_\delta) - \Pi((\vec{\mu}_\varepsilon - \vec{\mu}_\delta))) \le C\lim_{\varepsilon\to 0}\lim_{\delta\to 0}|\rho_\varepsilon - \rho_\delta| = 0. \end{aligned}$$

The last but one equality follows from the strong convergence of $\Pi\vec{\mu}_\varepsilon$, while the last inequality follows from (3.54). Hence, using the strict convexity of h_θ, this gives

$$\lim_{\varepsilon\to 0}\lim_{\delta\to 0}|\vec{\rho}_\varepsilon(x) - \vec{\rho}_\delta(x)| = 0 \quad \text{a.e. in } \Omega_{\eta,k}.$$

Consequently, using the weak lower semicontinuity of the $L^1(\Omega)$ norm,

$$\begin{aligned} \lim_{\delta\to 0}\int_{\Omega_\eta}|\vec{\rho}_\delta - \vec{\rho}|\,dx &\le \lim_{\delta\to 0}\int_{\Omega_{\eta,\kappa}}|\vec{\rho}_\delta - \vec{\rho}|\,dx + C\kappa \\ &\le \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{\Omega_{\eta,\kappa}}|\vec{\rho}_\varepsilon - \vec{\rho}_\delta|\,dx + C\kappa = C\kappa. \end{aligned}$$

The limit $\kappa \to 0$ gives

$$\lim_{\delta\to 0}\int_{\Omega_\eta}|\vec{\rho}_\delta - \vec{\rho}|\,dx = 0.$$

Thanks to the uniform equiintegrability of $(\vec{\rho}_\delta)$, this limit implies (3.51).

Relation (3.52) can be proved by using the approach presented before. Indeed, since $\vec{\rho}$ and $\Pi(\vec{\mu})$ are bounded a.e. in Ω, then $\rho(x) > 0$ implies the existence of some $i \in \{1, \dots, N\}$ such that $\rho_i(x) > 0$. But then the whole vector $\vec{\mu}(x)$ must be bounded and consequently, $\rho_j(x) > 0$ for all $j = 1, \dots, N$, which equals (3.52). □

To finish the proof of Theorem 3.1, recall that

$$\vec{\mu_\delta} = \partial_{\vec{\rho}} h_{\theta_\delta}(\vec{\rho_\delta}) \to \partial_{\vec{\rho}} h_\theta(\vec{\rho}) =: \vec{\mu} \quad \text{a.e. in } \Omega \setminus \{\rho = 0\}.$$

Since we cannot exclude that $\rho = 0$ on a set of positive measure, we redefine $\vec{\mu}$ and set $\vec{\mu} := 0$ on $\{\rho = 0\}$. On the other hand, we know that

$$\Pi\Big(\frac{\vec{\mu_\delta}}{\theta_\delta}\Big) \to \Pi(\vec{q}) \quad \text{a.e. in } \Omega,$$

and therefore also

$$\Pi(\vec{\mu_\delta}) = \theta_\delta \Pi\Big(\frac{\vec{\mu_\delta}}{\theta_\delta}\Big) \to \theta\Pi(\vec{q}) \quad \text{a.e. in } \Omega.$$

Therefore, $\Pi(\vec{\mu}) = \theta\Pi(\vec{q})$ a.e. in $\Omega \setminus \{\rho = 0\}$. This shows that $\sum_{j=1}^N M_{ij}\nabla q_j = \sum_{j=1}^N M_{ij} \times\nabla(\mu_j/\theta)$ as required in (1.29).

4. The approximate scheme

4.1. **Auxiliary results.** The following property is needed in the construction of the approximate solution.

Lemma 4.1. *Let the free energy h_θ fulfil Hypothesis (H6) and the pressure p satisfy Hypothesis (H7). Then for any sequence $(\theta^{(n)})_{n\in\mathbb{N}} \subset (0,\infty)$, $(\vec{\mu}^{(n)})_{n\in\mathbb{N}} \subset \mathbb{R}^N$, if both $\theta^{(n)}$ and $\rho^{(n)}$ are bounded and $\min_{1\le i\le N}\mu_i^{(n)} \to \infty$, then $\rho^{(n)} \to 0$.*

Proof. It follows from (1.18), (1.23), and Hypothesis (H7) that

$$\rho \min_{1\le i\le N} \mu_i \le \sum_{i=1}^N \rho_i\mu_i = p + h_\theta(\vec{\rho}) \le C(1 + \rho\theta + \rho^\gamma + \rho^{5(\gamma+\nu-\eta)/6} + \theta^{5(3\beta-\eta)/6}),$$

which immediately implies the statement. □

The following lemma plays an important role in the construction of the approximate solutions.

Lemma 4.2. *Let the free energy h_θ fulfil Hypothesis (H6). Then for every $\overline{\rho} > 0$ and $w \in L^\infty(\Omega)$, the algebraic equation*

$$\int_\Omega \sum_{i=1}^N \frac{\partial h_\theta^*}{\partial \mu_i}(\theta q_0 \vec{1})\,dx = |\Omega|\overline{\rho}, \quad \theta := e^w, \quad \vec{1} = (1,\dots,1) \in \mathbb{R}^N, \tag{4.1}$$

has a unique solution $q_0 \in \mathbb{R}$. Moreover, for every $\overline{\rho} > 0$ and $i = 1,\dots,N$, the mappings

$$\widetilde{q}_0 : L^\infty(\Omega) \to \mathbb{R}, \quad w \mapsto q_0,$$

$$\widetilde{\mathfrak{R}}_i : L^\infty(\Omega) \to L^\infty(\Omega), \quad w \mapsto \frac{\partial h^*_{\exp(w)}}{\partial \mu_i}\big(\exp(w)\widetilde{q}_0(w)\vec{1}\big)$$

are continuous.

Proof. Set $\partial_i = \partial/\partial\mu_i$ and $\partial^2_{ij} = \partial/\partial\mu_i\partial\mu_j$. For every $\theta_0 > 0$, the function $f_{\theta_0}(q_0) = \sum_{i=1}^N \partial_i h^*_{\theta_0}(\theta_0 q_0\vec{1})$ is strictly increasing, since $f'_{\theta_0}(q_0) = \theta_0\sum_{i,j=1}^N \partial^2_{ij}h^*_{\theta_0}(\theta_0 q_0\vec{1}) > 0$, due to the convexity of h^*_θ. It follows that $F_\theta(q_0) = \int_\Omega\sum_{i=1}^N \partial_i h^*_\theta(\theta q_0\vec{1})\,dx$ is strictly increasing, provided that $\theta \in L^\infty(\Omega)$ is uniformly positive in Ω. Thus, (4.1) has at most one solution $q_0 \in \mathbb{R}$. To prove the existence of a solution to (4.1), we need to show that $\lim_{q_0\to-\infty} F_\theta(q_0) = 0$ and $\lim_{q_0\to+\infty} F_\theta(q_0) = \infty$.

For $\theta = e^w$, $w \in L^\infty(\Omega)$, let θ_m, $\theta_M > 0$ be such that $\theta_m \le \theta \le \theta_M$ a.e. in Ω. The fact that for $\theta_0 > 0$, the mapping $\lambda \in \mathbb{R} \mapsto \sum_{i=1}^N \partial_i h^*_{\theta_0}(\lambda\vec{1}) \in \mathbb{R}$ is increasing, implies that

$$0 \le \int_\Omega \sum_{i=1}^N \partial_i h^*_\theta(\theta_m q_0\vec{1})\,dx \le F_\theta(q_0) \le \int_\Omega \sum_{i=1}^N \partial_i h^*_\theta(\theta_M q_0\vec{1})\,dx.$$

Therefore it is sufficient to show that

$$\lim_{\lambda\to-\infty}\int_\Omega \sum_{i=1}^N \partial_i h^*_\theta(\lambda\vec{1})\,dx = 0, \quad \lim_{\lambda\to+\infty}\int_\Omega \sum_{i=1}^N \partial_i h^*_\theta(\lambda\vec{1})\,dx = +\infty.$$

By the monotonicity of $\lambda \mapsto \sum_{i=1}^N \partial_i h^*_{\theta(x)}(\lambda\vec{1})$ for $\lambda \in \mathbb{R}$ and a.e. $x \in \Omega$, the dominated convergence theorem, and Fatou's lemma, it is sufficient to show that

$$\lim_{\lambda\to-\infty} \sum_{i=1}^N \partial_i h^*_{\theta_0}(\lambda\vec{1}) = 0, \quad \lim_{\lambda\to+\infty} \sum_{i=1}^N \partial_i h^*_{\theta_0}(\lambda\vec{1}) = \infty \quad \text{for all } \theta_0 > 0. \tag{4.2}$$

Actually, the limits in (4.2) follow from the monotonicity of $\lambda \mapsto \sum_{i=1}^N \partial_i h^*_{\theta_0}(\lambda\vec{1})$, Assumption (1.21), Lemma 4.1, and the fact that $\vec{\mu} = \nabla h_{\theta_0}(\vec{\rho})$ is the inverse mapping of $\vec{\rho} = \nabla h^*_{\theta_0}(\vec{\mu})$. This means that (4.1) has exactly one solution $q_0 \in \mathbb{R}$.

To prove the second part of the lemma, let $(w_n)_{n\in\mathbb{N}} \subset L^\infty(\Omega)$ be such that $w_n \to w$ strongly in $L^\infty(\Omega)$. Define $\theta_n = e^{w_n}$, $\theta = e^w$. Clearly, there exist constants L_1, $L_2 > 0$ such that $L_1 \le \theta_n \le L_2$ a.e. in Ω, $n \in \mathbb{N}$. The previous argument allows us to deduce that the corresponding sequence $(q_{0,n})$ is bounded in $\mathbb{R}$ and therefore (up to subsequences) convergent to a suitable $q_0 \in \mathbb{R}$. The fact that $h^*_\theta \in C^2((0,\infty)^N)$ depends smoothly on θ implies that $\nabla h^*_{\theta_n}(\theta_n q_{0,n}\vec{1}) \to \nabla h^*_\theta(\theta q_0 \vec{1})$ in $L^\infty(\Omega)$ and therefore the limits θ, q_0 satisfy (4.1). In particular, q_0 is uniquely determined by $w = \log\theta$. This finishes the proof. $\square$

4.2. **Formulation of the approximate equations.** In this subsection, we specify the approximate system, leading to a sequence of smooth, approximate solutions.

4.2.1. *Internal energy balance.* We replace the total energy balance (1.28) by the internal energy balance (2.3), whose weak formulation reads as

$$\begin{aligned}\int_\Omega \Big(-\rho e \boldsymbol{v} + \kappa(\theta)\nabla\theta + \sum_{i=1}^N M_i \nabla\left(\frac{\mu_i}{\theta}\right)\Big) \cdot \nabla\phi_0 \, dx \\ + \int_\Omega (p \operatorname{div} \boldsymbol{v} - \mathbb{S} : \nabla\boldsymbol{v})\phi_0 \, dx + \alpha_2 \int_{\partial\Omega} \phi_0(\theta - \theta_0)\, ds = 0 \quad \text{for all } \phi_0 \in H^1(\Omega).\end{aligned}$$

4.2.2. *Parameters of the approximate problem.* For the sake of clarity, we state here the parameters employed in the approximate scheme. A more detailed explanation is given in the part that follows:

$\varepsilon \in (0,1)$:	lower-order regularization in all equations;
$\xi \in (0,1)$:	higher-order regularization in the internal energy equation;
$\delta \in (0,\xi/2)$:	higher-order regularization in the partial mass densities equations;
$\chi \in (0,1)$:	quasilinear regularization in the partial mass densities equations;
$n \in \mathbb{N}$:	Galerkin approximation in the momentum equation;
$\eta \in (0,1)$:	regularization in the free energy and heat conductivity.

4.2.3. *Levels of approximations.* Let $(\boldsymbol{v}^{(k)})_{k\in\mathbb{N}} \subset \{\boldsymbol{v} \in C^\infty(\overline{\Omega}),\ \boldsymbol{v}\cdot\boldsymbol{\nu} = 0 \text{ on } \partial\Omega\}$ be a complete orthonormal system for $L^2(\Omega)$ and let X_n be the subspace of $L^2(\Omega)$ generated by $\boldsymbol{v}^{(1)}, \ldots, \boldsymbol{v}^{(n)}$. Since X_n has dimension $n < \infty$, the quantity

$$K(n) := \sup_{\mathbf{u}\in X_n\setminus\{\mathbf{0}\}} \frac{\int_\Omega |D^2(|\mathbf{u}|^2)|^2\, dx}{\int_\Omega |\mathbf{u}|^4\, dx} \tag{4.3}$$

is finite. Let $\overline{\rho} > 0$ be an arbitrary positive constant (which in the end will be the mean density of the solution). Let $q_0 = \widetilde{q}_0(\log\theta)$, $\vec{\mathfrak{R}} = (\mathfrak{R}_1, \ldots, \mathfrak{R}_N)$, $\mathfrak{R}_i = \widetilde{\mathfrak{R}}_i(\log\theta)$ $(i = 1, \ldots, N)$, $\mathfrak{R} = \sum_{j=1}^N \mathfrak{R}_j$, defined in Lemma 4.2. Note that q_0 is actually a constant.

Let $L \geq \max\{\beta, 3\beta - 2, \gamma + \nu, 2\beta_0\}$ be arbitrary, where $\nu > 0$ is as in Lemma 2.3 and $\beta_0 \geq 0$ is as in Hypothesis (H6), formula (1.22). We introduce the "regularized" free energy for the approximate system,

$$h_\theta^{(\eta)}(\vec{\rho}) = h_\theta(\vec{\rho}) + \eta\overline{n}^L + \eta\sum_{i=1}^N f_{\alpha_0}(\rho_i) + \eta|\vec{\rho}|^2, \tag{4.4}$$

where $\alpha_0 \in (\frac{1}{2}, 1)$ is the same as in Hypothesis (H6), formula (1.22), and

$$f_{\alpha_0}(s) = \frac{s^{2(1-\alpha_0)}}{2(1-\alpha_0)(1-2\alpha_0)}, \qquad s \geq 0.$$

All other thermodynamic quantities appearing in the approximate equations that are defined in terms of the free energy are modified correspondingly:

$$\begin{aligned}
\mu_i^{(\eta)} &= \mu_i + \eta L\frac{\overline{n}^{L-1}}{m_i} + \eta f'_{\alpha_0}(\rho_i) + 2\eta\rho_i, \quad i = 1, \ldots, N, \\
p^{(\eta)} &= p + \eta(L-1)\overline{n}^L + \eta\sum_{i=1}^N \frac{\rho_i^{2(1-\alpha_0)}}{2(1-\alpha_0)} + \eta|\vec{\rho}|^2, \\
\rho e^{(\eta)} &= \rho e + \eta\overline{n}^L + \eta\sum_{i=1}^N f_{\alpha_0}(\rho_i) + \eta|\vec{\rho}|^2, \\
\rho s^{(\eta)} &= \rho s.
\end{aligned}$$

We point out that $h_\theta^{(\eta)}$ satisfies the property from Lemma 4.1 and Hypothesis (H6); in particular, (1.22) holds with the same constants, since the mixed second-order derivatives of the free energy are not changed by the terms added in (4.4).

We also need to regularize the heat conductivity:

$$\kappa^{(\eta)}(\theta) = \kappa(\theta) + \eta\theta^L. \tag{4.5}$$

4.2.4. *Approximate problem.* The approximate equations involve $q_1, \ldots, q_N, \log\theta \in H^2(\Omega)$, $\boldsymbol{v} \in X_n$ as unknown and are given as follows:

$$\begin{aligned}
0 &= \sum_{i=1}^N \int_\Omega \bigg(-\rho_i\boldsymbol{v} + \sum_{j=1}^N M_{ij}\nabla q_j + M_i\nabla\Big(\frac{1}{\theta}\Big)\bigg) \cdot \nabla\phi_i\,dx - \sum_{i=1}^N \int_\Omega r_i\phi_i\,dx \\
&\quad + \delta\sum_{i=1}^N \int_\Omega D^2 q_i : D^2\phi_i\,dx + \varepsilon\sum_{i=1}^N \int_\Omega (\rho_i - \mathfrak{R}_i)\phi_i\,dx \qquad (4.6)\\
&\quad + \varepsilon^3\sum_{i=1}^N \int_\Omega \big((1+\chi|\nabla q_i|^2)\nabla q_i\cdot\nabla\phi_i + (1+|q_i-q_0|^{1/2})(q_i-q_0)\phi_i\big)\,dx;
\end{aligned}$$

$$
\begin{aligned}
0 &= \int_{\partial\Omega} \alpha_1 \boldsymbol{u}\cdot\boldsymbol{v}\,ds + \int_\Omega (-\rho\boldsymbol{v}\otimes\boldsymbol{v} + \mathbb{S}) : \nabla\boldsymbol{u}\,dx - \int_\Omega (p\operatorname{div}\boldsymbol{u} + \rho\boldsymbol{b}\cdot\boldsymbol{u})\,dx \\
&\quad + \varepsilon^3 \sum_{i=1}^d\sum_{j=1}^N \int_\Omega v_i\Big((1+\chi|\nabla q_j|^2)\nabla q_j\cdot\nabla u_i + \frac{1}{2}(1+|q_j-q_0|^{1/2})(q_j-q_0)u_i\Big)\,dx \qquad (4.7)\\
&\quad + \varepsilon\int_\Omega \rho\boldsymbol{v}\cdot\boldsymbol{u}\,dx + \delta K(n)\int_\Omega |\boldsymbol{v}|^2\boldsymbol{v}\cdot\boldsymbol{u}\,dx;
\end{aligned}
$$

$$
\begin{aligned}
0 &= \int_\Omega \Big(-\rho e\boldsymbol{v} + \kappa^{(\eta)}(\theta)\nabla\theta + \sum_{i=1}^N M_i\nabla q_i\Big)\cdot\nabla\phi_0\,dx + \int_\Omega (p\operatorname{div}\boldsymbol{v} - \mathbb{S}:\nabla\boldsymbol{v})\phi_0\,dx \\
&\quad + \int_{\partial\Omega}\alpha_2(\theta-\theta_0)\phi_0\,ds + \delta\int_\Omega (1+\theta)D^2(\log\theta) : D^2\phi_0\,dx \qquad (4.8)\\
&\quad + \xi\int_\Omega \big((1+\theta)(1+|\nabla\log\theta|^2)\nabla(\log\theta)\cdot\nabla\phi_0 + (\log\theta)\phi_0\big)\,dx - \varepsilon\int_\Omega \mathfrak{R}\frac{|v|^2}{2}\phi_0\,dx
\end{aligned}
$$

for all $\phi_1,\ldots,\phi_N\in H^2(\Omega)$, $\mathbf{u}\in X_n$, and $\phi_0\in H^2(\Omega)$. Given $q_1,\ldots,q_n$, θ, we define

$$
\vec\rho := \nabla(h_\theta^{(\eta)})^*(\theta\vec q) \quad\text{and}\quad \rho := \sum_{i=1}^n \rho_i. \qquad (4.9)
$$

The system will be reformulated as a fixed-point problem for an operator $F : X\times[0,1]\to X$ defined by means of a linearized problem.

4.3. **Existence of a solution to the approximate equations.** We formulate (4.6)–(4.8) as a fixed-point problem for a suitable operator.

4.3.1. *Reformulation as a fixed-point problem.* Define the spaces

$$
V = W^{1,4}(\Omega;\mathbb{R}^N)\times X_n\times W^{1,4}(\Omega), \quad V_0 = H^2(\Omega;\mathbb{R}^N)\times X_n\times H^2(\Omega).
$$

Let $(\vec q^*,\boldsymbol{v}^*,w^*)\in V$ and $\sigma\in[0,1]$ be arbitrary. Noting that $\vec\rho(\mu^*,\theta^*)$ is defined as in (4.9)) and $(\rho e)^* = (\rho e)(\vec\rho^*,\theta^*)$ is defined by means of (1.18), we set for $i,j=1,\ldots,N$:

$$
\begin{aligned}
&\theta^* = e^{w^*}, \quad \mu_i^* = \theta^* q_i^*, \quad \vec\rho^* = \vec\rho(\vec\mu^*,\theta^*), \quad \rho^* = \sum_{k=1}^N \rho_k^*, \quad p^* = p(\rho^*,\theta^*), \\
&M_i^* = M_i(\vec\rho^*,\theta^*), \quad M_{ij}^* = M_{ij}(\vec\rho^*,\theta^*), \quad r_i^* = r_i(\Pi(\vec\mu^*/\theta^*),\theta^*).
\end{aligned}
$$

The task is to find a solution $(\vec q,\boldsymbol{v},w)\in V_0$ to the linear system

(4.10)

$$
\begin{aligned}
0 &= \sigma\sum_{i=1}^N\int_\Omega \Big(-\rho_i^*\boldsymbol{v} + \sum_{j=1}^N M_{ij}^*\nabla q_j^* - M_i^*\frac{\nabla w^*}{\theta^*}\Big)\cdot\nabla\phi_i\,dx - \sigma\sum_{i=1}^N\int_\Omega r_i^*\phi_i\,dx \\
&\quad + \delta\sum_{i=1}^N\int_\Omega D^2 q_i : D^2\phi_i\,dx + \sigma\varepsilon\sum_{i=1}^N\int_\Omega (\rho_i^* - \widetilde{\mathfrak{R}}_i(w))\phi_i\,dx
\end{aligned}
$$

$$+ \varepsilon^3 \sum_{i=1}^N \int_\Omega \big((1+\chi|\nabla q_i^*|^2)\nabla q_i\cdot\nabla\phi_i + (1+|q_i^*-\widetilde{q}_0(w)|^{1/2})(q_i-\widetilde{q}_0(w))\phi_i\big)\,dx;$$

(4.11)

$$\begin{aligned} 0 &= \int_{\partial\Omega}\alpha_1 \boldsymbol{u}\cdot\boldsymbol{v}\,ds + \int_\Omega \mathbb{S}(\theta^*,\mathbb{D}(v)):\nabla\boldsymbol{u}\,dx \\ &\quad + \sigma\int_\Omega \left((-\rho^*\boldsymbol{v}^*\otimes\boldsymbol{v}^*):\nabla\boldsymbol{u} - p^*\operatorname{div}\boldsymbol{u} - \rho^*\boldsymbol{b}\cdot\boldsymbol{u}\right)\,dx \\ &\quad + \varepsilon^3\sum_{i=1}^d\sum_{j=1}^N\int_\Omega v_i^*\Big((1+\chi|\nabla q_j^*|^2)\nabla q_j\cdot\nabla u_i + \frac{1}{2}(1+|q_j-\widetilde{q}_0(w)|^{1/2})(q_j-\widetilde{q}_0(w))u_i\Big)\,dx \\ &\quad + \sigma\varepsilon\int_\Omega \rho^*\boldsymbol{v}\cdot\boldsymbol{u}\,dx + \delta K(n)\int_\Omega |\boldsymbol{v}^*|^2\boldsymbol{v}\cdot\boldsymbol{u}\,dx; \end{aligned}$$

(4.12)

$$\begin{aligned} 0 &= \sigma\int_\Omega\Big(-(\rho e)^*\boldsymbol{v}^* + \kappa^{(\eta)}(\theta^*)\theta^*\nabla w^* + \sum_{i=1}^N M_i^*\nabla q_i^*\Big)\cdot\nabla\phi_0\,dx \\ &\quad + \sigma\int_\Omega (p^*\operatorname{div}\boldsymbol{v}^* - \mathbb{S}^*:\nabla\boldsymbol{v}^*)\phi_0\,dx + \sigma\int_{\partial\Omega}\alpha_2(\theta^*-\theta_0^*)\phi_0\,ds \\ &\quad + \xi\int_\Omega\left((1+e^{w^*})(1+|\nabla w^*|^2)\nabla w\cdot\nabla\phi_0 + w\phi_0\right)\,dx \\ &\quad - \sigma\varepsilon\int_\Omega \widetilde{\mathfrak{R}}(w^*)\frac{|\boldsymbol{v}^*|^2}{2}\phi_0\,dx + \delta\int_\Omega (1+e^{w^*})D^2 w : D^2\phi_0\,dx \end{aligned}$$

for all $\phi_1,\ldots,\phi_N\in H^2(\Omega)$, $\boldsymbol{u}\in X_n$, and $\phi_0\in H^2(\Omega)$.

The Lax–Milgram lemma ensures the existence of a unique solution $(\vec{q},\boldsymbol{v},w)\in H^2(\Omega;\mathbb{R}^N)$ $\times X_n\times H^2(\Omega)$ to (4.10)–(4.12). More precisely, first we solve (4.12) for w, then insert w in (4.10) and solve this equation for $\vec{q}$, and finally insert both $\vec{q}$ and w in (4.11) and solve this equation for $\boldsymbol{v}$. As a consequence, we have the mapping

$$F: V\times[0,1]\to V,\quad ((\vec{q}^*,\boldsymbol{v}^*,w^*),\sigma)\mapsto(\vec{q},\boldsymbol{v},w).$$

We aim to show that F has a fixed point in V. Then this fixed point solves (4.6)–(4.8). To this end, we apply the Leray–Schauder theorem.

The operator F has the following properties:

(1) $F(\cdot,0)$ is constant. Clearly, if $\sigma=0$ then $w=0$, which implies that $\vec{q}=\widetilde{q}_0(0)\vec{1}$, which in turn implies that $\boldsymbol{v}=\boldsymbol{0}$.
(2) $F: V\times[0,1]\to V$ is continuous. Standard arguments show that F is sequentially continuous.
(3) $F: V\times[0,1]\to V$ is compact. Indeed, the compactness of F follows from the fact that the image of F is bounded in V_0 (consequence of Lax–Milgram's Lemma), the compact Sobolev embedding $H^2(\Omega)\hookrightarrow W^{1,4}(\Omega)$ (which holds, e.g., for $d=3$), and the fact that X_n is finite dimensional.

It remains to prove that the set

$$\{(\vec{q},\boldsymbol{v},w)\in V\ :\ F((\vec{q},\boldsymbol{v},w),\sigma)=(\vec{q},\boldsymbol{v},w)\}$$

is bounded in V uniformly with respect to $\sigma\in[0,1]$. If this is shown, the existence of a fixed point for F in V follows from the Leray–Schauder theorem.

Let $(\vec{q},\boldsymbol{v},w)\in V_0$ satisfy $F((\vec{q},\boldsymbol{v},w),\sigma)=(\vec{q},\boldsymbol{v},w)$ for some $\sigma\in[0,1]$. This means that the following relations hold:

(4.13)

$$\begin{aligned}0&=\sigma\sum_{i=1}^N\int_\Omega\Big(-\rho_i\boldsymbol{v}+\sum_{j=1}^N M_{ij}\nabla q_j-M_i\frac{\nabla w}{\theta}\Big)\cdot\nabla\phi_i\,dx-\sigma\sum_{i=1}^N\int_\Omega r_i\phi_i\,dx\\&\quad+\delta\sum_{i=1}^N\int_\Omega D^2q_i:D^2\phi_i\,dx+\sigma\varepsilon\sum_{i=1}^N\int_\Omega(\rho_i-\mathfrak{R}_i)\phi_i\,dx\\&\quad+\varepsilon^3\sum_{i=1}^N\int_\Omega\big((1+\chi|\nabla q_i|^2)\nabla q_i\cdot\nabla\phi_i+(1+|q_i-q_0|^{1/2})(q_i-q_0)\phi_i\big)\,dx;\end{aligned}$$

(4.14)

$$\begin{aligned}0&=\int_{\partial\Omega}\alpha_1\boldsymbol{u}\cdot\boldsymbol{v}\,ds+\int_\Omega\mathbb{S}:\boldsymbol{u}\,dx+\sigma\int_\Omega(-\rho\boldsymbol{v}\otimes\boldsymbol{v}):\nabla\boldsymbol{u}\,dx-\sigma\int_\Omega(p\operatorname{div}\boldsymbol{u}+\rho\boldsymbol{b}\cdot\boldsymbol{u})\,dx\\&\quad+\varepsilon^3\sum_{i=1}^3\sum_{j=1}^N\int_\Omega v_i\Big((1+\chi|\nabla q_j|^2)\nabla q_j\cdot\nabla u_i+\frac12(1+|q_j-q_0|^{1/2})(q_j-q_0)u_i\Big)\,dx\\&\quad+\sigma\varepsilon\int_\Omega\rho\boldsymbol{v}\cdot\boldsymbol{u}\,dx+\delta K(n)\int_\Omega|\boldsymbol{v}|^2\boldsymbol{v}\cdot\boldsymbol{u}\,dx;\end{aligned}$$

(4.15)

$$\begin{aligned}0&=\sigma\int_\Omega\Big(-\rho e\boldsymbol{v}+\kappa^{(\eta)}(\theta)\theta\nabla w+\sum_{i=1}^N M_i\nabla q_i\Big)\cdot\nabla\phi_0\,dx+\sigma\int_\Omega(p\operatorname{div}\boldsymbol{v}-\mathbb{S}:\nabla\boldsymbol{v})\phi_0\,dx\\&\quad+\sigma\int_{\partial\Omega}\alpha_2(\theta-\theta_0)\phi_0\,ds+\delta\int_\Omega(1+e^w)D^2w:D^2\phi_0\,dx\\&\quad-\sigma\varepsilon\int_\Omega\mathfrak{R}\frac{|\boldsymbol{v}|^2}{2}\phi_0\,dx+\xi\int_\Omega\big((1+e^w)(1+|\nabla w|^2)\nabla w\cdot\nabla\phi_0+w\phi_0\big)\,dx\end{aligned}$$

for all $\phi_1,\ldots,\phi_N\in H^2(\Omega)$, $\boldsymbol{u}\in X_n$, and $\phi_0\in H^2(\Omega)$. We need to show that $(\vec{q},\boldsymbol{v},w)$ can be uniformly bounded in V with respect to σ. To find such estimate, we derive global entropy and energy inequalities.

4.3.2. *Global entropy inequality for approximate solutions.* Let us choose $\phi_i=q_i-q_0$ ($i=1,\ldots,N$) and $\phi_0=-\exp(-w)=-1/\theta$ in (4.13) and (4.15), respectively, and sum the equations. By arguing like in the derivation of (2.7), we find that

$$
\begin{aligned}
&\sigma\int_\Omega\bigg(\sum_{i,j=1}^N M_{ij}\nabla q_i\cdot\nabla q_j+\kappa^{(\eta)}(\theta)|\nabla w|^2-\sum_{i=1}^N r_iq_i+\frac{1}{\theta}\mathbb{S}:\nabla\boldsymbol{v}\bigg)\,dx\\
&\quad+\delta\sum_{i=1}^N\|D^2q_i\|_{L^2(\Omega)}^2+\varepsilon^3\sum_{i=1}^N\int_\Omega\big(\chi|\nabla q_i|^4+|\nabla q_i|^2+|q_i-q_0|^{5/2}+(q_i-q_0)^2\big)\,dx\\
&\quad+\sigma\varepsilon\int_\Omega\mathfrak{R}\frac{|\boldsymbol{v}|^2}{2\theta}\,dx+\sigma\varepsilon\int_\Omega\sum_{i=1}^N(\rho_i-\mathfrak{R}_i)(q_i-q_0)\,dx+R=\sigma\int_{\partial\Omega}\alpha_2\frac{\theta-\theta_0}{\theta}\,ds,
\end{aligned}\tag{4.16}
$$

where we defined

$$
\begin{aligned}
R&=\int_\Omega(1+\theta)\Big(\delta D^2(\log\theta):D^2\Big(-\frac{1}{\theta}\Big)+\xi(1+|\nabla\log\theta|^2)\nabla(\log\theta)\cdot\nabla\Big(-\frac{1}{\theta}\Big)\Big)\,dx\\
&\quad+\xi\int_\Omega\frac{1}{\theta}\log\frac{1}{\theta}\,dx.
\end{aligned}
$$

We deduce from the construction of q_0 and $\mathfrak{R}_i$ (Lemma 4.2) and the convexity of $h_\theta^{(\eta)}$ that

$$
\sum_{i=1}^N(\rho_i-\mathfrak{R}_i)(q_i-q_0)=\frac{1}{\theta}\sum_{i=1}^N(\rho_i-\mathfrak{R}_i)\bigg(\frac{\partial h_\theta^{(\eta)}}{\partial\rho_i}(\vec{\rho})-\frac{\partial h_\theta^{(\eta)}}{\partial\rho_i}(\vec{\mathfrak{R}})\bigg)\geq 0.
$$

Furthermore, straightforward computations and the assumption $\delta\leq\xi/2$ show that

$$
R\geq c\delta\int_\Omega(1+\theta)\theta^{-3}|D^2\theta|^2\,dx+\frac{\xi}{2}\int_\Omega\bigg((1+\theta^{-1})|\nabla\log\theta|^4+\frac{1}{\theta}\log\frac{1}{\theta}\bigg)\,dx\tag{4.17}
$$

for some constant $c>0$. Putting (4.16)–(4.17) together gives the *global entropy inequality*:

$$
\begin{aligned}
&\sigma\int_\Omega\bigg(\sum_{i,j=1}^N M_{ij}\nabla q_i\cdot\nabla q_j+\kappa^{(\eta)}(\theta)|\nabla\log\theta|^2-\sum_{i=1}^N r_iq_i+\frac{1}{\theta}\mathbb{S}:\nabla\boldsymbol{v}\bigg)\,dx\\
&\quad+\varepsilon^3\sum_{i=1}^N\int_\Omega(\chi|\nabla q_i|^4+|\nabla q_i|^2+|q_i-q_0|^{5/2}+(q_i-q_0)^2)\,dx+\delta\int_\Omega(1+\theta)\theta^{-3}|D^2\theta|^2\,dx\\
&\quad+\delta\sum_{i=1}^N\|D^2q_i\|_{L^2(\Omega)}^2+\xi\int_\Omega\bigg((1+\theta^{-1})|\nabla\log\theta|^4+\frac{1}{\theta}\log\frac{1}{\theta}\bigg)\,dx+\sigma\int_{\partial\Omega}\alpha_2\frac{\theta_0-\theta}{\theta}\,ds\leq 0.
\end{aligned}\tag{4.18}
$$

4.3.3. *(Almost) mass conservation.* Choosing $\phi_i=1$ ($i=1,\ldots,N$) in (4.13) leads to

$$
\sigma\int_\Omega\sum_{i=1}^N(\rho_i-\mathfrak{R}_i)\,dx=\varepsilon^2\int_\Omega\sum_{i=1}^N(1+|q_i-q_0|^{1/2})(q_0-q_i)\,dx.
$$

However, $\int_\Omega \sum_{i=1}^N \mathfrak{R}_i dx = |\Omega|\overline{\rho}$ by construction (see Lemma 4.2), thus (4.18) shows that

$$\sigma\left|\frac{1}{\Omega}\int_\Omega \rho\,dx - \overline{\rho}\right| \le C\varepsilon^{1/5}. \tag{4.19}$$

4.3.4. *Global total energy inequality for approximate solutions.* We choose now $\phi_i = -\frac{1}{2}|\boldsymbol{v}|^2$ $(i = 1, \ldots, N)$, $\boldsymbol{u} = \boldsymbol{v}$, $\phi_0 = 1$ in (4.13)–(4.15) and sum the equations. By arguing like in the derivation of (2.8), we obtain

$$\begin{aligned}&\int_{\partial\Omega} \alpha_1|\boldsymbol{v}|^2\,ds + \sigma\int_{\partial\Omega}\alpha_2(\theta-\theta_0)\,ds + \xi\int_\Omega \log\theta\,dx + \sigma\varepsilon\int_\Omega \rho\frac{|\boldsymbol{v}|^2}{2}\,dx + (1-\sigma)\int_\Omega \mathbb{S}:\nabla\boldsymbol{v}\,dx \\ &\quad = \sigma\int_\Omega \rho\boldsymbol{b}\cdot\boldsymbol{v}\,dx - \delta K(n)\int_\Omega |\boldsymbol{v}|^4\,dx + \delta\int_\Omega D^2\bigg(\sum_{i=1}^N q_i\bigg) : D^2\frac{|\boldsymbol{v}|^2}{2}\,dx.\end{aligned} \tag{4.20}$$

It follows from (4.18) that

$$-\xi\int_\Omega \log\theta\,dx \le \xi\int_{\{\theta<1\}}\log\frac{1}{\theta}\,dx \le \xi\int_\Omega \frac{1}{\theta}\log\frac{1}{\theta}\,dx \le C,$$

while the Cauchy–Schwarz inequality, definition (4.3), and (4.18) yield

$$\begin{aligned}\delta\left|\int_\Omega D^2\bigg(\sum_{i=1}^N q_i\bigg) : D^2\frac{|\boldsymbol{v}|^2}{2}\,dx\right| &\le \frac{\delta}{2}\sum_{i=1}^N \|D^2 q_i\|_{L^2(\Omega)}\|D^2(|\boldsymbol{v}|^2)\|_{L^2(\Omega)} \\ &\le \frac{\delta}{2}\sqrt{K(n)}\sum_{i=1}^N \|D^2 q_i\|_{L^2(\Omega)}\|\boldsymbol{v}\|_{L^4(\Omega)}^2 \le \frac{\delta}{8}\sum_{i=1}^N \|D^2 q_i\|_{L^2(\Omega)}^2 + \frac{\delta}{2}K(n)\int_\Omega |\boldsymbol{v}|^4\,dx \\ &\le C + \frac{\delta}{2}K(n)\int_\Omega |\boldsymbol{v}|^4\,dx.\end{aligned}$$

Therefore, (4.20) leads to the *global total energy inequality* (for approximate solutions):

$$\begin{aligned}&\int_{\partial\Omega}\alpha_1|\boldsymbol{v}|^2\,ds + \sigma\int_{\partial\Omega}\alpha_2(\theta-\theta_0)\,ds + \frac{\delta K(n)}{2}\int_\Omega |\boldsymbol{v}|^4\,dx \\ &\qquad + \sigma\varepsilon\int_\Omega \rho\frac{|\boldsymbol{v}|^2}{2}\,dx \le C + \sigma\int_\Omega \rho\boldsymbol{b}\cdot\boldsymbol{v}\,dx.\end{aligned} \tag{4.21}$$

The right-hand side of (4.21) is estimated as

$$\sigma\left|\int_\Omega \rho\boldsymbol{b}\cdot\boldsymbol{v}\,dx\right| \le C\sigma\int_\Omega \rho|\boldsymbol{v}|\,dx \le C\frac{\sigma}{\varepsilon}\int_\Omega \rho\,dx + \frac{\sigma\varepsilon}{4}\int_\Omega \rho|\boldsymbol{v}|^2\,dx,$$

which, together with (4.19), (4.21), gives the estimate

$$\int_{\partial\Omega}\alpha_1|\boldsymbol{v}|^2\,ds + \sigma\int_{\partial\Omega}\alpha_2(\theta-\theta_0)\,ds + \frac{\delta K(n)}{2}\int_\Omega |\boldsymbol{v}|^4\,dx + \sigma\varepsilon\int_\Omega \rho\frac{|\boldsymbol{v}|^2}{2}\,dx \le C(1+\varepsilon^{-1}). \tag{4.22}$$

4.3.5. *Conclusion: existence for the approximate problem.* Inequalities (4.18), (4.19), and (4.22) yield a σ-uniform bound for $(\vec{q}, \boldsymbol{v}, \log\theta)$ in the space $V = W^{1,4}(\Omega;\mathbb{R}^N) \times X_n \times W^{1,4}(\Omega)$. Leray–Schauder's fixed-point theorem implies the existence of a fixed point $(\vec{q}, \boldsymbol{v}, \log\theta)$ for $F(\cdot,1)$, that is, a solution $(\vec{q}, \boldsymbol{v}, \log\theta) \in H^2(\Omega;\mathbb{R}^N) \times X_n \times H^2(\Omega)$ to (4.6)–(4.8). Moreover, estimates (4.18), (4.19), and (4.22) hold with $\sigma = 1$.

4.4. **Uniform estimates for approximate solutions.** An analogous version of Lemma 2.2 holds also for the approximate solutions. The proof employs (4.18), (4.21) with $\sigma = 1$ in place of (2.7), and (2.8) and is basically identical to the proof of Lemma 2.2. In short, (2.11)–(2.13) are fulfilled also by the approximate solutions built in the previous subsection. Here, we state some additional estimates that are satisfied by the approximate solutions and that follow immediately from (4.18), (4.21), and Lemma 4.2:

$$\|D^2\vec{q}\|_{L^2(\Omega)} \le C\delta^{-1/2}, \tag{4.23}$$

$$\|\nabla\vec{q}\|_{L^4(\Omega)} \le C\varepsilon^{-3/4}\chi^{-1/4}, \tag{4.24}$$

$$\|\nabla\vec{q}\|_{L^2(\Omega)} + \|\vec{q} - q_0\vec{1}\|_{L^2(\Omega)} \le C\varepsilon^{-3/2}, \tag{4.25}$$

$$\|\vec{q} - q_0\vec{1}\|_{L^{5/2}(\Omega)} \le C\varepsilon^{-6/5}, \tag{4.26}$$

$$\|\theta^{-1/2}D^2\log\theta\|_{L^2(\Omega)} + \|D^2\log\theta\|_{L^2(\Omega)} \le C\delta^{-1/2}, \tag{4.27}$$

$$\|\theta^{-1/4}\nabla\log\theta\|_{L^4(\Omega)} + \|\log\theta\|_{W^{1,4}(\Omega)} \le C\xi^{-1/4}, \tag{4.28}$$

$$\|\boldsymbol{v}\|_{L^4(\Omega)} \le C|\delta K(n)|^{-1/4}, \tag{4.29}$$

$$\|\sqrt{\rho}\boldsymbol{v}\|_{L^2(\Omega)} \le C\varepsilon^{-1/2}, \quad i = 1,\ldots,N, \tag{4.30}$$

$$|q_0| \le C(\varepsilon), \tag{4.31}$$

$$\|\mathfrak{R}_i\|_{L^1(\Omega)} \le \overline{\rho}, \quad i = 1,\ldots,N, \tag{4.32}$$

$$\|\theta^{L/2}\|_{H^1(\Omega)} \le C\eta^{-1/2}. \tag{4.33}$$

We prove an estimate for $\nabla\vec{\rho} \in L^2(\Omega;\mathbb{R}^{N\times 3})$. On the one hand, Young's inequality gives

$$\sum_{i=1}^N \nabla\rho_i \cdot \nabla\mu_i = \sum_{i=1}^N \nabla\rho_i \cdot (\theta\nabla q_i + q_i\nabla\theta) \le \frac{\eta}{2}|\nabla\vec{\rho}|^2 + \frac{\|\theta\|^2_{L^\infty(\Omega)}}{2\eta}\big(|\nabla\vec{q}|^2 + |\vec{q}|^2|\nabla\log\theta|^2\big).$$

On the other hand, we have

$$\begin{aligned}\sum_{i=1}^N \nabla\rho_i \cdot \nabla\mu_i &= \sum_{i=1}^N \nabla\rho_i \cdot \left(\sum_{j=1}^N \frac{\partial^2 h_\theta^{(\eta)}}{\partial\rho_i\partial\rho_j}\nabla\rho_j + \frac{\partial^2 h_\theta^{(\eta)}}{\partial\rho_i\partial\theta}\nabla\theta\right) \\ &\ge \sum_{i,j=1}^N \frac{\partial^2 h_\theta^{(\eta)}}{\partial\rho_i\partial\rho_j}\nabla\rho_i\cdot\nabla\rho_j - \frac{\eta}{2}\sum_{i=1}^N \rho_i^{-2\alpha_0}|\nabla\rho_i|^2 - \frac{|\nabla\log\theta|^2}{2\eta}\|\theta\|^2_{L^\infty(\Omega)}\sum_{i=1}^N \rho_i^{2\alpha_0}\left|\frac{\partial^2 h_\theta^{(\eta)}}{\partial\rho_i\partial\theta}\right|^2.\end{aligned}$$

We conclude from the these inequalities, Hypothesis (H6), more precisely (1.22), and (4.4) that

$$\begin{aligned}&\eta|\nabla\vec\rho|^2+\eta\sum_{i=1}^N\rho_i^{-2\alpha_0}|\nabla\rho_i|^2\\&\quad\le C(\|\theta\|_{L^\infty(\Omega)})\frac{|\nabla\log\theta|^2}{\eta}(1+|\vec\rho|^{2\beta_0})+C\frac{\|\theta\|^2_{L^\infty(\Omega)}}{\eta}(|\nabla\vec q|^2+|\vec q|^2|\nabla\log\theta|^2).\end{aligned}$$

As $\vec q$ and $\log\theta$ are essentially bounded functions, also $\vec\rho$ is essentially bounded (this bound is uniform with respect to δ and n). Therefore, we deduce from the previous estimate and bound (4.28) that

$$\|\nabla\vec\rho\|^2_{L^2(\Omega)}\le C(\xi,\eta)\big(1+\|\nabla\vec q\|^2_{L^2(\Omega)}+\|\vec q-q_0\vec 1\|^2_{L^4(\Omega)}+q_0^2\big).$$

In view of Lemma 4.2 and the ε-uniform $L^\infty(\Omega)$ bound for $\log\theta$, $q_0=\widetilde q_0(\log\theta)$ is bounded in ε, so we infer from (4.25) the desired gradient bound for $\vec\rho$:

$$\|\nabla\vec\rho\|_{L^2(\Omega)}\le C\varepsilon^{-3/2}, \tag{4.34}$$

where the constant $C>0$ is independent of ε.

5. Weak sequential compactness for approximate solutions

5.1. **Limits $\delta\to 0$, $n\to\infty$.** The bounds (2.11)–(2.13) and (4.23)–(4.32) allow us to extract subsequences (not relabeled) such that, as $\delta\to 0$,

$$\begin{aligned}&\vec q^{(\delta)}\rightharpoonup\vec q\quad\text{weakly in }W^{1,4}(\Omega;\mathbb R^N),\quad \vec q^{(\delta)}\to\vec q\quad\text{strongly in }L^\infty(\Omega;\mathbb R^N),\\&\log\theta^{(\delta)}\rightharpoonup\log\theta\quad\text{weakly in }W^{1,4}(\Omega),\quad\log\theta^{(\delta)}\to\log\theta\quad\text{strongly in }L^\infty(\Omega),\\&\boldsymbol v^{(\delta)}\rightharpoonup\boldsymbol v\quad\text{weakly in }H^1(\Omega;\mathbb R^3),\quad\boldsymbol v^{(\delta)}\to\boldsymbol v\quad\text{strongly in }L^{6-\nu}(\Omega;\mathbb R^3)\text{ for all }\nu>0,\\&\delta|D^2\vec q^{(\delta)}|+\delta K(n)|\boldsymbol v^{(\delta)}|^3+\delta(1+\theta^{(\delta)})|D^2\log\theta^{(\delta)}|\to 0\quad\text{strongly in }L^{4/3}(\Omega).\end{aligned}$$

The L^∞ bound of $\log\theta^{(\delta)}$ implies that $\theta^{(\delta)}$ is bounded away from zero and infinity. The $W^{1,4}(\Omega)$ bound yields the strong convergence of (a subsequence of) $\theta^{(\delta)}$ and thus the convergence $\log\theta^{(\delta)}\to\log\theta$ a.e. and uniformly. As a consequence of the strong $L^\infty(\Omega)$ convergence of $\vec q^{(\delta)}$ and $\log\theta^{(\delta)}$, we have

$$\vec\rho^{(\delta)}\to\vec\rho\quad\text{strongly in }L^\infty(\Omega;\mathbb R^N),\qquad\min_{i=1,\ldots,N}\operatorname*{ess\,inf}_\Omega\rho_i>0.$$

The previous convergences are sufficient to pass to the limit $\delta\to 0$ in (4.6)–(4.8) and to obtain an analogous system of equations without the terms proportional to δ. The terms that are nonlinear in $\nabla\vec q$, $\nabla\log\theta$ are monotone, so we can use Minty's monotonicity technique to identify the limit of these terms.

The limit $n\to\infty$ is carried out in the same way as the limit $\delta\to 0$. We have chosen not to take the two limits simultaneously because of the term $\delta K(n)\int_\Omega|\boldsymbol v|^2\boldsymbol v\cdot\boldsymbol u\,dx$ in (4.14), which vanishes as $\delta\to 0$ ($n\in\mathbb N$ fixed) but not necessarily when both $\delta\to 0$, $n\to\infty$. The two limits could be carried out simultaneously provided that one assumes that $\delta K(n)\to 0$.

After taking the limits $\delta \to 0$, $n \to \infty$ (in this order), we are left with the following system:

$$0 = \sum_{i=1}^N \int_\Omega \bigg(-\rho_i \boldsymbol{v} + \sum_{j=1}^N M_{ij}\nabla q_j + M_i \nabla\Big(\frac{1}{\theta}\Big)\bigg)\cdot\nabla\phi_i\,dx - \sum_{i=1}^N\int_\Omega r_i\phi_i\,dx \tag{5.1}$$
$$+ \varepsilon^3 \sum_{i=1}^N \int_\Omega \big((1+\chi|\nabla q_i|^2)\nabla q_i\cdot\nabla\phi_i + (1+|q_i-q_0|^{1/2})(q_i-q_0)\phi_i\big)\,dx$$
$$+ \varepsilon\sum_{i=1}^N\int_\Omega(\rho_i - \mathfrak{R}_i)\phi_i\,dx;$$

$$0 = \int_{\partial\Omega}\alpha_1\boldsymbol{u}\cdot\boldsymbol{v}\,ds + \int_\Omega(-\rho\boldsymbol{v}\otimes\boldsymbol{v} + \mathbb{S}) : \nabla\boldsymbol{u}\,dx - \int_\Omega(p\operatorname{div}\boldsymbol{u} + \rho\boldsymbol{b}\cdot\boldsymbol{u})\,dx \tag{5.2}$$
$$+ \varepsilon^3\sum_{i=1}^d\sum_{j=1}^N\int_\Omega v_i\Big((1+\chi|\nabla q_j|^2)\nabla q_j\cdot\nabla u_i + \frac{1}{2}(1+|q_j-q_0|^{1/2})(q_j-q_0)u_i\Big)\,dx$$
$$+ \varepsilon\int_\Omega\rho\boldsymbol{v}\cdot\boldsymbol{u}\,dx;$$

$$0 = \int_\Omega\bigg(-\rho e\boldsymbol{v} + \kappa(\theta)\nabla\theta + \sum_{i=1}^N M_i\nabla q_i\bigg)\cdot\nabla\phi_0\,dx + \int_\Omega(p\operatorname{div}\boldsymbol{v} - \mathbb{S}:\nabla\boldsymbol{v})\phi_0\,dx \tag{5.3}$$
$$+ \int_{\partial\Omega}\alpha_2(\theta-\theta_0)\phi_0\,ds - \varepsilon\int_\Omega\mathfrak{R}\frac{|\boldsymbol{v}|^2}{2}\phi_0\,dx$$
$$+ \varepsilon\int_\Omega\big((1+\theta)(1+|\nabla\log\theta|^2)\nabla(\log\theta)\cdot\nabla\phi_0 + (\log\theta)\phi_0\big)\,dx$$

for all $\phi_1,\ldots,\phi_N \in W^{1,4}(\Omega)$, $\boldsymbol{u}\in W^{1,12}(\Omega;\mathbb{R}^3)$, and $\phi_0\in W^{1,4}(\Omega)$.

5.2. **Limit** $\chi \to 0$**.** Since the estimate of $\nabla\vec{q}^{(\chi)}$ in $L^4(\Omega;\mathbb{R}^{N\times 3})$ blows up when $\chi\to 0$, we lose the control of $\vec\rho$ in $L^\infty(\Omega;\mathbb{R}^N)$. Therefore, we need to establish χ-independent estimates of ρ, which is the goal of the following lemma.

5.2.1. *Estimate for ρ via the Bogovskii operator.* The following result is the analogue of Lemma 2.3 for the approximate equations.

Lemma 5.1. *Let L, η be as in* (4.4) *and* $0<\zeta\le L/11$*. It holds that*

$$\int_\Omega\rho^{\gamma+\zeta}dx + \eta\int_\Omega\rho^{L+\zeta}dx \le C(1+\varepsilon^{(L+\zeta)/(2L)}), \tag{5.4}$$

where $C>0$ is independent of η and ε.

Proof. Let $\mathcal{B}$ be the Bogovskii operator (see Theorem A.1). Choosing $\boldsymbol{u} = \mathcal{B}(\rho^\zeta - \langle\rho^\zeta\rangle)$ in (5.2) and carrying out the computations contained in Lemma 2.3, it follows that

$$\int_\Omega p\rho^\zeta\,dx \le C(1+\|\rho\|^\lambda_{L^{\gamma+\zeta}(\Omega)}) + G, \quad \text{where} \tag{5.5}$$

$$G = \varepsilon^3 \sum_{i=1}^{d} \sum_{j=1}^{N} \int_\Omega v_i \Big((1+\chi|\nabla q_j|^2)\nabla q_j \cdot \nabla u_i + \frac{1}{2}(1+|q_i - q_0|^{1/2})(q_j - q_0)u_i \Big)\, dx + \varepsilon \int_\Omega \rho \boldsymbol{v} \cdot \boldsymbol{u}\, dx.$$

Note that due to our regularized problem, the computations in Lemma 2.3 are performed for the choice $\beta = \gamma = L$, hence it leads to the restriction $\zeta \le (3L-2)L/(3L+2)$.

The terms in G remain to be estimated. It holds that

$$\begin{aligned} |G| &\le C\varepsilon^3 \chi \|\boldsymbol{v}\|_{L^6(\Omega)} \|\nabla \vec{q}\|^3_{L^4(\Omega)} \|\nabla \boldsymbol{u}\|_{L^{12}(\Omega)} + C\varepsilon^3 \|\boldsymbol{v}\|_{L^6(\Omega)} \|\nabla \vec{q}\|_{L^2(\Omega)} \|\nabla \boldsymbol{u}\|_{L^3(\Omega)} \\ &\quad + C\varepsilon^3 \|\boldsymbol{v}\|_{L^6(\Omega)} \|\vec{q} - q_0\vec{1}\|_{L^2(\Omega)} \|\boldsymbol{u}\|_{L^3(\Omega)} + C\varepsilon^3 \|\boldsymbol{v}\|_{L^6(\Omega)} \|\vec{q} - q_0\vec{1}\|^{3/2}_{L^{5/2}(\Omega)} \|\boldsymbol{u}\|_{L^{30/7}(\Omega)} \\ &\quad + C\varepsilon \|\sqrt{\rho}\|_{L^2(\Omega)} \|\sqrt{\rho} v\|_{L^2(\Omega)} \|\boldsymbol{u}\|_{L^\infty(\Omega)}. \end{aligned}$$

From (4.24)–(4.26), (4.30), quasi mass conservation (4.19), as well as the uniform H^1 bound (2.11) for $\boldsymbol{v}$ and Sobolev's embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we infer that

$$\begin{aligned} |G| &\le C\varepsilon^{3/4}\chi^{1/4} \|\nabla \boldsymbol{u}\|_{L^{12}(\Omega)} + C\varepsilon^{3/2} \|\nabla \boldsymbol{u}\|_{L^3(\Omega)} + C\varepsilon^{3/2} \|\boldsymbol{u}\|_{L^3(\Omega)} + C\varepsilon^{1/2} \|\boldsymbol{u}\|_{L^\infty(\Omega)} \\ &\le C\varepsilon^{1/2} \|\boldsymbol{u}\|_{W^{1,12}(\Omega)}. \end{aligned}$$

It follows from Theorem A.1 that $\|\boldsymbol{u}\|_{W^{1,12}(\Omega)} \le C\|\rho^\zeta - \langle \rho^\zeta \rangle\|_{L^{12}(\Omega)}$, such that

$$|G| \le C\varepsilon^{1/2}\big(1 + \|\rho\|^\zeta_{L^{12\zeta}(\Omega)}\big).$$

Therefore, (5.5) implies that

$$\int_\Omega p\rho^\zeta\, dx \le C(1 + \|\rho\|^\lambda_{L^{\gamma+\zeta}(\Omega)}) + C\varepsilon^{1/2} \|\rho\|^\zeta_{L^{12\zeta}(\Omega)}.$$

Since $p \ge c\rho^\gamma + c\eta\rho^L$, while $\lambda < \gamma + \zeta$, we find that

$$\int_\Omega \rho^{\gamma+\zeta} dx + \eta \int_\Omega \rho^{L+\zeta} dx \le C + C\varepsilon^{1/2} \left(\int_\Omega \rho^{L+\zeta} dx \right)^{\frac{\zeta}{L+\zeta}},$$

which leads to (5.4). This finishes the proof of the lemma. □

Remark 5.2. Due to our assumption $L \ge 2\beta_0$, we still control $\nabla\vec{\rho}$ in $L^2(\Omega; \mathbb{R}^{N\times 2})$ independently of χ; we use the argument between (4.33) and (4.34).

5.2.2. *Limit passage in the equations.* At this point, since we have a (ε, χ)-uniform bound for ρ (given by (5.4) and Remark 5.2), we can take the limit $\chi \to 0$. Moreover, we still have a uniform $H^1(\Omega)$ bound for $\vec{q}$ and a uniform $W^{1,4}(\Omega)$ bound for $\log\theta$, so we deduce via compact Sobolev embedding that for $\chi \to 0$ (up to subsequences),

$$\vec{q}^{(\chi)} \to \vec{q} \quad \text{strongly in } L^{6-\nu}(\Omega; \mathbb{R}^N) \text{ for all } \nu > 0, \quad \log\theta^{(\chi)} \to \log\theta \quad \text{strongly in } L^\infty(\Omega).$$

As before, it is not difficult to justify that $\theta^{(\chi)} \to \theta$ a.e. in Ω. We deduce from the continuity of the free energy h_θ that $\vec{\rho}^{(\chi)}$ is also a.e. convergent in Ω. Bound (5.4) then implies that

$$\vec{\rho}^{(\chi)} \to \vec{\rho} \quad \text{strongly in } L^r(\Omega; \mathbb{R}^N) \quad \text{for all } r < \frac{12}{11}L.$$

The limit $\chi \to 0$ in (5.1)–(5.3) is carried out in a similar way as the previous limits $\delta \to 0$, $n \to \infty$. This leads to the following limiting system:

(5.6)
$$0 = \sum_{i=1}^N \int_\Omega \bigg(-\rho_i \boldsymbol{v} + \sum_{j=1}^N M_{ij} \nabla q_j + M_i \nabla \Big(\frac{1}{\theta} \Big) \bigg) \cdot \nabla \phi_i \, dx - \sum_{i=1}^N \int_\Omega r_i \phi_i \, dx$$
$$+ \varepsilon^3 \sum_{i=1}^N \int_\Omega \big(\nabla q_i \cdot \nabla \phi_i + (1 + |q_i - q_0|^{1/2})(q_i - q_0)\phi_i \big) \, dx + \varepsilon \sum_{i=1}^N \int_\Omega (\rho_i - \mathfrak{R}_i)\phi_i \, dx;$$

(5.7)
$$0 = \int_{\partial\Omega} \alpha_1 \boldsymbol{u} \cdot \boldsymbol{v} \, ds + \int_\Omega (-\rho \boldsymbol{v} \otimes \boldsymbol{v} + \mathbb{S}) : \nabla \boldsymbol{u} \, dx - \int_\Omega (p \operatorname{div} \boldsymbol{u} + \rho \boldsymbol{b} \cdot \boldsymbol{u}) \, dx$$
$$+ \varepsilon^3 \sum_{i=1}^d \sum_{j=1}^N \int_\Omega v_i \bigg(\nabla q_j \cdot \nabla u_i + \frac{1}{2}(1 + |q_j - q_0|^{1/2})(q_j - q_0) u_i \bigg) \, dx + \varepsilon \int_\Omega \rho \boldsymbol{v} \cdot \boldsymbol{u} \, dx;$$

(5.8)
$$0 = \int_\Omega \bigg(-\rho e \boldsymbol{v} + \kappa(\theta) \nabla \theta + \sum_{i=1}^N M_i \nabla q_i \bigg) \cdot \nabla \phi_0 \, dx + \int_\Omega (p \operatorname{div} \boldsymbol{v} - \mathbb{S} : \nabla \boldsymbol{v}) \phi_0 \, dx$$
$$+ \int_{\partial\Omega} \alpha_2 (\theta - \theta_0) \phi_0 \, ds - \varepsilon \int_\Omega \mathfrak{R} \frac{|\boldsymbol{v}|^2}{2} \, dx$$
$$+ \xi \int_\Omega \big((1 + \theta)(1 + |\nabla \log \theta|^2) \nabla(\log \theta) \cdot \nabla \phi_0 + (\log \theta)\phi_0 \big) \, dx$$

for all $\phi_1, \ldots, \phi_N \in H^1(\Omega) \cap L^\infty(\Omega)$, $\boldsymbol{u} \in W^{1,3}(\Omega; \mathbb{R}^3)$, and $\phi_0 \in W^{1,4}(\Omega)$.

5.2.3. *Entropy and total energy balance equations.* We will derive the balance equations for the entropy and total energy. It was not possible to derive the latter equation as long as the p-Laplacian regularization in terms of $\vec{q}$ was included in the mass densities equation.

Let $\psi \in W^{1,\infty}(\Omega)$ with $\psi \geq 0$ be arbitrary. By choosing $\phi_i = (q_i - q_0)\psi$ in (5.6) and $\phi_0 = -\frac{1}{\theta}\psi$ in (5.8) and proceeding as in the derivation of (4.18), we obtain

$$\int_\Omega \bigg(\rho s \boldsymbol{v} + \sum_{i=1}^N q_i \bigg(\sum_{j=1}^N M_{ij} \nabla q_j + M_i \nabla \frac{1}{\theta} \bigg) - \frac{1}{\theta} \bigg(\kappa(\theta) \nabla \theta + \sum_{i=1}^N M_i \nabla q_i \bigg) \bigg) \cdot \nabla \psi \, dx$$
$$+ \int_\Omega \bigg(\sum_{i,j=1}^N M_{ij} \nabla q_i \cdot \nabla q_j + \kappa^{(\eta)}(\theta) |\nabla \log \theta|^2 - \sum_{i=1}^N r_i q_i + \frac{1}{\theta} \mathbb{S} : \nabla \boldsymbol{v} \bigg) \psi \, dx \tag{5.9}$$
$$+ \varepsilon^3 \int_\Omega \sum_{i=1}^N \nabla q_i \cdot (q_i - q_0) \nabla \psi \, dx - \xi \int_\Omega (1 + \theta^{-1})(1 + |\nabla \log \theta|^2) \nabla \log \theta \cdot \nabla \psi \, dx$$
$$= \int_{\partial\Omega} \alpha_2 \frac{\theta - \theta_0}{\theta} \psi \, ds - \xi \int_\Omega \frac{1}{\theta} \log \frac{1}{\theta} \psi \, dx.$$

Next, let $\varphi \in W^{1,\infty}(\Omega)$. By choosing $\phi_i = -\frac{1}{2}|\boldsymbol{v}|^2\varphi$ in (5.6), $\boldsymbol{u} = \boldsymbol{v}\varphi$ in (5.7), and $\phi_0 = \varphi$ in (5.8) and proceeding as in the derivation of (4.21), it follows that

$$\begin{aligned} &\int_\Omega \big(-\rho E\boldsymbol{v} - \boldsymbol{Q} + \mathbb{S}\boldsymbol{v} - p\boldsymbol{v}\big)\cdot\nabla\varphi\,dx + \int_{\partial\Omega}(\alpha_1|\boldsymbol{v}|^2 + \alpha_2(\theta-\theta_0))\varphi\,ds \\ &\quad + \xi\int_\Omega (1+\theta)(1+|\nabla\log\theta|^2)\nabla(\log\theta)\cdot\nabla\varphi\,dx \\ &\quad + \varepsilon\int_\Omega \rho\frac{|\boldsymbol{v}|^2}{2}\varphi\,dx + \xi\int_\Omega \log\theta\varphi\,dx = \int_\Omega \rho\boldsymbol{b}\cdot\boldsymbol{v}\varphi\,dx. \end{aligned} \tag{5.10}$$

5.3. **Limit $\varepsilon \to 0$.** When taking the limit $\varepsilon \to 0$, we can argue as in Subsection 3.1 and prove that $\boldsymbol{v}^{(\varepsilon)}$ is (up to a subsequence) strongly convergent in $L^{6-\delta}(\Omega;\mathbb{R}^3)$, $\theta^{(\varepsilon)}$ is strongly convergent in $L^{3\beta-\delta}(\Omega)$ for all $\delta > 0$, and $\rho^{(\varepsilon)}$ is weakly convergent in $L^{L+\nu}(\Omega)$. On the other hand, $\log\theta^{(\varepsilon)}$ is strongly convergent in $L^\infty(\Omega)$, thanks to (4.28), which is better than the convergence property of θ in Subsection 3.1. As in Subsection 3.1, we can also show the convergence of the various terms appearing in equations (5.6), (5.7), (5.9), and (5.10), which do not involve the regularization. In this subsection, we only show that the additional ε-dependent regularizing terms vanish in the limit $\varepsilon \to 0$.

Taking into account estimates (4.25), (4.26), (4.32) and the uniform $L^1(\Omega)$ bounds for ρ, we see that for every $\phi_1,\dots,\phi_N \in H^1(\Omega)\cap L^\infty(\Omega)$,

$$\varepsilon^3\sum_{i=1}^N\int_\Omega \big(\nabla q_i\cdot\nabla\phi_i + (1+|q_i-q_0|^{1/2})(q_i-q_0)\phi_i\big)\,dx + \varepsilon\sum_{i=1}^N\int_\Omega(\rho_i - \mathfrak{R}_i)\phi_i\,dx \to 0$$

as $\varepsilon \to 0$. Furthermore, it follows from (4.25), (4.26) and the uniform bounds for ρ and $\boldsymbol{v}$ in $L^{6/5}(\Omega)$ and $L^6(\Omega)$, respectively, that, for every $\boldsymbol{u} \in W^{1,3}(\Omega)$,

$$\varepsilon^3\sum_{i=1}^d\sum_{j=1}^N\int_\Omega v_i\bigg(\nabla q_j\cdot\nabla u_i + \frac{1}{2}(1+|q_j-q_0|^{1/2})(q_j-q_0)u_i\bigg)\,dx + \varepsilon\int_\Omega \rho\boldsymbol{v}\cdot\boldsymbol{u}\,dx \to 0$$

as $\varepsilon \to 0$. Estimates (4.25) and (4.26) imply that

$$\varepsilon^3\int_\Omega\sum_{i=1}^N \nabla q_i\cdot(q_i-q_0)\nabla\psi\,dx \to 0 \quad \text{as } \varepsilon\to 0.$$

Finally, the uniform $L^{L+\nu}(\Omega)$ bound for ρ and the $L^6(\Omega;\mathbb{R}^3)$ bound for $\boldsymbol{v}$ give

$$\varepsilon\int_\Omega \rho\frac{|\boldsymbol{v}|^2}{2}\varphi\,dx \to 0 \quad \text{as } \varepsilon \to 0.$$

5.3.1. *Effective viscous flux.* We want to show that Lemma 3.2 holds for approximate solutions. For this, we only need to check that the div-curl Lemma can be applied. It follows from (5.7) that, in the distributional sense,

$$\operatorname{div}(\rho^{(\varepsilon)}\boldsymbol{v}^{(\varepsilon)}\otimes\boldsymbol{v}^{(\varepsilon)} - \mathbb{T}^{(\varepsilon)}) - \rho^{(\varepsilon)}\boldsymbol{b}^{(\varepsilon)}$$

$$= \varepsilon^3 \operatorname{div}\left(\nabla \sum_{j=1}^N q_j \boldsymbol{v}^{(\varepsilon)}\right) - \frac{1}{2}\varepsilon^3 \sum_{j=1}^N (1+|q_i^{(\varepsilon)} - q_0^{(\varepsilon)}|^{1/2})(q_i^{(\varepsilon)} - q_0^{(\varepsilon)})v^{(\varepsilon)} - \varepsilon\rho^{(\varepsilon)}\boldsymbol{v}^{(\varepsilon)}.$$

By arguing as in the previous subsection, we see that the right-hand side of this identity tends to zero strongly in $W^{-1,3/2}(\Omega)$. Hence, $\operatorname{div}(\rho^{(\varepsilon)}\boldsymbol{v}^{(\varepsilon)}\otimes\boldsymbol{v}^{(\varepsilon)} - \mathbb{T}^{(\varepsilon)})$ is relatively compact in $W^{-1,r}(\Omega)$ for some $r>1$.

We claim that $\operatorname{div}(\rho^{(\varepsilon)}\boldsymbol{v}^{(\varepsilon)})$ is relatively compact in $W^{-1,r}(\Omega)$ for some $r>1$. Indeed, we infer from (5.6) that

$$\operatorname{div}(\rho^{(\varepsilon)}\boldsymbol{v}^{(\varepsilon)}) = \varepsilon^3 \sum_{i=1}^N \Delta q_i - \varepsilon^3 \sum_{i=1}^N (1+|q_i - q_0|^{1/2})(q_i - q_0) - \varepsilon(\rho - \mathfrak{R}).$$

Once again, by arguing as in the previous subsection, we find that $\operatorname{div}(\rho^{(\varepsilon)}\boldsymbol{v}^{(\varepsilon)}) \to 0$ strongly in $W^{-1,3/2-\delta}(\Omega)$ for every $\delta>0$. We conclude that Lemma 3.2 effectively holds for the approximate solutions constructed in the previous subsections.

5.3.2. *Estimates based on the convexity of the free energy.* Lemmata 3.3–3.5 also hold for the approximate solutions, as they only require the properties of the free energy stated in the introduction, which still hold true for the regularized free energy introduced in (4.4), and the estimates in Lemmata 2.2–2.4, which are satisfied by the approximate solutions.

5.3.3. *Renormalized continuity equation.* We prove that the weak limit ρ of $(\rho^{(\varepsilon)})$ satisfies the renormalized continuity equation (3.42) by mimicking the proof of Lemma 3.6. Choosing $\phi_i = T_k'(\rho^{(\varepsilon)})\psi$ ($i=1,\ldots,N$), $\psi \in W^{1,\infty}(\Omega)$, in (5.6) (this is possible since $\rho^{(\varepsilon)} \in H^1(\Omega)$ thanks to (4.34)) yields

$$-\int_\Omega T_k(\rho^{(\varepsilon)})\boldsymbol{v}^{(\varepsilon)}\cdot\nabla\psi\,dx + \int_\Omega \big(T_k'(\rho^{(\varepsilon)})\rho^{(\varepsilon)} - T_k(\rho^{(\varepsilon)})\big)\operatorname{div}\boldsymbol{v}^{(\varepsilon)}\psi\,dx = R_k^{(\varepsilon)}(\psi), \tag{5.11}$$

where

$$\begin{aligned} R_k^{(\varepsilon)}(\psi) = -\varepsilon^3 \sum_{i=1}^N \int_\Omega \Big(\nabla q_i^{(\varepsilon)}\cdot\nabla(T_k'(\rho^{(\varepsilon)})\psi) + (1+|q_i^{(\varepsilon)} - q_0^{(\varepsilon)}|^{1/2})(q_i^{(\varepsilon)} - q_0^{(\varepsilon)})T_k'(\rho^{(\varepsilon)})\psi\Big)\,dx \\ - \varepsilon \sum_{i=1}^N \int_\Omega (\rho_i^{(\varepsilon)} - \mathfrak{R}_i^{(\varepsilon)})T_k'(\rho^{(\varepsilon)})\psi\,dx. \end{aligned}$$

Taking the limit inferior $\varepsilon\to 0$ in both sides of (5.11) leads to

$$\left| -\int_\Omega \overline{T_k(\rho^{(\varepsilon)})}\boldsymbol{v}\cdot\nabla\psi\,dx + \int_\Omega \overline{\big(T_k'(\rho^{(\varepsilon)})\rho^{(\varepsilon)} - T_k(\rho^{(\varepsilon)})\big)\operatorname{div}\boldsymbol{v}^{(\varepsilon)}}\psi\,dx\right| \le \liminf_{\varepsilon\to 0}|R_k^{(\varepsilon)}(\psi)|. \tag{5.12}$$

Thanks to estimates (4.25) and (4.26), we infer that

$$\begin{aligned} \liminf_{\varepsilon\to 0}|R_k^{(\varepsilon)}(\psi)| &\le \liminf_{\varepsilon\to 0}\varepsilon^3 \int_\Omega |\nabla\vec{q}^{(\varepsilon)}||\nabla\rho^{(\varepsilon)}||T''(\rho^{(\varepsilon)})||\psi|\,dx \\ &\le \frac{C}{k}\liminf_{\varepsilon\to 0}\varepsilon^3 \int_{\{k\le\rho^{(\varepsilon)}\le 3k\}} |\nabla\vec{q}^{(\varepsilon)}||\nabla\rho^{(\varepsilon)}||\psi|\,dx. \end{aligned}$$

We replace ψ by $b'(\overline{T_k(\rho^{(\varepsilon)})})\psi$ in (5.12) and exploit the definition of the truncation operator T_k to find that

$$\begin{aligned}(5.13)\qquad &\bigg|-\int_\Omega b\big(\overline{T_k(\rho^{(\varepsilon)})}\big)\boldsymbol{v}\cdot\nabla\psi\,dx+\int_\Omega \operatorname{div}\boldsymbol{v}\big(\overline{T_k(\rho^{(\varepsilon)})}b'\big(\overline{T_k(\rho^{(\varepsilon)})}\big)-b\big(\overline{T_k(\rho^{(\varepsilon)})}\big)\big)\psi\,dx\\ &\quad-\int_\Omega b'\big(\overline{T_k(\rho^{(\varepsilon)})}\big)\overline{(T_k(\rho^{(\varepsilon)})-\rho^{(\varepsilon)}T_k'(\rho^{(\varepsilon)}))\operatorname{div}\boldsymbol{v}^{(\varepsilon)}}\psi\,dx\bigg|\\ &\le\frac{C}{k}\liminf_{\varepsilon\to0}\varepsilon^3\int_{\{k\le\rho^{(\varepsilon)}\le3k\}}|\nabla\vec{q}^{(\varepsilon)}||\nabla\rho^{(\varepsilon)}|\big|b'\big(\overline{T_k(\rho^{(\varepsilon)})}\big)\big||\psi|\,dx.\end{aligned}$$

It is proved in Lemma 3.6 that the left-hand side of (5.13) converges to the left-hand side of the renormalized continuity equation (3.42) as $k\to\infty$. Therefore, it is sufficient to prove that the right-hand side of (5.13) tends to zero as $k\to\infty$. Indeed, since b' is bounded and (4.25) and (4.34) hold, it follows that

$$\begin{aligned}\liminf_{\varepsilon\to0}\varepsilon^3&\int_{\{k\le\rho^{(\varepsilon)}\le3k\}}|\nabla\vec{q}^{(\varepsilon)}||\nabla\rho^{(\varepsilon)}||b'(\overline{T_k(\rho^{(\varepsilon)})})||\psi|\,dx\\ &\le\liminf_{\varepsilon\to0}\varepsilon^3\|\nabla\vec{q}^{(\varepsilon)}\|_{L^2(\Omega)}\|\nabla\rho^{(\varepsilon)}\|_{L^2(\Omega)}\le C,\end{aligned}$$

meaning that the right-hand side of (5.13) tends to zero as $k\to\infty$. We conclude that the weak limit $(\rho,\boldsymbol{v})$ of $(\rho^{(\varepsilon)},\boldsymbol{v}^{(\varepsilon)})$ satisfies the renormalized continuity equation (3.42).

5.3.4. *Limit $\varepsilon\to0$: conclusion.* Lemmata 3.7 and 3.8 hold as they are not influenced by the approximation. We conclude that $\rho_i^{(\varepsilon)}\to\rho_i$ a.e. in Ω.

5.4. **Limit $\xi\to0$.** Due to the uniform bounds in Lemmata 2.2 and 5.1 (fulfilled by the approximating sequence), we deduce that, up to subsequences, $\Pi\vec{q}^{(\xi)}\to\Pi\vec{q}$, $\log\theta^{(\xi)}\to\log\theta$, $(\theta^{(\xi)})^{\beta/2}\to\theta^{\beta/2}$ weakly in $H^1(\Omega)$ and strongly in $L^{6-\delta}(\Omega)$ for every $\delta>0$, and $\rho^{(\xi)}\rightharpoonup\rho$ weakly in $L^{L+\nu}(\Omega)$, as $\xi\to0$. Moreover, the a.e. convergence of $\rho^{(\xi)}$ is proved in an analogous way as in the compactness part, since no ξ-dependent regularizing terms appear in either the linear momentum or the partial mass densities equations. We only need to show that the regularizing terms in the entropy balance equation (5.9) and in the total energy balance equation (5.10) vanish in the limit $\xi\to0$.

To this end, we note that, for any $\psi\in W^{1,\infty}(\Omega)$ with $\psi\ge0$, we have

$$\limsup_{\xi\to0}\bigg(-\xi\int_\Omega\frac{1}{\theta}\log\frac{1}{\theta}\psi\,dx\bigg)\le\limsup_{\xi\to0}\xi\int_{\{\theta\ge1\}}\frac{\log\theta}{\theta}\psi\,dx=0.$$

Next, we point out that (4.28) and (2.12) imply that

$$\|\theta^{-1/4}\|_{W^{1,4}(\Omega)}\le C\xi^{-1/4},$$

which, thanks to Gagliardo–Nirenberg inequality and the $L^1(\Omega)$ bound for $1/\theta$, leads to

$$\|\theta^{-1/4}\|_{L^\infty(\Omega)}\le C\|\theta^{-1/4}\|_{L^4(\Omega)}^{1/4}\|\theta^{-1/4}\|_{W^{1,4}(\Omega)}^{3/4}\le C\xi^{-3/16}.$$

We deduce from this estimate and (4.28) that

$$\xi \int_\Omega (1+\theta^{-1})(1+|\nabla \log\theta|^2)\nabla\log\theta\cdot\nabla\psi\,dx \to 0 \quad \text{as } \xi\to 0.$$

It follows from (4.33), Sobolev's embedding, and assumption $L\ge\gamma+\nu>4/3$ that

$$\|\theta\|_{L^4(\Omega)}^{L/2} \le C\|\theta\|_{L^{3L}(\Omega)}^{L/2} = C\|\theta^{L/2}\|_{L^6(\Omega)} \le C\|\theta^{L/2}\|_{H^1(\Omega)} \le C(\eta).$$

Together with (4.28) and the uniform $L^2(\Omega)$ bound for $\log\theta$, this imply that

$$\xi\int_\Omega (1+\theta)(1+|\nabla\log\theta|^2)\nabla\log\theta\cdot\nabla\varphi\,dx + \xi\int_\Omega \log\theta\varphi\,dx \to 0 \qquad \text{as } \xi\to 0.$$

This finishes the part related to the limit $\xi\to 0$.

5.5. **Limit $\eta\to 0$.** As in the previous limit, it holds (up to subsequences) that $\Pi\vec{q}^{(\eta)}\to\Pi\vec{q}$, $\log\theta^{(\eta)}\to\log\theta$, $(\theta^{(\eta)})^{\beta/2}\to\theta^{\beta/2}$ weakly in $H^1(\Omega)$ and strongly in $L^{6-\delta}(\Omega)$ for any $\delta>0$ as $\eta\to 0$. Lemma 2.3 can be shown to hold true as in the compactness part, since the regularizing terms in the linear momentum equations have been removed in the limit $\varepsilon\to 0$. The proof of the a.e. convergence of $\rho^{(\eta)}$ is identical to that one in the compactness part; in particular, Lemma 3.8 holds, as it only employs the properties of the free energy stated in Hypothesis (H6).

We show that the regularizing term, coming from the heat conductivity $\kappa^{(\eta)}$ defined in (4.5), vanishes in the entropy and total energy balance equations in the limit $\eta\to 0$. The strongest of these terms is $\eta(\theta^{(\eta)})^L\nabla\theta^{(\eta)}$, which appears in the total energy balance. It follows from (4.33) and (2.12)–(2.13) via the Gagliardo–Nirenberg inequality and assumption $L\ge 3\beta-2$ that, as $\eta\to 0$,

$$\begin{aligned}\eta\int_\Omega (\theta^{(\eta)})^L|\nabla\theta^{(\eta)}|\,dx &\le C\eta\|(\theta^{(\eta)})^{(L+2)/2}\|_{L^2(\Omega)}\|\nabla(\theta^{(\eta)})^{L/2}\|_{L^2(\Omega)}\\ &\le C\eta\|(\theta^{(\eta)})^{\beta/2}\|_{L^2(\Omega)}^{\lambda(L+2)/\beta}\|(\theta^{(\eta)})^{L/2}\|_{H^1(\Omega)}^{1+(1-\lambda)(L+2)/2}\to 0,\end{aligned}$$

where $\lambda=2\beta(L-1)/((L-\beta)(L+2))\in[0,1]$ satisfies $1+(1-\lambda)(L+2)/2<2$ if and only if $\beta>1$. This is the condition under which the total energy balance can be obtained. On the other hand, it is possible to see that the term $\eta(\theta^{(\eta)})^L\nabla\log\theta^{(\eta)}$, which appears in the entropy balance, tends to zero strongly as $\eta\to 0$ for $\beta>2/3$.

Concerning the regularization in the mass densities, Lemma 5.1 yields (remember that we have already taken the limit $\varepsilon\to 0$) for some $\zeta>0$,

$$\int_\Omega \rho^{L+\zeta}\,dx \le C\eta^{-1},$$

As a consequence, the regularizing terms in the pressure p and internal energy density ρe tend to zero strongly in $L^1(\Omega)$ as $\eta\to 0$. The entropy density ρs does not contain regularizing terms. Finally, let us turn our attention to the chemical potentials

$$\mu_i^{(\eta)} = \partial_{\rho_i}h_{\theta^{(\eta)}}(\vec{\rho}^{(\eta)}) + \eta L\frac{(\overline{n}^{(\eta)})^{L-1}}{m_i} + \eta f'_{\alpha_0}(\rho_i^{(\eta)}) + 2\eta\rho_i^{(\eta)}, \quad i=1,\ldots,N.$$

Clearly, $\eta L(\overline{n}^{(\eta)})^{L-1}/m_i + 2\eta\rho_i^{(\eta)} \to 0$ strongly in $L^1(\Omega)$. The only problematic term is $\eta f'_{\alpha_0}(\rho_i^{(\eta)})$, as it might be singular for $\rho_i \to 0$. Since $\alpha_0 < 1$, it follows that $f'_{\alpha_0}(s)$: $\mathbb{R}^+ \to \mathbb{R}$ has a finite limit for $s \to 0^+$, so $\rho_i^{(\eta)}\mu_i^{(\eta)}$ is a.e. convergent in Ω. We know from (3.52) that, up to sets of measure zero, $\cup_{i=1}^N\{\rho_i = 0\} = \{\rho = 0\}$, which implies (together with the strong convergence of $\vec{\rho}^{(\eta)}$, $\theta^{(\eta)}$) that the a.e. limit $\vec{\mu}$ of $\vec{\mu}^{(\eta)}$ is equal to $\nabla h_\theta(\vec{\rho})$ on $\{\rho > 0\}$. This finishes the proof of Theorem 1.5.

Remark 5.3 (Integral of source term in partial mass balance)**.** By integrating the partial mass balance equation (1.1) and taking into account the boundary conditions (1.4), (1.5), we find that $\int_\Omega r_i dx = 0$ for $i = 1, \ldots, N$. This seemingly additional constraint is, in fact, satisfied by the solution constructed in Theorem 1.5. Indeed, for $j = 1, \ldots, N$, choose $\phi_i = \delta_{ij}$ (being the Kronecker delta) in (4.6) and use estimates (4.19), (4.25), (4.26), and (4.32). A straightforward computation yields

$$\left| \int_\Omega r_j(\Pi(\vec{\mu}^{(\varepsilon)}/\theta^{(\varepsilon)}), \theta^{(\varepsilon)})\, dx \right| \le C(\varepsilon + \varepsilon^{6/5}) \le C\varepsilon.$$

The strong convergence of $\Pi(\vec{\mu}^{(\varepsilon)}/\theta^{(\varepsilon)})$, $\theta^{(\varepsilon)}$ and Hypothesis (H5) imply that $\int_\Omega r_j(\Pi(\vec{\mu}/\theta), \theta)\, dx = 0$ in the limit $\varepsilon \to 0$. □

Appendix A. Auxiliary results

For the convenience of the reader, we collect some results needed in this paper. For the first lemma, which is proved in [28, Theorem 10.11], we introduce the space

$$L_0^p(\Omega) = \left\{ u \in L^p(\Omega) : \int_\Omega u\, dx = 0 \right\}, \quad 1 < p < \infty.$$

Theorem A.1 (Bogovskii operator)**.** *Let $\Omega \subset \mathbb{R}^d$ ($d \ge 2$) be a bounded domain with Lipschitz boundary and let $1 < p < \infty$. Then there exists a bounded linear operator $\mathcal{B} : L_0^p(\Omega) \to W_0^{1,p}(\Omega)$ such that $\operatorname{div}\mathcal{B}(u) = u$ in the sense of distributions for all $u \in L_0^p(\Omega)$, and there exists $C > 0$ such that for all $u \in L_0^p(\Omega)$,*

$$\|\mathcal{B}(u)\|_{W^{1,p}(\Omega)} \le C\|u\|_{L^p(\Omega)}.$$

Next, we recall the div-curl lemma, developed by Murat and Tartar and proved in, e.g., [12, Theorem 10.21]. We introduce the curl of a vector-valued function $\boldsymbol{w} = (w_j)$ by setting $(\operatorname{curl}\boldsymbol{w})_{ij} = \partial w_j/\partial x_i - \partial w_i/\partial x_j$.

Lemma A.2 (Div-curl lemma)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set, $1/p + 1/q = 1/r < 1$, and $s > 1$. Let $\boldsymbol{u}_\delta \rightharpoonup \boldsymbol{u}$ weakly in $L^p(\Omega;\mathbb{R}^n)$ and $\boldsymbol{w} \rightharpoonup \boldsymbol{w}$ weakly in $L^q(\Omega;\mathbb{R}^n)$ as $\delta \to 0$. If $(\operatorname{div}\boldsymbol{u}_\delta)$ is precompact in $W^{-1,s}(\Omega)$ and $(\operatorname{curl}\boldsymbol{w}_\delta)$ is precompact in $W^{-1,s}(\Omega;\mathbb{R}^{n\times n})$, then*

$$\boldsymbol{u}_\delta \cdot \boldsymbol{w}_\delta \rightharpoonup \boldsymbol{u} \cdot \boldsymbol{w} \quad \textit{weakly in } L^r(\Omega) \quad \textit{as } \delta \to 0.$$

We need the following version of the Korn inequality, proved e.g. in [18, Lemma 2.1].

Lemma A.3 (Korn's inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ be a bounded, not axially symmetric domain with Lipschitz boundary. Let the stress tensor fulfil Hypothesis (H3). Then there exists $C > 0$, only depending on d and Ω, such that for all $\boldsymbol{u} \in W^{1,2}(\Omega;\mathbb{R}^d)$ with $\boldsymbol{u} \cdot \boldsymbol{\nu} = 0$ on $\partial\Omega$,*

$$\|\boldsymbol{u}\|_{W^{1,2}(\Omega)}^2 \leq C \int_\Omega \frac{\mathbb{S}(\theta,\nabla\boldsymbol{u}) : \nabla\boldsymbol{u}}{\theta}\,dx, \quad \|\boldsymbol{u}\|_{W^{1,2}(\Omega)}^2 \leq C \int_\Omega \mathbb{S}(\theta,\nabla\boldsymbol{u}) : \nabla\boldsymbol{u}\,dx.$$

The following result is a slight generalization of Fatou's lemma.

Lemma A.4. *Let $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ be a bounded domain, and let $(\mathbb{M}_k) \subset L^\infty(\Omega;\mathbb{R}^{N\times N})$ be a sequence of symmetric, positive semi-definite matrices and $(\vec{v_k}) \subset L^2(\Omega;\mathbb{R}^N)$ be such that*

$$\vec{v}_k \rightharpoonup \vec{v} \quad \text{weakly in } L^2(\Omega;\mathbb{R}^N), \quad \mathbb{M}_k \to \mathbb{M} \quad \text{a.e. in } \Omega$$

as $k \to \infty$. Then

$$\liminf_{k\to\infty} \int_\Omega \vec{v}_k \cdot \mathbb{M}_k \vec{v}_k\,dx \geq \int_\Omega \vec{v} \cdot \mathbb{M}\vec{v}\,dx.$$

Proof. Since $(\mathbb{M}_k)$ converges a.e., Egorov's theorem implies that there exists $\Omega^{(1)} \subset \Omega$ such that $|\Omega \setminus \Omega^{(1)}| < 1/2$ and $\mathbb{M}_k \to \mathbb{M}$ uniformly in $\Omega^{(1)}$. We consider

$$\int_\Omega \vec{v}_k \cdot \mathbb{M}_k \vec{v}_k\,dx \geq \int_{\Omega^{(1)}} \vec{v}_k \cdot \mathbb{M}_k \vec{v}_k\,dx = \int_{\Omega^{(1)}} \vec{v}_k \cdot (\mathbb{M}_k - \mathbb{M})\vec{v}_k\,dx + \int_{\Omega^{(1)}} \vec{v}_k \cdot \mathbb{M}\vec{v}_k\,dx.$$

As $\mathbb{M}_k \to \mathbb{M}$ uniformly in $\Omega^{(1)}$ and $(\vec{v}_k)$ is bounded in $L^2(\Omega;\mathbb{R}^N)$, it follows that

$$\liminf_{k\to\infty} \int_\Omega \vec{v}_k \cdot \mathbb{M}_k \vec{v}_k\,dx \geq \liminf_{k\to\infty} \int_{\Omega^{(1)}} \vec{v}_k \cdot \mathbb{M}\vec{v}_k\,dx \geq \int_{\Omega^{(1)}} \vec{v} \cdot \mathbb{M}\vec{v}\,dx. \tag{A.1}$$

The last inequality is a consequence of the weak lower semicontinuity of strongly continuous convex functionals. Applying Egorov's theorem again, there exists $\Omega^{(1)} \subset \Omega^{(2)} \subset \Omega$ such that $|\Omega \setminus \Omega^{(2)}| < 1/4$ and $\mathbb{M}_k \to \mathbb{M}$ uniformly in $\Omega^{(2)}$. Then (A.1) also holds with $\Omega^{(1)}$ replaced by $\Omega^{(2)}$. We obtain an increasing sequence $\Omega^{(m)}$ of subsets of Ω such that $|\Omega \setminus \Omega^{(m)}| < 2^{-m}$ and $\mathbb{M}_k \to \mathbb{M}$ uniformly in $\Omega^{(m)}$. Replacing $\Omega^{(1)}$ by $\Omega^{(m)}$ in (A.1) and taking the supremum for $m \geq 1$ leads to

$$\liminf_{k\to\infty} \int_\Omega \vec{v}_k \cdot \mathbb{M}_k \vec{v}_k\,dx \geq \sup_{m\geq 1} \int_{\Omega^{(m)}} \vec{v} \cdot \mathbb{M}\vec{v}\,dx.$$

Since $\vec{v} \cdot \mathbb{M}\vec{v} \geq 0$ a.e. in Ω and $(\Omega^{(m)})$ is an increasing sequence of subsets of Ω such that $\lim_{m\to\infty} |\Omega \setminus \Omega^{(m)}| = 0$, we conclude the proof. □

Lemma A.5. *Let $\mathbb{P} \in \mathbb{R}^{N\times N}$ be a matrix such that*

$$\operatorname{span}\{\vec{p}^1, \vec{p}^2, \ldots, \vec{p}^N\} = \operatorname{span}\{\vec{1}\}^\perp,$$

where $\vec{p}^i$ is the i-th column of the matrix $\mathbb{P}$. Introduce the matrix $\mathbb{A} \in \mathbb{R}^{N\times N}$ with $A_{ij} = \delta_{ij} - \delta_{Nj}$ for i, $j = 1, \ldots, N$. Then there exists a constant $C > 0$ such that for all $\vec{v} \in \mathbb{R}^N$,

$$|\mathbb{P}\vec{v}| \geq C|\mathbb{A}\vec{v}|,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$.

Proof. It follows from our assumptions that $\ker(\mathbb{A}) = \ker(\mathbb{P}) = \operatorname{span}\{\vec{1}\} =: V$. Choose $\varepsilon > 0$ and consider for $\vec{v} \in \mathbb{R}^N$,

$$|\mathbb{P}\vec{v}|^2 - \varepsilon|\mathbb{A}\vec{v}|^2 = \vec{v} \cdot (\mathbb{P}^T\mathbb{P} - \varepsilon\mathbb{A}^T\mathbb{A})\vec{v}.$$

The matrix $\mathbb{M}_\varepsilon := \mathbb{P}^T\mathbb{P} - \varepsilon\mathbb{A}^T\mathbb{A}$ is clearly symmetric and $\mathbb{M}_\varepsilon\vec{w} = \vec{0}$ for all $\vec{w} \in V$. However, the matrix $\mathbb{P}$ is nonsingular on $V^\perp$, thus

$$\inf_{\vec{v}\in V^\perp,\,|\vec{v}|=1} |\mathbb{P}\vec{v}|^2 > 0.$$

Now, it is enough to choose

$$\varepsilon < \frac{\inf_{\vec{v}\in V^\perp,\,|\vec{v}|=1} |\mathbb{P}\vec{v}|^2}{\sup_{|\vec{v}|=1} |\mathbb{A}\vec{v}|^2}.$$

Then $\mathbb{M}_\varepsilon$ is positive definite, and the claim follows. □

Remark A.6. Note that if the vector $\vec{v}$ depends on a spatial variable x, then

$$|\mathbb{P}\nabla\vec{v}| \geq C|\mathbb{A}\nabla\vec{v}| \quad \text{for all } \vec{v} \in \mathbb{R}^N.$$

Appendix B. Example for the free energy

We show that the free energy density

$$h_\theta(\vec{\rho}) = \theta \sum_{i=1}^N \frac{\rho_i}{m_i} \log \frac{\rho_i}{m_i} + \left(\sum_{i=1}^n \frac{\rho_i}{m_i}\right)^\gamma - c_W \theta \log \theta, \tag{B.1}$$

where $\gamma > 1$, satisfies Hypothesis (H6). We compute the partial derivatives

$$\mu_i = \partial_i h_\theta(\vec{\rho}) = \frac{\theta}{m_i}\left(1 + \log \frac{\rho_i}{m_i}\right) + \frac{\gamma}{m_i}\left(\sum_{j=1}^N \frac{\rho_i}{m_i}\right)^{\gamma-1},$$

$$\partial_{ij}^2 h_\theta(\vec{\rho}) = \frac{\gamma(\gamma-1)}{m_i m_j}\left(\sum_{k=1}^N \frac{\rho_k}{m_k}\right)^{\gamma-2} + \frac{\theta}{\rho_i m_i}\delta_{ij},$$

$$\partial_\theta \partial_i h_\theta(\vec{\rho}) = \frac{1}{m_i}\left(1 + \log \frac{\rho_i}{m_i}\right),$$

where $\partial_i = \partial/\partial\rho_i$, $\partial_{ij}^2 = \partial^2/(\partial\rho_i\partial\rho_j)$, and $\partial_\theta = \partial/\partial\theta$. An induction argument with respect to N shows that the Hessian is positive definite for $\vec{\rho} \in \mathbb{R}_+^N$ and $\theta \in \mathbb{R}_+$. Moreover, $h_\theta(\vec{\rho})$ is smooth with respect to θ in this region and satisfies growth conditions (1.22), (1.23), (1.24) as well as (1.21). We write the Hessian $D^2 h_\theta(\vec{\rho})$ as the sum of the two matrices $\mathbb{A}$ and $\mathbb{B}$ with coefficients

$$\mathbb{A}_{ij} = \frac{\gamma(\gamma-1)}{m_i m_j}\left(\sum_{j=1}^N \frac{\rho_i}{m_i}\right)^{\gamma-2}, \quad \mathbb{B}_{ij} = \frac{\theta\delta_{ij}}{\rho_i m_i}.$$

By construction of h_θ^*, we have $D^2 h_\theta^*(\vec{\mu}) = D^2 h_\theta(\vec{\rho})^{-1}|_{\vec{\rho}=Dh_\theta^*(\vec{\mu})}$. Therefore,

$$D^2 h_\theta^*(\vec{\rho}) = (\mathbb{A} + \mathbb{B})^{-1} = \mathbb{B}^{-1/2}\big(\mathbb{I} + \mathbb{B}^{-1/2}\mathbb{A}\mathbb{B}^{-1/2}\big)^{-1}\mathbb{B}^{-1/2}. \tag{B.2}$$

We set $\mathbb{M} := \mathbb{B}^{-1/2}\mathbb{A}\mathbb{B}^{-1/2}$. We are able to compute $\mathbb{M}$ explicitly by using the power series of the function $(1+z)^{-1}$. Since

$$\mathbb{M}_{ij} = \frac{\gamma(\gamma-1)}{\theta}\bigg(\sum_{j=1}^N \frac{\rho_i}{m_i}\bigg)^{\gamma-2}\frac{\sqrt{\rho_i\rho_j}}{\sqrt{m_i m_j}} = \frac{\gamma(\gamma-1)}{\theta}\frac{\sqrt{\rho_i\rho_j}}{\sqrt{m_i m_j}}n^{\gamma-2},$$

where $n = \sum_{i=1}^N n_i = \sum_{i=1}^N \rho_i/m_i$, the coefficients $(\mathbb{M}^\ell)_{ij}$ of the matrix $\mathbb{M}^\ell$ with $\ell \in \mathbb{N}$ read as

$$(\mathbb{M}^\ell)_{ij} = \gamma^\ell(\gamma-1)^\ell\theta^{-\ell}\frac{\sqrt{\rho_i\rho_j}}{\sqrt{m_i m_j}}n^{\ell(\gamma-1)-1}.$$

Therefore, we obtain

$$\begin{aligned}
\big((\mathbb{I}+\mathbb{M})^{-1}\big)_{ij} &= \delta_{ij} + \sum_{\ell=1}^\infty(-1)^\ell(\mathbb{M}^\ell)_{ij} \\
&= \delta_{ij} + \frac{\sqrt{\rho_i\rho_j}}{\sqrt{m_i m_j}}n^{-1}\bigg(\sum_{\ell=0}^\infty(-1)^\ell\gamma^\ell(\gamma-1)^\ell\theta^{-\ell}n^{\ell(\gamma-1)} - 1\bigg) \\
&= \delta_{ij} + \frac{\sqrt{\rho_i\rho_j}}{\sqrt{m_i m_j}}n^{-1}\bigg(\frac{1}{1+\gamma(\gamma-1)\theta^{-1}n^{\gamma-1}} - 1\bigg) \\
&= \delta_{ij} - \frac{\sqrt{\rho_i\rho_j}}{\sqrt{m_i m_j}}\frac{\gamma(\gamma-1)\theta^{-1}n^{\gamma-2}}{1+\gamma(\gamma-1)\theta^{-1}n^{\gamma-1}}.
\end{aligned}$$

Since $h_\theta(\vec\rho)$ and $D^2h_\theta^*(\vec\rho)$ are smooth in θ for $\theta \in \mathbb{R}^+$, this yields (1.19). We deduce from (B.2) that

$$\partial_{ij}^2 h_\theta^*(\vec\mu) = \frac{m_i\rho_i}{\theta}\delta_{ij} - \frac{\gamma(\gamma-1)\theta^{-1}n^{\gamma-2}}{1+\gamma(\gamma-1)\theta^{-1}n^{\gamma-1}}\frac{\rho_i\rho_j}{\theta}.$$

If $\rho \le R$ then $n \le R/\min_{i=1,\ldots,N} m_i$ and we see that $|\theta\partial_{ij}^2 h_\theta^*(\vec\mu)| \le C(R)$ for some constant $C(R) > 0$ depending on R. Since $\rho = \sum_{i=1}^N \rho_i = \sum_{i=1}^N \partial_i h_\theta^*(\vec\mu)$, it remains to show that $\theta\sum_{i,j=1}^N \partial_{ij}^2 h_\theta^*(\vec\mu) \le K_2\rho$. This follows from

$$\theta\sum_{i,j=1}^N \partial_{ij}^2 h_\theta^*(\vec\mu) = \frac{\theta}{\theta}\sum_{i=1}^N m_i\rho_i - \frac{\theta}{\theta}\frac{\gamma(\gamma-1)n^{\gamma-2}\rho^2}{1+\gamma(\gamma-1)\theta^{-1}n^{\gamma-1}} \le \Big(\max_{i=1,\ldots,N} m_i\Big)\rho =: K_2\rho.$$

This shows that the free energy density (B.1) satisfies Hypothesis (H6).

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