ENTROPY-DISSIPATIVE DISCRETIZATION OF NONLINEAR DIFFUSION EQUATIONS AND DISCRETE BECKNER INEQUALITIES

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Abstract. The time decay of fully discrete finite-volume approximations of porous-medium and fast-diffusion equations with Neumann or periodic boundary conditions is proved in the entropy sense. The algebraic or exponential decay rates are computed explicitly. In particular, the numerical scheme dissipates all zeroth-order entropies which are dissipated by the continuous equation. The proofs are based on novel continuous and discrete generalized Beckner inequalities. Furthermore, the exponential decay of some first-order entropies is proved in the continuous and discrete case using systematic integration by parts. Numerical experiments in one and two space dimensions illustrate the theoretical results and indicate that some restrictions on the parameters seem to be only technical.

1. Introduction

This paper is concerned with the time decay of fully discrete finite-volume solutions to the nonlinear diffusion equation

$$u_t = \Delta(u^\beta) \quad \text{in } \Omega,\ t > 0, \quad u(\cdot, 0) = u_0 \quad \text{in } \Omega, \tag{1}$$

and with the relation to discrete generalized Beckner inequalities. Here, $\beta > 0$ and $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) is a bounded domain. When $\beta > 1$, (1) is called the porous-medium equation, describing the flow of an isentropic gas through a porous medium [35]. Equation (1) with $\beta < 1$ is referred to as the fast-diffusion equation, which appears, for instance, in plasma physics with $\beta = \frac{1}{2}$ [5] or in semiconductor theory with $0 < \beta < 1$ [26]. We impose homogeneous Neumann boundary conditions

$$\nabla(u^\beta) \cdot \nu = 0 \quad \text{on } \partial\Omega,\ t > 0, \tag{2}$$

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where ν denotes the unit normal exterior vector to $\partial\Omega$, or multiperiodic boundary conditions (i.e. Ω equals the torus $\mathbb{T}^d$). Let us denote by m the Lebesgue measure in $\mathbb{R}^d$ or $\mathbb{R}^{d-1}$; we assume for simplicity that $\mathrm{m}(\Omega) = 1$. For existence and uniqueness results for the porous-medium equation in the whole space or under suitable boundary conditions, we refer to the monograph [35].

In the literature, there exist many numerical schemes for nonlinear diffusion equations related to (1). Numerical techniques include (mixed) finite-element methods [1, 15, 33], finite-volume approximations [19, 32], high-order relaxation ENO-WENO schemes [11], or particle methods [30]. In these references, also stability and numerical convergence properties are proved.

The preservation of the structure of diffusion equations is a very important property of a numerical scheme. For instance, ideas employed for hyperbolic conservation laws were extended to degenerate diffusion equations, like the porous-medium equation, which may behave like hyperbolic ones in the regions of degeneracy [31]. Positivity-preserving schemes for nonlinear fourth-order equations were thoroughly investigated in the context of lubrication-type equations [3, 37] and quantum diffusion equations [25]. Entropy-consistent finite-volume finite-element schemes for the fourth-order thin-film equation were suggested by Grün and Rumpf [22]. For quantum diffusion models, an entropy-dissipative relaxation-type finite-difference discretization was investigated by Carrillo et al. [9]. Furthermore, entropy-dissipative schemes for electro-reaction-diffusion systems were derived by Glitzky and Gärtner [20]. However, it seems that there does not exist any systematic study on entropy-dissipative discretizations for (1) and the time decay of their discrete solutions.

The main aim of this paper is to provide some results on this time decay and to give estimates on the decay rates. To this end, we adapt the proofs for the continuous case to the discrete situation. Only those proofs are chosen which can be directly "translated" in a finite-volume context. Therefore, a significant part of this paper is concerned with the continuous case; however, the corresponding proofs will be also used in the discrete situation. In the following, we detail the main methods and results.

Our first objective is to prove that the finite-volume scheme for (1)-(2), defined in (30), dissipates the discrete versions of the functionals

$$E_\alpha[u] = \frac{1}{\alpha+1}\left(\int_\Omega u^{\alpha+1}dx - \left(\int_\Omega u dx\right)^{\alpha+1}\right), \tag{3}$$

$$F_\alpha[u] = \frac{1}{2}\int_\Omega |\nabla u^{\alpha/2}|^2 dx, \quad \alpha > 0. \tag{4}$$

In fact, we will prove (algebraic or exponential) convergence rates at which the discrete functionals converge to zero as $t \to \infty$. We call E_α a zeroth-order entropy and F_α a first-order entropy. The functional F_1 is known as the Fisher information, used in mathematical statistics and information theory [16]. Our analysis of the decay rates of the entropies will be guided by the entropy-dissipation method. An essential ingredient of this technique is a functional inequality relating the entropy to the entropy dissipation [2, 8]. For the diffusion equation (1), this relation is realized by the Beckner inequality [4].

The entropy-dissipation method was applied to (1) in the whole space to prove the decay of the solutions to the asymptotic self-similar profile in, e.g., [10, 12]. The convergence towards the constant steady state on the one-dimensional torus was proved in [7]. However, we are not aware of general entropy decay estimates for solutions to (1) to the constant steady state, even in the continuous case. The reason might be that generalizations to the Beckner inequality, needed to relate the entropy dissipation to the entropy, are missing. As our second aim, we prove these generalized Beckner inequalities and provide some decay estimates for E_α and F_α along trajectories of (1).

This paper splits into two parts. The first part is concerned with the proof of generalized Beckner inequalities and the decay rates for the continuous case. The second—and main—part is the "translation" of these results to an implicit Euler finite-volume discretization of (1). In the following, we summarize our main results.

The *first result* is the proof of the generalized Beckner inequality

$$\int_\Omega |f|^q dx - \left(\int_\Omega |f|^{1/p} dx\right)^{pq} \le C_B(p,q)\|\nabla f\|^q_{L^2(\Omega)}, \tag{5}$$

where $f \in H^1(\Omega)$ and $0 < q \le 2$, $pq \ge 1$. In the case $q = 2$, we require that $\frac{1}{2} - \frac{1}{d} \le p \le 1$. The constant $C_B(p,q) > 0$ only depends on p, q, and the constant of the Poincaré-Wirtinger inequality (see Lemma 2 for details). The usual Beckner inequality [4] is recovered for $q = 2$; see Remark 3 for a comparison of related Beckner inequalities in the literature. The proof is elementary and only employs the Poincaré-Wirtinger inequality. By using a discrete version of this inequality (see [6]), the proof can be easily "translated" to derive the discrete generalized Beckner inequality

$$\int_\Omega |f|^q dx - \left(\int_\Omega |f|^{1/p} dx\right)^{pq} \le C_b(p,q)|f|^q_{1,2,\mathcal{T}},$$

where f is a function which is constant on each cell of the finite-volume triangulation $\mathcal{T}$ of Ω and $|\cdot|_{1,2,\mathcal{T}}$ is the discrete H^1-seminorm; see Section 3.1 and Lemma 12 for details.

The *second result* is the time decay of the entropies E_α and F_α along trajectories of (1). Differentiating $E_\alpha[u(t)]$ with respect to time and employing the Beckner inequality (5), we show for $\beta > 1$ that

$$\frac{dE_\alpha}{dt}[u(t)] \le CE_\alpha[u(t)]^{(\alpha+\beta)/(\alpha+1)}, \quad t > 0,$$

where $C > 0$ depends on α, β, and $C_B(p,q)$. By a nonlinear Gronwall inequality, this implies the algebraic decay of $u(t)$ to equilibrium in the entropy sense; see Theorem 5. If the solution is positive and $0 < \alpha \le 1$, the above inequality becomes

$$\frac{dE_\alpha}{dt}[u(t)] \le C(u_0)E_\alpha[u(t)], \quad t > 0,$$

which results in an exponential decay rate; see Theorem 6.

The first-order entropies $F_\alpha[u(t)]$ decay exponentially fast (for positive solutions) for all (α,β) lying in the strip $-2 \le \alpha - 2\beta \le 1$ (one-dimensional case) or in the region M_d, which is illustrated in Figure 1 below (multi-dimensional case); see Theorems 7 and 8. The proof is based on systematic integration by parts [23]. In order to avoid boundary integrals arising from the iterated integrations by parts, these results are valid only if $\Omega = \mathbb{T}^d$. Notice that all these results are new.

The *third—and main—result* is the "translation" of the continuous decay rates to the finite-volume approximation. We obtain the same results for a discrete version of E_α in Theorems 14 (algebraic decay) and 15 (exponential decay). The situation is different for the first-order entropies F_α. The reason is that it is very difficult to "translate" the iterated integrations by parts to iterated summations by parts since there is no discrete nonlinear chain rule. For the zeroth-order entropies, this is done by exploiting the convexity of the mapping $x \mapsto x^{\alpha+1}$. For the first-order entropies, we employ the concavity of the discrete version of dF_α/dt with respect to the time approximation parameter. We prove in Theorem 16 that for $1 \leq \alpha \leq 2$ and $\beta = \alpha/2$, the discrete first-order entropy is monotone (multi-dimensional case) and decays exponentially fast (one-dimensional case). We stress the fact that this is the first result in the literature on the decay of discrete first-order entropies.

Throughout this paper, we assume that the solutions to (1) are smooth and positive such that we can perform all the computations and integrations by parts. In particular, we avoid any technicalities due to the degeneracy ($\beta > 1$) or singularity ($\beta < 1$) in (1). Most of our results can be generalized to nonnegative weak solutions by using a suitable approximation scheme but details are left to the reader.

In this paper, we do not develop an efficient implementation and we do not perform a convergence analysis, since the scheme is rather standard. Our aim is of more theoretical interest. In fact, our results on the discrete decay rates contribute to the aim of developing and analyzing *structure-preserving* numerical schemes, and this is the main originality of the present work.

The paper is organized as follows. In Section 2, we investigate the continuous case. We prove two novel generalized Beckner inequalities in Section 2.1, the algebraic and exponential decay of $E_\alpha[u]$ in Section 2.2, and the exponential decay of $F_\alpha[u]$ in Section 2.3. The discrete situation is analyzed in Section 3. After introducing the finite-volume scheme in Section 3.1, the algebraic and exponential decay rates for the discrete version of $E_\alpha[u]$ is shown in Section 3.3, and the exponential decay of the discrete version of $F_\alpha[u]$ is proved in Section 3.4. In Section 4, we illustrate the theoretical results by numerical experiments in one and two space dimensions. They indicate that some of the restrictions on the parameters (α, β) seem to be only technical. In the appendix, a discrete nonlinear Gronwall lemma and some auxiliary inequalities are proved.

2. The continuous case

It is convenient to analyze first the continuous case before extending the ideas to the discrete situation. We prove new convex Sobolev inequalities and algebraic and exponential decay rates of the solutions to (1).

2.1. **Generalized Beckner inequalities.** We assume in this subsection that $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) is a bounded domain such that the Poincaré-Wirtinger inequality

$$\|f - \bar{f}\|_{L^2(\Omega)} \leq C_P \|\nabla f\|_{L^2(\Omega)} \tag{6}$$

for all $f \in H^1(\Omega)$ holds, where $\bar{f} = \mathrm{m}(\Omega)^{-1} \int_\Omega f dx$ and $C_P > 0$ only depends on d and Ω. This is the case if, for instance, Ω has the cone property [29, Theorem 8.11] or if $\partial\Omega$ is locally Lipschitz continuous [36, Theorem 1.3.4]. Suppose that $\mathrm{m}(\Omega) = 1$ (to shorten the proof). Before stating our main result, we prove the following lemma.

Lemma 1 (Generalized Poincaré-Wirtinger inequality). *Let $0 < q \leq 2$ and $f \in H^1(\Omega)$. Then*

$$\|f\|^q_{L^2(\Omega)} \leq C_P^q \|\nabla f\|^q_{L^2(\Omega)} + \|f\|^q_{L^q(\Omega)} \tag{7}$$

holds, where $C_P > 0$ is the constant of the Poincaré-Wirtinger inequality (6).

Proof. Let first $1 \leq q \leq 2$. The Poincaré-Wirtinger inequality (6)

$$\|f\|^2_{L^2(\Omega)} - \|f\|^2_{L^1(\Omega)} = \|f - \bar{f}\|^2_{L^2(\Omega)} \leq C_P^2 \|\nabla f\|^2_{L^2(\Omega)} \tag{8}$$

together with the Hölder inequality leads to

$$\|f\|^2_{L^2(\Omega)} \leq C_P^2 \|\nabla f\|^2_{L^2(\Omega)} + \|f\|^2_{L^q(\Omega)}. \tag{9}$$

Here we use the assumption $\mathrm{m}(\Omega) = 1$. Since $q/2 \leq 1$, it follows that

$$\|f\|^q_{L^2(\Omega)} \leq \left(C_P^2 \|\nabla f\|^2_{L^2(\Omega)} + \|f\|^2_{L^q(\Omega)}\right)^{q/2} \leq C_P^q \|\nabla f\|^q_{L^2(\Omega)} + \|f\|^q_{L^q(\Omega)},$$

which equals (7).

Next, let $0 < q < 1$. We claim that

$$a^{q/2} - a^{q-1} b^{1-q/2} \leq (a - b)^{q/2} \quad \text{for all } a \geq b > 0. \tag{10}$$

Indeed, setting $c = b/a$, this inequality is equivalent to

$$1 - c^{1-q/2} \leq (1 - c)^{q/2} \quad \text{for all } 0 < c \leq 1.$$

The function $g(c) = 1 - c^{1-q/2} - (1 - c)^{q/2}$ for $c \in [0, 1]$ satisfies $g(0) = g(1) = 0$ and $g''(c) = (q/2)(1 - q/2)(c^{-1-q/2} + (1 - c)^{q/2-2}) \geq 0$ for $c \in (0, 1)$, which implies that $g(c) \leq 0$, proving (10). Applying (10) to $a = \|f\|^2_{L^2(\Omega)}$ and $b = \|f\|^2_{L^1(\Omega)}$ and using (8), we find that

$$\|f\|^q_{L^2(\Omega)} - \|f\|^{2(q-1)}_{L^2(\Omega)} \|f\|^{2-q}_{L^1(\Omega)} \leq \left(\|f\|^2_{L^2(\Omega)} - \|f\|^2_{L^1(\Omega)}\right)^{q/2} \leq C_P^q \|\nabla f\|^q_{L^2(\Omega)}. \tag{11}$$

In order to get rid of the L^1 norm, we employ the interpolation inequality

$$\|f\|_{L^1(\Omega)} = \int_\Omega |f|^\theta |f|^{1-\theta} dx \leq \|f\|^\theta_{L^q(\Omega)} \|f\|^{1-\theta}_{L^2(\Omega)}, \tag{12}$$

where $\theta = q/(2 - q) < 1$. Since $(2 - q)\theta = q$ and $(2 - q)(1 - \theta) = 2(1 - q)$, (11) becomes

$$\|f\|^q_{L^2(\Omega)} - \|f\|^q_{L^q(\Omega)} \leq C_P^q \|\nabla f\|^q_{L^2(\Omega)},$$

which is the desired inequality. □

Lemma 2 (Generalized Beckner inequality I)**.** *Let $d \geq 1$, $0 < q < 2$, $pq \geq 1$ or $q = 2$, $\frac{1}{2} - \frac{1}{d} \leq p \leq 1$ ($0 < p \leq 1$ if $d \leq 2$), and let $f \in H^1(\Omega)$. Then the generalized Beckner inequality*

$$\int_\Omega |f|^q dx - \left(\int_\Omega |f|^{1/p} dx\right)^{pq} \leq C_B(p,q)\|\nabla f\|^q_{L^2(\Omega)} \tag{13}$$

holds, where

$$C_B(p,q) = \frac{2(pq-1)C_P^q}{2-q} \quad \text{if } q < 2, \quad C_B(p,2) = C_P^2 \quad \text{if } q = 2,$$

and $C_P > 0$ is the constant of the Poincaré-Wirtinger inequality (6)*.*

Remark 3. The case $q = 2$ corresponds to the usual Beckner inequality [4]

$$\int_\Omega |f|^2 dx - \left(\int_\Omega |f|^{2/r} dx\right)^r \leq C_B(p,2)\|\nabla f\|^2_{L^2(\Omega)},$$

where $1 \leq r = 2p \leq 2$. It is shown in [14] that the constant $C_B(p,2)$ can be related to the lowest positive eigenvalue of a Schrödinger operator if Ω is convex. On the one-dimensonal torus, the generalized Beckner inequality (13) for $p > 0$ and $0 < q < 2$ has been derived in [7]. In the multi-dimensional situation, the special case $p = 2/q$ was proved in [13]. In this work, it was also shown that (13) with $q > 2$ and $p = 2/q$ cannot be true unless the Lebesgue measure dx is replaced by the Dirac measure. In the limit $pq \to 1$, (13) leads to a generalized logarithmic Sobolev inequality (see (15) below). If $q = 2$ in this limit, the usual logarithmic Sobolev inequality [21] is obtained. □

Proof of Lemma 2. Let first $q = 2$. Then the Beckner inequality is a consequence of the Poincaré-Wirtinger inequality (6) and the Jensen inequality:

$$C_P^2\|\nabla f\|^2_{L^2(\Omega)} \geq \|f - \bar{f}\|^2_{L^2(\Omega)} = \|f\|^2_{L^2(\Omega)} - \|f\|^2_{L^1(\Omega)} \geq \int_\Omega f^2 dx - \left(\int_\Omega |f|^{2/r} dx\right)^r,$$

where $1 - \frac{2}{d} \leq r \leq 2$ ($0 < r \leq 2$ if $d \leq 2$). The lower bound for r ensures that the embedding $H^1(\Omega) \hookrightarrow L^{2/r}(\Omega)$ is continuous. The choice $p = r/2 \in [\frac{1}{2} - \frac{1}{d}, 1]$ yields the formulation (13).

Next, let $0 < q < 2$. The first part of the proof is inspired by the proof of Proposition 2.2 in [13]. Taking the logarithm of the interpolation inequality

$$\|f\|_{L^r(\Omega)} \leq \|f\|^{\theta(r)}_{L^q(\Omega)}\|f\|^{1-\theta(r)}_{L^2(\Omega)},$$

where $q \leq r \leq 2$ and $\theta(r) = q(2-r)/(r(2-q))$, gives

$$F(r) := \frac{1}{r}\log\int_\Omega |f|^r dx - \frac{\theta(r)}{q}\log\int_\Omega |f|^q dx - \frac{1-\theta(r)}{2}\log\int_\Omega |f|^2 dx \leq 0.$$

The function $F : [q,2] \to \mathbb{R}$ is nonpositive, differentiable and $F(q) = 0$. Therefore, $F'(q) \leq 0$, which equals

$$-\frac{1}{q^2}\log\int_\Omega |f|^q dx + \frac{1}{q}\left(\int_\Omega |f|^q dx\right)^{-1}\int_\Omega |f|^q \log|f| dx$$

$$+ \theta'(q)\left(\frac{1}{2}\log\int_\Omega |f|^2dx - \frac{1}{q}\log\int_\Omega |f|^q dx\right) \le 0.$$

We multiply this inequality by $q^2\int_\Omega |f|^q dx$ to obtain

$$\int_\Omega |f|^q \log\frac{|f|^q}{\|f\|^q_{L^q(\Omega)}}dx \le \frac{2}{2-q}\|f\|^q_{L^q(\Omega)}\log\frac{\|f\|^q_{L^2(\Omega)}}{\|f\|^q_{L^q(\Omega)}}. \tag{14}$$

Then, we employ Lemma 1 and the inequality $\log(x+1)\le x$ for $x\ge 0$ to infer that

$$\|f\|^q_{L^q(\Omega)}\log\frac{\|f\|^q_{L^2(\Omega)}}{\|f\|^q_{L^q(\Omega)}} \le \|f\|^q_{L^q(\Omega)}\log\left(\frac{C_P^q\|\nabla f\|^q_{L^2(\Omega)}}{\|f\|^q_{L^q(\Omega)}}+1\right) \le C_P^q\|\nabla f\|^q_{L^2(\Omega)}.$$

Combining this inequality and (14), we conclude the generalized logarithmic Sobolev inequality

$$\int_\Omega |f|^q\log\frac{|f|^q}{\|f\|^q_{L^q(\Omega)}}dx \le \frac{2C_P^q}{2-q}\|\nabla f\|^q_{L^2(\Omega)}. \tag{15}$$

The generalized Beckner inequality (13) is derived by extending slightly the proof of [27, Corollary 1]. Let

$$G(r) = r\log\int_\Omega |f|^{q/r}dx, \quad r\ge 1.$$

The function G is twice differentiable with

$$G'(r) = \left(\int_\Omega |f|^{q/r}dx\right)^{-1}\left(\int_\Omega |f|^{q/r}dx\log\int_\Omega |f|^{q/r}dx - \frac{q}{r}\int_\Omega |f|^{q/r}\log|f|dx\right),$$

$$G''(r) = \frac{q^2}{r^3}\left(\int_\Omega |f|^{q/r}dx\right)^{-2}\left(\int_\Omega |f|^{q/r}dx\int_\Omega |f|^{q/r}(\log|f|)^2dx - \left(\int_\Omega |f|^{q/r}\log|f|dx\right)^2\right).$$

The Cauchy-Schwarz inequality shows that $G''(r)\ge 0$, i.e., G is convex. Consequently, $r\mapsto e^{G(r)}$ is also convex and $r\mapsto H(r) = -(e^{G(r)}-e^{G(1)})/(r-1)$ is nonincreasing on $(1,\infty)$, which implies that

$$H(r) \le \lim_{t\to 1}H(t) = -e^{G(1)}G'(1) = \int_\Omega |f|^q\log\frac{|f|^q}{\|f\|^q_{L^q(\Omega)}}dx.$$

This inequality is equivalent to

$$\frac{1}{r-1}\left(\int_\Omega |f|^q dx - \left(\int_\Omega |f|^{q/r}dx\right)^r\right) \le \int_\Omega |f|^q\log\frac{|f|^q}{\|f\|^q_{L^q(\Omega)}}dx. \tag{16}$$

Combining this inequality and the generalized logarithmic Sobolev inequality (15), it follows that

$$\int_\Omega |f|^q dx - \left(\int_\Omega |f|^{q/r}dx\right)^r \le \frac{2(r-1)C_P^q}{2-q}\|\nabla f\|^q_{L^2(\Omega)}$$

for all $0<q<2$ and $r\ge 1$. Setting $p := r/q$, this proves (13) for all $pq = r\ge 1$. □

For the proof of exponential decay rates, we need the following variant of the Beckner inequality.

Lemma 4 (Generalized Beckner inequality II). *Let* $0 < q < 2$, $pq \geq 1$ *and* $f \in H^1(\Omega)$. *Then*

$$\|f\|_{L^q(\Omega)}^{2-q}\left(\int_\Omega |f|^q dx - \left(\int_\Omega |f|^{1/p}dx\right)^{pq}\right) \leq C_B'(p,q)\|\nabla f\|_{L^2(\Omega)}^2, \tag{17}$$

where

$$C_B'(p,q) = \begin{cases} \dfrac{q(pq-1)C_P^2}{2-q} & \text{if } 1 \leq q < 2, \\ (pq-1)C_P^2 & \text{if } 0 < q < 1. \end{cases}$$

Proof. By (14), it holds that for all $0 < q < 2$,

$$\int_\Omega |f|^q \log \frac{|f|^q}{\|f\|_{L^q(\Omega)}^q} dx \leq \frac{q}{2-q}\|f\|_{L^q(\Omega)}^q \log \frac{\|f\|_{L^2(\Omega)}^2}{\|f\|_{L^q(\Omega)}^2}.$$

Then, for $q > 1$, the Poincaré-Wirtinger inequality in the version (9) and the inequality $\log(x+1) \leq x$ for $x \geq 0$ yield

$$\|f\|_{L^q(\Omega)}^q \log \frac{\|f\|_{L^2(\Omega)}^2}{\|f\|_{L^q(\Omega)}^2} \leq \|f\|_{L^q(\Omega)}^q \log\left(C_P^2 \frac{\|\nabla f\|_{L^2(\Omega)}^2}{\|f\|_{L^q(\Omega)}^2} + 1\right) \leq C_P^2 \|f\|_{L^q(\Omega)}^{q-2}\|\nabla f\|_{L^2(\Omega)}^2. \tag{18}$$

Taking into account (16), the conclusion follows for $q > 1$.

Let $0 < q \leq 1$. Suppose that the following inequality holds:

$$\|f\|_{L^q(\Omega)}^2 + \frac{2-q}{q}C_P^2\|\nabla f\|_{L^2(\Omega)}^2 - \|f\|_{L^2(\Omega)}^2 \geq 0. \tag{19}$$

This implies that, by (16) and for $r = pq$,

$$\begin{aligned}\int_\Omega |f|^q dx - \left(\int_\Omega |f|^{q/r}dx\right)^r &\leq \frac{(pq-1)q}{2-q}\|f\|_{L^q(\Omega)}^q \log \frac{\|f\|_{L^2(\Omega)}^2}{\|f\|_{L^q(\Omega)}^2} \\ &\leq \frac{(pq-1)q}{2-q}\|f\|_{L^q(\Omega)}^q \log\left(\frac{(2-q)C_P^2}{q}\frac{\|\nabla f\|_{L^2(\Omega)}^2}{\|f\|_{L^q(\Omega)}^2} + 1\right) \\ &\leq (pq-1)C_P^2\|\nabla f\|_{L^2(\Omega)}^2\|f\|_{L^q(\Omega)}^{q-2},\end{aligned}$$

which shows the desired Beckner inequality.

It remains to prove (19). For this, we employ the Poincaré-Wirtinger inequality (8)

$$C_P^2\|\nabla f\|_{L^2(\Omega)}^2 \geq \|f\|_{L^2(\Omega)}^2 - \|f\|_{L^1(\Omega)}^2$$

and the interpolation inequality (12) in the form

$$\|f\|_{L^q(\Omega)}^2 \geq \|f\|_{L^1(\Omega)}^{2/\theta}\|f\|_{L^2(\Omega)}^{2(\theta-1)/\theta}, \quad \theta = \frac{q}{2-q} \leq 1,$$

to obtain

$$\begin{aligned}\|f\|_{L^q(\Omega)}^2 &+ \frac{2-q}{q}C_P^2\|\nabla f\|_{L^2(\Omega)}^2 - \|f\|_{L^2(\Omega)}^2 \\ &\geq \|f\|_{L^1(\Omega)}^{2/\theta}\|f\|_{L^2(\Omega)}^{2(\theta-1)/\theta} + \left(\frac{2-q}{q} - 1\right)\|f\|_{L^2(\Omega)}^2 - \frac{2-q}{q}\|f\|_{L^1(\Omega)}^2.\end{aligned}$$

We interpret the right-hand side as a function G of $\|f\|_{L^1(\Omega)}^2$. Then, setting $A = \|f\|_{L^2(\Omega)}^2$,

$$\begin{aligned} G(y) &= y^{1/\theta}A^{1-1/\theta} + \frac{2(1-q)}{q}A - \frac{2-q}{q}y, \\ G'(y) &= \frac{1}{\theta}y^{1/\theta-1}A^{1-1/\theta} - \frac{2-q}{q}, \\ G''(y) &= \frac{1}{\theta}\left(\frac{1}{\theta} - 1\right)y^{1/\theta-2}A^{1-1/\theta} \geq 0,\end{aligned}$$

Therefore, G is a convex function which satisfies $G(A) = 0$ and $G'(A) = 0$. This implies that G is a nonnegative function on $\mathbb{R}^+$ and in particular, $G(\|f\|_{L^1(\Omega)}^2) \geq 0$. This proves (19), completing the proof. □

2.2. **Zeroth-order entropies.** Let u be a smooth solution to (1)-(2) and let $u_0 \in L^\infty(\Omega)$, $\inf_\Omega u_0 \geq 0$ in Ω. By the maximum principle, $0 \leq \inf_\Omega u_0 \leq u(t) \leq \sup_\Omega u_0$ in Ω for $t \geq 0$. First, we prove algebraic decay rates for $E_\alpha[u]$, defined in (3).

Theorem 5 (Polynomial decay for E_α)**.** *Let $\alpha > 0$ and $\beta > 1$. Let u be a smooth solution to* (1)-(2) *and $u_0 \in L^\infty(\Omega)$ with $\inf_\Omega u_0 \geq 0$. Then*

$$E_\alpha[u(t)] \leq \frac{1}{(C_1t + C_2)^{(\alpha+1)/(\beta-1)}}, \quad t \geq 0,$$

where

$$C_1 = \frac{4\alpha\beta(\beta-1)}{(\alpha+1)(\alpha+\beta)^2}\left(\frac{\alpha+1}{C_B(p,q)}\right)^{(\alpha+\beta)/(\alpha+1)}, \quad C_2 = E_\alpha[u_0]^{-(\beta-1)/(\alpha+1)},$$

and $C_B(p,q) > 0$ is the constant in the Beckner inequality for $p = (\alpha+\beta)/2$ and $q = 2(\alpha+1)/(\alpha+\beta)$.

Proof. We apply Lemma 2 with $p = (\alpha+\beta)/2$ and $q = 2(\alpha+1)/(\alpha+\beta)$. The assumptions on α and β guarantee that $0 < q < 2$ and $pq > 1$. Then, with $f = u^{(\alpha+\beta)/2}$,

$$E_\alpha[u] = \frac{1}{\alpha+1}\left(\int_\Omega u^{\alpha+1}dx - \left(\int_\Omega udx\right)^{\alpha+1}\right) \leq \frac{C_B(p,q)}{\alpha+1}\left(\int_\Omega |\nabla u^{(\alpha+\beta)/2}|^2dx\right)^{(\alpha+1)/(\alpha+\beta)}.$$

Now, computing the derivative,

$$\frac{dE_\alpha}{dt} = -\int_\Omega \nabla u^\alpha \cdot \nabla u^\beta dx = -\frac{4\alpha\beta}{(\alpha+\beta)^2}\int_\Omega |\nabla u^{(\alpha+\beta)/2}|^2dx$$

$$\leq -\frac{4\alpha\beta}{(\alpha+\beta)^2}\left(\frac{\alpha+1}{C_B(p,q)}\right)^{(\alpha+\beta)/(\alpha+1)} E_\alpha[u]^{(\alpha+\beta)/(\alpha+1)}.$$

An integration of this inequality gives the assertion. □

Next, we show exponential decay rates.

Theorem 6 (Exponential decay for E_α)**.** *Let u be a smooth solution to* (1)-(2), $0<\alpha\leq 1$, $\beta>0$, $u_0\in L^\infty(\Omega)$ *with* $\inf_\Omega u_0\geq 0$. *Then*

$$E_\alpha[u(t)]\leq E_\alpha[u_0]e^{-\Lambda t},\quad t\geq 0.$$

The constant Λ is given by

$$\Lambda=\frac{4\alpha\beta}{C_B(\frac12(\alpha+1),2)(\alpha+1)}\inf_\Omega(u_0^{\beta-1})\geq 0,$$

for $\beta>0$ and

$$\Lambda=\frac{4\alpha\beta(\alpha+1)}{C_B'(p,q)(\alpha+\beta)^2}\|u_0\|_{L^1(\Omega)}^{\beta-1},$$

for $\beta>1$. Here, $C_B(\frac12(\alpha+1),2)$ and $C_B'(p,q)$ are the constants in the Beckner inequalities (13) *and* (17)*, respectively, with $p=(\alpha+\beta)/2$ and $q=2(\alpha+1)/(\alpha+\beta)$.*

Proof. Let $\beta>0$. We compute

$$\begin{aligned}\frac{dE_\alpha}{dt}&=-\frac{4\alpha\beta}{(\alpha+1)^2}\int_\Omega u^{\beta-1}|\nabla u^{(\alpha+1)/2}|^2dx\\ &\leq -\frac{4\alpha\beta}{(\alpha+1)^2}\inf_\Omega(u_0^{\beta-1})\int_\Omega|\nabla u^{(\alpha+1)/2}|^2dx.\end{aligned}$$

By the Beckner inequality (13) with $p=(\alpha+1)/2$, $q=2$, and $f=u^{(\alpha+1)/2}$, we find that

$$\frac{dE_\alpha}{dt}\leq -\frac{4\alpha\beta}{C_B(p,2)(\alpha+1)}\inf_\Omega(u_0^{\beta-1})E_\alpha,$$

and Gronwall's lemma proves the claim. The restriction $p\leq 1$ in Lemma 2 is equivalent to $\alpha\leq 1$.

Next, let $\beta>1$. By Lemma 4, with $p=(\alpha+\beta)/2$, $q=2(\alpha+1)/(\alpha+\beta)$, and $f=u^{(\alpha+\beta)/2}$, it follows that

$$\|u\|_{L^{\alpha+1}(\Omega)}^{\beta-1}\left(\int_\Omega u^{\alpha+1}dx-\left(\int_\Omega udx\right)^{\alpha+1}\right)\leq C_B'(p,q)\int_\Omega|\nabla u^{(\alpha+\beta)/2}|^2dx.$$

Hence, we can estimate

$$\begin{aligned}\frac{dE_\alpha}{dt}=-\frac{4\alpha\beta}{(\alpha+\beta)^2}\int_\Omega|\nabla u^{(\alpha+\beta)/2}|^2dx&\leq -\frac{4\alpha\beta(\alpha+1)}{(\alpha+\beta)^2}\frac{\|u\|_{L^{\alpha+1}(\Omega)}^{\beta-1}}{C_B'(p,q)}E_\alpha[u]\\ &\leq -\frac{4\alpha\beta(\alpha+1)}{(\alpha+\beta)^2}\frac{\|u_0\|_{L^1(\Omega)}^{\beta-1}}{C_B'(p,q)}E_\alpha[u],\end{aligned}$$

and Gronwall's lemma gives the conclusion. Note that in the last step of the inequality we used that $\|u\|_{L^{\alpha+1}(\Omega)} \geq \|u\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)}$. □

2.3. **First-order entropies.** In this section, we consider the diffusion equation (1) on the torus $\Omega = \mathbb{T}^d$. We prove the exponential decay for the first-order entropies (4).

Theorem 7 (Exponential decay of F_α in one space dimension)**.** *Let u be a smooth solution to* (1) *on the one-dimensional torus $\Omega = \mathbb{T}$. Let $u_0 \in L^\infty(\Omega)$ with $\inf_\Omega u_0 \geq 0$ and let $\alpha, \beta > 0$ satisfy $-2 \leq \alpha - 2\beta < 1$. Then*

$$F_\alpha[u(t)] \leq F_\alpha[u_0]e^{-\Lambda t}, \quad 0 \leq t \leq T,$$

where

$$\Lambda = \frac{2\beta}{C_P^2}\inf_\Omega(u_0^{\alpha+\beta-\gamma-1})\inf_\Omega(u_0^{\gamma-\alpha}) \geq 0, \quad \gamma = \frac{2}{3}(\alpha+\beta-1),$$

where $C_P > 0$ is the Poincaré constant in (6)*.*

Proof. We extend slightly the entropy construction method of [23]. The time derivative of the entropy reads as

$$\begin{aligned}\frac{dF_\alpha}{dt} &= \frac{\alpha}{2}\int_\Omega (u^{\alpha/2})_x(u^{\alpha/2-1}u_t)_x dx = -\frac{\alpha}{2}\int_\Omega (u^{\alpha/2})_{xx}u^{\alpha/2-1}(u^\beta)_{xx}dx \\ &= -\frac{\alpha^2\beta}{4}\int_\Omega u^{\alpha+\beta-1}\left(\left(\frac{\alpha}{2}-1\right)(\beta-1)\xi_G^4 + \left(\frac{\alpha}{2}+\beta-2\right)\xi_G^2\xi_L + \xi_L^2\right)dx,\end{aligned}$$

where we introduced

$$\xi_G = \frac{u_x}{u}, \quad \xi_L = \frac{u_{xx}}{u}.$$

This integral is compared to

$$\int_\Omega u^{\alpha+\beta-\gamma-1}(u^{\gamma/2})_{xx}^2 dx = \frac{\gamma^2}{4}\int_\Omega u^{\alpha+\beta-1}\left(\left(\frac{\gamma}{2}-1\right)^2\xi_G^4 + (\gamma-2)\xi_G^2\xi_L + \xi_L^2\right)dx,$$

where, compared to the method of [23], $\gamma \neq 0$ gives an additional degree of freedom in the calculations. In the one-dimensional situation, there is only one relevant integration-by-parts rule:

$$0 = \int_\Omega (u^{\alpha+\beta-4}u_x^3)_x dx = \int_\Omega u^{\alpha+\beta-1}\big((\alpha+\beta-4)\xi_G^4 + 3\xi_G^2\xi_L\big)dx.$$

We introduce the polynomials

$$S_0(\xi) = \left(\frac{\alpha}{2}-1\right)(\beta-1)\xi_G^4 + \left(\frac{\alpha}{2}+\beta-2\right)\xi_G^2\xi_L + \xi_L^2, \tag{20}$$

$$D_0(\xi) = \left(\frac{\gamma}{2}-1\right)^2\xi_G^4 + (\gamma-2)\xi_G^2\xi_L + \xi_L^2, \tag{21}$$

$$T(\xi) = (\alpha+\beta-4)\xi_G^4 + 3\xi_G^2\xi_L,$$

where $\xi = (\xi_G, \xi_L)$. We wish to show that there exist numbers $c, \gamma \in \mathbb{R}$ ($\gamma \neq 0$) and $\kappa > 0$ such that

$$S(\xi) = S_0(\xi) + cT(\xi) - \kappa D_0(\xi) \geq 0 \quad \text{for all } \xi \in \mathbb{R}^2.$$

The polynomial S can be written as $S(\xi) = a_1\xi_G^4 + a_2\xi_G^2\xi_L + (1-\kappa)\xi_L^2$, where

$$a_1 = -\frac{1}{4}(\gamma-2)^2\kappa + (\alpha+\beta-4)c + \frac{1}{2}(\alpha-2)(\beta-1),$$
$$a_2 = -(\gamma-2)\kappa + 3c + \frac{1}{2}(\alpha+2\beta-4).$$

Therefore, the maximal value for κ is $\kappa = 1$. Let $\kappa = 1$. Then we need to eliminate the mixed term $\xi_G^2\xi_L$. The solution of $a_2 = 0$ is given by $c = -\frac{1}{6}(\alpha+2\beta-2\gamma)$, which leads to

$$a_1 = -\frac{1}{4}\left(\gamma - \frac{2}{3}(\alpha+\beta-1)\right)^2 - \frac{1}{18}(\alpha-2\beta-1)(\alpha-2\beta+2).$$

Choosing $\gamma = \frac{2}{3}(\alpha+\beta-1)$ to maximize a_1, we find that $a_1 \geq 0$ and hence $S(\xi) \geq 0$ if and only if $-2 \leq \alpha - 2\beta \leq 1$.

Using the Poincaré inequality (6) and the maximum principle, we obtain

$$\begin{aligned}
\frac{dF_\alpha}{dt} &= -\frac{\alpha^2\beta}{4}\int_\Omega u^{\alpha+\beta-1}S_0(\xi)dx = -\frac{\alpha^2\beta}{4}\int_\Omega u^{\alpha+\beta-1}(S_0(\xi)+cT(\xi))dx \\
&\leq -\frac{\alpha^2\beta}{4}\int_\Omega u^{\alpha+\beta-1}D_0(\xi)dx = -\frac{\alpha^2\beta}{\gamma^2}\int_\Omega u^{\alpha+\beta-\gamma-1}(u^{\gamma/2})_{xx}^2dx \\
&\leq -\frac{\alpha^2\beta}{\gamma^2}\inf_{\Omega\times(0,\infty)}(u^{\alpha+\beta-\gamma-1})\int_\Omega (u^{\gamma/2})_{xx}^2dx \\
&\leq -\frac{\alpha^2\beta}{\gamma^2C_P^2}\inf_\Omega(u_0^{\alpha+\beta-\gamma-1})\int_\Omega (u^{\gamma/2})_x^2dx \\
&\leq -\frac{2\beta}{C_P^2}\inf_\Omega(u_0^{\alpha+\beta-\gamma-1})\inf_\Omega(u_0^{\gamma-\alpha})F_\alpha.
\end{aligned}$$

For the last inequality, we use that $(u^{\gamma/2})_x = \frac{\gamma}{\alpha}u^{(\gamma-\alpha)/2}(u^{\alpha/2})_x$, which cancels out the ratio α^2/γ^2. An application of the Gronwall's lemma finishes the proof. □

We turn to the multi-dimensional case.

Theorem 8 (Exponential decay of F_α in several space dimensions)**.** *Let u be a smooth solution to* (1) *on the torus $\Omega = \mathbb{T}^d$. Let $u_0 \in L^\infty(\Omega)$ with $\inf_\Omega u_0 > 0$ and let*

$$(\alpha,\beta) \in M_d = \big\{(\alpha,\beta)\in\mathbb{R}^2 : (2-2\alpha+2\beta-d+\alpha d)(4-4\beta-2d+\alpha d+2\beta+2\beta d) > 0 \\ \text{and } (\alpha-2\beta-1)(\alpha-2\beta+2) < 0\big\}$$

(see Figure 1). Then there exists $\Lambda > 0$, depending on α, β, d, u_0, and Ω such that

$$F_\alpha[u(t)] \leq F_\alpha[u_0]e^{-\Lambda t}, \quad t \geq 0.$$

Proof. The time derivative of the first-order entropy becomes

$$\frac{dF_\alpha}{dt} = -\frac{\alpha}{2}\int_\Omega u^{\alpha/2-1}\Delta(u^{\alpha/2})\Delta(u^\beta)dx = -\frac{\alpha^2\beta}{4}\int_\Omega u^{\alpha+\beta-1}S_0dx, \tag{22}$$

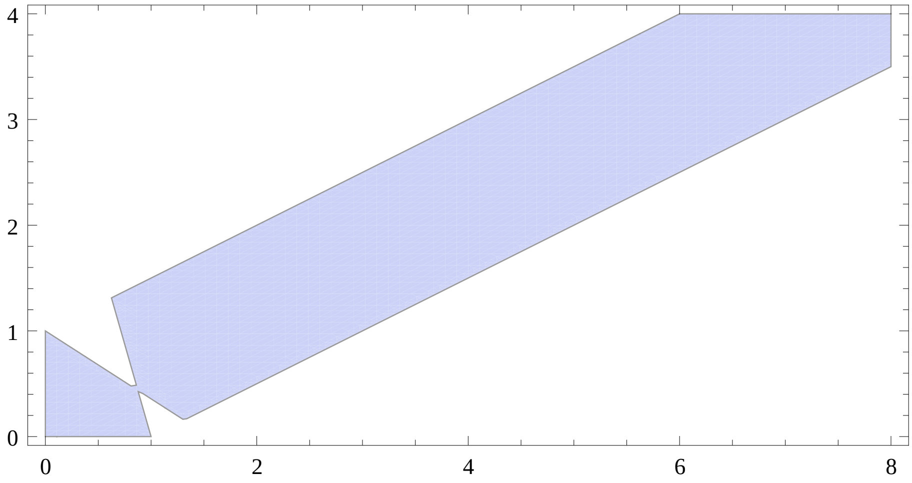

FIGURE 1. Illustration of the set M_d, defined in Theorem 8, for $d = 9$.

where S_0 is defined in (20) with the (scalar) variables $\xi_G = |\nabla u|/u$, $\xi_L = \Delta u/u$. We compare this integral to

$$\int_\Omega u^{\alpha+\beta-\gamma-1}(\Delta(u^{\gamma/2}))^2 dx = \frac{\gamma^2}{4}\int_\Omega u^{\alpha+\beta-1} D_0 dx,$$

where D_0 is as in (21) and $\gamma \neq 0$. In contrast to the one-dimensional case, we employ two integration-by-parts rules:

$$0 = \int_\Omega \operatorname{div}\left(u^{\alpha+\beta-4}|\nabla u|^2 \nabla u\right) dx = \int_\Omega u^{\alpha+\beta-1} T_1 dx,$$
$$0 = \int_\Omega \operatorname{div}\left(u^{\alpha+\beta-3}(\nabla^2 u - \Delta \mathbb{I}) \cdot \nabla u\right) dx = \int_\Omega u^{\alpha+\beta-1} T_2 dx,$$

where

$$T_1 = (\alpha+\beta-4)\xi_G^4 + 2\xi_{GHG} + \xi_G^2 \xi_L,$$
$$T_2 = (\alpha+\beta-3)\xi_{GHG} - (\alpha+\beta-3)\xi_G^2\xi_L + \xi_H^2 - \xi_L^2,$$

and $\xi_{GHG} = u^{-3}\nabla u^\top \nabla^2 u \nabla u$, $\xi_H = u^{-1}\|\nabla^2 u\|$. Here, $\|\nabla^2 u\|$ denotes the Frobenius norm of the hessian.
In order to compare $\nabla^2 u$ and Δu, we employ Lemma 2.1 of [24]:

$$\|\nabla^2 u\|^2 \geq \frac{1}{d}(\Delta u)^2 + \frac{d}{d-1}\left(\frac{\nabla u^\top \nabla^2 u \nabla u}{|\nabla u|^2} - \frac{\Delta u}{d}\right)^2.$$

Therefore, there exists $\xi_R \in \mathbb{R}$ such that

$$\xi_H^2 = \frac{\xi_L^2}{d} + \frac{d}{d-1}\left(\frac{\xi_{GHG}}{\xi_G^2} - \frac{1}{d}\xi_L\right)^2 + \xi_R^2 = \frac{\xi_L^2}{d} + \frac{d}{d-1}\xi_S^2 + \xi_R^2,$$

where we introduced $\xi_S = \xi_{GHG}/\xi_G^2 - \xi_L/d$. Rewriting the polynomials T_1 and T_2 in terms of $\xi = (\xi_G, \xi_L, \xi_S, \xi_R) \in \mathbb{R}^4$ leads to:

$$\begin{aligned}
T_1(\xi) &= (\alpha+\beta-4)\xi_G^4 + \frac{2+d}{d}\xi_G^2\xi_L + 2\xi_G^2\xi_S, \\
T_2(\xi) &= \frac{1-d}{d}(\alpha+\beta-3)\xi_G^2\xi_L + \frac{1-d}{d}\xi_L^2 + \xi_S\xi_G^2(\alpha+\beta-3) + \frac{d}{d-1}\xi_S^2 + \xi_R^2.
\end{aligned}$$

We wish to find c_1, c_2, $\gamma \in \mathbb{R}$ ($\gamma \neq 0$) and $\kappa > 0$ such that

$$S(\xi) = S_0(\xi) + c_1T_1(\xi) + c_2T_2(\xi) - \kappa D_0(\xi) \geq 0 \quad \text{for all } \xi \in \mathbb{R}^4.$$

The polynomial S can be written as

$$\begin{aligned}
S(\xi) &= a_1\xi_G^4 + a_2\xi_G^2\xi_L + a_3\xi_L^2 + a_4\xi_G^2\xi_S + a_5\xi_S^2 + c_2\xi_R^2, \text{ where} \\
a_1 &= \left(\frac{\alpha}{2}-1\right)(\beta-1) + (\alpha+\beta-4)c_1 - \left(\frac{\gamma}{2}-1\right)^2\kappa, \\
a_2 &= \frac{\alpha}{2} + \beta - 2 + \left(\frac{2}{d}+1\right)c_1 - (\alpha+\beta-3)\frac{d-1}{d}c_2 - (\gamma-2)\kappa, \\
a_3 &= 1 + \frac{1-d}{d}c_2 - \kappa, \\
a_4 &= 2c_1 + (\alpha+\beta-3)c_2, \\
a_5 &= \frac{d}{d-1}c_2.
\end{aligned}$$

We remove the variable ξ_R by requiring that $c_2 \geq 0$. The remaining polyomial can be reduced to a quadratic polynomial by setting $x = \xi_L/\xi_G^2$ and $y = \xi_S/\xi_G^2$:

$$S(x,y) \geq a_1 + a_2x + a_3x^2 + a_4y + a_5y^2 \geq 0 \quad \text{for all } x,\, y \in \mathbb{R}. \tag{23}$$

This quadratic decision problem can be solved by employing the computer algebra system `Mathematica`. The result of the command

```
Resolve[ForAll[{x, y}, Exists[{C1, C2, kappa, gamma},
a1 + a2*x + a3*x^2 + a4*y + a5*y^2 >= 0 && kappa > 0
&& gamma != 0]], Reals]
```

gives all $(\alpha, \beta) \in \mathbb{R}^2$ such that there exist c_1, c_2, $\gamma \in \mathbb{R}$ ($\gamma \neq 0$) and $\kappa > 0$ such that (23) holds. This region equals the set M_d, defined in the statement of the theorem.

Similar to the one-dimensional case, we infer that

$$\frac{dF_\alpha}{dt} \leq -\frac{\alpha^2\beta\kappa}{4}\int_\Omega u^{\alpha+\beta-1}D_0(\xi)dx = -\frac{\alpha^2\beta\kappa}{\gamma^2}\int_\Omega u^{\alpha+\beta-\gamma-1}(\Delta u^{\gamma/2})^2dx.$$

Thus, proceeding as in the proof of Theorem 7 and using the identity

$$\int_\Omega (\Delta f)^2dx = \int_\Omega \|\nabla^2 f\|^2dx$$

for smooth functions f (which can be obtained by integration by parts twice), we obtain

$$\frac{dF_\alpha}{dt} \le -\frac{2\beta\kappa}{C_P^2} \inf_\Omega (u_0^{\alpha+\beta-\gamma-1}) \inf_\Omega (u_0^{\gamma-\alpha}) F_\alpha.$$

Gronwall's lemma concludes the proof. □

Remark 9. Under the additional constraints $a_2 = a_3 = 0$, we are able to solve the decision problem (23) without the help of the computer algebra system. The solution set, however, is slightly smaller than M_d which is obtained from `Mathematica` without these constraints. Indeed, we can compute c_1 and c_2 from the equations $a_2 = a_3 = 0$ giving

$$c_1 = \frac{d}{d+2}\left(\frac{\alpha}{2} - 1 + \kappa(1+\gamma-\alpha-\beta)\right), \quad c_2 = \frac{d(1-\kappa)}{d-1}.$$

The decision problem (23) reduces to

$$a_1 + a_4 y + a_5 y^2 \ge 0 \quad \text{for all } y \in \mathbb{R}.$$

If $\kappa < 1$, it holds $c_2 > 0$ and consequently, $a_5 > 0$. Therefore, the above polynomial is nonnegative for all $y \in \mathbb{R}$ if it has no real roots, i.e., if

$$0 \le 4a_1a_5 - a_4^2 = q_0 + q_1\gamma + q_2\gamma^2$$

for some $\gamma \neq 0$, where (for $d > 1$)

$$q_2 = -\frac{d^2\kappa}{(d+2)^2(d-1)^2}\big(3d(d-4)\kappa + (d+2)^2\big) < 0,$$

and q_0, q_1 are polynomials depending on d, α, β, and κ. The above problem is solvable if and only if there exist real roots, i.e. if

$$0 \le q_1^2 - 4q_0q_2 = \frac{4\kappa(1-\kappa)}{(d+2)^2(d-1)^2}(s_0 + s_1\kappa + s_2\kappa^2),$$

where

$$\begin{aligned} s_0 &= -d(5d-8) + 6d(d-1)\alpha + 2d(d+2)\beta + 2(d+2)\alpha\beta - (2d^2+1)\alpha^2 - (d+2)^2\beta^2, \\ s_1 &= 2d(3d-4) - 2d(4d-3)\alpha - 4d(d+1)\beta + 2d(3d-5)\alpha\beta + 2d(d+1)\alpha^2 \\ &\quad - 2d(d-6)\beta^2, \\ s_2 &= -d^2(\alpha+\beta-1)^2. \end{aligned}$$

We set $f(\kappa) = s_0 + s_1\kappa + s_2\kappa^2$. We have to find $0 < \kappa < 1$ such that $f(\kappa) \ge 0$. Since $s_2 \le 0$, this is possible if $f(\kappa)$ possesses two (not necessarily distinct) real roots κ_0 and κ_1 and if at least one of these roots is between zero and one. Since $f(1) = -(d-1)^2(\alpha-2\beta)^2 \le 0$, there are only two possibilities for κ_0 and κ_1: either $\kappa_0 \le 0 \le \kappa_1 \le 1$ or $0 \le \kappa_0 \le \kappa_1 \le 1$. The first case holds if $f(0) = s_0 \ge 0$, the second one if

$$f'(0) = s_1 \ge 0, \quad f'(1) = s_1 + 2s_2 \le 0, \tag{24}$$

$$\begin{aligned} s_1^2 - 4s_0s_2 &= -4d^2(\alpha - 2\beta + 2)(\alpha - 2\beta - 1)(4 - 2d + d\alpha + 2d\beta) \\ &\quad \times (2 - d + (d-2)\alpha + 2\beta) \ge 0. \end{aligned} \tag{25}$$

The set of all $(\alpha, \beta) \in \mathbb{R}^2$ fulfilling these conditions is illustrated in Figure 2. □

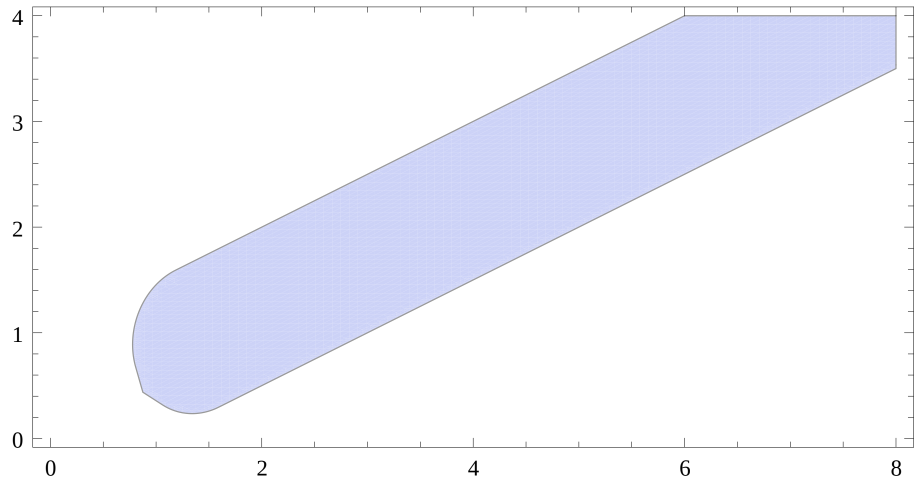

FIGURE 2. Set of all (α, β) fulfilling $s_0 \geq 0$, (24), and (25) for $d = 9$.

3. The discrete case

We introduce the finite-volume scheme and prove discrete versions of the generalized Beckner inequality as well as the discrete decay rates.

3.1. **Notations and finite-volume scheme.** Let Ω be an open bounded polyhedral subset of $\mathbb{R}^d$ ($d \geq 2$) with Lipschitz boundary and $\mathrm{m}(\Omega) = 1$. An admissible mesh of Ω is given by a family $\mathcal{T}$ of control volumes (open and convex polyhedral subsets of Ω with positive measure); a family $\mathcal{E}$ of relatively open parts of hyperplanes in $\mathbb{R}^d$ which represent the faces of the control volumes; and a family of points $(x_K)_{K\in\mathcal{T}}$ which satisfy Definition 9.1 in [17]. This definition implies that the straight line between two neighboring centers of cells (x_K, x_L) is orthogonal to the edge $\sigma = K|L$ between the two control volume K and L. For instance, triangular meshes in $\mathbb{R}^2$ satisfy the admissibility condition if all angles of the triangles are smaller than $\pi/2$ [17, Examples 9.1]. Voronoi meshes in $\mathbb{R}^d$ are also admissible meshes [17, Examples 9.2].

We distinguish the interior faces $\sigma \in \mathcal{E}_{\mathrm{int}}$ and the boundary faces $\sigma \in \mathcal{E}_{\mathrm{ext}}$. Then the union $\mathcal{E}_{\mathrm{int}} \cup \mathcal{E}_{\mathrm{ext}}$ equals the set of all faces $\mathcal{E}$. For a control volume $K \in \mathcal{T}$, we denote by $\mathcal{E}_K$ the set of its faces, by $\mathcal{E}_{\mathrm{int},K}$ the set of its interior faces, and by $\mathcal{E}_{\mathrm{ext},K}$ the set of edges of K included in $\partial\Omega$.

Furthermore, we denote by d the distance in $\mathbb{R}^d$. We assume that the family of meshes satisfies the following regularity requirement: There exists $\xi > 0$ such that for all $K \in \mathcal{T}$ and all $\sigma \in \mathcal{E}_{\mathrm{int},K}$ with $\sigma = K|L$, it holds

$$\mathrm{d}(x_K, \sigma) \geq \xi \mathrm{d}(x_K, x_L). \tag{26}$$

This hypothesis is needed to apply a discrete Poincaré inequality; see Lemma 11. Introducing for $\sigma \in \mathcal{E}$ the notation

$$d_\sigma = \begin{cases} \mathrm{d}(x_K, x_L) & \text{if } \sigma \in \mathcal{E}_{\mathrm{int}},\ \sigma = K|L, \\ \mathrm{d}(x_K, \sigma) & \text{if } \sigma \in \mathcal{E}_{\mathrm{ext},K}, \end{cases}$$

we define the transmissibility coefficient

$$\tau_\sigma = \frac{\mathrm{m}(\sigma)}{d_\sigma}, \quad \sigma \in \mathcal{E}.$$

The size of the mesh is denoted by $\triangle x = \max_{K \in \mathcal{T}} \mathrm{diam}(K)$. Let $T > 0$ be some final time and M_T the number of time steps. Then the time step size and the time points are given by, respectively, $\triangle t = \frac{T}{M_T}$, $t^k = k\triangle t$ for $0 \le k \le M_T$. We denote by $\mathcal{D}$ an admissible space-time discretization of $\Omega_T = \Omega \times (0, T)$ composed of an admissible mesh $\mathcal{T}$ of Ω and the values $\triangle t$ and M_T.
A classical finite volume scheme provides an approximate solution which is constant on each cell of the mesh and on each time interval. Let $X(\mathcal{T})$ be the linear space of functions $\Omega \to \mathbb{R}$ which are constant on each cell $K \in \mathcal{T}$. We define on $X(\mathcal{T})$ the discrete L^p norm, discrete $W^{1,p}$ seminorm, and discrete $W^{1,p}$ norm by, respectively,

$$\begin{aligned} \|u\|_{0,p,\mathcal{T}} &= \left(\int_\Omega |u|^p dx \right)^{1/p} = \left(\sum_{K \in \mathcal{T}} \mathrm{m}(K) |u_K|^p \right)^{1/p}, \\ |u|_{1,p,\mathcal{T}} &= \left(\sum_{\substack{\sigma \in \mathcal{E}_{\mathrm{int}}, \\ \sigma = K|L}} \frac{\mathrm{m}(\sigma)}{d_\sigma^{p-1}} |u_K - u_L|^p \right)^{1/p}, \\ \|u\|_{1,p,\mathcal{T}} &= \|u\|_{0,p,\mathcal{T}} + |u|_{1,p,\mathcal{T}}, \end{aligned}$$

where $u \in X(\mathcal{T})$, $u = u_K$ in $K \in \mathcal{T}$, and $1 \le p < \infty$. The discrete entropies for $u \in X(\mathcal{T})$ are defined analogously to the continuous case:

$$E_\alpha^d[u] = \frac{1}{\alpha + 1} \left(\sum_{K \in \mathcal{T}} \mathrm{m}(K) u_K^{\alpha+1} - \left(\sum_{K \in \mathcal{T}} \mathrm{m}(K) u_K \right)^{\alpha+1} \right), \tag{27}$$

$$F_\alpha^d[u] = \frac{1}{2} |u^{\alpha/2}|_{1,2,\mathcal{T}}^2. \tag{28}$$

We are now in the position to define the finite-volume scheme of (1)-(2). Let $\mathcal{D}$ be a finite-volume discretization of Ω_T. The initial datum is approximated by its L^2 projection on control volumes:

$$u^0 = \sum_{K \in \mathcal{T}} u_K^0 \mathbf{1}_K, \quad \text{where } u_K^0 = \frac{1}{\mathrm{m}(K)} \int_K u_0(x) dx, \tag{29}$$

and $\mathbf{1}_K$ is the characteristic function on K. Then $\|u^0\|_{0,1,\mathcal{T}} = \|u_0\|_{L^1(\Omega)}$. The numerical scheme reads as follows:

$$\mathrm{m}(K)\frac{u_K^{k+1} - u_K^k}{\triangle t} + \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}},\\ \sigma=K|L}} \tau_\sigma\big((u_K^{k+1})^\beta - (u_L^{k+1})^\beta\big) = 0, \tag{30}$$

for all $K \in \mathcal{T}$ and $k = 0, \ldots, M_T - 1$. This scheme is based on a fully implicit Euler discretization in time and a finite-volume approach for the volume variable. The Neumann boundary conditions (2) are taken into account as the sum in (30) applies only on the interior edges. The implicit scheme allows us to establish discrete entropy-dissipation estimates which would not be possible with an explicit scheme.

We summarize in the next proposition the classical results of existence, uniqueness and stability of the solution to the finite volume scheme (29)-(30).

Proposition 10. *Let $u_0 \in L^\infty(\Omega)$, $m \geq 0$, $M \geq 0$ such that $m \leq u_0 \leq M$ in Ω. Let $\mathcal{T}$ be an admissible mesh of Ω. Then the scheme* (29)-(30) *admits a unique solution $(u_K^k)_{K\in\mathcal{T},\,0\leq k\leq M_T}$ satisfying*

$$m \leq u_K^k \leq M, \quad \textit{for all } K \in \mathcal{T},\ 0 \leq k \leq M_T,$$
$$\sum_{K\in\mathcal{T}} \mathrm{m}(K)u_K^k = \|u_0\|_{L^1(\Omega)}, \quad \textit{for all } 0 \leq k \leq M_T.$$

We refer, for instance, to [17] and [18] for the proof of this proposition.

3.2. **Discrete generalized Beckner inequalities.** The decay properties rely on discrete generalized Beckner inequalities which follow from the discrete Poincaré-Wirtinger inequality [6, Theorem 5]:

Lemma 11 (Discrete Poincaré-Wirtinger inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open bounded polyhedral set and let $\mathcal{T}$ be an admissible mesh satisfying the regularity constraint* (26)*. Then there exists a constant $C_p > 0$ only depending on d and Ω such that for all $f \in X(\mathcal{T})$,*

$$\|f - \bar{f}\|_{0,2,\mathcal{T}} \leq \frac{C_p}{\xi^{1/2}}|f|_{1,2,\mathcal{T}} \tag{31}$$

where $\bar{f} = \int_\Omega f dx$ (recall that $m(\Omega) = 1$) and ξ is defined in (26)*.*

Lemma 12 (Discrete generalized Beckner inequality I)**.** *Let $0 < q < 2$, $pq > 1$ or $q = 2$ and $0 < p \leq 1$, and $f \in X(\mathcal{T})$. Then*

$$\int_\Omega |f|^q dx - \left(\int_\Omega |f|^{1/p} dx\right)^{pq} \leq C_b(p,q)|f|_{1,2,\mathcal{T}}^q$$

holds, where

$$C_b(p,q) = \frac{2(pq-1)C_p^q}{(2-q)\xi^{q/2}} \quad \textit{if } q < 2, \quad C_b(p,2) = \frac{C_p^2}{\xi} \quad \textit{if } q = 2.$$

C_p is the constant in the discrete Poincaré-Wirtinger inequality, and ξ is defined in (26)*.*

Proof. The proof follows the lines of the proof of Lemma 2 noting that in the discrete (finite-dimensional) setting, we do not need anymore the lower bound on p. Indeed, if $q = 2$, the conclusion results from the discrete Poincaré-Wirtinger inequality (31) and the Jensen inequality. If $q < 2$, let $f \in X(\mathcal{T})$. Then we have from (16) and (14)

$$\begin{aligned}\int_\Omega |f|^q dx - \left(\int_\Omega |f|^{1/p} dx\right)^{pq} &\le (pq-1)\int_\Omega |f|^q \log \frac{|f|^q}{\|f\|^q_{0,q,\mathcal{T}}} dx \\ &\le \frac{2(pq-1)}{2-q}\|f\|^q_{0,q,\mathcal{T}} \log \frac{\|f\|^q_{0,2,\mathcal{T}}}{\|f\|^q_{0,q,\mathcal{T}}}. \end{aligned} \tag{32}$$

We employ the discrete Poincaré-Wirtinger inequality (31),

$$\|f\|^2_{0,2,\mathcal{T}} - \|f\|^2_{0,1,\mathcal{T}} = \|f - \bar{f}\|^2_{0,2,\mathcal{T}} \le C_p^2 \xi^{-1} |f|^2_{1,2,\mathcal{T}},$$

which implies, as in the proof of Lemma 1 (see (9)), that

$$\|f\|^q_{0,2,\mathcal{T}} \le C_p^q \xi^{-q/2} |f|^q_{1,2,\mathcal{T}} + \|f\|^q_{0,q,\mathcal{T}}.$$

After inserting this inequality into (32) to replace $\|f\|_{0,2,\mathcal{T}}$ and using $\log(x+1) \le x$ for $x \ge 0$, the lemma follows. □

The following result is proved exactly as in Lemma 4 using the discrete Poincaré-Wirtinger inequality (31) instead of (6).

Lemma 13 (Discrete generalized Beckner inequality II)**.** *Let* $0 < q < 2$*,* $pq \ge 1$*, and* $f \in X(\mathcal{T})$*. Then*

$$\|f\|^{2-q}_{0,q,\mathcal{T}} \left(\int_\Omega |f|^q dx - \left(\int_\Omega |f|^{1/p} dx\right)^{pq}\right) \le C_b'(p,q)|f|^2_{1,2,\mathcal{T}}$$

holds, where

$$C_b'(p,q) = \begin{cases} \dfrac{q(pq-1)C_p^2}{(2-q)\xi} & \text{if } 1 \le q < 2, \\ \dfrac{(pq-1)C_p^2}{\xi} & \text{if } 0 < q < 1, \end{cases}$$

C_p *is the constant in the discrete Poincaré-Wirtinger inequality, and* ξ *is defined in* (26).

3.3. **Zeroth-order entropies.** We prove a result which is the discrete analogue of Theorem 5. Recall that the discrete entropies $E^d_\alpha[u^k]$ are defined in (27).

Theorem 14 (Polynomial decay of E^d_α)**.** *Let* $\alpha > 0$ *and* $\beta > 1$*. Let* $(u^k_K)_{K\in\mathcal{T},k\ge 0}$ *be a solution to the finite-volume scheme* (30) *with* $\inf_{K\in\mathcal{T}} u^0_K \ge 0$*. Then*

$$E^d_\alpha[u^k] \le \frac{1}{(c_1 t^k + c_2)^{(\alpha+1)/(\beta-1)}}, \quad k \ge 0,$$

where

$$c_1 = (\beta-1)\left(\frac{(\alpha+1)(\alpha+\beta)^2}{4\alpha\beta}\left(\frac{C_b(p,q)}{\alpha+1}\right)^{(\alpha+\beta)/(\alpha+1)} + (\alpha+\beta)\triangle t E^d_\alpha[u^0]^{(\alpha+1)/(\beta-1)}\right)^{-1},$$

$c_2 = E_\alpha^d[u^0]^{-(\beta-1)/(\alpha+1)}$,

and $C_b(p,q)$ *for* $p=(\alpha+\beta)/2$ *and* $q = 2(\alpha+1)/(\alpha+\beta)$ *is defined in Lemma 12.*

Proof. The idea is to "translate" the proof of Theorem 5 to the discrete case. To this end, we use the elementary inequality $y^{\alpha+1} - x^{\alpha+1} \le (\alpha+1)y^\alpha(y-x)$, which follows from the convexity of the mapping $x \mapsto x^{\alpha+1}$, and scheme (30):

$$\begin{aligned} E_\alpha^d[u^{k+1}] - E_\alpha^d[u^k] &= \frac{1}{\alpha+1}\sum_{K\in\mathcal{T}} \mathrm{m}(K)\big((u_K^{k+1})^{\alpha+1} - (u_K^k)^{\alpha+1}\big) \\ &\le \sum_{K\in\mathcal{T}} \mathrm{m}(K)(u_K^{k+1})^\alpha(u_K^{k+1} - u_K^k) \\ &\le -\triangle t \sum_{K\in\mathcal{T}} \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}},\\ \sigma=K|L}} \tau_\sigma (u_K^{k+1})^\alpha\big((u_K^{k+1})^\beta - (u_L^{k+1})^\beta\big). \end{aligned}$$

Rearranging the sum leads to

$$E_\alpha^d[u^{k+1}] - E_\alpha^d[u^k] \le -\triangle t \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}},\\ \sigma=K|L}} \tau_\sigma\big((u_K^{k+1})^\alpha - (u_L^{k+1})^\alpha\big)\big((u_K^{k+1})^\beta - (u_L^{k+1})^\beta\big). \tag{33}$$

Then, employing the inequality in Lemma 19 (see the appendix), it follows that

$$\begin{aligned} E_\alpha^d[u^{k+1}] - E_\alpha^d[u^k] &\le -\frac{4\alpha\beta\triangle t}{(\alpha+\beta)^2} \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}},\\ \sigma=K|L}} \tau_\sigma\big((u_K^{k+1})^{(\alpha+\beta)/2} - (u_L^{k+1})^{(\alpha+\beta)/2}\big)^2 \\ &\le -\frac{4\alpha\beta\triangle t}{(\alpha+\beta)^2}|(u^{k+1})^{(\alpha+\beta)/2}|_{1,2,\mathcal{T}}^2, \end{aligned}$$

and applying Lemma 12 with $p=(\alpha+\beta)/2$, $q=2(\alpha+1)/(\alpha+\beta)$, and $f=(u^{k+1})^{(\alpha+\beta)/2}$,

$$E_\alpha^d[u^{k+1}] - E_\alpha^d[u^k] \le -\frac{4\alpha\beta\triangle t}{(\alpha+\beta)^2}\left(\frac{\alpha+1}{C_b(p,q)}\right)^{(\alpha+\beta)/(\alpha+1)} E_\alpha^d[u^{k+1}]^{(\alpha+\beta)/(\alpha+1)}.$$

The discrete nonlinear Gronwall lemma (see Corollary 18 in the appendix) with

$$\tau = \frac{4\alpha\beta\triangle t}{(\alpha+\beta)^2}\left(\frac{\alpha+1}{C_b(p,q)}\right)^{(\alpha+\beta)/(\alpha+1)}, \quad \gamma = \frac{\alpha+\beta}{\alpha+1} > 1,$$

implies that

$$E_\alpha^d[u^k] \le \frac{1}{(E_\alpha^d[u^0]^{1-\gamma} + c_1 t^k)^{1/(\gamma-1)}}, \quad k\ge 0,$$

where $c_1 = (\gamma-1)/(1+\gamma\tau E_\alpha^d[u^0]^{\gamma-1})$. Finally, computing c_1 shows the result. □

The discrete analogue to Theorem 6 is as follows.

Theorem 15 (Exponential decay for E^d_α)**.** *Let $(u^k_K)_{K\in\mathcal{T},k\geq 0}$ be a solution to the finite-volume scheme* (30) *and let $0<\alpha\leq 1$, $\beta>0$, $\inf_{K\in\mathcal{T}} u^0_K\geq 0$. Then*

$$E^d_\alpha[u^k]\leq E^d_\alpha[u^0]e^{-\lambda t^k},\quad k\geq 0.$$

The constant λ is given by

$$\lambda=\frac{4\alpha\beta}{C_b(\frac12(\alpha+1),2)(\alpha+1)}\inf_{K\in\mathcal{T}}\big((u^0_K)^{\beta-1}\big)\geq 0,$$

for $\beta>0$, and

$$\lambda=\frac{4\alpha\beta(\alpha+1)}{C_b'(p,q)(\alpha+\beta)^2}\|u^0\|^{\beta-1}_{0,1,\mathcal{T}}$$

for $\beta>1$. Here $C_b'(p,q)>0$ is the constant from Lemma 13 with $p=(\alpha+\beta)/2$ and $q=2(\alpha+1)/(\alpha+\beta)$.

Proof. Let $\alpha\leq 1$ and $\beta>0$. As in the proof of Theorem 14, we find that (see (33))

$$E^d_\alpha[u^{k+1}]-E^d_\alpha[u^k]\leq -\triangle t\sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}}\tau_\sigma\big((u^{k+1}_K)^\alpha-(u^{k+1}_L)^\alpha\big)\big((u^{k+1}_K)^\beta-(u^{k+1}_L)^\beta\big).$$

Employing Corollary 20 (see the appendix), we obtain

$$\begin{aligned}E^d_\alpha[u^{k+1}]-E^d_\alpha[u^k]&\leq -\frac{4\alpha\beta\triangle t}{(\alpha+1)^2}\sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}}\tau_\sigma\min\big\{(u^{k+1}_K)^{\beta-1},(u^{k+1}_L)^{\beta-1}\big\}\\
&\quad\times\big((u^{k+1}_K)^{(\alpha+1)/2}-(u^{k+1}_L)^{(\alpha+1)/2}\big)^2\\
&\leq -\frac{4\alpha\beta\triangle t}{(\alpha+1)^2}\inf_{K\in\mathcal{T}}(u^{k+1}_K)^{\beta-1}|(u^{k+1})^{(\alpha+1)/2}|^2_{1,2,\mathcal{T}}\\
&\leq -\frac{4\alpha\beta\triangle t}{C_b(\frac12(\alpha+1),2)(\alpha+1)}\inf_{K\in\mathcal{T}}(u^0_K)^{\beta-1}E^d_\alpha[u^{k+1}],\end{aligned}$$

where we have used Lemma 12 with $p=(\alpha+1)/2$, $q=2$, and $f=u^{(\alpha+1)/2}$. Now, the Gronwall lemma shows the claim.

Next, let $\beta>1$. As in the proof of Theorem 14, we find that

$$E^d_\alpha[u^{k+1}]-E^d_\alpha[u^k]\leq -\frac{4\alpha\beta\triangle t}{(\alpha+\beta)^2}|(u^{k+1})^{(\alpha+1)/2}|^2_{1,2,\mathcal{T}}.$$

We apply Lemma 13 with $p=(\alpha+\beta)/2$, $q=2(\alpha+1)/(\alpha+\beta)$, and $f=u^{(\alpha+\beta)/2}$ to obtain

$$\begin{aligned}E^d_\alpha[u^{k+1}]-E^d_\alpha[u^k]&\leq -\frac{4\alpha\beta(\alpha+1)\triangle t}{(\alpha+\beta)^2}\frac{\|u^{k+1}\|^{\beta-1}_{0,\alpha+1,\mathcal{T}}}{C_b'(p,q)}E^d_\alpha[u^{k+1}]\\
&\leq -\frac{4\alpha\beta(\alpha+1)\triangle t}{(\alpha+\beta)^2}\frac{\|u^0\|^{\beta-1}_{0,1,\mathcal{T}}}{C_b'(p,q)}E^d_\alpha[u^{k+1}].\end{aligned}$$

Then Gronwall's lemma finishes the proof. ☐

3.4. **First-order entropies.** We consider the diffusion equation (1) on the half open unit cube $[0,1)^d \subset \mathbb{R}^d$ with multiperiodic boundary conditions (this is topologically equivalent to the torus $\mathbb{T}^d$). By identifying "opposite" faces on $\partial\Omega$, we can construct a family of control volumes and a family of edges in such a way that every face is an interior face. Then cells with such identified faces are neighboring cells.

Theorem 16 (Exponential decay of F_α^d)**.** *Let $(u_K^k)_{K\in\mathcal{T},\,k\geq 0}$ be a solution to the finite-volume scheme* (30) *with $\Omega = \mathbb{T}^d$ and $\inf_{K\in\mathcal{T}} u_K^0 \geq 0$. Then, for all $1\leq\alpha\leq 2$ and $\beta=\alpha/2$,*

$$F_\alpha^d[u^{k+1}] \leq F_\alpha^d[u^k], \quad k\in\mathbb{N}.$$

Furthermore, if $d=1$ and the grid is uniform with N subintervals,

$$F_\alpha^d[u^k] \leq F_\alpha^d[u_0]e^{-\lambda t^k},$$

where $\lambda = 4\beta\sin^2(\pi/N)\min_i((u_i^0)^{2(\beta-1)}) \geq 0$.

Proof. The difference $G_\alpha = F_\alpha^d[u^{k+1}] - F_\alpha^d[u^k]$ can be written as

$$G_\alpha = \frac{1}{2}\sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} \tau_\sigma\left(\big((u_K^{k+1})^{\alpha/2}-(u_L^{k+1})^{\alpha/2}\big)^2 - \big((u_K^k)^{\alpha/2}-(u_L^k)^{\alpha/2}\big)^2\right).$$

Introducing $a_K = (u_K^{k+1}-u_K^k)/\tau$, we find that

$$G_\alpha = \frac{1}{2}\sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} \tau_\sigma\left(\big((u_K^{k+1})^{\alpha/2}-(u_L^{k+1})^{\alpha/2}\big)^2 - \big((u_K^{k+1}-\tau a_K)^{\alpha/2}-(u_L^{k+1}-\tau a_L)^{\alpha/2}\big)^2\right).$$

We claim that G_α is concave with respect to τ. Indeed, we compute

$$\begin{aligned}
\frac{\partial G_\alpha}{\partial\tau} &= \frac{\alpha}{2}\sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} \tau_\sigma\big((u_K^{k+1}-\tau a_K)^{\alpha/2}-(u_L^{k+1}-\tau a_L)^{\alpha/2}\big) \\
&\quad\times\big((u_K^{k+1}-\tau a_K)^{\alpha/2-1}a_K-(u_L^{k+1}-\tau a_L)^{\alpha/2-1}a_L\big), \\
\frac{\partial^2 G_\alpha}{\partial\tau^2} &= -\frac{\alpha^2}{4}\sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} \tau_\sigma\big((u_K^{k+1}-\tau a_K)^{\alpha/2-1}a_K-(u_L^{k+1}-\tau a_L)^{\alpha/2-1}a_L\big)^2 \\
&\quad-\frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right)\sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} \tau_\sigma\big((u_K^{k+1}-\tau a_K)^{\alpha/2}-(u_L^{k+1}-\tau a_L)^{\alpha/2}\big) \\
&\quad\times\big((u_K^{k+1}-\tau a_K)^{\alpha/2-2}a_K^2-(u_L^{k+1}-\tau a_L)^{\alpha/2-2}a_L^2\big).
\end{aligned}$$

Replacing $u_K^{k+1}-\tau a_K$, $u_L^{k+1}-\tau a_L$ by u_K^k, u_L^k, respectively, the second derivative becomes

$$\frac{\partial^2 G_\alpha}{\partial\tau^2} = -\frac{\alpha^2}{4}\sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} \tau_\sigma\big((u_K^k)^{\alpha/2-1}a_K-(u_L^k)^{\alpha/2-1}a_L\big)^2$$

$$- \frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right) \sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} \tau_\sigma\big((u_K^k)^{\alpha/2}-(u_L^k)^{\alpha/2}\big)\big((u_K^k)^{\alpha/2-2}a_K^2-(u_L^k)^{\alpha/2-2}a_L^2\big)$$
$$= -\frac{\alpha}{4}\sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} \tau_\sigma(c_1a_K^2+c_2a_Ka_L+c_3a_L^2),$$

where

$$c_1 = (\alpha-2)\big((u_K^k)^{\alpha/2}-(u_L^k)^{\alpha/2}\big)(u_K^k)^{\alpha/2-2}+\alpha(u_K^k)^{\alpha-2},$$
$$c_2 = -2\alpha(u_K^k)^{\alpha/2-1}(u_L^k)^{\alpha/2-1},$$
$$c_3 = -(\alpha-2)\big((u_K^k)^{\alpha/2}-(u_L^k)^{\alpha/2}\big)(u_L^k)^{\alpha/2-2}+\alpha(u_L^k)^{\alpha-2}.$$

We show that the quadratic polynomial in the variables a_K and a_L is nonnegative for all u_K^k and u_L^k. This is the case if and only if $c_1\ge 0$ and $4c_1c_3-c_2^2\ge 0$. The former condition is equivalent to

$$2(\alpha-1)(u_K^k)^{\alpha-2} \ge (\alpha-2)(u_K^k)^{\alpha/2-2}(u_L^k)^{\alpha/2},$$

which is true for $1\le\alpha\le 2$. After an elementary computation, the latter condition becomes

$$4c_1c_3-c_2^2 = 8(\alpha-1)(2-\alpha)(u_K^k)^{\alpha/2-2}(u_L^k)^{\alpha/2-2}\big((u_K^k)^{\alpha/2}-(u_L^k)^{\alpha/2}\big)^2 \ge 0$$

for $1\le\alpha\le 2$. This proves the concavity of $\tau\mapsto G_\alpha(\tau)$.

A Taylor expansion and $G_\alpha(0)=0$ leads to

$$G_\alpha(\tau) \le G_\alpha(0)+\tau\frac{\partial G_\alpha}{\partial\tau}(0)$$
$$= \frac{\alpha\tau}{2}\sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} \tau_\sigma\big((u_K^{k+1})^{\alpha/2}-(u_L^{k+1})^{\alpha/2}\big)\big((u_K^{k+1})^{\alpha/2-1}a_K-(u_L^{k+1})^{\alpha/2-1}a_L\big)$$
$$= \frac{\alpha\tau}{2}\sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} \tau_\sigma\big((u_K^{k+1})^{\alpha/2}-(u_L^{k+1})^{\alpha/2}\big)(u_K^{k+1})^{\alpha/2-1}a_K$$
$$+ \frac{\alpha\tau}{2}\sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} \tau_\sigma\big((u_L^{k+1})^{\alpha/2}-(u_K^{k+1})^{\alpha/2}\big)(u_L^{k+1})^{\alpha/2-1}a_L.$$

Replacing a_K and a_L by scheme (30) and rearranging the terms, we infer that

$$G_\alpha(\triangle t) = -\frac{\alpha\triangle t}{2\mathrm{m}(K)}\sum_{K\in\mathcal{T}}\sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}}\tau_\sigma\sum_{\substack{\widetilde\sigma\in\mathcal{E}_{\rm int},\\ \widetilde\sigma'=K|M}}\tau_{\widetilde\sigma}(u_K^{k+1})^{\alpha/2-1}$$
$$\times\big((u_K^{k+1})^\beta-(u_M^{k+1})^\beta\big)\big((u_K^{k+1})^{\alpha/2}-(u_L^{k+1})^{\alpha/2}\big). \tag{34}$$

Note that the expression on the right-hand side is the discrete counterpart of the integral

$$-\frac{\alpha}{2}\int_\Omega u^{\alpha/2-1}(u^\beta)_{xx}(u^{\alpha/2})_{xx}dx,$$

appearing in (22). The condition $\alpha = 2\beta$ implies immediately the monotonicity of $k \mapsto F_\alpha^d[u^k]$.

For the proof of the second statement, let $d = 1$ and decompose the interval Ω in N subintervals $K_1, \ldots, K_N$ of length $h > 0$. Because of the periodic boundary conditions, we may set $u_{N+1}^k = u_0^k$ and $u_{-1}^k = u_N^k$, where u_i^k is the approximation of the mean value of $u(\cdot, t^k)$ on the subinterval K_i, $i = 1, \ldots, N$. We rewrite (34) for $\alpha = 2\beta$ in one space dimension:

$$G_{2\beta}(\tau) \le -\frac{\beta\tau}{2h} \sum_{i=1}^N \bigg(\sum_{j \in \{i-1, i+1\}} (u_i^{k+1})^{\beta-1} \big((u_i^{k+1})^\beta - (u_j{}^{k+1})^\beta \big) \bigg)^2$$

$$\le -\frac{\beta\tau}{2h} \min_{i=1,\ldots,N} \big((u_i^{k+1})^{2(\beta-1)} \big) \sum_{i=1}^N (z_i - z_{i-1})^2,$$

where $z_i = (u_i^{k+1})^\beta - (u_{i+1}^{k+1})^\beta$. The periodic boundary conditions imply that $\sum_{i=1}^N z_i = 0$. Hence, we can employ the discrete Wirtinger inequality in [34, Theorem 1] to obtain

$$G_{2\beta}(\tau) \le -\frac{2\beta\tau}{h} \sin^2 \frac{\pi}{N} \min_{i=1,\ldots,N} \big((u_i^k)^{2(\beta-1)} \big) \sum_{i=1}^N z_i^2$$

$$= -\frac{4\beta\tau}{h} \sin^2 \frac{\pi}{N} \min_{i=1,\ldots,N} \big((u_i^k)^{2(\beta-1)} \big) F_\alpha^d[u^{k+1}].$$

By the discrete maximum principle, $\max_i (u_i^{k+1})^{2(1-\beta)} \le \max_i (u_i^0)^{2(1-\beta)}$ which is equivalent to $\min_i (u_i^{k+1})^{\beta-1} \ge \min_i (u_i^0)^{\beta-1}$. Therefore,

$$F_\alpha^d[u^{k+1}] - F_\alpha^d[u^k] = G_{2\beta}(\triangle t) \le -\frac{4\beta\triangle t}{h} \sin^2 \frac{\pi}{N} \min_{i=1,\ldots,N} \big((u_i^0)^{2(\beta-1)} \big) F_\alpha^d[u^{k+1}],$$

and Gronwall's lemma finishes the proof. □

4. Numerical experiments

We illustrate the time decay of the solutions to the discretized porous-medium ($\beta = 2$) and fast-diffusion equation ($\beta = 1/2$) in one and two space dimensions.

First, let $\beta = 2$. We recall that the Barenblatt profile

$$u_B(x,t) = (t+t_0)^{-A} \Big(C - \frac{B(\beta-1)}{2\beta} \frac{|x-x_0|^2}{(t+t_0)^{2B}} \Big)_+^{1/(\beta-1)}$$

is a special solution to the porous-medium equation in the whole space. (Here, z_+ denotes the positive part of a function $z_+ := \max\{0, z\}$.) The constants are given by

$$A = \frac{d}{d(\beta-1)+2}, \quad B = \frac{1}{d(\beta-1)+2},$$

and C is typically determined by the initial datum via $\int_\Omega u(x,t)dx = \int_\Omega u(x,0)dx$. We choose $C = B(\beta-1)(2\beta)^{-1}(t_1+t_0)^{-2B}|x_1-x_0|^2$, where $t_1 > 0$ is the smallest time for which $u(x_1, t_1) = 0$.

In the one-dimensional situation, we choose $\Omega = (0,1)$ with homogeneous Neumann boundary conditions and a uniform grid $(x_i, t^j) \in [0,1] \times [0,0.2]$ with $1 \le i \le 50$ and $0 \le j \le 1000$, i.e., the space grid size is $\triangle x = 0.02$ and the time step size equals $\triangle t = 2 \cdot 10^{-4}$. We have chosen a very small time step size for a smoother graphical presentation of the solution, but the implicit scheme clearly also works for time step sizes of the order of $\triangle x$ and for smaller values of $\triangle x$. The initial datum is given by the Barenblatt profile $u_B(\cdot, 0)$ with $x_0 = 0.5$, $x_1 = 1$ and $t_0 = 0.01$. The constant C is computed by using $t_1 = 0.1$, which yields $C \approx 0.091$. For $0 \le t \le 0.1$, the analytical solution corresponds to the Barenblatt profile.

The time decay of the zeroth- and first-order entropies are depicted in Figure 3 in semi-logarithmic scale for various values of α. The decay rates are exponential for sufficiently large times, even for $\alpha > 1$ (compare to Theorem 15) and for $\alpha \neq 2\beta$ (see Theorem 16), which indicates that the conditions imposed in these theorems are technical. For small times, the decay seems to be faster than the decay in the large-time regime. This fact has been already observed in [7, Remark 4]. There is a significant change in the decay rate of the first-order entropies F_α^d for times around $t_1 = 0.1$. Indeed, the positive part of the discrete solution, which approximates the Barenblatt profile u_B for $t < t_1$, arrives the boundary and does not approximate u_B anymore. The change is more apparent for $\alpha < 1$.

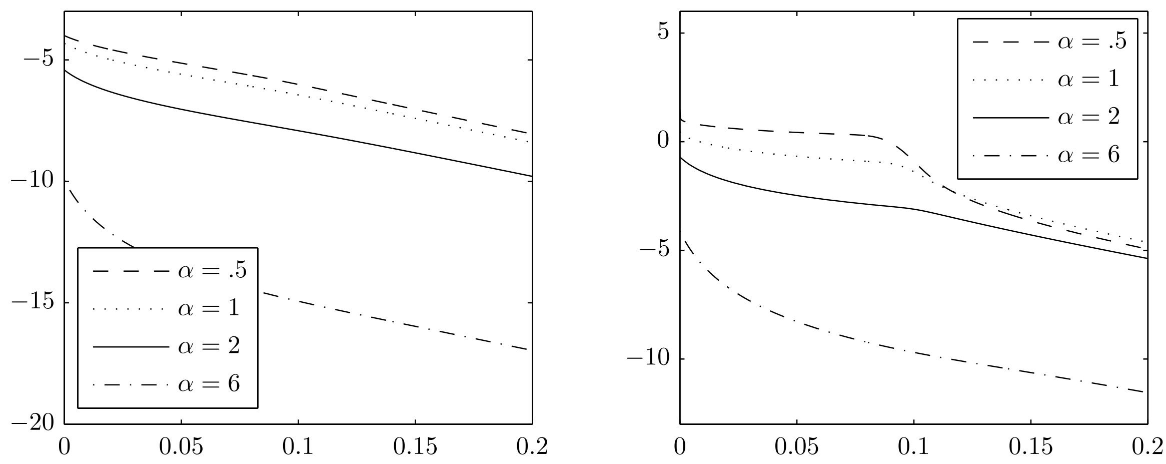

FIGURE 3. The natural logarithm of the entropies $\log(E_\alpha^d[u](t))$ (left) and $\log(F_\alpha^d[u](t))$ (right) versus time for different values of α ($\beta = 2$, $d = 1$).

Next, we investigate the two-dimensional situation (still with $\beta = 2$). The domain $\Omega = (0,1)^2$ is divided into 144 quadratic cells each of which consists of four control volumes (see Figure 4). Again we employ the Barenblatt profile as the initial datum, choosing $t_0 = 0.01$, $t_1 = 0.1$, and $x_0 = (0.5, 0.5)$, and impose homogeneous boundary conditions. The time step size equals $\triangle t = 8 \cdot 10^{-4}$.

In Figure 5, the time evolution of the (logarithmic) zeroth- and first-order entropies are presented. Again, the decay seems to be exponential for large times, even for values of α

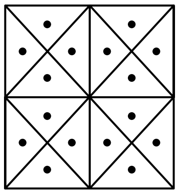

FIGURE 4. Four of the 144 cells used for the two-dimensional finite-volume scheme.

not covered by the theoretical results. At time $t = t_1$, the profile reaches the boundary of the domain. In contrast to the one-dimensional situation, since the radially symmetric profile does not reach the boundary everywhere at the same time, the time decay rate of F_α^d does not change as distinct as in Figure 3.

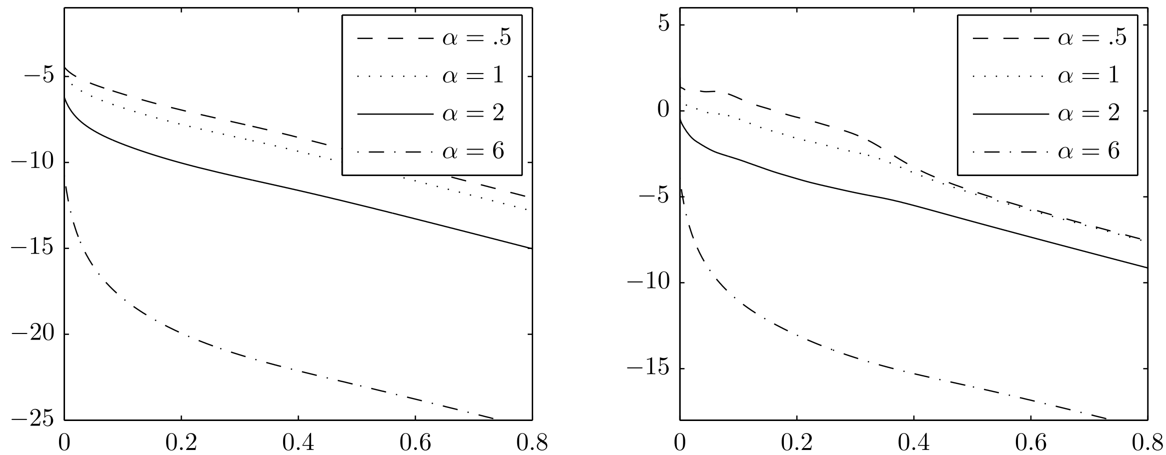

FIGURE 5. The natural logarithm of the entropies $\log(E_\alpha^d[u](t))$ (left) and $\log(F_\alpha^d[u](t))$ (right) versus time for different values of α ($\beta = 2$, $d = 2$).

Let $\beta = 1/2$. The one-dimensional interval $\Omega = (0, 1)$ is discretized as before using 51 grid points and the time step size is $\triangle t = 2 \cdot 10^{-4}$. We impose homogeneous Neumann boundary conditions. As initial datum, we choose the following truncated polynomial $u_0(x) = C((x_0 - x)(x - x_1))_+^2$, where $x_0 = 0.3$, $x_1 = 0.7$, and $C = 3000$. In the two-dimensional box $\Omega = (0, 1)^2$, we employ the discretization described above and the initial datum $u_0(x) = C(R^2 - |x - x_0|^2)_+^2$, where $R = 0.2$, $x_0 = (0.5, 0.5)$ and again $C = 3000$.

In the fast-diffusion case $\beta < 1$, we do not expect significant changes in the decay rate since the initial values propagate with infinite speed. This expectation is supported by the numerical results presented in Figures 6 and 7. For a large range of values of α, the decay rate is exponential, at least for large times. Interestingly, the rate seems to approach almost the same value for $\alpha \in \{0.5, 1, 2\}$ in Figure 7.

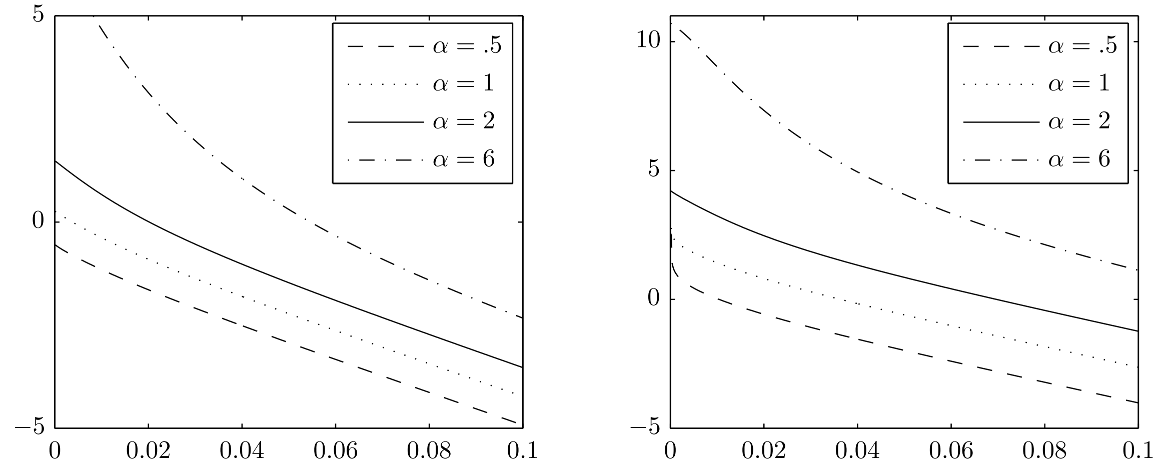

FIGURE 6. The natural logarithm of the entropies $\log(E^d_\alpha[u](t))$ (left) and $\log(F^d_\alpha[u](t))$ (right) versus time for different values of α ($\beta = 1/2$, $d = 1$).

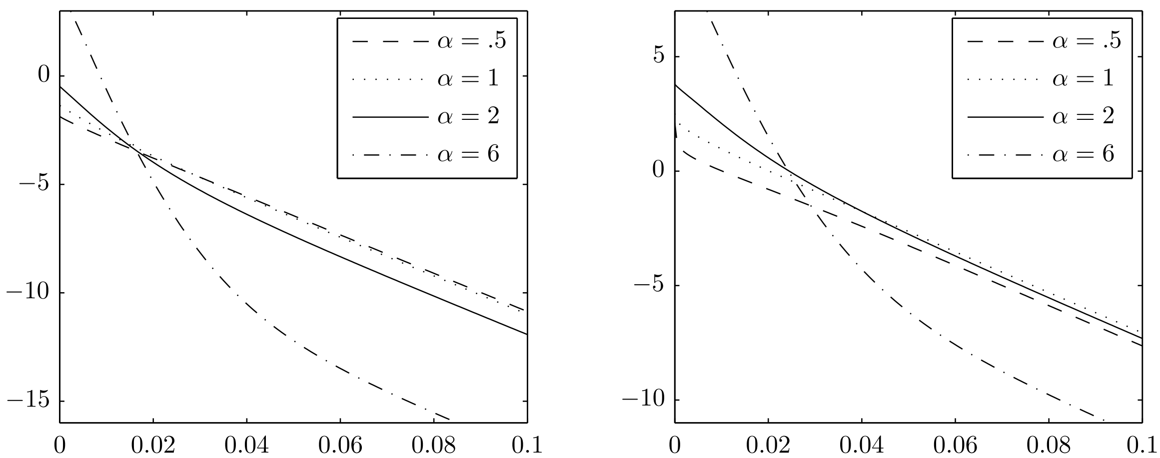

FIGURE 7. The natural logarithm of the entropies $\log(E^d_\alpha[u](t))$ (left) and $\log(F^d_\alpha[u](t))$ (right) versus time for different values of α ($\beta = 1/2$, $d = 2$).

APPENDIX A. SOME TECHNICAL LEMMAS

A.1. **Discrete Gronwall lemmas.** First, we prove a rather general discrete nonlinear Gronwall lemma.

Lemma 17 (Discrete nonlinear Gronwall lemma)**.** *Let $f \in C^1([0,\infty))$ be a positive, nondecreasing, and convex function such that $1/f$ is locally integrable. Define*

$$w(x) = \int_1^x \frac{dz}{f(z)}, \quad x \geq 0.$$

Let (x_n) be a sequence of nonnegative numbers such that $x_{n+1} - x_n + f(x_{n+1}) \leq 0$ for $n \in \mathbb{N}_0$. Then

$$x_n \leq w^{-1}\left(w(x_0) - \frac{n}{1+f'(x_0)}\right), \quad n \in \mathbb{N}.$$

Notice that the function w is strictly increasing such that its inverse is well defined.

Proof. Since f is nondecreasing and (x_n) is nonincreasing, we obtain

$$w(x_{n+1}) - w(x_n) = \int_{x_n}^{x_{n+1}} \frac{dz}{f(z)} \leq \frac{x_{n+1}-x_n}{f(x_n)}.$$

The sequence (x_n) satisfies $f(x_{n+1})/(x_{n+1}-x_n) \geq -1$. Therefore,

$$\begin{aligned} w(x_{n+1}) - w(x_n) &\leq \Big(\frac{f(x_{n+1})}{x_{n+1}-x_n} + \frac{f(x_n)-f(x_{n+1})}{x_{n+1}-x_n}\Big)^{-1} \\ &\leq \Big(-1 - \frac{f(x_n)-f(x_{n+1})}{x_n - x_{n+1}}\Big)^{-1}. \end{aligned}$$

By the convexity of f, $f(x_n) - f(x_{n+1}) \leq f'(x_n)(x_n - x_{n+1}) \leq f'(x_0)(x_n - x_{n+1})$, which implies that

$$w(x_{n+1}) - w(x_n) \leq (-1 - f'(x_0))^{-1}.$$

Summing this inequality from $n = 0$ to $N-1$, where $N \in \mathbb{N}$, yields

$$w(x_N) \leq w(x_0) - \frac{N}{1+f'(x_0)}.$$

Applying the inverse function of w shows the lemma. □

The choice $f(x) = \tau K x^\gamma$ for some $\gamma > 1$ in Lemma 17 lead to the following result.

Corollary 18. *Let (x_n) be a sequence of nonnegative numbers satisfying*

$$x_{n+1} - x_n + \tau x_{n+1}^\gamma \leq 0, \quad n \in \mathbb{N},$$

where $K > 0$ and $\gamma > 1$. Then

$$x_n \leq \frac{1}{\big(x_0^{1-\gamma} + c\tau n\big)^{1/(\gamma-1)}}, \quad n \in \mathbb{N},$$

where $c = (\gamma-1)/(1+\gamma\tau x_0^{\gamma-1})$.

A.2. **Some inequalities.** We show some inequalities in two variables.

Lemma 19. *Let α, $\beta > 0$. Then, for all x, $y \geq 0$,*

$$(y^\alpha - x^\alpha)(y^\beta - x^\beta) \geq \frac{4\alpha\beta}{(\alpha+\beta)^2}(y^{(\alpha+\beta)/2} - x^{(\alpha+\beta)/2})^2. \tag{35}$$

Proof. If $y = 0$, inequality (35) holds. Let $y \neq 0$ and set $z = (x/y)^\beta$. Then the inequality is proved if for all $z \geq 0$,

$$f(z) = (1 - z^{\alpha/\beta})(1 - z) - \frac{4\alpha\beta}{(\alpha+\beta)^2}(1 - z^{(\alpha+\beta)/2\beta})^2 \geq 0.$$

We differentiate f twice:

$$f'(z) = -1 - \frac{\alpha}{\beta}z^{\alpha/\beta-1} + \frac{(\alpha-\beta)^2}{\beta(\alpha+\beta)}z^{\alpha/\beta} + \frac{4\alpha}{\alpha+\beta}z^{(\alpha+\beta)/2\beta},$$

$$f''(z) = \frac{\alpha(\alpha-\beta)}{\beta}z^{\alpha/2\beta-3/2}\Big(- \frac{1}{\beta}z^{\alpha/2\beta-1/2} + \frac{\alpha-\beta}{\beta(\alpha+\beta)}z^{\alpha/2\beta+1/2} + \frac{2}{\alpha+\beta}\Big).$$

Then $f(1) = 0$ and $f'(1) = 0$. Thus, if we show that f is convex, the assertion follows. In order to prove the convexity of f, we define

$$g(z) = -\frac{1}{\beta}z^{\alpha/2\beta-1/2} + \frac{\alpha-\beta}{\beta(\alpha+\beta)}z^{\alpha/2\beta+1/2} + \frac{2}{\alpha+\beta}.$$

Then $g(1) = 0$ and it holds

$$g'(z) = \frac{\alpha-\beta}{2\beta^2}z^{\alpha/2\beta-3/2}(-1+z),$$

and therefore, $g'(1) = 0$. Now, if $\alpha > \beta$, $g(0) = 2/(\alpha+\beta) > 0$, and g is decreasing in $[0, 1]$ and increasing in $[1, \infty)$. Thus, $g(z) \geq 0$ for all $z \geq 0$. If $\alpha < \beta$ then $g(0+) = -\infty$, and g is increasing in $[0, 1]$ and decreasing in $[1, \infty)$. Hence, $g(z) \leq 0$ for $z \geq 0$. Independently of the sign of $\alpha - \beta$, we obtain

$$f''(z) = \frac{\alpha(\alpha-\beta)}{\beta}z^{\alpha/2\beta-3/2}g(z) \geq 0$$

for all $z \geq 0$, which shows the convexity of f. ☐

Corollary 20. *Let α, $\beta > 0$. Then, for all x, $y \geq 0$,*

$$(y^\beta - x^\beta)(y^\alpha - x^\alpha) \geq \frac{4\alpha\beta}{(\alpha+1)^2}\min\{x^{\beta-1}, y^{\beta-1}\}(y^{(\alpha+1)/2} - x^{(\alpha+1)/2})^2.$$

Proof. We assume without restriction that $y > x$. Then we apply Lemma 19 to $\beta = 1$:

$$(y^\beta - x^\beta)(y^\alpha - x^\alpha) = \frac{y^\beta - x^\beta}{y - x}(y^\alpha - x^\alpha)(y - x) \geq \frac{4\alpha}{(\alpha+1)^2}\frac{y^\beta - x^\beta}{y - x}(y^{(\alpha+1)/2} - x^{(\alpha+1)/2})^2.$$

Since

$$y^\beta - x^\beta = \beta\int_x^y t^{\beta-1}dt \geq \beta\min\{x^{\beta-1}, y^{\beta-1}\}(y - x),$$

the conclusion follows. ☐

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