HIGH-FRICTION LIMITS OF EULER FLOWS FOR MULTICOMPONENT SYSTEMS

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Abstract. The high-friction limit in Euler-Korteweg equations for fluid mixtures is analyzed. The convergence of the solutions towards the zeroth-order limiting system and the first-order correction is shown, assuming suitable uniform bounds. Three results are proved: The first-order correction system is shown to be of Maxwell-Stefan type and its diffusive part is parabolic in the sense of Petrovskii. The high-friction limit towards the first-order Chapman-Enskog approximate system is proved in the weak-strong solution context for general Euler-Korteweg systems. Finally, the limit towards the zeroth-order system is shown for smooth solutions in the isentropic case and for weak-strong solutions in the Euler-Korteweg case. These results include the case of constant capillarities and multicomponent quantum hydrodynamic models.

1. Introduction

Multicomponent flows appear in many applications including sedimentation, dialysis, electrolysis, and ion transport [22]. These flows may be described by Euler or Euler-Korteweg equations for the various species, coupled through interaction forces proportional to the difference of the partial velocities. The equations can be simplified when the interaction is strong, leading in the zeroth-order limit to the Euler equations for the partial particle densities and the common velocity and in the first-order correction to diffusive systems of Maxwell-Stefan type coupled with the momentum balance equation for the barycentric velocity. While such relaxation and high-friction limits are widely explored in mono-species situations, there are no results for multicomponent Euler-Korteweg flows. The aim of this paper is to compute the Chapman-Enskog expansion and to justify the expansion via a relative entropy approach, extending results for the mono-species case to fluid mixtures [10, 15, 16].

We consider the following Euler-Korteweg equations for multicomponent fluids,

$$\partial_t \rho_i + \operatorname{div}(\rho_i v_i) = 0, \tag{1}$$

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$$\partial_t(\rho_i v_i) + \operatorname{div}(\rho_i v_i \otimes v_i) = -\rho_i \nabla \frac{\delta \mathcal{E}}{\delta \rho_i}(\boldsymbol{\rho}) - \frac{1}{\varepsilon}\sum_{j=1}^n b_{ij}\rho_i\rho_j(v_i - v_j), \tag{2}$$

where $i = 1, \ldots, n$, $x \in \mathbb{R}^3$, $t > 0$, and $\boldsymbol{\rho} = \boldsymbol{\rho}(x,t) = (\rho_1, \ldots, \rho_n)(x,t)$. The initial conditions are

$$\rho_i(\cdot, 0) = \rho_i^0, \quad v_i(\cdot, 0) = v_i^0 \quad \text{in } \mathbb{R}^3, \ i = 1, \ldots, n.$$

The variables ρ_i are the partial densities and v_i the partial velocities. The parameters $b_{ij} \geq 0$ model the interaction of the ith and jth components with a strength that is measured by $\varepsilon > 0$. Model (1)-(2) belongs to the general realm of multicomponent fluid mixtures whose thermodynamical structure has been extensively analyzed; see, e.g., [3, 18, 19] and references therein. On the other hand, we adopt the mathematical structure espoused in [10], in that the dynamics of the flow is determined by the functional $\mathcal{E}(\boldsymbol{\rho})$ of potential energy, with $\delta\mathcal{E}/\delta\rho_i$ standing for the variational derivatives with respect to the partial densities ρ_i. Several isothermal models fit into this framework. In this work, we consider energies of the form

$$\mathcal{E}(\boldsymbol{\rho}) = \int_{\mathbb{R}^3} \sum_{i=1}^n F_i(\rho_i, \nabla\rho_i) dx.$$

For instance, when $F_i = h_i(\rho_i)$ for some (convex) function h_i we obtain the equations of multicomponent system of gas dynamics with friction. When

$$F_i = h_i(\rho_i) + \frac{1}{2}\kappa_i(\rho_i)|\nabla\rho_i|^2, \tag{3}$$

we obtain the multicomponent Euler-Korteweg system

$$\partial_t(\rho_i v_i) + \operatorname{div}(\rho_i v_i \otimes v_i) = \operatorname{div} S_i[\rho_i] - \frac{1}{\varepsilon}\sum_{j=1}^n b_{ij}\rho_i\rho_j(v_i - v_j),$$

where

$$S_i[\rho_i] := \bigg(- p_i(\rho_i) - \frac{1}{2}\big(\rho_i\kappa_i'(\rho_i) + \kappa_i(\rho_i)\big)|\nabla\rho_i|^2 + \operatorname{div}(\rho_i\kappa_i(\rho_i)\nabla\rho_i)\bigg)\mathbb{I} - \kappa_i(\rho_i)\nabla\rho_i \otimes \nabla\rho_i$$

is the stress tensor associated with the ith component and $p_i(\rho_i) = \rho_i h_i'(\rho_i) - h_i(\rho_i)$ is the partial pressure. A special case is the selection $\kappa_i(\rho_i) = k_i/(4\rho_i)$ with $k_i = \text{const.}$, which yields the multicomponent quantum hydrodynamic system with friction,

$$\partial_t(\rho_i v_i) + \operatorname{div}(\rho_i v_i \otimes v_i) + \nabla p_i(\rho_i) = \frac{1}{2}k_i\rho_i\nabla\bigg(\frac{\Delta\sqrt{\rho_i}}{\sqrt{\rho_i}}\bigg) - \frac{1}{\varepsilon}\sum_{j=1}^n b_{ij}\rho_i\rho_j(v_i - v_j),$$

used to describe quantum effects in semiconductors [12] or multicomponent quantum plasmas [17]. The dependence of F_i on the density (and its gradient) of the ith component is crucial; the general case leads to mixed terms like $\partial F_i/\partial\rho_j$ that we cannot control.

The interaction term (the last term in (2)) has an alignment effect on the partial velocities, and we expect that all partial velocities are the same in the high-friction limit $\varepsilon \to 0$,

leading to the zeroth-order limit system

$$\partial_t\bar\rho_i + \operatorname{div}(\bar\rho_i\bar v) = 0, \quad \partial_t(\bar\rho\bar v) + \operatorname{div}(\bar\rho\bar v\otimes\bar v) = -\sum_{i=1}^n \bar\rho_i\nabla\frac{\delta\mathcal{E}}{\delta\rho_i}(\bar{\boldsymbol\rho}) \tag{4}$$

for $i=1,\ldots,n$, where $\bar{\boldsymbol\rho}=(\bar\rho_1,\ldots,\bar\rho_n)$, while $\bar\rho=\sum_{i=1}^n\bar\rho_i$ stands for the total density. In the first-order correction, the solution $(\boldsymbol\rho^\varepsilon,\boldsymbol v^\varepsilon)=(\rho_i^\varepsilon,v_i^\varepsilon)_{i=1,\ldots,n}$ to the hyperbolic relaxation system (1)-(2) is expected to be close to the hyperbolic-diffusive system

$$\partial_t\widehat\rho_i^\varepsilon + \operatorname{div}(\widehat\rho_i^\varepsilon\widehat v^\varepsilon) = \varepsilon\operatorname{div}\sum_{j=1}^n D_{ij}^\varepsilon(\widehat{\boldsymbol\rho}^\varepsilon)\nabla\frac{\delta\mathcal{E}}{\delta\rho_j}(\widehat{\boldsymbol\rho}^\varepsilon), \tag{5}$$

$$\partial_t(\widehat\rho^\varepsilon\widehat v^\varepsilon) + \operatorname{div}(\widehat\rho^\varepsilon\widehat v^\varepsilon\otimes\widehat v^\varepsilon) = -\sum_{i=1}^n\widehat\rho_i^\varepsilon\nabla\frac{\delta\mathcal{E}}{\delta\rho_i}(\widehat{\boldsymbol\rho}^\varepsilon) \tag{6}$$

for $i=1,\ldots,n$, where $\widehat{\boldsymbol\rho}^\varepsilon=(\widehat\rho_1^\varepsilon,\ldots,\widehat\rho_n^\varepsilon)$ and $\widehat\rho^\varepsilon=\sum_{i=1}^n\widehat\rho_i^\varepsilon$. When the barycentric velocity $\widehat v^\varepsilon$ vanishes, we recover the Maxwell-Stefan equations analyzed in, e.g., [2, 5, 14].

Before stating our main results, we review the state of the art. The structure of relaxation systems and their relaxation limits were first explored for examples [7] and later for general systems [4, 8, 21, 28]. We call the limit $\varepsilon\to 0$ a relaxation limit if the time scale is of order $O(1/\varepsilon)$. Rigorous relaxation limits in the mono-species Euler equations towards the heat or porous-medium equation were proved, using energy estimates [9], the relative entropy approach [15], or convergence in Besov spaces [24]. The relaxation limit in non-isentropic flows was analyzed in, e.g., [23, 25].

When the potential energy $\mathcal{E}$ also depends on the gradient of the particle density, system (1)-(2) is of Euler-Korteweg type. The relaxation (or high-friction) limit in these equations for single species was studied in [16] for monotone pressures (i.e. convex energies) and in [11] for non-monotone pressures. Giesselmann et al. [10] proved stability theorems for the Euler-Korteweg system between a weak and a strong solution and for the Navier-Stokes-Korteweg system.

All these results concern the mono-species case. Relaxation limits in multi-species systems were proved in the Euler-Poisson equations for electrons and positively charged ions in plasmas or semiconductors [13]. At the zeroth order, such a limit leads to equations (4). First-order corrections can be derived by a Chapman-Enskog expansion or Maxwell-iteration technique. This was done in the Euler system with temperature [19], leading to equations for multitemperature mixtures in nonequilibrium thermodynamics. The Chapman-Enkog expansion was validated in [26, 27] in the isentropic case, proving an error estimate for the difference of the solutions of equations (1)-(2) and (5)-(6). Another validation was recently presented by Boudin et al. [6] by applying the formalism of Chen, Levermore, and Liu [8]. However, no results seem to be available in the literature for high-friction limits in Euler-Korteweg systems.

In this paper, we prove the convergence of solutions to (1)-(2) towards the limit system (4) and the first-order correction system (5)-(6). The main results can be sketched as follows:

1. We compute the Chapman-Enskog expansion leading to (5)-(6) and show that (5) has a gradient-flow structure (Lemma 2). Moreover, when the barycentric velocity $\widehat{v}^\varepsilon$ vanishes, the system is proved to be parabolic in the sense of Petrovskii (Lemma 3).
2. Assume that the functional (3) satisfies some convexity conditions. For weak solutions to the relaxation system (1)-(2) and strong solutions to the approximate system (5)-(6) with uniform bounds on the velocities, assuming that the difference of the initial data is of order $O(\varepsilon^2)$, we prove that

$$\chi(t) := \int_{\mathbb{R}^3}\sum_{i=1}^n\left(\frac{1}{2}\rho_i^\varepsilon|v_i^\varepsilon-\widehat{v}_i^\varepsilon|^2+(\rho_i^\varepsilon-\widehat{\rho}_i^\varepsilon)^2+\frac{1}{2\kappa_i(\rho_i^\varepsilon)}|\kappa_i(\rho_i^\varepsilon)\nabla\rho_i^\varepsilon-\kappa_i(\widehat{\rho}_i^\varepsilon)\nabla\widehat{\rho}_i^\varepsilon|^2\right)(t)dx$$
$$\le C(\chi(0)+\varepsilon^2)$$

uniformly in $t\in(0,T)$ for some constant $C>0$ independent of ε, see Theorem 7.

3a. Isentropic case: Smooth solutions to (1)-(2) converge towards a smooth solution to the limit system (4) in the sense

$$\sup_{0<t<T}\int_{\mathbb{R}^3}\sum_{i=1}^n\big((\rho_i^\varepsilon-\bar{\rho}_i)^2+|v_i^\varepsilon-\bar{v}|^2\big)dx\to 0\quad\text{as }\varepsilon\to 0,$$

if the initial relative entropy converges to zero; see Theorem 9.

3b. Euler-Korteweg case with functional (3): Weak solutions to (1)-(2) converge towards a strong solution to the limit system (4) in the sense

$$\chi(t) := \int_{\mathbb{R}^3}\sum_{i=1}^n\left(\frac{1}{2}\rho_i^\varepsilon|v_i^\varepsilon-\bar{v}|^2+(\rho_i^\varepsilon-\bar{\rho}_i)^2+\frac{1}{2\kappa_i(\rho_i^\varepsilon)}|\kappa_i(\rho_i^\varepsilon)\nabla\rho_i^\varepsilon-\kappa_i(\bar{\rho}_i)\nabla\bar{\rho}_i|^2\right)(t)dx$$
$$\le C(\chi(0)+\varepsilon)$$

uniformly in $t\in(0,T)$ for some constant $C>0$ independent of ε; see Theorem 11.

For these results, we need that the functions ρ_i^ε are uniformly bounded away from vacuum as well as h_i and $-1/\kappa_i$ are convex. The case of the multicomponent quantum hydrodynamic system and the system with constant capillarities are included.

The idea of the proofs is to estimate the relative entropy between two solutions

$$\mathcal{E}_{\rm tot}(\boldsymbol{\rho},\boldsymbol{m}|\widehat{\boldsymbol{\rho}},\widehat{\boldsymbol{m}})(t)=\int_{\mathbb{R}^3}\sum_{i=1}^n\left(F_i(\rho_i,\nabla\rho_i|\widehat{\rho}_i,\nabla\widehat{\rho}_i)+\frac{1}{2}\rho_i|v_i-\widehat{v}_i|^2\right)(t)dx,$$

where $\boldsymbol{m}=(m_1,\ldots,m_n)$ with $m_i=\rho_i v_i$, $\widehat{\boldsymbol{m}}=(\widehat{m}_1,\ldots,\widehat{m}_n)$ with $\widehat{m}_i=\widehat{\rho}_i\widehat{v}_i$, and $F_i(\rho_i,\nabla\rho_i|\widehat{\rho}_i,\nabla\widehat{\rho}_i)$ is the relative potential energy density, defined by

$$F_i(\rho_i,\nabla\rho_i|\widehat{\rho}_i,\nabla\widehat{\rho}_i)=F_i-\widehat{F}_i-\frac{\partial\widehat{F}_i}{\partial\rho_i}(\rho_i-\widehat{\rho}_i)-\frac{\partial\widehat{F}_i}{\partial\nabla\rho_i}\cdot\nabla(\rho_i-\widehat{\rho}_i),$$

with $F_i=F_i(\rho_i,\nabla\rho_i)$ and $\widehat{F}_i=F_i(\widehat{\rho}_i,\nabla\widehat{\rho}_i)$. This functional satisfies a relative entropy inequality, proved in Proposition 6 for solutions to (1)-(2) and (5)-(6) and in Proposition 10 for solutions to (1)-(2) and (4). The relative entropy approach has the advantage of being very elementary and to be able to treat weak solutions to the original system [10, 16].

For the proof of the high-friction limit in the isentropic case, we apply the general relaxation result in [21] which is also based on the relative entropy approach. We show that the framework is sufficiently general to include multicomponent Euler flows with friction.

The paper is organized as follows. The formal Chapman-Enskog expansion as well as the proof of parabolicity of the first-order correction system are performed in section 2. Section 3 is devoted to the rigorous proof of the Chapman-Enskog expansion in the Euler-Korteweg case. The high-friction limit in both the isentropic and Euler-Korteweg case is shown in section 4.

2. Formal asymptotics

In this section we perform a Chapman-Enskog asymptotic analysis to system (1)-(2) as $\varepsilon \to 0$. As a preparation, we analyze the solvability properties of the linear system

$$\sum_{j=1}^n b_{ij}\rho_i\rho_j(v_i - v_j) = d_i, \quad i = 1, \ldots, n\,, \tag{7}$$

and the associated homogeneous system

$$\sum_{j=1}^n b_{ij}\rho_i\rho_j(v_i - v_j) = 0, \quad i = 1, \ldots, n. \tag{8}$$

The key hypothesis for (8), to be assumed in the whole manuscript, reads as

(N) Let $(b_{ij}) \in \mathbb{R}^{n\times n}$ be a symmetric matrix with nonnegative coefficients, $b_{ij} \geq 0$. For any $\rho_1, \ldots, \rho_n > 0$, system (8) has the one-dimensional null space span$\{\mathbf{1}\}$, where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$.

By setting $B_{ij} = b_{ij}\rho_i\rho_j$, we rewrite (8) in the form

$$\sum_{j=1}^n B_{ij}(v_i - v_j) = 0, \quad i = 1, \ldots, n. \tag{9}$$

If the coefficients B_{ij} are symmetric and strictly positive, $B_{ij} > 0$ for $i \neq j$, then hypothesis **(N)** is automatically satisfied. Indeed, due to the symmetry of (B_{ij}),

$$\begin{aligned}\sum_{i,j=1}^n B_{ij}(v_i - v_j)\cdot v_i &= \frac{1}{2}\sum_{i,j=1}^n B_{ij}(v_i - v_j)\cdot v_i + \frac{1}{2}\sum_{i,j=1}^n B_{ji}(v_j - v_i)\cdot v_j \\ &= \frac{1}{2}\sum_{i,j=1}^n B_{ij}|v_i - v_j|^2.\end{aligned}$$

If (9) is satisfied, it follows that $v_i = v_j$ for all $i \neq j$, and the null space of system (8) is the linear span of the vector $\mathbf{1}$. This conclusion cannot be guaranteed if some b_{ij} vanish, which makes necessary assumption **(N)**. The assumption guarantees that there are no extraneous conservation laws associated to the frictional coefficients b_{ij}, beyond the conservation of mass and total momentum.

2.1. **Solution of a linear system.** In the sequel, we will need to solve the linear system

$$-\sum_{j=1}^n b_{ij}\rho_i\rho_j(u_i-u_j) = d_i \quad \text{for } i=1,\ldots,n, \quad \text{subject to } \sum_{i=1}^n \rho_i u_i = 0. \tag{10}$$

We give a semi-explicit solution to such systems, recalling the notation $B_{ij} = b_{ij}\rho_i\rho_j$.

Lemma 1. *Let $d_1,\ldots,d_n \in \mathbb{R}^3$ satisfy $\sum_{i=1}^n d_i = 0$, $\rho_1,\ldots,\rho_n > 0$, and $(B_{ij}) \in \mathbb{R}^{n\times n}$ be a symmetric matrix satisfying $B_{ij} \ge 0$ for all $i,j=1,\ldots,n$. We suppose that all solutions to the homogeneous system*

$$\sum_{j=1}^n B_{ij}(u_i-u_j) = 0, \quad i=1,\ldots,n, \tag{11}$$

lie in the space $\mathrm{span}\{\mathbf{1}\}$. *Then the system*

$$-\sum_{j=1}^n B_{ij}(u_i-u_j) = d_i \quad \textit{for } i=1,\ldots,n, \quad \textit{subject to } \sum_{i=1}^n \rho_i u_i = 0, \tag{12}$$

has the unique solution

$$\rho_i u_i = -\sum_{j,k=1}^{n-1}\left(\delta_{ij}\rho_i - \frac{\rho_i\rho_j}{\rho}\right)\tau_{jk}^{-1}d_k, \quad \rho_n u_n = -\sum_{j=1}^{n-1}\rho_j u_j, \tag{13}$$

where $i=1,\ldots,n$, $\rho = \sum_{i=1}^n \rho_i > 0$ and $(\tau_{ij}^{-1}) \in \mathbb{R}^{(n-1)\times(n-1)}$ is the inverse of a regular submatrix, obtained from reordering the matrix $(\tau_{ij}) \in \mathbb{R}^{n\times n}$ of rank $n-1$ with coefficients

$$\tau_{ij} = \delta_{ij}\sum_{k=1}^n B_{ik} - B_{ij}, \quad i,j=1,\ldots,n.$$

Proof. We proceed similarly as in [27, Section 4]. The idea is to formulate the linear system in $n-1$ equations and to invert the resulting linear system semi-explicitly. First, we notice that we can write (11) as

$$\sum_{j=1}^n \tau_{ij}u_j = 0 \quad \text{for } i=1,\ldots,n,$$

where

$$\tau_{ij} = -b_{ij}\rho_i\rho_j \text{ for } i\neq j \quad \text{and} \quad \tau_{ii} = -\sum_{j=1,\,j\neq i}^n \tau_{ij}. \tag{14}$$

Since we assumed that all solutions to this system lie in the space $\mathrm{span}\{\mathbf{1}\}$, the matrix $(\tau_{ij}) \in \mathbb{R}^{n\times n}$ has rank $n-1$. Thus, there exists an invertible submatrix $\tau = (\tau_{ij}) \in \mathbb{R}^{(n-1)\times(n-1)}$ (possibly after reordering of the indices).

The linear system (12) can be formulated in terms of the first $n-1$ variables. Indeed, since $\sum_{j=1}^n \tau_{ij} = 0$, we find that

$$-d_i = \sum_{j=1}^n \tau_{ij} u_j = \sum_{j=1}^{n-1} \tau_{ij} u_j + \tau_{in} u_n = \sum_{j=1}^{n-1} \tau_{ij} u_j - \sum_{j=1}^{n-1} \tau_{ij} u_n = \sum_{j=1}^{n-1} \tau_{ij}(u_j - u_n).$$

Using the property $\rho_n u_n = -\sum_{k=1}^{n-1} \rho_k u_k$, it follows that

$$-d_i = \sum_{j=1}^{n-1} \tau_{ij} \biggl(u_j + \frac{1}{\rho_n} \sum_{k=1}^{n-1} \rho_k u_k \biggr) = \sum_{j,k=1}^{n-1} \tau_{ij} \biggl(\frac{1}{\rho_j} \delta_{jk} + \frac{1}{\rho_n} \biggr) \rho_k u_k = \sum_{j,k=1}^{n-1} \tau_{ij} Q_{jk} \rho_k u_k, \tag{15}$$

where $Q_{ij} = \delta_{ij} \rho_j^{-1} + \rho_n^{-1}$ for $i, j = 1, \ldots, n-1$.

The matrix $Q = (Q_{ij}) \in \mathbb{R}^{(n-1)\times(n-1)}$ is invertible with inverse (Q_{ij}^{-1}), where $Q_{ij}^{-1} = \delta_{ij}\rho_j - \rho_i \rho_j / \rho$ for $i, j = 1, \ldots, n-1$. Indeed, a straightforward computation shows that

$$\begin{aligned}
\sum_{k=1}^{n-1} Q_{ik} Q_{kj}^{-1} &= \sum_{k=1}^{n-1} \biggl(\frac{1}{\rho_k} \delta_{ik} + \frac{1}{\rho_n} \biggr) \biggl(\delta_{kj} \rho_j - \frac{\rho_k \rho_j}{\rho} \biggr) \\
&= \delta_{ij} + \frac{\rho_j}{\rho_n} - \frac{\rho_j}{\rho} - \frac{\rho_j}{\rho_n \rho} \sum_{k=1}^{n-1} \rho_k = \delta_{ij}, \\
\sum_{k=1}^{n-1} Q_{ik}^{-1} Q_{kj} &= \sum_{k=1}^{n-1} \biggl(\delta_{ik} \rho_k - \frac{\rho_i \rho_k}{\rho} \biggr) \biggl(\frac{1}{\rho_k} \delta_{kj} + \frac{1}{\rho_n} \biggr) \\
&= \delta_{ij} - \frac{\rho_i}{\rho} + \frac{\rho_i}{\rho_n} - \frac{\rho_i}{\rho \rho_n} \sum_{k=1}^{n-1} \rho_k = \delta_{ij}.
\end{aligned}$$

Thus, the matrix product τQ is invertible with inverse $Q^{-1}\tau^{-1}$, and we infer that

$$\rho_i u_i = -\sum_{j,k=1}^{n-1} Q_{ij}^{-1} \tau_{jk}^{-1} d_k = -\sum_{j,k=1}^{n-1} \biggl(\delta_{ij} \rho_i - \frac{\rho_i \rho_j}{\rho} \biggr) \tau_{jk}^{-1} d_k, \quad i = 1, \ldots, n-1.$$

This ends the proof. □

2.2. **Formal derivation of the Chapman-Enskog expansion.** We perform a formal Chapman-Enskog expansion of (1)-(2) in the high-friction regime, i.e. for small $\varepsilon > 0$. We introduce the moments

$$\rho = \sum_{i=1}^n \rho_i, \quad \rho v = \sum_{i=1}^n \rho_i v_i,$$

and the relative velocities $u_i = v_i - v$ for $i = 1, \ldots, n$. This corresponds to a change of variables $(v_1, \ldots, v_n) \mapsto (v, u_1, \ldots, u_n)$. Then system (1)-(2) becomes

$$\partial_t \rho_i + \operatorname{div}(\rho_i u_i + \rho_i v) = 0, \tag{16}$$

$$\partial_t(\rho_i u_i + \rho_i v) + \operatorname{div}\big(\rho_i(u_i+v)\otimes(u_i+v)\big) = -\rho_i\nabla\frac{\delta\mathcal{E}}{\delta\rho_i}(\boldsymbol{\rho}) - \frac{1}{\varepsilon}\sum_{j=1}^n b_{ij}\rho_i\rho_j(u_i-u_j), \tag{17}$$

subject to the constraint

$$\sum_{i=1}^n \rho_i u_i = \sum_{i=1}^n \rho_i(v_i - v) = \sum_{i=1}^n \rho_i v_i - \rho v = 0. \tag{18}$$

The objective is to derive an effective equation in the spirit of the Chapman-Enskog expansion for the high-friction dynamics of system (16)-(17) subject to (18). For this, we introduce the Hilbert expansion

$$\begin{aligned} \rho_i &= \rho_i^0 + \varepsilon\rho_i^1 + \varepsilon^2\rho_i^2 + O(\varepsilon^3), \\ u_i &= u_i^0 + \varepsilon u_i^1 + \varepsilon^2 u_i^2 + O(\varepsilon^3), \\ v &= v^0 + \varepsilon v^1 + O(\varepsilon^2). \end{aligned} \tag{19}$$

Inserting this expansion into $\rho = \sum_{i=1}^n \rho_i$, we find that

$$\rho = \rho^0 + \varepsilon\rho^1 + O(\varepsilon^2), \quad \text{where } \rho^0 := \sum_{i=1}^n \rho_i^0,\ \rho^1 := \sum_{i=1}^n \rho_i^1, \tag{20}$$

and the constraint (18) leads to

$$0 = \sum_{i=1}^n \rho_i u_i = \sum_{i=1}^n \rho_i^0 u_i^0 + \varepsilon\sum_{i=1}^n\big(\rho_i^0 u_i^1 + \rho_i^1 u_i^0\big) + O(\varepsilon^2).$$

Equating terms of the same order gives

$$\sum_{i=1}^n \rho_i^0 u_i^0 = 0, \quad \sum_{i=1}^n\big(\rho_i^0 u_i^1 + \rho_i^1 u_i^0\big) = 0. \tag{21}$$

Next, we insert the Hilbert expansion (19) into system (16)-(17) and identify terms of the same order:

• Terms of order $O(1/\varepsilon)$:

$$\sum_{j=1}^n b_{ij}\rho_i^0\rho_j^0(u_i^0 - u_j^0) = 0, \quad i = 1,\ldots,n. \tag{22}$$

• Terms of order $O(1)$:

$$\partial_t\rho_i^0 + \operatorname{div}(\rho_i^0 u_i^0 + \rho_i^0 v^0) = 0, \tag{23}$$

$$\begin{aligned} \partial_t(\rho_i^0 u_i^0 + \rho_i^0 v^0) &+ \operatorname{div}\big(\rho_i^0(u_i^0+v^0)\otimes(u_i^0+v^0)\big) \\ &= -\rho_i^0\nabla\frac{\delta\mathcal{E}}{\delta\rho_i}(\boldsymbol{\rho}^0) - \sum_{j=1}^n b_{ij}\rho_i^0\rho_j^0(u_i^1-u_j^1) - \sum_{j=1}^n b_{ij}(\rho_i^1\rho_j^0 + \rho_i^0\rho_j^1)(u_i^0-u_j^0). \end{aligned} \tag{24}$$

• Terms of order $O(\varepsilon)$:

$$\partial_t\rho_i^1 + \operatorname{div}\big(\rho_i^1(u_i^0+v^0) + \rho_i^0(u_i^1+v^1)\big) = 0, \tag{25}$$

$$\begin{aligned}(26)\qquad &\partial_t\big(\rho_i^1(u_i^0+v^0)+\rho_i^0(u_i^1+v^1)\big)+\operatorname{div}\big(\rho_i^1(u_i^0+v^0)\otimes(u_i^0+v^0)\\ &\quad+\rho_i^0(u_i^1+v^1)\otimes(u_i^0+v^0)+\rho_i^0(u_i^0+v^0)\otimes(u_i^1+v^1)\big)\\ &=-\rho_i^1\nabla\frac{\delta\mathcal{E}}{\delta\rho_i}(\boldsymbol{\rho}^0)-\rho_i^0\nabla\bigg(\sum_{j=1}^n\frac{\delta^2\mathcal{E}}{\delta\rho_i\delta\rho_j}(\boldsymbol{\rho}^0)\rho_j^1\bigg)\\ &\quad-\sum_{j=1}^n b_{ij}\Big(\rho_i^0\rho_j^0(u_i^2-u_j^2)+(\rho_i^1\rho_j^0+\rho_i^0\rho_j^1)(u_i^1-u_j^1)\\ &\quad+(\rho_i^0\rho_j^2+\rho_i^1\rho_j^1+\rho_i^2\rho_j^0)(u_i^0-u_j^0)\Big).\end{aligned}$$

First, we consider equations (22) of order $O(1/\varepsilon)$. By assumption **(N)** on page 5, the first constraint in (21), and Lemma 1, we deduce that $u_i^0=0$ for $i=1,\dots,n$, which simplifies equations (23)-(26). Then, summing (24) from $i=1,\dots,n$ and using the symmetry of (b_{ij}), $(\rho_1^0,\dots,\rho_n^0,v^0)$ can be determined by solving the closed system

$$(27)\qquad \partial_t\rho_i^0+\operatorname{div}(\rho_i^0v^0)=0,$$

$$(28)\qquad \partial_t\bigg(\sum_{i=1}^n\rho_i^0v^0\bigg)+\operatorname{div}\bigg(\sum_{i=1}^n\rho_i^0v^0\otimes v^0\bigg)=-\sum_{i=1}^n\rho_i^0\nabla\frac{\delta\mathcal{E}}{\delta\rho_i}(\boldsymbol{\rho}^0).$$

It follows from (24) that $u_1^1,\dots,u_n^1$ satisfy the linear system

$$(29)\qquad -\sum_{j=1}^n b_{ij}\rho_i^0\rho_j^0(u_i^1-u_j^1)=d_i^0,$$

$$\text{where}\quad d_i^0=\partial_t(\rho_i^0v^0)+\operatorname{div}(\rho_i^0v^0\otimes v^0)+\rho_i^0\nabla\frac{\delta\mathcal{E}}{\delta\rho_i}(\boldsymbol{\rho}^0).$$

Since $u_i^0=0$, the second constraint in (21) becomes $\sum_{i=1}^n\rho_i^0u_i^1=0$. Moreover, (28) is equivalent to $\sum_{i=1}^n d_i^0=0$, which ensures the solvability of (29). By Lemma 1, there exists a unique solution $(u_1^1,\dots,u_n^1)$ to (29).

Next, we focus on the terms (25)-(26) of order $O(\varepsilon)$. We rewrite these equations using $u_i^0=0$ and the constraint $\sum_{i=1}^n\rho_i^0u_i^1=0$ as

$$(30)\qquad \partial_t\rho_i^1+\operatorname{div}(\rho_i^1v^0+\rho_i^0v^1)=-\operatorname{div}(\rho_i^0u_i^1),$$

$$\begin{aligned}(31)\qquad &\partial_t\bigg(\sum_{i=1}^n\rho_i^1v^0+\sum_{i=1}^n\rho_i^0v^1\bigg)+\operatorname{div}\bigg(\sum_{i=1}^n\rho_i^1v^0\otimes v^0+\sum_{i=1}^n\rho_i^0(v^1\otimes v^0+v^0\otimes v^1)\bigg)\\ &\quad=-\sum_{i=1}^n\bigg\{\rho_i^1\nabla\frac{\delta\mathcal{E}}{\delta\rho_i}(\boldsymbol{\rho}^0)+\rho_i^0\nabla\bigg(\sum_{j=1}^n\frac{\delta^2\mathcal{E}}{\delta\rho_i\delta\rho_j}(\boldsymbol{\rho}^0)\rho_j^1\bigg)\bigg\}.\end{aligned}$$

This is a closed system providing $(\rho_1^1,\dots,\rho_n^1,v^1)$.

The last task is to reconstruct the effective equations that are valid asymptotically up to order $O(\varepsilon^2)$. We are adding (27) and ε times (30) as well as (28) and ε times (31):

$$\partial_t(\rho_i^0+\varepsilon\rho_i^1)+\operatorname{div}\big(\rho_i^0v^0+\varepsilon(\rho_i^1v^0+\rho_i^0v^1)\big)=-\varepsilon\operatorname{div}(\rho_i^0u_i^1),$$

$$\partial_t\big(\rho^0 v^0 + \varepsilon(\rho^1 v^0 + \rho^0 v^1)\big) + \operatorname{div}\big(\rho^0 v^0 \otimes v^0 + \varepsilon(\rho^1 v^0 \otimes v^0 + \rho^0 v^1 \otimes v^0 + \rho^0 v^0 \otimes v^1)\big)$$
$$= -\sum_{i=1}^n \rho_i^0 \nabla \frac{\delta \mathcal{E}}{\delta \rho_i}(\boldsymbol{\rho}^0) - \varepsilon \sum_{i=1}^n \left\{ \rho_i^1 \nabla \frac{\delta \mathcal{E}}{\delta \rho_i}(\boldsymbol{\rho}^0) + \rho_i^0 \nabla \bigg(\sum_{j=1}^n \frac{\delta^2 \mathcal{E}}{\delta \rho_i \delta \rho_j}(\boldsymbol{\rho}^0) \rho_j^1 \bigg) \right\},$$

where ρ^0 and ρ^1 are defined in (20). With the notation

$$\rho_i^\varepsilon = \rho_i^0 + \varepsilon \rho_i^1 + O(\varepsilon^2), \quad u_i^\varepsilon = u_i^0 + \varepsilon u_i^1 + O(\varepsilon^2),$$
$$v^\varepsilon = v^0 + \varepsilon v^1 + O(\varepsilon^2), \quad \rho^\varepsilon = \sum_{i=1}^n \rho_i^\varepsilon,$$

and recalling that $u_i^0 = 0$, we infer that $(\rho_1^\varepsilon, \ldots, \rho_n^\varepsilon, v^\varepsilon)$ satisfies

$$\partial_t \rho_i^\varepsilon + \operatorname{div}(\rho_i^\varepsilon v^\varepsilon) = -\operatorname{div}(\rho_i^\varepsilon u_i^\varepsilon) + O(\varepsilon^2),$$
$$\partial_t(\rho^\varepsilon v^\varepsilon) + \operatorname{div}(\rho^\varepsilon v^\varepsilon \otimes v^\varepsilon) = -\sum_{i=1}^n \rho_i^\varepsilon \nabla \frac{\delta \mathcal{E}}{\delta \rho_i}(\boldsymbol{\rho}^\varepsilon) + O(\varepsilon^2).$$

It remains to reconstruct the formula determining $(u_1^\varepsilon, \ldots, u_n^\varepsilon)$. We deduce from (29) that

$$-\sum_{j=1}^n b_{ij} \rho_i^\varepsilon \rho_j^\varepsilon (u_i^\varepsilon - u_j^\varepsilon) = -\varepsilon \sum_{j=1}^n b_{ij} \rho_i^0 \rho_j^0 (u_i^1 - u_j^1) + O(\varepsilon^2) = \varepsilon d_i^0 + O(\varepsilon^2). \tag{32}$$

The variables d_i^0 can be expressed in terms of $\boldsymbol{\rho}^0$ only. Indeed, since $\partial_t \rho_i^0 + \operatorname{div}(\rho_i^0 v^0) = 0$ and $\partial_t \rho^0 + \operatorname{div}(\rho^0 v^0) = 0$, it follows that

$$\begin{aligned} d_i^0 &= \big(\partial_t \rho_i^0 + \operatorname{div}(\rho_i^0 v^0)\big) v^0 + \rho_i^0 \big(\partial_t v^0 + v^0 \cdot \nabla v^0\big) + \rho_i^0 \nabla \frac{\delta \mathcal{E}}{\delta \rho_i}(\boldsymbol{\rho}^0) \\ &= \rho_i^0 \big(\partial_t v^0 + v^0 \cdot \nabla v^0\big) + \rho_i^0 \nabla \frac{\delta \mathcal{E}}{\delta \rho_i}(\boldsymbol{\rho}^0) \\ &= \frac{\rho_i^0}{\rho^0} \big(\partial_t(\rho^0 v^0) + \operatorname{div}(\rho^0 v^0 \otimes v^0)\big) + \rho_i^0 \nabla \frac{\delta \mathcal{E}}{\delta \rho_i}(\boldsymbol{\rho}^0) \\ &= -\frac{\rho_i^0}{\rho^0} \sum_{j=1}^n \rho_j^0 \nabla \frac{\delta \mathcal{E}}{\delta \rho_j}(\boldsymbol{\rho}^0) + \rho_i^0 \nabla \frac{\delta \mathcal{E}}{\delta \rho_i}(\boldsymbol{\rho}^0), \end{aligned}$$

where in the last step we have used (28). This motivates us to define

$$d_i^\varepsilon := -\frac{\rho_i^\varepsilon}{\rho^\varepsilon} \sum_{j=1}^n \rho_j^\varepsilon \nabla \frac{\delta \mathcal{E}}{\delta \rho_j}(\boldsymbol{\rho}^\varepsilon) + \rho_i^\varepsilon \nabla \frac{\delta \mathcal{E}}{\delta \rho_i}(\boldsymbol{\rho}^\varepsilon). \tag{33}$$

Hence, we can formulate (32) as

$$-\sum_{j=1}^n b_{ij} \rho_i^\varepsilon \rho_j^\varepsilon (u_i^\varepsilon - u_j^\varepsilon) = \varepsilon d_i^\varepsilon + O(\varepsilon^2).$$

The constraints $\sum_{i=1}^n \rho_i^0 u_i^0 = 0$ and $\sum_{j=1}^n \rho_i^0 u_i^1 = 0$ from (21) imply that

$$\sum_{i=1}^n \rho_i^\varepsilon u_i^\varepsilon = \sum_{i=1}^n \rho_i^0 u_i^0 + \varepsilon \sum_{i=1}^n \rho_i^0 u_i^1 + O(\varepsilon^2) = O(\varepsilon^2).$$

As the functions ρ_i^ε, v^ε, and u_i^ε are defined only up to order $O(\varepsilon^2)$, we may set $\sum_{i=1}^n \rho_i^\varepsilon u_i^\varepsilon = 0$ up to that order.

We summarize our calculations. The functions $(\boldsymbol{\rho}^\varepsilon, v^\varepsilon) = (\rho_1^\varepsilon, \ldots, \rho_n^\varepsilon, v^\varepsilon)$ satisfy up to order $O(\varepsilon^2)$ the effective equations

$$\partial_t \rho_i^\varepsilon + \operatorname{div}(\rho_i^\varepsilon v^\varepsilon) = -\operatorname{div}(\rho_i^\varepsilon u_i^\varepsilon), \tag{34}$$

$$\partial_t(\rho^\varepsilon v^\varepsilon) + \operatorname{div}(\rho^\varepsilon v^\varepsilon \otimes v^\varepsilon) = -\sum_{i=1}^n \rho_i^\varepsilon \nabla \frac{\delta \mathcal{E}}{\delta \rho_i}(\boldsymbol{\rho}^\varepsilon), \tag{35}$$

where $\rho^\varepsilon = \sum_{i=1}^n \rho_i^\varepsilon$, and $\boldsymbol{u}^\varepsilon = (u_1^\varepsilon, \ldots, u_n^\varepsilon)$ is the unique solution to

$$-\sum_{j=1}^n b_{ij} \rho_i^\varepsilon \rho_j^\varepsilon (u_i^\varepsilon - u_j^\varepsilon) = \varepsilon d_i^\varepsilon, \quad \sum_{j=1}^n \rho_j^\varepsilon u_j^\varepsilon = 0, \tag{36}$$

for $i = 1, \ldots, n$, where d_i^ε is defined in (33).

2.3. **Gradient-flow structure and parabolicity.** We show that the effective equations have a formal gradient-flow structure and, if the total mass is constant, a parabolic structure in the sense of Petrovskii [1]. First, we reformulate system (34)-(36).

Lemma 2 (Gradient-flow structure)**.** *System* (34)-(36) *can be rewritten as*

$$\partial_t \rho_i^\varepsilon + \operatorname{div}(\rho_i^\varepsilon v^\varepsilon) = \varepsilon \operatorname{div} \sum_{j=1}^n D_{ij}^\varepsilon \nabla \frac{\delta \mathcal{E}}{\delta \rho_j}(\boldsymbol{\rho}^\varepsilon),$$

$$\partial_t(\rho^\varepsilon v^\varepsilon) + \operatorname{div}(\rho^\varepsilon v^\varepsilon \otimes v^\varepsilon) = -\sum_{i=1}^n \rho_i^\varepsilon \nabla \frac{\delta \mathcal{E}}{\delta \rho_i}(\boldsymbol{\rho}^\varepsilon),$$

where $i = 1, \ldots, n$, $\rho^\varepsilon = \sum_{i=1}^n \rho_i^\varepsilon$, *and*

$$D^\varepsilon = G(Q^\varepsilon)^{-1}(\tau^\varepsilon)^{-1}(Q^\varepsilon)^{-1} G^\top \in \mathbb{R}^{n \times n},$$

where $(Q^\varepsilon)^{-1} \in \mathbb{R}^{(n-1)\times(n-1)}$ *has the coefficients* $(Q^\varepsilon)_{ij}^{-1} = \delta_{ij}\rho_i^\varepsilon - \rho_i^\varepsilon \rho_j^\varepsilon / \rho^\varepsilon$, $(\tau^\varepsilon)^{-1}$ *is the inverse of the* $(n-1) \times (n-1)$ *matrix introduced in Lemma 1, and* $G = (G_{ij}) \in \mathbb{R}^{n \times (n-1)}$ *is defined by* $G_{ii} = 1$, $G_{ni} = -1$ *for* $i = 1, \ldots, n-1$, *and* $G_{ij} = 0$ *elsewhere.*

Proof. In view of Lemma 1, the solution to (36) can be expressed as

$$\rho_i^\varepsilon u_i^\varepsilon = -\varepsilon \sum_{j,k=1}^{n-1} \left(\delta_{ij} \rho_i^\varepsilon - \frac{\rho_i^\varepsilon \rho_j^\varepsilon}{\rho^\varepsilon} \right) (\tau^\varepsilon)_{jk}^{-1} d_k^\varepsilon, \quad i = 1, \ldots, n-1, \tag{37}$$

where $(\tau^\varepsilon)^{-1} = ((\tau^\varepsilon)^{-1}_{jk})$ is the inverse of a regular matrix in $\mathbb{R}^{(n-1)\times(n-1)}$ whose coefficients only depend on $b_{ij}\rho_i^\varepsilon\rho_j^\varepsilon$. We wish to reformulate d_i^ε in terms of $\boldsymbol{\rho}^\varepsilon$. For this, we compute, using $\rho_n^\varepsilon = \rho^\varepsilon - \sum_{j=1}^{n-1}\rho_j^\varepsilon$,

$$\begin{aligned}
d_i^\varepsilon &= \rho_i^\varepsilon\nabla\frac{\delta\mathcal{E}}{\delta\rho_i}(\boldsymbol{\rho}^\varepsilon) - \frac{\rho_i^\varepsilon}{\rho^\varepsilon}\sum_{j=1}^n\rho_j^\varepsilon\nabla\frac{\delta\mathcal{E}}{\delta\rho_j}(\boldsymbol{\rho}^\varepsilon) \\
&= \sum_{j=1}^{n-1}\left(\delta_{ij}\rho_i^\varepsilon - \frac{\rho_i^\varepsilon\rho_j^\varepsilon}{\rho^\varepsilon}\right)\nabla\frac{\delta\mathcal{E}}{\delta\rho_j}(\boldsymbol{\rho}^\varepsilon) - \frac{\rho_i^\varepsilon\rho_n^\varepsilon}{\rho^\varepsilon}\nabla\frac{\delta\mathcal{E}}{\delta\rho_n}(\boldsymbol{\rho}^\varepsilon) \\
&= \sum_{j=1}^{n-1}(Q^\varepsilon)^{-1}_{ij}\nabla\frac{\delta\mathcal{E}}{\delta\rho_j}(\boldsymbol{\rho}^\varepsilon) - \frac{\rho_i^\varepsilon}{\rho^\varepsilon}\left(\rho^\varepsilon - \sum_{j=1}^{n-1}\rho_j^\varepsilon\right)\nabla\frac{\delta\mathcal{E}}{\delta\rho_n}(\boldsymbol{\rho}^\varepsilon) \\
&= \sum_{j=1}^{n-1}(Q^\varepsilon)^{-1}_{ij}\nabla\frac{\delta\mathcal{E}}{\delta\rho_j}(\boldsymbol{\rho}^\varepsilon) - \sum_{j=1}^{n-1}\left(\delta_{ij}\rho_i^\varepsilon - \frac{\rho_i^\varepsilon\rho_j^\varepsilon}{\rho^\varepsilon}\right)\nabla\frac{\delta\mathcal{E}}{\delta\rho_n}(\boldsymbol{\rho}^\varepsilon) \\
&= \sum_{j=1}^{n-1}(Q^\varepsilon)^{-1}_{ij}\nabla\left(\frac{\delta\mathcal{E}}{\delta\rho_j}(\boldsymbol{\rho}^\varepsilon) - \frac{\delta\mathcal{E}}{\delta\rho_n}(\boldsymbol{\rho}^\varepsilon)\right).
\end{aligned}$$

Inserting this expression into (37) gives

$$\begin{aligned}
\rho_i^\varepsilon u_i^\varepsilon &= -\varepsilon\sum_{j,k,\ell=1}^{n-1}(Q^\varepsilon)^{-1}_{ij}(\tau^\varepsilon)^{-1}_{jk}(Q^\varepsilon)^{-1}_{k\ell}\nabla\left(\frac{\delta\mathcal{E}}{\delta\rho_\ell}(\boldsymbol{\rho}^\varepsilon) - \frac{\delta\mathcal{E}}{\delta\rho_n}(\boldsymbol{\rho}^\varepsilon)\right) \\
&= -\varepsilon\sum_{\ell=1}^{n-1}\widetilde{D}_{i\ell}^\varepsilon\nabla\left(\frac{\delta\mathcal{E}}{\delta\rho_\ell}(\boldsymbol{\rho}^\varepsilon) - \frac{\delta\mathcal{E}}{\delta\rho_n}(\boldsymbol{\rho}^\varepsilon)\right), \qquad (38)
\end{aligned}$$

with $\widetilde{D}_{ij}^\varepsilon$ the elements of the invertible matrix $\widetilde{D}^\varepsilon = (Q^\varepsilon)^{-1}(\tau^\varepsilon)^{-1}(Q^\varepsilon)^{-1} \in \mathbb{R}^{(n-1)\times(n-1)}$. Finally, setting $D^\varepsilon := G\widetilde{D}^\varepsilon G^\top$, we can formulate (38) as

$$\rho_i^\varepsilon u_i^\varepsilon = -\varepsilon\sum_{j=1}^n D_{ij}^\varepsilon\nabla\frac{\delta\mathcal{E}}{\delta\rho_j}(\boldsymbol{\rho}^\varepsilon), \quad i = 1,\ldots,n. \qquad (39)$$

Note that in this writing, the last row of the matrix expresses the constraint $\rho_n u_n = -\sum_{j=1}^{n-1}\rho_j u_j$. We finish the proof after inserting this expression into (34). □

Let $v^\varepsilon = 0$. Then the sum of (34) over $i = 1,\ldots,n$ yields, because of $\sum_{i=1}^n\rho_i^\varepsilon u_i^\varepsilon = 0$, $\partial_t\rho^\varepsilon = 0$. Thus, ρ^ε does not depend on time and is fixed by the initial total mass. It is sufficient to consider $\widetilde{\boldsymbol{\rho}}^\varepsilon := (\rho_1^\varepsilon,\ldots,\rho_{n-1}^\varepsilon)$ since the last component can be recovered from $\rho_n^\varepsilon = \rho^\varepsilon - \sum_{i=1}^{n-1}\rho_i^\varepsilon$. Accordingly, the energy can be formulated as a function of the variable $\widetilde{\boldsymbol{\rho}}^\varepsilon$:

$$\widetilde{\mathcal{E}}(\widetilde{\boldsymbol{\rho}}^\varepsilon) := \mathcal{E}\left(\rho_1^\varepsilon,\ldots,\rho_{n-1}^\varepsilon,\rho^\varepsilon - \sum_{i=1}^{n-1}\rho_i^\varepsilon\right). \qquad (40)$$

Lemma 3 (Parabolicity in the sense of Petrovskii)**.** *Let $(\boldsymbol{\rho}^\varepsilon, v^\varepsilon)$ be a solution to* (34)-(35) *with $v^\varepsilon = 0$ and let $\boldsymbol{u}^\varepsilon$ be a solution to* (36)*. Suppose that $\mathcal{E}(\boldsymbol{\rho}^\varepsilon)$ is strictly convex. Then $\boldsymbol{\rho}^\varepsilon$ solves*

$$\partial_t \rho_i^\varepsilon = \varepsilon \operatorname{div} \sum_{j=1}^{n-1} \widetilde{D}_{ij}^\varepsilon \nabla \frac{\delta \widetilde{\mathcal{E}}}{\delta \rho_j}(\widetilde{\boldsymbol{\rho}}^\varepsilon), \quad i = 1, \ldots, n-1, \tag{41}$$

the matrix $\widetilde{D}^\varepsilon = (\widetilde{D}_{ij}^\varepsilon)$ is positive definite, and the energy $\widetilde{\mathcal{E}}$ is a Lyapunov functional along solutions to (41)*:*

$$\frac{d\widetilde{\mathcal{E}}}{dt} = -\varepsilon \int_{\mathbb{R}^3} \sum_{i,j=1}^n \widetilde{D}_{ij}^\varepsilon \nabla \frac{\delta \widetilde{\mathcal{E}}}{\delta \rho_i} \cdot \nabla \frac{\delta \widetilde{\mathcal{E}}}{\delta \rho_j} dx \le 0.$$

Moreover, if $\rho_i^\varepsilon > 0$ for $i = 1, \ldots, n$, all eigenvalues of $\widetilde{D}^\varepsilon \widetilde{\mathcal{E}}''$ are real and positive (here, $\widetilde{\mathcal{E}}'' = d^2\widetilde{\mathcal{E}}/d\widetilde{\boldsymbol{\rho}}^2$ is the Hessian of the energy $\widetilde{\mathcal{E}}$). This means that (41) *is parabolic in the sense of Petrovskii.*

A second-order system is called *parabolic in the sense of Petrovskii* if the real parts of the eigenvalues of the diffusion matrix are positive; see [1, Remark 4.2a].

Proof. Since the variational derivative of $\widetilde{\mathcal{E}}$ equals

$$\frac{\delta \widetilde{\mathcal{E}}}{\delta \rho_i}(\widetilde{\boldsymbol{\rho}}^\varepsilon) = \frac{\delta \mathcal{E}}{\delta \rho_i}(\boldsymbol{\rho}^\varepsilon) - \frac{\delta \mathcal{E}}{\delta \rho_n}(\boldsymbol{\rho}^\varepsilon), \quad i = 1, \ldots, n-1,$$

expression (38) in the proof of Lemma 2 shows that for $i = 1, \ldots, n-1$,

$$\varepsilon \sum_{j=1}^n D_{ij}^\varepsilon \nabla \frac{\delta \mathcal{E}}{\delta \rho_j}(\boldsymbol{\rho}^\varepsilon) = \varepsilon \sum_{j=1}^{n-1} \widetilde{D}_{ij}^\varepsilon \nabla \left(\frac{\delta \mathcal{E}}{\delta \rho_j}(\boldsymbol{\rho}^\varepsilon) - \frac{\delta \mathcal{E}}{\delta \rho_n}(\boldsymbol{\rho}^\varepsilon) \right) = \varepsilon \sum_{j=1}^{n-1} \widetilde{D}_{ij}^\varepsilon \nabla \frac{\delta \widetilde{\mathcal{E}}}{\delta \rho_j}(\widetilde{\boldsymbol{\rho}}^\varepsilon),$$

proving (41). Next, we show that $\widetilde{D}^\varepsilon$ is positive definite. As (b_{ij}) is a symmetric matrix with nonnegative entries (by assumption **(N)** on page 5), the matrix

$$\tau_{ij}^\varepsilon = \delta_{ij} \sum_{k=1}^n b_{ik} \rho_i^\varepsilon \rho_k^\varepsilon - b_{ij} \rho_i^\varepsilon \rho_j^\varepsilon$$

is symmetric, diagonally dominant, and has real nonnegative diagonal elements. Therefore, (τ_{ij}^ε) is positive semidefinite. We know from the proof of Lemma 1 that there exists an invertible $(n-1) \times (n-1)$ submatrix $(\tau^\varepsilon)_{ij}^{-1}$. This submatrix is symmetric, positive semidefinite, and invertible, so all its eigenvalues must be positive and, in fact, it is positive definite. Moreover, since $(Q^\varepsilon)^{-1}$ is regular, $\widetilde{D}^\varepsilon = (Q^\varepsilon)^{-1}(\tau^\varepsilon)^{-1}(Q^\varepsilon)^{-1}$ is positive definite.

It remains to show that $\widetilde{D}^\varepsilon \widetilde{\mathcal{E}}''$ has only real and positive eigenvalues. We claim that $\widetilde{\mathcal{E}}''$ is positive definite. To see this, we calculate (dropping the superindex ε)

$$\widetilde{\mathcal{E}}'' = \frac{d}{d\widetilde{\boldsymbol{\rho}}} \left(\frac{d\widetilde{\mathcal{E}}}{d\boldsymbol{\rho}} \frac{d\boldsymbol{\rho}}{d\widetilde{\boldsymbol{\rho}}} \right) = \left(\frac{d\boldsymbol{\rho}}{d\widetilde{\boldsymbol{\rho}}} \right)^\top \frac{d^2\widetilde{\mathcal{E}}}{d\boldsymbol{\rho}^2} \left(\frac{d\boldsymbol{\rho}}{d\widetilde{\boldsymbol{\rho}}} \right) + \frac{d\widetilde{\mathcal{E}}}{d\boldsymbol{\rho}} \frac{d^2\boldsymbol{\rho}}{d\widetilde{\boldsymbol{\rho}}^2}.$$

Since $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_{n-1}, \rho - \sum_{i=1}^{n-1} \rho_i)$, we have

$$\frac{d\boldsymbol{\rho}}{d\widetilde{\boldsymbol{\rho}}} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & 0 & & 1 \\ -1 & -1 & \cdots & -1 \end{pmatrix} \in \mathbb{R}^{n\times(n-1)},$$

and $d^2\boldsymbol{\rho}/d\widetilde{\boldsymbol{\rho}}^2$ vanishes since the transformation $\widetilde{\boldsymbol{\rho}} \mapsto \boldsymbol{\rho}$ is linear. By the strict convexity of $\widetilde{\mathcal{E}}$, there exists $\kappa > 0$ such that for any $z = (z_1, \ldots, z_{n-1}) \in \mathbb{R}^{n-1}$,

$$z^\top \widetilde{\mathcal{E}}'' z = z^\top \left(\frac{d\boldsymbol{\rho}}{d\widetilde{\boldsymbol{\rho}}}\right)^\top \frac{d^2\widetilde{\mathcal{E}}}{d\boldsymbol{\rho}^2}\left(\frac{d\boldsymbol{\rho}}{d\widetilde{\boldsymbol{\rho}}}\right) z \ge \kappa \left|\frac{d\boldsymbol{\rho}}{d\widetilde{\boldsymbol{\rho}}} z\right|^2 = \kappa \sum_{i=1}^{n-1} z_i^2 + \kappa \left(\sum_{i=1}^{n-1} z_i\right)^2 \ge \kappa |z|^2.$$

This shows that $\widetilde{\mathcal{E}}''$ is symmetric and positive definite. Since also $\widetilde{D}^\varepsilon$ is symmetric and positive definite, Proposition 6.1 of [20] implies that the eigenvalues of $\widetilde{D}^\varepsilon \widetilde{\mathcal{E}}''$ are real and positive. □

3. Justification of the Chapman-Enskog expansion

In this section, we justify the validity of the Chapman-Enskog expansion performed in section 2.2. We recall that the energy is the sum of the partial energies depending on the partial densities and their gradients,

$$\mathcal{E}(\boldsymbol{\rho}) = \int_{\mathbb{R}^3} \sum_{i=1}^{n} F_i(\rho_i, \nabla\rho_i) dx. \tag{42}$$

It includes Euler-Korteweg models with the partial energy density (3). Under this hypothesis, it is shown in [10, formula (2.25)] that the force term in (2) can be written as the divergence of a stress tensor S_i:

$$-\rho_i \nabla \frac{\delta \mathcal{E}}{\delta \rho_i}(\boldsymbol{\rho}) = \operatorname{div} S_i(\boldsymbol{\rho}), \quad i = 1, \ldots, n, \tag{43}$$

where

$$\begin{aligned} &S_i(\boldsymbol{\rho}) = -s_i(\rho_i, \nabla\rho_i)\mathbb{I} + \operatorname{div} r_i(\rho_i, \nabla\rho_i)\mathbb{I} - H_i(\rho_i, \nabla\rho_i), \quad \text{and} \\ &s_i(\rho_i, q_i) = \rho_i \frac{\partial F_i}{\partial \rho_i}(\rho_i, q_i) + q_i \cdot \frac{\partial F_i}{\partial q_i}(\rho_i, q_i) - F_i(\rho_i, q_i), \\ &r_i(\rho_i, q_i) = \rho_i \frac{\partial F_i}{\partial q_i}(\rho_i, q_i), \\ &H_i(\rho_i, q_i) = q_i \otimes \frac{\partial F_i}{\partial q_i}(\rho_i, q_i), \end{aligned} \tag{44}$$

and $q_i = \nabla\rho_i$, $\mathbb{I}$ is the unit matrix in $\mathbb{R}^{3\times3}$.

We consider weak solutions to the original system (1)-(2),

$$\partial_t \rho_i^\varepsilon + \operatorname{div}(\rho_i^\varepsilon v_i^\varepsilon) = 0, \quad i = 1, \ldots, n, \tag{45}$$

$$\partial_t(\rho_i^\varepsilon v_i^\varepsilon) + \operatorname{div}(\rho_i^\varepsilon v_i^\varepsilon \otimes v_i^\varepsilon) = -\rho_i^\varepsilon \nabla \frac{\delta \mathcal{E}}{\delta \rho_i}(\boldsymbol{\rho}^\varepsilon) - \frac{1}{\varepsilon}\sum_{j=1}^n b_{ij}\rho_i^\varepsilon \rho_j^\varepsilon (v_i^\varepsilon - v_j^\varepsilon), \tag{46}$$

and strong solutions to the approximate system (34)-(35),

$$\partial_t \widehat{\rho}_i^\varepsilon + \operatorname{div}(\widehat{\rho}_i^\varepsilon \widehat{v}^\varepsilon) = -\operatorname{div}(\widehat{\rho}_i^\varepsilon \widehat{u}_i^\varepsilon), \quad i = 1, \ldots, n, \tag{47}$$

$$\partial_t(\widehat{\rho}^\varepsilon \widehat{v}^\varepsilon) + \operatorname{div}(\widehat{\rho}^\varepsilon \widehat{v}^\varepsilon \otimes \widehat{v}^\varepsilon) = -\sum_{j=1}^n \widehat{\rho}_j^\varepsilon \nabla \frac{\delta \mathcal{E}}{\delta \rho_j}(\widehat{\boldsymbol{\rho}}^\varepsilon), \quad \widehat{\rho}^\varepsilon = \sum_{j=1}^n \widehat{\rho}_j^\varepsilon, \tag{48}$$

where $(\widehat{u}_1^\varepsilon, \ldots, \widehat{u}_n^\varepsilon)$ solves (36),

$$-\sum_{j=1}^n b_{ij}\widehat{\rho}_i^\varepsilon \widehat{\rho}_j^\varepsilon (\widehat{u}_i^\varepsilon - \widehat{u}_j^\varepsilon) = \varepsilon \widehat{d}_i^\varepsilon, \quad \sum_{j=1}^n \widehat{\rho}_j^\varepsilon \widehat{u}_j^\varepsilon = 0, \tag{49}$$

and $\widehat{d}_i^\varepsilon$ is given by (33),

$$\widehat{d}_i^\varepsilon = -\frac{\widehat{\rho}_i^\varepsilon}{\widehat{\rho}^\varepsilon}\sum_{j=1}^n \widehat{\rho}_j^\varepsilon \nabla \frac{\delta \mathcal{E}}{\delta \rho_j}(\widehat{\boldsymbol{\rho}}^\varepsilon) + \widehat{\rho}_i^\varepsilon \nabla \frac{\delta \mathcal{E}}{\delta \rho_i}(\widehat{\boldsymbol{\rho}}^\varepsilon).$$

Our aim is to show that the difference of the solutions of (45)-(46) and (47)-(48) converges to zero as $\varepsilon \to 0$ in a certain sense; see Theorem 7 below.

Lemma 2 shows that system (47)-(48) can be written without the variable $\widehat{u}_i^\varepsilon$ as a diffusion system. However, the current formulation is more convenient to verify the convergence result. In the sequel, we replace $-\rho_i \nabla(\delta\mathcal{E}/\delta\rho_i)$ by $\operatorname{div} S_i$ using (43).

3.1. **Preparations.** We reformulate the approximate system (47)-(48) in a form that resembles the original system (45)-(46) with an error term:

Lemma 4. *Setting $\widehat{v}_i^\varepsilon = \widehat{v}^\varepsilon + \widehat{u}_i^\varepsilon$, system (47)-(48) is equivalent to*

$$\partial_t \widehat{\rho}_i^\varepsilon + \operatorname{div}(\widehat{\rho}_i^\varepsilon \widehat{v}_i^\varepsilon) = 0, \tag{50}$$

$$\partial_t(\widehat{\rho}_i^\varepsilon \widehat{v}_i^\varepsilon) + \operatorname{div}(\widehat{\rho}_i^\varepsilon \widehat{v}_i^\varepsilon \otimes \widehat{v}_i^\varepsilon) = -\widehat{\rho}_i^\varepsilon \nabla \frac{\delta \mathcal{E}}{\delta \rho_i}(\widehat{\boldsymbol{\rho}}^\varepsilon) - \frac{1}{\varepsilon}\sum_{j=1}^n b_{ij}\widehat{\rho}_i^\varepsilon \widehat{\rho}_j^\varepsilon (\widehat{v}_i^\varepsilon - \widehat{v}_j^\varepsilon) + \widehat{R}_i^\varepsilon, \tag{51}$$

where the remainder $\widehat{R}_i^\varepsilon$ is given by

$$\widehat{R}_i^\varepsilon := -\widehat{v}^\varepsilon \operatorname{div}(\widehat{\rho}_i^\varepsilon \widehat{u}_i^\varepsilon) + \partial_t(\widehat{\rho}_i^\varepsilon \widehat{u}_i^\varepsilon) + \operatorname{div}(\widehat{\rho}_i^\varepsilon \widehat{u}_i^\varepsilon \otimes \widehat{v}^\varepsilon + \widehat{\rho}_i^\varepsilon \widehat{v}^\varepsilon \otimes \widehat{u}_i^\varepsilon) + \operatorname{div}(\widehat{\rho}_i^\varepsilon \widehat{u}_i^\varepsilon \otimes \widehat{u}_i^\varepsilon). \tag{52}$$

Proof. Equation (50) follows directly from (47) and the definition $\widehat{v}_i^\varepsilon = \widehat{v}^\varepsilon + \widehat{u}_i^\varepsilon$. We write the evolution of the momentum in a similar format as (46),

$$\partial_t(\widehat{\rho}_i^\varepsilon \widehat{v}_i^\varepsilon) + \operatorname{div}(\widehat{\rho}_i^\varepsilon \widehat{v}_i^\varepsilon \otimes \widehat{v}_i^\varepsilon) = -\widehat{\rho}_i^\varepsilon \nabla \frac{\delta \mathcal{E}}{\delta \rho_i}(\widehat{\boldsymbol{\rho}}^\varepsilon) - \frac{1}{\varepsilon}\sum_{j=1}^n b_{ij}\widehat{\rho}_i^\varepsilon \widehat{\rho}_j^\varepsilon (\widehat{v}_i^\varepsilon - \widehat{v}_j^\varepsilon) + \widehat{R}_i^\varepsilon,$$

where $\widehat{R}_i^\varepsilon$ contains the remaining terms:

$$\widehat{R}_i^\varepsilon = \partial_t(\widehat{\rho}_i^\varepsilon\widehat{v}^\varepsilon) + \operatorname{div}(\widehat{\rho}_i^\varepsilon\widehat{v}^\varepsilon\otimes\widehat{v}^\varepsilon) + \widehat{\rho}_i^\varepsilon\nabla\frac{\delta\mathcal{E}}{\delta\rho_i}(\widehat{\boldsymbol{\rho}}^\varepsilon) + \frac{1}{\varepsilon}\sum_{j=1}^n b_{ij}\widehat{\rho}_i^\varepsilon\widehat{\rho}_j^\varepsilon(\widehat{v}_i^\varepsilon-\widehat{v}_j^\varepsilon)$$
$$+ \partial_t(\widehat{\rho}_i^\varepsilon\widehat{u}_i^\varepsilon) + \operatorname{div}(\widehat{\rho}_i^\varepsilon\widehat{u}_i^\varepsilon\otimes\widehat{v}^\varepsilon + \widehat{\rho}_i^\varepsilon\widehat{v}^\varepsilon\otimes\widehat{u}_i^\varepsilon) + \operatorname{div}(\widehat{\rho}_i^\varepsilon\widehat{u}_i^\varepsilon\otimes\widehat{u}_i^\varepsilon). \tag{53}$$

It remains to show that this expression equals (52). The last three terms are already in the desired form. By (49), we have

$$\frac{1}{\varepsilon}\sum_{j=1}^n b_{ij}\widehat{\rho}_i^\varepsilon\widehat{\rho}_j^\varepsilon(\widehat{v}_i^\varepsilon-\widehat{v}_j^\varepsilon) = \frac{\widehat{\rho}_i^\varepsilon}{\widehat{\rho}^\varepsilon}\sum_{j=1}^n\widehat{\rho}_j^\varepsilon\nabla\frac{\delta\mathcal{E}}{\delta\rho_j}(\widehat{\boldsymbol{\rho}}^\varepsilon) - \widehat{\rho}_i^\varepsilon\nabla\frac{\delta\mathcal{E}}{\delta\rho_i}(\widehat{\boldsymbol{\rho}}^\varepsilon).$$

Therefore, we can replace the third and fourth terms in $\widehat{R}_i^\varepsilon$ by

$$\frac{\widehat{\rho}_i^\varepsilon}{\widehat{\rho}^\varepsilon}\sum_{j=1}^n\widehat{\rho}_j^\varepsilon\nabla\frac{\delta\mathcal{E}}{\delta\rho_j}(\widehat{\boldsymbol{\rho}}^\varepsilon). \tag{54}$$

We reformulate the first and second terms in $\widehat{R}_i^\varepsilon$. Adding (47) over $i=1,\ldots,n$ and using $\sum_{j=1}^n\widehat{\rho}_j^\varepsilon\widehat{u}_j^\varepsilon=0$, we deduce that $\partial_t\widehat{\rho}^\varepsilon+\operatorname{div}(\widehat{\rho}^\varepsilon\widehat{v}^\varepsilon)=0$. This equation and (47), (48) show that

$$\begin{aligned}
\partial_t(\widehat{\rho}_i^\varepsilon\widehat{v}^\varepsilon) + \operatorname{div}(\widehat{\rho}_i^\varepsilon\widehat{v}^\varepsilon\otimes\widehat{v}^\varepsilon) &= \big(\partial_t\widehat{\rho}_i^\varepsilon + \operatorname{div}(\widehat{\rho}_i^\varepsilon\widehat{v}^\varepsilon)\big)\widehat{v}^\varepsilon + \widehat{\rho}_i^\varepsilon\big(\partial_t\widehat{v}^\varepsilon+\widehat{v}^\varepsilon\cdot\nabla\widehat{v}^\varepsilon\big)\\
&= -\operatorname{div}(\widehat{\rho}_i^\varepsilon\widehat{u}_i^\varepsilon)\widehat{v}^\varepsilon + \frac{\widehat{\rho}_i^\varepsilon}{\widehat{\rho}^\varepsilon}\big(\partial_t(\widehat{\rho}^\varepsilon\widehat{v}^\varepsilon) - (\partial_t\widehat{\rho}^\varepsilon)\widehat{v}^\varepsilon + \widehat{\rho}^\varepsilon\widehat{v}^\varepsilon\cdot\nabla\widehat{v}^\varepsilon\big)\\
&= -\operatorname{div}(\widehat{\rho}_i^\varepsilon\widehat{u}_i^\varepsilon)\widehat{v}^\varepsilon + \frac{\widehat{\rho}_i^\varepsilon}{\widehat{\rho}^\varepsilon}\big(\partial_t(\widehat{\rho}^\varepsilon\widehat{v}^\varepsilon) + \operatorname{div}(\widehat{\rho}^\varepsilon\widehat{v}^\varepsilon\otimes\widehat{v}^\varepsilon)\big)\\
&= -\operatorname{div}(\widehat{\rho}_i^\varepsilon\widehat{u}_i^\varepsilon)\widehat{v}^\varepsilon - \frac{\widehat{\rho}_i^\varepsilon}{\widehat{\rho}^\varepsilon}\sum_{j=1}^n\widehat{\rho}_j^\varepsilon\nabla\frac{\delta\mathcal{E}}{\delta\rho_j}(\widehat{\boldsymbol{\rho}}^\varepsilon),
\end{aligned}$$

where we used (48) in the last step. The last term cancels with (54), showing that (53) reduces to (52). □

We need later the explicit expressions of the variational derivatives of $\mathcal{E}$ and S_i.

Lemma 5 (Variational derivatives of $\mathcal{E}$)**.** *Let $\mathcal{E}$ be given by* (42)*. Then, for test functions ψ_i and ϕ_i,*

$$\sum_{i=1}^n\left\langle\frac{\delta\mathcal{E}}{\delta\rho_i}(\boldsymbol{\rho}),\psi_i\right\rangle = \int_{\mathbb{R}^3}\sum_{i=1}^n\left(\frac{\partial F_i}{\partial\rho_i}(\rho_i,\nabla\rho_i)\psi_i + \frac{\partial F_i}{\partial q_i}(\rho_i,\nabla\rho_i)\cdot\nabla\psi_i\right)dx,$$
$$\sum_{i=1}^n\left\langle\!\!\left\langle\frac{\delta^2\mathcal{E}}{\delta\rho_i^2}(\boldsymbol{\rho}),(\psi_i,\phi_i)\right\rangle\!\!\right\rangle = \sum_{i=1}^n\int_{\mathbb{R}^3}(\phi_i,\nabla\phi_i)\begin{pmatrix}\partial^2F_i/\partial\rho_i^2 & \partial^2F_i/\partial\rho_i\partial q_i\\ \partial^2F_i/\partial\rho_i\partial q_i & \partial^2F_i/\partial q_i^2\end{pmatrix}\begin{pmatrix}\psi_i\\ \nabla\psi_i\end{pmatrix}dx,$$

Proof. We compute the first variational derivative with respect to the test function $\boldsymbol{\psi} = (\psi_1, \dots, \psi_n)$:

$$\sum_{i=1}^n \left\langle \frac{\delta \mathcal{E}}{\delta \rho_i}(\boldsymbol{\rho}), \psi_i \right\rangle = \frac{d}{d\tau} \mathcal{E}(\boldsymbol{\rho} + \tau \boldsymbol{\psi})\Big|_{\tau=0} = \frac{d}{d\tau} \int_{\mathbb{R}^3} \sum_{i=1}^n F_i(\rho_i + \tau\psi_i, \nabla \rho_i + \tau \nabla \psi_i) dx \Big|_{\tau=0}$$
$$= \int_{\mathbb{R}^3} \sum_{i=1}^n \left(\frac{\partial F_i}{\partial \rho_i}(\rho_i, \nabla \rho_i)\psi_i + \frac{\partial F_i}{\partial q_i}(\rho_i, \nabla \rho_i) \cdot \nabla \psi_i \right) dx.$$

Next, we calculate the second variational derivative, where $\boldsymbol{\phi} = (\phi_1, \dots, \phi_n)$:

$$\sum_{i=1}^n \left\langle\!\!\left\langle \frac{\delta^2 \mathcal{E}}{\delta \rho_i^2}(\boldsymbol{\rho}), (\psi_i, \phi_i) \right\rangle\!\!\right\rangle = \frac{d}{d\tau} \left\langle \sum_{i=1}^n \frac{\delta \mathcal{E}}{\delta \rho_i}(\boldsymbol{\rho} + \tau \boldsymbol{\phi}), \psi_i \right\rangle \Big|_{\tau=0}$$
$$= \frac{d}{d\tau} \int_{\mathbb{R}^3} \sum_{i=1}^n \left(\frac{\partial F_i}{\partial \rho_i}\big(\rho_i + \tau\phi_i, \nabla(\rho_i + \tau\phi_i)\big)\psi_i \right.$$
$$\left. - \frac{\partial F_i}{\partial q_i}\big(\rho_i + \tau\phi_i, \nabla(\rho_i + \tau\phi_i)\big) \cdot \nabla \psi_i \right) dx \Big|_{\tau=0}$$
$$= \sum_{i=1}^n \left(\frac{\partial^2 F_i}{\partial \rho_i^2}(\boldsymbol{\rho})\phi_i\psi_i + \psi_i \frac{\partial^2 F_i}{\partial \rho_i \partial q_i}(\boldsymbol{\rho}) \cdot \nabla \phi_i + \phi_i \frac{\partial^2 F_i}{\partial q_i \partial \rho_i}(\boldsymbol{\rho}) \cdot \nabla \psi_i \right.$$
$$\left. + \frac{\partial^2 F_i}{\partial q_i^2} : (\nabla \phi_i \otimes \nabla \psi_i) \right) dx$$
$$= \sum_{i=1}^n \int_{\mathbb{R}^3} (\phi_i, \nabla \phi_i) \begin{pmatrix} \partial^2 F_i/\partial \rho_i^2 & \partial^2 F_i/\partial \rho_i \partial q_i \\ \partial^2 F_i/\partial \rho_i \partial q_i & \partial^2 F_i/\partial q_i^2 \end{pmatrix} \begin{pmatrix} \psi_i \\ \nabla \psi_i \end{pmatrix} dx.$$

This finishes the proof. □

Next, we define the relative potential energy

$$\mathcal{E}(\boldsymbol{\rho}|\widehat{\boldsymbol{\rho}}) = \mathcal{E}(\boldsymbol{\rho}) - \mathcal{E}(\widehat{\boldsymbol{\rho}}) - \sum_{i=1}^n \left\langle \frac{\delta \mathcal{E}}{\delta \rho_i}(\widehat{\boldsymbol{\rho}}), \rho_i - \widehat{\rho}_i \right\rangle.$$

Taking $\psi_i = \phi_i = \rho_i - \widehat{\rho}_i$ in the above lemma leads to the formula

$$\mathcal{E}(\boldsymbol{\rho}|\widehat{\boldsymbol{\rho}}) = \int_{\mathbb{R}^3} \sum_{i=1}^n F_i(\rho_i, \nabla \rho_i | \widehat{\rho}_i, \nabla \widehat{\rho}_i) dx.$$

We also define the total energy

$$\mathcal{E}_{\rm tot}(\boldsymbol{\rho}, \boldsymbol{m}) = \mathcal{E}(\boldsymbol{\rho}) + \int_{\mathbb{R}^3} \sum_{i=1}^n \frac{1}{2}\rho_i |v_i|^2 dx = \int_{\mathbb{R}^3} \sum_{i=1}^n \left(F_i(\rho_i, \nabla \rho_i) + \frac{1}{2}\rho_i |v_i|^2 \right) dx \tag{55}$$

and the relative total energy

$$\mathcal{E}_{\rm tot}(\boldsymbol{\rho}, \boldsymbol{m} | \widehat{\boldsymbol{\rho}}, \widehat{\boldsymbol{m}}) = \mathcal{E}_{\rm tot}(\boldsymbol{\rho}, \boldsymbol{m}) - \mathcal{E}_{\rm tot}(\widehat{\boldsymbol{\rho}}, \widehat{\boldsymbol{m}}) - \sum_{i=1}^n \left\langle \frac{\delta \mathcal{E}_{\rm tot}}{\delta \rho_i}(\widehat{\boldsymbol{\rho}}, \widehat{\boldsymbol{m}}), \rho_i - \widehat{\rho}_i \right\rangle$$

$$-\sum_{i=1}^n \left\langle \frac{\delta \mathcal{E}_{\rm tot}}{\delta m_i}(\widehat{\boldsymbol{\rho}},\widehat{\boldsymbol{m}}), \rho_i v_i - \widehat{\rho}_i\widehat{v}_i \right\rangle = \int_{\mathbb{R}^3} \sum_{i=1}^n \left(F_i(\rho_i,\nabla\rho_i|\widehat{\rho}_i,\nabla\widehat{\rho}_i) + \frac{1}{2}\rho_i|v_i-\widehat{v}_i|^2\right)dx. \tag{56}$$

3.2. **Relative energy inequality.** We compare a weak solution to the original system (45)-(46) with a strong solution to the approximate system (50)-(51) via a relative energy inequality. First, we make precise the notion of weak solution to the original system.

Definition 1 (Weak and dissipative weak solutions)**.** *A function $(\boldsymbol{\rho}^\varepsilon,\boldsymbol{v}^\varepsilon)$ is called a* weak solution *to* (45)-(46) *if for all $i=1,\ldots,n$,*

$$\begin{aligned}
&0\le \rho_i^\varepsilon \in C^0([0,\infty);L^1(\mathbb{R}^3)), \quad \rho_i^\varepsilon v_i^\varepsilon \in C^0([0,\infty);L^1(\mathbb{R}^3;\mathbb{R}^3)),\\
&\rho_i^\varepsilon v_i^\varepsilon\otimes v_i^\varepsilon,\ H_i^\varepsilon \in L^1_{\rm loc}([0,\infty)\times\mathbb{R}^3;\mathbb{R}^{3\times 3}),\\
&s_i^\varepsilon \in L^1_{\rm loc}([0,\infty)\times\mathbb{R}^3), \quad r_i^\varepsilon\in L^1_{\rm loc}([0,\infty)\times\mathbb{R}^3;\mathbb{R}^3),
\end{aligned}$$

and $(\boldsymbol{\rho}^\varepsilon,\boldsymbol{v}^\varepsilon)$ solves for $\psi_i\in C_0^\infty([0,\infty);C^\infty(\mathbb{R}^3))$ and $\phi_i\in C_0^2([0,\infty);C^\infty(\mathbb{R}^3;\mathbb{R}^3))$,

$$\begin{aligned}
&-\int_0^\infty\int_{\mathbb{R}^3}\sum_{i=1}^n(\rho_i^\varepsilon\partial_t\psi_i+\rho_i^\varepsilon v_i^\varepsilon\cdot\nabla\psi_i)dxdt = \int_{\mathbb{R}^3}\sum_{i=1}^n\rho_i^\varepsilon(x,0)\psi_i(x,0)dx,\\
&-\int_0^\infty\int_{\mathbb{R}^3}\sum_{i=1}^n\big(\rho_i^\varepsilon v_i^\varepsilon\cdot\partial_t\phi_i+\rho_i^\varepsilon v_i^\varepsilon\otimes v_i^\varepsilon:\nabla\phi_i+s_i^\varepsilon\operatorname{div}\phi_i+r_i^\varepsilon\cdot\nabla\operatorname{div}\phi_i+H_i^\varepsilon:\nabla\phi_i\big)dxdt\\
&\quad=\int_{\mathbb{R}^3}(\rho_i^\varepsilon v_i^\varepsilon)(x,0)\cdot\phi_i(x,0)dx-\frac{1}{\varepsilon}\int_0^\infty\int_{\mathbb{R}^3}\sum_{i,j=1}^n b_{ij}\rho_i^\varepsilon\rho_j^\varepsilon(v_i^\varepsilon-v_j^\varepsilon)\cdot\phi_i dxdt.
\end{aligned}$$

Moreover, if additionally $\sum_{i=1}^n(F_i(\rho_i^\varepsilon,\nabla\rho_i^\varepsilon)+\frac12\rho_i^\varepsilon|v_i^\varepsilon|^2)\in C^0([0,\infty);L^1(\mathbb{R}^3))$ and the integrated energy inequality

$$\begin{aligned}
&-\int_0^\infty\mathcal{E}_{\rm tot}(\boldsymbol{\rho}^\varepsilon(t),\boldsymbol{m}^\varepsilon(t))\theta'(t)dt+\frac{1}{2\varepsilon}\int_0^\infty\int_{\mathbb{R}^3}\sum_{i,j=1}^n b_{ij}\rho_i^\varepsilon\rho_j^\varepsilon|v_i^\varepsilon-v_j^\varepsilon|^2\theta(t)dxdt\\
&\qquad\le\mathcal{E}_{\rm tot}(\boldsymbol{\rho}^\varepsilon(0),\boldsymbol{m}^\varepsilon(0))\theta(0)
\end{aligned}\tag{57}$$

holds for any $\theta\in W^{1,\infty}([0,\infty))$ compactly supported in $[0,\infty)$, then we call $(\boldsymbol{\rho}^\varepsilon,\boldsymbol{v}^\varepsilon)$ a dissipative weak solution.

We impose the following assumption:

(A1) The dissipative weak solution $(\boldsymbol{\rho}^\varepsilon,\boldsymbol{v}^\varepsilon)$ to (45)-(46) has finite total mass and finite total energy, i.e., for any $T>0$, there exists a constant $K>0$ independent of ε such that

$$\sup_{0<t<T}\int_{\mathbb{R}^3}\sum_{i=1}^n\rho_i^\varepsilon dx\le K, \quad \sup_{0<t<T}\int_{\mathbb{R}^3}\sum_{i=1}^n\left(F_i(\rho_i^\varepsilon,\nabla\rho_i^\varepsilon)+\frac12\rho_i^\varepsilon|v_i^\varepsilon|^2\right)dx\le K.$$

We proceed by establishing the relative energy inequality.

Proposition 6 (Relative energy inequality)**.** *Let $(\boldsymbol{\rho}^\varepsilon, \boldsymbol{v}^\varepsilon)$ be a dissipative weak solution to* (45)-(46) *satisfying* (**A1**)*, let $(\widehat{\boldsymbol{\rho}}^\varepsilon, \widehat{v}^\varepsilon)$ be a strong solution to* (47)*,* (48)*,* (49) *such that $\widehat{\rho}_i^\varepsilon > 0$ in $\mathbb{R}^3$, $t > 0$, and let assumption* (**N**) *on page 5 holds. Then*

$$
\begin{aligned}
&\mathcal{E}_{\rm tot}(\boldsymbol{\rho}^\varepsilon, \boldsymbol{m}^\varepsilon|\widehat{\boldsymbol{\rho}}^\varepsilon, \widehat{\boldsymbol{m}}^\varepsilon)(t) + \frac{1}{2\varepsilon}\int_0^t\int_{\mathbb{R}^3}\sum_{i,j=1}^n b_{ij}\rho_i^\varepsilon\rho_j^\varepsilon\big|(v_i^\varepsilon - v_j^\varepsilon) - (\widehat{v}_i^\varepsilon - \widehat{v}_j^\varepsilon)\big|^2 dxds \\
&\quad\le \mathcal{E}_{\rm tot}(\boldsymbol{\rho}^\varepsilon, \boldsymbol{m}^\varepsilon|\widehat{\boldsymbol{\rho}}^\varepsilon, \widehat{\boldsymbol{m}}^\varepsilon)(0) - \int_0^t\int_{\mathbb{R}^3}\sum_{i=1}^n \rho_i^\varepsilon(v_i^\varepsilon - \widehat{v}_i^\varepsilon)\otimes(v_i^\varepsilon - \widehat{v}_i^\varepsilon) : \nabla\widehat{v}_i^\varepsilon dxds \\
&\qquad - \int_0^t\int_{\mathbb{R}^3}\sum_{i=1}^n \Big(s_i(\rho_i^\varepsilon, \nabla\rho_i^\varepsilon|\widehat{\rho}_i^\varepsilon, \nabla\widehat{\rho}_i^\varepsilon)\,\mathrm{div}\,\widehat{v}_i^\varepsilon + r_i(\rho_i^\varepsilon, \nabla\rho_i^\varepsilon|\widehat{\rho}_i^\varepsilon, \nabla\widehat{\rho}_i^\varepsilon)\cdot\nabla\,\mathrm{div}\,\widehat{v}^\varepsilon \\
&\qquad + H_i(\rho_i^\varepsilon, \nabla\rho_i^\varepsilon|\widehat{\rho}_i^\varepsilon, \nabla\widehat{\rho}_i^\varepsilon) : \nabla\widehat{v}_i^\varepsilon\Big)dxds - \int_0^t\int_{\mathbb{R}^3}\sum_{i=1}^n \frac{\rho_i^\varepsilon}{\widehat{\rho}_i^\varepsilon}\widehat{R}_i^\varepsilon\cdot(v_i^\varepsilon - \widehat{v}_i^\varepsilon)dxds \\
&\qquad - \frac{1}{\varepsilon}\int_0^t\int_{\mathbb{R}^3}\sum_{i,j=1}^n b_{ij}\rho_i^\varepsilon(\rho_j^\varepsilon - \widehat{\rho}_j^\varepsilon)(v_i^\varepsilon - \widehat{v}_i^\varepsilon)\cdot(\widehat{v}_i^\varepsilon - \widehat{v}_j^\varepsilon)dxds, \qquad (58)
\end{aligned}
$$

where s_i, r_i, and H_i are defined in (44) *and $\widehat{R}_i^\varepsilon$ is defined in* (52)*, and the relative stresses are given by*

$$
g_i(\rho_i^\varepsilon, q_i^\varepsilon|\widehat{\rho}_i^\varepsilon, \widehat{q}_i^\varepsilon) = g_i(\rho_i^\varepsilon, q_i^\varepsilon) - g_i(\widehat{\rho}_i^\varepsilon, \widehat{q}_i^\varepsilon) - \frac{\partial g_i}{\partial\rho_i}(\widehat{\rho}_i^\varepsilon, \widehat{q}_i^\varepsilon)(\rho_i^\varepsilon - \widehat{\rho}_i^\varepsilon) - \frac{\partial g_i}{\partial q_i}(\widehat{\rho}_i^\varepsilon, \widehat{q}_i^\varepsilon)\cdot(q_i^\varepsilon - \widehat{q}_i^\varepsilon),
$$

where $q_i^\varepsilon = \nabla\rho_i^\varepsilon, \widehat{q}_i^\varepsilon = \nabla\widehat{\rho}_i^\varepsilon$ and g_i represents s_i, r_i, and H_i.

Proof. The proof is similar to the proof of Theorem 1 in [10], but we need to take care of the friction terms. To simplify the notation, we drop the superscript ε. Recall that the relative total energy $\mathcal{E}_{\rm tot}(\boldsymbol{\rho}, \boldsymbol{m}|\widehat{\boldsymbol{\rho}}, \widehat{\boldsymbol{m}})$ defined by (56) has four parts, $\mathcal{E}_{\rm tot}(\boldsymbol{\rho}, \boldsymbol{m})$, $-\mathcal{E}_{\rm tot}(\widehat{\boldsymbol{\rho}}, \widehat{\boldsymbol{m}})$, $-\sum_{i=1}^n\langle(\delta\mathcal{E}_{\rm tot}/\delta\rho_i)(\widehat{\boldsymbol{\rho}}, \widehat{\boldsymbol{m}}),\ \rho_i - \widehat{\rho}_i\rangle$, and $-\sum_{i=1}^n\langle(\delta\mathcal{E}_{\rm tot}/\delta m_i)(\widehat{\boldsymbol{\rho}}, \widehat{\boldsymbol{m}}), \rho_i v_i - \widehat{\rho}_i\widehat{v}_i\rangle$. We first give the energy inequalities for the first two terms and then use the weak formulations to calculate the last two terms.

Step 1: The energy inequalities. Introducing the test function

$$
\theta(s) = \begin{cases} 1 & \text{for } 0 \le s < t, \\ (t-s)/\delta + 1 & \text{for } t \le s < t+\delta, \\ 0 & \text{for } s > t+\delta, \end{cases} \qquad (59)
$$

in the integrated energy inequality (57) and passing to the limit $\delta \to 0$, we obtain

$$
\mathcal{E}_{\rm tot}(\boldsymbol{\rho}(t), \boldsymbol{m}(t)) + \frac{1}{2\varepsilon}\int_0^t\int_{\mathbb{R}^3} b_{ij}\rho_i\rho_j|v_i - v_j|^2 dxds \le \mathcal{E}_{\rm tot}(\boldsymbol{\rho}(0), \boldsymbol{m}(0)). \qquad (60)
$$

To show the energy identity for the strong solution $(\widehat{\boldsymbol{\rho}}^\varepsilon, \widehat{\boldsymbol{v}}^\varepsilon)$, we write (51) in nonconservative form:

$$
\partial_t\widehat{v}_i + \widehat{v}_i\cdot\nabla\widehat{v}_i = -\nabla\frac{\delta\mathcal{E}}{\delta\rho_i}(\widehat{\boldsymbol{\rho}}) - \frac{1}{\varepsilon}\sum_{j=1}^n b_{ij}\widehat{\rho}_j(\widehat{v}_i - \widehat{v}_j) + \frac{\widehat{R}_i}{\widehat{\rho}_i}.
$$

We multiply this equation by $\widehat{\rho}_i\widehat{v}_i$, multiply (50) by $\frac{1}{2}|\widehat{v}_i^\varepsilon|^2$, and add the resulting equations:

$$\frac{1}{2}\partial_t(\widehat{\rho}_i|\widehat{v}_i|^2) + \frac{1}{2}\operatorname{div}(\widehat{\rho}_i\widehat{v}_i|\widehat{v}_i|^2) = -\widehat{\rho}_i\widehat{v}_i\cdot\nabla\frac{\delta\mathcal{E}}{\delta\rho_i}(\widehat{\boldsymbol{\rho}}) - \frac{1}{\varepsilon}\widehat{v}_i\cdot\sum_{j=1}^n b_{ij}\widehat{\rho}_i\widehat{\rho}_j(\widehat{v}_i-\widehat{v}_j) + \widehat{v}_i\cdot\widehat{R}_i. \tag{61}$$

Furthermore, we deduce from (50) that

$$\frac{d}{dt}\mathcal{E}(\widehat{\boldsymbol{\rho}}) = \sum_{i=1}^n\left\langle\frac{\delta\mathcal{E}}{\delta\rho_i}(\widehat{\boldsymbol{\rho}}),\partial_t\widehat{\rho}_i\right\rangle = -\sum_{i=1}^n\left\langle\frac{\delta\mathcal{E}}{\delta\rho_i}(\widehat{\boldsymbol{\rho}}),\operatorname{div}(\widehat{\rho}_i\widehat{v}_i)\right\rangle = \int_{\mathbb{R}^3}\sum_{i=1}^n\nabla\frac{\delta\mathcal{E}}{\delta\rho_i}(\widehat{\boldsymbol{\rho}})\cdot(\widehat{\rho}_i\widehat{v}_i)dx.$$

Integrating (61), summing over $i=1,\ldots,n$, and inserting the previous identity yields

$$\frac{d}{dt}\bigg(\mathcal{E}(\widehat{\boldsymbol{\rho}}) + \frac{1}{2}\int_{\mathbb{R}^3}\sum_{i=1}^n\widehat{\rho}_i|\widehat{v}_i|^2dx\bigg) = -\frac{1}{\varepsilon}\int_{\mathbb{R}^3}\sum_{i,j=1}^n b_{ij}\widehat{\rho}_i\widehat{\rho}_j(\widehat{v}_i-\widehat{v}_j)\cdot\widehat{v}_i dx + \int_{\mathbb{R}^3}\sum_{i=1}^n\widehat{R}_i\cdot\widehat{v}_i dx.$$

The symmetry of (b_{ij}) and integration of the above equality over $(0,t)$ lead to the following energy equality:

$$\begin{aligned}\mathcal{E}_{\rm tot}(\widehat{\boldsymbol{\rho}}(t),\widehat{\boldsymbol{m}}(t)) + \frac{1}{2\varepsilon}\int_0^t\int_{\mathbb{R}^3}\sum_{i,j=1}^n b_{ij}\widehat{\rho}_i\widehat{\rho}_j|\widehat{v}_i-\widehat{v}_j|^2dxds \\ = \mathcal{E}_{\rm tot}(\widehat{\boldsymbol{\rho}}(0),\widehat{\boldsymbol{m}}(0)) + \int_0^t\int_{\mathbb{R}^3}\sum_{i=1}^n\widehat{R}_i\cdot\widehat{v}_i dxds.\end{aligned} \tag{62}$$

Step 2: Equation for the difference. We proceed to calculate

$$-\sum_{i=1}^n\left\langle\frac{\delta\mathcal{E}_{\rm tot}}{\delta\rho_i}(\widehat{\boldsymbol{\rho}},\widehat{\boldsymbol{m}}),\rho_i-\widehat{\rho}_i\right\rangle \quad\text{and}\quad -\sum_{i=1}^n\left\langle\frac{\delta\mathcal{E}_{\rm tot}}{\delta m_i}(\widehat{\boldsymbol{\rho}},\widehat{\boldsymbol{m}}),\rho_i v_i-\widehat{\rho}_i\widehat{v}_i\right\rangle.$$

Following the definition of the weak solutions to (45)-(46) and (50)-(51), the differences of the solutions $(\rho_i-\widehat{\rho}_i, v_i-\widehat{v}_i)$ satisfy

$$\begin{aligned}&-\int_0^\infty\int_{\mathbb{R}^3}\sum_{i=1}^n\big((\rho_i-\widehat{\rho}_i)\partial_s\psi_i + (\rho_i v_i-\widehat{\rho}_i\widehat{v}_i)\cdot\nabla\psi_i\big)dxds \\ &\qquad = \int_{\mathbb{R}^3}\sum_{i=1}^n\big(\rho_i(x,0)-\widehat{\rho}_i(x,0)\big)\psi_i(x,0)dx, \\ &-\int_0^\infty\int_{\mathbb{R}^3}\sum_{i=1}^n\big((\rho_i v_i-\widehat{\rho}_i\widehat{v}_i)\cdot\partial_s\phi_i + (\rho_i v_i\otimes v_i-\widehat{\rho}_i\widehat{v}_i\otimes\widehat{v}_i):\nabla\phi_i \\ &\qquad\quad + (s_i-\widehat{s}_i)\operatorname{div}\phi_i + (H_i-\widehat{H}_i):\nabla\phi_i + (r_i-\widehat{r}_i):\nabla\operatorname{div}\phi_i\big)dxds \\ &\qquad = \int_{\mathbb{R}^3}\sum_{i=1}^n((\rho_i v_i)(x,0)-(\widehat{\rho}_i\widehat{v}_i)(x,0))\phi_i(x,0)dx \\ &\qquad\quad -\frac{1}{\varepsilon}\int_0^\infty\int_{\mathbb{R}^3}\sum_{i,j=1}^n b_{ij}\big(\rho_i\rho_j(v_i-v_j)-\widehat{\rho}_i\widehat{\rho}_j(\widehat{v}_i-\widehat{v}_j)\big)\cdot\phi_i dxds\end{aligned}$$

$$-\int_0^\infty \int_{\mathbb{R}^3} \sum_{i=1}^n \widehat{R}_i \cdot \phi_i dxds,$$

where $s_i = s_i(\rho_i, \nabla\rho_i)$, $\widehat{s}_i = s_i(\widehat{\rho}_i, \nabla\widehat{\rho}_i)$, and similar for the other quantities. Taking the test functions

$$\psi_i(s) = \theta(s)\bigg(\frac{\partial \widehat{F}_i}{\partial \rho_i} - \operatorname{div}\frac{\partial \widehat{F}_i}{\partial q_i} - \frac{1}{2}|\widehat{v}_i|^2\bigg)(s), \quad \phi_i(s) = \theta(s)\widehat{v}_i(s),$$

where θ is defined in (59) and $\widehat{F}_i = F_i(\widehat{\rho}_i, \nabla\widehat{\rho}_i)$, the sum of the above equations becomes

$$\begin{aligned}
&\sum_{i=1}^n \bigg(\bigg\langle \frac{\delta \mathcal{E}_{\rm tot}}{\delta \rho_i}(\widehat{\boldsymbol{\rho}}, \widehat{\boldsymbol{m}}), \rho_i - \widehat{\rho}_i \bigg\rangle + \bigg\langle \frac{\delta \mathcal{E}_{\rm tot}}{\delta m_i}(\widehat{\boldsymbol{\rho}}, \widehat{\boldsymbol{m}}), \rho_i v_i - \widehat{\rho}_i\widehat{v}_i \bigg\rangle\bigg)\bigg|_0^t \\
&\quad = \int_{\mathbb{R}^3} \sum_{i=1}^n \bigg(\frac{\partial \widehat{F}_i}{\partial \rho_i} - \operatorname{div}\frac{\partial \widehat{F}_i}{\partial q_i} - \frac{1}{2}|\widehat{v}_i|^2\bigg)(\rho_i - \widehat{\rho}_i)\bigg|_0^t dx + \int_{\mathbb{R}^3} \sum_{i=1}^n (\rho_i v_i - \widehat{\rho}_i\widehat{v}_i)\cdot\widehat{v}_i\bigg|_0^t dx \\
&\quad = \int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \bigg\{ \partial_s\bigg(\frac{\partial \widehat{F}_i}{\partial \rho_i} - \operatorname{div}\frac{\partial \widehat{F}_i}{\partial q_i} - \frac{1}{2}|\widehat{v}_i|^2\bigg)(\rho_i - \widehat{\rho}_i) \\
&\qquad + (\rho_i v_i - \widehat{\rho}_i\widehat{v}_i)\cdot\nabla\bigg(\frac{\partial \widehat{F}_i}{\partial \rho_i} - \operatorname{div}\frac{\partial \widehat{F}_i}{\partial q_i} - \frac{1}{2}|\widehat{v}_i|^2\bigg)\bigg\} dxds \\
&\qquad + \int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \big((\rho_i v_i - \widehat{\rho}_i\widehat{v}_i)\cdot\partial_s\widehat{v}_i + (\rho_i v_i \otimes v_i - \widehat{\rho}_i\widehat{v}_i\otimes\widehat{v}_i) : \nabla\widehat{v}_i\big) dxds \\
&\qquad + \int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \big((s_i - \widehat{s}_i)\operatorname{div}\widehat{v}_i + (H_i - \widehat{H}_i) : \nabla\widehat{v}_i + (r_i - \widehat{r}_i)\cdot\nabla\operatorname{div}\widehat{v}_i\big) dxds \\
&\qquad - \frac{1}{\varepsilon}\int_0^t \int_{\mathbb{R}^3} \sum_{i,j=1}^n b_{ij}\big(\rho_i\rho_j(v_i - v_j) - \widehat{\rho}_i\widehat{\rho}_j(\widehat{v}_i - \widehat{v}_j)\big)\cdot\widehat{v}_i dxds \\
&\qquad - \int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \widehat{R}_i\cdot\widehat{v}_i dxds \\
&\quad =: I_1 + I_2 + I_3 + I_4 + I_5. \qquad (63)
\end{aligned}$$

We reorganize the term I_1 as follows:

$$\begin{aligned}
&I_1 = I_{11} + I_{12} + I_{13}, \quad \text{where} \\
&I_{11} = \int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \partial_s\bigg(\frac{\partial \widehat{F}_i}{\partial \rho_i} - \operatorname{div}\frac{\partial \widehat{F}_i}{\partial q_i}\bigg)(\rho_i - \widehat{\rho}_i) dxds, \\
&I_{12} = \int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n (\rho_i v_i - \widehat{\rho}_i\widehat{v}_i)\cdot\nabla\bigg(\frac{\partial \widehat{F}_i}{\partial \rho_i} - \operatorname{div}\frac{\partial \widehat{F}_i}{\partial q_i}\bigg) dxds,
\end{aligned}$$

$$I_{13} = \int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \bigg(-\frac{1}{2}\partial_s\big(|\widehat{v}_i|^2\big)(\rho_i - \widehat{\rho}_i) - \frac{1}{2}(\rho_i v_i - \widehat{\rho}_i\widehat{v}_i)\cdot\nabla\big(|\widehat{v}_i|^2\big)\bigg)dxds,$$

Step 3: Calculation of I_{11} and I_{12}. Using (50), we obtain:

$$\begin{aligned}
I_{11} &= \int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \bigg\{ \bigg(\frac{\partial^2 \widehat{F}_i}{\partial \rho_i^2}\partial_s\widehat{\rho}_i + \frac{\partial^2\widehat{F}_i}{\partial\rho_i\partial q_i}\cdot\partial_s\nabla\widehat{\rho}_i\bigg)(\rho_i-\widehat{\rho}_i) \\
&\quad - \bigg(\operatorname{div}\bigg(\frac{\partial^2\widehat{F}_i}{\partial q_i\partial\rho_i}\partial_s\widehat{\rho}_i\bigg) + \operatorname{div}\bigg(\frac{\partial^2\widehat{F}_i}{\partial q_i^2}\cdot\partial_s\nabla\widehat{\rho}_i\bigg)\bigg)(\rho_i-\widehat{\rho}_i)\bigg\}dxds \\
&= -\int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \bigg\{ \bigg(\frac{\partial^2 \widehat{F}_i}{\partial \rho_i^2}\operatorname{div}(\widehat{\rho}_i\widehat{v}_i) + \frac{\partial^2\widehat{F}_i}{\partial\rho_i\partial q_i}\nabla\operatorname{div}(\widehat{\rho}_i\widehat{v}_i)\bigg)(\rho_i-\widehat{\rho}_i) \\
&\quad - \bigg(\operatorname{div}\bigg(\frac{\partial^2\widehat{F}_i}{\partial q_i\partial\rho_i}\operatorname{div}(\widehat{\rho}_i\widehat{v}_i)\bigg) + \operatorname{div}\bigg(\frac{\partial^2\widehat{F}_i}{\partial q_i^2}\cdot\nabla\operatorname{div}(\widehat{\rho}_i\widehat{v}_i)\bigg)\bigg)(\rho_i-\widehat{\rho}_i)\bigg\}dxds \\
&= -\int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \bigg(\frac{\partial^2 \widehat{F}_i}{\partial \rho_i^2}\operatorname{div}(\widehat{\rho}_i\widehat{v}_i)(\rho_i-\widehat{\rho}_i) + \frac{\partial^2\widehat{F}_i}{\partial\rho_i\partial q_i}\cdot\nabla\operatorname{div}(\widehat{\rho}_i\widehat{v}_i)(\rho_i-\widehat{\rho}_i) \\
&\quad + \frac{\partial^2\widehat{F}_i}{\partial q_i\partial\rho_i}\cdot\nabla(\rho_i-\widehat{\rho}_i)\operatorname{div}(\widehat{\rho}_i\widehat{v}_i) + \frac{\partial^2\widehat{F}_i}{\partial q_i^2} : \big(\nabla\operatorname{div}(\widehat{\rho}_i\widehat{v}_i)\otimes\nabla(\rho_i-\widehat{\rho}_i)\big)\bigg)dxds. \qquad (64)
\end{aligned}$$

We claim that the second-order derivatives of F_i can be related to the functional derivative of S_i. Indeed, we take the variational derivative of the weak formulation of (43),

$$\bigg\langle \frac{\delta\mathcal{E}}{\delta\rho_i}(\widehat{\boldsymbol{\rho}}), \operatorname{div}(\widehat{\rho}_i\phi_i)\bigg\rangle = -\int_{\mathbb{R}^3}\widehat{\rho}_i\nabla\frac{\delta\mathcal{E}}{\delta\rho_i}(\widehat{\boldsymbol{\rho}})\cdot\phi_i dx = -\int_{\mathbb{R}^3} S_i(\widehat{\boldsymbol{\rho}_i}) : \nabla\phi_i dx$$

for some test function ϕ_i. Let $\boldsymbol{\psi} = (\psi_1,\ldots,\psi_n)$ be another test function. Then the limit $\tau\to 0$ in

$$\begin{aligned}
\frac{1}{\tau}\bigg\langle \frac{\delta\mathcal{E}}{\delta\rho_i}(\widehat{\boldsymbol{\rho}}+\tau\boldsymbol{\psi}) - \frac{\delta\mathcal{E}}{\delta\rho_i}(\widehat{\boldsymbol{\rho}}), \operatorname{div}(\widehat{\rho}_i\phi_i)\bigg\rangle + \frac{1}{\tau}\bigg\langle \frac{\delta\mathcal{E}}{\delta\rho_i}(\widehat{\boldsymbol{\rho}}+\tau\boldsymbol{\psi}), \operatorname{div}\big((\widehat{\rho}_i+\tau\psi_i)\phi_i\big) - \operatorname{div}(\widehat{\rho}_i\phi_i)\bigg\rangle \\
= -\frac{1}{\tau}\int_{\mathbb{R}^3}\big(S_i(\widehat{\rho}_i+\tau\psi_i) - S_i(\widehat{\rho}_i)\big) : \nabla\phi_i dx
\end{aligned}$$

and summation over $i=1,\ldots,n$ leads to

$$\begin{aligned}
\sum_{i=1}^n\bigg\langle\!\!\bigg\langle \frac{\delta^2\mathcal{E}}{\delta\rho_i^2}(\widehat{\boldsymbol{\rho}}), \big(\operatorname{div}(\widehat{\rho}_i\phi_i),\psi_i\big)\bigg\rangle\!\!\bigg\rangle + \sum_{i=1}^n\bigg\langle \frac{\delta\mathcal{E}}{\delta\rho_i}(\widehat{\boldsymbol{\rho}}), \operatorname{div}(\psi_i\phi_i)\bigg\rangle \\
= -\sum_{i=1}^n\int_{\mathbb{R}^3}\bigg\langle \frac{\delta S_i}{\delta\rho_i}(\widehat{\boldsymbol{\rho}}),\psi_i\bigg\rangle : \nabla\phi_i dx.
\end{aligned}$$

Inserting the expressions for the variational derivatives from Lemma 5 and choosing $\phi_i = \widehat{v}_i$ and $\psi_i = \rho_i - \widehat{\rho}_i$, we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \sum_{i=1}^n \bigg(\frac{\partial^2 \widehat{F}_i}{\partial \rho_i^2} \operatorname{div}(\widehat{\rho}_i \widehat{v}_i)(\rho_i - \widehat{\rho}_i) + \frac{\partial^2 \widehat{F}_i}{\partial \rho_i \partial q_i} \cdot \nabla(\operatorname{div}(\widehat{\rho}_i \widehat{v}_i))(\rho_i - \widehat{\rho}_i) \\
&\qquad + \frac{\partial^2 \widehat{F}_i}{\partial q_i \partial \rho_i} \cdot \nabla(\rho_i - \widehat{\rho}_i) \operatorname{div}(\widehat{\rho}_i \widehat{v}_i) + \frac{\partial^2 \widehat{F}_i}{\partial q_i^2} : \big(\nabla(\operatorname{div}(\widehat{\rho}_i \widehat{v}_i)) \otimes \nabla(\rho_i - \widehat{\rho}_i) \big) \bigg) dx \\
&\qquad - \int_{\mathbb{R}^3} \sum_{i=1}^n \nabla \bigg(\frac{\partial \widehat{F}_i}{\partial \rho_i} - \operatorname{div} \frac{\partial \widehat{F}_i}{\partial q_i} \bigg) \cdot ((\rho_i - \widehat{\rho}_i) \widehat{v}_i) dx \\
&= \int_{\mathbb{R}^3} \sum_{i=1}^n \bigg\{ \bigg(\frac{\partial \widehat{s}_i}{\partial \rho_i} (\rho_i - \widehat{\rho}_i) + \frac{\partial \widehat{s}_i}{\partial q_i} \cdot \nabla(\rho_i - \widehat{\rho}_i) \bigg) \operatorname{div} \widehat{v}_i \\
&\qquad + \bigg(\frac{\partial \widehat{r}_i}{\partial \rho_i} (\rho_i - \widehat{\rho}_i) + \frac{\partial \widehat{r}_i}{\partial q_i} \cdot \nabla(\rho_i - \widehat{\rho}_i) \bigg) \cdot \nabla \operatorname{div} \widehat{v}_i \\
&\qquad + \bigg(\frac{\partial \widehat{H}_i}{\partial \rho_i} (\rho_i - \widehat{\rho}_i) + \frac{\partial \widehat{H}_i}{\partial q_i} \cdot \nabla(\rho_i - \widehat{\rho}_i) \bigg) : \nabla \widehat{v}_i \bigg\} dx. \qquad (65)
\end{aligned}$$

The first four terms on the left-hand side correspond, up to the sign, to the right-hand side of (64). Using

$$\begin{aligned}
&- \int_{\mathbb{R}^3} \sum_{i=1}^n \nabla \bigg(\frac{\partial \widehat{F}_i}{\partial \rho_i} - \operatorname{div} \frac{\partial \widehat{F}_i}{\partial q_i} \bigg) \cdot ((\rho_i - \widehat{\rho}_i) \widehat{v}_i) dx + \int_{\mathbb{R}^3} \sum_{i=1}^n (\rho_i v_i - \widehat{\rho}_i \widehat{v}_i) \cdot \nabla \bigg(\frac{\partial \widehat{F}_i}{\partial \rho_i} - \operatorname{div} \frac{\partial \widehat{F}_i}{\partial q_i} \bigg) dx \\
&\qquad = \int_{\mathbb{R}^3} \sum_{i=1}^n \nabla \bigg(\frac{\partial \widehat{F}_i}{\partial \rho_i} - \operatorname{div} \frac{\partial \widehat{F}_i}{\partial q_i} \bigg) \cdot \rho_i (v_i - \widehat{v}_i) dx,
\end{aligned}$$

we find that

$$\begin{aligned}
I_{11} + I_{12} &= \int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \nabla \bigg(\frac{\partial \widehat{F}_i}{\partial \rho_i} - \operatorname{div} \frac{\partial \widehat{F}_i}{\partial q_i} \bigg) \cdot \rho_i (v_i - \widehat{v}_i) dx ds \\
&\quad - \int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \bigg\{ \bigg(\frac{\partial \widehat{s}_i}{\partial \rho_i} (\rho_i - \widehat{\rho}_i) + \frac{\partial \widehat{s}_i}{\partial q_i} \cdot \nabla(\rho_i - \widehat{\rho}_i) \bigg) \operatorname{div} \widehat{v}_i \\
&\quad + \bigg(\frac{\partial \widehat{r}_i}{\partial \rho_i} (\rho_i - \widehat{\rho}_i) + \frac{\partial \widehat{r}_i}{\partial q_i} \cdot \nabla(\rho_i - \widehat{\rho}_i) \bigg) \cdot \nabla \operatorname{div} \widehat{v}_i \\
&\quad + \bigg(\frac{\partial \widehat{H}_i}{\partial \rho_i} (\rho_i - \widehat{\rho}_i) + \frac{\partial \widehat{H}_i}{\partial q_i} \cdot \nabla(\rho_i - \widehat{\rho}_i) \bigg) : \nabla \widehat{v}_i \bigg\} dx ds. \qquad (66)
\end{aligned}$$

Step 4: Calculation of I_{13} and I_2. The sum of I_{13} and I_2 is

$$I_{13} + I_2 = \int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \bigg(- \frac{1}{2} \partial_s \big(|\widehat{v}_i|^2 \big) (\rho_i - \widehat{\rho}_i) - \frac{1}{2} (\rho_i v_i - \widehat{\rho}_i \widehat{v}_i) \cdot \nabla \big(|\widehat{v}_i|^2 \big) \bigg) dx ds$$

$$
\begin{aligned}
&+\int_0^t\int_{\mathbb{R}^3}\sum_{i=1}^n\big((\rho_i v_i-\widehat{\rho}_i\widehat{v}_i)\cdot\partial_s\widehat{v}_i+(\rho_i v_i\otimes v_i-\widehat{\rho}_i\widehat{v}_i\otimes\widehat{v}_i):\nabla\widehat{v}_i\big)dxds\\
&=\int_0^t\int_{\mathbb{R}^3}\sum_{i=1}^n\big(-\widehat{v}_i\otimes(\rho_i v_i-\widehat{\rho}_i\widehat{v}_i)+(\rho_i v_i\otimes v_i-\widehat{\rho}_i\widehat{v}_i\otimes\widehat{v}_i)\big):\nabla\widehat{v}_i dxds\\
&\quad+\int_0^t\int_{\mathbb{R}^3}\sum_{i=1}^n\rho_i(v_i-\widehat{v}_i)\partial_s\widehat{v}_i dxds. \qquad (67)
\end{aligned}
$$

Observing that (51) reads in nonconservative form as

$$
\partial_t\widehat{v}_i+\widehat{v}_i\cdot\nabla\widehat{v}_i=-\nabla\bigg(\frac{\partial\widehat{F}_i}{\partial\rho_i}-\operatorname{div}\frac{\partial\widehat{F}_i}{\partial q_i}\bigg)-\frac{1}{\varepsilon}\sum_{j=1}^n b_{ij}\widehat{\rho}_j(\widehat{v}_i-\widehat{v}_j)+\frac{\widehat{R}_i}{\widehat{\rho}_i},
$$

it follows that

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3}\sum_{i=1}^n\rho_i(v_i-\widehat{v}_i)\cdot\partial_s\widehat{v}_i dxds\\
&=\int_0^t\int_{\mathbb{R}^3}\sum_{i=1}^n\rho_i(v_i-\widehat{v}_i)\cdot\bigg(-\widehat{v}_i\cdot\nabla\widehat{v}_i-\nabla\bigg(\frac{\partial\widehat{F}_i}{\partial\rho_i}-\operatorname{div}\frac{\partial\widehat{F}_i}{\partial q_i}\bigg)\cdot(\rho_i(v_i-\widehat{v}_i))\\
&\quad-\frac{1}{\varepsilon}\sum_{j=1}^n b_{ij}\widehat{\rho}_j(\widehat{v}_i-\widehat{v}_j)+\frac{\widehat{R}_i}{\widehat{\rho}_i}\bigg)dxds\\
&=\int_0^t\int_{\mathbb{R}^3}\sum_{i=1}^n\bigg(-\rho_i(v_i-\widehat{v}_i)\otimes\widehat{v}_i:\nabla\widehat{v}_i-\nabla\bigg(\frac{\partial\widehat{F}_i}{\partial\rho_i}-\operatorname{div}\frac{\partial\widehat{F}_i}{\partial q_i}\bigg)\\
&\quad-\frac{1}{\varepsilon}\sum_{j=1}^n b_{ij}\rho_i\widehat{\rho}_j(v_i-\widehat{v}_i)\cdot(\widehat{v}_i-\widehat{v}_j)+\frac{\rho_i}{\widehat{\rho}_i}(v_i-\widehat{v}_i)\cdot\widehat{R}_i\bigg)dxds. \qquad (68)
\end{aligned}
$$

Substituting the above formula into (67) leads to

$$
\begin{aligned}
I_{13}+I_2&=\int_0^t\int_{\mathbb{R}^3}\sum_{i=1}^n\rho_i(v_i-\widehat{v}_i)\otimes(v_i-\widehat{v}_i):\nabla\widehat{v}_i dxds\\
&\quad-\int_0^t\int_{\mathbb{R}^3}\sum_{i=1}^n\nabla\bigg(\frac{\partial\widehat{F}_i}{\partial\rho_i}-\operatorname{div}\frac{\partial\widehat{F}_i}{\partial q_i}\bigg)\cdot\rho_i(v_i-\widehat{v}_i)dxds\\
&\quad-\frac{1}{\varepsilon}\int_0^t\int_{\mathbb{R}^3}\sum_{i,j=1}^n b_{ij}\rho_i\widehat{\rho}_j(v_i-\widehat{v}_i)\cdot(\widehat{v}_i-\widehat{v}_j)dxds\\
&\quad+\int_0^t\int_{\mathbb{R}^3}\sum_{i=1}^n\frac{\rho_i}{\widehat{\rho}_i}(v_i-\widehat{v}_i)\cdot\widehat{R}_i dxds. \qquad (69)
\end{aligned}
$$

Step 5: Calculation of I_4. We collect the terms in I_4 and the friction term in (69):

$$\begin{aligned}
&\frac{1}{\varepsilon}\sum_{i,j=1}^n \Big(\big(-b_{ij}\rho_i\rho_j(v_i-v_j) + \widehat{\rho}_i\widehat{\rho}_j(\widehat{v}_i-\widehat{v}_j)\big)\cdot\widehat{v}_i - b_{ij}\rho_i\widehat{\rho}_j(v_i-\widehat{v}_i)\cdot(\widehat{v}_i-\widehat{v}_j)\Big) \\
&\quad = \frac{1}{\varepsilon}\sum_{i,j=1}^n b_{ij}\rho_i\rho_j(v_i-v_j)\cdot(v_i-\widehat{v}_i) - \frac{1}{\varepsilon}\sum_{i,j=1}^n b_{ij}\rho_i\rho_j(v_i-v_j)\cdot v_i \\
&\quad\quad + \frac{1}{\varepsilon}\sum_{i,j=1}^n b_{ij}\widehat{\rho}_i\widehat{\rho}_j(\widehat{v}_i-\widehat{v}_j)\cdot\widehat{v}_i - \frac{1}{\varepsilon}\sum_{i,j=1}^n b_{ij}\rho_i\widehat{\rho}_j(v_i-\widehat{v}_i)\cdot(\widehat{v}_i-\widehat{v}_j). \qquad (70)
\end{aligned}$$

By the symmetry of (b_{ij}), the second and the third term on the right-hand side become

$$\begin{aligned}
-\frac{1}{\varepsilon}\sum_{i,j=1}^n b_{ij}\rho_i\rho_j(v_i-v_j)\cdot v_i &= -\frac{1}{2\varepsilon}\sum_{i,j=1}^n b_{ij}\rho_i\rho_j|v_i-v_j|^2, \\
\frac{1}{\varepsilon}\sum_{i,j=1}^n b_{ij}\widehat{\rho}_i\widehat{\rho}_j(\widehat{v}_i-\widehat{v}_j)\cdot\widehat{v}_i &= \frac{1}{2\varepsilon}\sum_{i,j=1}^n b_{ij}\widehat{\rho}_i\widehat{\rho}_j|\widehat{v}_i-\widehat{v}_j|^2.
\end{aligned}$$

We write the last term on the right-hand side of (70) as

$$\begin{aligned}
-\frac{1}{\varepsilon}\sum_{i,j=1}^n b_{ij}\rho_i\widehat{\rho}_j(v_i-\widehat{v}_i)\cdot(\widehat{v}_i-\widehat{v}_j) &= \frac{1}{\varepsilon}\sum_{i,j=1}^n b_{ij}\rho_i(\rho_j-\widehat{\rho}_j)(v_i-\widehat{v}_i)\cdot(\widehat{v}_i-\widehat{v}_j) \\
&\quad - \frac{1}{\varepsilon}\sum_{i,j=1}^n b_{ij}\rho_i\rho_j(v_i-\widehat{v}_i)\cdot(\widehat{v}_i-\widehat{v}_j).
\end{aligned}$$

The last term can be combined with the first term on the right-hand side of (70):

$$\begin{aligned}
&\frac{1}{\varepsilon}\sum_{i,j=1}^n b_{ij}\rho_i\rho_j(v_i-v_j)\cdot(v_i-\widehat{v}_i) - \frac{1}{\varepsilon}\sum_{i,j=1}^n b_{ij}\rho_i\rho_j(v_i-\widehat{v}_i)\cdot(\widehat{v}_i-\widehat{v}_j) \\
&\quad = \frac{1}{\varepsilon}\sum_{i,j=1}^n b_{ij}\rho_i\rho_j\big((v_i-v_j)-(\widehat{v}_i-\widehat{v}_j)\big)\cdot(v_i-\widehat{v}_i) \\
&\quad = \frac{1}{2\varepsilon}\sum_{i,j=1}^n b_{ij}\rho_i\rho_j|(v_i-v_j)-(\widehat{v}_i-\widehat{v}_j)|^2.
\end{aligned}$$

Then, combining these results, we conclude from (70) that

$$\begin{aligned}
&I_4 - \frac{1}{\varepsilon}\int_0^t\int_{\mathbb{R}^3}\sum_{i,j=1}^n b_{ij}\rho_i\widehat{\rho}_j(v_i-\widehat{v}_i)\cdot(\widehat{v}_i-\widehat{v}_j)dxds \\
&\quad = -\frac{1}{\varepsilon}\int_0^t\int_{\mathbb{R}^3}\sum_{i,j=1}^n \Big(\big(b_{ij}\rho_i\rho_j(v_i-v_j) - \widehat{\rho}_i\widehat{\rho}_j(\widehat{v}_i-\widehat{v}_j)\big)\cdot\widehat{v}_i
\end{aligned}$$

$$
\begin{aligned}
&+ b_{ij}\rho_i\widehat{\rho}_j(v_i-\widehat{v}_i)\cdot(\widehat{v}_i-\widehat{v}_j)\Big)dxds \\
&= \frac{1}{2\varepsilon}\int_0^t\int_{\mathbb{R}^3}\sum_{i,j=1}^n b_{ij}\rho_i\rho_j|(v_i-v_j)-(\widehat{v}_i-\widehat{v}_j)|^2dxds \\
&\quad - \frac{1}{2\varepsilon}\int_0^t\int_{\mathbb{R}^3}\sum_{i,j=1}^n b_{ij}\rho_i\rho_j|v_i-v_j|^2 + \frac{1}{2\varepsilon}\int_0^t\int_{\mathbb{R}^3}\sum_{i,j=1}^n b_{ij}\widehat{\rho}_i\widehat{\rho}_j|\widehat{v}_i-\widehat{v}_j|^2dxds \\
&\quad + \frac{1}{\varepsilon}\int_0^t\int_{\mathbb{R}^3}\sum_{i,j=1}^n b_{ij}\rho_i(\rho_j-\widehat{\rho}_j)(v_i-\widehat{v}_i)\cdot(\widehat{v}_i-\widehat{v}_j)dxds. \qquad (71)
\end{aligned}
$$

Finally, we insert (66), (69), and (71) into (63) and then subtract the resulting (63) and equation (62) from (60) to arrive at (58). □

3.3. **Convergence of the Chapman-Enskog expansion.** We proceed to justify the Chapman-Enskog expansion using the relative entropy identity. We place a series of assumptions:

(A2) The strong solution $(\widehat{\rho}_i^\varepsilon,\widehat{v}^\varepsilon)$ to (47)-(48) satisfies for $\widehat{v}_i^\varepsilon=\widehat{v}^\varepsilon+\widehat{u}_i^\varepsilon$ with $\widehat{u}_i^\varepsilon$ being a solution of (49): There exists a constant $C>0$ such that for all $\varepsilon>0$ and $i=1,\ldots,n$,

$$\|\nabla\widehat{v}_i^\varepsilon\|_{L^\infty([0,T];L^\infty(\mathbb{R}^3))}+\|\nabla\operatorname{div}\widehat{v}_i^\varepsilon\|_{L^\infty([0,T];L^\infty(\mathbb{R}^3))}\le C.$$

(A3) The strong solution $\widehat{\rho}_i^\varepsilon$ to (47)-(48) satisfies: There are constants $K>\kappa>0$ such that for all $\varepsilon>0$, $x\in\mathbb{R}^3$, $t\in(0,T)$, and $i=1,\ldots,n$,

$$\kappa\le\widehat{\rho}_i^\varepsilon(x,t)\le K.$$

(A4) Let $F_i(\rho_i,q_i)=h_i(\rho_i)+\frac{1}{2}\kappa_i(\rho_i)|q_i|^2$, where h_i and κ_i are C^3 functions and there exists a constant $\alpha>0$ such that for all $i=1,\ldots,n$ and $\rho_i\ge0$,

$$h_i''(\rho_i)\ge\alpha,\quad \kappa_i(\rho_i)\kappa_i''(\rho_i)-2\kappa_i'(\rho_i)^2\ge0,\quad \kappa_i(\rho_i)>0.$$

(A5) The dissipative weak solution $(\boldsymbol{\rho}^\varepsilon,\boldsymbol{v}^\varepsilon)$ satisfies that ρ_i^ε are uniformly bounded in $L^\infty([0,T];L^\infty(\mathbb{R}^3))$ and there are constants $K>\kappa>0$ such that

$$\kappa\le\rho_i^\varepsilon\le K\quad\text{in }\mathbb{R}^3,\ 0<t<T.$$

Hypothesis **(A1)** concerns the family of dissipative weak solutions which is assumed to satisfy the uniform bounds **(A5)**. Hypotheses **(A2)** and **(A3)** concern the family of strong solutions to the target system (47)-(48).

Hypothesis **(A4)** is a structural hypothesis on the model. It is in particular satisfied for $\kappa_i(\rho_i)=\rho_i^s$ with $s\in[-1,0]$ for $\rho_i>0$. The important special cases $s=-1$ (corresponding to the quantum hydrodynamic system) and $s=0$ (corresponding to constant capillarity) are included.

Theorem 7. *Let $(\boldsymbol{\rho}^\varepsilon, \boldsymbol{v}^\varepsilon)$ be a dissipative weak solution to (45)-(46) satisfying assumption* **(A1)** *and* **(A5)**, *and let $(\widehat{\boldsymbol{\rho}}^\varepsilon, \widehat{v}^\varepsilon)$ be a strong solution to (47)-(48) satisfying assumptions* **(A2)**-**(A4)**. *Furthermore, let assumption* **(N)** *on page 5 hold and let $T > 0$. We introduce*

$$\chi(t) = \int_{\mathbb{R}^3} \sum_{i=1}^n \left(\frac{1}{2}\rho_i^\varepsilon |v_i^\varepsilon - \widehat{v}_i^\varepsilon|^2 + (\rho_i^\varepsilon - \widehat{\rho}_i^\varepsilon)^2 + \frac{1}{2\kappa_i(\rho_i^\varepsilon)} |\kappa_i(\rho_i^\varepsilon)\nabla\rho_i^\varepsilon - \kappa_i(\widehat{\rho}_i^\varepsilon)\nabla\widehat{\rho}_i^\varepsilon|^2 \right)(t)dx.$$

Then there exists a constant $C > 0$ such that for all $\varepsilon > 0$ and $t \in (0, T)$,

$$\chi(t) \le C(\chi(0) + \varepsilon^2), \quad t \in (0, T).$$

In particular, if $\chi(0) \to 0$ as $\varepsilon \to 0$, we have

$$\sup_{t\in(0,T)} \chi(t) \to 0 \quad \text{as } \varepsilon \to 0.$$

Proof. We apply the relative energy inequality (58). First, we relate the total relative entropy to $\chi(t)$. The superscript ε is dropped for simplicity of calculations. The relative potential is

$$\begin{aligned} F_i(\rho_i, q_i|\widehat{\rho}_i, \widehat{q}_i) &= F_i(\rho_i, q_i) - F_i(\widehat{\rho}_i, \widehat{q}_i) - \frac{\partial F_i}{\partial \rho_i}(\widehat{\rho}_i, \widehat{q}_i)(\rho_i - \widehat{\rho}_i) - \frac{\partial F_i}{\partial q_i}(\widehat{\rho}_i, \widehat{q}_i)\cdot(q_i - \widehat{q}_i) \\ &= h_i(\rho_i|\widehat{\rho}_i) + \left(\frac{1}{2}\kappa_i(\rho_i)|q_i|^2\right)(\rho_i, q_i|\widehat{\rho}_i, \widehat{q}_i). \end{aligned} \tag{72}$$

The second term on the right-hand side of the above equation is calculated in detail as follows:

$$\begin{aligned} &\left(\frac{1}{2}\kappa_i(\rho_i)|q_i|^2\right)(\rho_i, q_i|\widehat{\rho}_i, \widehat{q}_i) \\ &\quad= \frac{1}{2}\kappa_i(\rho_i)|q_i|^2 - \frac{1}{2}\kappa_i(\widehat{\rho}_i)|\widehat{q}_i|^2 - \frac{1}{2}\kappa_i'(\widehat{\rho}_i)|\widehat{q}_i|^2(\rho_i - \widehat{\rho}_i) - \kappa_i(\widehat{\rho}_i)\widehat{q}_i(q_i - \widehat{q}_i) \\ &\quad= \frac{1}{2\kappa_i(\rho_i)}(\kappa_i^2(\rho_i)|q_i|^2 - 2\kappa_i(\widehat{\rho}_i)\kappa_i(\rho_i)q_i\cdot\widehat{q}_i + \kappa_i^2(\widehat{\rho}_i)|\widehat{q}_i|^2) \\ &\qquad+ \frac{1}{2}|\widehat{q}_i|^2\left(-\frac{\kappa_i^2(\widehat{\rho}_i)}{\kappa_i(\rho_i)} + \kappa_i(\widehat{\rho}_i) - \kappa_i'(\widehat{\rho}_i)(\rho_i - \widehat{\rho}_i)\right) \\ &\quad= \frac{1}{2\kappa_i(\rho_i)}|\kappa_i(\rho_i)q_i - \kappa_i(\widehat{\rho}_i)\widehat{q}_i|^2 + \frac{\kappa_i^2(\widehat{\rho}_i)|\widehat{q}_i|^2}{2}\left(-\frac{1}{\kappa_i(\rho_i)} + \frac{1}{\kappa_i(\widehat{\rho}_i)} - \frac{\kappa_i'(\widehat{\rho}_i)}{\kappa_i^2(\widehat{\rho}_i)}(\rho_i - \widehat{\rho}_i)\right) \\ &\quad= \frac{1}{2\kappa_i(\rho_i)}|\kappa_i(\rho_i)q_i - \kappa_i(\widehat{\rho}_i)\widehat{q}_i|^2 + \frac{\kappa_i^2(\widehat{\rho}_i)|\widehat{q}_i|^2}{2}\left(-\frac{1}{\kappa_i}\right)(\rho_i|\widehat{\rho}_i). \end{aligned} \tag{73}$$

Assumption **(A4)** implies that

$$\begin{aligned} \left(-\frac{1}{\kappa_i}\right)(\rho_i|\widehat{\rho}_i) &= -\frac{1}{\kappa_i(\rho_i)} + \frac{1}{\kappa_i(\widehat{\rho}_i)} - \frac{\kappa_i'(\widehat{\rho}_i)}{\kappa_i^2(\widehat{\rho}_i)}|\widehat{q}_i|^2(\rho_i - \widehat{\rho}_i) \\ &= \int_0^1\int_0^\tau -\frac{2(\kappa_i')^2 - \kappa_i\kappa_i''}{\kappa_i^3}(s\rho_i + (1-s)\widehat{\rho}_i))dsd\tau(\rho_i - \widehat{\rho}_i)^2 \ge 0. \end{aligned}$$

Due to assumption **(A4)**, the Taylor expansion of $h_i(\rho_i|\widehat{\rho}_i)$ gives

$$\begin{aligned} h_i(\rho_i|\widehat{\rho}_i) &= h_i(\rho_i) - h_i(\widehat{\rho}_i) - h_i'(\widehat{\rho}_i)(\rho_i-\widehat{\rho}_i) \\ &= \int_0^1\int_0^\tau h_i''(s\rho_i+(1-s)\widehat{\rho}_i)dsd\tau(\rho_i-\widehat{\rho}_i)^2 \ge C|\rho_i-\widehat{\rho}_i|^2. \end{aligned}$$

It follows that, for some $C>0$ independent of ε,

$$F_i(\rho_i,q_i|\widehat{\rho}_i,\widehat{q}_i) \ge C|\rho_i-\widehat{\rho}_i|^2 + \frac{1}{2\kappa_i(\rho_i)}|\kappa_i(\rho_i)q_i-\kappa_i(\widehat{\rho}_i)\widehat{q}_i|^2.$$

We deduce that

$$\mathcal{E}_{\rm tot}(\boldsymbol{\rho}^\varepsilon,\boldsymbol{m}^\varepsilon|\widehat{\boldsymbol{\rho}}^\varepsilon,\widehat{\boldsymbol{m}}^\varepsilon) = \int_{\mathbb{R}^3}\sum_{i=1}^n\left(F_i(\rho_i^\varepsilon,\nabla\rho_i^\varepsilon|\widehat{\rho}_i^\varepsilon,\nabla\widehat{\rho}_i^\varepsilon) + \frac{1}{2}\rho_i^\varepsilon|v_i^\varepsilon-\widehat{v}_i^\varepsilon|^2\right)dx \ge C\chi(t).$$

We turn to the right-hand side of the energy inequality (58). We write $J_1,\ldots,J_4$ for the four integrals on the right-hand side of (58). Thanks to assumption (A2),

$$\begin{aligned} J_1 &= -\int_0^t\int_{\mathbb{R}^3}\sum_{i=1}^n \rho_i^\varepsilon(v_i^\varepsilon-\widehat{v}_i^\varepsilon)\otimes(v_i^\varepsilon-\widehat{v}_i^\varepsilon):\nabla\widehat{v}_i^\varepsilon dxds \\ &\le C\int_0^t\int_{\mathbb{R}^3}\sum_{i=1}^n\rho_i^\varepsilon|v_i^\varepsilon-\widehat{v}_i^\varepsilon|^2dxds \le C\int_0^t\chi(s)ds. \end{aligned} \tag{74}$$

To estimate J_2, we first calculate the stress tensors using (44) and obtain

$$\begin{aligned} s_i(\rho_i,q_i) &= p_i(\rho_i) + \frac{1}{2}(\kappa_i(\rho_i)+\rho_i\kappa_i'(\rho_i))|q_i|^2,\ p_i(\rho_i) = \rho_i h_i'(\rho_i) - h_i(\rho_i), \\ r_i(\rho_i,q_i) &= \rho_i\kappa_i(\rho_i)q_i, \\ H_i(\rho_i,q_i) &= \kappa_i(\rho_i)q_i\otimes q_i. \end{aligned}$$

For $s_i(\rho_i,q_i|\widehat{\rho}_i,\widehat{q}_i)$, we first split $s_i(\rho_i,q_i)$ as

$$s_i(\rho_i,q_i) = p_i(\rho_i) + \frac{1}{2}\kappa_i(\rho_i)|q_i|^2 + A_i(\rho_i,q_i), \quad A_i(\rho_i,q_i) = \frac{1}{2}\rho_i\kappa_i'(\rho_i)|q_i|^2.$$

Due to assumption **(A4)**, p_i'' is a continuous function. Furthermore, thanks to assumptions **(A3)** and **(A5)**, $s\rho_i+(1-s)\widehat{\rho}_i$ is bounded for $s\in[0,1]$, so $p_i''(s\rho_i+(1-s)\widehat{\rho}_i))$ is bounded. The relative pressure becomes

$$p_i(\rho_i|\widehat{\rho}_i) = \int_0^1\int_0^\tau p_i''(s\rho_i+(1-s)\widehat{\rho}_i)dsd\tau(\rho_i-\widehat{\rho}_i)^2 \le C|\rho_i-\widehat{\rho}_i|^2.$$

For $A_i(\rho_i,q_i|\widehat{\rho}_i,\widehat{q}_i)$, we can replace $\kappa_i(\rho_i)$ in the calculations of $(\frac{1}{2}\kappa_i(\rho_i)|q_i|^2)(\rho_i,q_i|\widehat{\rho}_i,\widehat{q}_i)$ by $\rho_i\kappa_i'(\rho_i)$ to get

$$A_i(\rho_i,q_i|\widehat{\rho}_i,\widehat{q}_i) = \frac{1}{2\rho_i\kappa_i'(\rho_i)}|\rho_i\kappa_i'(\rho_i)q_i-\widehat{\rho}_i\kappa_i'(\widehat{\rho}_i)\widehat{q}_i|^2 + \frac{|\widehat{q}_i|^2\widehat{\rho}_i^{\,2}(\kappa_i'(\widehat{\rho}_i))^2}{2}\left(-\frac{1}{\rho_i\kappa_i'}\right)(\rho_i|\widehat{\rho}_i). \tag{75}$$

The first term on the right-hand side can be estimated as follows:

$$\begin{aligned}
&\frac{1}{2\rho_i\kappa_i'(\rho_i)}|\rho_i\kappa_i'(\rho_i)q_i-\widehat{\rho}_i\kappa_i'(\widehat{\rho}_i)\widehat{q}_i)|^2\\
&\quad=\frac{1}{2\rho_i\kappa_i'(\rho_i)}\Big|\frac{\rho_i\kappa_i'(\rho_i)}{\kappa_i(\rho_i)}(\kappa_i(\rho_i)q_i-\kappa_i(\widehat{\rho}_i)\widehat{q}_i)+\left(\frac{\rho_i\kappa_i'(\rho_i)\kappa_i(\widehat{\rho}_i)}{\kappa_i(\rho_i)}-\widehat{\rho}_i\kappa_i'(\widehat{\rho}_i)\right)\widehat{q}_i\Big|^2\\
&\quad\le\frac{\rho_i\kappa_i'(\rho_i)}{2\kappa_i^2(\rho_i)}|\kappa_i(\rho_i)q_i-\kappa_i(\widehat{\rho}_i)\widehat{q}_i|^2+\frac{\kappa_i^2(\widehat{\rho}_i)\widehat{q}_i^2}{2\rho_i\kappa_i'(\rho_i)}\left|\frac{\rho_i\kappa_i'(\rho_i)}{\kappa_i(\rho_i)}-\frac{\widehat{\rho}_i\kappa_i'(\widehat{\rho}_i)}{\kappa_i(\widehat{\rho}_i)}\right|^2\\
&\quad\le\frac{C}{\kappa_i(\rho_i)}|\kappa_i(\rho_i)q_i-\kappa_i(\widehat{\rho}_i)\widehat{q}_i|^2+C|\rho_i-\widehat{\rho}_i|^2.
\end{aligned}$$

We use assumption **(A5)** in the first item of the last inequality to obtain an upper bound on $\rho_i\kappa_i'(\rho_i)/\kappa_i(\rho_i)$. Assumptions **(A3)** and **(A5)** are used to estimate the second item. By the same assumptions, a Taylor expansion of the last term on the right-hand side of (75) leads to

$$\left(-\frac{1}{\rho_i\kappa_i'}\right)(\rho_i|\widehat{\rho}_i)\le C|\rho_i-\widehat{\rho}_i|^2.$$

We thus have

$$s_i(\rho_i,q_i|\widehat{\rho}_i,\widehat{q}_i)\le C|\rho_i-\widehat{\rho}_i|^2+\frac{1}{2\kappa_i(\rho_i)}|\kappa_i(\rho_i)q_i-\kappa_i(\widehat{\rho}_i)\widehat{q}_i|^2. \tag{76}$$

Observe that

$$\begin{aligned}
&r_i(\rho_i,q_i|\widehat{\rho}_i,\widehat{q}_i)\\
&\quad=\rho_i\kappa_i(\rho_i)q_i-\widehat{\rho}_i\kappa_i(\widehat{\rho}_i)\widehat{q}_i-(\kappa_i(\widehat{\rho}_i)+\widehat{\rho}_i\kappa_i'(\widehat{\rho}_i))\widehat{q}_i(\rho_i-\widehat{\rho}_i)-\widehat{\rho}_i\kappa_i(\widehat{\rho}_i)(q_i-\widehat{q}_i)\\
&\quad=(\rho_i\kappa_i(\rho_i)-\widehat{\rho}_i\kappa_i(\widehat{\rho}_i))q_i-\kappa_i(\widehat{\rho}_i)\widehat{q}_i(\rho_i-\widehat{\rho}_i)-\widehat{\rho}_i\kappa_i'(\widehat{\rho}_i)\widehat{q}_i(\rho_i-\widehat{\rho}_i)\\
&\quad=\frac{\rho_i\kappa_i(\rho_i)-\widehat{\rho}_i\kappa_i(\widehat{\rho}_i)}{\kappa_i(\rho_i)}(\kappa_i(\rho_i)q_i-\kappa_i(\widehat{\rho}_i)\widehat{q}_i)\\
&\qquad+\widehat{\rho}_i\kappa_i^2(\widehat{\rho}_i)\widehat{q}_i(-\frac{1}{\kappa_i(\rho_i)}+\frac{1}{\kappa_i(\widehat{\rho}_i)}-\frac{\kappa_i'(\widehat{\rho}_i)}{\kappa_i^2(\widehat{\rho}_i)}(\rho_i-\widehat{\rho}_i))\\
&\quad\le\frac{C}{\kappa_i(\rho_i)}|\rho_i\kappa_i(\rho_i)-\widehat{\rho}_i\kappa_i(\widehat{\rho}_i)|^2+\frac{C}{\kappa_i(\rho_i)}|\kappa_i(\rho_i)q_i-\kappa_i(\widehat{\rho}_i)\widehat{q}_i|^2+\widehat{\rho}_i\kappa_i^2(\widehat{\rho}_i)\widehat{q}_i(-\frac{1}{\kappa_i})(\rho_i|\widehat{\rho}_i)\\
&\quad\le\frac{C}{\kappa_i(\rho_i)}|\kappa_i(\rho_i)q_i-\kappa_i(\widehat{\rho}_i)\widehat{q}_i|^2+C|\rho_i-\widehat{\rho}_i|^2,
\end{aligned}\tag{77}$$

where we used assumptions **(A3)** and **(A5)** to show the boundness of $(1/\kappa_i)(\rho_i)$ and $(-1/\kappa_i)(\rho_i|\widehat{\rho}_i)$. They are also used to estimate $|\rho_i\kappa_i(\rho_i)-\widehat{\rho}_i\kappa_i(\widehat{\rho}_i)|^2\le C|\rho_i-\widehat{\rho}_i|^2$.

Next, focusing on the term $H_i(\rho_i,q_i)$,

$$\begin{aligned}
H_i(\rho_i,q_i|\widehat{\rho}_i,\widehat{q}_i)&=\kappa_i(\rho_i)q_i\otimes q_i-\kappa_i(\widehat{\rho}_i)\widehat{q}_i\otimes\widehat{q}_i-\kappa_i'(\widehat{\rho}_i)\widehat{q}_i\otimes\widehat{q}_i(\rho_i-\widehat{\rho}_i)\\
&\quad-\kappa_i(\widehat{\rho}_i)(q_i-\widehat{q}_i)\otimes\widehat{q}_i-\kappa_i(\widehat{\rho}_i)\widehat{q}_i\otimes(q_i-\widehat{q}_i)\\
&=\frac{1}{\kappa_i(\rho_i)}(\kappa_i(\rho_i)q_i-\kappa_i(\widehat{\rho}_i)\widehat{q}_i)\otimes(\kappa_i(\rho_i)q_i-\kappa_i(\widehat{\rho}_i)\widehat{q}_i)
\end{aligned}$$

$$
\begin{aligned}
&+ \kappa_i^2(\widehat{\rho}_i)\widehat{q}_i \otimes \widehat{q}_i\bigg(- \frac{1}{\kappa_i(\rho_i)} + \frac{1}{\kappa_i(\widehat{\rho}_i)} - \frac{\kappa_i'(\widehat{\rho}_i)}{\kappa_i^2(\widehat{\rho}_i)}|\widehat{q}_i|^2(\rho_i - \widehat{\rho}_i)\bigg) \\
&\le \frac{1}{\kappa_i(\rho_i)}|\kappa_i(\rho_i)q_i - \kappa_i(\widehat{\rho}_i)\widehat{q}_i|^2 + \kappa_i^2(\widehat{\rho}_i)|\widehat{q}_i|^2(-\frac{1}{\kappa_i})(\rho_i|\widehat{\rho}_i) \\
&\le \frac{1}{\kappa_i(\rho_i)}|\kappa_i(\rho_i)q_i - \kappa_i(\widehat{\rho}_i)\widehat{q}_i|^2 + C|\rho_i - \widehat{\rho}_i|^2. \qquad (78)
\end{aligned}
$$

Combining (76), (77), and (78) and using assumption **(A2)**, we deduce that

$$
\begin{aligned}
J_2 &= -\int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \Big(s_i(\rho_i^\varepsilon, \nabla\rho_i^\varepsilon|\widehat{\rho}_i^\varepsilon, \nabla\widehat{\rho}_i^\varepsilon)\operatorname{div}\widehat{v}_i^\varepsilon + r_i(\rho_i^\varepsilon, \nabla\rho_i^\varepsilon|\widehat{\rho}_i^\varepsilon, \nabla\widehat{\rho}_i^\varepsilon)\cdot\nabla\operatorname{div}\widehat{v}_i^\varepsilon \\
&\quad + H_i(\rho_i^\varepsilon, \nabla\rho_i^\varepsilon|\widehat{\rho}_i^\varepsilon, \nabla\widehat{\rho}_i^\varepsilon) : \nabla\widehat{v}_i^\varepsilon\Big)dxds \\
&\le C\int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \left((\rho_i^\varepsilon - \widehat{\rho}_i^\varepsilon)^2 + \frac{1}{\kappa_i(\rho_i^\varepsilon)}|\kappa_i(\rho_i^\varepsilon)\nabla\rho_i^\varepsilon - \kappa_i(\widehat{\rho}_i^\varepsilon)\nabla\widehat{\rho}_i^\varepsilon|^2\right) dxds \\
&\le C\int_0^t \chi(s)ds. \qquad (79)
\end{aligned}
$$

From equation (39) we have

$$
\widehat{\rho}_i^\varepsilon\widehat{u}_i^\varepsilon = -\varepsilon\sum_{j=1}^n D_{ij}(\widehat{\boldsymbol{\rho}}^\varepsilon)\nabla\frac{\delta\mathcal{E}}{\delta\rho_j}(\widehat{\boldsymbol{\rho}}^\varepsilon).
$$

Hence, by definition (52) and upon using assumptions **(A3)** and (**A1**), we see that $\widehat{R}_i^\varepsilon$ is of order $O(\varepsilon)$ and that

$$
\begin{aligned}
J_3 &= -\int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \frac{\rho_i^\varepsilon}{\widehat{\rho}_i^\varepsilon}\widehat{R}_i^\varepsilon\cdot(v_i^\varepsilon - \widehat{v}_i^\varepsilon)dxds \\
&\le C\int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \rho_i^\varepsilon|v_i^\varepsilon - \widehat{v}_i^\varepsilon|^2 dxds + C\int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \rho_i^\varepsilon\left(\frac{\widehat{R}_i^\varepsilon}{\widehat{\rho}_i^\varepsilon}\right)^2 dxds \\
&\le C\int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \rho_i^\varepsilon|v_i^\varepsilon - \widehat{v}_i^\varepsilon|^2 dxds + C\varepsilon^2 t.
\end{aligned}
$$

Also $\widehat{v}_i^\varepsilon - \widehat{v}_j^\varepsilon = \widehat{u}_i^\varepsilon - \widehat{u}_j^\varepsilon$ is of order ε, so the last term J_4 is estimated using assumption **(A5)** by

$$
\begin{aligned}
J_4 &= -\frac{1}{\varepsilon}\int_0^t \int_{\mathbb{R}^3} \sum_{i,j=1}^n b_{ij}\rho_i^\varepsilon(\rho_j^\varepsilon - \widehat{\rho}_j^\varepsilon)(v_i^\varepsilon - \widehat{v}_i^\varepsilon)\cdot(\widehat{v}_i^\varepsilon - \widehat{v}_j^\varepsilon)dxds \\
&\le C\int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n b_{ij}\rho_i^\varepsilon|\rho_j^\varepsilon - \widehat{\rho}_j^\varepsilon||v_i^\varepsilon - \widehat{v}_i^\varepsilon|dxds
\end{aligned}
$$

$$\le C\int_0^t\int_{\mathbb{R}^3}\sum_{i=1}^n \rho_i^\varepsilon|v_i^\varepsilon-\widehat{v}_i^\varepsilon|^2dxds + C\int_0^t\int_{\mathbb{R}^3}\sum_{i,j=1}^n \rho_i^\varepsilon|\rho_j^\varepsilon-\widehat{\rho}_j^\varepsilon|^2dxds$$
$$\le \int_0^t \chi(s)ds.$$

Putting these estimates together, we arrive at

$$\chi(t)+\frac{1}{2\varepsilon}\int_0^t\int_{\mathbb{R}^3}\sum_{i,j=1}^n b_{ij}\rho_i^\varepsilon\rho_j^\varepsilon\big|(v_i^\varepsilon-v_j^\varepsilon)-(\widehat{v}_i^\varepsilon-\widehat{v}_j^\varepsilon)\big|^2dxds$$
$$\le C\chi(0)+C\int_0^t\chi(s)ds+C\varepsilon^2t.$$

Then Gronwall's inequality gives $\chi(t)\le C(\chi(0)+\varepsilon^2)e^{CT}$, finishing the proof. □

Remark 8. The assumption $h''(\rho_i)\ge\alpha$ is not needed if we assume that $\kappa_i(\rho_i)\kappa_i''(\rho_i)-2\kappa_i'(\rho_i)^2\ge\alpha$ and $|\nabla\widehat{\rho}_i|$ is bounded away from zero for any $i=1,\ldots,n$, because the second term on the right-hand side of (73) controls $|\rho_i-\widehat{\rho}_i|^2$.

The case of quantum hydrodynamics, $\kappa_i(\rho_i)=k_i/(4\rho_i)$ is included in the above proof. Indeed, $\chi(t)$ is taken to be

$$\chi(t)=\int_{\mathbb{R}^3}\sum_{i=1}^n\left(\frac{1}{2}\rho_i^\varepsilon|v_i^\varepsilon-\widehat{v}_i^\varepsilon|^2+(\rho_i^\varepsilon-\widehat{\rho}_i^\varepsilon)^2+\frac{2\rho_i^\varepsilon}{k_i}\left|\frac{\nabla\rho_i^\varepsilon}{\rho_i^\varepsilon}-\frac{\nabla\widehat{\rho}_i^\varepsilon}{\widehat{\rho}_i^\varepsilon}\right|^2\right)(t)dx.$$

The condition in assumption **(A4)** becomes

$$\kappa_i(\rho_i)\kappa_i''(\rho_i)-2\kappa_i'(\rho_i)^2=0,$$

but one needs the assumption $h_i''(\rho_i)\ge\alpha$ to derive the bounds for $|\rho_i^\varepsilon-\widehat{\rho}_i^\varepsilon|^2$. The use of the nonlinear quadratic term $(2\rho_i^\varepsilon/k_i)|\nabla\rho_i^\varepsilon/\rho_i^\varepsilon-\nabla\widehat{\rho}_i^\varepsilon/\widehat{\rho}_i^\varepsilon|^2$ is crucial to obtain the estimate.

Finally, for the case of constant capillarity, $\kappa_i(\rho_i)=k_i$, we conclude that $\kappa_i(\rho_i)\kappa_i''(\rho_i)-2\kappa_i'(\rho_i)^2=0$, such that assumption **(A4)** is satisfied. Thus, Theorem 7 also holds in this case.

4. Justification of the high-friction limit

We recall the original system (45)-(46):

$$\partial_t\rho_i^\varepsilon+\operatorname{div}(\rho_i^\varepsilon v_i^\varepsilon)=0, \tag{80}$$

$$\partial_t(\rho_i^\varepsilon v_i^\varepsilon)+\operatorname{div}(\rho_i^\varepsilon v_i^\varepsilon\otimes v_i^\varepsilon)=\operatorname{div}S_i(\boldsymbol{\rho})-\frac{1}{\varepsilon}\sum_{j=1}^n b_{ij}\rho_i^\varepsilon\rho_j^\varepsilon(v_i^\varepsilon-v_j^\varepsilon), \tag{81}$$

where $\operatorname{div}S_i=-\rho_i\nabla(\delta\mathcal{E}/\delta\rho_i)$. The limiting system for $\varepsilon\to0$ becomes

$$\partial_t\bar{\rho}_i+\operatorname{div}(\bar{\rho}_i\bar{v})=0, \tag{82}$$

$$\partial_t(\bar{\rho}\bar{v})+\operatorname{div}(\bar{\rho}\bar{v}\otimes\bar{v})=\operatorname{div}\bar{S}, \tag{83}$$

where $\bar{S} = \sum_{i=1}^n S_i(\bar{\rho}_i)$, $\bar{\rho} = \sum_{i=1}^n \bar{\rho}_i$, and $\bar{\rho}\bar{v} = \sum_{i=1}^n \bar{\rho}_i\bar{v}_i$. Indeed, system (82)-(83) corresponds to the zeroth-order Chapman-Enskog expansion (27)-(28). In this section, we verify the limit $\varepsilon \to 0$ rigorously, analyzing the isentropic case $F_i(\rho_i, q_i) = h_i(\rho_i)$ and the Korteweg case $F_i(\rho_i, q_i) = h_i(\rho_i) + \frac{1}{2}\kappa_i(\rho_i)|q_i|^2$ separately.

4.1. **High-friction limit in the isentropic case.** We consider the case when the energy density only depends on the particle density (and not on its gradients),

$$\mathcal{E}(\boldsymbol{\rho}) = \int_{\mathbb{R}^3} \sum_{i=1}^n F_i(\rho_i)dx, \quad F_i = h_i(\rho_i).$$

We prove the relaxation limit $\varepsilon \to 0$ in (80)-(81) by applying the general result of [21]. Noting that $\rho_i\nabla(\delta\mathcal{E}/\delta\rho_i) = \nabla p_i(\rho_i)$, where

$$p_i(\rho_i) = \rho_i h_i'(\rho_i) - h_i(\rho_i)$$

is the partial pressure, we can formulate (80)-(81) as the system of balance laws

$$\partial_t U^\varepsilon + \operatorname{div} F(U^\varepsilon) = \frac{1}{\varepsilon}R(U^\varepsilon), \tag{84}$$

where $U^\varepsilon = (\boldsymbol{\rho}^\varepsilon, \boldsymbol{m}^\varepsilon)$, $\boldsymbol{m}^\varepsilon = (\rho_i^\varepsilon v_i^\varepsilon)_{i=1,\ldots,n}$,

$$F(U^\varepsilon) = \begin{pmatrix} \rho_i^\varepsilon v_i^\varepsilon \\ \rho_i^\varepsilon v_i^\varepsilon \otimes v_i^\varepsilon + p_i(\rho_i^\varepsilon) \end{pmatrix}_{i=1,\ldots,n} \in \mathbb{R}^{2n},$$

$$R(U^\varepsilon) = \begin{pmatrix} 0 \\ -\sum_{j=1}^n b_{ij}\rho_i^\varepsilon\rho_j^\varepsilon(v_i^\varepsilon - v_j^\varepsilon) \end{pmatrix}_{i=1,\ldots,n} \in \mathbb{R}^{2n}.$$

The (formal) relaxation limit $\varepsilon \to 0$ leads to $R(U) = 0$, where $U = \lim_{\varepsilon\to 0} U^\varepsilon$. This implies that all limit velocities are the same, $v := v_i$ for $i = 1, \ldots, n$. Thus, the limit equations are expected to be

$$\partial_t\rho_i + \operatorname{div}(\rho_i v) = 0, \quad \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla p = 0,$$

for $i = 1, \ldots, n$, where $\rho = \sum_{i=1}^n \rho_i$ and $p = \sum_{i=1}^n p_i$. This system can be written as the conservation law

$$\partial_t u + \operatorname{div} f(u) = 0, \tag{85}$$

where $u = (\boldsymbol{\rho}, m)$, $m = \rho v$, and $f(u) = (\rho_1 v, \ldots, \rho_n v, \rho v \otimes v + p)$. System (84) has an entropy

$$\eta(U) = \sum_{i=1}^n \left(h_i(\rho_i) + \frac{1}{2}\rho_i|v_i|^2\right),$$

satisfying $\partial_t \int_{\mathbb{R}^3} \eta(U)dx \le 0$. We introduce the relative entropy density

$$\eta(U^\varepsilon|\bar{U}) = \sum_{i=1}^n \left(h_i(\rho_i^\varepsilon|\bar{\rho}_i) + \frac{1}{2}\rho_i^\varepsilon|v_i^\varepsilon - \bar{v}|^2\right),$$

where $h_i(\rho_i^\varepsilon|\bar{\rho}_i) = h_i(\rho_i^\varepsilon) - h_i(\bar{\rho}_i) - h_i'(\bar{\rho}_i)(\rho_i^\varepsilon - \bar{\rho}_i)$ and $\bar{U} = (\bar{\rho}_1, \ldots, \bar{\rho}_n, \bar{\rho}_1\bar{v}, \ldots, \bar{\rho}_n\bar{v})$.

Theorem 9 (Relaxation limit in the isentropic case)**.** *Assume that* **(N)** *on page 5 holds and that the function $h_i : [0,\infty) \to \mathbb{R}$ is uniformly convex on $(0,\infty)$ for all $i = 1,\dots,n$. Let $U^\varepsilon = (\boldsymbol{\rho}^\varepsilon, \boldsymbol{v}^\varepsilon)$ be a smooth solution to* (80)-(81) *or* (84) *and let $\bar{u} = (\bar{\boldsymbol{\rho}}, \bar{\rho}\bar{v})$ be a smooth solution to* (82)-(83) *or* (85)*. We suppose that there exists $\kappa > 0$ such that $\rho_i^\varepsilon, \bar{\rho}_i \geq \kappa > 0$ in $\mathbb{R}^3 \times (0,T)$ for all $i = 1,\dots,n$. Then for any $r > 0$, there exist $s > 0$ and $C > 0$ independent of ε such that for all $t \in (0,T)$,*

$$\int_{\{|x|<r\}} \eta(U^\varepsilon|\bar{U})(x,t)dx \leq C\bigg(\int_{\{|x|<r+st\}} \eta(U^\varepsilon|\bar{U})(x,0)dx + \varepsilon\bigg).$$

In particular, if

$$\lim_{\varepsilon\to 0} \int_{\{|x|<r+st\}} \eta(U^\varepsilon|\bar{U})(x,0)dx = 0$$

then

$$\lim_{\varepsilon\to 0} \sup_{0<t<T} \int_{\mathbb{R}^3} \sum_{i=1}^n \big((\rho_i^\varepsilon - \bar{\rho}_i)^2 + |v_i^\varepsilon - \bar{v}_i|^2\big)dx = 0.$$

Proof. As mentioned above, the result follows after applying Theorem 3.1 in [21]. To this end, we need to verify the structural conditions (h1)-(h7) of [21].

(h1) There exists a projection matrix $\mathbb{P} : \mathbb{R}^{2n} \to \mathbb{R}^{n+1}$ satisfying $\operatorname{rank}(\mathbb{P}) = n+1$ and $\mathbb{P}(R(U)) = 0$ for all $U \in \mathbb{R}^{2n}$. This matrix relates the variables u and U and is given by

$$u = \mathbb{P}U, \quad \mathbb{P} = \begin{pmatrix} \mathbb{I}_n & \mathbb{O}_n \\ 0,\dots,0 & 1,\dots,1 \end{pmatrix},$$

where $\mathbb{I}_n$ is the unit matrix of $\mathbb{R}^{n\times n}$, $\mathbb{O}_n$ is the zero matrix in $\mathbb{R}^{n\times n}$- It holds for all $U = (\boldsymbol{\rho}, \boldsymbol{m})$, $(\mathbb{P}R(U))_i = 0$ for $i = 1,\dots,n$ and

$$(\mathbb{P}R(U))_{n+1} = -\sum_{j,k=1}^n b_{jk}\rho_j\rho_k(v_j - v_k) = 0.$$

(h2) The equilibrium solutions to $R(U) = 0$, called $M(u)$, satisfies $\mathbb{P}M(u) = u$. The equilibrium solutions are given by $M(u) = (\rho_1,\dots,\rho_n,\rho_1 v,\dots,\rho_n v)$, since $(\mathbb{P}M(u))_i = \rho_i$ for $i = 1,\dots,n$ and $(\mathbb{P}M(u))_{n+1} = \sum_{j=1}^n \rho_j v = \rho v$.

(h3) The nondegeneracy conditions

$$\dim\ker(R_U(M(u))) = n+1, \quad \dim\operatorname{ran}(R_U(M(u))) = n-1$$

hold, where $R_U = dR/dU$. This can be verified by a straightforward computation.

(h4), (h5) There exists an entropy density $\eta : \mathbb{R}^{2n} \to \mathbb{R}$ which is convex and satisfies $\eta_U F_U = J_U$ and $\eta_U \cdot R(U) \leq 0$, where J is the flux vector. We choose

$$\eta(U) = \sum_{i=1}^n \bigg(h_i(\rho_i) + \frac{1}{2}\rho_i|v_i|^2\bigg), \quad J(U) = \sum_{i=1}^n \bigg(\rho_i h_i'(\rho_i)v_i + \frac{1}{2}\rho_i|v_i|^2 v_i\bigg).$$

Then the inequality is a consequence of the energy inequality (60).

(h6) The solution u to (85) has the entropy-flux pair

$$\eta(M(u)) = \sum_{i=1}^n h_i(\rho_i) + \frac{1}{2}\rho|v|^2, \quad J(M(u)) = \sum_{i=1}^n \rho_i h_i'(\rho_i)v + \frac{1}{2}\rho|v|^2 v.$$

This follows from (62) with $\widehat{\rho}_i, \widehat{v}_i$ replaced by $\bar{\rho}_i, \bar{v}$.

(h7) The following inequality holds:

$$-\big(\eta_U(U) - \eta_U(M(u)\big) \cdot \big(R(U) - R(M(u))\big) \geq \nu|U - M(u)|^2.$$

The inequality in (h7) amounts to proving

$$\frac{1}{2}\sum_{i,j=1}^n b_{ij}\rho_i\rho_j|v_i - v_j|^2 \geq \nu \sum_{i=1}^n \rho_i^2|v_i - v|^2\,. \tag{86}$$

The proof of this statement is motivated by the analysis in [27]. First, note that $\partial\eta/\partial\rho_i = h_i'(\rho_i) - \frac{1}{2}|v_i|^2$ and $\partial\eta/\partial m_i = v_i$, where $m_i = \rho_i v_i$. Taking into account that $R(M(u)) = 0$, we have

$$\begin{aligned}
&-\big(\eta_U(U) - \eta_U(M(u))\big) \cdot \big(R(U) - R(M(u))\big) \\
&\quad = \sum_{i=1}^n (v_i - v) \cdot \sum_{j=1}^n b_{ij}\rho_i\rho_j(v_i - v_j) = \frac{1}{2}\sum_{i,j} b_{ij}\rho_i\rho_j|v_i - v_j|^2.
\end{aligned}$$

For the proof of (86), let $v_i = v + u_i$, and we reformulate the left-hand side of the inequality in (h7) as

$$\begin{aligned}
&-\big(\eta_U(U) - \eta_U(M(u))\big) \cdot \big(R(U) - R(M(u))\big) \\
&\quad = \sum_{i,j=1}^n b_{ij}\rho_i\rho_j(u_i - u_j) \cdot u_i = \sum_{i,j=1}^n \tau_{ij} u_i \cdot u_j,
\end{aligned}$$

where $\tau_{ij} = \delta_{ij}\sum_{k=1}^n b_{ik}\rho_i\rho_k - b_{ij}\rho_i\rho_j$ as in (14). Since (τ_{ij}) is not positive definite, inequality (86) does not follow directly. The idea is to use the fact that there exists a submatrix $(\tau_{ij}) \in \mathbb{R}^{(n-1)\times(n-1)}$ that is positive definite; see the proof of Lemma 3. Recalling the properties $Q_{ij} = \delta_{ij}/\rho_i + 1/\rho_n$, $\sum_{i=1}^n \rho_i u_i = 0$, $\sum_{i=1}^n \tau_{ij} = 0$, and (15), we compute

$$\begin{aligned}
-\big(\eta_U(U) - \eta_U(M(u))\big) \cdot \big(R(U) - R(M(u))\big) &= \sum_{i=1}^n u_i \sum_{j,k=1}^{n-1} \tau_{ij} Q_{jk}\rho_k u_k \\
&= \sum_{i=1}^{n-1} u_i \sum_{j,k=1}^{n-1} \tau_{ij} Q_{jk}\rho_k u_k + u_n \sum_{j,k=1}^{n-1} \tau_{nj} Q_{jk}\rho_k u_k \\
&= \sum_{i=1}^{n-1} u_i \sum_{j,k=1}^{n-1} \tau_{ij} Q_{jk}\rho_k u_k - \sum_{\ell=1}^{n-1} \frac{\rho_\ell u_\ell}{\rho_n} \sum_{j,k=1}^{n-1} \bigg(-\sum_{m=1}^{n-1} \tau_{mj}\bigg) Q_{jk}\rho_k u_k \\
&= \sum_{i,j,k,\ell=1}^{n-1} \rho_\ell u_\ell \bigg(\frac{\delta_{i\ell}}{\rho_\ell} + \frac{1}{\rho_n}\bigg)\tau_{ij} Q_{jk}\rho_k u_k
\end{aligned}$$

$$= \sum_{i,j,k,\ell=1}^{n-1} \rho_\ell u_\ell Q_{i\ell}\tau_{ij}Q_{jk}(\rho_k u_k) = W^T Q^\top \tau Q W,$$

where $W = (\rho_1 u_1, \dots, \rho_{n-1}u_{n-1})^\top$. Since $(\tau_{ij}) \in \mathbb{R}^{(n-1)\times(n-1)}$ is positive definite and Q is invertible, $Q^\top \tau Q$ is also positive definite. We infer that there exists a constant $\mu > 0$ such that

$$-\big(\eta_U(U) - \eta_U(M(u))\big) \cdot \big(R(U) - R(M(u))\big) \geq \mu |W|^2 = \mu \sum_{i=1}^{n-1} |\rho_i u_i|^2.$$

We claim that we may sum from $i = 1$ to n using another constant. Indeed, we infer from

$$|\rho_n u_n|^2 = \Big| -\sum_{i=1}^{n-1} \rho_i u_i \Big|^2 \leq (n-1) \sum_{i=1}^{n-1} |\rho_i u_i|^2$$

that

$$\sum_{i=1}^{n} |\rho_i u_i|^2 = \sum_{i=1}^{n-1} |\rho_i u_i|^2 + |\rho_n u_n|^2 \leq n \sum_{i=1}^{n-1} |\rho_i u_i|^2$$

and therefore,

$$-\big(\eta_U(U) - \eta_U(M(u))\big) \cdot \big(R(U) - R(M(u))\big) \geq \frac{\mu}{n} \sum_{i=1}^{n} |\rho_i u_i|^2,$$

and the result follows with $\nu = \mu/n$. □

4.2. **High-friction limit in the Euler-Korteweg case.** We next justify the relaxation limit $\varepsilon \to 0$ for energies F_i depending on the particle density and its gradient. We place the assumption:

(A6) $\bar{u} = (\bar{\boldsymbol{\rho}}, \bar{\rho}\bar{v})$ is a smooth solution to (82)-(83) satisfying $\bar{u}$, $\partial_t \bar{u}$, $\nabla \bar{u}$, $D^2\bar{u}$, $D^3\bar{\rho} \in L^\infty([0,T]; L^\infty(\mathbb{R}^3))$.

Proposition 10 (Relative energy inequality)**.** *Let $(\boldsymbol{\rho}^\varepsilon, \boldsymbol{v}^\varepsilon)$ be a dissipative weak solution to* (80)-(81) *satisfying assumption* **(A1)** *on page 18 and let $(\bar{\boldsymbol{\rho}}, \bar{v})$ be a smooth solution to* (82)-(83) *satisfying assumption* **(A6)**. *Let assumption* **(N)** *on page 5 hold. Then*

$$\begin{aligned}
&\mathcal{E}_{\rm tot}(\boldsymbol{\rho}, \boldsymbol{m}|\bar{\boldsymbol{\rho}}, \bar{\boldsymbol{m}})(t) + \frac{1}{2\varepsilon}\int_0^t \int_{\mathbb{R}^3} \sum_{i,j=1}^n b_{ij}\rho_i^\varepsilon \rho_j^\varepsilon |v_i^\varepsilon - v_j^\varepsilon|^2 dxds \\
&\quad \leq \mathcal{E}_{\rm tot}(\boldsymbol{\rho}^\varepsilon, \boldsymbol{m}^\varepsilon|\bar{\boldsymbol{\rho}}, \bar{\boldsymbol{m}})(0) - \int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \rho_i^\varepsilon (v_i^\varepsilon - \bar{v}) \otimes (v_i^\varepsilon - \bar{v}) : \nabla \bar{v} dxds \\
&\quad - \int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \Big(s_i(\rho_i^\varepsilon, \nabla\rho_i^\varepsilon|\bar{\rho}_i, \nabla\bar{\rho}_i) \operatorname{div} \bar{v} + r_i(\rho_i^\varepsilon, \nabla\rho_i^\varepsilon|\bar{\rho}_i, \nabla\bar{\rho}_i) \cdot \nabla \operatorname{div} \bar{v} \\
&\quad + H_i(\rho_i^\varepsilon, \nabla\rho_i^\varepsilon|\bar{\rho}_i, \nabla\bar{\rho}_i) : \nabla\bar{v} \Big) dxds
\end{aligned}$$

$$-\int_0^t\int_{\mathbb{R}^3}\sum_{i=1}^n \rho_i^\varepsilon(v_i^\varepsilon-\bar v)\cdot\left(\frac{\operatorname{div}\bar S}{\bar\rho}-\frac{\operatorname{div}\bar S_i}{\bar\rho_i}\right)dxds. \tag{87}$$

Proof. The calculation is similar to the proof of Proposition 6. We can replace $\widehat{\rho_i^\varepsilon}$, $\widehat{v_i^\varepsilon}$ by $\bar\rho_i$, $\bar v_i$ in (58). To obtain the relative energy inequality, we need further to write the equation for $\bar\rho_i\bar v$ into the same form as (51) and replace $\widehat{R_i^\varepsilon}$ by $\bar R_i$, which is given by

$$\partial_t(\bar\rho_i\bar v)+\operatorname{div}(\bar\rho_i\bar v\otimes\bar v)=-\bar\rho_i\nabla\frac{\delta\mathcal{E}}{\delta\bar\rho_i}(\bar{\boldsymbol\rho})-\frac{1}{\varepsilon}\sum_{j=1}^n b_{ij}\bar\rho_i\bar\rho_j(\bar v-\bar v)+\bar R_i.$$

Using (82) and (83), $\bar R_i$ can be calculated as

$$\begin{aligned}\bar R_i&=(\partial_t\bar\rho_i+\operatorname{div}(\bar\rho_i\bar v))\cdot\bar v+\bar\rho_i(\partial_t\bar v+\bar v\cdot\nabla\bar v)+\bar\rho_i\nabla\frac{\delta\mathcal{E}}{\delta\bar\rho_i}\\&=\frac{\bar\rho_i}{\bar\rho}(\partial_t(\bar\rho\bar v)+\nabla\cdot(\bar\rho\bar v\otimes\bar v))+\bar\rho_i\nabla\frac{\delta\mathcal{E}}{\delta\bar\rho_i}\\&=\frac{\bar\rho_i}{\bar\rho}\operatorname{div}\bar S-\operatorname{div}\bar S_i.\end{aligned}$$

Replacing $\widehat{R_i^\varepsilon}$ with the above equation, (58) becomes (87). Notice that $\widehat{v_i^\varepsilon}-\widehat{v_j^\varepsilon}$ reduces to $\bar v-\bar v=0$ and the last term in (58) vanishes. □

Theorem 11 (Relaxation limit in the Korteweg case)**.** *Let $(\boldsymbol\rho^\varepsilon,\boldsymbol v^\varepsilon)$ be a dissipative weak solution to* (80)-(81) *satisfying* **(A1)** *on page 18 and* **(A5)** *on page 26 and let $(\bar{\boldsymbol\rho},\bar v)$ be a strong solution to* (82)-(83) *satisfying* **(A6)** *on page 35. Suppose that for some constants $K>\kappa>0$, we have the uniform bounds $\kappa\le\rho_i^\varepsilon(x,t)\le K$ and $\bar\rho_i(x,t)\ge\kappa$ for all $(x,t)\in\mathbb{R}^3\times(0,T)$ and $i=1,\ldots,n$. Furthermore, let assumption* **(N)** *hold. We fix $T>0$ and set, as in Theorem 7,*

$$\chi(t)=\int_{\mathbb{R}^3}\sum_{i=1}^n\left(\frac{1}{2}\rho_i^\varepsilon|v_i^\varepsilon-\bar v|^2+(\rho_i^\varepsilon-\bar\rho_i)^2+\frac{1}{2\kappa_i(\rho_i^\varepsilon)}|\kappa_i(\rho_i^\varepsilon)\nabla\rho_i^\varepsilon-\kappa_i(\bar\rho_i)\nabla\bar\rho_i|^2\right)(t)dx.$$

Then there exists a constant $C>0$ such that for all $\varepsilon>0$ and $t\in(0,T)$,

$$\chi(t)\le C(\chi(0)+\varepsilon),\quad t\in(0,T).$$

In particular, if $\chi(0)\to0$ as $\varepsilon\to0$, we have

$$\sup_{t\in(0,T)}\chi(t)\to0\quad\text{as }\varepsilon\to0.$$

Proof. We estimate the integrals on the right-hand side of the relative entropy inequality (87). The second and third terms can be estimated in the same way as (74) and (79), and they are bounded by $C\int_0^t\chi(s)ds$. We split the last term on the right-hand side of (87) into two parts:

$$-\int_0^t\int_{\mathbb{R}^3}\sum_{i=1}^n\rho_i^\varepsilon(v_i^\varepsilon-\bar v)\cdot\left(\frac{\operatorname{div}\bar S}{\bar\rho}-\frac{\operatorname{div}\bar S_i}{\bar\rho_i}\right)dxds=L_1+L_2,$$

where

$$L_1 = -\int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \rho_i^\varepsilon (v_i^\varepsilon - v^\varepsilon) \cdot \left(\frac{\operatorname{div} \bar{S}}{\bar{\rho}} - \frac{\operatorname{div} \bar{S}_i}{\bar{\rho}_i} \right) dxds,$$
$$L_2 = -\int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \rho_i^\varepsilon (v^\varepsilon - \bar{v}) \cdot \left(\frac{\operatorname{div} \bar{S}}{\bar{\rho}} - \frac{\operatorname{div} \bar{S}_i}{\bar{\rho}_i} \right) dxds.$$

We infer that

$$\begin{aligned} L_1 &= \int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \rho_i^\varepsilon (v_i^\varepsilon - v^\varepsilon) \frac{\operatorname{div} \bar{S}_i}{\bar{\rho}_i} dxds \\ &\le \frac{\nu}{2\varepsilon} \int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n (\rho_i^\varepsilon)^2 |v_i^\varepsilon - v^\varepsilon|^2 dxds + C\varepsilon \int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \left(\frac{\operatorname{div} \bar{S}_i}{\bar{\rho}_i} \right)^2 dxds \\ &\le \frac{\nu}{2\varepsilon} \int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n (\rho_i^\varepsilon)^2 |v_i^\varepsilon - v^\varepsilon|^2 dxds + C\varepsilon t. \end{aligned}$$

Using (86), we conclude that

$$L_1 \le \frac{1}{4\varepsilon} \int_0^t \int_{\mathbb{R}^3} \sum_{i,j=1}^n b_{ij} \rho_i^\varepsilon \rho_j^\varepsilon |v_i^\varepsilon - v_j^\varepsilon|^2 + C\varepsilon t.$$

To estimate L_2, recall that $\bar{S} = \sum_{i=1}^n \bar{S}_i$ and $\rho^\varepsilon = \sum_{i=1}^n \rho_i^\varepsilon$, yielding

$$\begin{aligned} L_2 &= -\int_0^t \int_{\mathbb{R}^3} (v^\varepsilon - \bar{v}) \cdot \sum_{i=1}^n \left(\frac{\rho^\varepsilon}{\bar{\rho}} \operatorname{div} \bar{S}_i - \frac{\rho_i^\varepsilon}{\bar{\rho}_i} \operatorname{div} \bar{S}_i \right) dxds \\ &= -\int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \left(\frac{1}{\bar{\rho}} - \frac{\rho_i^\varepsilon}{\bar{\rho}_i \rho^\varepsilon} \right) \rho^\varepsilon (v^\varepsilon - \bar{v}) \cdot (\operatorname{div} \bar{S}_i) dxds \\ &\le \int_0^t \int_{\mathbb{R}^3} \rho^\varepsilon |v^\varepsilon - \bar{v}|^2 dxds + C \int_0^t \int_{\mathbb{R}^3} \rho^\varepsilon \sum_{i=1}^n \left(\frac{1}{\bar{\rho}} - \frac{\rho_i^\varepsilon}{\bar{\rho}_i \rho^\varepsilon} \right)^2 dxds. \end{aligned} \tag{88}$$

To estimate the first term on the right-hand side, we need the uniform lower and upper bounds for ρ_i^ε:

$$\rho^\varepsilon |v^\varepsilon - \bar{v}|^2 = \frac{1}{\rho^\varepsilon} \left| \sum_{i=1}^n \rho_i^\varepsilon (v_i^\varepsilon - \bar{v}) \right|^2 \le \frac{n}{\rho^\varepsilon} \sum_{i=1}^n (\rho_i^\varepsilon)^2 |v_i^\varepsilon - \bar{v}|^2 \le \frac{nK}{\kappa} \sum_{i=1}^n \rho_i^\varepsilon |v_i^\varepsilon - \bar{v}|^2.$$

The last term in (88) can be estimated according to

$$\sum_{i=1}^n \left(\frac{1}{\bar{\rho}} - \frac{\rho_i^\varepsilon}{\bar{\rho}_i \rho^\varepsilon} \right)^2 = \sum_{i=1}^n \left(\frac{\rho^\varepsilon - \bar{\rho}}{\rho^\varepsilon \bar{\rho}} + \frac{\bar{\rho}_i - \rho_i^\varepsilon}{\rho^\varepsilon \bar{\rho}_i} \right)^2 \le C \sum_{i=1}^n (\rho_i^\varepsilon - \bar{\rho}_i)^2.$$

Therefore,

$$L_2 \le C\int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n \big(\rho_i^\varepsilon |v_i^\varepsilon - \bar{v}|^2 + (\rho_i^\varepsilon - \bar{\rho}_i)^2\big) dxds \le C\int_0^t \chi(s)ds.$$

Finally, as in the proof of Theorem 7, $\mathcal{E}_{\rm tot}(\boldsymbol{\rho}^\varepsilon, \boldsymbol{m}^\varepsilon|\bar{\boldsymbol{\rho}}, \bar{\boldsymbol{m}})(t) \ge C\chi(t)$. We conclude that

$$\chi(t) + \frac{1}{4\varepsilon}\int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^n b_{ij}\rho_i^\varepsilon \rho_j^\varepsilon |v_i^\varepsilon - v_j^\varepsilon|^2 dxds \le \chi(0) + C\int_0^t \chi(s)ds + C\varepsilon t. \tag{89}$$

An application of Gronwall's lemma then finishes the proof. □

Remark 12. In the previous proof, the interaction term involving b_{ij} was crucial to estimate the term L_1. The symmetry of (b_{ij}) enables us to control the kinetic energy by the interaction energy,

$$\int_{\mathbb{R}^3} \sum_{i=1}^n \rho_i^2 |v_i - v|^2 dx \le \frac{1}{2\nu}\int_{\mathbb{R}^3} \sum_{i=1}^n b_{ij}\rho_i\rho_j |v_i - v_j|^2.$$

In the single component case, the interaction energy vanishes, and we recover Theorem 3 in [10].

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