CONVEX SOBOLEV INEQUALITIES DERIVED FROM ENTROPY DISSIPATION

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Abstract. We study families of convex Sobolev inequalities, which arise as entropy-dissipation relations for certain linear Fokker-Planck equations. Extending the ideas recently developed by the first two authors, a refinement of the Bakry-Émery method is established, which allows us to prove non-trivial inequalities even in situations where the classical Bakry-Émery criterion fails.

The main application of our theory concerns the linearized fast diffusion equation in dimensions $d \geq 1$, which admits a Poincaré, but no logarithmic Sobolev inequality. We calculate bounds on the constants in the interpolating convex Sobolev inequalities, and prove that these bounds are sharp on a specified range. In dimension $d = 1$, our estimates improve the corresponding results that can be obtained by the measure-theoretic techniques of Barthe and Roberto. As a by-product, we give a short and elementary alternative proof of the sharp spectral gap inequality first obtained by Denzler and McCann. In further applications of our method, we prove convex Sobolev inequalities for a mean field model for the redistribution of wealth in a simple market economy, and the Lasota model for blood cell production.

1. Introduction

This article is concerned with upper bounds on the optimal constant $C_p > 0$ in specific families of convex Sobolev inequalities of Beckner type,

$$\frac{1}{2-p}\left[\int_\Omega w^2\,d\mu - \Big(\int_\Omega w^p\,d\mu\Big)^{2/p}\right] \leq C_p \int_\Omega D|\nabla w|^2\,d\mu \qquad (1 \leq p < 2). \tag{1}$$

Above, μ is a probability measure with smooth Lebesgue density $f_\infty(x) := d\mu/dx$ on the domain $\Omega \subset \mathbb{R}^d$, the diffusion coefficient $D : \Omega \to \mathbb{R}$ is strictly positive, and $w \in L^2(\Omega;\mu)$ is a smooth, positive function. Such a family of inequalities has first been derived [Bec89] for the d-dimensional Gaussian measure on $\Omega = \mathbb{R}^d$ with $f_\infty(x) = (2\pi)^{-2/d}\exp(-|x|^2/2)$ and $D(x) \equiv 1$: the corresponding optimal constant is $C_p \equiv 1$ for all $1 \leq p < 2$. Subsequently, variants of (1) have been proven for different measures μ, see e.g. [LO00], and with various subtle refinements [AD05, BD06, BG09].
Let us recall some of the motivations to study the inequalities (1). First of all, they form an interpolating family of increasingly (with p) sharp estimates, starting from the Poincaré inequality at $p = 1$. In particular, if the C_p are uniformly bounded, i.e. $C_2 := \limsup_{p\uparrow 2} C_p$ is finite, then the family (1) is "completed" at $p = 2$ by the logarithmic Sobolev inequality,

$$\int_\Omega w^2 \log\Big(\frac{w^2}{\|w\|^2_{L^2(\Omega;\mu)}}\Big)\,d\mu \leq 2C_2 \int_\Omega D|\nabla w|^2\,d\mu. \tag{2}$$

Inequality (2) is of paramount importance in various contexts in probability theory, mathematical physics, geometric evolution equations etc.
A second motivation is that (1) characterizes very precisely the concentration of the measure μ [Bar01, LO00]. Assume for the moment that $D \equiv 1$ in (1). Then boundedness of C_p for $p \uparrow 2$ implies (2) and thus Gaussian concentration. On the other hand, if C_p diverges, but $\limsup_{p\uparrow 2}(2-p)^{2/r-1}C_p < \infty$ for some $r \in [1,2]$, then μ is concentrated like $\exp(-Kx^r)$.

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A third motivation is the intimate relation between (1) and the rate of equilibration of the μ-ergodic semigroup on $L^q(\Omega;\mu)$ generated by the linear diffusion operator $\mathbf{L}$ with (recall that f_∞ denotes the density of μ)

$$\mathbf{L}[u] = D\Delta u - Q\cdot\nabla u, \quad Q(x) := -D(x)\nabla\log f_\infty(x) - \nabla D(x). \tag{3}$$

Validity of (2) is equivalent to hypercontractivity of this semigroup. But even if (2) fails, the convex inequalities (1) guarantee that the semigroup is exponentially contracting in any $L^q(\Omega;\mu)$ with $1 < q \leq 2$, i.e.,

$$\exp\Big(\frac{2t}{C_{2/q}}\Big)\bigg[\int_\Omega \big(\exp(t\mathbf{L})[u]\big)^q\,d\mu - \Big(\int_\Omega u\,d\mu\Big)^q\bigg] \quad \text{is non-increasing.} \tag{4}$$

The constant C_p for $1 \leq p < 2$ thus determines the time scale for exponential relaxation in $L^{2/p}(\Omega;\mu)$. For example, $C_p \equiv 1$ for the Gaussian measure means that the speed of contraction in the associated (standard) Ornstein-Uhlenbeck process is independent of the considered $L^q(\Omega;\mu)$-space. We remark that there is an extensive amount of recent results on the relation between "exotic" functional inequalities (among which the inequalities (1) are the most basic ones) and the long-time behaviour of linear and also non-linear diffusion semigroups; the interested reader is referred to [BCR06, BG09, CGG07, Cha04, DGW08, DNS08] and the numerous references therein.
Despite the illustrated broad interest in the inequalities (1) from probability theory, mathematical physics, and partial differential equations, there is only a very limited number of specific measures μ, for which the optimal constant C_p has been calculated so far. A collection of sharp inequalities for measures μ on *discrete* sets Ω is provided in [BT]. Apparently, the only examples for sharp inequalities on *continuous* domains Ω are those in which μ admits a logarithmic Sobolev inequality (2) with a constant C_2 equal to the *sharp* Poincaré-constant C_1; it then follows that $C_p = C_1$ is optimal for $1 < p \leq 2$. To our knowledge, we give the first proof of (1) with optimal constants C_p (on an explicit range $1 \leq p \leq \hat{p} < 2$) in a situation where the logarithmic Sobolev inequality fails.
To serve justice, we emphasize that although the constants C_p have not been calculated explicitly, powerful and profound tools have been developed to estimate them. First of all, there are numerous perturbative results along the lines of the original Holley-Stroock argument, see e.g. [AD05]. Second, there exist several approaches to prove (1) with measure theoretic tools. Based on a classical result by Muckenhoupt [Muc72], a connection between (1) and Hardy inequalities was established by Bobkov and Götze [BG99]. In dimension $d = 1$, this leads to estimates on C_p from above and from below. The method has been refined later by Barthe and Roberto [BR03]; we recall a particular result of their work in Theorem 7 in the appendix. Finally we mention a recent approach (which is closest in spirit to the one employed here) on basis of the Bakry-Émery method: in [DNS08], the optimal constant C_p is related to the spectral gap of an associated (p-dependent) Schrödinger operator.
The mentioned methods apply in very general situations; however, the obtained bounds on C_p are usually either quite rough or implicitly characterized. In particular, the bounds from [BR03] can only be approximated numerically in specific examples, and the Schrödinger problem from [DNS08] is typically not much easier to handle than the original inequality (1). Our approach is complementary to this as we work with quite elementary tools directly on the examples, exploiting their specific structure.
To derive our results, we adapt the the Bakry-Émery method [BE85], i.e. we prove (1) by estimating the entropy dissipation of the associated semigroup generated by $\mathbf{L}$ from (3). Originally [BE85], the method has been used to show that if the celebrated *Bakry-Émery condition* $\Gamma_2 \geq \lambda_{BE}\Gamma$ is satisfied with some $\lambda_{BE} > 0$ (where Γ and Γ_2 are the first two Gamma operators, see (17)&(18) in section 2.3), then (1) holds with $C_p = 1/\lambda_{BE}$ uniformly on $1 \leq p < 2$. In our examples, the logarithmic Sobolev inequality and hence also the Bakry-Émery condition may be violated. We refine the method of proof from [BE85] by avoiding the *pointwise* Bakry-Émery condition. Instead, we work directly on the p-dependent *integral expression* for the (second) entropy dissipation. In its core, our method relies on "clever" manipulations of the dissipation term using integration by parts. Since the selected examples exhibit an extremely nice algebraic structure, it is possible

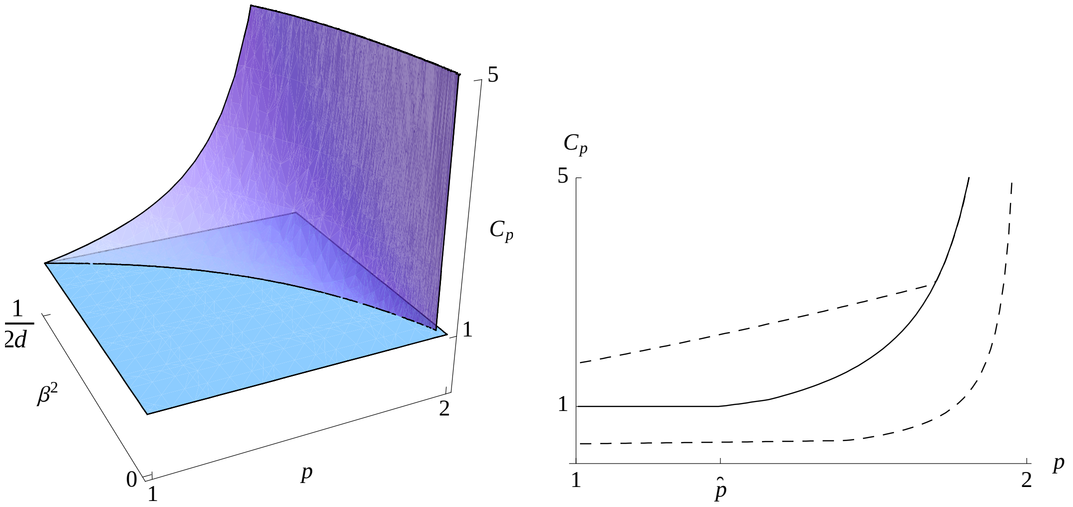

Figure 1. Estimates on C_p in (1) for the linearized fast-diffusion equation. *Left*: Our estimate on C_p in dimension $d = 5$. In particular, $C_p \equiv 1$ when (6) holds. *Right.* Comparison of our estimate on C_p (solid line) in dimension $d = 1$ for $\beta^2 = 1/4$ with the upper and lower bounds (dashed lines) corresponding to [BR03]. Notice that $C_p \equiv 1$ for $1 \le p \le \hat{p} = \hat{p} = 33/25$.

to perform these manipulations in a systematic, computer-assisted manner. A comment on this aspect of our work is given in section 3.3.

We will now define the specific examples we are dealing with and state our main results.

Linearized fast-diffusion equation. Inequality (1) with[1]

$$d\mu_\beta(x) = Z_\beta^{-1}\big(\alpha^2 + \beta^2|x|^2\big)^{-1-1/(2\beta^2)}dx \quad \text{and} \quad D_\beta(x) = \alpha^2 + \beta^2|x|^2,$$

with appropriate positive constants α and β, is associated via (3) to the linearized rescaled fast-diffusion equation,

$$\partial_t u = \mathbf{L}_\beta[u] := D_\beta \Delta u - x \cdot \nabla u, \tag{5}$$

see section 5 for details. In the limit $\beta \downarrow 0$, the measure μ_β converges weakly to a Gaussian measure μ_0, for which (1) holds with $C_p \equiv 1$. For every $\beta > 0$, the logarithmic Sobolev inequality is lost, and $C_p \uparrow \infty$ as $p \uparrow 2$. In Theorem 3, we quantify this loss by estimating C_p from above.

We remark that the long-time asymptotics of the full non-linear fast diffusion equation have been investigated at least since the 1980's, and are now essential understood, see [BBDGV09] and the references therein. Functional inequalities associated to the linearized operator $\mathbf{L}_\beta$ play a decisive role in this analysis. Over the years, these inequalities and the asymptotics of the linearized equation (5) have become a field of study on their own right, see e.g. [CT07, CLMT02, BBDGV07, BDGV09, DM05]. Previous results concerning the specific family (1) amount to the calculation of the optimal Poincaré constant C_1 — along with a complete spectral decomposition of $\mathbf{L}_\beta$ — in [DM05]. To our knowledge, the estimates for C_p with $1 < p < 2$ presented here are novel.

Our result is depicted in Figure 1 (left). The precise value of the bound on C_p is of secondary interest; the important finding is that (1) continues to hold with $C_p = 1$ even for positive β, provided that $\beta^2 < 1/(2d)$ and

$$1 \le p \le \hat{p} := 2 - \frac{(4 + d\beta^2)\beta^2}{(1 - 2d\beta^2) + (4 + d\beta^2)\beta^2}. \tag{6}$$

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[1] In our notation, we neglect the dependence of μ_β, D_β etc. on α, since all of our results are independent of its value, as long as $\alpha > 0$.

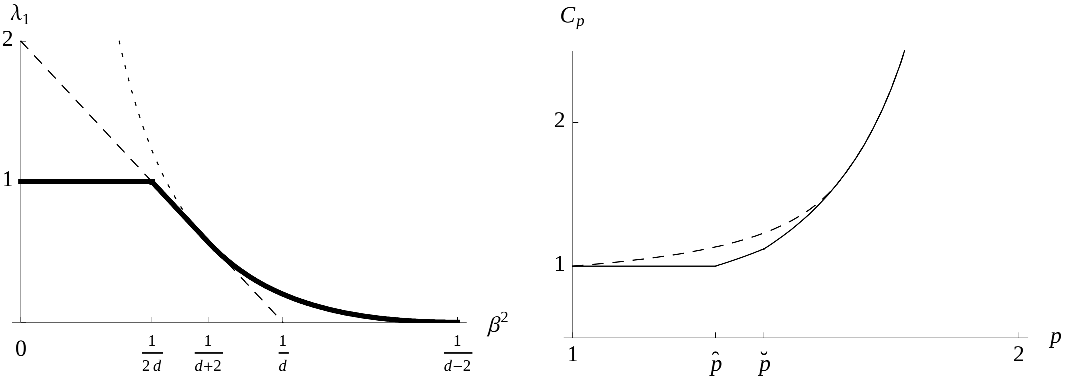

FIGURE 2. *Left:* Spectral gap $\lambda_1 = 1/C_1$ (solid line) of the operator $\mathbf{L}_\beta$ associated to linearized fast diffusion equation in dimension $d = 5$ as a function of β^2. *Right:* Comparison between our estimate on the constants C_p for the linearized fast diffusion equation at $\beta^2 = 1/4$ (solid line) and the wealth distribution model at $\theta = 1/4$ (dashed line). On the horizontal axis, $\hat{p} = 33/25$ and $\check{p} = 10/7$.

Hence, those convex inequalities in (1), which are sufficiently close to the Poincaré inequality, retain their constant under the perturbation $\mu_0 \to \mu_\beta$. Moreover, $C_p = 1$ is the *optimal* constant in (1) on the given p-range, since $C_1 = 1$ is known [DM05] to be the optimal constant in the Poincaré inequality, and $C_p \geq C_1$ by classical results, see Lemma 6 in the appendix.

For $\beta^2 < 1/(2d)$ and p outside of the range (6), another classical estimate provides $C_p \leq (2-\hat{p})/(2-p)$. In Theorem 4, we present a quantitative improvement of this bound in dimension $d = 1$. In Figure 1 (right), our estimate on C_p (fat line) is compared to the upper and lower bounds (dashed lines) obtained by numerical evaluation of the formulas from [BR03]. Our estimates coincide with their upper bounds for p close to two, and constitute a genuine improvement for smaller values of p.

The family (1) still persists for $1/(2d) < \beta^2 < 1/(d-2)$, now with $C_p > 1$ for all $1 \leq p < 2$. In Theorem 3, we estimate the value of C_1 in the associated Poincaré inequality, or equivalently, the width $\lambda_1 := 1/C_1$ of the spectral gap of $\mathbf{L}_\beta$. The result is shown as a solid line in Figure 2 (left). The gap width that we compute agrees with the (optimal) one obtained in [DM05], which is remarkable since our approach relies on much more elementary tools.

Wealth distribution model. The next example concerns a measure on the positive half-line $\Omega = (0, \infty)$. Inequalities (1) are investigated with

$$f_\infty(x) = Z_\theta^{-1} e^{-1/(\theta x)} x^{-2-1/\theta} \quad \text{and} \quad D_\theta(x) = \theta x^2,$$

where $\theta > 0$. The respective evolution equation

$$\partial_t u = \mathbf{L}_\theta[u] := \theta x^2 u_{xx} - (x-1) u_x, \tag{7}$$

appears as the grazing collisions limit of a kinetic model for wealth redistribution in a simple market economy [PT06]. The algebraically fat (Pareto) tail for $x \to \infty$ represents the riches accumulated by a small high society. We refer to, e.g., [BM00, Sol98] for a derivation and relevant references. To our knowledge, neither the associated inequalities (1) nor the long-time asymptotics of solutions to (7) have been investigated before.

The situation is very similar to that of the one-dimensional linearized fast-diffusion equation, upon replacing β^2 by θ. This might be surprising since the relation between the standard Ornstein-Uhlenbeck process and (7) in the limit $\theta \downarrow 0$ is not obvious. The slow decay of μ_θ as $x \to \infty$ causes the failure of the logarithmic Sobolev inequality (2), but we are able to prove a Poincaré inequality for arbitrary $\theta > 0$ in Theorem 5. The spectral gap of $\mathbf{L}_\theta$ amounts to $\lambda_1 = 1/C_1 = 1$ for $\theta \leq 1$, and to $\lambda_1 = (1+\theta)^2/(2\theta)$ for $\theta > 1$, which is precisely as for $\mathbf{L}_\beta$ from (5) with $\beta^2 = \theta$. On the other hand, the behavior of the associated convex inequalities (1) is apparently different. When proving non-trivial estimates on C_p for $\theta < 1$ in Theorem 5, we do not achieve $C_p \equiv 1$ for any $p < 2$; but instead, we prove (1) with

$$C_p \leq \Big(1 - \theta\frac{p-1}{2-p}\Big)^{-1} \tag{8}$$

for $1 \leq p \leq 2/(1+\theta)$, see Figure 2 (right). Observe that the bound in (8) is monotonically convergent to one for each exponent $p < 2$ as $\theta \downarrow 0$. Consequently, also here we recover in the limit exponential contraction of the $L^q(\Omega;\mu)$-semigroup at a q-independent rate.

Lasota model. Finally, we consider (1) for

$$f_\infty(x) = Z_\sigma^{-1} x^{\sigma-1} e^{-\sigma x} dx \quad \text{and} \quad D_\sigma(x) = \frac{x}{\sigma} \tag{9}$$

on $\Omega = (0,\infty)$, where $\sigma > 1/2$. The measure (9) is known as the Lasota production function [Las77], and it is related to the dynamics of a blood cell population [GM90]. Moreover, it appears as the invariant measure of

$$\partial_t u = \mathbf{L}_\sigma[u] := \sigma^{-1} x u_{xx} - (x-1)u_x, \tag{10}$$

which is one of the basic models[2] for a general equilibrium asset price [CIR85]. The underlying stochastic differential equation is nowadays called a CIR process. We remark that stochastic processes like (10) have already been studied by Feller [Fel51] in the fifties, but apparently, the $L^q(\Omega;\mu)$-contraction rates have not been calculated before.

Despite the apparent similarity between (10) and the economy model (7), the contractivity properties of the associated semigroup are quite different. Thanks to the exponentially small tail of μ_σ, there is indeed an associated logarithmic Sobolev inequality (2), with $C_2 = 2$, which yields a priori $C_p \leq 2$ in (1). In Theorem 6, we improve on the constants C_p in the range $1 \leq p < 2$. In particular, we show that $C_1 = 1$ for all $\sigma \geq 1/2$.

Plan of the paper. In section 2 below, we collect various facts — most of them classical — about the original Bakry-Émery approach and its generalizations. In section 3 we formulate the problem of proving (1) in algebraic terms. The remainder of the paper is devoted to applications of our method. In section 4, we give a proof of (1) under the hypothesis that the Bakry-Émery condition holds. The linearized fast diffusion equation is treated in section 5 (in dimension $d \geq 2$) and 6 (in dimension $d = 1$). Sections 7 and 8 are devoted to the wealth redistribution and Lasota model, respectively. In the appendix, we recall some known properties of the constants C_p, among them a result from [BR03].

2. PRELIMINARIES

2.1. **General assumptions.** Although we are mainly interested in the three specific families of inequalities mentioned in the introduction, we shall formulate the main strategy in rather general terms and make several assumptions that will simplify the subsequent analysis.

We start with a change of notations. In the proofs, it is more convenient to use an equivalent representation of the convex Sobolev inequalities (1), namely

$$k_q\bigg[\int_\Omega u^q \, d\mu - \Big(\int_\Omega u \, d\mu\Big)^q\bigg] \leq 2(q-1)\int_\Omega D\big|\nabla\big(u^{q/2}\big)\big|^2 \, d\mu. \tag{11}$$

The original form (1) is equivalent to (11) upon defining $p := 2/q$, $w := u^{q/2}$ and $C_p := q/k_q$. The seemingly non-canonical choice of the parameters q and k_q in (11) will be advantageous in the subsequent calculations. Whereas $C_{2/q}$ is proportional to the *time scale* of exponential convergence in (4), k_q is related to the *convergence rate.* Moreover, k_q possesses the useful interpolation property discussed in secion 3.2.

Next, we impose regularity conditions on Ω, μ, D and u:

A1 The domain $\Omega \subset \mathbb{R}^d$ is bounded and convex with smooth boundary $\partial\Omega$.

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[2] We emphasize that by abuse of notation, σ in (10) denotes the *inverse* of the volatility in the associated CIR process.

- A2 The probability measure μ possesses a smooth Lebesgue density $f_\infty \in C^\infty(\overline{\Omega})$, which is strictly positive, $\inf_\Omega f_\infty > 0$. Moreover, the diffusion coefficient $D \in C^\infty(\overline{\Omega})$ is smooth and strictly positive, $\inf_\Omega D > 0$.
- A3 The function u is *regular* in the sense that $u \in C^\infty(\overline{\Omega})$, $\inf_\Omega u > 0$ and $\mathbf{n} \cdot \nabla u = 0$ on $\partial\Omega$, where $\mathbf{n}$ denotes the outward normal vector.

Notice that $w = u^{q/2}$ in (11) is regular in the sense of assumption A3 if and only if u is regular. A comment is due concerning the applicability to our main examples, which are naturally posed on unbounded domains and with merely non-negative D. For the linearized fast diffusion equation (5), assumptions A1 and A2 are satisfied for any smoothly bounded and convex domain $\Omega \subset \mathbb{R}^d$ and the respective restriction μ_Ω of μ, defined by $\mu_\Omega(A) = \mu(A)/\mu(\Omega)$ for all measureable sets $A \subset \Omega$. Notice that the density function $f_{\infty,\Omega}$ of the restricted measure μ_Ω satisfies $\nabla \log f_{\infty,\Omega} = \nabla \log f_\infty$, so the operator $\mathbf{L}$ in (3) is formally the same on *any* domain Ω. Then Theorem 3 yields (11) for all regular u with constants k_q independent of Ω. By standard approximation arguments, this allows to conclude (11) on $\Omega = \mathbb{R}^d$ for all $u \in L^q(\mathbb{R}^d; \mu)$. Likewise, let $\Omega = (a, b) \subset \mathbb{R}$ with $0 < a < b < \infty$ for the evolution of wealth (7) or the Lasota model (10). The Ω-independent results of Theorems 5 and 6 extend to $\Omega = \mathbb{R}_+$.

2.2. Entropy functionals for diffusion semigroups.

The convex Sobolev inequalities (11) will be derived as entropy-dissipation relations for the diffusion equation

$$\partial_t u(t) = \mathbf{L}[u(t)] = D\Delta u(t) - Q \cdot \nabla u(t), \tag{12}$$

where $\mathbf{L}$ has already been introduced in (3). Upon imposing homogeneous Neumann boundary conditions,

$$\mathbf{n} \cdot \nabla u = 0 \text{ on } \partial\Omega, \tag{13}$$

$\mathbf{L}$ extends to a self-adjoint, strongly elliptic differential operator on $L^2(\Omega; \mu)$, with dense domain $\mathrm{Dom}(\mathbf{L}) \subset L^2(\Omega; \mu)$. Indeed, integrating by parts, it follows for regular u and v that

$$\int_\Omega \mathbf{L}[u] v \, d\mu = -\int_\Omega D\nabla u \cdot \nabla v \, d\mu = \int_\Omega u \mathbf{L}[v] \, d\mu,$$

which shows that $\mathbf{L}$ is symmetric and non-positive. Its kernel consists exactly of the constant functions. In view of the regularity assumptions A1 and A2 above, standard semigroup theory [Hen81] applies to $\mathbf{L}$ and shows that it generates an analytic semigroup on each $L^q(\Omega; \mu)$ with $1 < q \leq 2$. For any given regular initial datum $U \in L^q(\Omega; \mu)$, we denote the respective time-dependent solution to (12) by $u(t) = \exp(t\mathbf{L})[U]$.

Remark 1. *In applications, the evolution is typically formulated as a* Fokker-Planck equation *for the Lebesgue density*

$$f(t) = u(t) f_\infty \in L^1(\Omega; dx), \tag{14}$$

rather than in terms of u. It is easily seen that f satisfies the L^2-dual form of (12)*,*

$$\partial_t f = \nabla \cdot \big(Df \nabla \log(f/f_\infty)\big) = \Delta(Df) + \nabla \cdot (Qf) \tag{15}$$

subject to variational boundary conditions $\mathbf{n} \cdot \nabla(f/f_\infty) = 0$ on $\partial\Omega$. Notice that $f(t) \equiv f_\infty$ is the unique stationary solution to (15) *of unit mass.*

The *relative entropy functionals* $\mathbf{E}_q$ are defined for $1 < q \leq 2$ as multiples of the integral expression on the left-hand side of (11),

$$\mathbf{E}_q[u] = \int_\Omega \phi_q(u) \, d\mu - \phi_q\left(\int_\Omega u \, d\mu\right) \quad \text{with} \quad \phi_q(s) = \frac{q}{4(q-1)} s^q. \tag{16}$$

Again, the choice of the pre-factor is non-canonical but simplifies subsequent calculations. In particular, $\mathbf{E}_2$ is half of the variance,

$$\mathrm{Var}[u] := 2\mathbf{E}_2[u] = \int_\Omega u^2 \, d\mu - \Big(\int_\Omega u \, d\mu\Big)^2,$$

and in the limit $q \downarrow 1$, one recovers the logarithmic entropy functional,

$$\mathbf{E}_1[u] = \frac{1}{4}\int_\Omega u(x)\log\frac{u(x)}{\int_\Omega u(x)\,d\mu(x)}\,d\mu(x).$$

The functionals $\mathbf{E}_q$ are non-negative and convex on the set of positive functions $u:\Omega\to\mathbb{R}_+$. They vanish exactly on the constant functions, i.e. on the kernel of $\mathbf{L}$.

2.3. **Convex Sobolev inequalities.** The first two *iterated gradients* are the bi-linear forms Γ and Γ_2 defined on regular functions U and V by

$$\Gamma[U,V] = \frac{1}{2}\big(\mathbf{L}[UV] - U\mathbf{L}[V] - \mathbf{L}[U]V\big) = D\nabla U\cdot\nabla V, \tag{17}$$

$$\Gamma_2[U,V] = \frac{1}{2}\big(\mathbf{L}\Gamma[U,V] - \Gamma[U,\mathbf{L}[V]] - \Gamma[\mathbf{L}[U],V]\big). \tag{18}$$

For brevity, we shall also write $\Gamma[U]=\Gamma[U,U]$ and $\Gamma_2[U]=\Gamma_2[U,U]$. In terms of these quantities, one of the principal results from the Bakry-Émery theory [BE85] reads as follows.

Theorem 1. *Assume that for some $\lambda_{BE}>0$, the Bakry-Émery condition*

$$\Gamma_2[U] \geq \lambda_{BE}\Gamma[U] \quad \textit{pointwise on } \Omega \tag{19}$$

is satisfied for all regular functions U. Then the convex Sobolev inequalities (11) *hold with $k_q = q\lambda_{BE}$.*

This theorem appears as a consequence of the theory we develop below; a short proof is presented in section 4. At this point, our goal is to formulate a weaker, yet more practical condition than (19) that allows to conclude convex Sobolev inequalities (11). Following the original ideas from [BE85], we introduce the *first and second entropy production* $\mathbf{I}_q$ and $\mathbf{J}_q$, respectively, by

$$\mathbf{I}_q[U] = -\frac{d}{dt}\Big|_{t=0^+}\mathbf{E}_q\big[\exp(t\mathbf{L})[U]\big], \qquad \mathbf{J}_q[U] = \frac{d^2}{dt^2}\Big|_{t=0^+}\mathbf{E}_q\big[\exp(t\mathbf{L})[U]\big].$$

We define further $\psi_q(s)=s^{q/2}$; then $(\psi_q')^2=\phi_q''$, with ϕ_q from (16). Using the chain rule property $\Gamma[\varphi(u),v]=\varphi'(u)\Gamma[u,v]$ for smooth functions φ and regular u, v, it follows that

$$\mathbf{I}_q[u] = -\int_\Omega \phi_q'(u)\mathbf{L}[u]\,d\mu = \int_\Omega \Gamma\big[\phi_q'(u),u\big]\,d\mu = \int_\Omega \phi_q''(u)\Gamma[u]\,d\mu = \int_\Omega \Gamma[\psi_q(u)]\,d\mu.$$

Now, observing that $\mathbf{L}[\psi_q(u)] = \psi_q'(u)\mathbf{L}[u] + \psi_q''(u)\Gamma[u]$, we obtain

$$\begin{aligned}\mathbf{J}_q[u] &= -2\int_\Omega \Gamma\big[\psi_q(u),\psi_q'(u)\mathbf{L}[u]\big]\,d\mu = -2\int_\Omega \mathbf{L}[\psi_q(u)]\psi_q'(u)\mathbf{L}[u]\,d\mu\\ &= 2\int_\Omega \mathbf{L}[\psi_q(u)]\Big(\mathbf{L}[\psi_q(u)] - \frac{\psi_q''(u)}{\psi_q'(u)^2}\Gamma[\psi_q(u)]\Big)\,d\mu.\end{aligned}$$

With $w:=\psi_q(u)=u^{q/2}$, these calculations yield the following explicit representations,

$$\mathbf{I}_q[u] = \int_\Omega D|\nabla w|^2\,d\mu = \int_\Omega D\big|\nabla\big(u^{q/2}\big)\big|^2\,d\mu, \tag{20}$$

$$\begin{aligned}\mathbf{J}_q[u] = \frac{2}{q}\int_\Omega w^2\Big[&qD^2\Big(\frac{\Delta w}{w}\Big)^2 + (2-q)D^2\Big(\frac{\Delta w}{w}\Big)\Big|\frac{\nabla w}{w}\Big|^2 - 2qD\Big(\frac{\Delta w}{w}\Big)Q\cdot\Big(\frac{\nabla w}{w}\Big)\\ &- (2-q)DQ\cdot\frac{\nabla w}{w}\Big|\frac{\nabla w}{w}\Big|^2 + q\Big(Q\cdot\Big(\frac{\nabla w}{w}\Big)\Big)^2\Big]\,d\mu.\end{aligned} \tag{21}$$

In particular, inequality (11) at $q\in(1,2]$ is nothing but the dissipation relation

$$k_q\mathbf{E}_q[u] \leq \frac{q}{2}\mathbf{I}_q[u]. \tag{22}$$

The essential ingredient of the method is the following.

Lemma 1. *Assume that the first entropy production is estimated by the second one as follows,*

$$k_q \mathbf{I}_q[u] \leq \frac{q}{2} \mathbf{J}_q[u]. \tag{23}$$

Then the convex Sobolev inequality (22) *holds.*

Instead of passing from (23) further to the pointwise condition (19), we shall work with (23) directly on the integral level. As will be made clear in section 3 below, integration by parts provides a surprisingly powerful tool to prove (23) and calculate $k_q > 0$ explicitly even in situations in which (19) fails.

We emphasize that the proof of Theorem 1 in [BE85] only uses the weaker integral condition (23), and (19) has been derived *a posteriori* as a sufficient condition, using that

$$\mathbf{I}_q[u] = \int_\Omega \left(\frac{\Gamma[\phi_q'(u)]}{\phi_q''(u)} \right) d\mu \quad \text{and} \quad \mathbf{J}_q[u] \geq \int_\Omega \left(\frac{\Gamma_2[\phi_q'(u)]}{\phi_q''(u)} \right) d\mu. \tag{24}$$

It is well known that the use of integral criteria instead of condition (19) allows to improve the constants in certain logarithmic Sobolev inequalities, see for instance [Led92, Rot86]. In a recent article [BG09], it has been shown that the integral criterion

$$k_q \int_\Omega \left(\frac{\Gamma[\phi_q'(u)]}{\phi_q''(u)} \right) d\mu \leq \frac{q}{2} \int_\Omega \left(\frac{\Gamma_2[\phi_q'(u)]}{\phi_q''(u)} \right) d\mu,$$

which is slightly stronger than (23) but weaker than (19), does not only imply the convex inequalities (11), but also suffices to prove the *refined entropy inequalities*

$$k_q \left[\int_\Omega u^q \, d\mu - \Big(\int_\Omega u \, d\mu \Big)^{2(q-1)} \Big(\int_\Omega u^q \, d\mu \Big)^{2/q-1} \right] \leq q(q-1)^2 \int_\Omega D \big| \nabla \big(u^{q/2} \big) \big|^2 \, d\mu.$$

On the other hand, it appears that integral criteria of the form (23) have never been employed to improve the constants k_q for $1 < q < 2$ in specific examples of (11).

Proof of Lemma 1. Substitute $u(t) = \exp(-t\mathbf{L})[U]$ into (23) and apply Gronwall's lemma to conclude that

$$\mathbf{I}_q[u(t)] \leq \mathbf{I}_q[U] \exp\Big(-\frac{2k_q}{q} t \Big). \tag{25}$$

Implicitly, we have used that $\mathbf{I}_q[u(t)] \to \mathbf{I}_q[U]$ as $t \downarrow 0$. This is justified since U is regular by assumption and thus belongs to $\mathrm{Dom}(\mathbf{L})$. Semigroup theory implies that the first time derivative

$$\mathbf{I}[u(t)] = \frac{d}{dt} \mathbf{E}_q[u(t)] = \int_\Omega \phi_q'(u(t)) \mathbf{L}[u(t)] \, dx$$

is right-continuous at $t = 0$.

Integration of (25) with respect to time yields

$$\mathbf{E}_q[u(t)] = \int_t^\infty \mathbf{I}_q[u(t')] \, dt' \leq \frac{q}{2k_q} \mathbf{I}_q[U] \exp\Big(-\frac{2k_q}{q} t \Big), \tag{26}$$

which in particular implies (22), taking $t = 0$. We have used implicitly that $\mathbf{E}_q[u(t)] \to 0$ as $t \to \infty$. This is justified since (25) above implies in particular that $u(t)$ converges to a constant function in $L^q(\Omega; \mu)$. □

3. FORMULATION AS AN ALGEBRAIC PROBLEM

In this section, a formalism is introduced that helps to establish inequality (23) for particular examples of operators $\mathbf{L}$.

3.1. **Algebraic framework for integral manipulations.** For a given $w : \Omega \to \mathbb{R}_+$, which is regular in the sense of assumption A3, define the functions ξ_G, ξ_H and ξ_L by

$$\begin{cases} \xi_G(x) = w(x)^{-1}\nabla w(x) \in \mathbb{R}^d, \\ \xi_H(x) = w(x)^{-1}\nabla^2 w(x) \in \mathbb{R}^{d\times d}_{sym}, \\ \xi_L(x) = \operatorname{tr}\xi_H(x) = w(x)^{-1}\Delta w(x), \end{cases}$$

where $\mathbb{R}^{d\times d}_{sym}$ is the set of all symmetric $d\times d$ matrices and "tr" denotes the trace of a matrix. Using the explicit expression for $\mathbf{J}_q$ from (21), the desired inequality (23) can be represented as

$$\int_\Omega w^2\big(S_q - k_q D|\xi_G|^2\big)\,d\mu \ge 0, \tag{27}$$

where S_q is given by

$$S_q = qD^2\xi_L^2 + (2-q)D^2\xi_L|\xi_G|^2 - 2qD\xi_L(Q\cdot\xi_G) - (2-q)D(Q\cdot\xi_G)|\xi_G|^2 + q(Q\cdot\xi_G)^2. \tag{28}$$

The goal is to modify the integrand in (27) using integration by parts, so that the integrand becomes pointwise non-negative for $x\in\Omega$. As in [JM06], integration by parts is considered as the addition of a divergence under the integral. More precisely, we add suitable linear combinations of certain functions $T_i : \Omega \to \mathbb{R}$, defined in the following way:

$$w^2 f_\infty T_i = \nabla\cdot\big(w^2 f_\infty R_i\big). \tag{29}$$

For the functions $R_i : \Omega \to \mathbb{R}^d$, one needs to make a suitable ansatz. This is not quite as canonical as in [JM06], where spatially homogeneous evolution equations have been considered. The following choices have proven to be suitable for our applications.

$$\begin{cases} R_1 = D^2(\xi_H\cdot\xi_G - \xi_L\xi_G), \\ R_2 = D(Q\cdot\xi_G)\xi_G, \\ R_3 = D\nabla D|\xi_G|^2, \\ R_4 = D^2|\xi_G|^2\xi_G. \end{cases}$$

A straight-forward calculation yields explicit expressions for the T_i,

$$\begin{cases} T_1 = D^2\big(\|\xi_H\|^2 - \xi_L^2\big) - D(Q-\nabla D)\cdot\xi_H\cdot\xi_G + D(Q-\nabla D)\cdot\xi_G\xi_L, \\ T_2 = DQ\cdot\xi_H\cdot\xi_G + D\xi_L Q\cdot\xi_G + D\xi_G\cdot\nabla Q\cdot\xi_G - (Q\cdot\xi_G)^2, \\ T_3 = 2D\nabla D\cdot\xi_H\cdot\xi_G + (D\Delta D - \nabla D\cdot Q)|\xi_G|^2, \\ T_4 = 2D^2\xi_G\cdot\xi_H\cdot\xi_G + D^2\xi_L|\xi_G|^2 - D|\xi_G|^2(Q-\nabla D)\cdot\xi_G - D^2|\xi_G|^4. \end{cases} \tag{30}$$

Above, S_q and the T_i have been introduced as smooth real functions on Ω, defined in terms of w. In order to pass to an algebraic formulation of (27), consider for the moment ξ_G and ξ_H not as x-dependent functions, but merely as elements of $\mathbb{R}^d$ and $\mathbb{R}^{d\times d}_{sym}$, respectively. In view of the explicit representations in (28) and (30), respectively, S_q and the T_i can be canonically identified with functions $\mathcal{S}_q$ and $\mathcal{T}_i$ of $(x,\xi_G,\xi_H)\in\Omega\times\mathbb{R}^d\times\mathbb{R}^{d\times d}_{sym}$, which depend on the components of ξ_G and ξ_H in a polynomial way, and on x only indirectly via D and Q. In other words, we define $\mathcal{S}_q = \mathcal{S}_q(x;\xi_G,\xi_H)$ such that

$$S_q(x) = \mathcal{S}_q\big(x;\, w^{-1}(x)\nabla w(x),\, w(x)^{-1}\nabla^2 w(x)\big), \tag{31}$$

and likewise for the $\mathcal{T}_i$, which will be referred to as *shift polynomials* in the following. This interpretation allows us to give a sufficient condition for the analytical statement (27) in algebraic terms.

Lemma 2. *Assume that $\sigma\mathbf{n}\cdot\nabla D \le 0$ on $\partial\Omega$ for one consistent choice of $\sigma\in\{-1,+1\}$. Let $q\in(1,2]$ and $k_q>0$ be given. If there exist real constants c_1 to c_4 with $c_1\ge 0$ and $\sigma c_4 \ge 0$ such that*

$$\forall x\in\Omega:\ \forall\xi_G\in\mathbb{R}^d,\ \xi_H\in\mathbb{R}^{d\times d}_{sym}:\ \big(\mathcal{S}_q + c_1\mathcal{T}_1 + \cdots + c_4\mathcal{T}_4\big)(x;\xi_G,\xi_H) - k_q D(x)|\xi_G|^2 \ge 0, \tag{32}$$

then inequality (27) *follows, and so does the respective convex Sobolev inequality* (11)*, with (possibly non-optimal) constant k_q.*

Proof. The proof follows by the divergence theorem. Indeed, by definition of the T_i in (29),

$$\int_\Omega w^2 T_i \, d\mu = \int_\Omega \nabla \cdot \big(w^2 f_\infty R_i\big)\, dx = \int_{\partial\Omega} w^2 \mathbf{n} \cdot R_i \, d\mu_{\partial\Omega},$$

where $\mu_{\partial\Omega}$ is the measure on $\partial\Omega$ induced by μ. Hence (27) is equivalent to

$$\int_\Omega w^2 \big(S_q + c_1 T_1 + \cdots + c_4 T_4 - k_q D(x)|\xi_G|^2\big)\, d\mu \geq \int_{\partial\Omega} \mathbf{n} \cdot (c_1 R_1 + \cdots + c_4 R_4)\, d\mu_{\partial\Omega}. \tag{33}$$

If (32) holds, then the integral on the left-hand side of (33) has a pointwise non-negative integrand by the relation between S_q, T_i and $\mathcal{S}_q$, $\mathcal{T}_i$, see (31). Provided that the right-hand side, i.e. the contribution from the boundary integrals, is non-positive, then (33) is clearly true, and so is (27) by equivalence.

The boundary condition (13) implies that $\mathbf{n} \cdot \nabla w = 0$ on $\partial\Omega$. As an immediate consequence, the contributions from R_2 and R_4 vanish,

$$w^2 \mathbf{n} \cdot R_2 = D(Q \cdot \nabla w)(\mathbf{n} \cdot \nabla w) = 0, \quad w^2 \mathbf{n} \cdot R_4 = w^{-1} D^2 |\nabla w|^2 (\mathbf{n} \cdot \nabla w) = 0.$$

Elementary geometric considerations reveal that $\mathbf{n} \cdot \nabla w = 0$ implies $\mathbf{n} \cdot \nabla^2 w \cdot \nabla w \leq 0$ on the boundary $\partial\Omega$ of the convex domain Ω, so

$$\int_{\partial\Omega} w^2 \mathbf{n} \cdot R_1(\xi)\, d\mu_{\partial\Omega} = \int_{\partial\Omega} D^2 \mathbf{n} \cdot \nabla^2 w \cdot \nabla w \, d\mu_{\partial\Omega} \leq 0.$$

For $c_1 \geq 0$, the contribution from R_1 to the right hand side of (33) is thus non-positive. Finally,

$$c_3 \int_{\partial\Omega} w^2 \mathbf{n} \cdot R_3(\xi)\, d\mu_{\partial\Omega} = \int_{\partial\Omega} D(c_3 \mathbf{n} \cdot \nabla D) |\nabla w|^2 \, d\mu_{\partial\Omega}.$$

By assumption, $\sigma \mathbf{n} \cdot \nabla D \leq 0$ and $\sigma c_3 \geq 0$. Hence $c_3 \mathbf{n} \cdot \nabla D \leq 0$, and the last integral gives a non-positive contribution. □

Various simplifications can be made if $d = 1$. It suffices to consider two real-valued function ξ_1, ξ_2 defined by

$$\xi_1(x) = w(x)^{-1} \partial_x w(x), \quad \xi_2(x) = w(x)^{-1} \partial_x^2 w(x).$$

The expression for T_1 from (30) degenerates to $T_1 \equiv 0$. Finally, there is no need to impose sign restrictions on c_3 in (32) since the boundary condition (13) implies that $R_3 = 0$ on $\partial\Omega$. Introducing these simplifications into the above arguments, we obtain the following.

Lemma 3. *Assume $d = 1$. Let $q \in (1, 2]$ and $k_q > 0$ be given. If there exist real constants c_1, c_2 and c_3 such that*

$$\forall x \in \Omega : \forall \xi_1, \xi_2 \in \mathbb{R} : \big(\mathcal{S}_q + c_2 \mathcal{T}_2 + c_3 \mathcal{T}_3 + c_4 \mathcal{T}_4\big)(x; \xi_1, \xi_2) - k_q D(x) \xi_1^2 \geq 0, \tag{34}$$

then inequality (23) *follows, and so does* (11).

3.2. **An interpolation property.** As the polynomial $\mathcal{S}_q$ depends on q, so do in general the coefficients c_i and the constant k_q that are suitable for (32). Fortunately, the problem enjoys an interpolation property which reduces the effort of proving (32) on some interval $[q_-, q_+] \subset (1, 2]$ to proving it at the endpoints $q_\pm$.

Indeed, observe that the expression S_q in (28) depends on q in an affine manner. Hence, if inequality (27) is true with constants $k_{q_\pm}$ at $q = q_\pm$, respectively, then (27) is also true with constants

$$k_q = \frac{q - q_-}{q_+ - q_-} k_+ + \frac{q_+ - q}{q_+ - q_-} k_- \tag{35}$$

at any intermediate point $q \in [q_-, q_+]$. For a proof, simply take the respective convex combination of the inequalities at $q = q_-$ and $q = q_+$. As a direct consequence, the *optimal constants* k_q for (27) — which, however, might be smaller than the respective optimal constants in (11) — form a *concave* function with respect to $q \in (1, 2)$.

The interpolation property carries over from the integral formulation (27) to the algebraic problem (32). We state the corresponding result, which we shall frequently invoke in the following.

Lemma 4. *Assume that* $1 \leq q_- < q_+ \leq 2$*, and that* (32) *holds at* $q = q_+$ *and* $q = q_-$ *with respective constants* k_+ *and* k_-*. Then* (32) *holds for all* $q \in [q_-, q_+]$ *with respective constants* k_q *from* (35).

Proof. The polynomial $\mathcal{S}_q$ depend on q in an affine manner,

$$\mathcal{S}_q = \frac{q - q_-}{q_+ - q_-}\mathcal{S}_{q_+} + \frac{q_+ - q}{q_+ - q_-}\mathcal{S}_{q_-},$$

whereas $D|\xi_G|^2$ and the $\mathcal{T}_i$ are independent of q. Denote by $c_1^-, \ldots, c_4^-$ and $c_1^+, \ldots, c_4^+$ two quadruples of coefficients that make (32) true at $q = q_+$ and $q = q_-$, respectively. Let $c_i^q := [(q - q_-)c_i^+ + (q_+ - q)c_i^-]/(q_+ - q_-)$ be their affine interpolations. With k_q defined as in (35), it is now immediate to conclude that

$$\mathcal{S}_q + \sum_{i=1}^4 c_i^q \mathcal{T}_i - k_q D|\xi_G|^2 =$$
$$\frac{q - q_-}{q_+ - q_-}\big(\mathcal{S}_{q_+} + \sum_{i=1}^4 c_i^+ \mathcal{T}_i - k_+ D|\xi_G|^2\big) + \frac{q_+ - q}{q_+ - q_-}\big(\mathcal{S}_{q_-} + \sum_{i=1}^4 c_i^- \mathcal{T}_i - k_- D|\xi_G|^2\big)$$

is non-negative for all $q \in [q_-, q_+]$. □

3.3. **Computer-assisted solution of the algebraic problems.** A remarkable feature of the algebraic formulation obtained above is that the problems (32) and (34) can be tackled in a computer-aided way. In particular, the dependence of D and Q on x is polynomial in our examples. This makes (32) a *quantifier elimination problem* of real algebraic geometry, that is always solvable in an algorithmic way [Tar51]. This fact has already been exploited in [JM06] to obtain entropy-type Lyapunov functions for certain classes of homogeneous non-linear parabolic evolution equations of higher order. In the situation at hand, however, the spatial inhomogeneity introduced by D and Q leads to a much higher complexity of the quantifier elimination problem, which is no longer solvable directly by the software toolboxes that are currently available.

For our calculations, we resort to the so-called SOS method: a sufficient (but in general far from necessary) criterion for the non-negativity of a polynomial is the existence of a representation as a sum of squares (SOS) of other polynomials. Advanced software tools are available to determine SOS decompositions of parameterized polynomials of arbitrary degree and in arbitrarily many variables, as long as the coefficients depend linearly on the parameters. This is exactly the case in (32), where the parameters c_i are simply the coefficients in the linear combination of the polynomials $\mathcal{T}_i$.

To calculate the SOS decompositions, we have mainly employed the software package `YALMIP` [Loe04]. The results are purely numerical and cannot be turned into a proof directly. However, they give an invaluable indication on suitable choices for k_q and the parameters c_i, which in the actual proofs below seem to appear out of the blue.

4. Application in the Bakry-Émery setting

The developed machinery is now applied to provide a short proof of Theorem 1. First, we give a reformulation of the theorem to specify the hypotheses more precisely.

Theorem 2. *Assume that the domain* $\Omega \subset \mathbb{R}^d$*, the measure* μ *and the coefficient* $D : \Omega \to \mathbb{R}_+$ *satisfy assumptions A1 and A2. In dimensions* $d > 1$*, assume in addition that* $\mathbf{n} \cdot \nabla D \leq 0$ *on* $\partial\Omega$*. Then the Bakry-Émery condition* (19)*, i.e.* $\Gamma_2 \geq \lambda_{BE}\Gamma$*, is equivalent to*

$$\mathbf{M} := D\nabla Q + \frac{1}{4}(2 - d)\nabla D \otimes \nabla D + \frac{1}{2}\big(D\Delta D - |\nabla D|^2 - Q \cdot \nabla D\big)\mathbf{1} \geq \lambda_{BE} D\mathbf{1}. \tag{36}$$

If it holds with $\lambda_{BE} > 0$*, then* (32) *or* (34)*, respectively, is satisfied with* $k_q = q\lambda_{BE}$ *for all* $1 \leq q \leq 2$*. Consequently, the convex inequalities* (1) *hold with an optimal constant* $C_p \leq 1/\lambda_{BE}$*.*

For the proof, we shall derive the representation formula for $\mathbf{J}_q$ given in (24). Our argument is similar to that from the proof of the main theorem in [AMTU01], but our framework allows to reduce the relevant calculations to a couple of lines.

Proof. A tedious, but straightforward calculation reveals the following representation of Γ_2,

$$\Gamma_2[U](x) = \Pi\big(x; \nabla^2 U, \nabla U\big),$$

where the function

$$\Pi(x; \xi_G, \xi_H) := \big\| \, D\xi_H + \nabla D \otimes_s \xi_G - \tfrac{1}{2}(\nabla D \cdot \xi_G)\mathbf{1} \big\|^2 + \xi_G \cdot \mathbf{M} \cdot \xi_G \tag{37}$$

is a polynomial in $\xi_G \in \mathbb{R}^d$ and $\xi_H \in \mathbb{R}^{d\times d}_{sym}$, with coefficients depending on $x \in \mathbb{R}^d$. Here and below, $\zeta \otimes_s \eta \in \mathbb{R}^{d\times d}_{sym}$ denotes the symmetrized tensor product of ζ, $\eta \in \mathbb{R}^d$, i.e. $(\zeta \otimes_s \eta)_{ij} = \frac{1}{2}(\zeta_i\eta_j + \zeta_j\eta_i)$. Condition (36) obviously implies

$$\Pi(x; \xi_G, \xi_H) \geq \lambda_{BE} D(x)|\xi_G|^2, \tag{38}$$

and thus also (19), recalling that $\Gamma[U](x) = D(x)|\nabla U(x)|^2$. To prove the reverse implication, fix a point $\bar{x} \in \mathbb{R}^d$ and a vector $z \in \mathbb{R}^d$. Choose a regular function $\bar{U}$ with $\nabla \bar{U}(\bar{x}) = z$ and $D\nabla^2\bar{U}(\bar{x}) = \frac{1}{2}(z \cdot \nabla D)\mathbf{1} - z \otimes_s \nabla D$. Then the squared norm in (37) vanishes, which implies

$$z \cdot \mathbf{M}(\bar{x}) \cdot z = \Pi(\bar{x}; \nabla^2\bar{U}, \bar{U}) = \Gamma_2[\bar{U}, \bar{U}](\bar{x}) \geq \lambda_{BE}\Gamma[\bar{U}, \bar{U}](\bar{x}) = \lambda_{BE} D(\bar{x})|z|^2.$$

We are going to prove (32) by considering the two extremals $q = 1$ and $q = 2$ and applying the interpolation Lemma 4. First, let $q = 2$, and observe that

$$\mathcal{S}_2(x; \xi_G, \xi_H) = 2\big(D(x)\xi_L - Q(x) \cdot \xi_G\big)^2.$$

Use $c_1 = 2$, $c_2 = 2$, $c_3 = 1$, and $c_4 = 0$ to achieve

$$\big(\mathcal{S}_2 + c_1\mathcal{T}_1 + c_2\mathcal{T}_2 + c_3\mathcal{T}_3 + c_4\mathcal{T}_4\big)(x; \xi_G, \xi_H) = 2\Pi(x; \xi_G, \xi_H) \geq 2\lambda_{BE} D(x)|\xi_G|^2,$$

employing (38). Thus, (32) holds with $k_2 = 2\lambda_{BE}$. On the other hand, at $q = 1$,

$$\mathcal{S}_1(x; \xi_G, \xi_H) = \frac{1}{2}\mathcal{S}_2(x; \xi_G, \xi_H) + D(x)^2\xi_L|\xi_G|^2 - D(x)Q(x) \cdot \xi_G|\xi_G|^2.$$

Choosing $c_1 = 1$, $c_2 = 1$, $c_3 = 1/2$ and $c_4 = -1$, we infer that

$$\begin{aligned} \mathcal{S}(x; \xi_G, \xi_H) &:= \big(\mathcal{S}_1 + c_1\mathcal{T}_1 + c_2\mathcal{T}_2 + c_3\mathcal{T}_3 + c_4\mathcal{T}_4\big)(x; \xi_G, \xi_H) \\ &= \Pi(x; \xi_G, \xi_H) - 2D(x)\xi_G \cdot \xi_H \cdot \xi_G - D(x)|\xi_G|^2\nabla D(x) \cdot \xi_G + D^2|\xi_G|^4. \end{aligned}$$

The additional terms can now be combined with the squared norm in Π to form a different complete square, namely

$$\begin{aligned} \mathcal{S} &= \Pi - 2D\xi_G \cdot \big(D\xi_H + \nabla D \otimes_s \xi_G - \frac{1}{2}\nabla D \cdot \xi_G\mathbf{1}\big) \cdot \xi_G + D^2|\xi_G|^4 \\ &= \big\|D\xi_H - D\xi_G \otimes \xi_G + \nabla D \otimes_s \xi_G - \frac{1}{2}(\nabla D \cdot \xi_G)\mathbf{1}\big\|^2 + \xi_G \cdot \mathbf{M} \cdot \xi_G. \end{aligned}$$

Condition (36) obviously implies (32) with $k_1 = \lambda_{BE}$. Interpolation by means of (35) finishes the proof. ☐

5. APPLICATION TO THE LINEARIZED FAST-DIFFUSION EQUATION IN MULTIPLE DIMENSIONS

The linearly confined porous medium equation

$$\partial_t F = \Delta(F^m) + \nabla \cdot (xF) \quad \text{on } \Omega \subset \mathbb{R}^d \tag{39}$$

models the diffusive spreading of a particle concentration $F = F(t; x) \geq 0$ on Ω, under the influence of a linear force towards the origin. In contrast to the heat equation, the mobility of the particles is not constant, but is a function of the concentration itself. The most relevant case is $\Omega = \mathbb{R}^d$, when (39) is equivalent to the unconfined equation $\partial_t F = \Delta(F^m)$ upon self-similar rescaling. For an exhaustive introduction to the subject, the reader is referred to [Vaz07].

In (39), the parameter $m > 0$ controls the dependence of the particle mobility on the density. Here we are interested in the range $0 < m < 1$, called the *fast diffusion regime*. Equation (39) is known to possess mass-preserving solutions F when m lies above the critical value $m_c := 1 - 2/d$. The stationary solution to (39) is a pseudo-Barenblatt profile,

$$B_m(x) = \Big(C + \frac{1-m}{2m}|x|^2\Big)^{-1/(1-m)},$$

where the constant $C > 0$ controls the mass of B_m. Large-time asymptotics of solutions to (39) have been studied by a variety of authors; the most complete treatment can be found in [BBDGV09].

Here, we are only interested in the linear stability analysis around B_m. The linearization of (39) at $F = B_m$ reads

$$\partial_t f = \Delta\big((\alpha^2 + \beta^2|x|^2)f\big) + \nabla\cdot(xf), \quad \alpha^2 = mC > 0, \quad \beta^2 = \frac{1-m}{2} \in \Big(0, \frac{1}{d}\Big), \tag{40}$$

which is a Fokker-Planck equation of the form (15) with

$$D_\beta(x) = \alpha^2 + \beta^2|x|^2 \quad \text{and} \quad Q(x) = x.$$

The unique stationary state to (40) of unit mass is proportional to $D_\beta B_m$,

$$f_\infty(x) = \frac{1}{Z} D_\beta(x)^{-1-1/(2\beta^2)},$$

where the normalization constant $Z = Z_{\alpha,\beta,\Omega} = \int_\Omega D_\beta(x)^{-1-1/(2\beta^2)}dx$ is well-defined only if $\beta^2 < 1/(d-2)$. Define μ_β as the measure on Ω with density f_∞, and introduce u according to (14). Then $u(t)$ is a solution to (12) with the associated linear operator from (3),

$$\mathbf{L}_\beta[u] = (\alpha^2 + \beta^2|x|^2)\Delta u - x\cdot\nabla u.$$

For the proof of associated convex Sobolev inequalities (11) on $\Omega = \mathbb{R}^d$, we cannot resort to Theorem 2, due to the following observation.

Lemma 5. *The Bakry-Émery condition* (19) *is fulfilled on any bounded domain* $\Omega \subset \mathbb{R}^d$ *with some* $\lambda_{BE}^\Omega > 0$*, but the largest* Ω*-uniform constant is* $\lambda_{BE}^* = 0$*. In fact, the corresponding logarithmic Sobolev inequality* (2) *does not hold on* $\Omega = \mathbb{R}^d$*.*

Proof. The matrix $\mathbf{M}$ defined in (36) becomes

$$\mathbf{M}(x) = \alpha^2(1 + d\beta^2)\mathbf{1} + (d-2)\beta^4\big(|x|^2\mathbf{1} - x\otimes x\big),$$

which is non-negative definite at every $x \in \Omega$, since for arbitrary $z \in \mathbb{R}^d \setminus \{0\}$,

$$e_z\cdot\mathbf{M}(x)\cdot e_z = \alpha^2(1 + d\beta^2) + (d-2)\beta^4\big(|x|^2 - (x\cdot e_z)^2\big) > 0$$

by the Cauchy-Schwarz inequality, with $e_z = z/|z|$. On the other hand, at any fixed $x \in \Omega$, the choice $z = x$ yields

$$\frac{x\cdot\mathbf{M}(x)\cdot x}{D_\beta(x)|x|^2} = \frac{\alpha^2(1 + d\beta^2)}{D_\beta(x)},$$

which can be made arbitrarily small when Ω is sufficiently large. The largest Ω-uniform constant λ_{BE} in (19) is thus zero.

To show that the logarithmic Sobolev inequality (2) does not hold, consider the following H^1-smooth, radially symmetric function $w : \mathbb{R}^d \to \mathbb{R}_+$ with $w(x) = W(|x|)$, where $W : [0,\infty) \to \mathbb{R}_+$ is defined by

$$W(r) = \begin{cases} (r^{1+1/(2\beta^2)-d/2})/\log r & \text{for } r > e, \\ e^{1+1/(2\beta^2)-d/2} & \text{for } 0 \le r \le e. \end{cases}$$

By the change of variables $x = r\Theta$ with $r \ge 0$ and $\Theta \in \mathbb{S}^{d-1}$,

$$\begin{aligned}\int_{\mathbb{R}^d} w^2\,d\mu_\beta &= \frac{|\mathbb{S}^{d-1}|}{Z}\int_0^\infty W(r)^2(\alpha^2 + \beta^2 r^2)^{-1-1/(2\beta^2)}r^{d-1}\,dr \\ &\le A\Big(1 + \int_e^\infty \frac{dr}{r\,(\log r)^2}\Big) = A\Big(1 + \int_1^\infty \frac{dz}{z^2}\Big) < \infty,\end{aligned}$$

with some finite constant $A = A(\alpha,\beta) > 0$. Similarly, one finds

$$\begin{aligned}\int_{\mathbb{R}^d} D_\beta |\nabla w|^2\, d\mu_\beta &= \frac{|\mathbb{S}^{d-1}|}{Z}\int_e^\infty W'(r)^2(\alpha^2+\beta^2r^2)^{-1/(2\beta^2)}r^{d-1}\,dr \\ &\le A\int_e^\infty \big[\big(1+1/(2\beta^2)-d/2\big)-(\log r)^{-1}\big]^2\frac{dr}{r\,(\log r)^2} \\ &\le A\int_1^\infty \big[\big(1+1/(2\beta^2)-d/2\big)-z^{-1}\big]\frac{dz}{z^2}<\infty.\end{aligned}$$

On the other hand, there is a positive constant $a = a(\alpha,\beta) > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^d} w^2\log w^2\, d\mu_\beta &= \frac{|\mathbb{S}^{d-1}|}{Z}\int_e^\infty \big[W(r)^2\log W(r)^2\big](\alpha^2+\beta^2r^2)^{-1-1/(2\beta^2)}r^{d-1}\,dr \\ &\ge a\int_e^\infty \Big[(2+1/\beta^2-d)-\frac{\log(\log r)}{\log r}\Big]\frac{dr}{r\,\log r} \\ &\ge a\int_1^\infty \Big[(2+1/\beta^2-d)-\frac{\log z}{z}\Big]\frac{dz}{z}=+\infty,\end{aligned}$$

since $\beta^2 < 1/(d-2)$ and $(\log z)/z \to 0$ as $z \to \infty$. By standard arguments, w can be approximated by a sequence of bounded and C^∞-smooth functions $w_n : \mathbb{R}^d \to \mathbb{R}_+$ for which the right-hand side in (2) remains n-uniformly bounded whereas the left-hand side tends to infinity as $n \to \infty$. □

5.1. **Derivation of convex inequalities.** Despite the failure of the classical Bakry-Émery condition, our refinement of the method allows us to obtain convex functional inequalities.

Theorem 3. *Assume that $d \ge 2$ (the case $d = 1$ is covered by Theorem 4 below) and that*[3] *$\Omega \subset \mathbb{R}^d$ is such that $x \cdot \mathbf{n} \ge 0$ on $\partial\Omega$.*

(1) *Let $0 < \beta^2 < 1/(2d)$ or, equivalently, $1 - 1/d < m < 1$. Then criterion (32) is fulfilled with $k_q = q$ for all $\hat q_{d,\beta} \le q \le 2$, where*

$$\hat q_{d,\beta} := 1 + \frac{(4+d\beta^2)\beta^2}{2(1-2d\beta^2)+(4+d\beta^2)\beta^2} \in (1,2).$$

The optimal constant C_p in the corresponding convex Sobolev inequalities (1) is $C_p = 1$ for $1 \le p \le \hat p$ with $\hat p$ from (6), and satisfies $C_p \le (2-\hat p)/(2-p)$ for $\hat p < p < 2$.

(2) *Let $1/(2d) < \beta^2 < 1/(d-2)$, or equivalently $1 - 2/(d-2) < m < 1 - 1/d$. Then criterion (32) is fulfilled at $q = 2$ with*

$$k_2 = \begin{cases} 4(1-d\beta^2) & \text{if } 1/(2d) \le \beta^2 \le 1/(d+2), \\ (1-(d-2)\beta^2)^2/(2\beta^2) & \text{if } 1/(d+2) \le \beta^2 < 1/(d-2). \end{cases}$$

Consequently, the associated linear operator $\mathbf{L}_\beta$ possesses a spectral gap of width $\lambda_1 \ge k_2/2$, and the convex Sobolev inequalities (1) hold with an optimal constant $C_p \le 2/(2-p)k_2^{-1}$.

Remark 2. *Several comments are in order:*

- *The same estimates on the spectral gap (i.e. on k_2 and C_1) have been obtained — calculating the spectral decomposition of $\mathbf{L}_\beta$ — before in* [DM05]*; these estimates are sharp. The estimates for k_q with $1 < q < 2$ are novel.*
- *For $1 \le p \le \hat p$, $C_p = 1$ is clearly the optimal constant, since $C_1 = 1$ is sharp by the previous remark, and $C_p \ge C_1$ for $1 < p < 2$ by Lemma 6.*
- *Although the estimates for k_2 have been formulated for the entire range $0 \le \beta^2 < 1/(d-2)$ on which the steady state f_∞ of (40) defines a probability measure, we recall that equation (40) loses its interpretation as the linearization of (39) when $\beta^2 \ge 1/d$, i.e. $m \le m_c$.*

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[3] For example, one can choose Ω as a ball around the origin.

Proof. The results will be obtained by applying Lemma 2. To start with, we notice that $\mathbf{n}\cdot\nabla D_\beta(x) = 2\beta^2\mathbf{n}\cdot x > 0$ on $\partial\Omega$ by assumption, and thus we will need to choose $c_3 \le 0$. Substitution of $D_\beta(x) = \alpha^2 + \beta^2|x|^2$ and $Q(x) = x$ into (28) yields

$$\mathcal{S}_q = qD_\beta^2\xi_L^2 + (2-q)D_\beta^2\xi_L|\xi_G|^2 - 2qD_\beta\xi_L(x\cdot\xi_G) - (2-q)D_\beta|\xi_G|^2(x\cdot\xi_G) + q(x\cdot\xi_G)^2,$$

while the shift polynomials $\mathcal{T}_i$ defined in (30) specify to

$$\begin{cases} \mathcal{T}_1 = D_\beta^2(\|\xi_H\|^2 - \xi_L^2) + (1-2\beta^2)D_\beta\big(\xi_L(x\cdot\xi_G) - x\cdot\xi_H\cdot\xi_G\big), \\ \mathcal{T}_2 = D_\beta\xi_L(x\cdot\xi_G) + D_\beta x\cdot\xi_H\cdot\xi_G + D_\beta|\xi_G|^2 - (x\cdot\xi_G)^2, \\ \mathcal{T}_3 = 4\beta^2 D_\beta x\cdot\xi_H\cdot\xi_G - 2\beta^2|x|^2|\xi_G|^2 + 2d\beta^2 D_\beta|\xi_G|^2, \\ \mathcal{T}_4 = D_\beta^2\xi_L|\xi_G|^2 + 2D_\beta^2\xi_G\cdot\xi_H\cdot\xi_G - D_\beta^2|\xi_G|^4 - (1-2\beta^2)D_\beta(x\cdot\xi_G)|\xi_G|^2. \end{cases} \tag{41}$$

We start by showing (32) on the range $\beta^2 \le 1/(2d)$ for $\hat{q}_{d,\beta} \le q < 2$. Our strategy is to choose parameters c_1, c_2, c_4 such that the polynomial $\mathcal{S}$ with

$$\mathcal{S} := \mathcal{S}_q + c_1\mathcal{T}_1 + c_2\mathcal{T}_2 + c_4\mathcal{T}_4 - qD_\beta|\xi_G|^2 \tag{42}$$

becomes a product of $D_\beta(x)^2$ and a polynomial in ξ_G, ξ_H, and ξ_L, which is *independent* of x. In order to avoid the appearance of the product $x\cdot\xi_H\cdot\xi_G$, we choose

$$c_1 = \frac{c_2}{1-2\beta^2}.$$

Next, we study the remainder of $\mathcal{S}$ after polynomial division by D. Since most of the terms appearing in $\mathcal{S}_q$ and $\mathcal{T}_k$ are actually multiples of D_β or D_β^2, this remainder is easily calculated and reads as

$$\mathcal{S} \bmod D_\beta = (x\cdot\xi_G)^2(q - c_2).$$

It vanishes (identically in x and ξ_G) if and only if $c_2 = q$, and consequently $c_1 = q/(1-2\beta^2)$. Finally, c_4 is determined such that the remainder of $\mathcal{S}$ with respect to D_β^2 vanishes as well. With the above choices for c_1 and c_2,

$$\mathcal{S} \bmod D_\beta^2 = -\big((2-q) + (1-2\beta^2)c_4\big)D_\beta|\xi_G|^2(x\cdot\xi_G),$$

which vanishes if and only if $c_4 = -(2-q)/(1-2\beta^2)$. Substitute c_1, c_2 and c_4 into (42) to find

$$\mathcal{S} = \frac{D_\beta^2}{1-2\beta^2}\Big[(2-q)|\xi_G|^4 - 2(2-q)\big(\beta^2\xi_L|\xi_G|^2 + \xi_G\cdot\xi_H\cdot\xi_G\big) + q\big(\|\xi_H\|^2 - 2\beta^2\xi_L^2\big)\Big].$$

To prove the non-negativity of $\mathcal{S}$, we estimate the norm of ξ_H from below. By [JM08, Lemma 2.1], the inequality

$$\|M\|^2 \ge \frac{1}{d}(\operatorname{tr} M)^2 + \frac{d}{d-1}\Big(\frac{v\cdot M\cdot v}{|v|^2} - \frac{\operatorname{tr} M}{d}\Big)^2 \tag{43}$$

holds for any matrix $M \in \mathbb{R}^{d\times d}_{sym}$ and any vector $v \in \mathbb{R}^d$. Introducing accordingly

$$y = \frac{\xi_G\cdot\xi_H\cdot\xi_G}{|\xi_G|^4} - \frac{\xi_L}{d|\xi_G|^2}, \quad z = \frac{\xi_L}{d|\xi_G|^2},$$

we obtain from (43) that

$$\mathcal{S} \ge \frac{D_\beta^2|\xi_G|^4}{1-2\beta^2}\Big[(2-q) - 2(2-q)\big(y + (1+d\beta^2)z\big) + dq\big((d-1)^{-1}y^2 + (1-2d\beta^2)z^2\big)\Big].$$

The expression inside the square brackets is a quadratic polynomial in y and z of the special form

$$a_4y^2 + a_3z^2 + a_2y + a_1z + a_0 = a_4\Big(y + \frac{a_2}{2a_4}\Big)^2 + a_3\Big(z + \frac{a_1}{2a_3}\Big)^2 + \frac{a_0}{a_3a_4}\Big(a_3a_4 - \frac{a_1^2a_4}{4a_0} - \frac{a_2^2a_3}{4a_0}\Big).$$

Observe that $a_0 = 2 - q$, $a_3 = dq(1 - 2d\beta^2)$, and $a_4 = dq/(d-1)$ are positive quantities since $\hat{q}_{d,\beta} \le q < 2$ and $0 < \beta^2 < 1/(2d)$ with $d \ge 2$. Thus, $\mathcal{S}$ is a non-negative polynomial if

$$\begin{aligned} 0 &\le a_3 a_4 - \frac{a_1^2 a_4}{4a_0} - \frac{a_2^2 a_3}{4a_0} \\ &= \frac{dq}{d-1}\big(d(1-2d\beta^2)q - (1+d\beta^2)^2(2-q) - (d-1)(1-2d\beta^2)(2-q)\big). \end{aligned}$$

A straightforward calculation reveals that the expression inside the parenthesis is non-negative because $q \ge \hat{q}_{d,\beta}$, proving the criterion (32). By Lemma 2, the inequalities (11) hold with $k_q = q$ for $\hat{q}_{d,\beta} \le q \le 2$, and equivalently, the inequalities (1) hold with $C_p = 1$ for $1 \le p \le \hat{p} = 2/\hat{q}_{d,\beta}$. To conclude $C_p = (2-\hat{p})/(2-p)$ for $\hat{p} < p < 2$, simply invoke Lemma 6.

To prove the claim about the spectral gap for $1/(2d) < \beta^2 < 1/(d-2)$, let $q = 2$, and recall that

$$\mathcal{S}_2(\xi) = 2D^2\xi_L^2 - 4D\xi_L(x\cdot\xi_G) + 2(x\cdot\xi_G)^2.$$

First, assume $1/(2d) \le \beta^2 \le 1/(d+2)$. With $k_2 = 4(1-d\beta^2)$ and the choices

$$c_1 = \frac{2d}{d-1}, \quad c_2 = \frac{2}{d-1}(d-2+2d\beta^2), \quad c_3 = -\frac{2d\beta^2-1}{(d-1)\beta^2},$$

one finds that

$$\begin{aligned} \mathcal{S}(\xi) &= \mathcal{S}_2 + c_1\mathcal{T}_1 + c_2\mathcal{T}_2 + c_3\mathcal{T}_3 - 4(1-d\beta^2)D_\beta|\xi_G|^2 \\ &= \frac{2}{d-1}\Big[(2d\beta^2-1)\big(|x|^2|\xi_G|^2 - (x\cdot\xi_G)^2\big) + D_\beta^2\big(d\|\xi_H\|^2 - \xi_L^2\big)\Big]. \end{aligned}$$

This expression is non-negative; indeed, by the Cauchy-Schwarz inequality,

$$|x|^2|\xi_G|^2 \ge (x\cdot\xi_G)^2 \quad \text{and} \quad d\|\xi_H\|^2 \ge \xi_L^2.$$

This proves (32) with $q = 2$ on $1/(2d) \le \beta^2 \le 1/(d+2)$.

The remaining range $1/(d+2) < \beta^2 < 1/(d-2)$ has to be divided into two zones (for a reason that will become apparent later), namely above and below of

$$\beta_*^2 := (d-1)/(d^2-d+2). \tag{44}$$

First assume that $1/(d+2) < \beta^2 \le \beta_*^2$ (or $1/4 < \beta^2 < \infty$ if $d = 2$). Shift polynomials $\mathcal{T}_1$ to $\mathcal{T}_3$ are added to $\mathcal{S}_2$ in such a way that second-order derivatives only appear in the form $d\xi_H - \xi_L\mathbf{1}$. With

$$c_1 = \frac{2d}{d-1}, \quad c_2 = 2 + \frac{2(d-2)\beta^2}{d-1}, \quad c_3 = \frac{1}{2\beta^2}\Big[-1 + \Big(d + \frac{2}{d-1}\Big)\beta^2\Big], \tag{45}$$

one finds indeed that

$$\begin{aligned} \mathcal{S} &= \mathcal{S}_2 + c_1\mathcal{T}_1 + c_2\mathcal{T}_2 + c_3\mathcal{T}_3 - \frac{(1-(d-2)\beta^2)^2}{2\beta^2}D_\beta|\xi_G|^2 \\ &= \frac{1}{d-1}\Big[2D_\beta^2\big(d\|\xi_H\|^2 - \xi_L^2\big) + 2D_\beta\big((d+2)\beta^2 - 1\big)\big(x\cdot(d\xi_H - \xi_L\mathbf{1})\cdot\xi_G\big) \\ &\quad + \big((d-1) - (d^2-d+2)\beta^2\big)|x|^2|\xi_G|^2 - 2(d-2)\beta^2(x\cdot\xi_G)^2\Big] \\ &\quad + \frac{(d+2)^2\beta^4 - 1}{2\beta^2}D_\beta|\xi_G|^2. \end{aligned}$$

As $\beta^2 > 1/(d+2)$, the coefficient in front of $D_\beta|\xi_G|$ is non-negative, and we may thus estimate

$$\frac{(d+2)^2\beta^4 - 1}{2\beta^2}D_\beta|\xi_G|^2 \ge \frac{(d+2)^2\beta^4 - 1}{2}|x|^2|\xi_G|^2. \tag{46}$$

Further, employing the identity

$$\big(dA - (\operatorname{tr}A)\mathbf{1}\big) : \big(dB - (\operatorname{tr}B)\mathbf{1}\big) = d^2A : B - d(\operatorname{tr}A)(\operatorname{tr}B)$$

for all symmetric matrices A, $B \in \mathbb{R}^{d\times d}_{sym}$, it is easily verified that

$$\begin{aligned} K &:= \frac{1}{2d}\big\|2D_\beta\big(d\xi_H - \xi_L\mathbf{1}\big) + \big((d+2)\beta^2-1\big)\big(d\,x\otimes_s\xi_G - (x\cdot\xi_G)\mathbf{1}\big)\big\|^2 \\ &= 2D_\beta^2\big(d\|\xi_H\|^2 - \xi_L^2\big) + 2D_\beta\big((d+2)\beta^2-1\big)\big(x\cdot(d\xi_H - \xi_L\mathbf{1})\cdot\xi_G\big) \\ &\quad + \big((d+2)\beta^2-1\big)^2\Big(\frac{d}{4}|x|^2|\xi_G|^2 + \frac{d-2}{4}(x\cdot\xi_G)^2\Big). \end{aligned} \tag{47}$$

We can thus incorporate all second-order derivatives into K. Collecting the remaining terms and plugging into (46), we find

$$\mathcal{S} \ge \frac{1}{d-1}K + \frac{d-2}{4(d-1)}\big((d+2)^2\beta^4 - 2(d-2)\beta^2 + 1\big)\big(|x|^2|\xi_G|^2 - (x\cdot\xi_G)^2\big).$$

The coefficient of $|x|^2|\xi_G|^2 - (x\cdot\xi_G)^2 \ge 0$ is non-negative, since we have assumed $d \ge 2$, and clearly

$$(d+2)^2\beta^4 - 2(d-2)\beta^2 + 1 = 8d\beta^2 + \big((d-2)\beta^2 - 1\big)^2 \ge 0.$$

This shows criterion (32) for $1/(d+2) < \beta^2 \le \beta_*^2$. Unfortunately, yet another strategy is needed for $\beta_*^2 \le \beta^2 < 1/(d-2)$, since c_3 in (45) would be positive, and thus Lemma 2 is no longer applicable. On the remaining range, we choose $c_3 \equiv 0$ instead, and

$$c_1 = \frac{1-(d-2)\beta^2}{\beta^2}, \quad c_2 = d(1-(d-2)\beta^2).$$

In order to prove non-negativity of

$$\begin{aligned} \mathcal{S} &= \mathcal{S}_2 + c_1\mathcal{T}_1 + c_2\mathcal{T}_2 - \frac{(1-(d-2)\beta^2)^2}{2\beta^2}D_\beta|\xi_G|^2 \\ &= \frac{1-(d-2)\beta^2}{d\beta^2}\Big[D_\beta^2\big(d\|\xi_H\|^2 - \xi_L\big)^2 - (1-(d+2)\beta^2)D_\beta\big(dx\cdot\xi_H\cdot\xi_G - \xi_L(x\cdot\xi_G)\big)\Big] \\ &\quad + \frac{(d^2-d+2)\beta^2-(d+1)}{d\beta^2}\Big[D_\beta^2\xi_L^2 - (1+(d-2)\beta^2)D_\beta\xi_L(x\cdot\xi_G)\Big] \\ &\quad + \frac{(1-(d-2)\beta^2)((3d-2)\beta^2-1)}{2\beta^2}D_\beta|\xi_G|^2 + (d-2)(d\beta^2-1)(x\cdot\xi_G)^2, \end{aligned}$$

we follow a similar strategy as above, only that we build one additional complete square from the terms involving ξ_L and $x\cdot\xi_G$. The argument is similar to the one leading to (47). First, we combine the trivial estimate $D_\beta(x) \ge \beta^2|x|^2$ with the fact that the coefficient in front of $D_\beta|\xi_G|^2$ is non-negative since $1/(d+2) < \beta_*^2 \le \beta^2 \le 1/(d-2)$ and $d \ge 2$. Second, we absorb the first group of second-order terms into one square and complete another square with the terms involving ξ_L. Omitting the details, we find that

$$\begin{aligned} \mathcal{S} \ge{} & \frac{1-(d-2)\beta^2}{4d^2\beta^2}\big\|2D_\beta\big(d\xi_H - \xi_L\mathbf{1}\big) - (1-(d+2)\beta^2)\big(dx\otimes_s\xi_G - (x\cdot\xi_G)\mathbf{1}\big)\big\|^2 \\ &+ \frac{(d^2-d+2)\beta^2-(d-1)}{4d\beta^2}\big[2D_\beta\xi_L - (1+(d-2)\beta^2)(x\cdot\xi_G)\big]^2 \\ &+ \frac{1}{8\beta^2}(1-(d-2)\beta^2)(-1+2d\beta^2-(12-8d+d^2)\beta^4)\big(|x|^2|\xi_G|^2 - (x\cdot\xi_G)^2\big). \end{aligned}$$

The coefficients in front of the two complete squares are non-negative because $\beta_*^2 \le \beta^2 \le 1/(d-2)$, see (44). To prove that the contribution from the last line is also non-negative, observe that the quadratic expression in β^2 in the second bracket is positive at $\beta^2 = 1/(d+2)$ and $\beta^2 = 1/(d-2)$ (with respective values $8(d-2)/(d+2)^2$ and $8/(d-2)$) and behaves monotonically between those points (the respective derivatives are $4(5d-6)/(d+2) > 0$ and 12). Another application of the Cauchy-Schwarz inequality finishes the proof. □

6. APPLICATION TO THE LINEARIZED FAST DIFFUSION EQUATION IN $d = 1$

The results of section 5 can be improved in dimension $d = 1$.

Theorem 4. *Assume that* $d = 1$*. For arbitrary* $0 \le \beta^2 \le 1$*, define*

$$\check{q}_\beta = 1 + \frac{2\beta^2}{1+\beta^2}, \quad \hat{q}_\beta = \min\big(\hat{q}_{1,\beta}, 2\big) = \min\Big(1 + \frac{(4+\beta^2)\beta^2}{2+\beta^4}, 2\Big).$$

Then (34) *is satisfied with* $k_q = q$ *for* $\hat{q}_\beta \le q \le 2$*, and thus* $C_p = 1$ *is the optimal constant in* (1) *for* $1 \le p \le \hat{p}$ *with* $\hat{p}$ *from* (6)*. Moreover, for* $1 < q < \hat{q}_\beta$*, condition* (34) *is verified with*

$$k_q = (1+\beta^2)\frac{\hat{q}_\beta - q}{\hat{q}_\beta - \check{q}_\beta} + \hat{q}_\beta \frac{q - \check{q}_\beta}{\hat{q}_\beta - \check{q}_\beta}. \tag{48}$$

A comparison of our results (in terms of bounds on C_p in (1)) with the estimates obtained from [BR03] by means of Theorem 7 is given in Figure 1 (right) in the introduction.

Proof. The proof is an application of Lemma 3. Recall that ξ_1 and ξ_2 symbolize the functions w_x/w and w_{xx}/w, respectively. In view of (31),

$$\mathcal{S}_q = qD_\beta^2\xi_2^2 + (2-q)D_\beta^2\xi_2\xi_1^2 - 2qxD_\beta\xi_2\xi_1 - (2-q)xD_\beta\xi_1^3 + qx^2\xi_1^2.$$

Only two of the polynomials from (30) will be used, namely

$$\begin{cases} \mathcal{T}_2 = (D_\beta - x^2)\xi_1^2 + 2xD_\beta\xi_1\xi_2, \\ \mathcal{T}_4 = -D_\beta^2\xi_1^4 - (1-2\beta^2)D_\beta x\xi_1^3 + 3D_\beta^2\xi_2\xi_1^2. \end{cases}$$

Observe that for $q = 2$,

$$\mathcal{S}_2 + 2\mathcal{T}_2 = 2D_\beta^2\xi_2^2 + 2D_\beta\xi_1^2 \ge 2D_\beta\xi_1^2,$$

and thus, (34) holds with $k_2 = 2$. Next, consider the point $q = \hat{q}_\beta$ under the assumption $\hat{q}_\beta < 2$. The choice

$$c_2 = \hat{q}_\beta = \frac{(1+\beta^2)^2}{2+\beta^4} \quad \text{and} \quad c_4 = -\frac{2-\hat{q}_\beta}{1-2\beta^2}$$

eliminates both the cubic term ξ_1^3 and the product $\xi_1\xi_2$ in the sum $\mathcal{S}_{\hat{q}_\beta} + c_2\mathcal{T}_2 + c_4\mathcal{T}_4$:

$$\begin{aligned} \mathcal{S}_{\hat{q}_\beta} + c_2\mathcal{T}_2 + c_4\mathcal{T}_4 &= D_\beta^2\Big[\hat{q}_\beta\xi_2^2 + \frac{2(2-\hat{q}_\beta)(1+\beta^2)}{1-2\beta^2}\xi_1^2\xi_2 + \frac{2-\hat{q}_\beta}{1-2\beta^2}\xi_1^4\Big] + \hat{q}_\beta D_\beta\xi_1^2 \\ &= D_\beta^2\hat{q}_\beta\Big[\xi_2 + \frac{1}{1+\beta^2}\xi_1^2\Big]^2 + \hat{q}_\beta D_\beta\xi_1^2. \end{aligned}$$

Condition (34) thus follows with $k = \hat{q}_\beta > 0$. The interpolation Lemma 4 yields $k_q = q$ for $\hat{q}_\beta \le q \le 2$, corresponding to $C_p = 1$ for $1 \le p \le \hat{p}$.
On the other hand, for $q = \check{q}_\beta$, one has

$$\begin{aligned} \mathcal{S}_{\check{q}_\beta} = \frac{1}{1+\beta^2}\Big[&(1+3\beta^2)D_\beta^2\xi_2^2 + (1-\beta^2)D_\beta^2\xi_2\xi_1^2 - 2(1+3\beta^2)xD_\beta\xi_2\xi_1 \\ &- (1-\beta^2)xD_\beta\xi_1^3 + (1+3\beta^2)x^2\xi_1^2\Big]. \end{aligned}$$

The choices $c_2 = 1+\beta^2$ and $c_4 = -(1-\beta^2)/(1+\beta^2)$ yield

$$\begin{aligned} \mathcal{S}_{\check{q}_\beta} + c_2\mathcal{T}_2 + c_4\mathcal{T}_4 = \frac{1}{1+\beta^2}\Big[&(1+3\beta^2)D_\beta^2\xi_2^2 - 2(1-\beta^2)D_\beta\xi_1(D_\beta\xi_1 + \beta^2 x)\xi_2 \\ &+ (1-\beta^2)\xi_1^2\big((1-\beta^2)\beta^2x^2 + (D_\beta\xi_1 - \beta^2x)^2\big)\Big] + (1+\beta^2)D_\beta\xi_1^2. \end{aligned}$$

We have written the term inside the square brackets as a quadratic polynomial in ξ_2. A sufficient criterion for the non-negativity of the expression $a\xi_2^2 + b\xi_2 + c$ is that the leading coefficient a is positive and its discriminant $\Delta = 4ac - b^2$ is non-negative. Since $D_\beta(x)$ vanishes nowhere, $a = (1+3\beta^2)D_\beta(x)^2 > 0$ for all $x \in \Omega$. The discriminant

$$\Delta = 4\beta^2(1-\beta^2)D_\beta^2\xi_1^2\big[2D_\beta\xi_1 - (1+\beta^2)x\big]^2$$

is obviously non-negative since $\beta^2 \leq 1$. This proves (34) at $q = \check{q}_\beta$ with $k = 1 + \beta^2$. Another application of Lemma 4 shows that (34) is satisfied with (48) on $\check{q}_\beta \leq q \leq \hat{q}_\beta$. ☐

To conclude the discussion of the linearized fast diffusion equation, we comment on the associated convex Sobolev inequalities in terms of concentration estimates.

Corollary 1. *Define ν as the measure on $\mathbb{R}$ with density*

$$g_\infty(y) = \frac{1}{Z_\beta}\big(\cosh(\beta y)\big)^{-(1+\beta^2)/\beta^2}.$$

Then the convex Sobolev inequalities (11) *— with ν in place of μ and $D \equiv 1$ — hold, with the same k_q as in Theorem 4.*

Proof. For given α and β, let D, μ and f_∞ be defined as before. Introduce the new real variable

$$y(x) = \beta^{-1} \operatorname{arsinh}\Big(\frac{\beta x}{\alpha}\Big).$$

Observe that it satisfies

$$y'(x) = D_\beta(x)^{-1/2} \quad \text{and} \quad g_\infty(y(x))\, y'(x) = f_\infty(x). \tag{49}$$

Now, for a given regular function $v = v(y)$, define the function u by $u(x) = v(y(x))$. By the change-of-variables formula, the entropy of u with respect to μ equals that of v with respect to ν,

$$\int_{\mathbb{R}} \phi_q\big(u(x)\big) f_\infty(x)\, dx - \phi_q\Big(\int_{\mathbb{R}} u(x) f_\infty(x)\, dx\Big) = \int_{\mathbb{R}} \phi_q\big(v(y)\big) g_\infty(y)\, dy - \phi_q\Big(\int_{\mathbb{R}} v(y) g_\infty(y)\, dy\Big),$$

and likewise for the dissipation,

$$\int_{\mathbb{R}} D_\beta(x)\big(u(x)^{q/2}\big)_x^2 f_\infty(x)\, dx = \int_{\mathbb{R}} D_\beta(x) y_x(x)^2 \big(v(y(x))^{q/2}\big)_y^2 f_\infty(x)\, dx = \int_\Omega \big(v(y)^{q/2}\big)_y^2 g_\infty(y)\, dy.$$

Thus, the inequalities (11) for $D_\beta(x) = \alpha^2 + \beta^2 x^2$ and μ are equivalent to that for $D \equiv 1$ and ν. ☐

This corollary shows in particular that there is a family of convex Sobolev inequalities (1) — with the same optimal constants C_p as for the linearized fast diffusion equation — for $D \equiv 1$ and a measure ν that behaves like $d\nu(x)/dx \propto \exp(-cx)$ for $|x| \to \infty$. It follows that $(2-p)^a C_p$ remains bounded for $a = 1$ [Bar01], but diverges to $+\infty$ for any $a < 1$ [LO00] as $p \uparrow 2$. On the other hand, Theorem 4 and Lemma 6 imply (with $\check{p} := 2/\check{q}_\beta$)

$$C_p \leq \frac{2-\check{p}}{2-p} C_{\check{p}} = \frac{4\beta^2}{(1+\beta^2)^2}(2-p)^{-1}.$$

In combination, this gives a quite complete picture of the behavior of C_p as $p \uparrow 2$.

7. Application to the wealth distribution model

The following equation has been derived in the context of wealth distribution among agents in a simple market economy in [PT06] (see also [DMT08] for a general overview on recent mathematical results):

$$\partial_t f = \theta(x^2 f)_{xx} + \big((x-1)f\big)_x. \tag{50}$$

The value $f(x)$ should be understood as the density of agents in the market with wealth equal to x. A basic assumption of the model is the absence of debts, so the range of the wealth x is restricted to $\Omega = \mathbb{R}_+$. The parameter $\theta > 0$ is related to the agents' tendencies to spend money in binary trade interactions and to the intrinsic risk of the market. Roughly speaking, the smaller θ is, the stronger is the tendency of the model to develop a rich high society, whereas for large θ, wealth is quite equally distributed in the long-time limit.

Equation (50) is given in Fokker-Planck form (15). Its unique steady state of unit mass is

$$f_\infty(x) = \frac{1}{Z_\theta} e^{-1/(\theta x)} x^{-2-1/\theta}, \quad x > 0.$$

This function converges exponentially fast to zero as $x \downarrow 0$, but decays only algebraically for $x \to \infty$. Notice that the normalization constant Z_θ is well-defined for all $\theta > 0$.
Introducing $u(t)$ by (14), it is immediately seen that $u(t)$ satisfies the dual equation (12) with

$$D_\theta(x) = \theta x^2 \text{ and } Q(x) = x - 1.$$

In terms of the Bakry-Émery condition, the situation is similar to that of the linearized fast diffusion equation: in (36), one has

$$\mathbf{M}(x) = D_\theta Q_x + \frac{1}{4}(D_\theta)_x^2 + \frac{1}{2}\big(D_\theta (D_\theta)_{xx} - (D_\theta)_x^2 - Q(D_\theta)_x\big) = \theta x = x^{-1} D_\theta(x). \tag{51}$$

The infimum of this expression is positive on all finite intervals (a, b) with $0 \le a < b < \infty$. On the other hand, $\lambda_{BE} = 0$ is obviously the largest number such that $\mathbf{M} \ge \lambda_{BE} D_\theta$ uniformly on $\mathbb{R}_+$.

7.1. Derivation of convex inequalities.

Our refinement of the Bakry-Émery method allows us to prove convex Sobolev inequalities with Ω-independent constants.

Theorem 5. *Let $\theta > 0$. Then criterion (34) is satisfied for all $q \in (1, 2]$ with*

$$k_q = \begin{cases} \dfrac{(1+\theta)^2}{2\theta}(q-1) & \text{for } 1 < q \le \min(2, 1+\theta), \\ q\Big(1 - \dfrac{\theta}{2}\dfrac{2-q}{q-1}\Big) & \text{for } \min(2, 1+\theta) \le q \le 2, \end{cases} \tag{52}$$

giving rise to a family of non-trivial convex Sobolev inequalities (11). In particular, the linear operator $\mathbf{L}_\theta$ from (7) possesses a spectral gap at least of width

$$\lambda_1 = \begin{cases} 1 & \text{for } 0 < \theta < 1, \\ \dfrac{(1+\theta)^2}{4\theta} & \text{for } \theta \ge 1. \end{cases} \tag{53}$$

Proof. The relevant shift polynomials from (30) specify to

$$\begin{aligned} \mathcal{T}_2 &= \big((\theta-1)x^2 + 2x - 1\big)\xi_1^2 + 2\theta(x-1)x^2\xi_1\xi_2, \\ \mathcal{T}_3 &= \theta\big((\theta-1)x + 1\big)x\xi_1^2 + 2\theta^2 x^3 \xi_1\xi_2, \\ \mathcal{T}_4 &= 3\theta^2 x^4 \xi_2 \xi_1^2 + \theta\big(1 + (\theta-1)x\big)x^2\xi_1^3 - \theta^2 x^4 \xi_1^4. \end{aligned}$$

First, assume that $\theta \ge 1$ and observe that

$$\begin{aligned} \mathcal{S}_2 + 2\mathcal{T}_2 + \frac{\theta-1}{2\theta}\mathcal{T}_3 &= \big((\theta-1)x + (\theta^2+1)x^2\big)\xi_1^2 + 2\theta(\theta-1)x^3\xi_2\xi_1^2 + 2\theta^2x^4\xi_2^2 \\ &= \frac{(1+\theta)^2}{2\theta}D\xi_1^2 + \frac{x^2}{4}\big(2\theta x\xi_2 + (\theta-1)\xi_1\big)^2. \end{aligned}$$

This proves (34) at $q = 2$ with $k_2 = (1+\theta)^2/(2\theta)$. Furthermore, by the calculation of $\mathbf{M}$ in (51) above, (34) is satisfied at $q = 1$ with $k_1 = 0$; see (the proof of) Theorem 2. Hence, the interpolation Lemma 4 leads to (34) for arbitrary $1 < q < 2$ with k_q as in (52).
Now, let $0 < \theta < 1$ and assume $1 + \theta \le q \le 2$. With k_q as in (52),

$$\begin{aligned} &\mathcal{S}_q + q\mathcal{T}_2 - (2-q)\mathcal{T}_4 - k_q D\xi_1^2 \\ &\quad = \theta^2x^2\big[(2-q)\big(\frac{q}{2(q-1)}\xi_1^2 - 2x\xi_1^3 + x^2\xi_1^4\big) + qx^2\xi_2^2 - 2(2-q)x^2\xi_1^2\xi_2\big] \\ &\quad = \frac{\theta^2x^2}{q}\big[\frac{2-q}{2(q-1)}\big(q\xi_1 - 2(q-1)x\xi_1^2\big)^2 + x^2\big(q\xi_2 - (2-q)\xi_1^2\big)^2\big] \ge 0, \end{aligned}$$

implying (34). Interpolation by means of Lemma 4 extends the validity of (34) to $1 < q < 1 + \theta$, thus proving (52) and (53). □

8. APPLICATION TO THE LASOTA MODEL

Finally, we shall consider the following Fokker-Planck equation on $\Omega \subset \mathbb{R}_+$,

$$\partial_t f = \frac{1}{\sigma}(xf)_{xx} + \big((x-1)f\big)_x, \tag{54}$$

which arises in mathematical biology in connection with a model for blood cell production [GM90, Las77]; see also [Fel50] for an application to a diffusion problem in genetics. Moreover, (54) is the associated Fokker-Planck equation for the stochastic process (W_t denoting a Brownian motion)

$$dX_t = \sqrt{X_t/\sigma}\, dW_t + (1 - X_t)\, dt.$$

The latter has been introduced in financial mathematics [CIR85] as the description of the evolution of interest rates subject to a (stochastic) source of market risk.
Introducing $u(t)$ as in (14) with respect to the stationary solution

$$f_\infty(x) = \frac{d\mu_\sigma}{dx} = \frac{1}{Z_\sigma} x^{\sigma-1} e^{-\sigma x},$$

it follows that $u(t)$ satisfies (12) with

$$D_\sigma(x) = \frac{x}{\sigma} \quad \text{and} \quad Q(x) = x - 1.$$

The cases with $\sigma > 1$ are the most relevant ones,[4] since this is the situation where a norm-preserving non-negative solution to (54) exists on $\Omega = \mathbb{R}_+$, for arbitrary non-negative initial data $f_0 \in L^1(\Omega)$, which is such that both f and its flux vanish as $x \downarrow 0$ for all $t > 0$, see [Fel51]. On the other hand, we are able to prove convex Sobolev inequalities (1) for arbitrary $\sigma > 1/2$.
The associated Bakry-Émery condition (36) is satisfied with $\lambda_{BE} = 1/2$, independently of $\Omega = (a, b) \subset \mathbb{R}_+$. Indeed,

$$\mathbf{M}(x) = \frac{x}{\sigma} + \frac{1}{4\sigma^2} + \frac{1}{2}\Big(-\frac{1}{\sigma^2} - \frac{x-1}{\sigma}\Big) = \frac{x}{2\sigma} + \frac{2\sigma - 1}{4\sigma^2} = D_\sigma(x)\Big(\frac{1}{2} + \frac{2\sigma-1}{4\sigma x}\Big).$$

Theorem 2 applies and yields the Beckner inequalities (1) with $C_p = 2$ for all $1 \le p < 2$.

Remark 3. *There is no contradiction between the existence of a logarithmic Sobolev inequality and the density f_∞ being concentrated like $e^{-\sigma x}$ on $\mathbb{R}_+$, due to the influence of the coefficient function D_σ. In fact, the change of variables $x \mapsto y = \sqrt{2\sigma x}$ produces an equivalent evolution equation (12) with $\tilde{D} \equiv 1$ and $\tilde{f}_\infty(y) \propto \exp(-y^2)$.*

8.1. Improvement of the convex inequalities.

We improve the result above as follows.

Theorem 6. *For $\sigma > 1/2$ and $1 \le q \le 2$, define*

$$\theta_q = \frac{1}{2}\big((2\sigma-1)(q-1)+1\big) - \frac{1}{2}\sqrt{\big((2\sigma-1)(q-1)+1\big)^2 - q(2-q)} \ge 0. \tag{55}$$

Then condition (32) is satisfied with $k_q = q - \theta_q \ge q/2$ in the given range. In particular, there is a spectral gap of width $\lambda_1 = k_2/2 = 1$.

Remark 4. *The inequality $k_q \ge q/2$ becomes strict for $q > 1$, meaning that the estimate is a genuine improvement in comparison to $\tilde{k}_q = \lambda_{BE} q/2$ as predicted by the Bakry-Émery theory.*

Proof. To start with, observe that the square root in the definition of θ_q is well defined. Indeed, the expression under the root can be written as a quadratic polynomial in $\zeta = (2\sigma-1)(q-1) \ge 0$,

$$R = \zeta^2 + 2\zeta + (q-1)^2$$

which is obviously non-negative on the entire range $1 \le q \le 2$. Next, observe that $\theta_q \le q/2$, since this inequality is equivalent to $\zeta + 1 - q \le \sqrt{R}$, which in turn is true since $(\zeta+1)^2 - 2(\zeta+1)q + q^2 \le (\zeta+1)^2 - 2q + q^2$ for $\zeta \ge 0$ and $q > 0$. Moreover, it implies that $k_q = q - \theta_q \ge q/2$ as claimed.
The proof of condition (32) works without interpolation directly by representation of the polynomial $\mathcal{S} + \sum_i c_i \mathcal{T}_i$ as a sum of two squares.

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[4] Clearly, smooth and classical solutions exist on the finite intervals $(a, b) \subset \mathbb{R}$ with $0 < a < b < \infty$ considered here.

We turn to prove (34). In the situation at hand, (31) gives

$$\mathcal{S}_q = \frac{q}{\sigma^2}x^2\xi_2^2 + \frac{2-q}{\sigma^2}x^2\xi_2\xi_1^2 - \frac{2q}{\sigma}x(x-1)\xi_2\xi_1 - \frac{2-q}{\sigma}x(x-1)\xi_1^3 + q(x-1)^2\xi_1^2,$$

while (30) provides the shift polynomials

$$\begin{cases} \mathcal{T}_2(\xi) = \dfrac{2x(x-1)}{\sigma}\xi_1\xi_2 + \Big(\dfrac{x}{\sigma} - (x-1)^2\Big)\xi_1^2, \\ \mathcal{T}_3(\xi) = \dfrac{2x}{\sigma^2}\xi_1\xi_2 - \dfrac{1}{\sigma}(x-1)\xi_1^2, \\ \mathcal{T}_4(\xi) = \dfrac{3x^2}{\sigma^2}\xi_2\xi_1^2 + \Big(\dfrac{x}{\sigma^2} - \dfrac{x(x-1)}{\sigma}\Big)\xi_1^3 - \dfrac{x^2}{\sigma^2}\xi_1^4. \end{cases}$$

Choosing the parameters $c_2 = q$, $c_3 = \theta_q$, and $c_4 = q-2$, it follows that

$$\begin{aligned} \mathcal{S} &= \mathcal{S}_q + c_2\mathcal{T}_2 + c_3\mathcal{T}_3 + c_4\mathcal{T}_4 - k_q D\xi_1^2 \\ &= \frac{q}{\sigma^2}x^2\xi_2^2 - 2\frac{2-q}{\sigma^2}x^2\xi_1^2\xi_2 + \frac{2\theta_q}{\sigma^2}x\xi_1\xi_2 + \frac{2-q}{\sigma^2}x^2\xi_1^4 - \frac{2-q}{\sigma^2}x\xi_1^3 + \frac{\theta_q}{\sigma}\xi_1^2. \end{aligned}$$

This polynomial can be written as a sum of squares:

$$\begin{aligned} \mathcal{S} = {}& \frac{2-q}{\sigma^2}\Big(x\xi_1^2 - x\xi_2 - \frac{1}{2}\xi_1\Big)^2 + \frac{1}{8\sigma^2(q-1)}\big(4(q-1)x\xi_2 + (2\theta_q - 2 + q)\xi_1\big)^2 \\ & + \frac{1}{8\sigma^2(q-1)}\big[-4\theta_q^2 + 4\big((2\sigma-1)(q-1)+1\big)\theta_q - q(2-q)\big]\xi_1^2. \end{aligned}$$

By definition of θ_q in (55), the coefficient of ξ_1^2 vanishes, and we conclude that $\mathcal{S}$ is non-negative. □

9. Appendix

In this appendix, a few properties of the constants C_p in (1) are collected. First, we recall the following elementary relations between the values of C_p for different p.

Lemma 6. *The optimal constant* C_p *in* (1) *satisfies*

$$(2-q)C_q \le (2-p)C_p \quad \text{and} \quad C_1 \le C_p \tag{56}$$

for all $1 \le p \le q < 2$.

Proof. The first inequality in (56) follows by observing that the second integral on the left-hand side of (1) is non-decreasing with respect to p, while the other two integrals are independent of p. The second inequality is obtained by substituting $u(x) = \bar{u} + \epsilon v(x)$, where $\bar{u} > 0$ is a constant and $v \in C^\infty(\Omega) \cap L^2(\Omega; \mu)$ has zero average, into (1) and considering the limit $\epsilon \downarrow 0$. See e.g. [AD05] for details. □

Second, we recall that in one spatial dimension, there exist powerful tools from measure-capacity theory to prove convex Sobolev inequalities (1) and to estimate the optimal constants. Below, we state one particularly useful result from [BR03], which is based on a previous work by Bobkov and Götze [BG99]. We use the bounds on the optimal constant C_p derived by this approach as an indication for the quality of our own estimates in section 6.

Theorem 7. *Consider the convex inequalities* (1) *with optimal constant* C_p *on* $\Omega = \mathbb{R}$. *Assume that* D *and* μ *are symmetric with respect to* $x = 0$, *and that* μ *possesses a density* $f_\infty(x) = d\mu/dx$. *Then* $b_p \le (2-p)C_p \le 4B_p$ *for all* $1 \le p < 2$, *where*

$$\begin{aligned} b_p &:= \sup_{x>0} \mu[x,\infty)\Big(1 - \Big(1 + \frac{1}{2\mu[x,\infty)}\Big)^{1-2/p}\Big)\int_0^x \frac{dy}{D(y)f_\infty(y)}, \\ B_p &:= \sup_{x>0} \mu[x,\infty)\Big(1 - \Big(1 + \frac{(p-1)^{p/(p-2)}}{\mu[x,\infty)}\Big)^{1-2/p}\Big)\int_0^x \frac{dy}{D(y)f_\infty(y)}. \end{aligned}$$

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