WEAK-STRONG UNIQUENESS OF RENORMALIZED SOLUTIONS TO REACTION-CROSS-DIFFUSION SYSTEMS

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ABSTRACT. The weak-strong uniqueness for solutions to reaction-cross-diffusion systems in a bounded domain with no-flux boundary conditions is proved. The system generalizes the Shigesada-Kawasaki-Teramoto population model to an arbitrary number of species. The diffusion matrix is neither symmetric nor positive definite, but the system possesses a formal gradient-flow or entropy structure. No growth conditions on the source terms are imposed. It is shown that any renormalized solution coincides with a strong solution with the same initial data, as long as the strong solution exists. The proof is based on the evolution of the relative entropy modified by suitable cutoff functions.

1. Introduction

This paper is a continuation of our work [6], in which we proved the global existence of renormalized solutions to a class of reaction-cross-diffusion systems describing the evolution of population species. The reaction part does not obey any growth condition which makes it necessary to use the concept of renormalized solutions like in [17]. The uniqueness of weak solutions to cross-diffusion systems is a very delicate topic, and there are very few results only for special problems; we refer to [7] and references therein. In this work, we show a weak-strong uniqueness result for the population cross-diffusion system. This means that any renormalized solution coincides with a strong solution emanating from the same initial data as long as the latter exists. This paper generalizes the weak-strong uniqueness result of Fischer [18] for *semilinear* reaction-diffusion systems to *quasilinear* reaction-cross-diffusion problems.

More specifically, we consider the evolution of n population species with densities $u_i = u_i(x,t)$, $i=1,\ldots,n$, whose evolution is governed by the equations

$$\partial_t u_i - \operatorname{div}\left(\sum_{j=1}^n A_{ij}(u)\nabla u_j - u_i b_i\right) = f_i(u) \quad \text{in } \Omega,\ i=1,\ldots,n, \tag{1}$$

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where $A_{ij}(u)$ are the density-dependent diffusion coefficients, $u = (u_1, \ldots, u_n)$ is the density vector, $b_i \in \mathbb{R}^d$ is a given vector which describes the environmental potential acting on the ith species, $f_i(u)$ is a reaction term describing the population growth dynamics, and $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ is a bounded domain. We impose no-flux boundary and initial conditions,

$$\left(\sum_{j=1}^n A_{ij}(u)\nabla u_j - u_i b_i\right) \cdot \nu = 0 \quad \text{on } \partial\Omega, \quad u_i(\cdot, 0) = u_i^0 \quad \text{in } \Omega, \ i = 1, \ldots, n, \tag{2}$$

where ν is the exterior unit normal vector on $\partial\Omega$. The diffusion coefficients are given by

$$A_{ij}(u) = \delta_{ij}\left(a_{i0} + \sum_{k=1}^n a_{ik}u_k\right) + a_{ij}u_i, \quad i, j = 1, \ldots, n, \tag{3}$$

where $a_{i0} \geq 0$, $a_{ij} \geq 0$ for $i, j = 1, \ldots, n$, and δ_{ij} is the Kronecker delta. Observe that the diffusion matrix is generally neither symmetric nor positive definite, which constitutes a major difficulty in the analysis of the diffusion system. This problem is overcome by exploiting its entropy structure, which is explained below.

1.1. **State of the art.** System (1)-(3) has been suggested by Shigesada, Kawasaki, and Teramoto for $n = 2$ species to describe the segregation of populations [23]. The equations (for any $n \geq 2$) were derived from a random-walk on a lattice in the diffusion limit [25]. The global existence of nonnegative weak solutions to (1)-(3) for two species was proved in [4] for any coefficients $a_{ij} > 0$. This result was generalized to an arbitrary number of species in [5], under a growth condition on the source terms. This condition could be replaced by a weaker entropy-dissipation assumption, yielding the global existence of renormalized solutions [6].

The concept of renormalized solutions has been introduced by DiPerna and Lions for transport and Boltzmann equations [12, 13, 14]. The idea is to replace the solution u by a nonlinear function $\xi(u)$ with compact support. This concept was applied also to elliptic and parabolic problems (e.g. [2, 9]) and diffusion systems (e.g. [10, 17]).

Weak-strong uniqueness was established by Leray [20] for incompressible Navier-Stokes equations and by Dafermos [8] for conservation laws; see the review by Wiedemann [24] for more details. Later this concept has been applied to other fluid models, including measure-valued solutions [15, 19]; to magneto-viscoelastic flow equations [22]; and to gradient flows based on optimal transport [3]. As far as we know, there are very few works on the weak-strong uniqueness involving renormalized solutions. An example is the paper [16], where the weak-strong uniqueness for renormalized relaxed Lagrangian solutions to semi-geostrophic equations was shown, and the already mentioned work [18] by Fischer on the weak-strong uniqueness for renormalized solutions to reaction-diffusion systems.

The question of uniqueness of weak solutions to parabolic diffusion systems is extremely delicate. One of the first results is due to Alt and Luckhaus [1] for linear elliptic operators. Pham and Temam [21] proved a uniqueness result for the population system (1)-(3), but only for two species and assuming a positive definite diffusion matrix. Finally, Gajewski's uniqueness method was applied to a simplified volume-filling cross-diffusion system in [25].

Up to our knowledge, there does not exist any uniqueness result for generalized solutions to the population system (1)-(3) without the assumptions imposed in [21].

1.2. **Key ideas.** The analysis of (1)-(3) is based on its entropy structure. This means that under some conditions, there exists a convex Lyapunov functional, which is called an entropy and which yields gradient estimates. The entropy gives rise to a transformation to entropy variables that makes the transformed diffusion matrix positive semidefinite, thus reveiling the parabolic structure of the evolution system. For this result, we need two assumptions. The first one are entropy-dissipating source terms, which means that there exist numbers $\pi_1,\ldots,\pi_n>0$ and $\lambda_1,\ldots,\lambda_n\in\mathbb{R}$ such that

$$\sum_{i=1}^n \pi_i f_i(u)(\log u_i+\lambda_i)\le 0 \quad \text{for all } u\in(0,\infty)^n. \tag{4}$$

This condition implies the quasi-positivity of f_i which is necessary to conclude nonnegative solutions to (1). Note that we do not impose any growth restriction on the reaction terms, modeling possibly fast growing populations.

Condition (4) ensures that the entropy density

$$h(u)=\sum_{i=1}^n \pi_i h_i(u_i), \quad h_i(s)=s(\log s-1+\lambda_i)+e^{-\lambda_i}, \tag{5}$$

is a Lyapunov functional for the pure reaction system $\partial_t u_i=f_i(u)$ if $\pi_i=1$ for all $i=1,\ldots,n$. When the diffusion terms are present, a second assumption is needed, namely either the weak cross-diffusion condition

$$\eta:=\min_{i=1,\ldots,n}\left(a_{ii}-\frac14\sum_{j=1}^n\left(\sqrt{a_{ij}}-\sqrt{a_{ji}}\right)^2\right)>0, \tag{6}$$

or the detailed-balance condition

$$\pi_i a_{ij}=\pi_j a_{ji} \quad \text{for all } i,j=1,\ldots,n,\ i\neq j. \tag{7}$$

In the former case, we may choose $\pi_i=1$. For an interpretation of the detailed-balance condition, we refer to [5].

Under conditions (4) and either (6) or (7), the matrix product $A(u)h''(u)^{-1}$ is positive semidefinite (here, $h''(u)$ denotes the Hessian of $h(u)$), i.e. for any $z\in\mathbb{R}^n$,

$$z:A(u)h''(u)^{-1}z=\sum_{i,j=1}^n A_{ij}(u)u_jz_iz_j\ge\alpha_0\sum_{i=1}^n u_iz_i^2+2\eta_0\sum_{i=1}^n u_i^2z_i^2, \tag{8}$$

for some constants $\alpha_0,\eta_0>0$; see Lemma 4 below. As a consequence, the entropy $\int_\Omega h(u)dx$ is a Lyapunov functional along solutions to (1)-(3), and we obtain the so-called entropy inequality

$$\frac{d}{dt}\int_\Omega h(u)dx+C\int_\Omega\sum_{i=1}^n\left(|\nabla\sqrt{u_i}|^2+|\nabla u_i|^2\right)dx\le 0, \tag{9}$$

where the constant $C > 0$ depends on π_i and a_{ij}. Clearly, these assumptions are also needed for our uniqueness result. In fact, we need an additional condition on the reaction terms detailed in hypothesis (H2) below.

As in [18], the key idea of the uniqueness proof is the use of the relative entropy,

$$\begin{aligned} H(u|v) &= \int_\Omega \big(h(u) - h(v) - h'(v)\cdot(u-v)\big)dx \\ &= \sum_{i=1}^n \int_\Omega \big(u_i(\log u_i - 1) - u_i \log v_i + v_i\big)dx, \end{aligned}$$

which can be seen as a generalized distance between a renormalized solution u and a strong solution v. There is a relation between Gajewski's semimetric and the relative entropy; see the discussion in [7, Remark 4]. To simplify the following formal arguments (which are made rigourous in section 3), we set $b_i = 0$, $\lambda_i = 0$, and $\pi_i = 1$. A computation shows that

$$\begin{aligned} \frac{dH}{dt}(u|v) = &- \sum_{i,j=1}^n \int_\Omega A_{ij}(u)u_j \nabla \log\frac{u_i}{v_i}\cdot\nabla\log\frac{u_j}{v_j}dx \\ &- \sum_{i,j=1}^n \int_\Omega \left(A_{ij}(u)\frac{u_j}{v_j} - A_{ij}(v)\frac{u_i}{v_i}\right)\nabla v_j\cdot\nabla\log\frac{u_i}{v_i}dx \\ &+ \sum_{i=1}^n \int_\Omega \left(f_i(u)\log\frac{u_i}{v_i} + f_i(v)\left(1 - \frac{u_i}{v_i}\right)\right)dx =: G_1 + G_2 + G_3. \end{aligned} \tag{10}$$

The second term G_2 is a result of the strong coupling and does not appear in reaction-diffusion systems with diagonal and constant diffusion matrix as in [18]. The positive semidefiniteness property (8) shows that the first term G_1 can be estimated from below,

$$G_1 \le -2\eta_0 \sum_{i=1}^n \int_\Omega u_i^2 \left|\nabla\log\frac{u_i}{v_i}\right|^2 dx. \tag{11}$$

Using the special structure (3) of the diffusion matrix, the second term G_2 can be reformulated and estimated as

$$\begin{aligned} G_2 &= -\sum_{i,j=1}^n a_{ij}u_i(u_j - v_j)\nabla\log(v_iv_j)\cdot\nabla\log\frac{u_i}{v_i}dx \\ &\le C(v)\sum_{i,j=1}^n \int_\Omega |u_j - v_j|u_i\left|\nabla\log\frac{u_i}{v_i}\right|dx \\ &\le \eta_0\sum_{i=1}^n\int_\Omega u_i^2\left|\nabla\log\frac{u_i}{v_i}\right|^2 dx + C(v)\sum_{i=1}^n\int_\Omega |u_i - v_i|^2 dx. \end{aligned}$$

The first term on the right-hand side is absorbed by the right-hand side of (11). The convexity of $h(u)$ shows that the relative entropy is bounded from below by $\sum_{i=1}^n |u_i - v_i|^2$

(up to some constant), provided that u is bounded. In that situation, we infer that

$$\frac{dH}{dt}(u|v) \le C(v)H(u|v) + G_3, \quad t > 0.$$

Since we cannot prove the boundedness of u, we cannot use the relative entropy directly. We need to construct a modified entropy with cutoff for u_i, such that the previous arguments can be made rigorous. Note that this difficulty does not appear when the diffusion matrix is diagonal and constant, as in [18]. Indeed, then the term G_2 does not appear, and the only difficulty is to estimate the remaining term G_3.

The idea of Fischer [18] to estimate G_3 is to introduce the relative entropy with cutoff for v_i,

$$\widetilde{H}_K^L(u|v) = \sum_{i=1}^n \int_\Omega \big(u_i(\log u_i + \lambda_i - 1) - \widetilde{\varphi}_K^L(u)u_i(\log v_i + \lambda_i) + v_i\big)dx,$$

where $K > 3$, $L > 0$ and $\widetilde{\varphi}_K^L$ is a cutoff function which equals one if $\sum_{k=1}^n u_k \le L$ and vanishes if $\sum_{k=1}^n u_k > (L+e)^K$,

$$\widetilde{\varphi}_K^L(u) = \varphi\bigg(\frac{\log(\sum_{k=1}^n u_k + e) - \log(L+e)}{(K-1)\log(L+e)}\bigg),$$

$e = \exp(1)$ is the Euler number, and φ is a smooth cutoff such that $\varphi(s) = 1$ if $s \le 0$ and $\varphi(s) = 0$ if $s \ge 1$. The cutoff allows for the control of $\widetilde{\varphi}_K^L(u)f_i(u)\log(1/v_i)$, which appears in G_3 using $\widetilde{H}_K^L(u|v)$ instead of $H(u|v)$.

Unfortunately, this cutoff is not sufficient in the situation at hand, because of the strong coupling in G_1 and G_2. Compared to [18], we need two refinements. First, we introduce an additional cutoff:

$$\begin{aligned} H_{K,\varepsilon}^{M,L}(u|v) = \int_\Omega \sum_{i=1}^n \Big(&\varphi_K^M(u+\varepsilon I)(u_i+\varepsilon)\big(\log(u_i+\varepsilon) + \lambda_i - 1\big) \\ &- \varphi_K^L(u+\varepsilon I)(u_i+\varepsilon)(\log u_i + \lambda_i) + v_i\Big)dx, \end{aligned} \tag{12}$$

where $M > L$, $\varepsilon > 0$, and $I = (1,\ldots,1) \in \mathbb{R}^n$. The parameter ε is needed to control terms like $\log(u_i+\varepsilon)$ when $u_i = 0$. Second, the cutoff function involves the double logarithm:

$$\varphi_K^L(u) := \varphi\bigg(\frac{\log\log(\sum_{k=1}^n u_k + e) - \log\log(L+e)}{\log(K+1)}\bigg).$$

The additional logarithm slightly improves the estimates. Indeed, $|\partial_j\widetilde{\varphi}_K^L(u)|$ is bounded by $C/[K(\sum_{k=1}^n u_k + e)]$, while

$$|\partial_j\varphi_K^L(u)| \le \frac{C}{\log(K+1)(\sum_{k=1}^n u_k + e)\log(\sum_{k=1}^n u_k + e)}$$

for some constant $C > 0$.

These refinements allow us to estimate not only G_1 and G_2 but also G_3. Then we can pass to the limits $\varepsilon \to 0$ and $M \to \infty$, yielding, for sufficiently large $K > 0$,

$$\frac{dH_K^L}{dt}(u|v) \le C(K,L)H_K^L(u|v), \quad t > 0,$$

where $H_K^L(u|v) := \sum_{i=1}^n \int_\Omega \big(u_i(\log u_i + \lambda_i - 1) - \varphi_K^L(u)u_i(\log v_i + \lambda_i) + v_i\big)dx$. When u and v have the same initial data, we conclude for sufficiently large $L > 0$ that $H_K^L(u(t)|v(t)) = 0$ for all $t > 0$ and hence, by Lemma 8 below, $u(t) = u(t)$ for $t > 0$.

1.3. **Main results.**

First, we specify our notion of renormalized solution.

Definition 1. *We call* $u = (u_1, \ldots, u_n)$ *a* renormalized solution *to* (1)-(3) *if for all* $T > 0$, $u_i \in L^2(0,T;H^1(\Omega))$ *or* $\sqrt{u_i} \in L^2(0,T;H^1(\Omega))$, *and for any* $\xi \in C^\infty([0,\infty)^n)$ *satisfying* $\xi' \in C_0^\infty([0,\infty)^n;\mathbb{R}^n)$ *and* $\phi \in C_0^\infty(\overline{\Omega} \times [0,T))$, *it holds that*

$$\begin{aligned}
-\int_0^T \int_\Omega \xi(u)\partial_t\phi dxdt - \int_\Omega \xi(u^0)\phi(x,0)dx \\
= -\sum_{i,k=1}^n \int_0^T \int_\Omega \phi\partial_i\partial_k\xi(u)\bigg(\sum_{j=1}^n A_{ij}(u)\nabla u_j - u_ib_i\bigg)\cdot\nabla u_k dxdt \\
- \sum_{i=1}^n \int_0^T \int_\Omega \partial_i\xi(u)\bigg(\sum_{j=1}^n A_{ij}(u)\nabla u_j - u_ib_i\bigg)\cdot\nabla\phi dxdt \\
+ \sum_{i=1}^n \int_0^T \int_\Omega \phi\partial_i\xi(u)f_i(u)dxdt.
\end{aligned} \tag{13}$$

We impose the following hypotheses.

- (H1) Drift term: $b = (b_1, \ldots, b_n) \in L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^{n\times d}))$ for $i = 1, \ldots, n$.
- (H2) Reaction terms: (i) $f = (f_1, \ldots, f_n) : [0,\infty)^n \to \mathbb{R}^n$ is locally Lipschitz continuous; (ii) there exist numbers $\pi_1, \ldots, \pi_n > 0$ and $\lambda_1, \ldots, \lambda_n > 0$ such that
$$\sum_{i=1}^n \pi_i f_i(u)(\log u_i + \lambda_i) \le 0 \quad \text{for all } u \in (0,\infty)^n;$$
(iii) there exists $M_0 \in \mathbb{N}$ such that for all $u \in [0,\infty)^n$ with $\sum_{i=1}^n u_i \ge M_0$ it holds that $\sum_{i=1}^n f_i(u) \ge 0$.
- (H3) Initial data: $u^0 = (u_1^0, \ldots, u_n^0) \in L^\infty(\Omega;\mathbb{R}^n)$ such that $\inf_\Omega u_i^0 > 0$ for $i = 1, \ldots, n$.
- (H4) Diffusion coefficients: $a_{i0} > 0$, $a_{ii} > 0$ for $i = 1, \ldots, n$ and either the weak cross-diffusion condition (6) holds and $\pi_i = 1$ for $i = 1, \ldots, n$, or the detailed-balance condition (7) holds.

Remark 1. Under hypotheses (H1), (H2.i)-(H2.ii), (H3)-(H4), there exists a renormalized solution to (1)-(3) satisfying $u_i \ge 0$ in $\Omega \times (0,T)$ and $\int_\Omega h(u(t))dx < \infty$ for $t \in (0,T)$, and hence $u_i \in L^\infty(0,T;L^1(\Omega))$; see [6]. If $a_{i0} > 0$ and $a_{ii} > 0$ for $i = 1, \ldots, n$ then both functions u_i and $\sqrt{u_i}$ are in $L^2(0,T;H^1(\Omega))$. □

Remark 2. We discuss the assumptions. Hypotheses (H1) and (H3) are rather natural. Condition (H2.ii) with $\lambda_i = 0$ was also imposed in [10], and we already mentioned that it allows for the proof of the nonnegativity of the densities. Condition (H2.iii) on the positivity of $\sum_{i=1}^n f_i(u)$ may be surprising at first sight. It means that in the absence of diffusion effects and for large total population, the total population is still increasing. One would expect that an overcrowding effect will lead to a decrease of the total population, thus requiring $\sum_{i=1}^n f_i(u) \le 0$. However, in this situation, there is an upper bound for the reaction terms and we can apply standard methods. The situation becomes difficult when the total population is not limited. This makes a priori estimate impossible (and makes necessary the renormalization). An alternative condition is $|\sum_{i=1}^n f_i(u)| \le C(1+|u|^p)$ for all $u \in [0,\infty)^n$ and $p = 2+2/d$; see Remark 7. Finally, hypothesis (H4) is needed in the global existence analysis to show that system (1) has a certain parabolic structure; see Lemma 4 below. □

The main result of this paper reads as follows.

Theorem 3 (Weak-strong uniqueness)**.** *Let (H1)-(H4) hold. Suppose that u is a renormalized solution to* (1)-(3) *and v is a "strong" solution to* (1)-(3) *on some time interval $[0,T^*)$ with $T^* \le T$, in the following sense: There exist $C > c > 0$ such that*

$$c \le v_i(x,t) \le C \quad \text{for } (x,t) \in \Omega \times [0,T^*), \tag{14}$$

$$\|\partial_t v_i\|_{L^\infty(\Omega\times[0,T^*))} + \|\nabla v_i\|_{L^\infty(\Omega\times[0,T^*))} \le C, \tag{15}$$

and for any $s \in (0,T^*)$, $\phi \in C^\infty(\overline{\Omega} \times [0,s])$, and $i = 1,\ldots,n$,

$$\int_0^s\int_\Omega \phi\partial_t v_i dxdt = -\int_0^s\int_\Omega\bigg(\sum_{j=1}^n A_{ij}(v)\nabla v_j - v_i b_i\bigg)\cdot\nabla\phi dxdt + \int_0^s\int_\Omega \phi f_i(v)dxdt. \tag{16}$$

Then $u(x,s) = v(x,s)$ for $x \in \Omega$, $s \in (0,T^*)$.

The population model (1)-(3) can be derived from a random-walk on-lattice model with transition rates that depend linearly on the densities [25]. When the dependence is nonlinear (e.g. power functions), we obtain population models with coefficients $A_{ij}(u)$ that depend nonlinearly on u_k. These models were analyzed in, e.g., [11, 25]. However, it is unclear to what extent the weak-strong uniqueness result can be extended to this case, since the entropy density becomes a power function, and the construction of suitable cutoff functions is an open problem.

As explained in section 1.2, the proof of the theorem is highly technical, involving two approximation levels with parameters $\varepsilon > 0$, $M > 0$, and $K > 0$. The idea is to choose renormalizations $\xi(u)$ involving $\varphi_K^L(u)$ and $\varphi_K^M(u)$ in (13), respectively, and to estimate all occuring terms, leading to lengthy estimations. We summarize some auxiliary results in section 2 and present the proof of Theorem 3 in section 3.

2. Some auxiliary results

As explained in the introduction, the matrix $A(u)h''(u)^{-1}$ is positive semidefinite under hypothesis (H4). We recall the precise result.

Lemma 4. *Let hypothesis (H4) hold. Then for all* $z \in \mathbb{R}^n$,

$$z : A(u)h''(u)^{-1}z = \sum_{i,j=1}^n A_{ij}(u)u_j z_i z_j \ge \alpha_0 \sum_{i=1}^n u_i z_i^2 + 2\eta_0 \sum_{i=1}^n u_i^2 z_i^2,$$

where the coefficients of $A(u)$ *are given in* (3), $h(u)$ *is defined in* (5), $\alpha_0 = \min_{i=1,\ldots,n} \pi_i^{-1} a_{i0} > 0$, $\eta_0 = \eta$ *if* (6) *holds and* $\eta_0 = \min_{i=1,\ldots,n} \pi_i^{-1} a_{ii} > 0$ *if* (7) *holds.*

The weak formulation (13) is valid for test functions $\phi \in C_0^\infty(\overline{\Omega} \times [0,T))$. We wish to allow for test functions in $C^\infty(\overline{\Omega} \times [0,s])$ for some $s \in (0,T)$.

Lemma 5. *Let* u *be a renormalized solution to* (1)-(3) *and let* $s \in (0,T)$. *Then for any* $\xi \in C^\infty([0,\infty)^n)$ *with* $\xi' \in C_0^\infty([0,\infty)^n;\mathbb{R}^n)$ *and all* $\phi \in C^\infty(\overline{\Omega} \times [0,s])$,

$$\begin{aligned} -\int_0^s \int_\Omega \xi(u)\partial_t\phi dxdt + \int_\Omega \xi(u(x,s))\phi(x,s)dx - \int_\Omega \xi(u^0(x))\phi(x,0)dx \\ = -\sum_{i,k=1}^n \int_0^s \int_\Omega \phi\partial_i\partial_k\xi(u)\bigg(\sum_{j=1}^n A_{ij}(u)\nabla u_j - u_i b_i\bigg) \cdot \nabla u_k dxdt \\ - \sum_{i=1}^n \int_0^s \int_\Omega \partial_i\xi(u)\bigg(\sum_{j=1}^n A_{ij}(u)\nabla u_j - u_i b_i\bigg) \cdot \nabla\phi dxdt \\ + \sum_{i=1}^n \int_0^s \int_\Omega \phi\partial_i\xi(u) f_i(u) dxdt. \end{aligned} \tag{17}$$

This expression also holds for all $\phi \in W^{1,p}(\Omega \times (0,s))$ *with* $p > d+1$.

The proof of the lemma is the same as in step 1 of the proof of Lemma 11 in [6].

To define the cutoff function, let $\varphi \in C^\infty(\mathbb{R})$ be a nonincreasing function satisfying $\varphi(x) = 1$ for $x \le 0$ and $\varphi(x) = 0$ for $x \ge 1$ and let K, $L \in \mathbb{N}$ with $K \ge 3$. We define

$$\varphi_K^L(v) := \varphi\bigg(\frac{\log\log(\sum_{k=1}^n v_k + e) - \log\log(L+e)}{\log(K+1)}\bigg) \quad \text{for } v \in [0,\infty)^n, \tag{18}$$

where $e = \exp(1)$ is the Euler number. This function has the following properties.

Lemma 6. *It holds* $\varphi_K^L \in C_0^\infty([0,\infty)^n)$. *Let* $v \in [0,\infty)^n$. *Then*

- (L1) $0 \le \varphi_K^L(v) \le 1$ *for* $v \in [0,\infty)^n$.
- (L2) *If* $\sum_{k=1}^n v_k \le L$ *then* $\varphi_K^L(v) = 1$.
- (L3) *If* $\sum_{k=1}^n v_k > (L+e)^{K+1}$ *then* $\varphi_K^L(v) = 0$.
- (L4) *There exists* $C > 0$ *such that for* $v \in [0,\infty)^n$ *and* $j = 1,\ldots,n$,

$$|\partial_j\varphi_K^L(v)| \le \frac{C}{\log(K+1)(\sum_{k=1}^n v_k + e)\log(\sum_{k=1}^n v_k + e)}.$$

- (L5) *There exists* $C > 0$ *such that for* $v \in [0,\infty)^n$ *and* $i,j = 1,\ldots,n$,

$$|\partial_i\partial_j\varphi_K^L(v)| \le \frac{C}{\log(K+1)(\sum_{k=1}^n v_k + e)^2\log(\sum_{k=1}^n v_k + e)}.$$

Proof. If $\sum_{k=1}^n v_k \leq L$ then the argument of φ in definition (18) is negative which implies that $\varphi_K^L(v) = 0$, proving (L2). Next, $\varphi_K^L(v) = 0$ holds if and only if the argument of φ is equal or larger than one which is equivalent to

$$\log \frac{\log(\sum_{k=1}^n v_k + e)}{\log(L+e)} = \log\log\left(\sum_{k=1}^n v_k + e\right) - \log\log(L+e) \geq \log(K+1),$$

and, after taking the exponential, to $\sum_{k=1}^n v_k + e \geq (L+e)^{K+1}$. This holds true since we assumed that $\sum_{k=1}^n v_k > (L+e)^{K+1}$, showing (L3). Finally, (L4) and (L5) follow from

$$\begin{aligned}
\partial_j \varphi_K^L(v) &= \frac{\varphi'(z)}{\log(K+1)(\sum_{k+1}^n v_k + e)\log(\sum_{k=1}^n + e)}, \\
\partial_i\partial_j \varphi_K^L(v) &= \frac{\varphi''(z)}{(\log(K+1))^2(\sum_{k+1}^n v_k + e)^2(\log(\sum_{k=1}^n + e))^2} \\
&\quad - \frac{\varphi'(z)}{\log(K+1)(\sum_{k+1}^n v_k + e)^2 \log(\sum_{k=1}^n + e)} \\
&\quad - \frac{\varphi'(z)}{\log(K+1)(\sum_{k+1}^n v_k + e)^2(\log(\sum_{k=1}^n + e))^2},
\end{aligned}$$

where z is the argument of φ in definition (18), since $\log(K+1) > 1$. □

3. Proof of Theorem 3

Without loss of generality, we prove Theorem 3 by setting $\pi_i = 1$. This is not a restriction since these numbers only appear when applying Lemma 4 and do not change the analysis. We split the proof into several steps.

3.1. **Approximate entropy identity for $H_{K,\varepsilon}^{M,L}$.** We derive an integrated analog of the entropy identity (10) for the approximate entropy with cutoff (12). We choose $\phi \equiv 1$ and

$$\xi(u) = \varphi_K^M(u + \varepsilon I)\sum_{i=1}^n \Big((u_i + \varepsilon)\big(\log(u_i+\varepsilon) + \lambda_i - 1\big) + e^{-\lambda_i}\Big)$$

in (17), where $\varepsilon \in (0, 1/2)$ and we recall that $I = (1, \ldots, 1) \in \mathbb{R}^n$. Clearly, the derivative ξ' is an element of $C_0^\infty([0,\infty)^n; \mathbb{R}^n)$, as required. This gives the following identity for $s \in (0,T)$:

$$\int_\Omega \varphi_K^M(u+\varepsilon I)\bigg(\sum_{i=1}^n (u_i+\varepsilon)\big(\log(u_i+\varepsilon) + \lambda_i - 1\big) + e^{-\lambda_i}\bigg)dx\bigg|_0^s = G_1 + \cdots + G_6, \tag{19}$$

where

$$G_1 = -\sum_{i=1}^n \int_0^s \int_\Omega \varphi_K^M(u+\varepsilon I)\bigg(\sum_{j=1}^n A_{ij}(u)\nabla u_j - u_i b_i\bigg)\cdot \frac{\nabla u_i}{u_i + \epsilon}dxdt,$$

$$
\begin{aligned}
G_2 &= -\sum_{i,k=1}^n \int_0^s \int_\Omega \partial_i\partial_k\varphi_K^M(u+\varepsilon I)\sum_{\ell=1}^n \Big((u_\ell+\epsilon)\big(\log(u_\ell+\varepsilon)+\lambda_\ell-1\big)+e^{-\lambda_\ell}\Big) \\
&\qquad \times \bigg(\sum_{j=1}^n A_{ij}(u)\nabla u_j - u_ib_i\bigg)\cdot\nabla u_k dxdt, \\
G_3 &= -\sum_{i,k=1}^n \int_0^s \int_\Omega \partial_k\varphi_K^M(u+\varepsilon I)\big(\log(u_i+\epsilon)+\lambda_i\big)\bigg(\sum_{j=1}^n A_{ij}(u)\nabla u_j - u_ib_i\bigg)\cdot\nabla u_k dxdt, \\
G_4 &= -\sum_{i,k=1}^n \int_0^s \int_\Omega \partial_i\varphi_K^M(u+\varepsilon I)\big(\log(u_k+\varepsilon)+\lambda_k\big)\bigg(\sum_{j=1}^n A_{ij}(u)\nabla u_j - u_ib_i\bigg)\cdot\nabla u_k dxdt, \\
G_5 &= \sum_{i=1}^n \int_0^s \int_\Omega \partial_i\varphi_K^M(u+\varepsilon I)\sum_{\ell=1}^n \Big((u_\ell+\varepsilon)\big(\log(u_\ell+\varepsilon)+\lambda_\ell-1\big)+e^{-\lambda_\ell}\Big) f_i(u)dxdt, \\
G_6 &= \sum_{i=1}^n \int_0^s \int_\Omega \varphi_K^M(u+\varepsilon I)\big(\log(u_i+\varepsilon)+\lambda_i\big)f_i(u)dxdt.
\end{aligned}
$$

Next, we choose $\phi = \log v_i + \lambda_i \in W^{1,\infty}(\Omega\times(0,s))$ and $\xi(u) = (u_i+\varepsilon)\varphi_K^L(u+\varepsilon I)$ in (17). Then

$$
\begin{aligned}
&\int_\Omega (u_i+\varepsilon)\varphi_K^L(u+\varepsilon I)(\log v_i+\lambda_i)dx\Big|_0^s - \int_0^s\int_\Omega \frac{u_i+\varepsilon}{v_i}\varphi_K^L(u+\varepsilon I)\partial_t v_i dxdt \\
&= -\sum_{j=1}^n \int_0^s\int_\Omega \partial_j\varphi_K^L(u+\varepsilon I)(\log v_i+\lambda_i)\bigg(\sum_{\ell=1}^n A_{j\ell}(u)\nabla u_\ell - u_jb_j\bigg)\cdot\nabla u_i dxdt \\
&\quad - \sum_{j=1}^n \int_0^s\int_\Omega \partial_j\varphi_K^L(u+\varepsilon I)(\log v_i+\lambda_i)\bigg(\sum_{\ell=1}^n A_{i\ell}(u)\nabla u_\ell - u_ib_i\bigg)\cdot\nabla u_j dxdt \\
&\quad - \sum_{j,k=1}^n \int_0^s\int_\Omega (u_i+\varepsilon)\partial_j\partial_k\varphi_K^L(u+\varepsilon I)(\log v_i+\lambda_i) \\
&\qquad \times \bigg(\sum_{\ell=1}^n A_{j\ell}(u)\nabla u_\ell - u_jb_j\bigg)\cdot\nabla u_k dxdt \\
&\quad - \int_0^s\int_\Omega \varphi_K^L(u+\varepsilon I)\bigg(\sum_{\ell=1}^n A_{i\ell}(u)\nabla u_\ell - u_ib_i\bigg)\cdot\frac{\nabla v_i}{v_i}dxdt \\
&\quad - \sum_{j=1}^n \int_0^s\int_\Omega (u_i+\varepsilon)\partial_j\varphi_K^L(u+\varepsilon I)\bigg(\sum_{\ell=1}^n A_{j\ell}(u)\nabla u_\ell - u_jb_j\bigg)\cdot\frac{\nabla v_i}{v_i}dxdt \\
&\quad + \int_0^s\int_\Omega \varphi_K^L(u+\varepsilon I)(\log v_i+\lambda_i)f_i(u)dxdt
\end{aligned}
$$

$$+\sum_{j=1}^n\int_0^s\int_\Omega(u_i+\varepsilon)\partial_j\varphi_K^L(u+\varepsilon I)(\log v_i+\lambda_i)f_j(u)dxdt. \tag{20}$$

We wish to replace the second integral on the left-hand side. For this, we choose the test function $\phi=(u_i+\varepsilon)\varphi_K^L(u+\varepsilon I)/v_i-1\in L^2(0,s;H^1(\Omega))$ in the weak formulation (16) for v, giving

$$\begin{aligned}
&\int_0^s\int_\Omega\frac{u_i+\varepsilon}{v_i}\varphi_K^L(u+\varepsilon I)\partial_t v_i dxdt-\int_\Omega v_i dx\Big|_0^s \\
&\quad=-\int_0^s\int_\Omega\frac{\varphi_K^L(u+\varepsilon I)}{v_i}\bigg(\sum_{\ell=1}^n A_{i\ell}(v)\nabla v_\ell-v_ib_i\bigg)\cdot\nabla u_i dxdt \\
&\qquad-\sum_{j=1}^n\int_0^s\int_\Omega\frac{u_i+\varepsilon}{v_i}\partial_j\varphi_K^L(u+\varepsilon I)\bigg(\sum_{\ell=1}^n A_{i\ell}(v)\nabla v_\ell-v_ib_i\bigg)\cdot\nabla u_j dxdt \\
&\qquad+\int_0^s\int_\Omega\frac{u_i+\varepsilon}{v_i^2}\varphi_K^L(u+\varepsilon I)\bigg(\sum_{\ell=1}^n A_{i\ell}(v)\nabla v_\ell-v_ib_i\bigg)\cdot\nabla v_i dxdt \\
&\qquad+\int_0^s\int_\Omega\bigg(\frac{u_i+\varepsilon}{v_i}\varphi_K^L(u+\varepsilon I)-1\bigg)f_i(v)dxdt.
\end{aligned}$$

Then, replacing the second integral on the left-hand side of (20) by the previous expression, summing the resulting equation over $i=1,\ldots,n$ and multiplying it by -1, we obtain

$$-\int_\Omega\sum_{i=1}^n\Big(\varphi_K^L(u+\varepsilon I)(u_i+\varepsilon)(\log v_i+\lambda_i)-v_i\Big)dx\Big|_0^s=:I_1+\cdots+I_{12}, \tag{21}$$

where

$$\begin{aligned}
I_1&=\sum_{i,j=1}^n\int_0^s\int_\Omega\partial_j\varphi_K^L(u+\varepsilon I)(\log v_i+\lambda_i)\bigg(\sum_{\ell=1}^n A_{j\ell}(u)\nabla u_\ell-u_jb_j\bigg)\cdot\nabla u_i dxdt, \\
I_2&=\sum_{i,j=1}^n\int_0^s\int_\Omega\partial_j\varphi_K^L(u+\varepsilon I)(\log v_i+\lambda_i)\bigg(\sum_{\ell=1}^n A_{i\ell}(u)\nabla u_\ell-u_ib_i\bigg)\cdot\nabla u_j dxdt, \\
I_3&=\sum_{i,j,k=1}^n\int_0^s\int_\Omega(u_i+\varepsilon)\partial_j\partial_k\varphi_K^L(u+\varepsilon I)(\log v_i+\lambda_i)\bigg(\sum_{\ell=1}^n A_{j\ell}(u)\nabla u_\ell-u_jb_j\bigg)\cdot\nabla u_k dxdt, \\
I_4&=\sum_{i=1}^n\int_0^s\int_\Omega\varphi_K^L(u+\varepsilon I)\sum_{\ell=1}^n A_{i\ell}(u)\nabla u_\ell\cdot\frac{\nabla v_i}{v_i}dxdt, \\
I_5&=\sum_{i,j=1}^n\int_0^s\int_\Omega(u_i+\varepsilon)\partial_j\varphi_K^L(u+\varepsilon I)\bigg(\sum_{\ell=1}^n A_{j\ell}(u)\nabla u_\ell-u_jb_j\bigg)\cdot\frac{\nabla v_i}{v_i}dxdt, \\
I_6&=\sum_{i=1}^n\int_0^s\int_\Omega\varphi_K^L(u+\varepsilon I)\bigg(\sum_{\ell=1}^n A_{i\ell}(v)\nabla v_\ell-v_ib_i\bigg)\cdot\frac{\nabla u_i}{v_i}dxdt,
\end{aligned}$$

$$I_7 = \sum_{i,j=1}^n \int_0^s \int_\Omega (u_i+\varepsilon)\partial_j\varphi_K^L(u+\varepsilon I)\bigg(\sum_{\ell=1}^n A_{i\ell}(v)\nabla v_\ell - v_i b_i\bigg)\cdot\frac{\nabla u_j}{v_i}dxdt,$$

$$I_8 = -\sum_{i=1}^n \int_0^s \int_\Omega \varphi_K^L(u+\varepsilon I)(\log v_i + \lambda_i)f_i(u)dxdt,$$

$$I_9 = -\sum_{i,j=1}^n \int_0^s \int_\Omega (u_i+\varepsilon)\partial_j\varphi_K^L(u+\varepsilon I)(\log v_i + \lambda_i)f_j(u)dxdt,$$

$$I_{10} = -\sum_{i=1}^n \int_0^s \int_\Omega (u_i+\varepsilon)\varphi_K^L(u+\varepsilon I)\sum_{\ell=1}^n A_{i\ell}(v)\nabla v_\ell\cdot\frac{\nabla v_i}{v_i^2}dxdt,$$

$$I_{11} = -\sum_{i=1}^n \int_0^s \int_\Omega \bigg(\frac{u_i+\varepsilon}{v_i}\varphi_K^L(u+\varepsilon I) - 1\bigg)f_i(v)dxdt,$$

$$I_{12} = \varepsilon\sum_{i=1}^n \int_0^s \int_\Omega \varphi_K^L(u+\varepsilon I)\frac{b_i\cdot\nabla v_i}{v_i}dxdt.$$

Adding (19) and (21) gives the desired approximated entropy identity:

$$H_{K,\varepsilon}^{M,L}(u|v)\Big|_0^s + \sum_{i=1}^n e^{-\lambda_i}\int_\Omega \varphi_K^M(u+\varepsilon I)dx\Big|_0^s = G_1+\cdots+G_6+I_1+\cdots+I_{12}. \tag{22}$$

3.2. **Estimate of the reaction part.** We start by estimating the terms in (22) involving the reaction terms $f_i(u)$, namely G_6, I_8, I_9, and I_{11} (the remaining term G_5 will be treated later when we pass to the limits $\varepsilon\to 0$ and $M\to\infty$).

We split the integral G_6 into two parts:

$$\begin{aligned}
G_6 &= \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)>L\}}\varphi_K^M(u+\varepsilon I)\sum_{i=1}^n f_i(u)\big(\log(u_i+\varepsilon)+\lambda_i\big)dxdt \\
&\quad + \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)\le L\}}\varphi_K^M(u+\varepsilon I)\sum_{i=1}^n f_i(u)\big(\log(u_i+\varepsilon)+\lambda_i\big)dxdt \\
&=: G_{61} + G_{62},
\end{aligned}$$

where χ_A is the characteristic function on the set A. Adding and subtracting the term $f_i(u+\varepsilon I)$ and using condition (H2.ii) gives

$$\begin{aligned}
G_{61} &= \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)>L\}}\varphi_K^M(u+\varepsilon I)\sum_{i=1}^n f_i(u+\varepsilon I)\big(\log(u_i+\varepsilon)+\lambda_i\big)dxdt \\
&\quad + \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n(u_\ell+\epsilon)>L\}}\varphi_K^M(u+\varepsilon I)\sum_{i=1}^n \big(f_i(u)-f_i(u+\varepsilon I)\big)\big(\log(u_i+\varepsilon)+\lambda_i\big)dxdt \\
&\le \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)>L\}}\varphi_K^M(u+\varepsilon I)\sum_{i=1}^n \big(f_i(u)-f_i(u+\varepsilon I)\big)\big(\log(u_i+\varepsilon)+\lambda_i\big)dxdt.
\end{aligned}$$

We claim that for any $K > 0$, there exists $C(K) > 0$ such that for all $0 \le s \le K$, it holds that $|\log(s+\varepsilon)| \le C(K)(1-\log\varepsilon)$ (recall that $\varepsilon < 1/2$). Indeed, let $1/2 \le s \le K$. Then $\log\frac{1}{2} \le \log(s+\varepsilon) \le \log(K+1)$ and consequently $|\log(s+\varepsilon)| \le C(K)$ for $C(K) = \max\{\log 2, \log(K+1)\}$. If $0 \le s \le \frac{1}{2}$, we find that $|\log(s+\varepsilon)| = -\log(s+\varepsilon) \le -\log\varepsilon$, which shows the claim.

We know from (L3) that $\varphi_K^M(u+\varepsilon I)$ vanishes if $\sum_{\ell=1}^n u_\ell$ is large enough. This allows us to apply the local Lipschitz continuity of f_i from (H2). Therefore, using (L1), we infer that

$$G_{61} \le C(M,K,f)\varepsilon(1-\log\varepsilon). \tag{23}$$

For G_{62}, we observe that $M > L$ and (L2) imply that $\varphi_K^M(u+\varepsilon I) = 1$ in $\{\sum_{\ell=1}^n(u_\ell+\varepsilon) \le L\}$. Hence,

$$G_{62} = \int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)\le L\}}\sum_{i=1}^n f_i(u)\big(\log(u_i+\varepsilon)+\lambda_i\big)dxdt.$$

We wish to estimate this term together with the terms I_8, I_9, and I_{11}. Consider the integrands of G_{62}, I_8, and I_{11} in the set $\{\sum_{\ell=1}^n(u_\ell+\varepsilon) \le L\}$ (where it holds that $\varphi_K^L(u+\varepsilon I) = 1$):

$$\begin{aligned}
&f_i(u)\big(\log(u_i+\varepsilon)+\lambda_i\big) - f_i(u)\varphi_K^L(u+\varepsilon I)(\log v_i+\lambda_i) - f_i(v)\bigg(\frac{u_i+\varepsilon}{v_i}\varphi_K^L(u+\varepsilon I)-1\bigg) \\
&\quad = f_i(u)\log\frac{u_i+\epsilon}{v_i} - f_i(v)\bigg(\frac{u_i+\varepsilon}{v_i}-1\bigg) \\
&\quad = f_i(u)\bigg(\log\frac{u_i+\varepsilon}{v_i} - \frac{u_i+\varepsilon}{v_i}+1\bigg) + \big(f_i(u)-f_i(v)\big)\bigg(\frac{u_i+\epsilon}{v_i}-1\bigg).
\end{aligned}$$

Therefore, we need to estimate

$$\begin{aligned}
&G_{62}+I_8+I_9+I_{11} \\
&\quad\le \sum_{i=1}^n\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)\le L\}} f_i(u)\bigg(\log\frac{u_i+\varepsilon}{v_i} - \frac{u_i+\varepsilon}{v_i}+1\bigg)dxdt \\
&\qquad + \sum_{i=1}^n\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)\le L\}}\big(f_i(u)-f_i(v)\big)\bigg(\frac{u_i+\varepsilon}{v_i}-1\bigg)dxdt \\
&\qquad + \sum_{i=1}^n\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)> L\}} \\
&\qquad\quad \times\bigg(|f_i(u)\varphi_K^L(u+\varepsilon I)||\log v_i+\lambda_i| + |f_i(v)|\bigg(\frac{u_i+\varepsilon}{v_i}+1\bigg)\bigg)dxdt \\
&\qquad + \sum_{i,j=1}^n\int_0^s\int_\Omega \big|(u_i+\varepsilon)\partial_j\varphi_K^L(u+\varepsilon I)\big||f_j(u)||\log v_i+\lambda_i|dxdt \\
&\quad =: J_1+\cdots+J_4.
\end{aligned}$$

We first consider J_1. The elementary inequalities $-|s-1|^2 \le \log s - s + 1 \le 0$ for $s \ge 1$ and $-|s-1|^2/s \le \log s - s + 1 \le 0$ for $s \in (0,1)$ imply that (as shown in [18])

$$-\Big(1+\frac{1}{s}\Big)|s-1|^2 \le \log s - s + 1 \le 0 \quad \text{for } s > 0. \tag{24}$$

Furthermore, we use the local Lipschitz continuity of f_i and the quasi-positivity property $f_i(u) \ge 0$ for all $u \in [0,\infty)^n$ with $u_i = 0$ (as a consequence of (H2.ii)) to conclude that in the set $\{\sum_{\ell=1}^n u_\ell \le L\}$,

$$\begin{aligned} -f_i(u) &\le f_i(u_1,\ldots,u_{i-1},0,u_{i+1},\ldots,u_n) - f_i(u) \\ &\le |f_i(u_1,\ldots,u_{i-1},0,u_{i+1},\ldots,u_n) - f_i(u)| \le C(L,f_i)u_i. \end{aligned}$$

This allows us to estimate the integrand of J_1. Indeed, we obtain in $\{\sum_{\ell=1}^n u_\ell \le L\}$

$$\begin{aligned} f_i(u)\bigg(\log\frac{u_i+\varepsilon}{v_i} - \frac{u_i+\varepsilon}{v_i} + 1\bigg) &\le C(L,f_i)u_i\Big(1+\frac{v_i}{u_i+\varepsilon}\Big)\bigg|\frac{u_i+\varepsilon}{v_i} - 1\bigg|^2 \\ &= C(L,f_i)\Big(u_i + \frac{u_i}{u_i+\varepsilon}v_i\Big)\frac{1}{v_i^2}\big|(u_i-v_i)+\varepsilon\big|^2 \le C(L,f_i,v_i)\big(|u_i-v_i|^2+\varepsilon^2\big). \end{aligned}$$

This estimate also holds in $\{\sum_{\ell=1}^n (u_\ell+\varepsilon) \le L\}$ for $\varepsilon > 0$ since $\{\sum_{\ell=1}^n (u_\ell+\varepsilon) \le L\}$ is a subset of $\{\sum_{\ell=1}^n u_\ell \le L\}$. We deduce that

$$\begin{aligned} J_1 &\le \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell \le L\}} \sum_{i=1}^n C(L,f_i,v_i)\big(|u_i-v_i|^2+\varepsilon^2\big)dxdt \\ &\le C(L,f,v)\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell \le L\}} \sum_{i=1}^n |u_i-v_i|^2 dxdt + C(L,f,v,T,\Omega)\varepsilon^2. \end{aligned}$$

Using again $\{\sum_{\ell=1}^n (u_\ell+\varepsilon) \le L\} \subset \{\sum_{\ell=1}^n u_\ell \le L\}$ and the local Lipschitz continuity of f_i, it follows that

$$\begin{aligned} J_2 &\le \sum_{i=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell \le L\}} C(L,f_i,v_i)|u-v|\big(|u_i-v_i|+\varepsilon\big)dxdt \\ &\le C(L,f,v)\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell \le L\}} \sum_{i=1}^n |u_i-v_i|^2 dxdt + C(L,f,v,T,\Omega)\varepsilon. \end{aligned}$$

Taking into account (L3), we have $|f_i(u)\varphi_K^L(u+\epsilon I)| \le C(L,K,f_i)$ and thus

$$J_3 \le C(L,K,f,v)\sum_{i=1}^n \int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon) > L\}}\Big(1+\sum_{i=1}^n u_i\Big)dxdt.$$

Since $\partial_j\varphi_K^L(u+\varepsilon I) = 0$ for sufficiently large u, we can estimate as

$$J_4 \le C(L,K,f,v)\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon) > L\}} dxdt.$$

We conclude that

$$
\begin{aligned}
G_6 + I_8 + I_9 + I_{11} \le C(L,f,v) \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell \le L\}} \sum_{i=1}^n |u_i - v_i|^2 dxdt \\
+ C(L,K,f,v) \sum_{i=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon) > L\}} \Big(1 + \sum_{i=1}^n u_i\Big) dxdt \\
+ C(M,K,f)\varepsilon(1-\log\varepsilon) + C(L,f,v,T,\Omega)\varepsilon. \qquad (25)
\end{aligned}
$$

3.3. **Estimate of the cross-diffusion part.** We estimate only some terms involving the diffusion coefficients, namely G_1, I_4, I_6, and I_{10}. We split $G_1 = G_{11} + G_{12}$ in (19) and $I_6 = I_{61} + I_{61}$ in (21) into two parts, the cross-diffusion part and the drift part:

$$
\begin{aligned}
G_{11} &= -\sum_{i,j=1}^n \int_0^s \int_\Omega \varphi_K^M(u+\varepsilon I) A_{ij}(u)\nabla u_j \cdot \frac{\nabla u_i}{u_i+\varepsilon} dxdt, \\
G_{12} &= \sum_{i=1}^n \int_0^s \int_\Omega \varphi_K^M(u+\epsilon I) u_i b_i \cdot \frac{\nabla u_i}{u_i+\epsilon} dxdt, \\
I_{61} &= \sum_{i,j=1}^n \int_0^s \int_\Omega \varphi_K^L(u+\varepsilon I) A_{ij}(v)\nabla v_j \cdot \frac{\nabla u_i}{v_i} dxdt, \\
I_{62} &= -\sum_{i=1}^n \int_0^s \int_\Omega \varphi_K^L(u+\varepsilon I) v_i b_i \cdot \frac{\nabla u_i}{v_i} dxdt.
\end{aligned}
$$

We split Ω into the subsets $\{\sum_{\ell=1}^n (u_\ell+\varepsilon) \le L\}$ and $\{\sum_{\ell=1}^n (u_\ell+\varepsilon) > L\}$ and combine on the former set the terms $G_{11} + I_4$ and $I_{61} + I_{10}$. This yields

$$
\begin{aligned}
G_{11} + I_4 + I_{61} + I_{10} = -\sum_{i,j=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon) \le L\}} \bigg\{ A_{ij}(u)\nabla u_j \cdot \Big(\frac{\nabla u_i}{u_i+\varepsilon} - \frac{\nabla v_i}{v_i} \Big) \\
+ A_{ij}(v)\nabla v_j \cdot \Big(\frac{\nabla v_i}{v_i}\frac{u_i+\varepsilon}{v_i} - \frac{\nabla u_i}{v_i} \Big) \bigg\} dxdt \\
- \sum_{i,j=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon) > L\}} \varphi_K^M(u+\varepsilon I) A_{ij}(u)\nabla u_j \cdot \frac{\nabla u_i}{u_i+\varepsilon} dxdt \\
+ \sum_{i,j=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon) > L\}} \varphi_K^L(u+\varepsilon I) A_{ij}(u)\nabla u_j \cdot \frac{\nabla v_i}{v_i} dxdt \\
+ \sum_{i,j=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon) > L\}} \varphi_K^L(u+\varepsilon I) A_{ij}(v)\nabla v_j \cdot \frac{\nabla u_i}{v_i} dxdt \\
- \sum_{i,j=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon) > L\}} \varphi_K^L(u+\varepsilon I) A_{ij}(v)\nabla v_j \cdot \frac{\nabla v_i}{v_i}\frac{u_i+\varepsilon}{v_i} dxdt
\end{aligned}
$$

$$=: O_1 + \cdots + O_5. \tag{26}$$

The estimation of the expressions O_i is rather technical. We start with O_1.

Estimation of O_1. We add and subtract $A_{ij}(u+\varepsilon I)$ in O_1, which gives $O_1 = O_{11} + O_{12}$, where

$$\begin{aligned} O_{11} = &-\sum_{i,j=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon)\le L\}} \bigg\{ A_{ij}(u+\varepsilon I)\nabla u_j \cdot \bigg(\frac{\nabla u_i}{u_i+\varepsilon} - \frac{\nabla v_i}{v_i} \bigg) \\ &+ A_{ij}(v)\nabla v_j \cdot \bigg(\frac{\nabla v_i}{v_i}\frac{u_i+\varepsilon}{v_i} - \frac{\nabla u_i}{v_i} \bigg) \bigg\} dxdt, \\ O_{12} = &\sum_{i,j=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon)\le L\}} \big(A_{ij}(u+\varepsilon I) - A_{ij}(u) \big) \nabla u_j \cdot \bigg(\frac{\nabla u_i}{u_i+\varepsilon} - \frac{\nabla v_i}{v_i} \bigg) dxdt. \end{aligned}$$

Furthermore, we add and subtract the term $\nabla v_j/v_j$ in O_{11}. We find after a short computation that

$$\begin{aligned} O_{11} = &-\sum_{i,j=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon)\le L\}} A_{ij}(u+\varepsilon I)(u_j+\varepsilon) \\ &\times \bigg(\frac{\nabla u_i}{u_i+\varepsilon} - \frac{\nabla v_i}{v_i} \bigg) \cdot \bigg(\frac{\nabla u_j}{u_j+\varepsilon} - \frac{\nabla v_j}{v_j} \bigg) dxdt \\ &- \sum_{i,j=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon)\le L\}} A_{ij}(u+\varepsilon I)(u_j+\varepsilon) \frac{\nabla v_j}{v_j} \cdot \bigg(\frac{\nabla u_i}{u_i+\varepsilon} - \frac{\nabla v_i}{v_i} \bigg) dxdt \\ &- \sum_{i,j=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon)\le L\}} A_{ij}(v)\nabla v_j \cdot \bigg(\frac{\nabla v_i}{v_i}\frac{u_i+\varepsilon}{v_i} - \frac{\nabla u_i}{v_i} \bigg) dxdt \\ = &-\sum_{i,j=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon)\le L\}} A_{ij}(u+\varepsilon I)(u_j+\varepsilon) \\ &\times \bigg(\frac{\nabla u_i}{u_i+\varepsilon} - \frac{\nabla v_i}{v_i} \bigg) \cdot \bigg(\frac{\nabla u_j}{u_j+\varepsilon} - \frac{\nabla v_j}{v_j} \bigg) dxdt \\ &- \sum_{i,j=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon)\le L\}} \bigg(A_{ij}(u+\varepsilon I)\frac{u_j+\varepsilon}{v_j} - A_{ij}(v)\frac{u_i+\varepsilon}{v_i} \bigg) \\ &\times \nabla v_j \cdot \bigg(\frac{\nabla u_i}{u_i+\varepsilon} - \frac{\nabla v_i}{v_i} \bigg) dxdt \\ =: &\; O_{111} + O_{112}. \end{aligned}$$

It follows from the positive definiteness of $A(u)h''(u)^{-1}$ (Lemma 4) that

$$O_{111} \le -\alpha_0 \sum_{i=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon)\le L\}} (u_i+\varepsilon) \bigg| \frac{\nabla u_i}{u_i+\varepsilon} - \frac{\nabla v_i}{v_i} \bigg|^2 dxdt$$

$$-2\eta_0\sum_{i=1}^n\int_0^s\int_\Omega\chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)\le L\}}(u_i+\varepsilon)^2\left|\frac{\nabla u_i}{u_i+\varepsilon}-\frac{\nabla v_i}{v_i}\right|^2dxdt. \tag{27}$$

For the estimate of O_{112}, we use definition (3) of the coefficients A_{ij}. Some terms cancel in O_{112} and we end up with

$$\begin{aligned}
O_{112} = &-\sum_{i=1}^n\int_0^s\int_\Omega\chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)\le L\}}\sum_{k=1}^n a_{ik}(u_k-v_k+\varepsilon)\frac{u_i+\varepsilon}{v_i}\nabla v_i\cdot\left(\frac{\nabla u_i}{u_i+\varepsilon}-\frac{\nabla v_i}{v_i}\right)dxdt \\
&-\sum_{i,j=1}^n\int_0^s\int_\Omega\chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)\le L\}}a_{ij}\left((u_i+\varepsilon)\frac{u_j+\varepsilon}{v_j}-v_i\frac{u_i+\varepsilon}{v_i}\right) \\
&\times\nabla v_j\cdot\left(\frac{\nabla u_i}{u_i+\varepsilon}-\frac{\nabla v_i}{v_i}\right)dxdt \\
= &-\sum_{i,j=1}^n\int_0^s\int_\Omega\chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)\le L\}}a_{ij}(u_j-v_j+\varepsilon)(u_i+\varepsilon) \\
&\times\left(\frac{\nabla v_i}{v_i}+\frac{\nabla v_j}{v_j}\right)\cdot\left(\frac{\nabla u_i}{u_i+\varepsilon}-\frac{\nabla v_i}{v_i}\right)dxdt.
\end{aligned}$$

Using the regularity of v and Young's inequality, we find that

$$\begin{aligned}
O_{112} &\le C(v)\sum_{i,j=1}^n\int_0^s\int_\Omega\chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)\le L\}}|u_j-v_j+\varepsilon|(u_i+\varepsilon)\left|\frac{\nabla u_i}{u_i+\varepsilon}-\frac{\nabla v_i}{v_i}\right|dxdt \\
&\le \eta_0\sum_{i=1}^n\int_0^s\int_\Omega\chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)\le L\}}(u_i+\varepsilon)^2\left|\frac{\nabla u_i}{u_i+\varepsilon}-\frac{\nabla v_i}{v_i}\right|^2dxdt \\
&\quad + C(v)\sum_{i=1}^n\int_0^s\int_\Omega\chi_{\{\sum_{\ell=1}^n u_\ell\le L\}}|u_i-v_i|^2dxdt+C(v,T,\Omega)\varepsilon^2,
\end{aligned}$$

where in the last step we have used $\{\sum_{\ell=1}^n(u_\ell+\varepsilon)\le L\}\subset\{\sum_{\ell=1}^n u_\ell\le L\}$. The first term on the right-hand side can be absorbed by the second term on the right-hand side of estimate (27) for O_{111}, and combining the estimates, we obtain

$$O_{11}\le C(v)\sum_{i=1}^n\int_0^s\int_\Omega\chi_{\{\sum_{\ell=1}^n u_\ell\le L\}}|u_i-v_i|^2dxdt+C(v,T,\Omega)\varepsilon^2.$$

We turn to the estimate of O_{12}. Again using definition (3) of A_{ij}, it follows that

$$\begin{aligned}
O_{12} &= \varepsilon\sum_{i,j=1}^n\int_0^s\int_\Omega\chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)\le L\}}\left(\delta_{ij}\sum_{k=1}^n a_{ik}+a_{ij}\right)\nabla u_j\cdot\left(\frac{\nabla u_i}{u_i+\varepsilon}-\frac{\nabla v_i}{v_i}\right)dxdt \\
&\le C\varepsilon\sum_{i,j=1}^n\int_0^s\int_\Omega\chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)\le L\}}|\nabla u_j|\frac{|\nabla u_i|}{u_i+\varepsilon}dxdt
\end{aligned}$$

$$+ C(v)\varepsilon\sum_{i=1}^n\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)\le L\}}|\nabla u_i|dxdt.$$

We integrand of the first term on the right-hand side can be reformulated according to

$$\sqrt{\varepsilon}|\nabla u_j|\frac{|\nabla u_i|}{u_i+\varepsilon} = 2\sqrt{\frac{\varepsilon}{u_i+\varepsilon}}\sqrt{\frac{u_i}{u_i+\varepsilon}}|\nabla u_j||\nabla\sqrt{u_i}| \le 2|\nabla u_j||\nabla\sqrt{u_i}|, \tag{28}$$

and using Young's inequality, we deduce that

$$\begin{aligned} O_{12} &\le C(v)\sqrt{\varepsilon}\sum_{i=1}^n\int_0^s\int_\Omega|\nabla u_i|^2dxdt + C(v)\sqrt{\varepsilon}\sum_{i=1}^n\int_0^s\int_\Omega\big(1+|\nabla\sqrt{u_i}|^2\big)dxdt \\ &\le C(v,T,\Omega)\sqrt{\varepsilon}. \end{aligned}$$

Note that we need here the condition $a_{i0} > 0$ which yields an L^2 bound for $\nabla\sqrt{u_i}$ (see Remark 1).

We conclude the estimate of O_1 by adding the bounds for O_{11} and O_{12}:

$$O_1 \le C(v)\sum_{i=1}^n\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell\le L\}}|u_i-v_i|^2dxdt + C(v,T,\Omega)\sqrt{\varepsilon}.$$

Estimation of O_2. We add and subtract $A_{ij}(u+\varepsilon I)$ in definition (26) of O_2 and use the definition of A_{ij} to find that

$$\begin{aligned} O_2 &= -\sum_{i,j=1}^n\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)>L\}}\varphi_K^M(u+\varepsilon I)A_{ij}(u+\varepsilon I)\nabla u_j\cdot\frac{\nabla u_i}{u_i+\varepsilon}dxdt \\ &\quad + \sum_{i,j=1}^n\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)>L\}}\varphi_K^M(u+\varepsilon I)\big(A_{ij}(u+\varepsilon I)-A_{ij}(u)\big)\nabla u_j\cdot\frac{\nabla u_i}{u_i+\varepsilon}dxdt \\ &= -\sum_{i,j=1}^n\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)>L\}}\varphi_K^M(u+\varepsilon I)A_{ij}(u+\varepsilon I)(u_j+\varepsilon)\frac{\nabla u_j}{u_j+\varepsilon}\cdot\frac{\nabla u_i}{u_i+\varepsilon}dxdt \\ &\quad + \varepsilon\sum_{i,j=1}^n\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)>L\}}\varphi_K^M(u+\varepsilon I)\bigg(\delta_{ij}\sum_{k=1}^n a_{ik}+a_{ij}\bigg)\nabla u_j\cdot\frac{\nabla u_i}{u_i+\varepsilon}dxdt \\ &=: O_{21}+O_{22}. \end{aligned}$$

We employ the positive definiteness of $A(u+\varepsilon I)h''(u+\varepsilon I)^{-1}$ to estimate O_{21}:

$$\begin{aligned} O_{21} &\le -2\eta_0\sum_{i=1}^n\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)>L\}}\varphi_K^M(u+\varepsilon I)(u_i+\varepsilon)^2\bigg|\frac{\nabla u_i}{u_i+\varepsilon}\bigg|^2dxdt \\ &\le -2\eta_0\sum_{i=1}^n\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)>L\}}\varphi_K^M(u+\varepsilon I)|\nabla u_i|^2dxdt. \end{aligned}$$

For the estimate of O_{22}, we take into account (28) and use Young's inequality similarly as in the estimate of O_{12}:

$$\begin{aligned} O_{22} &\le C\varepsilon \sum_{i,j=1}^n \int_0^s \int_\Omega |\nabla u_j| \bigg| \frac{\nabla u_i}{u_i+\varepsilon} \bigg| dxdt \\ &\le C\sqrt{\varepsilon} \sum_{i=1}^n \int_0^s \int_\Omega \big(|\nabla u_i|^2 + |\nabla \sqrt{u_i}|^2 \big) dxdt \le C\sqrt{\varepsilon}. \end{aligned}$$

Adding the inequalities for O_{21} and O_{22} then gives

$$O_2 \le -2\eta_0 \sum_{i=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon)>L\}} \varphi_K^M(u+\varepsilon I) |\nabla u_i|^2 dxdt + C\sqrt{\varepsilon}.$$

Estimation of O_3, O_4, and O_5. We conclude from (L3) and Young's inequality that

$$\begin{aligned} O_3 &= \sum_{i,j=1}^n \int_0^s \int_\Omega \chi_{\{(L+e)^{K+1} > \sum_{\ell=1}^n (u_\ell+\varepsilon)>L\}} \varphi_K^L(u+\varepsilon I) A_{ij}(u) \nabla u_j \cdot \frac{\nabla v_i}{v_i} dxdt \\ &\le C(L,K,v) \sum_{i=1}^n \int_0^s \int_\Omega \chi_{\{(L+e)^{K+1} > \sum_{\ell=1}^n (u_\ell+\varepsilon)>L\}} |\nabla u_i| dxdt \\ &\le \frac{\eta_0}{2} \sum_{i=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon)>L\}} |\nabla u_i|^2 dxdt \\ &\quad + C(L,K,v) \sum_{i=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon)>L\}} dxdt. \end{aligned}$$

In a similar way, we can estimate

$$\begin{aligned} O_4 + O_5 &= \sum_{i,j=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon)>L\}} \varphi_K^L(u+\varepsilon I) A_{ij}(v) \nabla v_j \cdot \frac{\nabla u_i}{v_i} dxdt \\ &\quad - \sum_{i,j=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon)>L\}} \varphi_K^L(u+\varepsilon I) A_{ij}(v) \nabla v_j \cdot \frac{\nabla v_i}{v_i} \frac{u_i+\varepsilon}{v_i} dxdt \\ &\le C(v) \sum_{i=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon)>L\}} |\nabla u_i| dxdt \\ &\quad + C(v) \sum_{i=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon)>L\}} (u_i+1) dxdt \\ &\le \frac{\eta_0}{2} \sum_{i=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon)>L\}} |\nabla u_i|^2 dxdt \end{aligned}$$

$$+ C(v)\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)>L\}}\bigg(1+\sum_{i=1}^n u_i\bigg)dxdt.$$

Adding all the estimates for $O_1,\ldots,O_5$, we conclude from (26) that

$$\begin{aligned}
&G_{11}+I_4+I_{61}+I_{10} \\
&\quad\le -2\eta_0\sum_{i=1}^n\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)>L\}}\varphi_K^M(u+\varepsilon I)|\nabla u_i|^2dxdt \\
&\qquad + \eta_0\sum_{i=1}^n\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)>L\}}|\nabla u_i|^2dxdt \\
&\qquad + C(v)\sum_{i=1}^n\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell\le L\}}|u_i-v_i|^2dxdt \\
&\qquad + C(L,K,v)\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)>L\}}\bigg(1+\sum_{i=1}^n u_i\bigg)dxdt \\
&\qquad + C(v,T,\Omega)\sqrt{\varepsilon}.
\end{aligned} \tag{29}$$

3.4. **The limit** $\varepsilon\to 0$.

Inserting the estimates of the previous subsections and observing that the term I_{12} can be estimated as

$$I_{12}=\varepsilon\sum_{i=1}^n\int_0^s\int_\Omega \varphi_K^L(u+\varepsilon I)\frac{b_i\cdot\nabla v_i}{v_i}dxdt\le C(b,v,T,\Omega)\varepsilon,$$

we infer from (22) that

$$\begin{aligned}
&H_{K,\epsilon}^{M,L}(u|v)\Big|_0^s+\sum_{i=1}^n e^{-\lambda_i}\int_\Omega \varphi_K^M(u+\varepsilon I)dx\Big|_0^s \\
&\quad\le -2\eta_0\sum_{i=1}^n\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)>L\}}\varphi_K^M(u+\varepsilon I)|\nabla u_i|^2dxdt \\
&\qquad + \eta_0\sum_{i=1}^n\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)>L\}}|\nabla u_i|^2dxdt \\
&\qquad + C(L,K,f,v)\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n(u_\ell+\varepsilon)>L\}}\bigg(1+\sum_{i=1}^n u_i\bigg)dxdt \\
&\qquad + C(L,f,v)\sum_{i=1}^n\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell\le L\}}|u_i-v_i|^2dxdt \\
&\qquad + C(M,K)\varepsilon(1-\log\varepsilon)+C(L,b,f,v,T,\Omega)\sqrt{\varepsilon} \\
&\qquad + G_{12}+G_2+\cdots+G_5+I_1+I_2+I_3+I_5+I_{62}+I_7.
\end{aligned} \tag{30}$$

We pass to the limit $\varepsilon \to 0$ in this inequality. First, we consider the left-hand side. We split the integral of $H^{M,l}_{K,\varepsilon}$ into two parts and analyze each part separately. By the mean-value theorem, we have for some $\theta_i \in [0,1]$,

$$\begin{aligned}\sum_{i=1}^n (u_i+\varepsilon)\big(\log(u_i+\varepsilon)+\lambda_i-1\big)+\sum_{i=1}^n e^{-\lambda_i} &= h(u+\varepsilon I)\\ &= h(u)+\sum_{i=1}^n h_i'(u_i+\theta_i\varepsilon)\varepsilon = h(u)+\varepsilon\sum_{i=1}^n\big(\log(u_i+\theta_i\varepsilon)+\lambda_i\big)\\ &\le h(u)+\sum_{i=1}^n(u_i+1+\lambda_i)\in L^1(\Omega).\end{aligned}$$

Thus, together with the bound (L1), we can apply the dominated convergence theorem to infer that, as $\varepsilon \to 0$,

$$\begin{aligned}&\int_\Omega \varphi_K^M(u+\varepsilon I)\sum_{i=1}^n\big[(u_i+\varepsilon)\big(\log(u_i+\varepsilon)+\lambda_i-1\big)+e^{-\lambda_i}\big]dx\Big|_0^s\\ &\quad\to \int_\Omega\varphi_K^M(u)\sum_{i=1}^n\big[u_i\big(\log u_i+\lambda_i-1\big)+e^{-\lambda_i}\big]dx\Big|_0^s.\end{aligned}$$

Similarly, it follows from the uniform bound

$$\bigg|\sum_{i=1}^n\Big(\varphi_K^L(u+\varepsilon I)(u_i+\varepsilon)(\log v_i+\lambda_i)-v_i\Big)\bigg|\le C(v)\bigg(\sum_{i=1}^n u_i+1\bigg)\in L^1(\Omega)$$

that in the limit $\varepsilon\to 0$,

$$\begin{aligned}&\int_\Omega\sum_{i=1}^n\Big(\varphi_K^L(u+\varepsilon I)(u_i+\varepsilon)(\log v_i+\lambda_i)-v_i\Big)dx\Big|_0^s\\ &\quad\to\int_\Omega\sum_{i=1}^n\big(\varphi_K^L(u)u_i(\log v_i+\lambda_i)-v_i\big)dx\Big|_0^s.\end{aligned}$$

Consequently, the left-hand side of (30) converges as $\varepsilon\to 0$:

$$\begin{aligned}&H^{M,L}_{K,\varepsilon}(u|v)+\sum_{i=1}^n e^{-\lambda_i}\int_\Omega\varphi_K^M(u+\varepsilon I)dx\Big|_0^s\\ &\quad\to H^{M,L}_K(u|v)+\sum_{i=1}^n e^{-\lambda_i}\int_\Omega\varphi_K^M(u)dx\Big|_0^s,\end{aligned}\tag{31}$$

where

$$H^{M,L}_K(u|v):=\int_\Omega\sum_{i=1}^n\Big(\varphi_K^M(u)u_i(\log u_i+\lambda_i-1)-\varphi_K^L(u)u_i(\log v_i+\lambda_i)+v_i\Big)dx.$$

Next, we turn to the limit $\varepsilon \to 0$ on the right-hand side of (30). We observe that for a.e. $(x,t) \in \Omega \times (0,s)$,

$$\lim_{\varepsilon \to 0} \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon)>L\}}(x,t) = \chi_{\{\sum_{\ell=1}^n u_\ell \geq L\}}(x,t).$$

Then, by the dominated convergence theorem, we can pass to the limit $\varepsilon \to 0$ in the first three terms on the right-hand side of (30), leading to

$$\begin{aligned}
&\sum_{i=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon)>L\}} \varphi_K^M(u+\varepsilon I)|\nabla u_i|^2 dxdt \\
&\qquad \to \sum_{i=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell \geq L\}} \varphi_K^M(u)|\nabla u_i|^2 dxdt, \\
&\sum_{i=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon)>L\}} |\nabla u_i|^2 dxdt \to \sum_{i=1}^n \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell \geq L\}} |\nabla u_i|^2 dxdt, \\
&\int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n (u_\ell+\varepsilon)>L\}} \bigg(1 + \sum_{i=1}^n u_i\bigg) dxdt \to \int_0^s \int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell \geq L\}} \bigg(1 + \sum_{i=1}^n u_i\bigg) dxdt.
\end{aligned}$$

We perform the limit $\varepsilon \to 0$ in the remaining terms. By dominated cnvergence, we find that

$$\begin{aligned}
G_{12} &= \sum_{i=1}^n \int_0^s \int_\Omega \varphi_K^M(u+\varepsilon I)\frac{u_i}{u_i+\varepsilon} b_i \cdot \nabla u_i dxdt \to \sum_{i=1}^n \int_0^s \int_\Omega \chi_{\{u_i>0\}} \varphi_K^M(u) b_i \cdot \nabla u_i dxdt, \\
I_{62} &= -\sum_{i=1}^n \int_0^s \int_\Omega \varphi_K^L(u+\varepsilon I) b_i \cdot \nabla u_i dxdt \\
&\to -\sum_{i=1}^n \int_0^s \int_\Omega \varphi_K^L(u) b_i \cdot \nabla u_i dxdt = -\sum_{i=1}^n \int_0^s \int_\Omega \chi_{\{u_i>0\}} \varphi_K^L(u) b_i \cdot \nabla u_i dxdt.
\end{aligned}$$

Let us consider the integrand of G_3. Using the definition for A_{ij}, we obtain the pointwise convergence as $\varepsilon \to 0$:

$$\begin{aligned}
&\partial_k \varphi_K^M(u+\varepsilon I)\big(\log(u_i+\varepsilon)+\lambda_i\big)\bigg(\sum_{j=1}^n A_{ij}(u)\nabla u_j - u_i b_i\bigg) \cdot \nabla u_k \\
&\quad = 2\partial_k \varphi_K^M(u+\varepsilon I)\sqrt{u_i}\big(\log(u_i+\varepsilon)+\lambda_i\big)\bigg(a_{i0} + \sum_{\ell=1}^n a_{il} u_\ell\bigg)\nabla\sqrt{u_i} \cdot \nabla u_k \\
&\qquad + \partial_k \varphi_K^M(u+\varepsilon I) u_i \big(\log(u_i+\varepsilon)+\lambda_i\big)\bigg(\sum_{j=1}^n a_{ij}\nabla u_j - b_i\bigg) \cdot \nabla u_k \\
&\quad \to 2\partial_k \varphi_K^M(u)\sqrt{u_i}(\log u_i + \lambda_i)\bigg(a_{i0} + \sum_{\ell=1}^n a_{i\ell} u_\ell\bigg)\nabla\sqrt{u_i} \cdot \nabla u_k
\end{aligned}$$

$$+ \partial_k \varphi_K^M(u) u_i (\log u_i + \lambda_i)\bigg(\sum_{j=1}^n a_{ij}\nabla u_j - b_i\bigg)\cdot \nabla u_k.$$

Taking the modulus and summing over $i = 1, \dots, n$, the left-hand side is bounded from above by

$$C(M,K)\sum_{i,k=1}^n \big(|\nabla\sqrt{u_i}| + |\nabla u_i| + 1\big)|\nabla u_k| \le C(M,K)\sum_{i=1}^n \big(|\nabla u_i|^2 + |\nabla \sqrt{u_i}|^2 + 1\big),$$

which is an $L^1(\Omega\times(0,T))$ function. Therefore, we can use the dominated convergence theorem again to infer that

$$G_3 \to -2\sum_{i,k=1}^n \int_0^s\int_\Omega \partial_k\varphi_K^M(u)\sqrt{u_i}(\log u_i + \lambda_i)\bigg(a_{i0} + \sum_{\ell=1}^n a_{i\ell}u_\ell\bigg)\nabla\sqrt{u_i}\cdot\nabla u_k dxdt$$
$$- \sum_{i,k=1}^n \int_0^s\int_\Omega \partial_k\varphi_K^M(u)u_i(\log u_i + \lambda_i)\bigg(\sum_{j=1}^n a_{ij}\nabla u_j - b_i\bigg)\cdot\nabla u_k dxdt.$$

Similarly, the limit $\varepsilon\to 0$ in G_4 gives

$$G_4 \to -2\sum_{i,k=1}^n \int_0^s\int_\Omega \partial_i\varphi_K^M(u)\sqrt{u_k}(\log u_k + \lambda_k)\bigg(\sum_{j=1}^n A_{ij}(u)\nabla u_j - u_i b_i\bigg)\cdot\nabla\sqrt{u_k}dxdt.$$

The limit $\varepsilon\to 0$ in the remaining terms G_2, G_5, I_1, I_2, I_3, I_5, I_7 follows directly from property (L3) and the dominated convergence theorem. We conclude from (30) and (31) that

$$H_K^{M,L}(u|v)\Big|_0^s + \sum_{i=1}^n e^{-\lambda_i}\int_\Omega \varphi_K^M(u)dx\Big|_0^s \le P_1 + \cdots + P_{15}, \tag{32}$$

where

$$P_1 = -2\eta_0\sum_{i=1}^n\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell \ge L\}}\varphi_K^M(u)|\nabla u_i|^2 dxdt,$$

$$P_2 = \eta_0\sum_{i=1}^n\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell \ge L\}}|\nabla u_i|^2 dxdt,$$

$$P_3 = C(L,f,v)\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell \le L\}}\sum_{i=1}^n |u_i - v_i|^2 dxdt,$$

$$P_4 = C(L,K,f,v)\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell \ge L\}}\bigg(1 + \sum_{i=1}^n u_i\bigg)dxdt,$$

$$P_5 = \sum_{i=1}^n\int_0^s\int_\Omega \chi_{\{u_i>0\}}\big(\varphi_K^M(u) - \varphi_K^L(u)\big)b_i\cdot\nabla u_i dxdt,$$

$$\begin{aligned}
P_6 &= -\sum_{i,k=1}^n \int_0^s \int_\Omega \partial_i\partial_k\varphi_K^M(u) \sum_{\ell=1}^n \big(u_\ell(\log u_\ell + \lambda_\ell - 1) + e^{-\lambda_\ell}\big) \\
&\quad \times \bigg(\sum_{j=1}^n A_{ij}(u)\nabla u_j - u_ib_i\bigg) \cdot \nabla u_k dxdt, \\
P_7 &= -2\sum_{i,k=1}^n \int_0^s \int_\Omega \partial_k\varphi_K^M(u)\sqrt{u_i}(\log u_i + \lambda_i)\bigg(a_{i0} + \sum_{\ell=1}^n a_{i\ell}u_\ell\bigg)\nabla\sqrt{u_i}\cdot\nabla u_k dxdt, \\
P_8 &= -\sum_{i,k=1}^n \int_0^s \int_\Omega \partial_k\varphi_K^M(u)u_i(\log u_i + \lambda_i)\bigg(\sum_{j=1}^n a_{ij}\nabla u_j - b_i\bigg)\cdot\nabla u_k dxdt, \\
P_9 &= -2\sum_{i,k=1}^n \int_0^s \int_\Omega \partial_i\varphi_K^M(u)\sqrt{u_k}(\log u_k + \lambda_k)\bigg(\sum_{j=1}^n A_{ij}(u)\nabla u_j - u_ib_i\bigg)\cdot\nabla\sqrt{u_k}dxdt, \\
P_{10} &= \sum_{i=1}^n \int_0^s \int_\Omega \partial_i\varphi_K^M(u) \sum_{\ell=1}^n \big(u_\ell(\log u_\ell + \lambda_\ell - 1) + e^{-\lambda_\ell}\big) f_i(u) dxdt, \\
P_{11} &= \sum_{i,j=1}^n \int_0^s \int_\Omega \partial_j\varphi_K^L(u)(\log v_i + \lambda_i)\bigg(\sum_{\ell=1}^n A_{j\ell}(u)\nabla u_\ell - u_jb_j\bigg)\cdot\nabla u_i dxdt, \\
P_{12} &= \sum_{i,j=1}^n \int_0^s \int_\Omega \partial_j\varphi_K^L(u)(\log v_i + \lambda_i)\bigg(\sum_{\ell=1}^n A_{i\ell}(u)\nabla u_\ell - u_ib_i\bigg)\cdot\nabla u_j dxdt, \\
P_{13} &= \sum_{i,j,k=1}^n \int_0^s \int_\Omega u_i\partial_j\partial_k\varphi_K^L(u)(\log v_i + \lambda_i)\bigg(\sum_{\ell=1}^n A_{j\ell}(u)\nabla u_\ell - u_jb_j\bigg)\cdot\nabla u_k dxdt, \\
P_{14} &= \sum_{i,j=1}^n \int_0^s \int_\Omega u_i\partial_j\varphi_K^L(u)\bigg(\sum_{\ell=1}^n A_{j\ell}(u)\nabla u_\ell - u_jb_j\bigg)\cdot\frac{\nabla v_i}{v_i}dxdt, \\
P_{15} &= \sum_{i,j=1}^n \int_0^s \int_\Omega u_i\partial_j\varphi_K^L(u)\bigg(\sum_{\ell=1}^n A_{i\ell}(v)\nabla v_\ell - v_ib_i\bigg)\cdot\frac{\nabla u_j}{v_i}dxdt.
\end{aligned}$$

3.5. **The limit $M\to\infty$.** We perform the limit $M\to\infty$ in (32). Observe that the terms $P_2,\ldots,P_4$ and $P_{11},\ldots,P_{15}$ do not depend on M such that we need to pass to the limit only in the remaining terms. First, we consider the left-hand side of (32). Recall that

$$\begin{aligned}
H_\varepsilon^{M,L}(u|v) &+ \sum_{i=1}^n e^{-\lambda_i}\int_\Omega \varphi_K^M(u)dx \\
&= \int_\Omega \bigg(\varphi_K^M(u)h(u) - \sum_{i=1}^n \big(\varphi_K^L(u)u_i(\log v_i + \lambda_i) - v_i\big)\bigg)dx,
\end{aligned}$$

where $h(u)$ is defined in (5). Since $|\varphi_K^M(u)h(u)| \le h(u)$ and $\varphi_K^M(u) \to 1$ pointwise a.e. as $M \to \infty$, we infer from the dominated convergence theorem that

$$\int_\Omega \varphi_K^M(u)h(u)dx\Big|_0^s \to \int_\Omega h(u)dx\Big|_0^s = \int_\Omega \sum_{\ell=1}^n u_\ell(\log u_\ell + \lambda_\ell - 1)dx\Big|_0^s + \sum_{i=1}^n e^{-\lambda_i}\int_\Omega dx\Big|_0^s$$
$$= \int_\Omega \sum_{\ell=1}^n u_\ell(\log u_\ell + \lambda_\ell - 1)dx\Big|_0^s.$$

This shows that in the limit $M \to \infty$,

$$H_K^{M,L}(u|v)\Big|_0^s + \sum_{i=1}^n e^{-\lambda_i}\int_\Omega \varphi_K^M(u)dx\Big|_0^s \to H_K^L(u|v)\Big|_0^s$$
$$:= \int_\Omega \bigg(\sum_{\ell=1}^n u_\ell(\log u_\ell + \lambda_\ell - 1) - \sum_{i=1}^n \big(\varphi_K^L(u)u_i(\log v_i + \lambda_i) - v_i\big)\bigg)dx\Big|_0^s.$$

We turn to the terms on the right-hand side of (32). Clearly, as $M \to \infty$,

$$P_1 \to -2\eta_0\sum_{i=1}^n\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell \ge L\}}|\nabla u_i|^2 dxdt.$$

Recall that P_3 and P_4 do not depend on M. Furthermore,

$$P_5 \to \sum_{i=1}^n\int_0^s\int_\Omega \chi_{\{u_i>0\}}\big(1-\varphi_K^L(u)\big)b_i\cdot\nabla u_i dxdt.$$

We use (L5) to estimate the following part of the integrand of P_6:

$$\Big|\partial_i\partial_k\varphi_K^M(u)\big(u_\ell(\log u_\ell + \lambda_\ell - 1) + e^{-\lambda_\ell}\big)(1+u_j)\Big|$$
$$\le C(K)\frac{[u_\ell(\log u_\ell + \lambda_\ell - 1) + e^{-\lambda_\ell}](1+u_j)}{(\sum_{i=1}^n u_i + e)^2\log(\sum_{i=1}^n u_i + e)} \le C(K).$$

Thus, the integrand of P_6 is bounded from above by

$$C(K)\sum_{j=1}^n(|\nabla u_j|+1)|\nabla u_k| \le C(K)\sum_{j=1}^n\big(|\nabla u_j|^2+1\big) \in L^1(\Omega\times(0,T)).$$

We deduce from $\partial_i\partial_k\varphi_K^M(u) \to 0$ as $M \to \infty$ that

$$P_6 \to 0 \quad \text{as } M \to \infty.$$

We rewrite the term P_7 as

$$P_7 = -2\sum_{i,k=1}^n\int_0^s\int_\Omega \chi_{\{u_i\le 1\}}\partial_k\varphi_K^M(u)\sqrt{u_i}(\log u_i + \lambda_i)\bigg(a_{i0} + \sum_{\ell=1}^n a_{il}u_\ell\bigg)\nabla\sqrt{u_i}\cdot\nabla u_k dxdt$$

$$- \sum_{i,k=1}^n \int_0^s \int_\Omega \chi_{\{u_i>1\}} \partial_k \varphi_K^M(u)(\log u_i + \lambda_i)\bigg(a_{i0} + \sum_{\ell=1}^n a_{il}u_\ell\bigg)\nabla u_i \cdot \nabla u_k dxdt.$$

Since

$$\Big|\chi_{\{u_i\le 1\}}\partial_k\varphi_K^M(u)\sqrt{u_i}(\log u_i+\lambda_i)(1+u_j)\Big| \le \frac{C(K)(1+u_j)}{(\sum_{i=1}^n u_i+e)\log(\sum_{i=1}^n u_i+e)} \le C(K),$$

$$\Big|\chi_{\{u_i>1\}}\partial_k\varphi_K^M(u)(\log u_i+\lambda_i)(1+u_j)\Big| \le \frac{C(K)\chi_{\{u_i>1\}}(\log u_i+\lambda_i)(1+u_j)}{(\sum_{i=1}^n u_i+e)\log(\sum_{i=1}^n u_i+e)} \le C(K),$$

the integrand of P_7 is bounded from above by

$$C(K)\sum_{i,k=1}^n \big(|\nabla u_i| + |\nabla\sqrt{u_i}|\big)|\nabla u_k| \le C(K)\sum_{i=1}^n \big(|\nabla u_i|^2 + |\nabla\sqrt{u_i}|^2\big).$$

and the right-hand side is a function in $L^1(\Omega\times(0,T))$. We infer from $\lim_{M\to\infty}\partial_k\varphi_K^M(u)=0$ and Lebesgue's dominated convergence theorem that $P_7\to 0$ as $M\to\infty$. Similarly, we infer that

$$P_8 \to 0, \quad P_9 \to 0 \quad \text{as } M\to\infty.$$

It remains to estimate P_{10} as $P_{11},\ldots,P_{15}$ do not depend on M. For this, we make explicit the derivative $\partial_i\varphi_K^M(u)$:

$$P_{10} = \int_0^s\int_\Omega \chi_{\{\sum_{k=1}^n u_k\ge M\}}\varphi'\bigg(\frac{\log\log(\sum_{k=1}^n u_k+e) - \log\log(M+e)}{\log(K+1)}\bigg)$$
$$\times \frac{\sum_{\ell=1}^n[u_\ell(\log u_\ell+\lambda_\ell-1)+e^{-\lambda_\ell}]}{\log(K+1)(\sum_{k=1}^n u_k+e)\log(\sum_{k=1}^n u_k+e)}\sum_{i=1}^n f_i(u)dxdt.$$

According to condition (H2.iii), there exists $M_0\in\mathbb{N}$ such that for all $\sum_{i=1}^n u_i\ge M_0$, it holds that $\sum_{i=1}^n f_i(u)\ge 0$ if $M\ge M_0$, and hence from $\varphi'\le 0$ that $P_{10}\le 0$.

Remark 7. If we assume that $|\sum_{i=1}^n f_i(w)| \le C(1+|w|^p)$ for all $w\in[0,\infty)^n$, we can conclude that $P_{10}\to 0$ as $M\to\infty$. Indeed, it follows from the Gagliardo-Nirenberg inequality (as shown in [5, page 732]) that $u_i\in L^p(\Omega\times(0,T))$ with $p=2+2/d$. This implies that $\sum_{i=1}^n f_i(u)\in L^1(\Omega\times(0,T))$, and we deduce from $\lim_{M\to\infty}\chi_{\{\sum_{k=1}^n u_k\ge M\}(x,t)}=0$ and Lebesgue's dominated convergence theorem that $P_{10}\to 0$ as $M\to\infty$. □

In conclusion, we obtain from (32) in the limit $M\to\infty$,

$$H_K^L(u|v)\Big|_0^s \le Q_1+\cdots+Q_4+P_{11}+\cdots+P_{15}, \tag{33}$$

where

$$Q_1 = -\eta_0\sum_{i=1}^n\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell\ge L\}}|\nabla u_i|^2dxdt,$$

$$Q_2 = C(L,f,v)\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell\le L\}}\sum_{i=1}^n|u_i-v_i|^2dxdt,$$

$$Q_3 = C(L,K,f,v)\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell \ge L\}}\bigg(1+\sum_{i=1}^n u_i\bigg)dxdt,$$
$$Q_4 = \sum_{i=1}^n \int_0^s\int_\Omega \chi_{\{u_i>0\}}\big(1-\varphi_K^L(u)\big)b_i\cdot\nabla u_i dxdt,$$

and we recall that the terms $P_{11},\ldots,P_{15}$ are defined after (32).

3.6. **End of the proof.** We claim that the right-hand side of (33) can be bounded from above by $\int_0^s H_K^L(u|v)dt$ (up to a constant), which then allows for a Gronwall argument to conclude that $H_K^L(u|v)=0$. To this end, we estimate the terms Q_i and P_i.

The terms Q_2 and Q_3 can be bounded from above by a constant times the entropy $H_K^L(u|v)$. This was shown by Fischer in [18], and we recall his result for the convenience of the reader.

Lemma 8 (Lemma 9 in [18])**.** *There exists $L\in\mathbb{N}$ such that for all $K\in\mathbb{N}$,*

$$\int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell\ge L\}}\bigg(1+\sum_{i=1}^n u_i\bigg)dx \le 2H_K^L(u|v), \tag{34}$$
$$\int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell\le L\}}\sum_{i=1}^n |u_i-v_i|^2dx \le C(L)H_K^L(u|v). \tag{35}$$

Hence, we infer that

$$Q_2+Q_3\le C(L,K,f,v)\int_0^s H_K^L(u|v)dt.$$

It follows from (L1), (L2), Young's inequality, and Lemma 8 that

$$\begin{aligned}
Q_4 &= \sum_{i=1}^n \int_0^s\int_\Omega \chi_{\{u_i>0\}}\chi_{\{\sum_{\ell=1}^n u_\ell\ge L\}}\big(1-\varphi_K^L(u)\big)b_i\cdot\nabla u_i dxdt \\
&\le C(b)\sum_{i=1}^n\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell\ge L\}}|\nabla u_i|dxdt \\
&\le \frac{\eta_0}{2}\sum_{i=1}^n\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell\ge L\}}|\nabla u_i|^2dxdt + C(b)\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell\ge L\}}dxdt \\
&\le \frac{\eta_0}{2}\sum_{i=1}^n\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell\ge L\}}|\nabla u_i|^2dxdt + C(b)\int_0^s H_K^L(u|v)dt,
\end{aligned}$$

and the first term on the right-hand side can be absorbed by Q_1. In a similar way, using (L2), (L4), and Lemma 8, we have

$$\begin{aligned}
&P_{11}+P_{12} \\
&= \sum_{i,j=1}^n\int_0^s\int_\Omega \chi_{\{\sum_{\ell=1}^n u_\ell\ge L\}}\partial_j\varphi_K^L(u)(\log v_i+\lambda_i)\bigg(\sum_{\ell=1}^n A_{j\ell}(u)\nabla u_\ell - u_jb_j\bigg)\cdot\nabla u_i dxdt,
\end{aligned}$$

$$+\sum_{i,j=1}^n\int_0^s\int_\Omega\chi_{\{\sum_{\ell=1}^n u_\ell\ge L\}}\partial_j\varphi_K^L(u)(\log v_i+\lambda_i)\bigg(\sum_{\ell=1}^n A_{i\ell}(u)\nabla u_\ell-u_ib_i\bigg)\cdot\nabla u_jdxdt$$
$$\le\frac{C(v,b)}{\log(K+1)}\sum_{i,j=1}^n\int_0^s\int_\Omega\chi_{\{\sum_{\ell=1}^n u_\ell\ge L\}}|\nabla u_i|(|\nabla u_j|+1)dxdt$$
$$\le\frac{C(v,b)}{\log(K+1)}\sum_{i=1}^n\int_0^s\int_\Omega\chi_{\{\sum_{\ell=1}^n u_\ell\ge L\}}|\nabla u_i|^2dxdt+C(v,b)\int_0^s H_K^L(u|v)dt.$$

Furthermore, taking into account (L2), (L5), and Lemma 8,

$$\begin{aligned}P_{13}&=\sum_{i,j,k=1}^n\int_0^s\int_\Omega\chi_{\{\sum_{\ell=1}^n u_\ell\ge L\}}u_i\partial_j\partial_k\varphi_K^L(u)(\log v_i+\lambda_i)\\&\quad\times\bigg(\sum_{\ell=1}^n A_{j\ell}(u)\nabla u_\ell-u_jb_j\bigg)\cdot\nabla u_kdxdt\\&\le\frac{C(v,b)}{\log(K+1)}\sum_{i,j=1}^n\int_0^s\int_\Omega\chi_{\{\sum_{\ell=1}^n u_\ell\ge L\}}|\nabla u_i|(|\nabla u_j|+1)dxdt\\&\le\frac{C(v,b)}{\log(K+1)}\sum_{i=1}^n\int_0^s\int_\Omega\chi_{\{\sum_{\ell=1}^n u_\ell\ge L\}}|\nabla u_i|^2dxdt+C(v,b)\int_0^s H_K^L(u|v)dt.\end{aligned}$$

Finally, using (L2)-(L4) and estimating as before:

$$\begin{aligned}P_{14}+P_{15}&\le\frac{C(v,b)}{\log(K+1)}\sum_{i,j=1}^n\int_0^s\int_\Omega\chi_{\{(L+e)^{K+1}>\sum_{\ell=1}^n u_\ell\ge L\}}\big(u_i|\nabla u_j|+u_i+|\nabla u_j|\big)dxdt\\&\le\frac{C(v,b)}{\log(K+1)}\sum_{i=1}^n\int_0^s\int_\Omega\chi_{\{(L+e)^{K+1}>\sum_{\ell=1}^n u_\ell\ge L\}}|\nabla u_i|^2dxdt\\&\quad+\frac{C(v,b)}{\log(K+1)}\int_0^s\int_\Omega\chi_{\{(L+e)^{K+1}>\sum_{\ell=1}^n u_\ell\ge L\}}\bigg(1+\sum_{i=1}^n u_i^2\bigg)dxdt\\&\le\frac{C(v,b)}{\log(K+1)}\sum_{i=1}^n\int_0^s\int_\Omega\chi_{\{\sum_{\ell=1}^n u_\ell\ge L\}}|\nabla u_i|^2dxdt+C(L,K,v,b)\int_0^s H_K^L(u|v)dt.\end{aligned}$$

Summarizing, we infer from (33) that

$$\begin{aligned}H_K^L(u|v)\Big|_0^s&\le C(L,K,f,v,b)\int_0^s H_K^L(u|v)dt\\&\qquad+\bigg(-\frac{\eta_0}{2}+\frac{C(v,b)}{\log(K+1)}\bigg)\sum_{i=1}^n\int_0^s\int_\Omega\chi_{\{\sum_{\ell=1}^n u_\ell\ge L\}}|\nabla u_i|^2dxdt.\end{aligned}$$

Choosing $K \in \mathbb{N}$ sufficiently large, the second term on the right-hand side is nonpositive and consequently,

$$H_K^L(u|v)\Big|_0^s \le C(L,K,f,v)\int_0^s H_K^L(u|v)dt.$$

It remains to determine $L \in \mathbb{N}$. Since we assumed that the initial data u^0 is bounded, we choose $L \in \mathbb{N}$ such that $\sum_{i=1}^n u_i^0 < L$. Then $\varphi_K^L(u^0) = 1$ and $H_K^L(u(0)|v(0)) = H_K^L(u^0, u^0) = 0$. The Gronwall lemma shows that $H_K^L(u(s)|v(s)) = 0$ for all $s \in (0, T^*)$. We claim that this yields $u(s) = v(s)$ for $s \in (0, T^*)$. Indeed, by (35) in Lemma 8, it follows that $u_i(s) = v_i(s)$ in $\{\sum_{\ell=1}^n u_\ell \le L\}$ for all $i = 1, \ldots, n$ and $s \in (0, T^*)$. Furthermore, by (34) in Lemma 8, we have $\mathrm{meas}(\{\sum_{\ell=1}^n u_\ell \ge L\}) = 0$. Therefore, $u_i(s) = v_i(s)$ on Ω, which concludes the proof.

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