RIGOROUS MEAN-FIELD LIMIT AND CROSS DIFFUSION

LI CHEN, ESTHER S. DAUS, AND ANSGAR JÜNGEL

Abstract. The mean-field limit in a weakly interacting stochastic many-particle system for multiple population species in the whole space is proved. The limiting system consists of cross-diffusion equations, modeling the segregation of populations. The mean-field limit is performed in two steps: First, the many-particle system leads in the large population limit to an intermediate nonlocal diffusion system. The local cross-diffusion system is then obtained from the nonlocal system when the interaction potentials approach the Dirac delta distribution. The global existence of the limiting and the intermediate diffusion systems is shown for small initial data, and an error estimate is given.

1. Introduction

Cross-diffusion models are systems of quasilinear parabolic equations with a nondiagonal diffusion matrix. They arise in many applications in cell biology, multicomponent gas dynamics, population dynamics, etc. [16]. To understand the range of validity of these diffusion systems, it is important to derive them from first principles or from more general models. In the literature, cross-diffusion systems were derived from random walks on lattice models [29], the kinetic Boltzmann equation [2], reaction-diffusion systems [5, 14], or from stochastic many-particle systems [28]. We derive in this paper rigorously the n-species cross-diffusion system

$$\begin{aligned} &\partial_t u_i - \sigma_i \Delta u_i = \operatorname{div}\left(\sum_{j=1}^n a_{ij} u_i \nabla u_j\right) \quad \text{in } \mathbb{R}^d,\ t>0, \\ &u_i(0) = u_i^0, \quad i=1,\ldots,n, \end{aligned} \tag{1}$$

where $\sigma_i > 0$ and a_{ij} are real numbers, starting from a stochastic many-particle system for multiple species. System (1) describes the diffusive dynamics of populations subject to segregation effects modeled by the term on the right-hand side [10].

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1.1. **Setting of the problem.** We consider n subpopulations of interacting individuals moving in the whole space $\mathbb{R}^d$ with the particle numbers $N_i \in \mathbb{N}$, $i = 1, \ldots, n$. We take $N_i = N$ to simplify the notation. The individuals are represented by the stochastic processes $X_{\eta,i}^{k,N}(t)$ evolving according to

$$
\begin{aligned}
dX_{\eta,i}^{k,N}(t) &= -\sum_{j=1}^{n} \frac{1}{N} \sum_{\ell=1}^{N} \nabla V_{ij}^{\eta}\big(X_{\eta,i}^{k,N}(t) - X_{\eta,j}^{\ell,N}(t)\big)dt + \sqrt{2\sigma_i}dW_i^k(t), \\
X_{\eta,i}^{k,N}(0) &= \xi_i^k, \quad i = 1, \ldots, n,\ k = 1, \ldots, N,
\end{aligned} \tag{2}
$$

where $(W_i^k(t))_{t\geq 0}$ are d-dimensional Brownian motions, the initial data $\xi_i^1, \ldots, \xi_i^N$ are independent and identically distributed random variables with the common probability density function u_i^0, and the interaction potential V_{ij}^{η} is given by

$$
V_{ij}^{\eta}(x) = \frac{1}{\eta^d} V_{ij}\left(\frac{|x|}{\eta}\right), \quad x \in \mathbb{R}^d,\ i, j = 1, \ldots, n.
$$

Here, V_{ij} is a given smooth function and $\eta > 0$ the scaling parameter. The scaling is chosen in such a way that the L^1 norm of V_{ij}^{η} stays invariant and $V_{ij}^{\eta} \to a_{ij}\delta$ in the sense of distributions as $\eta \to 0$, where δ denotes the Dirac delta distribution.

The mean-field limit $N \to \infty$, $\eta \to 0$ has to be understood in the following sense. For fixed $\eta > 0$, the many-particle model (2) is approximated for $N \to \infty$ by the intermediate stochastic system

$$
\begin{aligned}
d\bar{X}_{\eta,i}^k(t) &= -\sum_{j=1}^{n} (\nabla V_{ij}^{\eta} * u_{\eta,j})(\bar{X}_{\eta,i}^k(t), t)dt + \sqrt{2\sigma_i}dW_i^k(t), \\
\bar{X}_{\eta,i}^k(0) &= \xi_i^k, \quad i = 1, \ldots, n,\ k = 1, \ldots, N,
\end{aligned} \tag{3}
$$

where $u_{j,\eta}$ is the probability density function of $\bar{X}_{j,\eta}^k$, satisfying the nonlocal diffusion system

$$
\begin{aligned}
\partial_t u_{\eta,i} &= \sigma_i \Delta u_{\eta,i} + \operatorname{div}\left(\sum_{j=1}^{n} u_{\eta,i} \nabla V_{ij}^{\eta} * u_{\eta,j}\right) \quad \text{in } \mathbb{R}^d,\ t > 0, \\
u_{\eta,i}(0) &= u_i^0, \quad i = 1, \ldots, n.
\end{aligned} \tag{4}
$$

Observe that the intermediate system depends on k only through the initial data. Then, passing to the limit $\eta \to 0$ in the intermediate system, the limit $\nabla V_{ij}^{\eta} * u_{\eta,j} \to a_{ij}\nabla u_j$ in L^2 leads to the limiting stochastic system

$$
\begin{aligned}
d\widehat{X}_i^k(t) &= -\sum_{j=1}^{n} a_{ij} \nabla u_j(\widehat{X}_i^k(t), t)dt + \sqrt{2\sigma_i}dW_i^k(t), \\
\widehat{X}_i^k(0) &= \xi_i^k, \quad i = 1, \ldots, n,\ k = 1, \ldots, N,
\end{aligned} \tag{5}
$$

and the law of $\widehat{X}_i^k$ is a solution to the limiting cross-diffusion system (1). The main result of this paper is the proof of the error estimate

$$\mathbb{E}\bigg(\sum_{i=1}^n \sup_{0<s<t} \sup_{k=1,\ldots,N} \big|X_{\eta,i}^{k,N}(s) - \widehat{X}_i^k(s)\big|\bigg) \le C(t)\eta,$$

if we choose η and N such that $\eta^{-(2d+4)} \le \varepsilon \log N$ holds and $\varepsilon > 0$ can be any small number.

1.2. **State of the art.** Mean-field limits were investigated intensively in the last decades to derive, for instance, reaction-diffusion equations [8] or McKean-Vlasov equations [6, 11] (also see the reviews [12, 15]). Oelschläger [21] considered in the 1980s a weakly interacting particle system of N particles and proved that in the limit $N \to \infty$, the stochastic system converges to a deterministic nonlinear process. Later, he generalized his approach to systems of reaction-diffusion equations [22].

The analysis of quasilinear diffusion systems started more recently. The chemotaxis system was derived by Stevens [28] from a stochastic many-particle system with a limiting procedure that is based on Oelschläger's work. Reaction-diffusion systems with nonlocal terms were derived in [17] as the mean-field limit of a master equation for a vanishing reaction radius; also see [13]. The two-species Maxwell-Stefan equations were found to be the hydrodynamic limit system of two-component Brownian motions with singular interactions [26]. Nonlocal Lotka-Volterra systems with cross diffusion were obtained in the large population limit of point measure-valued Markov processes by Fontbona and Méléard [9]. Moussa [20] then proved the limit from the nonlocal to the local diffusion system (but only for triangular diffusion matrices), which gives the Shigesada-Kawasaki-Teramoto cross-diffusion system. A derivation of a space discretized version of this system from a Markov chain model was presented in [7]. Another nonlocal mean-field model was analyzed in [3].

Our system (1) is different from the aforementioned Shigesada-Kawasaki-Teramoto system

$$\partial_t u_i = \Delta\bigg(u_i \sum_{j=1}^n a_{ij}u_j\bigg) = \operatorname{div}\bigg(\sum_{j=1}^n a_{ij}u_i\nabla u_j\bigg) + \operatorname{div}\bigg(\sum_{j=1}^n a_{ij}u_j\nabla u_i\bigg)$$

derived in [9, 20]. Our derivation produces the first term on the right-hand side. The reason for the difference is that in [9], the diffusion coefficient σ_i in (2) is assumed to depend on the convolutions $W_{ij} * u_j$ for some functions W_{ij} – yielding the last term in the previous equation –, while we have assumed a constant diffusion coefficient. It is still an open problem to derive the general Shigesada-Kawasaki-Teramoto system; the approach of Moussa [20] requires that $a_{ij} = 0$ for $j < i$.

System (1) was also investigated in the literature. A formal derivation from the intermediate diffusion system (4) was performed by Galiano and Selgas [10], while probabilistic representations of (1) were presented in [1]. A rigorous derivation from the stochastic many-particle system (2) is still missing in the literature. In this paper, we fill this gap by extending the technique of [4] to diffusion systems. Compared to [4], the argument to derive the uniform estimates is more involved and involves a nonlinear Gronwall argument (see Lemma 17 in the appendix).

The global existence of solutions to (1) for general initial data and coefficients $a_{ij} \geq 0$ is an open problem. The reason is that we do not know any entropy structure of (1). For the two-species system, Galiano and Selgas [10] proved the global existence of weak solutions in a bounded domain with no-flux boundary conditions under the condition $4a_{11}a_{22} > (a_{12}+a_{21})^2$. The idea of the proof is to show that $H(u) = \int u_i(\log u_i - 1)dx$ is a Lyapunov functional (entropy). The condition can be weakened to $a_{11}a_{22} > a_{12}a_{21}$ using the modified entropy $H_1(u) = \int(a_{21}u_1(\log u_1 - 1) + a_{12}u_2(\log u_2 - 1))dx$, but this is still a weak cross-diffusion condition.

We use the following notation throughout the paper. We write $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^s}$ for the norms of $L^p = L^p(\mathbb{R}^d)$ and $H^s = H^s(\mathbb{R}^d)$, respectively. Furthermore, $|u|^2 = \sum_{i=1}^n u_i^2$ for $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and $\|u\|_{L^p}^2 = \sum_{i=1}^n \|u_i\|_{L^p}^2$ for functions $u = (u_1, \ldots, u_n)$. We use the notation $u(t) = u(\cdot, t)$ for functions depending on x and t, and $C > 0$ is a generic constant whose value may change from line to line.

1.3. **Main results.** The first two results are concerned with the solvability of the nonlocal diffusion system (4) and the limiting cross-diffusion system (1). The existence results are needed for our main result, Theorem 3 below.

We impose the following assumptions on the interaction potential. Let $V_{ij} \in C_0^2(\mathbb{R}^d)$ be such that $\operatorname{supp}(V_{ij}) \subset B_1(0)$ for $i,j = 1, \ldots, n$. Then $V_{ij}^\eta(x) = \eta^{-d}V_{ij}(|x|/\eta)$ for $\eta > 0$ satisfies $\operatorname{supp}(V_{ij}^\eta) \subset B_\eta(0)$ and

$$a_{ij} := \int_{\mathbb{R}^d} V_{ij}(|x|)dx = \int_{\mathbb{R}^d} V_{ij}^\eta(x)dx, \quad i,j = 1, \ldots, n.$$

As the potential may be negative (and a_{ij} may be negative too), we introduce

$$A_{ij} := \|V_{ij}\|_{L^1} = \|V_{ij}^\eta\|_{L^1}, \quad i,j = 1, \ldots, n.$$

Proposition 1 (Existence for the nonlocal diffusion system)**.** *Let $u^0 = (u_1^0, \ldots, u_n^0) \in H^s(\mathbb{R}^d; \mathbb{R}^n)$ with $s > d/2 + 1$ and $u_i^0 \geq 0$ in $\mathbb{R}^d$ and assume that*

$$\|u^0\|_{H^s} \leq \frac{\sigma}{C^* \sum_{i,j=1}^n A_{ij}}, \tag{6}$$

where $\sigma = \min_{i=1,\ldots,n} \sigma_i > 0$ and $C^* > 0$ is a constant only depending on s and d. Then there exists a global solution $u_\eta = (u_{\eta,1}, \ldots, u_{\eta,n})$ to problem (4) such that $u_{\eta,i}(t) \geq 0$ in $\mathbb{R}^d$, $t > 0$, $u_\eta \in L^\infty(0,\infty; H^s(\mathbb{R}^d;\mathbb{R}^n) \cap L^2(0,\infty; H^{s+1}(\mathbb{R}^d;\mathbb{R}^n))$, and

$$\sup_{t>0} \|u_\eta(t)\|_{H^s} \leq \|u^0\|_{H^s}.$$

Moreover, if for some $0 < \gamma < \sigma$ the slightly stronger condition

$$\|u^0\|_{H^s} \leq \frac{\sigma - \gamma}{C^* \sum_{i,j=1}^n A_{ij}} \tag{7}$$

holds, then the solution is unique and

$$\sup_{t>0}\|u_\eta(t)\|_{H^s}^2 + \gamma\|\nabla u_\eta\|_{L^2(0,\infty;H^{s+1})}^2 \le \|u^0\|_{H^s}^2. \tag{8}$$

Since we do not use the structure of the equations, we can only expect the global existence of solutions for sufficiently small initial data. The proof of this result is based on the Banach fixed-point theorem and a priori estimates and is rather standard. We present it for completeness.

Proposition 2 (Existence for the limiting cross-diffusion system). *Let $u^0 \in H^s(\mathbb{R}^d;\mathbb{R}^n)$ with $s > d/2+1$ such that $u_i^0 \ge 0$ in $\mathbb{R}^d$ and (7) holds. Then there exists a unique global solution $u = (u_1,\dots,u_n)$ to problem (1) satisfying $u_i(t) \ge 0$ in $\mathbb{R}^d$, $t > 0$, $u \in L^\infty(0,\infty;H^s(\mathbb{R}^d;\mathbb{R}^n)) \cap L^2(0,\infty;H^{s+1}(\mathbb{R}^d;\mathbb{R}^n))$, and*

$$\sup_{t>0}\|u(t)\|_{H^s}^2 + \gamma\|u\|_{L^2(0,\infty;H^{s+1})}^2 \le \|u^0\|_{H^s}^2. \tag{9}$$

Moreover, let u_η be the solution to problem (4). Then the following error estimate holds for any $T > 0$:

$$\|u_\eta - u\|_{L^\infty(0,T;L^2)} + \|\nabla(u_\eta - u)\|_{L^2(0,T;L^2)} \le C(T)\eta \tag{10}$$

for some constant $C(T) > 0$.

The proposition is proved by performing the limit $\eta \to 0$ in (4) which is possible in view of the uniform estimate (8). The error estimate (10) follows from the uniform bounds and the smallness condition (6).

For our main result, we need to make precise the stochastic setting. Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge0},\mathcal{P})$ be a filtered probability space and let $(W_i^k(t))_{t\ge0}$ for $i = 1,\dots,n$, $k = 1,\dots,N$ be d-dimensional $\mathcal{F}_t$-Brownian motions that are independent of the random variables ξ_i^k. We assume that the Brownian motions are independent and that the initial data $\xi_i^1,\dots,\xi_i^N$ are independent and identically distributed random variables with the common probability density function u_i^0.

We prove in Section 4 that if $s > d/2+2$ and the initial density u^0 satisfies the smallness condition (6), the stochastic differential systems (2), (3), and (5) have pathwise unique strong solutions; also see Remark 11.

Theorem 3 (Error estimate for the stochastic system). *Under the aforementioned assumptions, let $s > d/2+2$ and let $X_{\eta,i}^{k,N}$ and $\widehat{X}_i^k$ be solutions to the problems (2) and (5), respectively. Furthermore, let $0 < \varepsilon < 1$ be sufficiently small and choose $N \in \mathbb{N}$ such that $\varepsilon \log N \ge \eta^{-2d-4}$. Then, for any $t > 0$,*

$$\mathbb{E}\bigg(\sum_{i=1}^n \sup_{0<s<t}\sup_{k=1,\dots,N}\big|X_{\eta,i}^{k,N}(s) - \widehat{X}_i^k(s)\big|\bigg) \le C(t)\eta,$$

where the constant $C(t)$ depends on t, n, $\|D^2V_{ij}\|_{L^\infty}$, and the initial datum u^0.

The idea of the proof is to derive error estimates for the differences $X^{k,N}_{\eta,i} - \bar{X}^k_{\eta,i}$ and $\bar{X}^k_{\eta,i} - \widehat{X}^k_i$ (where $\bar{X}^{k,N}_{\eta,i}$ solves (3)) and to use

$$\begin{aligned}\mathbb{E}\bigg(\sum_{i=1}^n \sup_{0<s<t}\sup_{k=1,\ldots,N}\big|X^{k,N}_{\eta,i} - \widehat{X}^k_i\big|\bigg) &\le \mathbb{E}\bigg(\sum_{i=1}^n \sup_{0<s<t}\sup_{k=1,\ldots,N}\big|X^{k,N}_{\eta,i} - \bar{X}^k_{\eta,i}\big|\bigg) \\ &\quad + \mathbb{E}\bigg(\sum_{i=1}^n \sup_{0<s<t}\sup_{k=1,\ldots,N}\big|\bar{X}^k_{\eta,i} - \widehat{X}^k_i\big|\bigg).\end{aligned}$$

The expectations on the right-hand side are estimated by taking the difference of the solutions to the corresponding stochastic differential equations, exploiting the Lipschitz continuity of ∇V^η_{ij}, and observing that $\|V^\eta_{ij} * \nabla u_j - a_{ij}\nabla u_j\|_{L^2(0,t;L^2)} \le C\eta$.

The paper is organized as follows. Sections 2 and 3 are concerned with the proof of Propositions 1 and 2, respectively. The existence of solutions to the stochastic systems is shown in Section 4. The main result (Theorem 3) is then proved in Section 5. Finally, the appendix recalls some auxiliary results needed in our analysis.

2. Existence for the nonlocal diffusion system (4)

We show Proposition 1 whose proof is split into several lemmas.

Lemma 4 (Local existence of solutions)**.** *Let $u^0 = (u^0_1, \ldots, u^0_n) \in H^s(\mathbb{R}^d;\mathbb{R}^n)$ with $s > d/2+1$ and $u^0_i \ge 0$ in $\mathbb{R}^d$. Then there exists $T^* > 0$ such that (4) possesses the unique solution $u_\eta = (u_{\eta,1}, \ldots, u_{\eta,n}) \in L^\infty(0,T^*;H^s(\mathbb{R}^d;\mathbb{R}^n)) \cap L^2(0,T^*;H^{s+1}(\mathbb{R}^d;\mathbb{R}^n))$ satisfying $u_{\eta,i}(t) \ge 0$ in $\mathbb{R}^d$ for $t > 0$. The time $T^* > 0$ depends on $\|u^0\|_{H^s}$ such that $T^* \to 0$ if $\|u^0\|_{H^s} \to \infty$.*

Proof. The idea is to apply the Banach fixed-point theorem. For this, we introduce

$$Y = \Big\{v \in L^\infty(0,T^*;H^s(\mathbb{R}^d;\mathbb{R}^n)) \cap L^2(0,T^*;H^{s+1}(\mathbb{R}^d;\mathbb{R}^n)) : \tag{11}$$

$$\sup_{0<t<T^*}\|v(\cdot,t)\|^2_{H^s} \le M := 1 + \|u^0\|^2_{H^s}\Big\}, \tag{12}$$

endowed with the metric $\mathrm{dist}(u,w) = \sup_{0<t<T^*}\|(u-w)(t)\|_{L^2}$, where $T^* > 0$ will be determined later. The fixed-point operator $S : Y \to Y$ is defined by $Sv = u$, where u is the unique solution to the Cauchy problem

$$\partial_t u_i = \sigma_i\Delta u_i + \operatorname{div}\bigg(\sum_{j=1}^n u_i^+ \nabla V^\eta_{ij} * v_j\bigg), \quad u_i(0) = u^0_i \text{ in } \mathbb{R}^d, \tag{13}$$

and $u_i^+ = \max\{0,u_i\}$. The existence of a unique solution $u \in C^0([0,T];H^s(\mathbb{R}^d)) \cap L^2(0,T;H^{s+1}(\mathbb{R}^d))$ to this linear advection-diffusion problem follows from semigroup theory since $u^0 \in H^s(\mathbb{R}^d;\mathbb{R}^n)$. Taking the test function $u_i^- = \min\{0,u_i\}$ in the weak formulation of (13) yields

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}(u_i^-)^2dx + \sigma_i\int_{\mathbb{R}^d}|\nabla u_i^-|^2dx = -\int_{\mathbb{R}^d}\sum_{j=1}^n u_i^+(\nabla V^\eta_{ij} * v_j)\cdot\nabla u_i^- dx.$$

Since $u_i^+\nabla u_i^- = 0$ in $\mathbb{R}^d$, we infer that $u_i^- = 0$ in $\mathbb{R}^d$, showing that $u_i(t)$ is nonnegative for all $t \in (0, T^*)$.

We prove that $\sup_{0<t<T^*} \|u(\cdot,t)\|_{H^s}^2 \le M$ for sufficiently small values of $T^* > 0$. Then $u \in Y$ and $S : Y \to Y$ is well defined. We apply the differential operator D^α for an arbitrary multi-index $\alpha \in \mathbb{N}^d$ of order $|\alpha| \le s$ to (13), multiply the resulting equation by $D^\alpha u_i$, and integrate over $\mathbb{R}^d$:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |D^\alpha u_i|^2 dx + \sigma_i \int_{\mathbb{R}^d} |\nabla D^\alpha u_i|^2 dx = -\int_{\mathbb{R}^d} \sum_{j=1}^n D^\alpha(u_i \nabla V_{ij}^\eta * v_j) \cdot \nabla D^\alpha u_i dx.$$

We sum these equations from $i = 1, \dots, n$, apply the Cauchy-Schwarz inequality to the integral on the right-hand side, and the Moser-type calculus inequality (Lemma 15):

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |D^\alpha u|^2 dx + \sigma \int_{\mathbb{R}^d} |\nabla D^\alpha u|^2 dx &\le \sum_{i,j=1}^n \big\|D^\alpha(u_i \nabla V_{ij}^\eta * v_j)\big\|_{L^2} \|\nabla D^\alpha u_i\|_{L^2} \\
&\le C(\varepsilon) \sum_{i,j=1}^n \big\|D^\alpha(u_i \nabla V_{ij}^\eta * v_j)\big\|_{L^2}^2 + \varepsilon n \|\nabla D^\alpha u\|_{L^2}^2 \\
&\le C(\varepsilon) \sum_{i,j=1}^n \Big(\|u_i\|_{L^\infty} \big\|D^s(\nabla V_{ij}^\eta * v_j)\big\|_{L^2} + \|D^s u_i\|_{L^2} \|\nabla V_{ij}^\eta * v_j\|_{L^\infty}\Big)^2 \\
&\quad + \varepsilon n \|\nabla D^\alpha u\|_{L^2}^2,
\end{aligned}$$

where we recall that $\sigma = \min_{i=1,\dots,n} \sigma_i > 0$ and ε is any positive number. The last term on the right-hand side can be absorbed by the second term on the left-hand side if $\varepsilon \le \sigma/(2n)$. Hence, summing over all multi-indices α of order $|\alpha| \le s$, using Young's convolution inequality (Lemma 16), and the inequality $\|\nabla V_{ij}^\eta\|_{L^1} \le C(\eta)$, we find that

$$\begin{aligned}
\frac{d}{dt}\|u\|_{H^s}^2 + \frac{\sigma}{2}\|\nabla u\|_{H^s}^2 &\le C \sum_{i,j=1}^n \Big(\|u\|_{L^\infty} \|\nabla V_{ij}^\eta\|_{L^1} \|D^s v_j\|_{L^2} + \|D^s u_i\|_{L^2} \|\nabla V_{ij}^\eta\|_{L^1} \|v_j\|_{L^\infty}\Big)^2 \\
&\le C(\eta) \|u\|_{H^s}^2 \|v\|_{H^s}^2,
\end{aligned}$$

where in the last step we have taken into account the continuous embedding $H^s(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$. As $v \in Y$ and consequently $\|v(t)\|_{H^s}^2 \le M$, we infer that

$$\frac{d}{dt}\|u\|_{H^s}^2 \le C(\eta) M \|u\|_{H^s}^2.$$

Gronwall's inequality then yields

$$\|u(t)\|_{H^s}^2 \le \|u^0\|_{H^s}^2 e^{C(\eta)Mt} = (M-1)e^{C(\eta)MT^*} \le M, \quad 0 < t < T^*,$$

if we choose $T^* > 0$ so small that $e^{C(\eta)MT^*} \le M/(M-1)$. We conclude that $u \in Y$.

Note that the time T^* depends on M and hence on u^0 in such a way that T^* becomes smaller if $\|u^0\|_{H^s}$ is large but $T^* > 0$ is bounded from below if $\|u^0\|_{H^s}$ is small.

It remains to show that the map $S:Y\to Y$ is a contraction, possibly for a smaller value of $T^*>0$. Let $v,w\in Y$ and take the difference of the equations satisfied by Sv and Sw, respectively:

$$\begin{aligned}&\partial_t\big((Sv)_i-(Sw)_i\big)-\sigma_i\Delta\big((Sv)_i-(Sw)_i\big)\\&\quad=\operatorname{div}\bigg(\sum_{j=1}^n\big((Sv)_i-(Sw)_i\big)\nabla V_{ij}^\eta*v_j\bigg)+\operatorname{div}\bigg(\sum_{j=1}^n(Sw)_i\nabla V_{ij}^\eta*(v_j-w_j)\bigg).\end{aligned}$$

Multiplying these equations by $(Sv)_i-(Sw)_i$, summing from $i=1,\ldots,n$, integrating over $\mathbb{R}^d$, and using the Cauchy-Schwarz inequality leads to

$$\begin{aligned}&\frac12\frac{d}{dt}\int_{\mathbb{R}^d}|Sv-Sw|^2dx+\sigma\int_{\mathbb{R}^d}|\nabla(Sv-Sw)|^2dx\\&\quad\le\sum_{i,j=1}^n\big\|((Sv)_i-(Sw)_i)\nabla V_{ij}^\eta*v_j\big\|_{L^2}\|\nabla((Sv)_i-(Sw)_i)\|_{L^2}\\&\qquad+\sum_{i,j=1}^n\big\|(Sw)_i\nabla V_{ij}^\eta*(v_j-w_j)\big\|_{L^2}\|\nabla((Sv)_i-(Sw)_i)\|_{L^2}.\end{aligned}$$

We deduce from Young's convolution inequality that

$$\begin{aligned}&\frac12\frac{d}{dt}\int_{\mathbb{R}^d}|Sv-Sw|^2dx+\frac{\sigma}{2}\int_{\mathbb{R}^d}|\nabla(Sv-Sw)|^2dx\\&\quad\le C(\sigma)\sum_{i,j=1}^n\big\|((Sv)_i-(Sw)_i)\nabla V_{ij}^\eta*v_j\big\|_{L^2}^2+C(\sigma)\sum_{i,j=1}^n\big\|(Sw)_i\nabla V_{ij}^\eta*(v_j-w_j)\big\|_{L^2}^2\\&\quad\le C(\sigma)\sum_{i,j=1}^n\|\nabla V_{ij}^\eta*v_j\|_{L^\infty}^2\|(Sv)_i-(Sw)_i\|_{L^2}^2\\&\qquad+C(\sigma)\max_{i=1,\ldots,n}\|(Sw)_i\|_{L^\infty}^2\sum_{i,j=1}^n\|\nabla V_{ij}^\eta*(v_j-w_j)\|_{L^2}^2\\&\quad\le C(\sigma)\sum_{i,j=1}^n\|\nabla V_{ij}^\eta\|_{L^1}^2\|v_j\|_{L^\infty}^2\|Sv-Sw\|_{L^2}^2\\&\qquad+C(\sigma)\|Sw\|_{L^\infty}^2\sum_{i,j=1}^n\|\nabla V_{ij}^\eta\|_{L^1}^2\|v_j-w_j\|_{L^2}^2.\end{aligned}$$

By definition of the metric on Y, we have shown that

$$\frac{d}{dt}\|Sv-Sw\|_{L^2}^2\le C_1(\sigma,\eta,M)\|Sv-Sw\|_{L^2}^2+C_2(\sigma,\eta,M)\|v-w\|_{L^2}^2.$$

The constants C_1 and C_2 depend on M (and hence on u^0) in such a way that they become larger if $\|u^0\|_{H^s}$ is large but they stay bounded for small values of $\|u^0\|_{H^s}$. Thus, because of $v(0) = w(0)$, Gronwall's inequality gives

$$\begin{aligned}\|Sv(t) - Sw(t)\|_{L^2}^2 &\le C_2(\sigma,\eta,M)\int_0^t e^{C_1(\sigma,\eta)(t-s)}\|v(s)-w(s)\|^2 ds \\ &\le C_2(\sigma,\eta,M)\big(e^{C_1(\sigma,\eta)t}-1\big)\sup_{0<s<t}\|v(s)-w(s)\|_{L^2}^2,\end{aligned}$$

and the definition of the metric leads to

$$\operatorname{dist}(Sv,Sw)^2 \le C_2(\sigma,\eta,M)\big(e^{C_1(\sigma,\eta,M)T^*}-1\big)\operatorname{dist}(v,w)^2.$$

Then, choosing $T^* > 0$ such that $C_2(\sigma,\eta,M)(e^{C_1(\sigma,\eta,M)T^*}-1) \le 1/2$ shows that $S : Y \to Y$ is a contraction. Again, T^* depends on u^0 but it is bounded from below for small values of $\|u^0\|_{H^s}$. Thus, we can apply the Banach fixed-point theorem, finishing the proof. □

Lemma 5 (A priori estimates)**.** *Let assumption* (7) *hold. For the local solution u_η to problem* (4)*, the uniform estimate* (8) *holds. In particular, the solution u_η can be extended to a global one.*

Proof. We proceed similarly as in the proof of Lemma 4. We choose α of order $|\alpha| \le s$, apply the operator D^α on both sides of (4), multiply the resulting equation by $D^\alpha u_{\eta,i}$, and integrate over $\mathbb{R}^d$. By the Cauchy-Schwarz inequality, the Moser-type calculus inequality, and Young's convolution inequality and writing u_i instead of $u_{\eta,i}$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}|D^\alpha u|^2 dx + \sigma\int_{\mathbb{R}^d}|\nabla D^\alpha u|^2 dx \\ &\quad\le \sum_{i,j=1}^n \big\|D^\alpha(u_i V_{ij}^\eta * \nabla u_j)\big\|_{L^2}\|\nabla D^\alpha u_i\|_{L^2} \\ &\quad\le C_M\sum_{i,j=1}^n \Big(\|u_i\|_{L^\infty}\|D^s(V_{ij}^\eta * \nabla u_j)\|_{L^2} + \|D^s u_i\|_{L^2}\|V_{ij}^\eta * \nabla u_j\|_{L^\infty}\Big)\|\nabla D^\alpha u_i\|_{L^2} \\ &\quad\le C_M\sum_{i,j=1}^n \Big(C\|u\|_{H^s}\|V_{ij}^\eta\|_{L^1}\|D^s\nabla u\|_{L^2} + \|D^s u\|_{L^2}\|V_{ij}^\eta\|_{L^1}\|\nabla u\|_{L^\infty}\Big)\|\nabla D^\alpha u\|_{L^2} \\ &\quad\le C^*\sum_{i,j=1}^n A_{ij}\|u\|_{H^s}\|\nabla u\|_{H^s}\|\nabla D^\alpha u\|_{L^2},\end{aligned}$$

where C_M is the constant from Lemma 15, $C^* > 0$ depends on C_M and the constant of the embedding $H^s(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$, and we have used $\|V_{ij}^\eta\|_{L^1} = A_{ij}$. Summation of all $|\alpha| \le s$ leads to

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^s}^2 + \sigma\|\nabla u\|_{H^s}^2 \le C^*\sum_{i,j=1}^n A_{ij}\|u\|_{H^s}\|\nabla u\|_{H^s}^2,$$

which can be written as

$$\frac{d}{dt}\|u\|_{H^s}^2 + 2\Big(\sigma - C^*\sum_{i,j=1}^n A_{ij}\|u\|_{H^s}\Big)\|\nabla u\|_{H^s}^2 \le 0. \tag{14}$$

This inequality holds for all $t \in [0,T]$, where $T < T^*$. By Lemma 17, applied to $f(t) = \|u(t)\|_{H^s}^2$, $g(t) = \|\nabla u(t)\|_{H^s}$, $a = \sigma$, and $b = C^* \sum_{i,j=1}^n A_{ij}$, we find that $\|u(t)\|_{H^s}^2 \le (a/b)^2$ for $t \in [0,T]$. Here, we use Assumption (6). We deduce that $(d/dt)\|u\|_{H^s}^2 \le 0$ and consequently $\|u(t)\|_{H^s} \le \|u^0\|_{H^s}$ for $t \in [0,T]$.

Now, we take $u(T)$ as the initial datum for problem (4). We deduce from Lemma 4 the existence of a solution u to (4) defined on $[T, T+T^*)$. Here, $T^* > 0$ can be chosen as the same end time as before since the norm of the initial datum $\|u(T)\|_{H^s}$ is not larger as $\|u^0\|_{H^s}$. Note that T^* becomes smaller only when the initial datum is larger in the H^s norm. Hence, $u(t)$ exists for $t \in [T, 2T]$ and inequality (14) holds. As before, we conclude from Lemma 17 that $\|u(t)\|_{H^s} \le \|u^0\|_{H^s}$ for $t \in [T, 2T]$. This argument can be continued, obtaining a global solution satisfying $\|u(t)\|_{H^s} \le \|u^0\|_{H^s}$ for all $t > 0$. Then, under the stronger assumption (7),

$$\frac{d}{dt}\|u\|_{H^s}^2 \le -2\bigg(\sigma - C^* \sum_{i,j=1}^n A_{ij}\|u^0\|_{H^s}\bigg)\|\nabla u\|_{H^s}^2 \le -\gamma\|\nabla u\|_{H^s}^2,$$

which leads to (8), finishing the proof. □

Lemma 6 (Uniqueness of solutions)**.** *Let assumption* (7) *hold. Then the solution to problem* (4) *is unique in the class of functions* $u \in L^\infty(0,\infty; H^s(\mathbb{R}^d;\mathbb{R}^n)) \cap L^2(0,\infty; H^{s+1}(\mathbb{R}^d;\mathbb{R}^n))$.

Proof. Let u and v be two solutions to (4) with the same initial data. We multiply the difference of the equations satisfied by u_i and v_i by $u_i - v_i$, sum from $i = 1, \ldots, n$, and integrate over $\mathbb{R}^d$. Then, for all $0 < t < T$ and some $T > 0$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}|u-v|^2dx + \sigma\int_{\mathbb{R}^d}|\nabla(u-v)|^2dx \\
&\quad\le \sum_{i,j=1}^n \big(\|u_i - v_i\|_{L^2}\|V_{ij}^\eta * u_j\|_{L^\infty} + \|v_i\|_{L^\infty}\|V_{ij}^\eta * \nabla(u_j - v_j)\|_{L^2}\big)\|\nabla(u_i - v_i)\|_{L^2} \\
&\quad\le \sum_{i,j=1}^n \big(\|u-v\|_{L^2}\|V_{ij}^\eta\|_{L^1}\|\nabla u\|_{L^\infty} + \|v\|_{L^\infty}\|V_{ij}^\eta\|_{L^1}\|\nabla(u-v)\|_{L^2}\big)\|\nabla(u-v)\|_{L^2} \\
&\quad\le \sum_{i,j=1}^n A_{ij}\big(\|u\|_{H^{s+1}}\|u-v\|_{L^2}\|\nabla(u-v)\|_{L^2} + \|v\|_{H^s}\|\nabla(u-v)\|_{L^2}^2\big).
\end{aligned}$$

By assumption, $\sum_{i,j=1}^n A_{ij}\|v\|_{H^s} \le \sigma - \gamma$ (since we supposed that $C^* \ge 1$). Thus, using the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}|u-v|^2dx + \sigma\int_{\mathbb{R}^d}|\nabla(u-v)|^2dx \\
&\quad\le C(\varepsilon)\bigg(\sum_{i,j=1}^n A_{ij}\|u\|_{H^{s+1}}\bigg)^2\|u-v\|_{L^2}^2 + \varepsilon n^2\|\nabla(u-v)\|_{L^2}^2 + (\sigma-\gamma)\|\nabla(u-v)\|_{L^2}^2
\end{aligned}$$

$$\le C(\varepsilon)\|u(t)\|_{H^{s+1}}^2\|u-v\|_{L^2}^2+\sigma\|\nabla(u-v)\|_{L^2}^2,$$

if we choose $\varepsilon \le \gamma/n^2$. Observe that the norm $\|u\|_{L^2(0,\infty;H^{s+1})}$ is bounded. This allows us to apply the Gronwall inequality, and together with the fact that $\|(u-v)(0)\|_{L^2}=0$, we infer that $\|(u-v)(t)\|_{L^2}=0$, concluding the proof. ☐

3. Existence for the cross-diffusion system (1)

We prove Proposition 2 whose proof is split into two lemmas.

Lemma 7 (Existence and uniqueness of solutions)**.** *Let the assumptions of Proposition 2 hold. Then there exists a unique solution to* (1) *satisfying* (9)*.*

Proof. Let u_η be the solution to (4). We prove that a subsequence of (u_η) converges to a solution to problem (1). In view of the uniform estimate (8), there exists a subsequence of (u_η), which is not relabeled, such that, as $\eta\to 0$,

$$u_\eta \rightharpoonup u \quad \text{weakly in } L^2(0,T;H^{s+1}(\mathbb{R}^d)). \tag{15}$$

We show that u is a weak solution to problem (1). First, we claim that

$$V_{ij}^\eta * \nabla u_{\eta,j} \rightharpoonup a_{ij}\nabla u_j \quad \text{weakly in } L^2(0,T;L^2(\mathbb{R}^d)).$$

To prove this statement, we observe that $V_{ij}^\eta * \psi \to a_{ij}\psi$ strongly in $L^2(0,T;L^2(\mathbb{R}^d))$ for any $\phi\in L^2(0,T;L^2(\mathbb{R}^d))$ [25, Theorem 9.10] and $\nabla u_{\eta,j}\rightharpoonup \nabla u_j$ weakly in $L^2(0,T;L^2(\mathbb{R}^d))$. Therefore, for all $\psi\in C_0^\infty(\mathbb{R}^d\times(0,T);\mathbb{R}^n)$, as $\eta\to 0$,

$$\begin{aligned}
&\left|\int_0^T\int_{\mathbb{R}^d}\big(V_{ij}^\eta*\nabla u_{\eta,j}-a_{ij}\nabla u_j\big)\cdot\psi dxdt\right| \\
&\quad=\left|\int_0^T\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}V_{ij}^\eta(x-y)\nabla u_{\eta,j}(y,t)dy\right)\cdot\psi(x,t)dxdt\right. \\
&\qquad\left.-\int_0^T\int_{\mathbb{R}^d}a_{ij}\nabla u_j(y,t)\cdot\psi(y,t)dydt\right| \\
&\quad\le\left|\int_0^T\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}V_{ij}^\eta(x-y)\psi(x,t)dx-a_{ij}\psi(y,t)\right)\cdot\nabla u_{\eta,j}(y,t)dydt\right| \\
&\qquad+|a_{ij}|\left|\int_0^T\int_{\mathbb{R}^d}(\nabla u_{\eta,j}-\nabla u_j)(y,t)\cdot\psi(y,t)dydt\right| \\
&\quad\le\|V_{ij}^\eta*\psi-a_{ij}\psi\|_{L^2(0,T;L^2(\mathbb{R}^d))}\|\nabla u_{\eta,j}\|_{L^2(0,T;L^2(\mathbb{R}^d))} \\
&\qquad+|a_{ij}|\left|\int_0^T\int_{\mathbb{R}^d}(\nabla u_{\eta,j}-\nabla u_j)(y,t)\cdot\psi(y,t)dydt\right|\to 0,
\end{aligned}$$

which proves the claim. Estimate (8) and the embedding $H^s(\mathbb{R}^d)\hookrightarrow L^\infty(\mathbb{R}^d)$ show that

$$\big\|u_{\eta,i}V_{ij}^\eta*\nabla u_{\eta,j}\big\|_{L^2(0,T;L^2)}\le\|u_{\eta,i}\|_{L^\infty(0,T;L^\infty)}\|V_{ij}^\eta\|_{L^1}\|\nabla u_{\eta,j}\|_{L^2(0,T;L^2)}\le C \tag{16}$$

and consequently, for any $T > 0$,

$$\|\partial_t u_{\eta,i}\|_{L^2(0,T;H^{-1})} \le \sigma_i \|\nabla u_{\eta,i}\|_{L^2(0,T;L^2)} + \sum_{j=1}^n \big\|u_{\eta,i} V_{ij}^\eta * \nabla u_{\eta,j}\big\|_{L^2(0,T;L^2)} \le C.$$

The weak formulation of (4) reads as

$$\int_0^T \langle \partial_t u_{\eta,i}, \phi\rangle \zeta(t)dt = \int_0^T \int_{\mathbb{R}^d} \bigg(\sigma_i \nabla u_{\eta,i} + \sum_{j=1}^n u_{\eta,i} \nabla V_{ij}^\eta * \nabla u_{\eta,j}\bigg) \cdot \nabla\phi dx \zeta(t) dt, \tag{17}$$

where $\phi \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp}(\phi) \subset B_R(0)$ and $\zeta \in C^\infty([0,T])$. Since the ball $B_R(0)$ is bounded and the embedding $H^1(B_R(0)) \hookrightarrow L^2(B_R(0))$ is compact, the Aubin-Lions lemma [27] gives the existence of a subsequence of (u_η), which is not relabeled, such that $u_\eta \to u$ strongly in $L^2(0,T;L^2(B_R(0)))$ as $\eta \to 0$, and the limit coincides with the weak limit in (15). We deduce that

$$u_{\eta,i} V_{ij}^\eta * \nabla u_{\eta,j} \rightharpoonup a_{ij} u_i \nabla u_j \quad \text{weakly in } L^1(0,T;L^1(B_R(0))).$$

Estimate (16) shows that this convergence even holds in $L^2(0,T;L^2(B_R(0)))$. We can perform the limit in (17), which shows that the limit u is a solution to the cross-diffusion problem (1). The uniform estimates (9) follow from (8) using the lower semicontinuity of the norm.

Next, we show the uniqueness of solutions. Let u and v be two solutions to (1) with the same initial data. Taking the difference of the equations satisfied by u_i and v_i, multiplying the resulting equation by $u_i - v_i$, summing from $i = 1, \ldots, n$, integrating over $\mathbb{R}^d$, and using the Cauchy-Schwarz inequality leads to

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |u-v|^2 dx + \sigma \int_{\mathbb{R}^d} |\nabla(u-v)|^2 dx \\ &\le \sum_{i,j=1}^n A_{ij} \|u\|_{H^{s+1}} \|u-v\|_{L^2} \|\nabla(u-v)\|_{L^2} + \sum_{i,j} A_{ij} \|v\|_{H^s} \|u-v\|_{L^2}^2. \end{aligned}$$

In view of estimate (9), this becomes

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |u-v|^2 dx + \frac{\sigma}{2}\int_{\mathbb{R}^d} |\nabla(u-v)|^2 dx \le C(t)\|u-v\|_{L^2}^2,$$

and the constant $C(t) > 0$ is integrable (as it depends on $\|u(t)\|_{H^{s+1}}$). Gronwall's inequality then implies that $(u-v)(t) = 0$ for $t > 0$. □

Lemma 8 (Error estimate)**.** *Let the assumptions of Proposition 2 hold. Let u be the solution to* (1) *and u_η be the solution to* (4). *Then the error estimate* (10) *holds.*

Proof. We take the difference of equations (4) and (1),

$$\partial_t(u_{\eta,i} - u_i) - \sigma_i \Delta(u_{\eta,i} - u_i) = \operatorname{div}\bigg(\sum_{j=1}^n u_{\eta,i} V_{ij}^\eta * \nabla u_{\eta,j} - \sum_{i=1}^n a_{ij} u_i \nabla u_j\bigg)$$

$$= \operatorname{div} \sum_{j=1}^n \Big((u_{\eta,i} - u_i) V_{ij}^\eta * \nabla u_{\eta,j} + u_i (V_{ij}^\eta * \nabla u_{\eta,j} - a_{ij} \nabla u_{\eta,j}) + a_{ij} u_i \nabla (u_{\eta,j} - u_j) \Big).$$

Multiplying this equation by $u_{\eta,i} - u_i$, summing from $i = 1, \ldots, n$, integrating over $\mathbb{R}^d$, using the Cauchy-Schwarz inequality, and the estimate $|a_{ij}| \le A_{ij}$, we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |u_\eta - u|^2 dx + \sigma \int_{\mathbb{R}^d} |\nabla (u_\eta - u)|^2 dx \\
&\quad \le \sum_{i,j=1}^n \|u_{\eta,i} - u_i\|_{L^2} \|V_{ij}^\eta * \nabla u_{\eta,j}\|_{L^\infty} \|\nabla (u_{\eta,i} - u_i)\|_{L^2} \\
&\qquad + \sum_{i,j=1}^n \|u_i\|_{L^\infty} \|V_{ij}^\eta * \nabla u_{\eta,j} - a_{ij} \nabla u_{\eta,j}\|_{L^2} \|\nabla (u_{\eta,j} - u_j)\|_{L^2} \\
&\qquad + \sum_{i,j=1}^n A_{ij} \|u_i\|_{L^\infty} \|\nabla (u_{\eta,j} - u_j)\|_{L^2} \|\nabla (u_{\eta,i} - u_i)\|_{L^2} \\
&\quad = I_1 + I_2 + I_3.
\end{aligned}$$

We estimate the right-hand side term by term. First, by the continuous embedding $H^s(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$,

$$\begin{aligned}
|I_1| &\le \sum_{i,j=1}^n \|u_{\eta,i} - u_i\|_{L^2} \|V_{ij}^\eta\|_{L^1} \|\nabla u_{\eta,j}\|_{L^\infty} \|\nabla (u_{\eta,i} - u_i)\|_{L^2} \\
&\le C(\gamma) \bigg(\sum_{i,j=1}^n A_{ij} \|\nabla u_\eta\|_{H^s} \bigg)^2 \|u_\eta - u\|_{L^2}^2 + \frac{\gamma}{4} \|\nabla (u_\eta - u)\|_{L^2}^2.
\end{aligned}$$

To estimate I_2, let $g \in L^2(\mathbb{R}^d; \mathbb{R}^n)$. Since $\operatorname{supp} V_{ij}^\eta \subset B_\eta(0)$, the mean-value theorem shows that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} \big(V_{ij}^\eta * \nabla u_{\eta,j} - a_{ij} \nabla u_{\eta,j} \big)(x) \cdot g(x) dx \right| \\
&\quad = \left| \int_{\mathbb{R}^d} \int_{B_\eta(0)} V_{ij}^\eta(y) \sum_{k=1}^d \left(\frac{\partial u_{\eta,j}}{\partial x_k}(x-y) - \frac{\partial u_{\eta,j}}{\partial x_k}(x) \right) g_k(x) dy dx \right| \\
&\quad = \left| \int_{\mathbb{R}^d} \int_{B_\eta(0)} V_{ij}^\eta(y) \bigg(\int_0^1 \sum_{k,\ell=1}^d \frac{\partial^2 u_{\eta,j}}{\partial x_k \partial x_\ell}(x - ry) y_\ell dr \bigg) g_k(x) dy dx \right| \\
&\quad \le \eta \int_0^1 \int_{B_\eta(0)} |V_{ij}^\eta(y)| \int_{\mathbb{R}^d} |D^2 u_{\eta,j}(x - ry)| \, |g(x)| dx dy dr \\
&\quad \le \eta \int_0^1 \int_{B_\eta(0)} |V_{ij}^\eta(y)| \, \|D^2 u_{\eta,j}(\cdot - ry)\|_{L^2} \|g\|_{L^2} dy dr \\
&\quad \le \eta \|V_{ij}^\eta\|_{L^1} \|D^2 u_{\eta,j}\|_{L^2} \|g\|_{L^2} = \eta A_{ij} \|D^2 u_{\eta,j}\|_{L^2} \|g\|_{L^2}.
\end{aligned}$$

This shows that

$$\|V_{ij}^\eta * \nabla u_{\eta,j} - a_{ij}\nabla u_{\eta,j}\|_{L^2} \le \eta C\|D^2 u_{\eta,j}\|_{L^2} \le \eta C \tag{18}$$

and consequently,

$$\begin{aligned}|I_2| &\le C(\gamma)\sum_{i,j=1}^n \|V_{ij}^\eta * \nabla u_{\eta,j} - a_{ij}\nabla u_{\eta,j}\|_{L^2}^2 + \frac{\gamma}{4}\|\nabla(u_\eta - u)\|_{L^2}^2 \\ &\le C(\gamma)\eta^2 + \frac{\gamma}{4}\|\nabla(u_\eta - u)\|_{L^2}^2.\end{aligned}$$

Finally, by Assumption (7),

$$|I_3| \le \bigg(\sum_{i,j=1}^n A_{ij}\|u\|_{H^{s+1}}\bigg)\|\nabla(u_\eta - u)\|_{L^2}^2 \le (\sigma-\gamma)\|\nabla(u_\eta - u)\|_{L^2}^2.$$

Therefore,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\int_{\mathbb{R}^d}|u_\eta - u|^2dx + \frac{\gamma}{2}\int_{\mathbb{R}^d}|\nabla(u_\eta - u)|^2dx \\ &\le C(\gamma)\bigg(\sum_{i,j=1}^n A_{ij}\|\nabla u_\eta\|_{H^s}\bigg)^2\|u_\eta - u\|_{L^2}^2 + C(\gamma)\eta^2 \\ &\le C(\gamma)(\sigma-\gamma)^2\|u_\eta - u\|_{L^2}^2 + C(\gamma)\eta^2,\end{aligned}$$

and Gronwall's lemma gives the conclusion. □

4. Existence of solutions to the stochastic systems

We prove the solvability of the stochastic ordinary differential systems (2), (3), and (5).

Lemma 9 (Solvability of the stochastic many-particle system)**.** *For any fixed $\eta > 0$, problem* (2) *has a pathwise unique strong solution $X_{\eta,i}^{k,N}$ that is $\mathcal{F}_t$-adapted.*

Proof. By assumption, the gradient ∇V_{ij}^η is bounded and Lipschitz continuous. Then [23, Theorem 5.2.1] or [24, Theorem 3.1.1] show that there exists a (up to $\mathbb{P}$-indistinguishability) pathwise unique strong solution to (2). □

Lemma 10 (Solvability of the nonlocal stochastic system)**.** *Let u_η be a solution to the nonlocal diffusion system* (4) *satisfying $|\nabla u_\eta| \in L^\infty(0,\infty;W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^n))$. Then problem* (3) *has a pathwise unique strong solution $\bar{X}_{\eta,i}^k$ with probability density function $u_{\eta,i}$.*

Remark 11. We have shown in Proposition 2 that if $u^0 \in H^s(\mathbb{R}^d;\mathbb{R}^n)$ and the smallness condition (6) holds, there exists a unique solution $u_\eta \in L^\infty(0,\infty;H^s(\mathbb{R}^d;\mathbb{R}^n))$. The regularity for u_η, required in Lemma 10, is fulfilled if $s > d/2+2$, thanks to the embedding $u_{\eta,i} \in L^\infty(0,\infty;H^s(\mathbb{R}^d)) \hookrightarrow L^\infty(0,\infty;W^{2,\infty}(\mathbb{R}^d))$ for $i = 1,\ldots,n$. □

Proof of Lemma 10. We proceed as in the proof of Lemma 3.2 of [4]. Let v be a solution to (4) satisfying $v(0) = u^0$ in $\mathbb{R}^d$, where u_i^0 is the density of ξ_i^k. By assumption, $\nabla V_{ij}^\eta * v_j = V_{ij}^\eta * \nabla v_j$ is bounded and Lipschitz continuous. Therefore,

$$d\bar{X}_{\eta,i}^k = -\sum_{j=1}^n (\nabla V_{ij}^\eta * v_j)(\bar{X}_{\eta,i}(t),t)dt + \sqrt{2\sigma_i}dW_i^k(t), \quad \bar{X}_{\eta,i}^k(0) = \xi_i^k,$$

has a pathwise unique strong solution $\bar{X}_{\eta,i}^k$. Let $u_{\eta,i}$ be the probability density function of $\bar{X}_{\eta,i}^k$ and let ϕ_i be a smooth test function. Then Itô's lemma implies that

$$\begin{aligned}
&\phi_i(\bar{X}_{\eta,i}^k(t),t) - \phi_i(\xi_i^k,0) \\
&\quad = \int_0^t \partial_s\phi_i(\bar{X}_{\eta,i}^k(s),s)ds - \sum_{j=1}^n \int_0^t (V_{ij}^\eta * \nabla v_j)(\bar{X}_{\eta,i}^k(s),s)\cdot\nabla\phi_i(\bar{X}_{\eta,i}^k(s),s)ds \\
&\qquad + \sigma_i\int_0^t \Delta\phi_i(\bar{X}_{\eta,i}^k(s),s)ds + \sqrt{2\sigma_i}\int_0^t \nabla\phi_i(\bar{X}_{\eta,i}^k(s),s)dW_i^k(s).
\end{aligned}$$

Applying the expectation

$$\mathbb{E}\big(\phi_i(\bar{X}_{\eta,i}^k(t),t)\big) = \int_{\mathbb{R}^d}\phi_i(x,t)u_{\eta,i}(x,t)dx$$

to the previous expression yields

$$\begin{aligned}
\int_{\mathbb{R}^d}\phi_i(x,t)u_{\eta,i}(x,t)dx &= \int_{\mathbb{R}^d}\phi_i(x,0)u_{\eta,i}(x,0)dx \\
&\quad + \int_0^t\int_{\mathbb{R}^d}\big(\partial_s\phi_i(x,s) + \sigma_i\Delta\phi_i(x,s)\big)u_{i,\eta}(x,s)dxds \\
&\quad - \sum_{j=1}^n\int_0^t\int_{\mathbb{R}^d}\nabla\phi_i(x,s)\cdot(V_{ij}^\eta * \nabla v_j)(x,s)u_{\eta,i}(x,s)dxds.
\end{aligned}$$

This is the weak formulation of

$$\partial_t u_{\eta,i} - \sigma_i\Delta u_{\eta,i} = \operatorname{div}\left(\sum_{j=1}^n u_{\eta,i}V_{ij}^\eta * \nabla v_j\right), \quad u_{\eta,i}(0) = u^0.$$

The unique solvability of problem (4) implies that the solution is $u_{\eta,i}$, and we obtain $v = u_\eta$. This finishes the proof. □

By the same technique, the solvability of the limiting stochastic system can be proved.

Lemma 12 (Solvability of the limiting stochastic system)**.** *Let u be the unique solution to problem* (1) *satisfying $|\nabla u| \in L^\infty(0,\infty;W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^n))$. Then there exists a pathwise unique strong solution $\widehat{X}_i^k$ with probability density function u_i.*

5. Proof of Theorem 3

First, we show an estimate for the difference $X^{k,N}_{\eta,i} - \bar{X}^k_{\eta,i}$.

Lemma 13. *Let the assumptions of Theorem 3 hold. Then, for any $t > 0$,*

$$\mathbb{E}\bigg(\sum_{i=1}^n \sup_{0<s<t} \sup_{k=1,\ldots,N} \big| X^{k,N}_{\eta,i}(s) - \bar{X}^k_{\eta,i}(s) \big|^2 \bigg) \le \frac{C(t)}{N^{1-C(t)\varepsilon}},$$

where the constant $C(t)$ depends on t, n, $\|D^2 V_{ij}\|_{L^\infty}$, and the initial datum u^0.

Proof. We set

$$S_t = \sum_{i=1}^n \sup_{0<s<t} \sup_{k=1,\ldots,N} \big| X^{k,N}_{\eta,i}(s) - \bar{X}^k_{\eta,i}(s) \big|^2.$$

The difference of equations (2) and (3), satisfied by $X^{k,N}_{\eta,i}$ and $\bar{X}^k_{\eta,i}$, respectively, equals

$$\begin{aligned} &X^{k,N}_{\eta,i}(t) - \bar{X}^k_{\eta,i}(t) \\ &\quad = -\int_0^t \sum_{j=1}^n \frac{1}{N} \sum_{\ell=1}^N \Big(\nabla V^\eta_{ij}\big(X^{k,N}_{\eta,i}(s) - X^{\ell,N}_{\eta,j}(s) \big) - (\nabla V^\eta_{ij} * u_{\eta,j})(\bar{X}^k_{\eta,i}(s), s) \Big) ds \end{aligned}$$

and thus,

$$\begin{aligned} \sum_{i=1}^n \sup_{k=1,\ldots,N} \big| X^{k,N}_{\eta,i}(s) - \bar{X}^\ell_{\eta,i}(s) \big|^2 &\le \sum_{i=1}^n \bigg(\int_0^t \frac{1}{N} \sum_{j=1}^n \sup_{k=1,\ldots,N} \\ &\quad \times \bigg| \sum_{\ell=1}^N \Big(\nabla V^\eta_{ij}\big(X^{k,N}_{\eta,i}(s) - X^{\ell,N}_{\eta,j}(s) \big) - (\nabla V^\eta_{ij} * u_{\eta,j})(\bar{X}^k_{\eta,i}(s), s) \Big) \bigg| ds \bigg)^2. \end{aligned}$$

Taking the supremum in $(0, t)$ and the expectation and using the Cauchy-Schwarz inequality with respect to t yields

$$\begin{aligned} \mathbb{E}(S_t) &\le \sum_{i=1}^n \frac{t}{N^2} \int_0^t \mathbb{E}\bigg(\sum_{j=1}^n \sup_{k=1,\ldots,N} \\ &\quad \times \bigg| \sum_{\ell=1}^N \Big(\nabla V^\eta_{ij}\big(X^{k,N}_{\eta,i}(s) - X^{\ell,N}_{\eta,j}(s) \big) - (\nabla V^\eta_{ij} * u_{\eta,j})(\bar{X}^k_{\eta,i}(s), s) \Big) \bigg| ds \bigg)^2 \\ &\le \sum_{i=1}^n \frac{t}{N^2} \int_0^t \bigg\{ \mathbb{E} \sum_{j=1}^n \sup_{k=1,\ldots,N} \\ &\quad \times \bigg| \sum_{\ell=1}^N \Big(\nabla V^\eta_{ij}\big(X^{k,N}_{\eta,i}(s) - X^{\ell,N}_{\eta,j}(s) \big) - \nabla V^\eta_{ij}\big(X^{k,N}_{\eta,i}(s) - \bar{X}^\ell_{\eta,j}(s) \big) \Big) \bigg|^2 \\ &\quad + \mathbb{E} \sum_{j=1}^n \sup_{k=1,\ldots,N} \bigg| \sum_{\ell=1}^N \Big(\nabla V^\eta_{ij}\big(X^{k,N}_{\eta,i}(s) - \bar{X}^\ell_{\eta,j}(s) \big) - \nabla V^\eta_{ij}\big(\bar{X}^k_{\eta,i}(s) - \bar{X}^\ell_{\eta,j}(s) \big) \Big) \bigg|^2 \end{aligned}$$

$$+\mathbb{E}\sum_{j=1}^n \sup_{k=1,\ldots,N}\bigg|\sum_{\ell=1}^N\Big(\nabla V_{ij}^\eta\big(\bar X_{\eta,i}^k(s)-\bar X_{\eta,j}^\ell(s)\big)-(\nabla V_{ij}^\eta * u_{\eta,j})(\bar X_{\eta,i}^k(s),s)\Big)\bigg|^2\bigg\}ds$$
$$=: J_1+J_2+J_3.$$

We estimate the terms J_1, J_2, and J_3 separately.

Let L_{ij}^η be the Lipschitz constant of ∇V_{ij}^η. Because of $V_{ij}^\eta(z)=\eta^{-d}V_{ij}(|z|/\eta)$, we obtain $L_{ij}^\eta=\eta^{-d-2}\|D^2V_{ij}\|_{L^\infty}$. Hence,

$$\begin{aligned}
|J_1| &\le \sum_{i=1}^n \frac{t}{N^2}\int_0^t \mathbb{E}\bigg(\sum_{j=1}^n (L_{ij}^\eta)^2\bigg(\sum_{\ell=1}^N \big|X_{\eta,j}^{\ell,N}(s)-\bar X_{\eta,j}^\ell(s)\big|\bigg)^2\bigg)ds \\
&\le t\sum_{i,j=1}^n (L_{ij}^\eta)^2\int_0^t \mathbb{E}\Big(\sup_{\ell=1,\ldots,N}\big|X_{\eta,j}^{\ell,N}(s)-\bar X_{\eta,j}^\ell(s)\big|^2\Big)ds \\
&\le \frac{tn}{\eta^{2d+4}}\sup_{i,j=1,\ldots,n}\|D^2V_{ij}\|_{L^\infty}^2\int_0^t \mathbb{E}(S_s)ds.
\end{aligned}$$

Furthermore, by similar arguments,

$$\begin{aligned}
|J_2| &\le \sum_{i=1}^n \frac{t}{N^2}\int_0^t \mathbb{E}\bigg(\sum_{j=1}^n (L_{ij}^\eta)^2\sup_{k=1,\ldots,N}\bigg(\sum_{\ell=1}^N \big|X_{\eta,i}^{k,N}(s)-\bar X_{\eta,j}^k(s)\big|\bigg)^2\bigg)ds \\
&\le t\sum_{i,j=1}^n (L_{ij}^\eta)^2\int_0^t \mathbb{E}\Big(\sup_{k=1,\ldots,N}\big|X_{\eta,i}^{k,N}(s)-\bar X_{\eta,j}^k(s)\big|^2\Big)ds \\
&\le \frac{tn}{\eta^{2d+4}}\sup_{i,j=1,\ldots,n}\|D^2V_{ij}\|_{L^\infty}^2\int_0^t \mathbb{E}(S_s)ds.
\end{aligned}$$

For the third term, we set

$$Z_{i,j}^{k,\ell}(s) := \nabla V_{ij}^\eta\big(\bar X_{\eta,i}^k(s)-\bar X_{\eta,j}^\ell(s)\big)-(\nabla V_{ij}^\eta * u_{\eta,j})(\bar X_{\eta,i}^k(s),s),$$

write the square as a product of two sums, and use the independence of $Z_{i,j}^{k,1},\ldots,Z_{i,j}^{k,N}$:

$$\begin{aligned}
|J_3| &= \sum_{i=1}^n \frac{t}{N^2}\int_0^t \mathbb{E}\sum_{j=1}^n\sup_{k=1,\ldots,N}\bigg|\sum_{\ell=1}^N Z_{i,j}^{k,\ell}(s)\bigg|^2 ds \\
&= \sum_{i,j=1}^n \frac{t}{N^2}\int_0^t \mathbb{E}\bigg(\sup_{k=1,\ldots,N}\sum_{\ell=1}^N Z_{i,j}^{k,\ell}(s)\cdot\sum_{m=1}^N Z_{i,j}^{k,m}(s)\bigg)ds \\
&\le \sum_{i,j=1}^n \frac{t}{N^2}\sum_{\ell,m=1}^N\int_0^t \sup_{k=1,\ldots,N}\mathbb{E}\Big(Z_{i,j}^{k,\ell}(s)\cdot Z_{i,j}^{k,m}(s)\Big)ds \\
&= \sum_{i,j=1}^n \frac{t}{N^2}\sum_{\ell=1}^N\int_0^t \sup_{k=1,\ldots,N}\mathbb{E}\big|Z_{i,j}^{k,\ell}(s)\big|^2 ds
\end{aligned}$$

$$+\sum_{i,j=1}^n \frac{t}{N^2}\sum_{\ell\neq m}\int_0^t \sup_{k=1,\ldots,N}\mathbb{E}\big(Z_{i,j}^{k,\ell}(s)\big)\mathbb{E}\big(Z_{i,j}^{k,m}(s)\big)ds.$$

We claim that the expectation of $Z_{i,j}^{k,\ell}$ vanishes. Indeed, since $\bar{X}_{\eta,i}^k$ and $\bar{X}_{\eta,j}^\ell$ are independent with distribution functions $u_{\eta,i}$ and $u_{\eta,j}$, the joint distribution is $u_{\eta,i}\otimes u_{\eta,j}$. This gives

$$\begin{aligned}\mathbb{E}\big(Z_{ij}^{k,\ell}(s)\big) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big(\nabla V_{ij}^\eta(\xi-x)-(\nabla V_{ij}^\eta * u_{\eta,j})(\xi,s)\big)u_{\eta,i}(\xi,s)u_{\eta,j}(x,s)dxd\xi \\ &= \int_{\mathbb{R}^d}u_{\eta,i}(\xi,s)\bigg(\int_{\mathbb{R}^d}\nabla V_{ij}^\eta(\xi-x)u_{\eta,j}(x,s)dx-(\nabla V_{ij}^\eta * u_{\eta,j})(\xi,s)\bigg)d\xi = 0.\end{aligned}$$

Therefore, using the estimates $|\nabla V_{ij}^\eta|\le C\eta^{-d-1}$, $\|\nabla V_{ij}^\eta\|_{L^2}\le C\eta^{-1}$, and consequently

$$\|\nabla V_{ij}^\eta * u_{\eta,j}\|_{L^\infty}\le\|\nabla V_{ij}^\eta\|_{L^2}\|u_{\eta,j}\|_{L^2}\le C\eta^{-1},$$

we deduce that

$$|J_3| = \sum_{i,j=1}^n\frac{t}{N^2}\sum_{\ell=1}^N\int_0^t\sup_{k=1,\ldots,N}\mathbb{E}\big|Z_{i,j}^{k,\ell}(s)\big|^2ds\le\frac{n^2}{N}\frac{Ct^2}{\eta^{2d+2}}.$$

Summarizing these estimations, we conclude that

$$\mathbb{E}(S_t)\le\frac{tn}{\eta^{2d+4}}\sup_{i,j=1,\ldots,n}\|D^2V_{ij}\|_{L^\infty}^2\int_0^t\mathbb{E}(S_s)ds+\frac{n^2}{N}\frac{Ct^2}{\eta^{2d+2}},$$

and, by Gronwall's inequality,

$$\mathbb{E}(S_t)\le\frac{Ct^2}{N\eta^{2d+2}}\exp\bigg(\frac{Ct}{\eta^{2d+4}}\bigg),\quad t>0.$$

For fixed $\varepsilon\in(0,1)$ and $\eta\in(0,1)$, we choose $N\in\mathbb{N}$ such that $\varepsilon\log N\ge\eta^{-2d-4}$. Using $\eta^{-2d-2}\le\eta^{-2d-4}\le\exp(C\eta^{-2d-4})$ for $C\ge1$, we obtain

$$\begin{aligned}\mathbb{E}(S_t) &\le\frac{C}{N\eta^{2d+2}}\exp\bigg(\frac{Ct}{\eta^{2d+4}}\bigg)\le\frac{C}{N}\exp\bigg(\frac{C(1+t)}{\eta^{2d+4}}\bigg)\\ &\le\frac{C}{N}\exp\big(C\varepsilon(1+t)\log N\big)\le CN^{-1+C\varepsilon(1+t)}.\end{aligned}$$

This proves the result. ☐

Next, we prove an estimate for the difference $\bar{X}_{\eta,i}^k-\widehat{X}_i^k$.

Lemma 14. *Let the assumptions of Theorem 3 hold and let $s>d/2+2$, $t>0$. Then*

$$\mathbb{E}\bigg(\sum_{i=1}^n\sup_{0<s<t}\sup_{k=1,\ldots,N}\big|\bar{X}_{\eta,i}^k(s)-\widehat{X}_i^k(s)\big|\bigg)\le C(t)\eta.$$

Proof. We use similar arguments as in the proof of Lemma 13. Taking the difference of equations (3) and (5), satisfied by $\bar{X}^k_{\eta,i}$ and $\widehat{X}^k_i$, respectively, and setting

$$U_t = \sum_{i=1}^n \sup_{0<s<t} \sup_{k=1,\ldots,N} \big|\bar{X}^k_{\eta,i}(s) - \widehat{X}^k_i(s)\big|,$$

if follows that

$$\begin{aligned}
\mathbb{E}(U_t) &\le \sum_{i=1}^n \int_0^t \mathbb{E}\bigg(\sum_{j=1}^n \sup_{k=1,\ldots,N} \Big|a_{ij}\nabla u_j(\widehat{X}^k_i(s),s) - (V^\eta_{ij} * \nabla u_{\eta,j})(\bar{X}^k_{\eta,i}(s),s)\Big|\bigg)ds \\
&\le \sum_{i=1}^n \int_0^t \mathbb{E}\bigg(\sum_{j=1}^n \sup_{k=1,\ldots,N} \Big|a_{ij}\nabla u_j(\widehat{X}^k_i(s),s) - (V^\eta_{ij} * \nabla u_j)(\widehat{X}^k_i(s),s)\Big|\bigg)ds \\
&\quad + \sum_{i=1}^n \int_0^t \mathbb{E}\bigg(\sum_{j=1}^n \sup_{k=1,\ldots,N} \Big|(V^\eta_{ij} * \nabla u_j)(\widehat{X}^k_{\eta,i}(s),s) - (V^\eta_{ij} * \nabla u_{\eta,j})(\widehat{X}^k_i(s),s)\Big|\bigg)ds \\
&\quad + \sum_{i=1}^n \int_0^t \mathbb{E}\bigg(\sum_{j=1}^n \sup_{k=1,\ldots,N} \Big|(V^\eta_{ij} * \nabla u_{\eta,j})(\widehat{X}^k_i(s),s) - (V^\eta_{ij} * \nabla u_{\eta,j})(\bar{X}^k_{\eta,i}(s),s)\Big|\bigg)ds \\
&=: K_1 + K_2 + K_3.
\end{aligned}$$

Using (18), the first two terms on the right-hand side are estimated according to

$$\begin{aligned}
K_1 &\le \sum_{i,j=1}^n \int_0^t \int_{\mathbb{R}^d} \big|\big(a_{ij}\nabla u_j(x,s) - (V^\eta_{ij} * \nabla u_j)(x,s)\big)u_i(x,s)\big| dxds \\
&\le n^2 \max_{i,j=1,\ldots,n} \big\|a_{ij}\nabla u_j - V^\eta_{ij} * \nabla u_j\big\|_{L^2(0,t;L^2)} \|u_i\|_{L^2(0,t;L^2)} \\
&\le C(n)\eta \|D^2 u\|_{L^2(0,t;L^2)} \le C\eta, \\
K_2 &\le \sum_{i,j=1}^n \int_0^t \int_{\mathbb{R}^d} \big|V^\eta_{ij} * \nabla(u_j - u_{\eta,j})u_i\big| dxds \\
&\le n^2 \max_{i,j=1,\ldots,n} \|V^\eta_{ij}\|_{L^1} \|\nabla(u_j - u_{\eta,j})\|_{L^2(0,t;L^2)} \|u_i\|_{L^2(0,t;L^2)} \le C\eta,
\end{aligned}$$

where we have used Lemma 16 (ii) and the error estimate from Lemma 13. Finally, the term K_3 can be controlled by

$$\begin{aligned}
K_3 &\le \sum_{i,j=1}^n \|\nabla(V^\eta_{ij,} * \nabla u_{\eta,j})\|_{L^\infty} \int_0^t \mathbb{E}(U_s)ds \\
&\le n^2 \max_{i,j=1,\ldots,n} \|V^\eta_{ij}\|_{L^1} \|D^2 u_\eta\|_{L^\infty} \int_0^t \mathbb{E}(U_s)ds \le C \int_0^t \mathbb{E}(U_s)ds.
\end{aligned}$$

We need the assumption $s > d/2 + 2$ for the continuous embedding $H^s(\mathbb{R}^d) \hookrightarrow W^{2,\infty}(\mathbb{R}^d)$, which allows us to estimate $D^2 u_\eta$ in $L^\infty(\mathbb{R}^d)$. This shows that

$$\mathbb{E}(U_t) \le C\eta + C\int_0^t \mathbb{E}(U_s)ds,$$

and Gronwall's inequality yields $\mathbb{E}(U_t) \le C(t)\eta$ for $t > 0$. The statement of the lemma follows after taking the supremum over $t > 0$. □

Lemmas 13 and 14 imply Theorem 3. Indeed, it follows that

$$\begin{aligned}
&\mathbb{E}\bigg(\sum_{i=1}^n \sup_{0<s<t}\sup_{k=1,\ldots,N}\big|X_{\eta,i}^{k,N}(s) - \widehat{X}_i^k(s)\big|\bigg) \\
&\le \mathbb{E}\bigg(\sum_{i=1}^n \sup_{0<s<t}\sup_{k=1,\ldots,N}\big|X_{\eta,i}^{k,N}(s) - \bar{X}_{\eta,i}^k(s)\big|\bigg) + \mathbb{E}\bigg(\sum_{i=1}^n \sup_{0<s<t}\sup_{k=1,\ldots,N}\big|\bar{X}_{\eta,i}^k(s) - \widehat{X}_i^k(s)\big|\bigg) \\
&\le CN^{(-1+C(t)\varepsilon)/2} + C(t)\eta \le C(t)\eta,
\end{aligned}$$

since the choice $\log N \ge 1/(\varepsilon\eta^{2d+4})$ is equivalent to

$$N^{(-1+C(t)\varepsilon)/2} \le \exp\bigg(\frac{1}{2\varepsilon}(-1+C(t)\varepsilon)\eta^{-2d-4}\bigg),$$

and the right-hand side is smaller than η possibly times a constant $C(t)$.

Appendix A. Some auxiliary results

We recall some auxiliary results.

Lemma 15 (Moser-type calculus inequality; Prop. 2.1 in [19])**.** *Let f, $g \in H^s(\mathbb{R}^d)$ with $s > d/2 + 1$ and let $\alpha \in \mathbb{N}^d$ be a multi-index of order $|\alpha| \le s$. Then, for some constant $C_M > 0$ only depending on s and d,*

$$\|D^\alpha(fg)\|_{L^2} \le C_M\big(\|f\|_{L^\infty}\|D^s g\|_{L^2} + \|g\|_{L^\infty}\|D^s f\|_{L^2}\big).$$

Lemma 16 (Young's convolution inequality; formula (7) on page 107 in [18])**.** (i) *Let $1 \le p, q \le \infty$ satisfying $1/p + 1/q = 1/r + 1$ and $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$. Then*

$$\|f * g\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q}.$$

(ii) *Let p, q, $r \ge 1$ satisfying $1/p + 1/q + 1/r = 2$ and $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, $h \in L^r(\mathbb{R}^d)$. Then, for some constant $C > 0$ only depending on p, q, r, and d,*

$$\bigg|\int_{\mathbb{R}^d}(f * g)h\,dx\bigg| \le C\|f\|_{L^p}\|g\|_{L^q}\|h\|_{L^r}.$$

Lemma 17 (Gronwall-type inequality)**.** *Let a, $b > 0$, $g \in C^0([0,T])$ with $g(t) \ge 0$ for $t \in [0,T]$ and $f : [0,T] \to [0,\infty)$ be absolutely continuous such that*

$$f'(t) \le -g(t)\big(a - b\sqrt{f(t)}\big) \quad \textit{for } t > 0$$

and $0 < f(0) \le (a/b)^2$. Then $f(t) \le (a/b)^2$ for all $t \in [0,T]$.

We present a proof of this lemma since we could not find a reference in the literature.

Proof. First, let $f(0) < (a/b)^2$. We claim that $f(t) < (a/b)^2$ for $t \in [0,T]$. Assume that there exists $t_0 \in [0,T]$ such that $f(t_0) \geq (a/b)^2$. By continuity, there exists $t_1 \in [0,t_0]$ such that $f(t_1) < (a/b)^2$ and $f'(t_1) > 0$. This leads to the contradiction $0 < f'(t_1) \leq -g(t_1)(a - b\sqrt{f(t_1)}) \leq 0$, proving the claim.

Since $f(t) < (a/b)^2$ for $t \in [0,T]$, the differential inequality can be written as

$$\frac{f'(t)}{a - b\sqrt{f(t)}} \leq -g(t), \quad t \in [0,T]. \tag{19}$$

Introduce

$$F(s) = \int_0^s \frac{d\sigma}{a - b\sqrt{\sigma}} = \int_0^{\sqrt{s}} \frac{2\tau d\tau}{a - b\tau} = -\frac{2}{b^2}\big(b\sqrt{s} + a\log(a - b\sqrt{s}) - a\ln a\big).$$

Then integrating (19) over $(0,t)$ leads to

$$F(f(t)) - F(f(0)) \leq -\int_0^t g(\tau)d\tau,$$

which, after a computation, is equivalent to

$$a - b\sqrt{f(t)} \geq \big(a - b\sqrt{f(0)}\big)\exp\left(\frac{b}{a}\sqrt{f(0)} - \frac{b}{a}\sqrt{f(t)} + \frac{b^2}{2a}\int_0^t g(\tau)d\tau\right).$$

Finally, we choose a sequence of initial data $f_0^\delta < (a/b)^2$ such that $f_0^\delta \to f_0 \leq (a/b)^2$ as $\delta \to 0$. To each f_0^δ, we associate a function f^δ satisfying the differential inequality. The proof shows that $f^\delta(t) < (a/b)^2$. In the limit $\delta \to 0$, this reduces to $f(t) \leq (a/b)^2$, where $f(t) = \lim_{\delta\to 0} f^\delta(t)$ for $t \in [0,T]$. ☐

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