Modeling of herding and wealth distribution in large markets

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Abstract The dynamics of the number of participants in a large market is described by nonlinear partial differential equations of kinetic and diffusive type. The results on the modeling, analysis, and numerical simulation of three market models are briefly reviewed. The interplay of the agents with external sources, herding phenomena, and irrationality of the individuals as well as the exchange of knowledge and wealth is explored mathematically. The focus lies on the mathematical understanding of the differential equations rather than on the modeling of real economic situations, aiming at identifying models which are able to produce the desired effects.

0.1 Introduction

The modeling of markets with a large number of agents became very vital in recent years with the aim to understand inefficient markets or irrational behavior of agents, for instance. The dynamics of such markets may be described by agent-based models, kinetic equations, or diffusive systems. Agent-based models specify the behavior of individuals by using elements of game theory and Monte-Carlo simulation techniques [31]. In kinetic modeling, the analogy with statistical mechanics is exploited: Interactions between market agents are interpreted as collisions between gas particles, and conservation laws for income and/or wealth may hold [26, 28]. Diffusive systems are often derived from kinetic equations in the so-called grazing collision limit, and they illustrate the behavior on a macroscopic level [33]. In this

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section, we summarize the results of [6, 14, 25] on kinetic and diffusive equations modeling socio-economic scenarios.

The first scenario is the herding in financial markets. Herding is characterized by a homogenization of the actions of the market participants, which behave at a certain time in the same way. Herding may lead to strong trends with low volatility of asset prices, but eventually also to abrupt corrections, so it promotes the occurrence of bubbles and crashes. Numerous socio-economic papers [4, 8, 30] and research in biological sciences [1, 19] show that herding interactions play a crucial role in social scenarios. Herding behavior is often irrational because people are not basing their decision on objective criteria.

A full understanding of herding behavior needs the ability to understand two levels: the microscopic one, which considers each individual of the crowd separately; and the macroscopic level, which deals with the group of individuals, i.e. the herd. The first level usually represents the individual as a particle, and a microscopic particle-type or mesoscopic kinetic description may be useful. The latter one may be represented by a density function depending (continuously) on space and time, leading to diffusive equations. We consider a diffusive herding model in Section 0.2 and a kinetic herding equation in Section 0.3.

The second scenario addressed in this review is the distribution of wealth. Most of the models in the literature are agent-based models [9], mean-field games [18], or kinetic equations [27]. In kinetic modeling, binary collisions are replaced by trades between agents by defining rules which specify how wealth is exchanged in trades. The output of the model are the statistics of the wealth distribution in the market. It turns out that in many models, the stationary profile has an overpopulated tail (called fat tail or Pareto tail), which is interpreted as the existence of an upper class of very wealthy people [27]. Pareto tails appear under various assumptions, assuming wealth conservation in the mean or pointwise wealth conservation [15].

Binary wealth exchange models go back to the work [3]. Later, the relation to statistical mechanics was highlighted [24], and strictly conservative exchange models were developed [10]. The strict conservation was relaxed in [12] to conservation in the mean. Our contribution is to combine wealth and knowledge of agents in a society and to examine the interaction of these qualities; see Section 0.4.

We stress the fact that the models that we are proposing and analyzing are quite simple. Certainly, the socio-economic behavior of real market agents is extremely complex and includes psychologic and social phenomena. Still, we believe that a large number of agents may be described to some extent in an averaged sense – at least in simplified situations. Our aim is to understand the mathematical phenomena arising from the new terms in the models rather than devising models that include as many features as possible. Our analysis shows which terms produce the desired effects and henceforth can be included in more realistic models. The hope is that this analysis helps to identify irregularities in (financial) markets or in societies and to lead to improved market regulations and counter-actions to avoid financial crashes.

0.2 A cross-diffusion herding model

A very simple model for herding behavior is given by the cross-diffusion system [25]

$$\partial_t u = \mathrm{div}(\nabla u - g(u)\nabla v), \quad \partial_t v = \mathrm{div}(\delta\nabla u + \kappa\nabla v) + f(u) - \alpha v, \tag{1}$$

where $u(x,t)$ represents the normalized density of individuals with information variable $x \in \Omega$ at time $t \geq 0$ ($\Omega \subset \mathbb{R}^d$ being a bounded domain), and $v(x,t)$ is an influence function which modifies the information state of the individuals. The influence function acts through the cross-diffusion term $g(u)\nabla v$ in the first equation in (1). We assume that the influence becomes weak if the number of individuals at a fixed state x is very low or close to the maximal value $u = 1$. Thus, we suppose that $g(0) = g(1) = 0$. The influence function is modified by diffusive effects also due to the random behavior of the agents with parameter $\delta > 0$, by the nonnegative source term $f(u)$, time relaxation with rate $\alpha > 0$, and diffusion with coefficient $\kappa > 0$. Our aim is to understand whether the above model exhibits herding phenomena, i.e. regions in which the density of the agents is very low or close to the maximal value.

The equations are supplemented by no-flux boundary and initial conditions:

$$\begin{aligned} &(\nabla u - g(u)\nabla v)\cdot\nu = 0, \quad (\delta\nabla u + \kappa\nabla v)\cdot\nu = 0 \quad \text{on } \partial\Omega, \\ &u(\cdot,0) = u_0, \quad v(\cdot,0) = v_0 \quad \text{in } \Omega,\ t > 0, \end{aligned} \tag{2}$$

where ν denotes the exterior unit normal vector to $\partial\Omega$.

If $\delta = 0$, system (1) correspond to a nonlinear chemotaxis Keller-Segel model, where u represents the cell density and v the concentration of the chemoattractant [20]. While the original Keller-Segel model exhibits finite-time blow-up of the solutions, the nonlinear mobility $g(u) = u(1-u)$ prevents blow up [34]. Equations (1) with $\delta > 0$ can be derived from stochastic partial differential equations describing interacting particles, at least for constant mobility functions $g(u)$ [17]. The case $\delta > 0$ and $g(u) = u$ was analyzed in [23]. A typical example in the present situation is $g(u) = u(1-u)$ since this function satisfies $g(0) = g(1) = 0$.

In the work [25], the following results have been obtained.

Existence of solutions. If f and g are smooth, bounded, nonnegative functions such that there exists $m \in (0,1)$ satisfying

$$g(0) = g(1) = 0, \quad \int_0^m \frac{ds}{g(s)} = \int_m^1 \frac{ds}{g(s)} = \infty, \tag{3}$$

and u_0, $v_0 \in L^\infty(\Omega)$, then there exists a global weak solution (u,v) to (1)-(2) satisfying $0 \leq u \leq 1$ in Ω, $t > 0$, as long as $\delta > -\kappa/\gamma$, where $\gamma = \max_{s\in[0,1]} g(s)$. The function $g(u) = u(1-u)$ satisfies (3).

The restriction on δ ensures that the real parts of the eigenvalues of the diffusion matrix from (1) are positive, such that the system is parabolic in the sense of Petrovskii and local existence of solutions can be expected [2]. The challenge is to prove the existence of global (weak) solutions. A key element of the proof is the observation that equations (1) admit a Lyapunov functional (called an entropy),

$$H(u,v) = \int_\Omega \left(h(u) + \frac{v^2}{2\delta_0} \right), \quad \text{where } h(u) = \int_m^s \int_m^\sigma \frac{dt}{g(t)} d\sigma,$$

and $\delta_0 = \delta$ if $\delta > 0$, $\delta_0 = \kappa/\gamma$ if $\delta < 0$. A computation shows that for $\delta > -\kappa/\gamma$, there exists $c_\delta > 0$ such that

$$\frac{dH}{dt} + c_\delta \int_\Omega \left(\frac{|\nabla u|^2}{g(u)} + \frac{|\nabla v|^2}{\delta_0^2} \right) dx \le c,$$

where $c_\delta > 0$ also depends on Ω, f, and g. The gradient estimate is needed to prove the compactness of the fixed-point operator needed to apply the Leray-Schauder fixed-point theorem [25]. The exponential decay of the solutions (in terms of $H(u,v)$) to a constant steady state holds for sufficiently large values of $\delta > 0$. We wish to understand what happens if δ becomes small positive or negative large. This is done via a bifurcation analysis.

Bifurcation analysis. Choosing δ as a bifurcation parameter, we can apply bifurcation theory to show that the stationary solutions bifurcate from the constant steady state (u^*, v^*) for $\delta \neq \delta_d := -\kappa/g(u^*)$. For this result, we employed in [25] the local bifurcation theory of Crandell and Rabinowitz and the global bifurcation theory for nonlinear Fredholm mappings from Shi and Wang [32]. The difficulty here is that (u^*, v^*) is not an isolated bifurcation branch as a function of δ, since fixing any initial mass, there is a family of homogeneous steady states with $u^* = \int_\Omega u(x)dx/\text{meas}(\Omega)$. For the numerical bifurcation analysis, this degeneracy is resolved by introducing a small relaxation term $\rho(u - u^*)$ in the first equation of (1) with very small $\rho > 0$ and by applying a homotopy continuation step to achieve solutions for $\rho = 0$.

Numerically, there exist local bifurcation points on the branch of homogeneous steady states if $\delta < \delta_d$ for sufficiently large α and if $\delta > \delta_d$ for sufficiently small α. The results have been obtained by using the software AUTO; for details we refer to [25]. Here, we only depict one stationary density u in Figure 1, showing that there is indeed a region in which the number of individuals with a certain information state is very small, which indicates some herding phenomenum.

0.3 A kinetic model with irrationality and herding

A second approach to model herding consists in using kinetic equations. We describe the evolution of the distribution function $f(x,w,t)$ of the agents depending on the rationality $x \in \mathbb{R}$ and the estimated asset value $w \in \mathbb{R}_+ := [0,\infty)$, assigned to the asset by an individual. The agent behaves rational when $x > 0$ and irrational when $x < 0$. The time evolution is given by the inhomogeneous kinetic equation

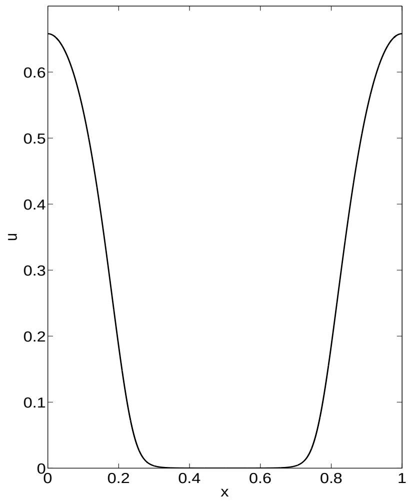


Fig. 1 Stationary density of individuals for the model (1) with parameters $\alpha = 0.001$, $\kappa = 1$, $\delta = 9$, and $\Omega = (0,50)$.

$$\partial_t f + (\Phi(x,w)f)_x = Q_I(f) + Q_H(f,f), \quad (x,w) \in \mathbb{R} \times \mathbb{R}_+,\ t > 0, \tag{4}$$

with the boundary and initial conditions

$$f(x,0,t) = 0,\ f(x,w,0) = f_0(x,w) \quad \text{for } (x,w) \in \mathbb{R} \times \mathbb{R}_+,\ t > 0. \tag{5}$$

The second term in (4) models the irrationality of the agents. When the asset price w lies within a certain range $|w - W| < R$ around a “fair” prize $W > 0$ which is determined by fundamentals, the agents are suppposed to behave more irrational because of psychological biases like overconfidence or limited attention [22]. This is modeled by a negative drift field $\Phi(x,w)$. When the asset value is outside of the “fair” prize region, it is believed to be driven by speculation. The agents will recognize this fact at a certain point and are becoming more rational. Thus, the drift field is positive. An example for such a function is

$$\Phi(x,w) = \begin{cases} -\delta\kappa & \text{if } |w-W| < R, \\ \kappa & \text{if } |w-W| \geq R, \end{cases} \tag{6}$$

where δ and κ are some positive numbers.

The first term on the right-hand side of (4) describes an interaction that is solely based on economic fundamentals, and the second term describes binary interactions of the agents modeling the exchange of information and possibly leading to herding. The precise modeling is as follows.

Public information and herding. Let w be the estimated asset value of an agent before the interaction and w^* the asset value after exchanging information with a public source. Similarly as in [11], the interaction is given by

$$w^* = w - \alpha P(|w-W|)(w-W) + \eta d(w), \tag{7}$$

where $P \in [0,1]$ measures the compromise propensity, $\alpha > 0$ measures the strength of this effect, η is a random variable with distribution μ with variance σ_I^2 and zero mean taking values in $\mathbb{R}$, and $d(w) \in [0,1]$ models the modification of the asset prize due to diffusion. For instance, we may choose $P(|w-W|)$ as the characteristic function $\mathbf{1}_{\{|w-W|<r\}}$ on $\{|w-W| < r\}$ for some $r > 0$. The above interaction rule means that if a market agent trusts an information source, she/he will update her/his estimated value to bring it closer to the one suggested by the information source. A rational investor is supposed to follow such a strategy.

The second interaction rule models herding effects by taking into account the interaction between an agent and other investors. We choose, similarly as in [33],

$$\begin{aligned} w^* &= w - \beta\gamma(v,w)(w-v) + \eta_1 d(w), \\ v^* &= v - \beta\gamma(v,w)(v-w) + \eta_2 d(v). \end{aligned} \tag{8}$$

Here, (w,v) and (w^*,v^*) denote the asset values of two arbitrary agents before and after the interaction, respectively. The constant $\beta \in [0,1/2]$ measures the attitude of the market participants to change their mind because of herding mechanisms, η_1, η_2 are random variables with the same distribution with variance σ_H^2 and zero mean, and the function d is as above. The function $\gamma \in [0,1]$ describes a socio-economic scenario where individuals are highly confident in the asset. In [13], the example $\gamma(v,w) = \mathbf{1}_{\{w<v\}} v f(w)$ is suggested, where f is nonincreasing, $f(0) = 1$, and $\lim_{w\to\infty} f(w) = 0$. The meaning of this choice is as follows: If an agent has an asset value w smaller than v, the function γ will push the agent to assume a higher value w^* than that one before the interaction. This means that the agent trusts other agents that assign a higher value. If w is larger than v, the agent hesitates to lower her/his asset value and nothing changes. For a discussion of the nonnegativity of w^* and v^*, we refer to [14, Section 2].

With the above interaction rules, we can define the interaction operators Q_I and Q_H in the weak form. Let $\phi(w) := \phi(x,w)$ be a regular test function and set $\Omega = \mathbb{R} \times \mathbb{R}_+$, $z = (x,w)$. Then

$$\begin{aligned} \int_\Omega Q_I(f)\phi(w)dz &= \frac{1}{\tau_H}\left\langle \int_{\mathbb{R}_+}\int_\Omega \big(\phi(w^*)-\phi(w)\big)M(W)f(x,w,t)dzdW \right\rangle, \\ \int_\Omega Q_H(f,f)\phi(w)dz &= \frac{1}{\tau_I}\left\langle \int_{\mathbb{R}_+}\int_\Omega \big(\phi(w^*)-\phi(w)\big)f(x,w,t)f(x,v,t)dzdW \right\rangle, \end{aligned}$$

where $\langle\cdot\rangle$ is the expectation value with respect to the random variable η and $M(W) \geq 0$ is a fixed background satisfying $\int_{\mathbb{R}_+} M(W)dW = 1$.

We have obtained in [14] the following results.

Grazing collision limit. The analysis of the Boltzmann equation (4) is quite involved, and we expect that its main features are contained in the limiting equation derived in the diffusion limit $(\alpha,\beta,\sigma_H^2,\sigma_I^2)\to 0$. More precisely, we scale the variables according to $t\mapsto\alpha t$ and $x\mapsto\alpha x$. Performing a Taylor expansion in the collision integrals and passing to the limit $(\alpha,\beta,\sigma_H^2,\sigma_I^2)\to 0$ such that $\lambda_I=\sigma_I^2/\alpha$ and $\lambda_H=\sigma_H^2/\alpha$ are fixed, the limiting equation for the function $g(x,w,t)$ reads as

$$\partial_t g+(\Phi(x,w)g)_x=(K[g]g+H(w))_w+(D(w)g)_{ww}, \tag{9}$$

where $(x,w)\in\mathbb{R}\times\mathbb{R}_+$, $t>0$, $D(w)=\frac{1}{2}(\lambda_I/\tau_I+\lambda_H\rho/\tau_H)d(w)^2$, $\rho=\int_\Omega f dz$,

$$\begin{aligned} K[g]&=\int_0^\infty \Gamma(v,w)g(v)dv, \quad \Gamma(v,w)=\frac{k}{\tau_H}\gamma(v,w)(v-w),\\ H(w)&=\frac{1}{\tau_I}\int_{\mathbb{R}_+}P(|w-W|)(w-W)M(W)dW. \end{aligned} \tag{10}$$

The equation is supplemented by the boundary and initial conditions

$$g(x,0,t)=0,\ g(x,w,0)=g_0(x,w) \quad \text{for } (x,w)\in\mathbb{R}\times\mathbb{R}_+,\ t>0. \tag{11}$$

Existence of weak solutions. Equation (9) is nonlinear , nonlocal, degenerate in w, and anisotropic in x (incomplete diffusion) and hence, its analysis is challenging. Partial diffusion may lead to singularity formation [21], and often solutions have very low regularity [16]. As the transport in x is linear in (9), our situation is better but still delicate. In particular, we need the hypothesis that $D(w)$ is strictly positive to get rid of the degeneracy in w. Assuming additionally that the functions in (10) are smooth, $\Gamma\geq 0$ and $\partial\Gamma/\partial w\leq 0$, and the initial datum g_0 is nonnegative and bounded, there exists a weak solution g to (9)-(11) such that $g\in L^2(0,T;H^1(\Omega))$, $\partial_t g\in L^2(0,T;H^1(\Omega)')$ and $0\leq g(x,w,t)\leq\|g_0\|_{L^\infty}e^{\lambda t}$ for $(x,w)\in\Omega$, $t>0$, for some $\lambda>0$ and for all $T>0$.

The idea of the proof is to regularize equation (9) by adding a second-order derivative with respect to x, to truncate the nonlinearity, and to solve the equation in the finite interval $w\in(0,R)$. Then we pass to the deregularization limit. The key step of the proof is the derivation of H^1 estimates uniform in the approximation parameters, which allow for the compactness argument. These estimates are derived by analyzing the differential equation satisfied by g_x and by making crucial use of the boundary conditions. For details, we refer to [14].

Numerical simulations. We illustrate the behavior of the solution to the kinetic model (4) numerically by using an operator splitting ansatz, i.e., we split (4) into a drift part and the collisional parts $\partial_t f=Q_I(f)/\tau_I$ and $\partial_t f=Q_H(f,f)/\tau_H$. The collisional parts are solved by using the interaction rules (7), (8), respectively, and a slightly modified Bird scheme [5]. The transport part $\partial_t f=(\Phi(x,w)f)_x$ is numerically solved by a flux-limited Lax-Wendroff/upwind scheme. The parameters and functions are chosen as follows: $\tau_H=\tau_I=1$ and

$$P(|w-W|)=1, \quad d(w)=4w(1-w), \quad \gamma(v,w)=\mathbf{1}_{\{w<v\}}v(1-w),$$

Φ is given by (6), and we choose the time-dependent background $W(t) = (\sin(t/200) + 0.5 * \exp(t/500))/30$. The time evolution of the first moment

$$m(f(t)) = \int_\Omega f(x,w,t)d(x,w)$$

is shown in Figure 2. The mean asset value stays within the range $[W(t)-R, W(t)+R]$ if $W(t)$ is increasing but it has the tendency to become larger than $W(t)+R$ if $W(t)$ is not varying much. Furthermore, if α is "small", $m(f(t))$ usually does not leave the interval $[W(t)-R, W(t)+R]$ (see [14]). Large values of α mean that the compromise propensity is larger and thus, herding may be become more likely.

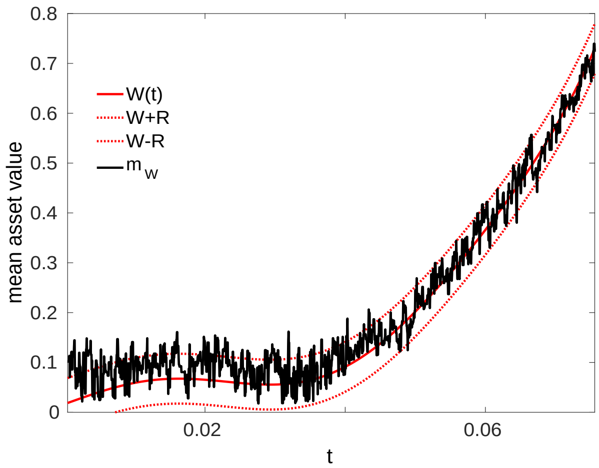


Fig. 2 Mean asset value $m(f(t))$ versus time for $\kappa = 1$, $\alpha = 0.3$, $\beta = 0.2$, $\delta = 1.5$, $R = 0.05$, and $\eta = \pm 0.061$.

0.4 A kinetic model with wealth and knowledge exchanges

The effect of wealth and knowledge exchange in a closed society may be described by kinetic equations. Let $f(x,w,t)$ be the distribution function depending on the knowledge variable $x \in \mathbb{R}_+$, the wealth $v \in \mathbb{R}_+$, and time $t > 0$. We assume that the evolution of f is given by the homogeneous Boltzmann-type equation

$$\partial_t f = Q_K(f,f) + Q_W(f,f), \quad (x,w) \in \mathbb{R}_+ \times \mathbb{R}_+,\ t > 0, \tag{12}$$

where the operators Q_K and Q_W model the interaction of the agents with respect to knowledge and wealth, respectively. The exchange rules, defining these operators, are as follows.

Let (x,v) and (y,w) denote the knowledges and wealths of two agents, respectively. The knowledges x^* and y^* after the interaction are, similarly as in (8), given by

$$x^* = x + \kappa(w)(y-x), \quad y^* = y + \kappa(v)(x-y),$$

where κ is a nondecreasing function of the wealth variable, modeling the confidence, i.e., agent (y,w) trusts agent (x,v) more if the latter agent is wealthier than the former one. The wealth values v^* and w^* after the interaction are defined by

$$v^* = (1-\gamma\Psi(x))v + \gamma\Psi(y)w, \quad w^* = \gamma\Psi(x)v + (1-\gamma\Psi(y))w,$$

where $\gamma \in (0,1)$ is fixed and $\Psi : \mathbb{R}_+ \to (0,1]$ is a nonincreasing continuous function of the knowledge variable. This rule is exactly that one used in [29] without the random risk parameter. The quantity $\gamma\Psi(x)$ can be understood as the saving/risk-taking propensity of agent (x,v). The monotonicity of Ψ means that the higher is the knowledge of an agent, the less risky is the wealth exchange for her/him. We observe that the microscopic total wealth is conserved during the exchange, $v^* + w^* = v + w$. With the above exchange rules, the interaction operators are defined in weak form, for some smooth test function ϕ, as

$$\int_{(\mathbb{R}_+)^2} Q_K(f,f)\phi dz_1 = \nu_K \int_{(\mathbb{R}_+)^4} \big(\phi(x^*,v) - \phi(x,v)\big) f(x,v,t) f(y,w,t) dz_1 dz_2,$$
$$\int_{(\mathbb{R}_+)^2} Q_W(f,f)\phi dz_2 = \nu_W \int_{(\mathbb{R}_+)^4} \big(\phi(x,v^*) - \phi(x,v)\big) f(x,v,t) f(y,w,t) dz_1 dz_2,$$

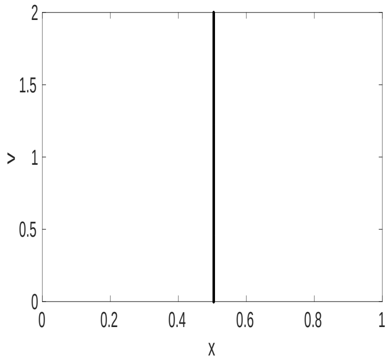
where ν_K, ν_W are some rate parameters and $dz_1 = dxdv$, $dz_2 = dydw$.

Existence of solutions. If Ψ is lower bounded by a positive constant and the initial datum $f_0 \in L^1(\mathbb{R}^2_+)$ is nonnegative, there exists a nonnegative solution $f \in L^\infty(0,T;L^1(\mathbb{R}^2_+))$ to (12), $f(x,v,0) = f_0(x,v)$ for $(x,v) \in \mathbb{R}^2_+$. This result is shown similarly as in [7]. The idea is to solve (12) iteratively, thus defining a sequence (f_n) which is bounded and satisfies $f_{n+1} \geq f_n$. The monotone convergence theorem then ensures the existence of a limit function which solves (12) in a distributional sense in time and in a weak sense in $L^1(\mathbb{R}^2_+)$.

Numerical simulations. Equation (12) is numerically solved by a particle method [5], approximating the distribution function by a sum of Dirac masses,

$$f(x,w,t) \approx \sum_{p=1}^{N} \delta_{(x_p(t),w_p(t))}(x,v),$$

where $N \in \mathbb{N}$ is the number of agents and $x_p(t)$, $w_p(t)$ are the knowledge and wealth of the pth agent at time $t > 0$, respectively. The simulations are repeated 30 times for $N = 2000$ agents with a fixed set of parameters, and the results are averaged. The simulations are performed until we have approximately reached the stationary state. The functions are defined by $\kappa(v) = 0.15$ and $\Psi(x) = (1+x)^{-\beta}$, and we have taken the parameters $\beta = 1$, $\gamma = 0.9$, and $\nu_k = \nu_w = 1$. Figure 3 shows the level set $f = 1$ of the stationary distribution function if only interactions for the knowledge Q_K (left figure) or for the wealth Q_W (right figure) are present, i.e., we have considered only one type of collisions in each simulation series. The collision rule for the knowledge induces a concentration of the agents at the average knowledge, which equals 0.5, having no effect on the wealth distribution. If only the wealth collision rule is applied, the agents aggregate again on a line, but they do not have the same wealth. The less informed agents are poorer, while the more informed are more wealthy. Choosing other values for β will not give a line but a curve, which allows for more flexibility in the modeling. For instance, for $\beta > 1$, the wealth increases superlinearly with the knowledge, i.e., even a small improvement of the knowledge leads to a significant increase of the wealth. Thus, Figure 3 (right) presents a situation which seems to be not unrealistic, giving rise to the hope that the model may be applicable to more complex socio-economic scenarios.



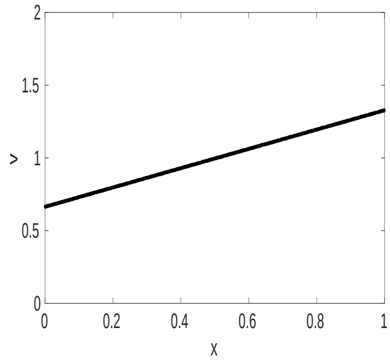


Fig. 3 Level sets for the stationary distribution function at $f = 1$ if only knowledge (left) or wealth (right) is exchanged.

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