ENTROPY-DISSIPATING FINITE-DIFFERENCE SCHEMES FOR NONLINEAR FOURTH-ORDER PARABOLIC EQUATIONS

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ABSTRACT. Structure-preserving finite-difference schemes for general nonlinear fourth-order parabolic equations on the one-dimensional torus are derived. Examples include the thin-film and the Derrida–Lebowitz–Speer–Spohn equations. The schemes conserve the mass and dissipate the entropy. The scheme associated to the logarithmic entropy also preserves the positivity. The idea of the derivation is to reformulate the equations in such a way that the chain rule is avoided. A central finite-difference discretization is then applied to the reformulation. In this way, the same dissipation rates as in the continuous case are recovered. The strategy can be extended to a multi-dimensional thin-film equation. Numerical examples in one and two space dimensions illustrate the dissipation properties.

1. INTRODUCTION

The design of numerical schemes that preserve the structure of the associated partial differential equations is an important task in numerical mathematics. In this paper, we develop new finite-difference approximations conserving the mass, preserving the positivity, and dissipating the entropy of nonlinear fourth-order parabolic equations of the type

$$(1) \qquad \partial_t u = -J_x, \quad J = u^\beta u_{xxx} + a u^{\beta-1} u_{xx} u_x + b u^{\beta-2} u_x^3 \quad \text{in } \mathbb{T},$$

where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the one-dimensional torus, $a, b \in \mathbb{R}$, and $\beta \geq 0$, together with the initial condition $u(0) = u^0$ in $\mathbb{T}$. We also discuss briefly the discretization of multi-dimensional equations; see Section 4.

A special case of (1) (with $a = b = 0$) is the thin-film equation

$$(2) \qquad \partial_t u = -(u^\beta u_{xxx})_x,$$

which models the flow of a thin liquid along a solid surface with film height $u(x, t)$ or the thin neck of a Hele-Shaw flow in the lubrication approximation [1]. The multi-dimensional version is given by $\partial_t u = -\operatorname{div}(u^\beta \nabla \Delta u)$ and its discretization will be discussed in Section 4. Another example (with $a = -2$, $b = 1$, $\beta = 0$) is the Derrida–Lebowitz–Speer–Spohn

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(DLSS) equation

$$(3) \qquad \partial_t u = -2\left(u\left(\frac{(\sqrt{u})_{xx}}{\sqrt{u}}\right)_x\right)_x = -\left(u_{xxx} - 2\frac{u_{xx}u_x}{u} + \frac{u_x^3}{u^2}\right)_x,$$

which arises as a scaling limit of interface fluctuations in spin systems [7] and describes the evolution of the electron density $u(x,t)$ in a quantum semiconductor under simplifying assumptions.

Equations (2) and (3) conserve the mass $\int_{\mathbb{T}} u(x,t)dx$ and preserve the positivity of the solution, even for the corresponding multi-dimensional versions [6, 14]. They also dissipate the entropy[1] in the sense that there exists a Lyapunov functional $S(t) = \int_{\mathbb{T}} s(u(x,t))dx$ with entropy density $s(u)$ such that the entropy production $-dS/dt$ provides gradient estimates. For instance, (2) and (3) dissipate the entropy density $s(u) = s_\alpha(u)$, where

$$s_\alpha(u) = \frac{u^\alpha}{\alpha(\alpha-1)} \quad \text{for } \alpha > 0, \ \alpha \neq 0, 1,$$
$$(4) \qquad s_0(u) = -\log u \quad \text{for } \alpha = 0,$$
$$s_1(u) = u(\log u - 1) \quad \text{for } \alpha = 1,$$

if $\frac{3}{2} \leq \alpha + \beta \leq 3$ (thin-film equation) and $0 \leq \alpha \leq \frac{3}{2}$ (DLSS equation); see [15, Section 4]. These bounds are optimal. We call $s_1(u)$ the Shannon entropy (density), $s_0(u)$ the logarithmic entropy, and $s_\alpha(u)$ for $\alpha \neq 0, 1$ a Rényi entropy. It holds that $s_\alpha(u) \to s_0(u)$ pointwise for $\alpha \to 0$ and $s_\alpha(u) \to u \log u$ pointwise for $\alpha \to 1$. We prefer to define $s_1(u)$ as in (4) to avoid the additional term in $d(u \log u)/du = \log u + 1$ which would complicate the subsequent computations.

The proof of the dissipation of the entropy is based on suitable integrations by parts and the chain rule [15]. On the discrete level, we face the problem that the chain rule is generally not available. We are aware of two general strategies.

The first strategy is to exploit the gradient-flow structure of the parabolic equation (if it exists). It involves only one integration by parts, and the discrete chain rule can be formulated by means of suitable mean functions. This idea was elaborated as the Discrete Variational Derivative Method for finite-difference approximations [11]. The gradient-flow formulation (with respect to the L^2-Wasserstein metric) yields a natural semi-discretization in time of the evolution using the minimizing movement scheme in finite-dimensional spaces from finite-volume or finite-difference approximations. These techniques were used in [8, 13, 19, 20] for the multi-dimensional DLSS equation. It allows for the proof of entropy dissipation of the Shannon entropy and the Fisher information $\int_{\mathbb{T}} (\sqrt{u})_x^2 dx$, but not for general Rényi entropies, since no Wasserstein gradient-flow formulation seems to be available for these functionals. The thin-film equation with $0 < \beta < 1$ is shown in [18] to be a gradient flow with respect to a weighted Wasserstein metric. In the work [22], a finite-difference scheme that dissipates the discrete H^1 norm of the solution to the one-dimensional thin-film equation was analyzed.

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[1]Strictly speaking, equation (1) *produces* entropy and *dissipates* energy, but the notion *entropy dissipation* seems to be common in numerical schemes.

The minimizing movement scheme is based on the implicit Euler method. We mention that higher-order time discretizations were investigated too, in the framework of semi-discrete problems; see [4] using the two-step BDF method and [17] using one-leg multistep generalizations. A generic framework for Galerkin methods in space and discontinuous Galerkin methods in time was presented in [9].

The second strategy uses time-continuous Markov chains on finite state spaces. Birth-death processes that define the Markov chain can be interpreted as a finite-volume discretization of a one-dimensional Fokker–Planck equation, and the dissipation of the discrete Shannon entropy can be proved. The nonlinear integrations by parts are reduced to a discrete Bochner-type inequality [5, 10, 16], which is obtained by identifying the Radon–Nikodym derivative of a measure involving the jump rates of the Markov chain [3, Section 2]. It seems that this idea is restricted to linear diffusion equations.

In this paper, we suggest a third strategy. The idea is to write the flux J as a combination of derivatives of the function $s'(u)$. This allows for integrations by parts that can be extended to the discrete level and it avoids the application of the chain rule. More precisely, we determine two functions A and B depending on $v := s'(u)$ and its derivatives such that $J = A_x - vB_x$. The function $s'(u)$ is known in thermodynamics as the chemical potential, and the formulation of the flux in terms of the chemical potential seems to be natural from a thermodynamic viewpoint. We apply this idea to fourth-order parabolic equations for the first time. It turns out that for $s = s_\alpha$ with $\alpha \neq 1$, we can write

$$A = \frac{u^{\alpha+\beta}}{(\alpha-1)^2 v}(\lambda_1 \xi_2 + \lambda_2 \xi_1^2), \quad B = \frac{u^{\alpha+\beta}}{(\alpha-1)^2 v^2}(\lambda_3 \xi_2 + \lambda_4 \xi_1^2),$$

where $\xi_1 = v_x/v$, $\xi_2 = v_{xx}/v$, and the coefficients λ_i depend on a, b, α, and β; see (10) and (11). Integrating by parts twice and using equation (1) gives for $S_\alpha = \int_{\mathbb{T}} s_\alpha(u)dx$:

$$\frac{dS_\alpha}{dt} = \int_{\mathbb{T}} Jv_x dx = -\int_{\mathbb{T}} \left(v_{xx}A - (vv_x)_x B\right)dx.$$

The task is to show that the integrand, written as a polynomial in (ξ_1, ξ_2), is nonnegative for all values of $(\xi_1, \xi_2) \in \mathbb{R}^2$. It follows from the product rule $(vv_x)_x = vv_{xx} + v_x^2$ that, for $\alpha \neq 1$,

$$\frac{dS_\alpha}{dt} = -\int_{\mathbb{T}} (\xi_2 vA - (\xi_2 + \xi_1^2)v^2 B)dx$$

(5)
$$= -\int_{\mathbb{T}} \frac{u^{\alpha+\beta}}{(\alpha-1)^2}\left((\lambda_1 - \lambda_3)\xi_2^2 + (\lambda_2 - \lambda_3 - \lambda_4)\xi_2\xi_1^2 - \lambda_4\xi_1^4\right)dx.$$

Under certain conditions on the parameters, we expect that the integrand is bounded from below by $u^{\alpha+\beta}(\xi_2^2 + \xi_1^4)$, up to a factor, which yields some gradient estimates.

On the discrete level, we imitate this idea: The flux $J = A_x - vB_x$ and the variables ξ_1, ξ_2 of the polynomials A and B is discretized by central finite differences. For this, let $\mathbb{T}_h = \mathbb{T}/(h\mathbb{Z})$ be a discrete torus with grid size $h > 0$ and define the scheme

(6) $\quad \partial_t u_i = -\frac{1}{h}(J_{i+1/2} - J_{i-1/2}), \quad J_{i+1/2} = \frac{1}{h}(A_{i+1} - A_i) - \frac{1}{2h}(v_{i+1} + v_i)(B_{i+1} - B_i),$

with the initial condition $u_i(0) = u^0(i)$ for $i \in \mathbb{T}_h$, where $u_i = u(i)$, $v_i = s'(u_i)$, and A_i and B_i are the polynomials A and B, evaluated at $i \in \mathbb{T}_h$, respectively; see (19). We show below that with the discrete entropy $S_\alpha^h = h \sum_{i \in \mathbb{T}_h} s_\alpha(u)$, the discrete analog of (5) becomes

$$\frac{dS_\alpha^h}{dt} = -h \sum_{i \in \mathbb{T}_h} \left(\xi_{2,i} v_i A_i - (v_i \xi_{2,i} + \xi_{1,i}^2) v_i^2 B_i \right),$$

where $\xi_{1,i}$, $\xi_{2,i}$ approximate $v_x(i)/v(i)$, $v_{xx}(i)/v(i)$, respectively. This yields exactly the polynomial of the continuous case. Thus, we obtain the same conditions on the parameters as for the continuous equation.

We still need a discrete analog of the product rule $(vv_x)_x = vv_{xx} + v_x^2$ to conclude. This is done by carefully choosing $\xi_{1,i}$ and $\xi_{2,i}$. Definition (20) ensures that $v_i^2(\xi_{2,i} + \xi_{1,i}^2) = (v_{i+1}^2 - 2v_i + v_{i-1}^2)/(2h^2)$ which approximates $\frac{1}{2}(v^2)_{xx} = vv_{xx} + v_x^2$. This choice is used in the central scheme (6) for $J_{i+1/2}$. Noncentral schemes require different definitions of $\xi_{1,i}$ and $\xi_{2,i}$; see Remark 7.

A drawback of our technique is that scheme (6) depends on the entropy to be dissipated. The scheme does not dissipate all admissible entropies. In applications, however, one usually wants to dissipate only that entropy which is physically relevant.

Our main results can be sketched as follows:

- Lyapunov functional: Let $\alpha \geq 0$, $\alpha \neq 1$ and assume that

(7) $$K(\alpha, \beta) := -2\alpha^2 + (3a - 4\beta + 9)\alpha - 2\beta^2 + (3a + 9)\beta - 9(a + b + 1) \geq 0.$$

  Then the continuous entropy S_α *and* the discrete entropy S_α^h are dissipated in the sense that $dS_\alpha/dt \leq 0$ and $dS_\alpha^h/dt \leq 0$, i.e., S_α and S_α^h are Lyapunov functionals for $u(t)$ and $u_i(t)$, respectively; see Theorems 2 and 5. Condition (7) is optimal for the thin-film and DLSS equations.

- Entropy dissipation: Let $\alpha \geq 0$, $\alpha \neq 1$ and assume that $K(\alpha, \beta) > 0$. Then there exists a constant $c(\alpha, \beta) > 0$ such that for all $t > 0$,

(8) $$\frac{dS_\alpha}{dt} + c(\alpha, \beta) \int_{\mathbb{T}} u^{\alpha+\beta}(\xi_2^2 + \xi_1^4) dx \leq 0,$$

(9) $$\frac{dS_\alpha^h}{dt} + c(\alpha, \beta) h \sum_{i \in \mathbb{T}_h} \bar{u}_i^{\alpha+\beta}(\xi_{2,i}^2 + \xi_{1,i}^4) \leq 0,$$

  where $\xi_1 = v_x/v$, $\xi_2 = v_{xx}/v$, $v = s_\alpha(u)$, $\bar{u}_i$ is an arbitrary average of (u_i), and $\xi_{1,i}$, $\xi_{2,i}$ are defined in (20). This result is proved in Theorems 2 and 5.

- Case $\alpha = 1$: For the case $\alpha = 1$, we need the formulation $J = A_x - wB_x$ instead of $J = A_x - vB_x$, where $w = s_0'(u) = -u^{-1}$ and $v = \log u$, since J generally does not depend on the logarithm. For details, see Proposition 8.

- Case $\alpha = 0$: We show that scheme (6) with $\alpha = 0$ possesses global positive solutions. This result is a consequence of the discrete entropy inequality and mass conservation, which imply that $h \sum_{i \in \mathbb{T}_h} (u_i(t) - \log u_i(t))$ is bounded for all $t > 0$. Consequently, $u_i(t) - \log u_i(t)$ is bounded for all $i \in \mathbb{T}_h$ and $t > 0$, and since the

function $s \mapsto s - \log s$ tends to infinity if either $s \to 0$ or $s \to \infty$, this proves that $u_i(t)$ is bounded from below and above. We refer to Proposition 6 for details.

- Multi-dimensional case: In principle, the multi-dimensional case can be treated using functions A and B with many variables. Practically, however, the computations are becoming too involved and it may be unclear how to discretize mixed derivatives. One idea to overcome this issue is to use scalar variables only, like $\xi_1 = |\nabla v|/v$ and $\xi_2 = \Delta v/v$. This allows us to treat the multi-dimensional thin-film equation; see Proposition 10.

The paper is organized as follows. We prove the continuous entropy inequality (8) in Section 2 and the discrete entropy inequality (9) in Section 3. A scheme for the multi-dimensional thin-film equation is proposed and analyzed in Section 4. Numerical simulations for the thin-film and DLSS equations in one space dimension and for a thin-film equation in two space dimensions are presented in Section 5.

2. GENERAL CONTINUOUS EQUATION

To prepare the discretization, we need to analyze the entropy dissipation properties of the continuous equation (1). We show first that J can be written as $A_x - vB_x$ with $v = s'_\alpha(u)$ and functions A and B which depend on v, v_x, and v_{xx}.

Lemma 1. *Let J be given as in* (1) *and s_α as in* (4) *and let $\alpha \geq 0$ satisfy $\alpha \neq 1$. Then $J = A_x - vB_x$, where*

$$A = u^{\alpha+\beta}\big(\lambda_1 u^{2-2\alpha}(s'_\alpha(u))_{xx} + \lambda_2(\alpha-1)u^{3-3\alpha}(s'_\alpha(u))_x^2\big)$$

$$= \frac{u^{\alpha+\beta}}{(\alpha-1)^2 v}\left(\lambda_1 \frac{v_{xx}}{v} + \lambda_2\left(\frac{v_x}{v}\right)^2\right),$$

$$B = u^{\alpha+\beta}\big(\lambda_3(\alpha-1)u^{3-3\alpha}(s'_\alpha(u))_{xx} + \lambda_4(\alpha-1)^2 u^{4-4\alpha}(s'_\alpha(u))_x^2\big)$$

$$= \frac{u^{\alpha+\beta}}{(\alpha-1)^2 v^2}\left(\lambda_3 \frac{v_{xx}}{v} + \lambda_4\left(\frac{v_x}{v}\right)^2\right)$$

and

$$(10) \qquad \begin{aligned} \lambda_1 &= \frac{-2\alpha^2 + (\beta+5)\alpha + a\beta - \beta^2 - a - 2b - 3}{(\alpha-1)(\beta-2\alpha+3)} + \frac{2(\alpha-1)}{\beta-2\alpha+3}\lambda_4, \\ \lambda_2 &= \frac{2\alpha^2 - (a+7)\alpha + 2a + b + 6}{(\alpha-1)(\beta-2\alpha+3)} + \frac{\beta-3\alpha+4}{\beta-2\alpha+3}\lambda_4, \\ \lambda_3 &= \frac{(a+1)\beta - \beta^2 - a - 2b}{(1-\alpha)(\beta-2\alpha+3)} + \frac{2(\alpha-1)}{\beta-2\alpha+3}\lambda_4, \end{aligned}$$

with $\lambda_4 \in \mathbb{R}$ being a free parameter.

The lemma shows in particular that the formulation $J = A_x - vB_x$ is not unique. This fact is used to optimize later the range of admissible parameters α, β, a, and b.

Proof. Let $\alpha > 0$ with $\alpha \neq 1$. A direct computation yields

$$A = \lambda_1 u^\beta u_{xx} + \left((\lambda_1 + \lambda_2)\alpha - 2\lambda_1 - \lambda_2\right)u^{\beta-1}u_x^2,$$
$$B = \lambda_3(\alpha - 1)u^{\beta-\alpha+1}u_{xx} + \left((\lambda_3 + \lambda_4)\alpha - 2\lambda_3 - \lambda_4\right)u^{\beta-\alpha}u_x^2.$$

Inserting these expressions into $A_x - vB_x$ and simplifying leads to

$$A_x - \frac{u^{\alpha-1}}{\alpha - 1}B_x = (\lambda_1 - \lambda_3)u^\beta u_{xxx}$$
$$+ \left((2\lambda_1 + 2\lambda_2 - \lambda_3 - 2\lambda_4)\alpha + (\lambda_1 - \lambda_3)\beta - 4\lambda_1 - 2\lambda_2 + 3\lambda_3 + 2\lambda_4\right)u^{\beta-1}u_{xx}u_x$$
$$+ \left((\lambda_3 + \lambda_4)\alpha^2 + (\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)\alpha\beta - (\lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_4)\alpha\right.$$
$$\left. + (-2\lambda_1 - \lambda_2 + 2\lambda_3 + \lambda_4)\beta + 2\lambda_1 + \lambda_2\right)u^{\beta-2}u_x^3.$$

We identify the coefficients with those in the expression (1) for J:

$$1 = \lambda_1 - \lambda_3,$$
$$a = (2\lambda_1 + 2\lambda_2 - \lambda_3 - 2\lambda_4)\alpha + (\lambda_1 - \lambda_3)\beta - 4\lambda_1 - 2\lambda_2 + 3\lambda_3 + 2\lambda_4,$$
$$b = (\lambda_3 + \lambda_4)\alpha^2 + (\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)\alpha\beta - (\lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_4)\alpha$$
$$+ (-2\lambda_1 - \lambda_2 + 2\lambda_3 + \lambda_4)\beta + 2\lambda_1 + \lambda_2.$$

The general solution of this linear system for $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, with free parameter $\lambda_4 \in \mathbb{R}$, gives (10).

Next, let $\alpha = 0$. The ansatz for A and B becomes

$$A = u^\beta\left(\lambda_1 u^2(-u^{-1})_{xx} - \lambda_2 u^3(-u^{-1})_x^2\right),$$
$$B = u^\beta\left(-\lambda_3 u^3(-u^{-1})_{xx} + \lambda_4 u^4(-u^{-1})_x^2\right).$$

Then

$$A_x - vB_x = (\lambda_1 - \lambda_3)u^\beta u_{xxx} + \left((\lambda_1 - \lambda_3)\beta - 4\lambda_1 - 2\lambda_2 + 3\lambda_3 + 2\lambda_4\right)u^{\beta-1}u_{xx}u_x$$
$$+ \left((-2\lambda_1 - \lambda_2 + 2\lambda_3 + \lambda_4)\beta + 2\lambda_1 + \lambda_2\right)u^{\beta-2}u_x^3.$$

Identifying the coefficients with those in (1) again gives a linear system for the parameters $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. The general solution reads as

$$\lambda_1 = \frac{\beta^2 - a\beta + a + 2b + 3}{\beta + 3} - \frac{2}{\beta + 3}\lambda_4,$$

(11)
$$\lambda_2 = -\frac{2a + b + 6}{\beta + 3} + \frac{\beta + 4}{\beta + 3}\lambda_4,$$

$$\lambda_3 = \frac{\beta^2 - a\beta - \beta + a + 2b}{\beta + 3} - \frac{2}{\beta + 3}\lambda_4,$$

These expressions are the same as (10) with $\alpha = 0$.   □

For the following theorem, we recall definition (4) of s_α and set $S_\alpha(u) = \int_\mathbb{T} s_\alpha(u)dx$.

Theorem 2. *Let u be a smooth positive solution to (1) and let $\alpha \geq 0$. If $K(\alpha, \beta) \geq 0$ (see definition (7)), then S_α is a Lyapunov functional, i.e. $dS_\alpha/dt \leq 0$ for all $t > 0$. If $K(\alpha, \beta) > 0$ then there exists $c(\alpha, \beta) > 0$ such that for all $t > 0$,*

$$\frac{dS_\alpha}{dt} + c(\alpha, \beta) \int_{\mathbb{T}} u^{\alpha+\beta}\big(u^{2(1-\alpha)}(u^{\alpha-1})^2_{xx} + u^{4(1-\alpha)}(u^{\alpha-1})^4_x\big)dx \leq 0$$

and there exists another constant $C(\alpha, \beta) > 0$ such that for all $t > 0$,

$$\frac{dS_\alpha}{dt} + C(\alpha, \beta) \int_{\mathbb{T}} u^{\alpha+\beta}\big(u^{-2}u^2_{xx} + u^{-4}u^4_x\big)dx \leq 0.$$

Proof. Let first $\alpha \geq 0$ with $\alpha \neq 1$. We calculate the time derivative of the entropy, using integration by parts twice:

$$\frac{dS_\alpha}{dt} = \int_{\mathbb{T}} s'_\alpha(u)\partial_t u\, dx = -\int_{\mathbb{T}} v J_x dx = \int_{\mathbb{T}} J v_x dx$$

$$= \int_{\mathbb{T}} (A_x - v B_x)v_x dx = -\int_{\mathbb{T}} \big(A v_{xx} - (v v_{xx} + v^2_x)B\big)dx$$

$$= -\int_{\mathbb{T}} \frac{u^{\alpha+\beta}}{(\alpha-1)^2}\left[(\lambda_1 - \lambda_3)\left(\frac{v_{xx}}{v}\right)^2 + (\lambda_2 - \lambda_3 - \lambda_4)\frac{v_{xx}}{v}\left(\frac{v_x}{v}\right)^2 - \lambda_4\left(\frac{v_x}{v}\right)^4\right]dx,$$

$$(12) \qquad = -\int_{\mathbb{T}} \frac{u^{\alpha+\beta}}{(\alpha-1)^2} P\left(\frac{v_x}{v}, \frac{v_{xx}}{v}\right)dx,$$

where

$$(13) \qquad\qquad P(\xi_1, \xi_2) = (\lambda_1 - \lambda_3)\xi_2^2 + (\lambda_2 - \lambda_3 - \lambda_4)\xi_2\xi_1 - \lambda_4\xi_1^4.$$

The right-hand side of (12) is nonpositive if $P(\xi_1, \xi_2) \geq 0$ for all $(\xi_1, \xi_2) \in \mathbb{R}^2$. Taking into account that $\lambda_1 - \lambda_3 = 1$, this is the case if and only if

$$-4\lambda_4 - (\lambda_2 - \lambda_3 - \lambda_4)^2 \geq 0.$$

In view of definition (10) of λ_2 and λ_3, we may interpret λ_4 as a free parameter und $\lambda_2 = \lambda_2(\lambda_4)$ and $\lambda_3 = \lambda_3(\lambda_4)$ as affine functions in λ_4. Therefore, we require that

$$(14) \qquad\qquad f(\lambda_4) := -4\lambda_4 - \big(\lambda_2(\lambda_4) - \lambda_3(\lambda_4) - \lambda_4\big)^2 \geq 0.$$

We choose the optimal value of λ_4 by computing the critical value of f:

$$0 = f'(\lambda_4) = \frac{2(-2\alpha^2 + (-3a + 8\beta + 3)\alpha - 3a\beta + \beta^2 + 9(a + b) - 15\beta)}{(\beta - 2\alpha + 3)^2}$$

$$- \frac{18(\alpha - 1)^2}{(\beta - 2\alpha + 3)^2}\lambda_4.$$

This yields

$$(15) \qquad\qquad \lambda_4 = \frac{-2\alpha^2 + (-3a + 8\beta + 3)\alpha - 3a\beta + \beta^2 + 9(a + b) - 15\beta}{9(\alpha - 1)^2}.$$

Inserting λ_4 in (14) leads to

$$0 \le f(\lambda_4) = \frac{4}{9(\alpha-1)^2}\big(-2\alpha^2 + (3a - 4\beta + 9)\alpha - 2\beta^2 + (3a+9)\beta - 9(a+b+1)\big),$$

which is equivalent to $K(\alpha, \beta) \ge 0$, see (7).

If $K(\alpha, \beta) > 0$, there exists $c_0(\alpha, \beta) > 0$ such that for all $(\xi_1, \xi_2) \in \mathbb{R}^2$, $P(\xi_1, \xi_2) \ge c_0(\alpha, \beta)(\xi_1^2 + \xi_1^4)$. Inserting this information in (12), we infer that

$$\frac{dS_\alpha}{dt} \le -c_0(\alpha, \beta) \int_\mathbb{T} \frac{u^{\alpha+\beta}}{(\alpha-1)^2}\left[\left(\frac{v_{xx}}{v}\right)^2 + \left(\frac{v_x}{v}\right)^4\right] dx$$

$$= -c_0(\alpha, \beta) \int_\mathbb{T} u^{\alpha+\beta-2}\left[\left(\frac{u_{xx}}{u}\right)^2 + 2(\alpha-2)\frac{u_{xx}}{u}\left(\frac{u_x}{u}\right)^2 + (2\alpha^2 - 6\alpha + 5)\left(\frac{u_x}{u}\right)^4\right] dx.$$

The discriminant equals

$$1 \cdot (2\alpha^2 - 6\alpha + 5) - (\alpha-2)^2 = (\alpha-1)^2,$$

and it is positive for all $\alpha \ne 1$. Therefore, there exists $k(\alpha) > 0$ such that

$$\frac{dS_\alpha}{dt} \le -c_0(\alpha, \beta)k(\alpha) \int_\mathbb{T} u^{\alpha+\beta}\left(\left(\frac{u_{xx}}{u}\right)^2 + \left(\frac{u_x}{u}\right)^4\right) dx,$$

and this gives the conclusion for $c(\alpha, \beta) = c_0(\alpha, \beta)/(\alpha-1)^2$ and $C(\alpha, \beta) = c_0(\alpha, \beta)k(\alpha)$ when $\alpha \ne 1$.

It remains to analyze the case $\alpha = 1$. Here, we cannot formulate the flux as $J = A_x - vB_x$ with $v = s_1'(u) = \log u$, since J does not contain logarithmic terms. Our idea is to write $J = A_x - wB_x$ with $w = -u^{-1}$ and functions A and B that depend on w, w_x, and w_{xx}. The time derivative of the entropy S_1 can be written in terms of w and its derivatives only, since the logarithmic term $v = \log u$ only appears with its derivatives.

The formulation $J = A_x - wB_x$ corresponds to the expression used for $\alpha = 0$. In fact, we have

$$A = u^\beta\big(\lambda_1 u^2(-u^{-1})_{xx} - \lambda_2 u^3(-u^{-1})_x^2\big) = \frac{u^\beta}{w}\left(\lambda_1 \frac{w_{xx}}{w} + \lambda_2\left(\frac{w_x}{w}\right)^2\right),$$

$$B = u^\beta\big(-\lambda_3 u^3(-u^{-1})_{xx} + \lambda_4 u^4(-u^{-1})_x^2\big) = \frac{u^\beta}{w^2}\left(\lambda_3 \frac{w_{xx}}{w} + \lambda_4\left(\frac{w_x}{w}\right)^2\right),$$

where $\lambda_1, \lambda_2, \lambda_3$ are given by (11) and λ_4 is a free parameter. As before, the time derivative becomes

$$\frac{dS_1}{dt} = \int_\mathbb{T} \partial_t u \log u \, dx = -\int_\mathbb{T} J_x v \, dx = \int_\mathbb{T} J v_x \, dx = -\int_\mathbb{T} \big(Av_{xx} - B(v_x w)_x\big) dx.$$

Set $\xi_1 = w_x/w$ and $\xi_2 = w_{xx}/w$. Since $v_{xx} = -w_{xx}/w + (w_x/w)^2 = -\xi_2 + \xi_1^2$ and $(v_x w)_x = -w_{xx} = -w\xi_2$, we obtain

$$\frac{dS_1}{dt} = -\int_\mathbb{T} \frac{u^\beta}{w}\big((\lambda_1\xi_2 + \lambda_2\xi_1^2)(-\xi_2 + \xi_1^2) + (\lambda_3\xi_2 + \lambda_4\xi_1^2)\xi_2\big) dx$$

$$= -\int_{\mathbb{T}} u^{\beta+1}\big(-(\lambda_1\xi_2 + \lambda_2\xi_1^2)(-\xi_2 + \xi_1^2) - (\lambda_3\xi_2 + \lambda_4\xi_1^2)\xi_2\big)dx$$

$$= -\int_{\mathbb{T}} u^{\beta+1}P_1(\xi_1, \xi_2)dx,$$

where

(16) $$P_1(\xi_1, \xi_2) = (\lambda_1 - \lambda_3)\xi_2^2 + (-\lambda_1 + \lambda_2 - \lambda_4)\xi_2\xi_1^2 - \lambda_2\xi_1^4.$$

Observe that $\lambda_1 - \lambda_3 = 1$. Thus, $dS_1/dt \geq 0$ if the polynomial P_1 is nonnegative for all $(\xi_1, \xi_2) \in \mathbb{R}^2$, which is the case if and only if

$$f_1(\lambda_4) := -4\lambda_2(\lambda_4) - \big(-\lambda_1(\lambda_4) + \lambda_2(\lambda_4) - \lambda_4\big)^2 \geq 0.$$

We choose the optimal value λ_4 from $f_1'(\lambda_4) = 0$, i.e.

$$\lambda_4 = \frac{1}{9}\big(\beta^2 - (3a + 14)\beta + 9(a + b) + 3\big).$$

Then $f(\lambda_4) \geq 0$ is equivalent to $K(1, \beta) \geq 0$.

Finally, if $K(1, \beta) > 0$, we have $P_1(\xi_1, \xi_2) \geq c_0(1, \beta)(\xi_2^2 + \xi_1^4)$ and consequently,

$$\frac{dS_1}{dt} \leq -c_0(1, \beta)\int_{\mathbb{T}} u^{\beta+1}\left(u^2\left(\frac{1}{u}\right)_{xx}^2 + u^4\left(\frac{1}{u}\right)_x^4\right)dx$$

$$= -c_0(1, \beta)\int_{\mathbb{T}} u^{\beta+1}\big(u^{-2}u_{xx}^2 - 4u^{-3}u_{xx}u_x^2 + 5u^{-4}u_x^4\big)dx.$$

The discriminant is positive and there exists $k(1) > 0$ such that $\xi_2^2 - 4\xi_2\xi_1^2 + 5\xi_1^4 \geq k(1)(\xi_2^2 + \xi_1^4)$. We infer that

$$\frac{dS_1}{dt} \leq -c_0(1, \beta)k(1)\int_{\mathbb{T}} u^{\beta+1}\big(u^{-2}u_{xx}^2 + u^{-4}u_x^4\big)dx,$$

which concludes the proof with $c(1, \beta) = c_0(1, \beta)$ and $C(1, \beta) = c_0(1, \beta)k(1)$. $\qquad\square$

Remark 3 (Examples). The DLSS equation corresponds to (1) if $a = -2$, $b = 0$, and $\beta = 0$. Then condition (7) becomes $0 < \alpha \leq \frac{3}{2}$, which is the optimal interval. Choosing $a = b = 0$, we obtain the thin-film equation, and condition (7) is equivalent to $\frac{3}{2} \leq \alpha + \beta \leq 3$, which again is the optimal parameter range. $\qquad\square$

Remark 4 (Systematic integration by parts). The result of Theorem 2 can be also derived by the method of systematic integration by parts of [15]. Our proof is taylored in such a way that it can be directly "translated" to the discrete level. Indeed, the method of [15] needs several chain rules that are not available on the discrete level and our technique needs to be used.

Still, systematic integration by parts and our strategy are strongly related. Systematic integration by parts means that we are adding so-called integration-by-parts formulas whose derivatives vanishe. In this way, we can derive (12),

$$\frac{dS_\alpha}{dt} = \int_{\mathbb{T}}(A_x - vB_x)v_x dx + c_1\int_{\mathbb{T}}(Av_x - Bvv_x)_x dx = -\int_{\mathbb{T}}(Av_{xx} - B(vv_x)_x)dx,$$

by choosing $c_1 = -1$. In the method of systematic integration by parts, we are adding a term of the type $c_2 \int_{\mathbb{T}} (u^{\alpha+\beta}(v_x/v)^3)_x dx$ and optimize c_2. By contrast, the constant c_1 is fixed, but we optimize λ_4 in the formulation of A and B. In both cases, just one parameter needs to be optimized. $\square$

3. General discretized equation

We "translate" the computations of the previous section to the discrete level. For this, we use the discrete entropy

$$S_\alpha^h(u) = h \sum_{i \in \mathbb{T}_h} s_\alpha(u_i), \quad \text{where } \alpha \geq 0, \ \alpha \neq 1, \tag{17}$$

where $u_i = u(i)$ for $i \in \mathbb{T}_h$ and s_α is defined in (4). We recall scheme (6):

$$\partial_t u_i = -\frac{1}{h}(J_{i+1/2} - J_{i-1/2}), \quad J_{i+1/2} = \frac{1}{h}(A_{i+1} - A_i) - \frac{1}{2h}(v_{i+1} + v_i)(B_{i+1} - B_i), \tag{18}$$

where the functions A_i and B_i are given by

$$A_i = \frac{\bar{u}_i^{\alpha+\beta}}{(\alpha-1)^2 v_i}(\lambda_1 \xi_{2,i} + \lambda_2(\alpha-1)\xi_{1,i}^2), \quad B_i = \frac{\bar{u}_i^{\alpha+\beta}}{(\alpha-1)^2 v_i^2}(\lambda_3 \xi_{2,i} + \lambda_4(\alpha-1)\xi_{1,i}^2), \tag{19}$$

$\bar{u}_i$ is an arbitrary average of u around the point ih, and $\xi_{2,i}$ and $\xi_{1,i}^2$ are discrete analogs of v_{xx}/v and $(v_x/v)^2$:

$$\xi_{2,i} := \frac{1}{v_i h^2}(v_{i+1} - 2v_i + v_{i-1}), \quad \xi_{1,i}^2 := \frac{1}{2v_i^2 h^2}\left((v_{i+1} - v_i)^2 + (v_i - v_{i-1})^2\right). \tag{20}$$

The parameters λ_i for $i = 1, 2, 3, 4$ are given by (10) and (15). For later use, we note that

$$\xi_{1,i}^2 = \frac{1}{2v_i^2 h^2}\left((v_{i+1}^2 - 2v_i^2 + v_{i-1}^2) - 2v_i(v_{i+1} - 2v_i + v_{i-1})\right)$$

$$= \frac{1}{2v_i^2 h^2}(v_{i+1}^2 - 2v_i^2 + v_{i-1}^2) - \xi_{2,i}$$

implies that

$$\frac{1}{2h^2}(v_{i+1}^2 - 2v_i^2 + v_{i-1}^2) = v_i^2(\xi_{2,i} + \xi_{1,i}^2), \quad \frac{1}{h^2}(v_{i+1} - 2v_i + v_{i-1}) = v_i \xi_{2,i}, \tag{21}$$

Recall definition (17) of S_α^h and condition (7) for $K(\alpha, \beta)$.

Theorem 5. *Let $(u_i)_{i \in \mathbb{T}_h}$ be a positive solution to (18)–(20) and let $\alpha \geq 0$, $\alpha \neq 1$. If $K(\alpha, \beta) \geq 0$ then $dS_\alpha^h/dt \leq 0$ for all $t > 0$. Moreover, if $K(\alpha, \beta) > 0$,*

$$\frac{dS_\alpha^h}{dt} + c(\alpha, \beta) h \sum_{i \in \mathbb{T}_h} \bar{u}_i^{\alpha+\beta}\left(\xi_{2,i}^2 + \xi_{1,i}^4\right) \leq 0$$

with the same constant $c(\alpha, \beta) > 0$ as in the proof of Theorem 2.

Proof. Let $\alpha > 0$, $\alpha \neq 1$. We compute the time derivative of the discrete entropy, using summation by parts twice:

$$\frac{dS_\alpha^h}{dt} = h \sum_{i \in \mathbb{T}_h} s_\alpha'(u) \partial_t u_i = -\sum_{i \in \mathbb{T}_h} v_i (J_{i+1/2} - J_{i-1/2}) = \sum_{i \in \mathbb{T}_h} (v_{i+1} - v_i) J_{i+1/2}$$

$$= \frac{1}{h} \sum_{i \in \mathbb{T}_h} (v_{i+1} - v_i) \left((A_{i+1} - A_i) - \frac{1}{2}(v_{i+1} + v_i)(B_{i+1} - B_i) \right)$$

$$= \frac{1}{h} \sum_{i \in \mathbb{T}_h} \left((v_{i+1} - v_i)(A_{i+1} - A_i) - \frac{1}{2}(v_{i+1}^2 - v_i^2)(B_{i+1} - B_i) \right)$$

$$= -\frac{1}{h} \sum_{i \in \mathbb{T}_h} \left((v_{i+1} - 2v_i + v_{i-1}) A_i - \frac{1}{2}(v_{i+1}^2 - 2v_i^2 + v_{i-1}^2) B_i \right).$$

In the last step, we recognize the discrete analog of the chain rule $(v_x v)_x = \frac{1}{2}(v^2)_{xx}$. Inserting (21), we find that

$$\frac{dS_\alpha^h}{dt} = -h \sum_{i \in \mathbb{T}_h} \left(\xi_{2,i} v_i A_i - (\xi_{2,i} + \xi_{1,i}^2) v_i^2 B_i \right)$$

$$= -h \sum_{i \in \mathbb{T}_h} \frac{\bar{u}_i^{\alpha+\beta}}{(\alpha-1)^2} \left((\lambda_1 - \lambda_3)\xi_{2,i}^2 + (\lambda_2 - \lambda_3 - \lambda_4)\xi_{2,i}\xi_{1,i}^2 - \lambda_4\xi_{1,i}^4 \right)$$

$$= -h \sum_{i \in \mathbb{T}_h} \frac{\bar{u}_i^{\alpha+\beta}}{(\alpha-1)^2} P(\xi_{1,i}, \xi_{2,i}),$$

where P is the same polynomial as in (13). The proof of Theorem 2 shows that $P(\xi_{1,i}, \xi_{2,i}) \geq 0$ if $K(\alpha, \beta) \geq 0$. Moreover, if the strict inequality $K(\alpha, \beta) > 0$ holds, $P(\xi_{1,i}, \xi_{2,i}) \geq c_0(\alpha, \beta)(\xi_{2,i}^2 + \xi_{1,i}^4)$ with the same constant as in Theorem 2, which translates into the inequality

$$\frac{dS_\alpha}{dt} \leq -c_0(\alpha, \beta) \sum_{i \in \mathbb{T}_h} \frac{\bar{u}_i^{\alpha+\beta}}{(\alpha-1)^2} (\xi_{2,i}^2 + \xi_{1,i}^4),$$

finishing the proof. $\square$

In Theorem 2, the existence of positive solutions is assumed. We show that such solutions exist globally, at least in case $\alpha = 0$.

Proposition 6. *Let $\alpha = 0$ and $u_i^0 > 0$ for $i \in \mathbb{T}_h$. Then there exists a global solution $(u_i)_{i \in \mathbb{T}_h}$ to scheme (18)–(20) and $u_i(0) = u_i^0$ for $i \in \mathbb{T}_h$ and constants $\kappa_1 \geq \kappa_0 > 0$ such that*

$$0 < \kappa_0 \leq u_i(t) \leq \kappa_1 \quad \text{for all } i \in \mathbb{T}_h, \ t > 0.$$

Proof. Scheme (18) is a system of ordinary differential equations. According to the Picard–Lindelöf theorem, there exists a unique local positive differentiable solution $(u_i)_{i \in \mathbb{T}_h}$ on the

maximal time interval $[0, T)$ for some $T > 0$. This solution can be extended to $[0, \infty)$ if the functions $u_i(t)$ are uniformly positive and bounded. By Theorem 5,

$$h \sum_{i \in \mathbb{T}_h} (-\log u_i(t)) \leq h \sum_{i \in \mathbb{T}_h} (-\log u_i^0),$$

Moreover, scheme (18) conserves the mass, $h \sum_{i \in \mathbb{T}_h} u_i(t) = h \sum_{i \in \mathbb{T}_h} u_i^0$. This shows that

$$h \sum_{i \in \mathbb{T}_h} (u_i(t) - \log u_i(t)) \leq h \sum_{i \in \mathbb{T}_h} (u_i^0 - \log u_i^0).$$

Since $s \mapsto s - \log s$ diverges for $s \to 0$ and $s \to \infty$, there exist constants $\kappa_0 > 0$ and $\kappa_1 > 0$ such that $\kappa_0 \leq u_i(t) \leq \kappa_1$ for all $i \in \mathbb{T}_h$ and $t > 0$. Therefore, we can extend the solution globally. $\qquad \square$

Remark 7 (Noncentral scheme for $J_{i+1/2}$). A more direct discrete analog of v_{xx}/v and v_x/v is given by

$$\xi_{2,i} = \frac{1}{v_i h^2}(v_{i+1} - 2v_i + v_{i-1}), \quad \xi_{1,i}^2 = \frac{1}{v_i^2 h^2}(v_i - v_{i-1})^2.$$

In this situation, the scheme for J needs to be noncentral,

$$J_{i+1/2} = \frac{1}{h}(A_{i+1} - A_i) - \frac{1}{h}v_i(B_{i+1} - B_i),$$

where A_i and B_i are as before. Indeed, it follows from summation by parts for $\alpha \geq 0$, $\alpha \neq 1$ that

$$\frac{dS_\alpha}{dt} = \frac{1}{h} \sum_{i \in \mathbb{T}_h} (v_{i+1} - v_i)\big((A_{i+1} - A_i) - v_i(B_{i+1} - B_i)\big)$$

$$= -\frac{1}{h} \sum_{i \in \mathbb{T}_h} \Big[(v_{i+1} - 2v_i + v_{i-1})A_i - \big((v_{i+1} - v_i)v_i - (v_i - v_{i-1})v_{i-1}\big)B_i\Big]$$

$$= -\frac{1}{h} \sum_{i \in \mathbb{T}_h} \Big[(v_{i+1} - 2v_i + v_{i-1})A_i - \big(v_i(v_{i+1} - 2v_i + v_{i-1}) + (v_i - v_{i-1})^2\big)B_i\Big]$$

$$= -h \sum_{i \in \mathbb{T}_h} \big(\xi_{2,i}v_i A_i - (\xi_{2,i} + \xi_{1,i}^2)v_i^2 B_i\big) = -h \sum_{i \in \mathbb{T}_h} \bar{u}_i^{\alpha+\beta}P(\xi_{1,i}, \xi_{2,i}),$$

and we can conclude as before. $\qquad \square$

The case $\alpha = 1$ has to be treated in a slightly different way. Since $\lim_{\alpha \to 1}(\alpha - 1)v_i = \lim_{\alpha \to 1} u_i^{\alpha-1} = 1$, we consider scheme

$$(22) \quad \partial_t u_i = -\frac{1}{h}(J_{i+1/2} - J_{i-1/2}), \quad J_{i+1/2} = \frac{1}{h}(A_{i+1} - A_i) - \frac{1}{2h}(w_{i+1} + w_i)(B_{i+1} - B_i),$$

with A_i and B_i defined via

$$(23) \qquad A_i = -\bar{u}_i^{\beta+1}(\lambda_1 \xi_{2,i} - \lambda_2 \xi_{1,i}^2), \quad B_i = \bar{u}_i^{\beta+1}(\lambda_3 \xi_{2,i} - \lambda_4 \xi_{1,i}^2),$$

and

$$\xi_{2,i} = \frac{1}{2h^2}\big((v_{i+1} - v_i)(w_{i+1} + w_i) - (v_i - v_{i-1})(w_i + w_{i-1})\big),$$

(24)
$$\xi_{1,i}^2 = \xi_{2,i} + \frac{1}{h^2}(v_{i+1} - 2v_i + v_{i-1}),$$

where $v_i = \log u_i$ and $w_i = -u_i^{-1}$. The parameters $\lambda_1, \dots, \lambda_4$ are given by (10) and (15).

Proposition 8. *Let $\alpha = 1$ and let $(u_i)_{i \in \mathbb{T}_h}$ be a positive solution to (22), (23), and (24). If $K(1, \beta) \geq 0$ then $dS_1^h/dt \leq 0$ for all $t > 0$. Moreover, if $K(1, \beta) > 0$,*

$$\frac{dS_1^h}{dt} + c(1, \beta)h \sum_{i \in \mathbb{T}_h} \bar{u}_i^{\beta+1}\big(\xi_{2,i}^2 + \xi_{1,i}^4\big) \leq 0,$$

with the same constant $c(1, \beta) > 0$ as in the proof of Theorem 2.

Proof. We have with $v_i = \log u_i$ and $w_i = -u_i^{-1}$:

$$\frac{dS_1^h}{dt} = \sum_{i \in \mathbb{T}_h}(v_{i+1} - v_i)J_{i+1/2}$$

$$= h^{-1}\sum_{i \in \mathbb{T}_h}(v_{i+1} - v_i)\left((A_{i+1} - A_i) - \frac{1}{2}(w_{i+1} + w_i)(B_{i+1} - B_i)\right)$$

$$= -h^{-1}\sum_{i \in \mathbb{T}_h}\bigg((v_{i+1} - 2v_i + v_{i-1})A_i$$

$$- \frac{1}{2}\big((v_{i+1} - v_i)(w_{i+1} + w_i) - (v_i - v_{i-1})(w_i + w_{i-1})\big)B_i\bigg).$$

By definition of $\xi_{1,i}$ and $\xi_{2,i}$,

$$\frac{1}{h^2}(v_{i+1} - 2v_i + v_{i-1}) = -\xi_{2,i} + \xi_{1,i}^2,$$

$$-\frac{1}{2h^2}\big((v_{i+1} - v_i)(w_{i+1} + w_i) - (v_i - v_{i-1})(w_i + w_{i-1})\big) = -\xi_{2,i}.$$

Therefore,

$$\frac{dS_1^h}{dt} = -h\sum_{i \in \mathbb{T}_h}\bar{u}_i^{\beta+1}\big((\lambda_1\xi_{2,i} - \lambda_2\xi_{1,i}^2)(-\xi_{2,i} + \xi_{1,i}^2) + (\lambda_3\xi_{2,i} - \lambda_4\xi_{1,i})\xi_{2,i}\big)$$

$$= -h\sum_{i \in \mathbb{T}_h}\bar{u}_i^{\beta+1}P_1(\xi_{1,i}, \xi_{2,i}),$$

where P_1 is the same polynomial as in (16). It is nonnegative if and only if $K(1, \beta) \geq 0$ holds. $\square$

4. Discretized multi-dimensional thin-film equation

The ideas of the previous section cannot be adapted in a straightforward way to the multi-dimensional setting, since there are many possibilities to choose the finite differences and the discrete variables. One idea is to employ only *scalar* variables like Δu, $|\nabla u|^2$, etc., similarly as for the method of systematic integration by parts of [15]. Still, there does not exist a general approach to define the scalar discrete variables, but we show in this section that the multi-dimensional case can be treated at least in principle. As an example, we consider the thin-film equation

$$\partial_t u = -\operatorname{div} J, \quad J = u^\beta \nabla \Delta u \quad \text{in } \mathbb{T}^d,$$

where $\mathbb{T}^d$ is the multi-dimensional torus and $\beta > 0$, and the logarithmic entropy

$$S_0(u) = \int_{\mathbb{T}^d} (-\log u) dx.$$

We show first that we can write $J = \nabla A - v\nabla B$, where $v = -u^{-1}$ and A, B are functions depending on $\Delta v/v$ and $|\nabla v|^2/v^2$.

Lemma 9. *It holds that $J = \nabla A - v\nabla B$, where*

$$A = u^\beta \big(\lambda_1 u^2 \Delta(-u^{-1}) - \lambda_2 u^3 |\nabla(-u^{-1})|^2\big) = \frac{u^\beta}{v}\left(\lambda_1 \frac{\Delta v}{v} + \lambda_2 \left|\frac{\nabla v}{v}\right|^2\right),$$

$$B = u^\beta \big(-\lambda_3 u^3 \Delta(-u^{-1}) + \lambda_4 u^4 |\nabla(-u^{-1})|^2\big) = \frac{u^\beta}{v^2}\left(\lambda_3 \frac{\Delta v}{v} + \lambda_4 \left|\frac{\nabla v}{v}\right|^2\right)$$

and the parameters λ_i are defined by

(25) $$\lambda_1 = \beta + 1, \quad \lambda_2 = -2(\beta + 1), \quad \lambda_3 = \beta, \quad \lambda_4 = -2\beta.$$

Proof. We compute

$$\nabla A + u^{-1}\nabla B = (\lambda_1 - \lambda_3)u^\beta \nabla \Delta u + \big(\beta\lambda_1 - (\beta + 1)\lambda_3\big)u^{\beta-1}\Delta u \nabla u$$
$$+ 2\big(-(2\lambda_1 + \lambda_2) + 2\lambda_3 + \lambda_4\big)u^{\beta-1}\nabla^2 u \nabla u$$
$$+ \big((1 - \beta)(2\lambda_1 + \lambda_2) + \beta(2\lambda_3 + \lambda_4)\big)|\nabla u|^2 \nabla u.$$

Here, $\nabla^2 u$ denotes the Hessian matrix of u. Identifying the coefficients of $\nabla A + u^{-1}\nabla B$ and $J = u^\beta \nabla \Delta u$ gives the linear system

$$\lambda_1 - \lambda_3 = 1, \qquad\qquad\qquad \beta\lambda_1 - (\beta + 1)\lambda_3 = 0,$$
$$-(2\lambda_1 + \lambda_2) + 2\lambda_3 + \lambda_4 = 0, \qquad (1 - \beta)(2\lambda_1 + \lambda_2) + \beta(2\lambda_3 + \lambda_4) = 0.$$

The unique solution is given by (25). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 10. *Let $\beta = 2$. Then $dS_0/dt \leq 0$ for all $t > 0$.*

Proof. The time derivative of S_0 becomes

$$\frac{dS_0}{dt} = \int_{\mathbb{T}^d} J \cdot \nabla v dx = \int_{\mathbb{T}^d} (\nabla A - v\nabla B) \cdot \nabla v dx$$

$$= -\int_{\mathbb{T}^d} \big(\Delta v A - (v\Delta v + |\nabla v|^2)B\big)dx$$

$$= -\int_{\mathbb{T}^d} u^\beta\left((\lambda_1 - \lambda_3)\left(\frac{\Delta v}{v}\right)^2 + (\lambda_2 - \lambda_3 - \lambda_4)\frac{\Delta v}{v}\left|\frac{\nabla v}{v}\right|^2 - \lambda_4\left|\frac{\nabla v}{v}\right|^4\right)dx.$$

The polynomial

(26) $$P_0(\xi_1, \xi_2) = (\lambda_1 - \lambda_3)\xi_2^2 + (\lambda_2 - \lambda_3 - \lambda_4)\xi_2\xi_1^2 - \lambda_4\xi_1^4$$

is nonnegative in $\mathbb{R}^2$ if and only if $-4\lambda_4 - (\lambda_2 - \lambda_3 - \lambda_4)^2 = -(\beta - 2)^2 \geq 0$. Hence, we need to assume that $\beta = 2$. $\qquad\square$

Remark 11 (Discussion)**.** The restriction $\beta = 2$ in the previous lemma comes from the fact that we do not have a free parameter to optimize the inequalities. One may overcome this issue by allowing A and B to depend on more variables or by assuming that A and B are matrix-valued with variables like $\nabla^2 u$ and $\nabla u \otimes \nabla u$ and to formulate $J = \operatorname{div} A - v\operatorname{div} B$. However, this leads to several parameters that need to be determined and eventually to complicated numerical schemes which seem to be less interesting in practice.

Another way to understand the restriction $\beta = 2$ is from systematic integration by parts. Indeed, computing

$$\frac{dS_0}{dt} = \int_{\mathbb{T}^d} u^\beta \nabla\Delta u \cdot \nabla(-u^{-1})dx - \int_{\mathbb{T}^d} \operatorname{div}(u^{\beta-2}\Delta u \nabla u)dx$$

$$= -\int_{\mathbb{T}^d} u^{\beta-2}(\Delta u)^2 dx - (\beta - 2)\int_{\mathbb{T}^d} u^{\beta-3}\Delta u|\nabla u|^2 dx,$$

we see that S_0 is a Lyapunov functional if the last term vanishes, which is the case if $\beta = 2$. This computation suggests the following generalization: Let $\beta \in (0, 2)$ and consider the Rényi entropy S_α. Then

$$\frac{dS_\alpha}{dt} = -\int_{\mathbb{T}^d} u^{\alpha+\beta-2}(\Delta u)^2 dx - (\alpha + \beta - 2)\int_{\mathbb{T}^d} u^{\alpha+\beta-3}\Delta u|\nabla u|^2 dx,$$

and the last term vanishes if $\alpha = 2 - \beta > 0$. As for the case $\beta = 2$ and $\alpha = 0$, which is discussed below, we expect that the generalization $\beta \in (0, 2)$ and $\alpha = 2 - \beta$ can be "translated" to the discrete case, but we leave the details to the interested reader. $\qquad\square$

We turn to the discrete setting. Let $\mathbb{T}_h^d$ be the discrete multi-dimensional torus, $u : \mathbb{T}_h^d \to \mathbb{R}$ be a function, and e_μ be the μ-th unit vector of $\mathbb{R}^d$. We write $u_i = u(i)$ for $i \in \mathbb{T}_h^d$ and we introduce the finite differences

$$\partial_\mu^+ u_i = \frac{1}{h}(u(i + he_\mu) - u(i)), \quad \partial_\mu^- u_i = \frac{1}{h}(u(i) - u(i - he_\mu)),$$

where $i \in \mathbb{T}_h^d$ and $\mu = 1, \ldots, d$. The discrete divergence of $F = (F_1, \ldots, F_d) : \mathbb{T}_h^d \to \mathbb{R}^d$ and the discrete gradient and Laplacian of $u : \mathbb{T}_h^d \to \mathbb{R}$ are defined by, respectively,

$$\operatorname{div}_h^+ F = \sum_{\mu=1}^d \partial_\mu^+ F_\mu, \quad (\nabla_h^\pm u)_\mu = \partial_\mu^\pm u, \quad \Delta_h u = \operatorname{div}_h^+ \nabla_h^- u.$$

where $\mu = 1, \ldots, d$. The discrete analogs of v_{xx}/v and $(v_x/v)^2$ are

$$\xi_{2,i} = \frac{\Delta_h v_i}{v_i}, \quad \xi_{1,i}^2 = \left| \frac{\nabla_h^+ v_i}{v_i} \right|^2,$$

where $v_i = -u_i^{-1}$. The numerical scheme reads as

$$\partial_t u_i = -\operatorname{div}_h^+ J_i, \quad J_i = \nabla_h^- A_i - v_i \nabla_h^- B_i,$$

(27) $$A_i = \frac{\bar{u}_i^\beta}{v_i}(\lambda_1 \xi_{2,i} + \lambda_2 \xi_{1,i}^2), \quad B_i = \frac{\bar{u}_i^\beta}{v_i^2}(\lambda_3 \xi_{2,i} + \lambda_4 \xi_{1,i}^2),$$

and the coefficients are as in (25).

Lemma 12. *Let $\beta = 2$ and $u_i^0 > 0$ for $i \in \mathbb{T}_h^d$. Then there exists a positive solution $(u_i)_{i \in \mathbb{T}_h^d}$ to (25) and (27), and it holds that $dS_0^h/dt \le 0$ for all $t > 0$.*

Proof. By the Picard–Lindelöf theorem, there exists a local smooth positive solution $(u_i)_{i \in \mathbb{T}_h^d}$. We use the summation-by-parts formula

$$\sum_{i \in \mathbb{T}_h^d} v_i \operatorname{div}_h^+ F_i = -\sum_{i \in \mathbb{T}_h^d} (\nabla_h^- v_i) \cdot F_i$$

to compute the time derivative of the entropy:

$$\frac{dS_0^h}{dt} = h \sum_{i \in \mathbb{T}_h^d} \partial_t u_i v_i = -h \sum_{i \in \mathbb{T}_h^d} \operatorname{div}_h^+ J_i v_i = h \sum_{i \in \mathbb{T}_h^d} J_i \cdot \nabla_h^- v_i$$

$$= h \sum_{i \in \mathbb{T}_h^d} (\nabla_h^- A_i - v_i \nabla_h^- B_i) \cdot \nabla_h^- v_i = -h \sum_{i \in \mathbb{T}_h^d} \left(\Delta_h v_i A_i - \operatorname{div}_h^+(v_i \nabla_h^- v_i) B_i \right).$$

The product rule reads as

$$\operatorname{div}_h^+(v_i \nabla_h^- v_i) = \sum_{\mu=1}^d \partial_\mu^+(v_i \partial_\mu^- v_i) = \frac{1}{h} \sum_{\mu=1}^d \partial_\mu^+ \big(v(i)(v(i) - v(i - he_\mu)) \big)$$

$$= \frac{1}{h^2} \sum_{\mu=1}^d \big[v(i + he_\mu)\big(v(i + he_\mu) - v(i)\big) - v(i)\big(v(i) - v(i - he_\mu)\big) \big]$$

$$= \frac{1}{h^2} \sum_{\mu=1}^d \big[v(i)\big(v(i + he_\mu) - 2v(i) + v(i - he_\mu)\big) + \big(v(i + he_\mu) - v(i)\big)^2 \big]$$

$$= v_i \Delta_h v_i + |\nabla_h^+ v_i|^2.$$

Therefore,

$$\frac{dS_0^h}{dt} = -h \sum_{i \in \mathbb{T}_h^d} \big(\Delta_h v_i A_i - (v_i \Delta_h v_i + |\nabla_h^+ v_i|^2) B_i \big)$$

$$= -h \sum_{i \in \mathbb{T}_h^d} \left(\xi_{2,i} v_i A_i - (\xi_{2,i} + \xi_{1,i}^2) v_i^2 B_i \right)$$

$$= -h \sum_{i \in \mathbb{T}_h^d} \left((\lambda_1 - \lambda_3)\xi_{2,i}^2 + (\lambda_2 - \lambda_3 - \lambda_4)\xi_{2,i}\xi_{1,i}^2 - \lambda_4\xi_{1,i}^4 \right)$$

$$= -h \sum_{i \in \mathbb{T}_h^d} \left(\xi_{2,i}^2 - (\beta + 2)\xi_{2,i}\xi_{1,i}^2 + 2\beta\xi_{1,i}^4 \right).$$

This gives the polynomial (26) which is nonnegative in $\mathbb{R}^2$ if and only if $\beta = 2$. Then $S_0^h(t) = S_0^h(0)$ for all $t > 0$, and we conclude as in the proof of Proposition 6 that $0 < c_0 \le u_i(t) \le c_1$ for all $i \in \mathbb{T}_h^d$ and $t > 0$, where $c_1 \ge c_0 > 0$ are some constants. Thus, the solution $u_i(t)$ can be extended to a global one. $\square$

5. NUMERICAL TESTS

We apply our scheme to the thin-film and DLSS equations on the torus in one and two space dimensions. The system of ordinary differential equations is solved by the command

`scipy.integrate.solve_ivp`

from the SciPy library, which uses the Backward Differentiation Formula (BDF) method of variable order or the implicit Runge–Kutta method of the Radau IIA family of order 5. We used the default values `atol`= 1e-3 for the absolute tolerance and `rtol`= 1e-6 for the relative tolerance. The local erroris computed according to `atol`+ rtol * abs(u).

5.1. **DLSS equation.** The DLSS equation is solved by scheme (18) using the logarithmic entropy:

$$\partial_t u_i = -\frac{1}{h}(J_{i+1/2} - J_{i-1/2}), \quad J_{i+1/2} = \frac{1}{h}(A_{i+1} - A_i) + \frac{1}{2h}(u_{i+1}^{-1} + u_i^{-1})(B_{i+1} - B_i),$$

$$A_i = \bar{u}_i\left(\frac{5}{3}\xi_{2,i} - \frac{7}{3}\xi_{1,i}^2\right), \quad B_i = \bar{u}_i u_i\left(-\frac{2}{3}\xi_{2,i} + \xi_{1,i}^2\right),$$

$$\xi_{1,i}^2 = \frac{\bar{u}_i^2}{2h^2}\left((u_{i+1}^{-1} - u_i^{-1})^2 + (u_i^{-1} - u_{i-1}^{-1})^2\right), \quad \xi_{2,i} = \frac{\bar{u}_i^2}{u_i h^2}(u_{i+1}^{-1} - 2u_i^{-1} + u_{i-1}^{-1}),$$

where $i \in \mathbb{T}_h$. Figure 1 shows the solution to the DLSS equation at various time steps using the initial datum $u^0(x) = \max\{10^{-10}, \cos(\pi x)^{16}\}$ and the space grid size $h = 1/100$. We see that the solution is not monotone, since it possesses at $x = 0.5$ and $t = 10^{-8}$ a local maximum. After some time, it approaches the constant steady state given by $\int_0^1 u^0(x)dx$.

The entropy decay for $\alpha = 0$ is illustrated in Figure 2 (left). We used the initial datum $u^0(x) = 2 - 10^{-6}$ for $x \in (0, 0.5)$ and $u^0(x) = 10^{-6}$ for $x \in (0.5, 1)$. We observe in the semi-logarithmic plot that the decay is exponential, as expected. The rate degrades for larger times when the ℓ^2 error dominates, i.e., when the grid is rather coarse.

The ℓ^2 error (in space and time) at time $t = 0.001$ is shown in Figure 2 (right), using the initial datum $u^0(x) = 1 + 0.5\sin(2\pi x)$ for $x \in (0, 1)$. As an explicit solution is not known,
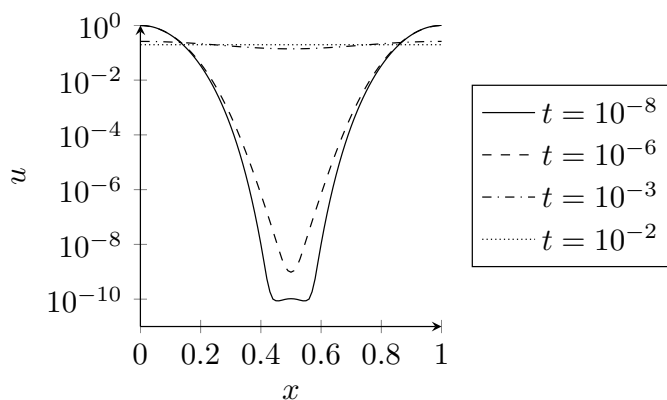
FIGURE 1. Evolution of the DLSS equation in a semi-logarithmic scale, using the initial datum $u^0(x) = \max\{10^{-10}, \cos(\pi x)^{16}\}$.

we use a numerical solution with $h = 1/2048$ as the reference solution. As expected, the convergence rate is roughly of second order.
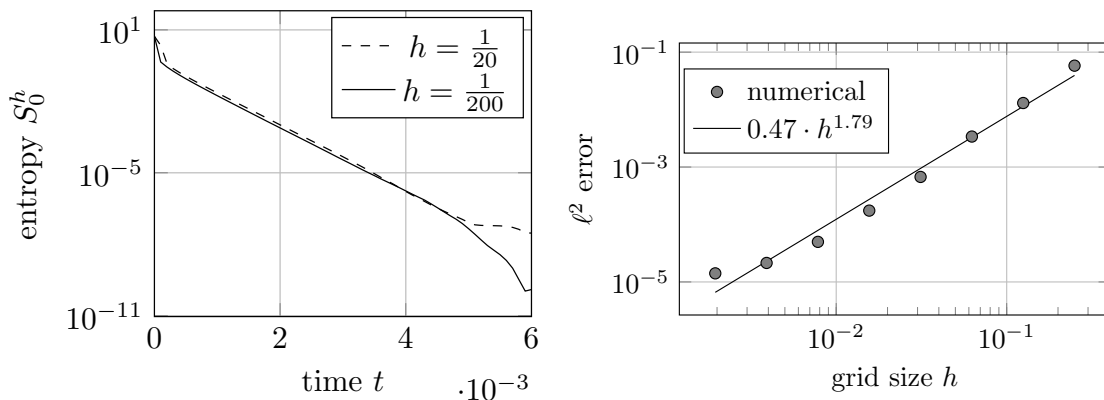

FIGURE 2. Left: Decay of the logarithmic entropy $s_0(u(t))$ for two different space grid sizes $h = 1/20$ and $h = 1/200$. Right: Convergence of the ℓ^2 error. The dots are the values from the numerical solution, the solid line is the regression curve.

5.2. **Thin-film equation.** The thin-film equation is solved by scheme (18) using the logarithmic entropy:

$$\partial_t u_i = -\frac{1}{h}(J_{i+1/2} - J_{i-1/2}), \quad J_{i+1/2} = \frac{1}{h}(A_{i+1} - A_i) + \frac{1}{2h}(u_{i+1}^{-1} + u_i^{-1})(B_{i+1} - B_i),$$

$$A_i = u_i^{\beta+1}\left(\frac{7\beta + 9}{9}\xi_{2,i} + \frac{\beta^2 - 14\beta - 18}{9}\xi_{1,i}^2\right),$$

$$B_i = u_i^{\beta+2}\left(-\frac{7\beta}{9}\xi_{2,i} + \frac{15\beta - \beta^2}{9}\xi_{1,i}^2\right),$$

$$\xi_{1,i}^2 = \frac{u_i^2}{2h^2}\big((u_{i+1}^{-1} - u_i^{-1})^2 + (u_i^{-1} - u_{i-1}^{-1})^2\big), \quad \xi_{2,i} = \frac{u_i}{h^2}(u_{i+1}^{-1} - 2u_i^{-1} + u_{i-1}^{-1}),$$

where $i \in \mathbb{T}_h$. The solutions at different times, emanating from the initial datum $u^0(x) = 1 + 0.5\sin(2\pi x)$, are shown in Figure 3, where we have chosen $\beta = 2$. Again, the solutions converge to the constant steady state. The decay of the logarithmic entropy is illustrated in Figure 4, using $\beta = 2$ and the initial datum $u^0(x) = 1 + (1 - 10^{-16})\sin(2\pi x)$ for $x \in (0,1)$. The decay rate is exponential over a large time interval.
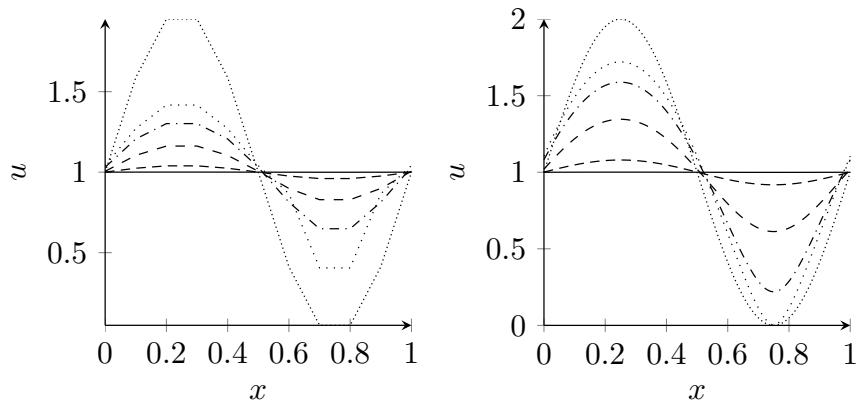

FIGURE 3. Evolution of the solution to the thin-film equation at times $t = 0$ (densely dotted), $t = 2 \cdot 10^{-4}$ (dotted), $t = 5 \cdot 10^{-4}$ (dash-dotted), $t = 1 \cdot 10^{-3}$ (dashed), $t = 2 \cdot 10^{-3}$ (densely dashed), and $t = 5 \cdot 10^{-3}$ (solid) and grid sizes $h = 1/10$ (left), $h = 1/200$ (right).
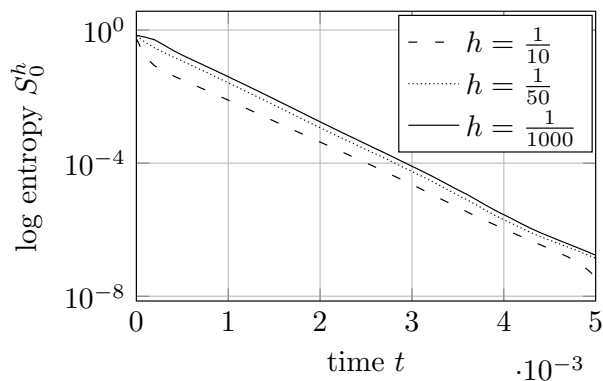

FIGURE 4. Decay of the logarithmic entropy $S_0(u(t))$ for various space grid sizes.

Finally, we present a numerical example in two space dimensions. As the initial datum, we choose a lantern picture with 77×100 pixels in gray scale; see Figure 5 (top left). The evolution of the discrete solution is shown in the remaining panels of Figure 5 for various times. The values $u = 0$ and $u = 1$ correspond to black and white, respectively. Because

of the periodic boundary conditions, we observe a small gray band at the lower right boundary. Interestingly, the solution shows a denoising effect, especially for $t = 3 \cdot 10^{-8}$. For larger times, the diffusion drives the solution to the constant steady state. These results are not surprising, as fourth-order parabolic equations have been suggested in the literature for image denoising. For instance, Bertozzi and Greer [2] analyzed

$$\partial_t u = -\operatorname{div}\big(g((\Delta u)^2)\nabla\Delta u\big),$$

where g is a diffusivity function, while Wei [21] considered

$$\partial_t u = -\operatorname{div}\big(g(|\nabla u|^2)\nabla\Delta u\big).$$

This model was generalized to fractional derivatives; see, e.g., [12]. An example is the equation

$$\partial_t u = -\operatorname{div}\big(g(-(\Delta)^{1-\varepsilon}u)\nabla\Delta u\big), \quad \varepsilon > 0,$$

which formally reduces to a general thin-film equation in the limiting case $\varepsilon = 1$. We do *not* claim that the thin-film equation is a good image denoising model; our numerical example is just a nice illustration.

Finally, we show the entropy decay of the two-dimensional example in Figure 6. The decay rate is exponential until approximately $t = 10^{-2}$. For later times, the numerical error dominates. Observe, however, that we obtain denoising for very small times, like $t = 10^{-9} \ldots 10^{-8}$, where the decay rate is still exponential.

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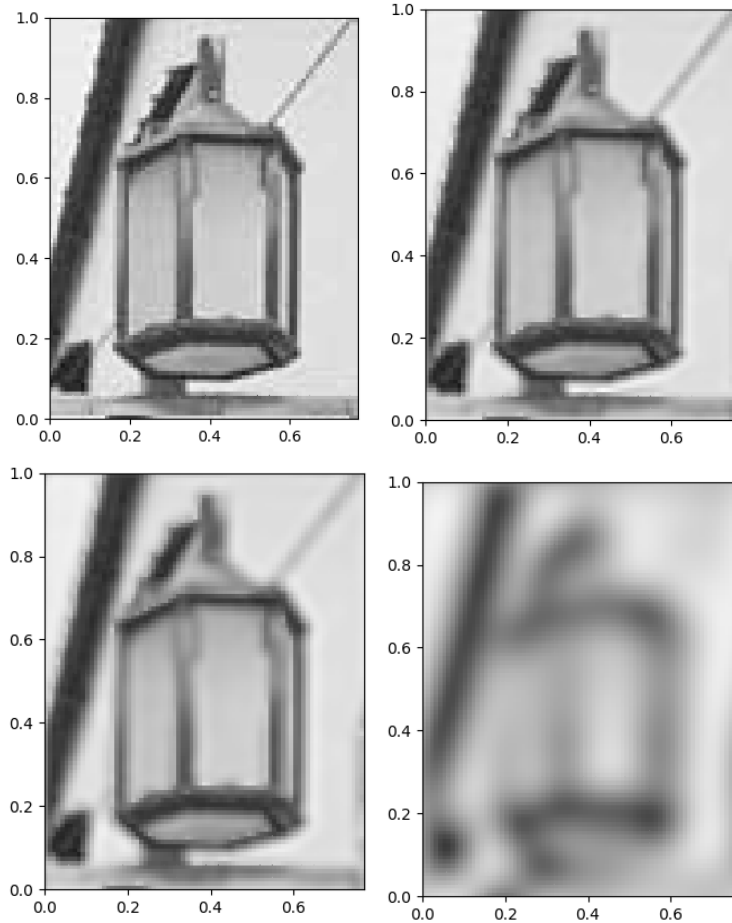
FIGURE 5. Evolution of the solution to the two-dimensional thin-film equation with $\beta = 2$, $t = 0$ (top left), $t = 3 \cdot 10^{-9}$ (top right), $t = 10^{-8}$ (bottom left), $t = 10^{-6}$ (bottom right).

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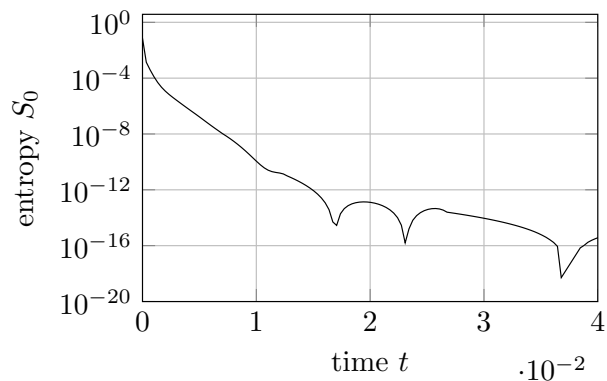
FIGURE 6. Decay of the logarithmic entropy $S_0(u(t))$ for various space grid sizes.

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