# CORRIGENDUM: CROSS DIFFUSION PREVENTING BLOW UP IN THE TWO-DIMENSIONAL KELLER-SEGEL MODEL

MARCEL BRAUKHOFF, XIUQING CHEN, AND ANSGAR JÜNGEL

Abstract. We correct the proof of Proposition 2.1 in the paper [1]. This proposition is used to prove the existence of global weak solutions to a Keller-Segel model with additional cross-diffusion.

In [1], the following result has been stated.

**Proposition 1** (Proposition 2.1 in [1]). *Let $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) be a bounded domain with $\partial\Omega \in C^{0,1}$, $T > 0$, and $s \geq 0$. Furthermore, let $(u_\varepsilon)$ be a sequence of nonnegative functions satisfying*

$$\|\sqrt{u_\varepsilon}\|_{L^2(0,T;H^1(\Omega))} + \|u_\varepsilon \log u_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} + \|\partial_t u_\varepsilon\|_{L^1(0,T;(H^s(\Omega))^*)} \leq C_0$$

*for some $C > 0$ independent of $\varepsilon$. Then, up to a subsequence, as $\varepsilon \to 0$,*

$$u_\varepsilon \to u \quad \text{strongly in } L^2(0,T;L^{d/(d-1)}(\Omega)).$$

The proof in [1] consists in showing first that $u_\varepsilon \to u$ strongly in $L^\infty(0,T;L^1(\Omega))$ as $\varepsilon \to 0$. However, this is generally wrong as the counter-example $u_\varepsilon(x,t) = \max\{0, 1-t/\varepsilon\}$ shows.[1] In the following, we give a corrected proof for Proposition 2.1.

*Proof.* The uniform estimate for $u_\varepsilon$ implies that $\nabla u_\varepsilon = 2\sqrt{u_\varepsilon}\nabla\sqrt{u_\varepsilon}$ is uniformly bounded in $L^2(0,T;L^1(\Omega))$. Thus, $(u_\varepsilon)$ is bounded in $L^2(0,T;W^{1,1}(\Omega))$. We observe that the embedding $W^{1,1}(\Omega) \hookrightarrow L^p(\Omega)$ is compact for all $1 < p < d/(d-1)$. Moreover, if $s \geq d/2$, the embedding $H^s(\Omega) \hookrightarrow L^{p^*}(\Omega)$, where $p^* = p/(p-1)$, and hence $L^p(\Omega) \hookrightarrow (H^s(\Omega))^*$ is continuous. Thus, we can apply the Aubin-Lions lemma with the spaces $W^{1,1}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow (H^s(\Omega))^*$. If $0 \leq s < d/2$, we have $(H^s(\Omega))^* \hookrightarrow (H^{d/2}(\Omega))^*$ and $\|\partial_t u_\varepsilon\|_{L^1(0,T;(H^{d/2}(\Omega))^*)} \leq C\|\partial_t u_\varepsilon\|_{L^1(0,T;(H^s(\Omega))^*)} \leq CC_0$, and the Aubin-Lions lemma can be applied with the spaces $W^{1,1}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow (H^{d/2}(\Omega))^*$. In both cases, there exists a subsequence of $(u_\varepsilon)$, which is not relabeled, such that $u_\varepsilon \to u$ strongly in $L^2(0,T;L^1(\Omega))$ and also a.e. in $\Omega \times (0,T)$.

Let $L > e$ be given, and set $v_\varepsilon^L = \min\{u_\varepsilon, L\}$ and $w_\varepsilon^L = \max\{u_\varepsilon - L, 0\}$. Then $u_\varepsilon = v_\varepsilon^L + w_\varepsilon^L$. It holds that $v_\varepsilon^L \to v^L = \min\{u, L\}$ a.e. and $w_\varepsilon^L \to w^L = \max\{u - L, 0\}$ a.e. with

*Date*: February 25, 2020.

The first and last authors acknowledge partial support from the Austrian Science Fund (FWF), grants P30000, W1245, and F65. The second author acknowledges support from the National Natural Science Foundation of China (NSFC), grant 11971072.

[1] We thank Martin Vetter, BSc for pointing out this example to us.

$u = v^L + w^L$. Then

$$\begin{aligned}\|u_\varepsilon - u\|_{L^2(0,T;L^{d/(d-1)}(\Omega))} &\le \|v_\varepsilon^L - v^L\|_{L^2(0,T;L^{d/(d-1)}(\Omega))} + \|w_\varepsilon^L\|_{L^2(0,T;L^{d/(d-1)}(\Omega))} \\ &\quad + \|w^L\|_{L^2(0,T;L^{d/(d-1)}(\Omega))} \\ &=: I_1 + I_2 + I_3. \qquad (1)\end{aligned}$$

We first estimate the terms $I_2$ and $I_3$. By the Hölder's inequality, we find that

$$\begin{aligned}\|\nabla w_\varepsilon^L\|^2_{L^2(0,T;L^1(\Omega))} &= \int_0^T \left( \int_{\{u_\varepsilon > L\}} |\nabla u_\varepsilon| dx \right)^2 dt \\ &= 4 \int_0^T \left( \int_{\{u_\varepsilon > L\}} |\sqrt{u_\varepsilon}||\nabla \sqrt{u_\varepsilon}| dx \right)^2 dt \\ &\le 4 \|\sqrt{u_\varepsilon}\|^2_{L^2(0,T;H^1(\Omega))} \left\| \int_{\{u_\varepsilon > L\}} u_\varepsilon dx \right\|_{L^\infty(0,T)} \\ &\le 4C_0^2 \left\| \int_{\{u_\varepsilon > L\}} u_\varepsilon \frac{\log u_\varepsilon}{\log L} dx \right\|_{L^\infty(0,T)} \le \frac{4C_0^3}{\log L}\end{aligned}$$

and

$$\begin{aligned}\|w_\varepsilon^L\|^2_{L^2(0,T;L^1(\Omega))} &= \int_0^T \left( \int_{\{u_\varepsilon > L\}} (u_\varepsilon - L) dx \right)^2 dt \le \int_0^T \left( \int_{\{u_\varepsilon > L\}} u_\varepsilon dx \right)^2 dt \\ &\le T \left\| \int_{\{u_\varepsilon > L\}} u_\varepsilon \frac{\log u_\varepsilon}{\log L} dx \right\|^2_{L^\infty(0,T)} \le \frac{TC_0^2}{\log^2 L} \le \frac{TC_0^2}{\log L}.\end{aligned}$$

Therefore,

$$\|w_\varepsilon^L\|_{L^2(0,T;W^{1,1}(\Omega))} \le \frac{2C_0^{3/2} + T^{1/2}C_0}{(\log L)^{1/2}}.$$

A similar way, it follows that

$$\|w^L\|_{L^2(0,T;W^{1,1}(\Omega))} \le \frac{2C_0^{3/2} + T^{1/2}C_0}{(\log L)^{1/2}}.$$

We conclude from the Sobolev imbedding $W^{1,1}(\Omega) \hookrightarrow L^{d/(d-1)}(\Omega)$ ($d \ge 2$) with the constant $C_d > 0$ that

$$I_2 + I_3 \le C_d \|w_\varepsilon^L\|_{L^2(0,T;W^{1,1}(\Omega))} + C_d \|w^L\|_{L^2(0,T;W^{1,1}(\Omega))} \le \frac{2C_d(2C_0^{3/2} + T^{1/2}C_0)}{(\log L)^{1/2}}.$$

For the estimate of $I_1$, we observe that, since $|v_\varepsilon^L(x,t) - v^L(x,t)|^{d/(d-1)} \le (2L)^{d/(d-1)}$ and $|v_\varepsilon^L(x,t) - v^L(x,t)|^{d/(d-1)} \to 0$ a.e. in $\Omega \times (0,T)$, the dominated convergence theorem implies that $\int_\Omega |v_\varepsilon^L(x,t) - v^L(x,t)|^{d/(d-1)} dx \to 0$ a.e. in $(0,T)$ and hence $\|v_\varepsilon^L(\cdot,t) - v^L(\cdot,t)\|_{L^{d/(d-1)}(\Omega)} \to 0$ a.e. in $(0,T)$. Moreover,

$$\left\| \|v_\varepsilon^L(\cdot,t) - v^L(\cdot,t)\|_{L^{d/(d-1)}(\Omega)} \right\|_{L^3(0,T)} \le 2L|\Omega|^{(d-1)/d} T^{1/3} =: C(L).$$

We claim that for any $L > e$, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, it holds that $I_1 < 1/\log L$. Indeed, set $f_\varepsilon^L(t) := \|v_\varepsilon^L(\cdot,t) - v^L(\cdot,t)\|_{L^{d/(d-1)}(\Omega)}$. Recall that $f_\varepsilon^L(t) \to 0$ a.e. in $(0,T)$ and $\|f_\varepsilon^L\|_{L^3(0,T)} \le C(L)$. For given $\delta > 0$, there exists $\eta > 0$ such that $C(L)^2\eta^{1/3} \le \delta$. We deduce for any $E \subset (0,T)$ such that $|E| \le \eta$ and Hölder's inequality that

$$\int_E |f_\varepsilon^L(t)|^2 dt \le \bigg( \int_E |f_\varepsilon^L(t)|^3 dt \bigg)^{2/3} |E|^{1/3} \le C(L)^2\eta^{1/3} \le \delta,$$

which shows that $(f_\varepsilon^L)$ is uniformly integrable. As convergence a.e. in $(0,T)$ implies convergence in measure in $(0,T)$, we can apply the Vitali convergence theorem to infer that $f_\varepsilon^L \to 0$ strongly in $L^2(0,T)$ as $\varepsilon \to 0$. Thus, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, we have $I_1 = \|f_\varepsilon^L\|_{L^2(0,T)} \le 1/\log L$. Therefore, for any $L > e$, there exists $\varepsilon_0(L) > 0$ such that for all $0 < \varepsilon < \varepsilon_0(L)$, we infer from (1) that

$$\|u_\varepsilon - u\|_{L^2(0,T;L^{d/(d-1)}(\Omega))} \le \frac{1 + 2C_d(2C_0^{3/2} + T^{1/2}C_0)}{(\log L)^{1/2}}.$$

Since $L > e$ is arbitrary, this ends the proof. □

## References

[1] S. Hittmeir and A. Jüngel. Cross diffusion preventing blow up in the two-dimensional Keller-Segel model. *SIAM J. Math. Anal.* 43 (2011), 997–1022.

Institute for Analysis and Scientific Computing, Vienna University of Technology, Wiedner Hauptstrasse 8–10, 1040 Wien, Austria
*E-mail address*: marcel.braukhoff@tuwien.ac.at

School of Mathematics (Zhuhai), Sun Yat-sen University, Zhuhai 519082, Guangdong Province, China
*E-mail address*: buptxchen@yahoo.com

Institute for Analysis and Scientific Computing, Vienna University of Technology, Wiedner Hauptstrasse 8–10, 1040 Wien, Austria
*E-mail address*: juengel@tuwien.ac.at