GLOBAL MARTINGALE SOLUTIONS TO A SEGREGATION CROSS-DIFFUSION SYSTEM WITH STOCHASTIC FORCING

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Abstract. The existence of a global martingale solution to a cross-diffusion system with multiplicative Wiener noise in a bounded domain with no-flux boundary conditions is shown. The model describes the dynamics of population densities of different species due to segregation cross-diffusion effects. The diffusion matrix is generally neither symmetric nor positive semidefinite. This difficulty is overcome by exploiting the Rao entropy structure. The existence proof uses a stochastic Galerkin method, uniform estimates from the Rao entropy inequality, and the Skorokhod–Jakubowski theorem. Furthermore, an exponential equilibration result is proved for sufficiently small Lipschitz constants of the noise by using the relative Rao entropy. Numerical tests illustrate the behavior of solutions in one space dimension for two and three population species.

1. Introduction

The segregation of population species can be described by cross-diffusion systems involving quadratic nonlinearities; see, e.g., [3]. This class of models was derived as the mean-field limit of moderately interacting particle systems [9]. In this paper, we analyze these systems taking into account the random influence of the environment. We assume that the dynamics of the population density u_i of the i-th species is modeled by

$$\mathrm{d}u_i(t) = \operatorname{div}\big(\delta\nabla u_i + u_i\nabla p_i(u)\big)\mathrm{d}t + \sum_{j=1}^n \sigma_{ij}(u)\mathrm{d}W_j(t), \quad p_i(u) = \sum_{j=1}^n a_{ij}u_j, \tag{1}$$

in a bounded domain $\mathcal{O} \subset \mathbb{R}^d$ ($d \geq 1$) with the initial and no-flux boundary conditions

$$u_i(0) = u_i^0 \quad \text{in } \mathcal{O}, \quad \nabla u_i \cdot \nu = 0 \quad \text{on } \partial\mathcal{O},\ t > 0,\ i = 1, \ldots, n, \tag{2}$$

where ν is the exterior unit normal vector to $\partial\mathcal{O}$, $W_1, \ldots, W_n$ are cylindrical Wiener processes in a Hilbert space U (see Section 2 for details), and $\delta > 0$, $a_{ij} \geq 0$ for $i, j = 1, \ldots, n$. The density u_i depends on the spatial variable $x \in \mathcal{O}$, the time $t \geq 0$, and the stochastic variable $\omega \in \Omega$. The terms $a_{ij}u_i\nabla u_j$ for $i \neq j$ are called cross-diffusion terms.

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The analysis of equations (1) is delicate already in the deterministic case, where $\sigma_{ij}(u) = 0$, since the diffusion matrix is generally neither symmetric nor positive semi-definite. The key idea of the analysis is to exploit the entropy structure associated to (1).

To explain this structure, we need two assumptions. First, we suppose that the diffusion matrix $(u_i a_{ij}) \in \mathbb{R}^{n\times n}$ has only eigenvalues with positive real part (for $u_i > 0$). This means that system (1) is parabolic in the sense of Petrovskii, which is a minimal condition for local solvability [1]. Second, we assume the existence of numbers $\pi_i > 0$ satisfying

$$\pi_i a_{ij} = \pi_j a_{ji} \quad \text{for all } i,j = 1,\dots,n,$$

which is the detailed-balance condition for the Markov chain associated to (a_{ij}), and $(\pi_1,\dots,\pi_n)$ is the corresponding reversible stationary measure. Both assumptions imply that the matrix $(\pi_i a_{ij})$ is symmetric positive definite. We introduce the so-called Rao entropy

$$H(u) = \int_{\mathcal{O}} h(u)\mathrm{d}x, \quad h(u) = \frac{1}{2}\sum_{i,j=1}^n \pi_i a_{ij} u_i u_j, \tag{3}$$

which is used as a diversity measure in population dynamics [18]. Observe that the entropy density $h(u)$ is not the sum of the individual entropies of the species, but it mixes the species. A formal computation, which is made rigorous below (see Lemma 4), shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}H(u(t)) + \delta\sum_{i,j=1}^n \pi_i a_{ij}\mathbb{E}\int_{\mathcal{O}} \nabla u_i\cdot\nabla u_j \mathrm{d}x + \sum_{i=1}^n \pi_i\mathbb{E}\int_{\mathcal{O}} u_i|\nabla p_i(u)|^2\mathrm{d}x = 0. \tag{4}$$

Since $(\pi_i a_{ij})$ is positive definite and $\delta > 0$, this yields an a priori estimate for ∇u_i in $L^2(\mathcal{O})$. The last integral can be interpreted as the kinetic energy of the system, with $\nabla p_i(u)$ being the partial velocity. Unfortunately, it does not provide any gradient estimate, and this is the reason why we have included the δ-terms. The main results of this paper are the existence of a global martingale solution to (1)–(2) and the exponential decay in expectation of the solution to its spatial average. These results are detailed in Section 2.

The analysis of stochastic cross-diffusion systems is rather recent. The existence of martingale solutions to cross-diffusion systems with a positive definite diffusion matrix (including nonlocal diffusion) was shown in [2]. A stochastic population cross-diffusion system with cubic nonlinearities was investigated in [12]. In these works, the quadratic energy structure allows for the use of a stochastic Galerkin method. Combining the theory of quasilinear parabolic equations with evolution semigroup methods, the authors of [14] proved the existence of a unique local pathwise mild solution to the Shigesada–Kawasaki–Teramoto system with stochastic forcing.

For the existence analysis of system (1), we use similar techniques as in [12]. The large-time behavior result is based on the relative entropy method (which becomes here a weighted L^2 norm). Both results are new.

The random influence may also be taken into account on the level of the fluxes, using a conservative noise of the type $\sum_{j=1}^n \operatorname{div}\sigma_{ij}(u)\mathrm{d}W_j$. This noise can be handled by considering the noise in $H^{-1}(\mathcal{O})$ and estimating in the norm of that space [10]. Writing the noise in Stratonovich form and assuming that $\operatorname{div}(\sigma(u)/u) = 0$, the conservative noise can be reformulated as an Itô noise plus a regularizing Laplacian term; see [17]. The authors of [13] pass to the kinetic formulation of the equation, which yields an equation in which the noise enters as a linear transport. Unfortunately, these techniques cannot be easily applied to equations (1) with arbitrary conservative noise.

2. Notation and main results

Let $\mathcal{O} \subset \mathbb{R}^d$ ($d \geq 1$) be a bounded domain and let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space endowed with a complete right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. We write $L^p(\Omega, \mathcal{F}; B)$ or simpler $L^p(\Omega; B)$ for the set of all $\mathcal{F}$-measurable B-valued random variables in a Banach space B, such that $\mathbb{E}\|u\|_B^p = \int_\Omega \|u(\omega)\|_B^p \mathbb{P}(\mathrm{d}\omega) < \infty$ ($1 \leq p < \infty$). Let U be a Hilbert space with orthonormal basis $(\eta_k)_{k\in\mathbb{N}}$. The space of Hilbert–Schmidt operators from U to $L^2(\mathcal{O})$ is defined by

$$\mathcal{L}_2(U; L^2(\mathcal{O})) = \left\{ F : U \to L^2(\mathcal{O}) \text{ linear, continuous} : \sum_{k=1}^\infty \|F\eta_k\|_{L^2(\mathcal{O})} < \infty \right\},$$

endowed with the norm $\|F\|_{\mathcal{L}_2(U;L^2(\mathcal{O}))} = (\sum_{k=1}^\infty \|F\eta_k\|^2_{L^2(\mathcal{O})})^{1/2}$. Let β_{jk} for $j = 1, \ldots, n$, $k \in \mathbb{N}$ be independent one-dimensional Brownian motions. There exists a Hilbert space $U_0 \supset U$ and a Hilbert–Schmidt embedding $J : U \to U_0$ such that the series $W_j = \sum_{k=1}^\infty \beta_{jk} J(\eta_k)$ converges in $L^\infty(0, T; L^2(\Omega; U_0))$. Moreover, $W_j(\omega) \in C^0([0, T); U_0)$ for a.e. $\omega \in \Omega$ [15, Prop. 2.5.2].

We impose the following hypotheses:

- (H1) Domain: Let $\mathcal{O} \subset \mathbb{R}^d$ ($1 \leq d \leq 3$) be a bounded domain with Lipschitz boundary.
- (H2) Initial datum: $u_i^0 \in L^2(\Omega, \mathcal{F}_0; L^2(\mathcal{O}))$ satisfies $u_i^0 \geq 0$ a.e. in $\mathcal{O}$, $\mathbb{P}$-a.s. for $i = 1, \ldots, n$.
- (H3) Diffusion coefficients: All eigenvalues of $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ have positive real parts and the detailed-balance condition holds, i.e., there exist $\pi_1, \ldots, \pi_n >$ such that
$$\pi_i a_{ij} = \pi_j a_{ji} \quad \text{for all } i, j = 1, \ldots, n.$$
- (H4) Noise coefficients: $\sigma_{ij} : L^2(\mathcal{O})\times\Omega \to \mathcal{L}_2(U; L^2(\mathcal{O}))$ is $\mathcal{B}(L^2(\mathcal{O})\otimes\mathcal{F}/\mathcal{B}(\mathcal{L}_2(U; L^2(\mathcal{O})))$-measurable and $\mathbb{F}$-adapted, and there exists $C_\sigma > 0$ such that for all $i, j = 1, \ldots, n$ and $u, v \in L^2(\mathcal{O})$,
$$\begin{aligned} \|\sigma_{ij}(u) - \sigma_{ij}(v)\|_{\mathcal{L}_2(U;L^2(\mathcal{O}))} &\leq C_\sigma \|u - v\|_{L^2(\mathcal{O})}, \\ \|\sigma_{ij}(u)\|_{\mathcal{L}_2(U;L^2(\mathcal{O}))} &\leq C_\sigma \|u_i\|_{L^2(\mathcal{O})}. \end{aligned}$$

Let us comment on these hypotheses. The restriction $d \leq 3$ is not essential; our proof works in any space dimension, but we need to choose a larger space to show the Aldous condition, needed for the tightness of the laws of the Galerkin solutions. Let $P = \operatorname{diag}(\pi_1, \ldots, \pi_n)$. Then, by Hypothesis (H3), PA is symmetric. Since the eigenvalues of PA are positive, we conclude that PA is positive definite. The linear growth condition in Hypothesis (H4) is needed to prove the $\mathbb{P}$-a.s. nonnegativity of u_i. It can be weakened to $\|\sigma_{ij}(u)\|_{\mathcal{L}_2(U;L^2(\mathcal{O}))} \leq C_\sigma(1 + \|u\|_{L^2(\mathcal{O})})$, but then we cannot prove the nonnegativity property.

Definition 1 (Martingale solution). *Let $T > 0$. A* global martingale solution *to* (1)–(2) *is a tuple $(\widetilde{V}, \widetilde{W}, \widetilde{u})$ such that*

- *$\widetilde{V} = (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, \widetilde{\mathbb{F}})$ is a complete probability space with a filtration $\widetilde{\mathbb{F}} = (\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$;*
- *$\widetilde{W} = (\widetilde{W}_1, \ldots, \widetilde{W}_n)$ is a cylindrical Wiener process in $U^n := U \times \cdots \times U$;*
- *$\widetilde{u} = (\widetilde{u}_1, \ldots, \widetilde{u}_n) : [0, T] \times \Omega \to L^2(\mathcal{O})$ is an $\mathbb{F}$-progressively measurable process such that for all $t \in [0, T]$ and $i = 1, \ldots, n$,*

$$\widetilde{u}_i \in L^2(\Omega; C^0([0, T]; L^2_w(\mathcal{O}))) \cap L^2(\Omega; L^2(0, T; H^1(\mathcal{O}))),$$

the law of $\widetilde{u}_i(0)$ is the same as for u_i^0, and $\widetilde{u}_i$ satisfies for $t \in [0, T]$ and $\phi \in H^1(\mathcal{O})$,

$$(\widetilde{u}_i(t), \phi)_{L^2(\mathcal{O})} = (\widetilde{u}_i(0), \phi)_{L^2(\mathcal{O})} - \int_0^t \int_{\mathcal{O}} \big(\delta\nabla\widetilde{u}_i(s) + \widetilde{u}_i(s)\nabla p_i(\widetilde{u}(s))\big) \cdot \nabla\phi \mathrm{d}x\mathrm{d}s$$
$$- \int_0^t \bigg(\sum_{j=1}^n \sigma_{ij}(\widetilde{u}(s))\mathrm{d}\widetilde{W}_j(s), \phi\bigg)_{L^2(\mathcal{O})} \quad \mathbb{P}\text{-}a.s.$$

Our first main result is as follows.

Theorem 1 (Existence of a global martingale solution). *Let $T > 0$ and let Hypotheses (H1)–(H4) hold. Then there exists a global martingale solution $(\widetilde{U}, \widetilde{W}, \widetilde{u})$ to* (1)–(2) *satisfying $\widetilde{u}_i(t) \geq 0$ a.e. in $\mathcal{O}$, $\widetilde{\mathbb{P}}$-a.s. for all $t \in [0, T]$, $i = 1, \ldots, n$.*

As mentioned in the introduction, the idea of the proof is the use of the stochastic Galerkin method combined with the entropy method. We project equations (1) on a Galerkin space with finite dimension $N \in \mathbb{N}$. The existence of a pathwise unique strong solution $u^N = (u_1^N, \ldots, u_n^N)$ (up to some stopping time) is shown by means of Banach's fixed-point theorem. Itô's lemma allows us to derive an entropy inequality; see (4). Since we introduced the δ-terms, we obtain estimates for u^N in $H^1(\mathcal{O})$ uniformly in N.

The tightness of the laws of (u^N) in the topological space Z_T, defined in (15), is proved by applying the criterion of [4]. We deduce from Skorokhod–Jakubowski's theorem that there exists a subsequence of (u^N), another probability space, and random variables $(\widetilde{u}^N, \widetilde{W}^N)$ having the same law as (u^N, W^N), and $(\widetilde{u}^N, \widetilde{W}^N)$ converges to $(\widetilde{u}, \widetilde{W})$ in the topology of Z_T. Because of the gradient estimates and compactness, we infer the strong convergence $\widetilde{u}^N \to \widetilde{u}$ in $L^2(\mathcal{O})$ a.s., and we can identify the limits in the nonlinearities. Finally, the a.s. nonnegativity follows from a stochastic Stampacchia truncation argument from [8].

For the second main result, we introduce the relative Rao entropy

$$H(u|\bar{u}) = H(u) - H(\bar{u}) - \frac{\delta H}{\delta u}(\bar{u}) \cdot (u - \bar{u}) = \frac{1}{2}\sum_{i,j=1}^n \pi_i a_{ij} \int_{\mathcal{O}} (u_i - \bar{u}_i)(u_j - \bar{u}_j)\mathrm{d}x,$$

where $\bar{u}$ is the solution to

$$\mathrm{d}\bar{u}_i(t) = \sum_{j=1}^n \bar{\sigma}_{ij}(u(t))\mathrm{d}W_j(t), \quad t > 0, \quad \bar{\sigma}_{ij}(u) = \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} \sigma_{ij}(u)\mathrm{d}x,$$

and $|\mathcal{O}|$ is the measure of $\mathcal{O}$. Recalling $P = \mathrm{diag}(\pi_1, \ldots, \pi_n)$ and $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, the relative entropy can be written as the weighted L^2 norm

$$H(u|\bar{u}) = \frac{1}{2}\|(PA)^{1/2}(u-\bar{u})\|^2_{L^2(\mathcal{O})},$$

Theorem 2 (Exponential time decay)**.** *Let Hypotheses (H1)–(H4) hold and let u be a martingale solution to* (1)–(2)*. Then there exists $c_0 > 0$ such that for all $0 < C_\sigma < c_0$ (see Hypothesis (H4)),*

$$\mathbb{E}H(u(t)|\bar{u}(t)) \le \mathbb{E}H(u^0|\bar{u}(0))e^{-\eta t}, \quad \eta := c_0^2 - C_\sigma^2, \ t > 0.$$

Usually, the large-time behavior of solutions to stochastic differential equations is analyzed by proving the existence and uniqueness of an invariant measure and studying the ergodicity of the equations. This can be done, for instance, by establishing the strong Feller property and the tightness of the laws of the solutions and by applying the Krylov–Bogolyubov's theorem [11, Chapter 11]. Here, this program is delicate since we are lacking semigroup properties. Therefore, we rely on estimates from the entropy method.

For the proof of Theorem 2, we apply Itô's lemma to the process $e^{-\eta t/2}(PA)^{1/2}(u-\bar{u})$, take the expectation, and use the Poincaré–Wirtinger inequality with constant $C_P > 0$ for the gradient term. This leads to (see Section 4)

$$\begin{aligned} &e^{\eta t}\mathbb{E}H(u(t)|\bar{u}(t)) - \mathbb{E}H(u(0)|\bar{u}(0)) - \eta\int_0^t e^{\eta s}\mathbb{E}H(u(s)|\bar{u}(s))\mathrm{d}s \\ &\quad \le -\frac{\delta\lambda}{C_P^2}\int_0^t \|u-\bar{u}\|^2_{L^2(\mathcal{O})}\mathrm{d}s + \frac{1}{2}\mathbb{E}\int_0^t \|(PA)^{1/2}(\sigma(u)-\bar{\sigma}(u))\|^2_{\mathcal{L}_2(U;L^2(\mathcal{O}))}\mathrm{d}s. \end{aligned} \tag{5}$$

The difficult part of the proof is the estimate of the last integral. Using the properties of $\sigma_{ij}(u(t))$ and $\bar{\sigma}_{ij}(u(t))$, we find that

$$\mathbb{E}\int_0^t \|(PA)^{1/2}(\sigma(u)-\bar{\sigma}(u))\|^2_{\mathcal{L}_2(U;L^2(\mathcal{O}))}\mathrm{d}s \le 2C_1C_\sigma^2\|u-\bar{u}\|^2_{L^2(\mathcal{O})}.$$

Thus choosing $C_\sigma > 0$ (see Hypothesis (H4)) and $\eta > 0$ sufficiently small, this integral can be absorbed by the first term on the right-hand side of (5), and we conclude the result.

3. Proof of Theorem 1

The proof of Theorem 1 is split into several steps.

3.1. **Stochastic Galerkin approximation.** Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathcal{O})$, which is orthogonal in $H^1(\mathcal{O})$. For each $N\in\mathbb{N}$, define the finite-dimensional subspace $H_N := \operatorname{span}(e_1,\ldots,e_N)$ of $L^2(\mathcal{O})$ and the corresponding projection $\Pi_N : L^2(\mathcal{O})\to H_N$, $\Pi_N(v)=\sum_{i=1}^N (v,e_i)_{L^2(\mathcal{O})}e_i$ for $v\in L^2(\mathcal{O})$. We consider system (1) projected on the subspace $H_N^n := H_n\times\cdots\times H_N$:

$$\mathrm{d}u_i^N(t) = \Pi_N \operatorname{div}\big(\delta\nabla u_i^N + (u_i^N)^+\nabla p_i(u^N)\big)\mathrm{d}t + \Pi_N\sum_{j=1}^n \sigma_{ij}(u^N)\mathrm{d}W_j(t), \quad t>0, \tag{6}$$

$$u_i^N(0) = \Pi_N(u_i^0), \quad i=1,\ldots,n, \tag{7}$$

where $z^+=\max\{0,z\}$ denotes the positive part. This truncation is necessary, since the Rao entropy does not provide nonnegative densities. The nonnegativity of the Galerkin limit u_i is proved in Section 3.5. Given $T>0$, we introduce the space $M_T := L^2(\Omega; C^0([0,T];H_N^n))$ with the norm $\|u\|_{M_T} = (\mathbb{E}\sup_{0<t<T}\|u(t)\|^2_{L^2(\mathcal{O})})^{1/2}$ for $u\in M_T$. For given $R>0$ and $u\in M_T$, we define the exit time $\tau_R=\inf\{t\in[0,T]:\|u(t)\|_{H^1(\mathcal{O})}>R\}$. Then $\{\omega\in\Omega : \tau_R(\omega)>t\}$ belongs to $\mathcal{F}_t$ for every $t\in[0,T]$, and τ_R is an $\mathbb{F}$-stopping time.

Lemma 3 (Local existence of u^N)**.** *Let $T>0$, $R>0$ and let the Hypotheses (H1)–(H4) hold. Then there exists a pathwise unique strong solution $u^N\in M_{T\wedge\tau_R}$ to (6)–(7) such that for any $t\in[0,T\wedge\tau_R]$, $i=1,\ldots,n$, and $\phi=(\phi_1,\ldots,\phi_n)\in H_N^n$,*

$$\begin{aligned}(u_i^N(t),\phi_i)_{L^2(\mathcal{O})} &= (u_i(0),\phi_i)_{L^2(\mathcal{O})} - \int_0^t\int_{\mathcal{O}}\big(\delta\nabla u_i^N(s) + (u_i^N)^+(s)\nabla p_i(u^N(s))\big)\cdot\nabla\phi_i \mathrm{d}x\mathrm{d}s \\ &\quad + \left(\sum_{j=1}^n\int_0^t \sigma_{ij}(u^N(s))\mathrm{d}W_j(s),\phi_i\right)_{L^2(\mathcal{O})} \quad \mathbb{P}\text{-a.s.}\end{aligned}$$

Proof. We use Banach's fixed-point theorem. Define the fixed-point operator $S: M_T\to M_T$ for $v\in M_T$ and $\psi\in H_N^n$ by

$$\begin{aligned}(S(v)(t),\psi)_{L^2(\mathcal{O})} &= \sum_{i=1}^n(\Pi_N u_i^0,\psi_i)_{L^2(\mathcal{O})} - \sum_{i=1}^n\int_0^t\int_{\mathcal{O}}\big(\delta\nabla v_i + v_i^+\nabla p_i(v)\big)\cdot\nabla\psi_i\mathrm{d}x\mathrm{d}s \\ &\quad + \sum_{i=1}^n\left(\sum_{j=1}^n\int_0^t\sigma_{ij}(v(s))\mathrm{d}W_j(s),\psi_i\right)_{L^2(\mathcal{O})} \quad \mathbb{P}\text{-a.s., } t\in[0,T].\end{aligned}$$

The aim is to show that S is a contraction on $M_{T\wedge\tau_R}$ for sufficiently small $T>0$.

First, we verify that S is a self-mapping. Let $v\in M_T$, $\psi\in H_N^n$, and $T_R:=T\wedge\tau_R$. Then

$$\begin{aligned}&\|(S(v),\psi)_{L^2(\mathcal{O})}\|^2_{L^2(\Omega;L^\infty(0,T_R))} \le I_1+\cdots+I_4, \quad \text{where} \\ &I_1 = \mathbb{E}\|u^0\|^2_{L^2(\mathcal{O})}\|\psi\|^2_{L^2(\mathcal{O})}, \\ &I_2 = \mathbb{E}\bigg(\sup_{0<t<T_R}\bigg|\sum_{i=1}^n\int_0^t\int_{\mathcal{O}}\delta\nabla v_i(s)\cdot\nabla\psi_i\mathrm{d}x\mathrm{d}s\bigg|\bigg)^2,\end{aligned} \tag{8}$$

$$I_3 = \mathbb{E}\bigg(\sup_{0<t<T_R}\bigg|\sum_{i,j=1}^n\int_0^t\int_{\mathcal{O}} a_{ij}v_i^+(s)\nabla v_j(s)\cdot\nabla\psi_i\mathrm{d}x\mathrm{d}s\bigg|\bigg)^2,$$

$$I_4 = \mathbb{E}\bigg(\sup_{0<t<T_R}\bigg|\sum_{i=1}^n\bigg(\sum_{j=1}^n\int_0^t\sigma_{ij}(v)\mathrm{d}W_j(s),\psi_i\bigg)_{L^2(\mathcal{O})}\bigg|\bigg)^2.$$

We estimate the term I_2, using the equivalence of the norms in H_N:

$$\begin{aligned}
I_2 &\le \delta^2 T\mathbb{E}\bigg(\sup_{0<t<T_R}\sum_{i=1}^n\int_0^t\bigg|\int_{\mathcal{O}}\nabla v_i(s)\cdot\nabla\psi_i\mathrm{d}x\bigg|^2\mathrm{d}s\bigg) \\
&\le \delta^2 TC\mathbb{E}\bigg(\int_0^{T_R}\|\nabla v(s)\|_{L^2(\mathcal{O})}^2\mathrm{d}s\bigg)\|\nabla\psi\|_{L^2(\mathcal{O})}^2 \\
&\le C(N)\delta^2T^2\mathbb{E}\bigg(\sup_{0<s<T_R}\|v(s)\|_{L^2(\mathcal{O})}^2\bigg)\|\psi\|_{L^2(\mathcal{O})}^2 = C(N)\delta^2T^2\mathbb{E}\|v\|_{M_{T_R}}^2\|\psi\|_{L^2(\mathcal{O})}^2,
\end{aligned}$$

where here and in the following $C>0$, $C_i>0$, etc. are constants independent of the solution, with values changing from line to line. The term I_3 is estimated in a similar way, taking into account that $\|\nabla v(s)\|_{L^2(\mathcal{O})}\le R$ for $s<T_R$:

$$\begin{aligned}
I_3 &\le CT\mathbb{E}\bigg(\int_0^{T_R}\|v(s)\|_{L^2(\mathcal{O})}^2\|\nabla v(s)\|_{L^2(\mathcal{O})}^2\mathrm{d}s\bigg)\|\nabla\psi\|_{L^\infty(\mathcal{O})}^2 \\
&\le C(N)R^2T^2\mathbb{E}\bigg(\sup_{0<s<T_R}\|v(s)\|_{L^2(\mathcal{O})}^2\bigg)\|\psi\|_{L^2(\mathcal{O})}^2,
\end{aligned}$$

Finally, by the Burkholder–Davis–Gundy inequality [16, Theorem 1.1.7] and (H4),

$$\begin{aligned}
I_4 &\le C\mathbb{E}\bigg(\int_0^{T_R}\|\sigma(v(s))\|_{\mathcal{L}_2(U;L^2(\mathcal{O}))}^2\mathrm{d}s\bigg)\|\psi\|_{L^2(\mathcal{O})}^2 \\
&\le C_\sigma^2C\mathbb{E}\bigg(\int_0^{T_R}\|v(s)\|_{L^2(\mathcal{O})}^2\mathrm{d}s\bigg)\|\psi\|_{L^2(\mathcal{O})}^2 \le C_\sigma^2C(N)T\|v\|_{M_{T_R}}^2\|\psi\|_{L^2(\mathcal{O})}^2.
\end{aligned}$$

Inserting these estimates into (8) leads to

$$\|S(v)\|_{M_{T_R}}^2 \le \mathbb{E}\|u^0\|_{L^2(\mathcal{O})}^2 + C(N,R)(T^2+T)\|v\|_{M_{T_R}}^2.$$

This shows that S is a self-mapping. Next, we prove that S is a contraction if $T>0$ is sufficiently small. The estimations are similar as above with the exception of the nonlinear diffusion part. Let $u,v\in M_T$ and $\psi\in H_N^n$. Then

$$\|(S(u)-S(v),\psi)_{L^2(\mathcal{O})}\|_{L^2(\Omega;L^\infty(0,T_R))}^2 \le I_5+I_6+I_7, \quad \text{where} \tag{9}$$

$$I_5 = \mathbb{E}\bigg(\sup_{0<t<T_R}\bigg|\sum_{i=1}^n\int_0^t\int_{\mathcal{O}}\delta\nabla(u_i-v_i)(s)\cdot\nabla\psi_i\mathrm{d}x\mathrm{d}s\bigg|\bigg)^2,$$

$$I_6 = \mathbb{E}\bigg(\sup_{0<t<T_R}\bigg|\sum_{i,j=1}^n\int_0^t\int_{\mathcal{O}}a_{ij}\big(u_i^+(s)\nabla u_j(s)-v_i^+(s)\nabla v_j(s)\big)\cdot\nabla\psi_i\mathrm{d}x\mathrm{d}s\bigg|\bigg)^2,$$

$$I_7 = \mathbb{E}\bigg(\sup_{0<t<T_R}\bigg|\sum_{i=1}^n\bigg(\int_0^t\big(\sigma_{ij}(u(s))-\sigma_{ij}(v(s))\big)\mathrm{d}W_j(s),\psi_i\bigg)_{L^2(\mathcal{O})}\bigg|\bigg)^2.$$

The terms I_5 and I_7 are estimated as I_2 and I_4, respectively, giving

$$I_5 \le C(N)\delta^2T^2\|u-v\|_{M_{T_R}}^2\|\psi\|_{L^2(\mathcal{O})}^2, \quad I_7 \le C(N)T\|u-v\|_{M_{T_R}}^2\|\psi\|_{L^2(\mathcal{O})}^2.$$

Writing $u_i^+\nabla u_j - v_i^+\nabla v_j = u_i^+\nabla(u_j-v_j) + (u_i^+-v_i^+)\nabla v_j$ and using the Lipschitz continuity of $z\mapsto z^+$, the remaining term becomes

$$\begin{aligned}
I_6 &\le C_AT\mathbb{E}\bigg(\int_0^{T_R}\|u(s)\|_{L^2(\mathcal{O})}^2\|\nabla(u-v)(s)\|_{L^2(\mathcal{O})}^2\mathrm{d}s\bigg)\|\nabla\psi\|_{L^\infty(\mathcal{O})}^2 \\
&\quad + C_AT\mathbb{E}\bigg(\int_0^{T_R}\|(u-v)(s)\|_{L^2(\mathcal{O})}^2\|\nabla v(s)\|_{L^2(\mathcal{O})}^2\mathrm{d}s\bigg)\|\nabla\psi\|_{L^\infty(\mathcal{O})}^2 \\
&\le C_AC(N)R^2T^2\mathbb{E}\bigg(\sup_{0<s<T_R}\|(u-v)(s)\|_{L^2(\mathcal{O})}^2\bigg)\|\psi\|_{L^2(\mathcal{O})}^2 \\
&= C(N,R)T^2\|u-v\|_{M_{T_R}}^2\|\psi\|_{L^2(\mathcal{O})}^2.
\end{aligned}$$

We conclude from (9) that

$$\|S(u)-S(v)\|_{M_{T_R}}^2 \le C(N,R)(T^2+T)\|u-v\|_{M_{T_R}}^2.$$

Then, choosing $T>0$ such that $C(N,R)(T^2+T)<1$, the mapping $S: M_{T_R}\to M_{T_R}$ is a contraction. By Banach's fixed-point theorem, there exists a unique fixed point $u^N\in M_{T_R}$ that solves (6)–(7) in $[0,T_R]$. □

3.2. **Uniform estimates.** We show that the solution u^N to (6)–(7) satisfies suitable uniform estimates. The following lemma is the key result of this subsection.

Lemma 4 (Entropy inequality for u^N)**.** *Let $T>0$ and let u^N be the pathwise unique strong solution to (6)–(7) on $[0,T\wedge\tau_R]$, constructed in Lemma 3. Then there exists a constant $C(\lambda,T)$, depending on λ and T, such that*

$$\begin{aligned}
&\mathbb{E}\bigg(\sup_{0<t<T}H(u^N(t))\bigg) + 2\delta\lambda\mathbb{E}\bigg(\int_0^T\int_{\mathcal{O}}|\nabla u^N(s)|^2\mathrm{d}x\mathrm{d}s\bigg) \\
&\quad + \sum_{i=1}^n\mathbb{E}\bigg(\int_0^T\int_{\mathcal{O}}(u_i^N)^+(s)|\nabla p_i(u^N(s))|^2\mathrm{d}x\mathrm{d}s\bigg) \le \mathbb{E}H(u^N(0))C(\lambda,T),
\end{aligned}$$

where $\lambda>0$ is the smallest eigenvalue of PA.

Proof. Since PA is symmetric positive definite, there exists $(PA)^{1/2}$ and the process $Y(t)=(PA)^{1/2}u^N(t)$ is well defined. Hence, we can write the entropy (3) as

$$H(u^N) = \frac12\sum_{i,j=1}^n\int_{\mathcal{O}}\pi_ia_{ij}u_i^Nu_j^N\mathrm{d}x = \frac12\int_{\mathcal{O}}(u^N)^T(PA)u^N\mathrm{d}x = \frac12\|(PA)^{1/2}u^N\|_{L^2(\mathcal{O})}^2,$$

Let $t_R = t \wedge \tau_R$ and $T_R = T \wedge \tau_R$. We apply Itô's lemma [15, Theorem 4.2.5] to $Y(t)$:

$$H(u^N(t_R)) = H(u^N(0)) + J_1 + \cdots + J_4, \quad \text{where} \tag{10}$$

$$J_1 = -\delta \sum_{i,j=1}^n \pi_i a_{ij} \int_0^{t_R} \int_{\mathcal{O}} \nabla u_i^N(s) \cdot \nabla u_j^N(s) \mathrm{d}x\mathrm{d}s \le -\delta\lambda \int_0^{t_R} \int_{\mathcal{O}} |\nabla u^N|^2 \mathrm{d}x\mathrm{d}s,$$

$$J_2 = -\sum_{i=1}^n \int_0^{t_R} \int_{\mathcal{O}} (u_i^N)^+(s) |\nabla p_i(u^N(s))|^2 \mathrm{d}x\mathrm{d}s \le 0,$$

$$J_3 = \frac{1}{2} \int_0^{t_R} \int_{\mathcal{O}} \mathrm{Tr}\left[\sigma(u^N(s)) \mathrm{D}^2 h(u^N(s)) \sigma(u^N(s))^*\right] \mathrm{d}x\mathrm{d}s,$$

$$J_4 = \sum_{i,j=1}^n \pi_i a_{ij} \int_{\mathcal{O}} \bigg(\int_0^{t_R} u_i^N(s) \sigma_{ij}(u^N(s)) \mathrm{d}W_j(s) \bigg) \mathrm{d}x.$$

Using $\mathrm{D}^2 h(u^N) = PA$ and Hypothesis (H4),

$$\begin{aligned} J_3 &= \frac{1}{2} \int_0^{t_R} \|(PA)^{1/2} \sigma(u^N(s))\|^2_{\mathcal{L}_2(U;L^2(\mathcal{O}))} \mathrm{d}s \le C \int_0^{t_R} \|\sigma(u^N(s))\|^2_{\mathcal{L}_2(U;L^2(\mathcal{O}))} \mathrm{d}s \\ &\le C C_\sigma^2 \int_0^{t_R} \|u^N(s)\|^2_{L^2(\mathcal{O})} \mathrm{d}s \le C(\lambda) \int_0^{t_R} H(u^N(s)) \mathrm{d}s. \end{aligned} \tag{11}$$

Inserting this estimate into (10), taking the supremum over $[0, T_R]$ and then the expectation, we find that

$$\begin{aligned} \mathbb{E}\bigg(\sup_{0<t<T_R} H(u^N(t)) \bigg) &\le \mathbb{E} H(u^N(0)) - \delta\lambda \mathbb{E}\bigg(\sup_{0<t<T_R} \int_0^t \int_{\mathcal{O}} |\nabla u^N(s)|^2 \mathrm{d}x\mathrm{d}s \bigg) \\ &\quad - \mathbb{E}\bigg(\sup_{0<t<T_R} \sum_{i=1}^n \int_0^t \int_{\mathcal{O}} (u_i^N)^+(s) |\nabla p_i(u^N(s))|^2 \mathrm{d}x\mathrm{d}s \bigg) \\ &\quad + C(\lambda) \mathbb{E}\bigg(\sup_{0<t<T_R} \int_0^t H(u^N(s)) \mathrm{d}s \bigg) + \mathbb{E}\bigg(\sup_{0<t<T_R} J_4(t) \bigg). \end{aligned} \tag{12}$$

We deduce from the Burkholder–Davis–Gundy inequality that

$$\begin{aligned} \mathbb{E}\bigg(\sup_{0<t<T_R} J_4(t) \bigg) &\le C \sum_{i,j,\ell=1}^n \mathbb{E}\bigg\{ \sup_{0<t<T_R} \int_0^t \sum_{k=1}^\infty \bigg(\int_{\mathcal{O}} u_j^N(s) \sigma_{i\ell}(u^N(s)) \eta_k \mathrm{d}x \bigg)^2 \mathrm{d}s \bigg\}^{1/2} \\ &\le C \sum_{i,j,\ell=1}^n \mathbb{E}\bigg\{ \sup_{0<t<T_R} \int_0^t \sum_{k=1}^\infty \|u_j^N(s)\|^2_{L^2(\mathcal{O})} \|\sigma_{i\ell}(u^N(s)) \eta_k\|^2_{L^2(\mathcal{O})} \mathrm{d}s \bigg\}^{1/2} \\ &\le C \mathbb{E}\bigg(\int_0^{T_R} \|u^N(s)\|^2_{L^2(\mathcal{O})} \|\sigma(u^N(s))\|^2_{\mathcal{L}_2(U;L^2(\mathcal{O}))} \mathrm{d}s \bigg)^{1/2}. \end{aligned}$$

Therefore, using Hypothesis (H4) and Young's inequality,

$$\begin{aligned}\mathbb{E}\bigg(\sup_{0<t<T_R} J_4(t)\bigg) &\le C_\sigma C\mathbb{E}\bigg\{\sup_{0<t<T_R}\|u^N(s)\|_{L^2(\mathcal{O})}\bigg(\int_0^{T_R}\|u^N(s)\|_{L^2(\mathcal{O})}^2\mathrm{d}s\bigg)^{1/2}\bigg\} \\ &\le C(\lambda)\mathbb{E}\bigg\{\sup_{0<t<T_R}\|(PA)^{1/2}u^N(s)\|_{L^2(\mathcal{O})}\bigg(\int_0^{T_R}\|(PA)^{1/2}u^N(s)\|_{L^2(\mathcal{O})}^2\mathrm{d}s\bigg)^{1/2}\bigg\} \\ &\le \frac12\mathbb{E}\bigg(\sup_{0<t<T_R} H(u^N(t))\bigg) + C(\lambda)\int_0^{T_R}\mathbb{E}\bigg(\sup_{0<s<t} H(u^N(s))\bigg)\mathrm{d}t.\end{aligned}$$

We insert this estimate into (12),

$$\begin{aligned}\frac12\mathbb{E}\bigg(\sup_{0<t<T_R} H(u^N(t))\bigg) &\le \mathbb{E}H(u^N(0)) - \delta\lambda\mathbb{E}\int_0^{T_R}\int_{\mathcal{O}}|\nabla u^N(s)|^2\mathrm{d}x\mathrm{d}s \\ &\quad - \sum_{i=1}^n\mathbb{E}\int_0^{T_R}\int_{\mathcal{O}}(u_i^N)^+(s)|\nabla p_i(u^N(s))|^2\mathrm{d}x\mathrm{d}s + C(\lambda)\int_0^{T_R}\mathbb{E}\bigg(\sup_{0<s<t} H(u^N(s))\bigg)\mathrm{d}t,\end{aligned} \tag{13}$$

and apply Gronwall's lemma to obtain

$$\mathbb{E}\bigg(\sup_{0<t<T_R} H(u^N(t))\bigg) \le \mathbb{E}H(u^N(0))e^{2C(\lambda)T}.$$

Hence, the right-hand side of (13) does not depend on the chosen sequence of stopping times τ_R, and we can pass to the limit $R\to\infty$, finishing the proof. □

Corollary 5 (Uniform estimates). *Let $T>0$ and let u^N be the pathwise unique strong solution to (6)–(7) on $[0,T\wedge\tau_R]$, constructed in Lemma 3. Then there exists a constant $C>0$, which is independent of N, such that for all $i=1,\ldots,n$,*

$$\mathbb{E}\bigg(\sup_{0<t<T}\|u_i^N(t)\|_{L^2(\mathcal{O})}^2\bigg) + \delta\lambda\mathbb{E}\bigg(\int_0^T\|\nabla u_i^N(t)\|_{L^2(\mathcal{O})}^2\mathrm{d}t\bigg) \le C.$$

We also need higher-order moment estimates.

Lemma 6 (Higher-order moment estimates). *Let $T>0$, $p>2$, and let u^N be the pathwise unique strong solution to (6)–(7) on $[0,T\wedge\tau_R]$, constructed in Lemma 3. Then there exists a constant $C>0$, which is independent of N, such that*

$$\mathbb{E}\bigg(\sup_{0<t<T}\|u^N(t)\|_{L^2(\mathcal{O})}^p\bigg) \le C.$$

Proof. Let $T_R=T\wedge\tau_R$. We start from (10), neglect the terms J_1 and J_2, take the supremum over $(0,T_R)$, raise the power $p/2$ on both sides, take the expectation, and use Hölder's inequality:

$$\mathbb{E}\bigg(\sup_{0<t<T_R} H(u^N(t))^{p/2}\bigg) \le C\mathbb{E}H(u^N(0))^{p/2} + J_5 + J_6, \quad \text{where} \tag{14}$$

$$J_5 = CT^{p/2-1}\mathbb{E}\int_0^{T_R}\|(PA)^{1/2}\sigma(u^N(s))\|_{\mathcal{L}_2(U;L^2(\mathcal{O}))}^p\mathrm{d}s,$$

$$J_6 = \mathbb{E}\bigg(\sup_{0<t<T_R}\bigg|\sum_{i,j,k=1}^n \pi_i a_{ij}\int_{\mathcal{O}}\int_0^t u_j^N(s)\sigma_{ik}(u^N(s))\mathrm{d}W_k(s)\mathrm{d}x\bigg|^{p/2}\bigg)$$

The term J_5 can be estimated as in (11):

$$J_5 \le C(\lambda)\mathbb{E}\int_0^{T_R} H(u^N(s))^{p/2}\mathrm{d}s \le C(\lambda)\int_0^{T_R}\mathbb{E}\bigg(\sup_{0<s<t} H(u^N(s))^{p/2}\bigg)\mathrm{d}t.$$

Similarly as in Lemma 4, we apply the Burkholder–Davis–Gundy inequality and Young's inequality to find that

$$\begin{aligned} J_6 &\le C\mathbb{E}\bigg(\int_0^{T_R}\|u^N(s)\|_{L^2(\mathcal{O})}^2\|\sigma(u^N(s))\|_{\mathcal{L}_2(U;L^2(\mathcal{O}))}^2\mathrm{d}s\bigg)^{p/4} \\ &\le CT^{1-p/2}\mathbb{E}\bigg\{\bigg(\sup_{0<s<T_R}\|u^N(s)\|_{L^2(\mathcal{O})}^p\bigg)^{1/2}\bigg(\int_0^{T_R}\|u^N(s)\|_{L^2(\mathcal{O})}^p\mathrm{d}s\bigg)^{1/2}\bigg\} \\ &\le \frac{1}{2}\mathbb{E}\bigg(\sup_{0<s<T_R} H(u^N(s))^{p/2}\bigg) + C(T,\lambda)\int_0^{T_R}\mathbb{E}\bigg(\sup_{0<s<t} H(u^N(s))^{p/2}\bigg)\mathrm{d}t. \end{aligned}$$

We insert these estimates into (14) and apply Gronwall's lemma to conclude that

$$\mathbb{E}\bigg(\sup_{0<t<T_R} H(u^N(t))^{p/2}\bigg) \le C(\lambda,T)\mathbb{E}H(u^N(0))^{p/2},$$

and the positive definiteness of $(PA)^{1/2}$ finishes the proof. □

3.3. **Tightness.** The tightness of the sequence of laws of u^N on a suitable subspace is proved similarly as in [12, Section 2.4]. We consider the following spaces:

- $C^0([0,T];H^3(\mathcal{O})')$ with the topology $\mathcal{T}_1$, induced by its canonical norm.
- $L^2_w(0,T;H^1(\mathcal{O}))$ is the space $L^2(0,T;H^1(\mathcal{O}))$ with the weak topology $\mathcal{T}_2$.
- $L^2(0,T;L^2(\mathcal{O}))$ with the topology $\mathcal{T}_3$ induced by its canonical norm.
- $C^0([0,T];L^2_w(\mathcal{O}))$ is the space of weakly continuous functions $v:[0,T]\to L^2(\mathcal{O})$ endowed with the weakest topology $\mathcal{T}_4$ such that for all $h\in L^2(\mathcal{O})$, the mappings
$$C^0([0,T];L^2_w(\mathcal{O}))\to C^0([0,T];\mathbb{R}),\quad u\mapsto (u(\cdot),h)_{L^2(\mathcal{O})},$$
are continuous.

We define the following space:

$$Z_T := C^0([0,T];H^3(\mathcal{O})')\cap L^2_w(0,T;H^1(\mathcal{O}))\cap L^2(0,T;L^2(\mathcal{O}))\cap C^0([0,T];L^2_w(\mathcal{O})), \tag{15}$$

endowed with the topology $\mathcal{T}$, which is the maximum of the topologies $\mathcal{T}_1,\ldots,\mathcal{T}_4$ of the corresponding spaces, i.e. the smallest topology containing $\cap_{i=1}^4\mathcal{T}_i$.

Lemma 7 (Tightness)**.** *Let u^N be the pathwise unique strong solution to (6)–(7), constructed in Lemma 3. Then the set of laws of (u^N) is tight on $(Z_T,\mathcal{T})$.*

Proof. We apply [6, Corollary 3.9] with $U = H^3(\mathcal{O})$, $V = H^1(\mathcal{O})$, and $H = L^2(\mathcal{O})$. Since $V \hookrightarrow H$ is compact, Corollary 5 shows that conditions (a) and (b) of [6, Corollary 3.9] are satisfied. It remains to verify that (u^N) satisfies the Aldous condition in $H^3(\mathcal{O})'$. For this, let $(\tau_N)_{N\in\mathbb{N}}$ be a sequence of $\mathbb{F}$-stopping times such that $0 \le \tau_N \le T$. Let $t \in [0,T]$, $i \in \{1,\ldots,n\}$, and $\phi \in H^3(\mathcal{O})$. We write (6) as

$$\begin{aligned} &\langle u_i^N(t),\phi\rangle = \langle K_1^N + K_2^N(t) + K_3^N(t),\phi\rangle, \quad \text{where} \\ &\langle K_1^N,\phi\rangle = (\Pi_N(u_i^0),\phi)_{L^2(\mathcal{O})}, \\ &\langle K_2^N(t),\phi\rangle = -\int_0^t\int_{\mathcal{O}} \Pi_N\bigg(\delta\nabla u_i^N(s) + \sum_{j=1}^n a_{ij}(u_i^N)^+(s)\nabla u_j^N(s)\bigg)\cdot\nabla\phi\mathrm{d}x\mathrm{d}s, \\ &\langle K_3^N(t),\phi\rangle = \sum_{j=1}^n\bigg(\int_0^t \Pi_N\big(\sigma_{ij}(u^N(s))\big)\mathrm{d}W_j(s),\phi\bigg)_{L^2(\mathcal{O})}, \end{aligned} \tag{16}$$

and $\langle\cdot,\cdot\rangle$ denotes the dual pairing between $H^3(\mathcal{O})'$ and $H^3(\mathcal{O})$. (If $d \ge 1$ is arbitrary, we replace $H^3(\mathcal{O})$ by $H^m(\mathcal{O})$, where $m \ge 1 + d/2$, $m \in \mathbb{N}$.) Let $\theta > 0$. Then, using the embedding $H^3(\mathcal{O}) \hookrightarrow W^{1,\infty}(\mathcal{O})$ for $d \le 3$ and the Cauchy–Schwarz inequality,

$$\begin{aligned} \mathbb{E}|\langle K_2^N(\tau_N+\theta) - K_2^N(\tau_N),\phi\rangle| &\le \mathbb{E}\int_{\tau_N}^{\tau_N+\theta}\bigg(\delta\|\nabla u_i(t)\|_{L^2(\mathcal{O})}\|\nabla\phi\|_{L^2(\mathcal{O})} \\ &\quad + \sum_{j=1}^n a_{ij}\|(u_i^N)^+(s)\|_{L^2(\mathcal{O})}\|\nabla u_j^N\|_{L^2(\mathcal{O})}\|\nabla\phi\|_{L^\infty(\mathcal{O})}\bigg)\mathrm{d}t \\ &\le \delta\theta^{1/2}\bigg(\mathbb{E}\int_{\tau_N}^{\tau_N+\theta}\|\nabla u_i(t)\|_{L^2(\mathcal{O})}^2\mathrm{d}t\bigg)^{1/2}\|\phi\|_{H^1(\mathcal{O})} \\ &\quad + C_A^{1/2}\theta^{1/2}\bigg\{\mathbb{E}\bigg(\sup_{0<t<T}\|u_i^N(t)\|_{L^2(\mathcal{O})}^2\bigg)\bigg\}^{1/2}\bigg\{\mathbb{E}\int_0^T\|\nabla u(t)\|_{L^2(\mathcal{O})}^2\mathrm{d}t\bigg\}^{1/2}\|\phi\|_{H^3(\mathcal{O})}. \end{aligned}$$

Therefore, in view of the estimates of Corollary 5,

$$\mathbb{E}|\langle K_2^N(\tau_N+\theta) - K_2^N(\tau_N),\phi\rangle| \le C\theta^{1/2}\|\phi\|_{H^3(\mathcal{O})}. \tag{17}$$

For the stochastic term, we use the Itô isometry and Hypothesis (H4):

$$\begin{aligned} &\mathbb{E}|\langle K_3^N(\tau_N+\theta) - K_3^N(\tau_N),\phi\rangle|^2 \\ &\quad\le \mathbb{E}\bigg(\int_{\tau_N}^{\tau_N+\theta}\|\sigma(u^N(t))\|_{\mathcal{L}_2(U;L^2(\mathcal{O}))}^2\mathrm{d}t\bigg)\|\phi\|_{L^2(\mathcal{O})}^2 \\ &\quad\le C_\sigma\theta\mathbb{E}\bigg(\sup_{0<t<T}\|u^N(t)\|_{L^2(\mathcal{O})}^2\bigg)\|\phi\|_{L^2(\mathcal{O})}^2 \le C\theta\|\phi\|_{L^2(\mathcal{O})}^2. \end{aligned} \tag{18}$$

Next, let ζ, $\varepsilon > 0$. We conclude from the Chebyshev inequality and (17) that

$$\mathbb{P}\big(\|K_2^N(\tau_N+\theta) - K_2^N(\tau_N)\|_{H^3(\mathcal{O})'} \ge \zeta\big)$$

$$\leq \frac{1}{\zeta}\mathbb{E}\|K_2^N(\tau_N+\theta)-K_2^N(\tau_N)\|_{H^3(\mathcal{O})'} \leq \frac{C\theta^{1/2}}{\zeta}.$$

Choosing $\theta_1 := (\zeta\varepsilon/C)^2$, we infer that

$$\sup_{N\in\mathbb{N}}\sup_{0<\theta<\theta_1}\mathbb{P}\big(\|K_2^N(\tau_N+\theta)-K_2^N(\tau_N)\|_{H^3(\mathcal{O})'}\geq\zeta\big)\leq\varepsilon.$$

In a similar way, it follows from the Chebyshev inequality and (18) that

$$\sup_{N\in\mathbb{N}}\sup_{0<\theta<\theta_2}\mathbb{P}\big(\|K_3^N(\tau_N+\theta)-K_3^N(\tau_N)\|_{L^2(\mathcal{O})}\geq\zeta\big)\leq\varepsilon.$$

We infer from (16) that

$$\sup_{N\in\mathbb{N}}\sup_{0<\theta<\min\{\theta_1,\theta_2\}}\mathbb{P}\big(\|u_i^N(\tau_N+\theta)-u_i^N(\tau_N)\|_{H^3(\mathcal{O})'}\geq\zeta\big)\leq 2\varepsilon.$$

This shows that (u_i^N) satisfies the Aldous condition in $H^3(\mathcal{O})'$. □

3.4. **Convergence of the approximate solutions.** According to [12, Theorem 23], the topological space $(Z_T,\mathcal{T})$ possesses a sequence of continuous funtions $Z_T\to\mathbb{R}$ that separates points of Z_T. We deduce from the theorem of Skorokhod–Jakubowski and the tightness of the laws of (u^N) on $(Z_T,\mathcal{T})$ that there exists a subsequence of (u^N), which is not relabeled, a probability space $(\widetilde\Omega,\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$, and, on this space, $Z_T\times C^0([0,T];U_0)$-valued random variables $(\widetilde u,\widetilde W)$, $(\widetilde u^N,\widetilde W^N)$ with $N\in\mathbb{N}$ such that $(\widetilde u^N,\widetilde W^N)$ has the same law as (u^N,W) on $\mathcal{B}(Z_T\times C^0([0,T];U_0)$ and

$$(\widetilde u^N,\widetilde W^N)\to(\widetilde w,\widetilde W)\quad\text{in } Z_T\times C^0([0,T];U_0)\ \mathbb{P}\text{-a.s. as } N\to\infty.$$

We deduce from the definition of Z_T that

$$\begin{aligned}\widetilde u^N &\to \widetilde u \quad \text{in } C^0([0,T];H^3(\mathcal{O})') \text{ and } L^2(0,T;L^2(\mathcal{O})),\\ \widetilde u^N &\rightharpoonup \widetilde u \quad \text{weakly in } L^2(0,T;H^1(\mathcal{O})),\\ \widetilde u^N &\to \widetilde u \quad \text{in } C^0([0,T];L^2_w(\mathcal{O})),\\ \widetilde W^N &\to \widetilde W \quad \text{in } C^0([0,T];U_0).\end{aligned}$$

The strong convergence of $(\widetilde u_i^N)$ implies that $(\widetilde u_i^N)^+\to\widetilde u_i^+$ in $L^2(0,T;L^2(\mathcal{O}))$ $\mathbb{P}$-a.s. Using Corollary 5 and Lemma 6 and arguing as in [12, Section 2.5], we can prove that for $p>2$,

$$\widetilde{\mathbb{E}}\bigg(\int_0^T\|\widetilde u(t)\|_{H^1(\mathcal{O})}^2\mathrm{d}t\bigg)\leq C,\quad \widetilde{\mathbb{E}}\bigg(\sup_{0<t<T}\|\widetilde u(t)\|_{L^2(\mathcal{O})}^p\bigg)\leq C.$$

Let $\widetilde{\mathbb{F}}$ and $\widetilde{\mathbb{F}}^N$ be the filtrations generated by $(\widetilde u,\widetilde W)$ and $(\widetilde u^N,\widetilde W^N)$, respectively. The arguments of [7, Prop. B4] allow us to show that these new random variables are actually stochastic processes. The progressive measurability of $\widetilde u^N$ is a consequence of [5, Appendix B]. Set $\widetilde W_j^{N,k}=(\widetilde W_j^N(t),\eta_k)_{L^2(\mathcal{O})}$. Then $\widetilde W_j^{N,k}(t)$ are independent, standard $\widetilde{\mathcal{F}}_t$-Wiener processes. Passing to the limit $N\to\infty$ in the characteristic function, we infer that $\widetilde W(t)$

are $\widetilde{F}_t$-martingales with the correct marginal distributions. Then, by Lévy's charaterization theorem, $\widetilde{W}(t)$ is a cylindrical Wiener process. For the following lemma, we introduce

$$H^3_\nu(\mathcal{O}) = \big\{v \in H^3(\mathcal{O}) : \nabla v \cdot \nu = 0 \text{ on } \partial\Omega\big\}.$$

Lemma 8. *Let $\phi \in L^2(\mathcal{O};\mathbb{R}^n)$ and $\psi \in H^3_\nu(\mathcal{O};\mathbb{R}^n)$. Then it holds for $i = 1,\ldots,n$ that*

$$\lim_{N\to\infty} \widetilde{\mathbb{E}} \int_0^T \big(\widetilde{u}^N(t) - \widetilde{u}(t), \phi\big)_{L^2(\mathcal{O})}\mathrm{d}t = 0, \quad \lim_{N\to\infty} \widetilde{\mathbb{E}}\big(\widetilde{u}^N(0) - \widetilde{u}(0), \phi\big)^2_{L^2(\mathcal{O})} = 0,$$

$$\begin{aligned}\lim_{N\to\infty} \widetilde{\mathbb{E}} \int_0^T \bigg| \int_0^t \int_{\mathcal{O}} \bigg(&\delta\nabla\widetilde{u}^N(s) + \sum_{j=1}^n a_{ij}(\widetilde{u}_i^N)^+(s)\nabla\widetilde{u}_j^N(s) \\ &- \delta\nabla\widetilde{u}(s) - \sum_{j=1}^n a_{ij}\widetilde{u}_i^+(s)\nabla\widetilde{u}_j(s)\bigg) \cdot \nabla\psi_i \mathrm{d}x\mathrm{d}s\bigg|\mathrm{d}t = 0,\end{aligned}$$

$$\lim_{N\to\infty} \widetilde{\mathbb{E}} \int_0^T \bigg|\sum_{j=1}^n \int_0^t \big(\sigma_{ij}(\widetilde{u}^N(s))\mathrm{d}\widetilde{W}_j^N(s) - \sigma_{ij}(\widetilde{u}(s))\mathrm{d}\widetilde{W}_j(s), \phi_i\big)_{L^2(\mathcal{O})}\bigg|^2 \mathrm{d}t = 0.$$

Proof. The proof of this lemma is very similar to that one for [12, Lemma 16], using the previous convergence results and Vitali's convergence theorem. The only difference is the convergence in the nonlinear terms $(\widetilde{u}_i^N)^+\nabla\widetilde{u}_j^N$. In fact, the strong convergence of $(\widetilde{u}_i^N)^+$ in $L^2(0,T;L^2(\mathcal{O}))$ and the weak convergence of $\nabla\widetilde{u}_j^N$ in $L^2(0,T;H^1(\mathcal{O}))$ imply that

$$\begin{aligned}\bigg| \int_0^t \int_{\mathcal{O}} &\big((\widetilde{u}_i^N)^+(s)\nabla\widetilde{u}_j^N(s) - \widetilde{u}_i^+(s)\nabla\widetilde{u}_j(s)\big) \cdot \nabla\psi_i \mathrm{d}x\mathrm{d}s\bigg| \\ &\le \|(\widetilde{u}_i^N)^+ - \widetilde{u}_i^+\|_{L^2(0,t;L^2(\mathcal{O})}\|\widetilde{u}_j^N\|_{L^2(0,t;H^1(\mathcal{O}))}\|\psi_i\|_{W^{1,\infty}(\mathcal{O})} \\ &\quad + \bigg|\int_0^t \int_{\mathcal{O}} (\widetilde{u}_i^N)^+(s)\nabla(\widetilde{u}_j^N - \widetilde{u}_i)(s) \cdot \nabla\psi_i \mathrm{d}x\mathrm{d}s\bigg| \to 0 \quad \mathbb{P}\text{-a.s. as } N \to \infty.\end{aligned}$$

Since $(\widetilde{u}_i^N)^+\nabla\widetilde{u}_j^N$ is uniformly bounded, Vitali's convergence theorem gives

$$\lim_{N\to\infty} \widetilde{\mathbb{E}} \int_0^T \bigg| \int_0^t \int_{\mathcal{O}} \big((\widetilde{u}_i^N)^+(s)\nabla\widetilde{u}_j^N(s) - \widetilde{u}_i^+(s)\nabla\widetilde{u}_j(s)\big) \cdot \nabla\psi_i \mathrm{d}x\mathrm{d}s\bigg|\mathrm{d}t = 0.$$

This finishes the proof. □

We consider the maps $L_i^N,\ L_i : Z_T \times C^0([0,T];U_0) \times H^3_\nu(\mathcal{O}) \to L^1(\widetilde{\Omega} \times (0,T))$ for $i = 1,\ldots,n$, defined by

$$\begin{aligned}L_i^N(\widetilde{u}^N, \widetilde{W}^N, \psi)(t) &= (\Pi_N(\widetilde{u}_i(0)), \psi)_{L^2(\mathcal{O})} \\ &\quad - \int_0^t \int_{\mathcal{O}} \Pi_N\bigg(\delta\nabla\widetilde{u}_i^N(s) + \sum_{j=1}^n a_{ij}(\widetilde{u}_i^N)^+(s)\nabla\widetilde{u}_j^N(s)\bigg) \cdot \nabla\psi_i \mathrm{d}x\mathrm{d}s \\ &\quad + \sum_{j=1}^n \bigg(\int_0^t \Pi_N\sigma_{ij}(\widetilde{u}^N(s))\mathrm{d}\widetilde{W}_j^N(s), \psi_i\bigg)_{L^2(\mathcal{O})},\end{aligned}$$

$$\begin{aligned}L_i(\widetilde{u},\widetilde{W},\psi)(t) = (\widetilde{u}_i(0),\psi_i)_{L^2(\mathcal{O})} \\ &- \int_0^t\int_{\mathcal{O}}\bigg(\delta\nabla\widetilde{u}_i(s) + \sum_{j=1}^n a_{ij}\widetilde{u}_i^+(s)\nabla\widetilde{u}_j(s)\bigg)\cdot\nabla\psi_i\mathrm{d}x\mathrm{d}s \\ &+ \sum_{j=1}^n\bigg(\int_0^t\sigma_{ij}(\widetilde{u}(s))\mathrm{d}\widetilde{W}_j(s),\psi_i\bigg)_{L^2(\mathcal{O})}\end{aligned}$$

The previous lemma implies the following result.

Corollary 9. *It holds for any $\phi\in L^2(\mathcal{O};\mathbb{R}^n)$ and $\psi\in H^3_\nu(\mathcal{O};\mathbb{R}^n)$ that*

$$\lim_{N\to\infty}\|(\widetilde{u}^N,\phi)_{L^2(\mathcal{O})} - (\widetilde{u},\phi)_{L^2(\mathcal{O})}\|_{L^2(\widetilde{\Omega}\times(0,T))} = 0,$$
$$\lim_{N\to\infty}\|L_i^N(\widetilde{u}^N,\widetilde{W}^N,\psi) - L_i(\widetilde{u},\widetilde{W},\psi)\|_{L^1(\widetilde{\Omega}\times(0,T))} = 0.$$

3.5. **End of the proof.** Since u^N is a strong solution to (6)–(7), we have

$$(u_i^N(t),\psi)_{L^2(\mathcal{O})} = L_i^N(u^N,W,\psi)(t) \quad \mathbb{P}\text{-a.s.}$$

for all $t\in[0,T]$, $\psi\in H^1(\mathcal{O})$, and $i=1,\ldots,n$. In particular,

$$\mathbb{E}\int_0^T\big|(u_i^N(t),\psi)_{L^2(\mathcal{O})} - L_i^N(u^N,W,\psi)(t)\big|\mathrm{d}t = 0.$$

As the laws of (u^N,W) and $(\widetilde{u}^N,\widetilde{W}^N)$ are the same,

$$\widetilde{\mathbb{E}}\int_0^T\big|(\widetilde{u}_i^N(t),\psi)_{L^2(\mathcal{O})} - L_i^N(\widetilde{u}^N,\widetilde{W}^N,\psi)(t)\big|\mathrm{d}t = 0.$$

By Corollary 9, we can pass to the limit $N\to\infty$, yielding

$$\widetilde{\mathbb{E}}\int_0^T\big|(\widetilde{u}_i(t),\psi)_{L^2(\mathcal{O})} - L_i(\widetilde{u},\widetilde{W},\psi)(t)\big|\mathrm{d}t = 0$$

for all $\psi\in H^3_\nu(\mathcal{O})$. Since the space $H^3_\nu(\mathcal{O})$ is dense in $H^1(\mathcal{O})$, this identity also holds for all $\psi\in H^1(\mathcal{O})$. This means that $(\widetilde{u}_i(t),\psi)_{L^2(\mathcal{O})} = L_i(\widetilde{u},\widetilde{W},\psi)(t)$ $\mathbb{P}$-a.s. for a.e. $t\in(0,T)$, for all $\psi\in H^1(\mathcal{O})$ and $i=1,\ldots,n$. Thus, setting $\widetilde{V}=(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{F}},\widetilde{\mathbb{P}})$, we conclude that $(\widetilde{V},\widetilde{u},\widetilde{W})$ is a martingale solution to (1)–(2).

It remains to verify that $\widetilde{u}$ is nonnegative componentwise in $\mathcal{O}\times(0,T)$ $\mathbb{P}$-a.s. This follows from the stochastic Stampacchia truncation argument developed in [8]. The idea is to approximate the test function $g(z)=z^-:=\max\{0,-z\}$ for $z\in\mathbb{R}$ by a smooth function g_ε with linear growth and to apply Itô's formula to $G_\varepsilon(v)=\int_{\mathcal{O}}g_\varepsilon(v(x))^2\mathrm{d}x$. It is proved in [12, Section 2.6] that the limit $N\to\infty$ and then $\varepsilon\to 0$ leads to

$$\mathbb{E}\|\widetilde{u}_i^-(t)\|^2_{L^2(\mathcal{O})} \le \mathbb{E}\|(u_i^0)^-\|^2_{L^2(\mathcal{O})} + \sum_{j=1}^n\mathbb{E}\int_0^t\|\sigma_{ij}(-\widetilde{u}_i^-(s))\|^2_{\mathcal{L}_2(U;L^2(\mathcal{O}))}\mathrm{d}s$$

$$\le C_\sigma \sum_{j=1}^n \mathbb{E}\int_0^t \|\widetilde{u}_i^-(s))\|_{L^2(\mathcal{O})}^2 \mathrm{d}s,$$

where we used that $\widetilde{u}_i^+\nabla\widetilde{u}_i^- = 0$. It follows from Gronwall's lemma that $\mathbb{E}\|\widetilde{u}_i^-(t)\|_{L^2(\mathcal{O})}^2 = 0$ for $t\in(0,T)$ and consequently $\widetilde{u}_i^((t)\ge 0$ in $\mathcal{O}$ $\mathbb{P}$-a.s. for a.e. $t\in(0,T)$ and all $i=1,\ldots,n$.

4. Proof of Theorem 2

We apply Itô's lemma [15, Theorem 4.2.5] to the process $t\mapsto e^{\eta t/2}(PA)^{1/2}(u^N-\bar{u}^N)(t)$, where $\eta>0$ will be determined later and we set $\bar{u}^N(t) := |\mathcal{O}|^{-1}\int_{\mathcal{O}} u^N(x,t)\mathrm{d}x$. Recall that $H(u^N|\bar{u}^N) = \frac{1}{2}\|(PA)^{-1/2}(u^N-\bar{u}^N)\|_{L^2(\mathcal{O})}^2$. Then $\bar{u}^N$ solves

$$\mathrm{d}\bar{u}_i^N(t) = \sum_{j=1}^n \bar\sigma_{ij}(u^N(t))\mathrm{d}W_j(t), \quad \bar\sigma_{ij}(u^N) = \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}}\sigma(u^N)\mathrm{d}x.$$

We compute

$$\begin{aligned}
&e^{\eta t}H(u^N(t)|\bar{u}^N(t)) - H(u^N(0)|\bar{u}^N(0)) - \eta\int_0^t e^{\eta s}H(u(s)|\bar{u}^N(s))\mathrm{d}s \\
&\quad = -\delta\sum_{i,j=1}^n \pi_i a_{ij}\int_0^t e^{\eta s}\int_{\mathcal{O}}\nabla u_i^N(s)\cdot\nabla u_j^N(s)\mathrm{d}x\mathrm{d}s \\
&\qquad - \sum_{i=1}^n \pi_i\int_0^t e^{\eta s}\int_{\mathcal{O}}(u_i^N)^+(s)|\nabla p_i(u^N(s))|^2\mathrm{d}x\mathrm{d}s \\
&\qquad + \frac{1}{2}\int_0^t e^{\eta s}\int_{\mathcal{O}}\operatorname{Tr}\big[(\sigma(u^N)-\bar\sigma(u^N))\mathrm{D}_u^2 H(u^N|\bar{u}^N)(\sigma(u^N)-\bar\sigma(u^N))\big](s)\mathrm{d}x\mathrm{d}s \\
&\qquad + \sum_{i,j,k=1}^n \pi_i a_{ij}\int_{\mathcal{O}}\int_0^t e^{\eta s}(u_j^N-\bar{u}_j^N)(s)\big(\sigma_{ik}(u^N(s))-\bar\sigma_{ik}(u^N(s))\big)\mathrm{d}W_k(s)\mathrm{d}x.
\end{aligned} \tag{19}$$

With $\lambda>0$ being the smallest eigenvalue of the positive definite matrix $(\pi_i a_{ij})$, the first term on the right-hand side is estimated according to

$$-\delta\sum_{i,j=1}^n \pi_i a_{ij}\int_0^t e^{\eta s}\int_{\mathcal{O}}\nabla u_i^N(s)\cdot\nabla u_j^N(s)\mathrm{d}s\mathrm{d}s \ge -\delta\lambda\int_0^t e^{\eta s}\int_{\mathcal{O}}|\nabla u^N(s)|^2\mathrm{d}x\mathrm{d}s.$$

Then, taking the expecation in (19), neglecting the second term on the right-hand side of (19), and observing that the stochastic integral vanishes, we obtain

$$\begin{aligned}
&e^{\eta t}\mathbb{E}H(u(t)|\bar{u}(t)) - \mathbb{E}H(u(0)|\bar{u}(0)) - \eta\int_0^t e^{\eta s}\mathbb{E}H(u(s)|\bar{u}(s))\mathrm{d}s \\
&\quad \le -\delta\lambda\int_0^t e^{\eta s}\int_{\mathcal{O}}|\nabla u(s)|^2\mathrm{d}x\mathrm{d}s + C\mathbb{E}\int_0^t e^{\eta s}\|(PA)^{1/2}(\sigma(u)-\bar\sigma(u))\|_{\mathcal{L}_2(U;L^2(\mathcal{O}))}^2\mathrm{d}s.
\end{aligned}$$

We estimate the last term on the right-hand side:

$$\begin{aligned}\|(PA)^{1/2}(\sigma(u)-\bar\sigma(u))\|^2_{\mathcal{L}_2(U;L^2(\mathcal{O}))} &= \sum_{k=1}^\infty \big\|(PA)^{1/2}\big(\sigma(u)-\bar\sigma(u)\big)\eta_k\big\|^2_{L^2(\mathcal{O})} \\ &\le C\sum_{i,j=1}^n\sum_{k=1}^\infty \|\sigma_{ij}(u(s))\eta_k-\bar\sigma_{ij}(u(s))\eta_k\|^2_{L^2(\mathcal{O})} \\ &= C\sum_{i,j=1}^n\sum_{k=1}^\infty\int_{\mathcal{O}}\bigg(\sigma_{ij}(u(s))\eta_k-\frac{1}{|\mathcal{O}|}\int_{\mathcal{O}}\sigma_{ij}(u(s))\eta_k\mathrm{d}y\bigg)^2\mathrm{d}x \\ &= \frac{C}{|\mathcal{O}|^2}\sum_{i,j=1}^n\sum_{k=1}^\infty\int_{\mathcal{O}}\bigg(\int_{\mathcal{O}}\big(\sigma_{ij}(u(x,s))\eta_k-\sigma_{ij}(u(s,y))\eta_k\big)\mathrm{d}y\bigg)^2\mathrm{d}x \\ &\le \frac{C}{|\mathcal{O}|}\sum_{i,j=1}^n\sum_{k=1}^\infty\int_{\mathcal{O}}\int_{\mathcal{O}}\big(\sigma_{ij}(u(x,s))\eta_k-\sigma_{ij}(u(s,y))\eta_k\big)^2\mathrm{d}y\mathrm{d}x,\end{aligned}$$

where we used the Cauchy–Schwarz inequality in the last step. Next, we add and subtract the space-independent term $\sigma_{ij}(\bar u(s))\eta_k$ and apply the triangle inequality:

$$\begin{aligned}&\|(PA)^{1/2}(\sigma(u)-\bar\sigma(\bar u))\|^2_{\mathcal{L}_2(U;L^2(\mathcal{O}))} \\ &\quad\le \frac{C}{|\mathcal{O}|}\sum_{i,j=1}^n\sum_{k=1}^\infty\int_{\mathcal{O}}\int_{\mathcal{O}}\big(\sigma_{ij}(u(x,s))\eta_k-\sigma_{ij}(\bar u(s))\eta_k\big)^2\mathrm{d}y\mathrm{d}x \\ &\qquad+\frac{C}{|\mathcal{O}|}\sum_{i,j=1}^n\sum_{k=1}^\infty\int_{\mathcal{O}}\int_{\mathcal{O}}\big(\sigma_{ij}(u(y,s))\eta_k-\sigma_{ij}(\bar u(s))\eta_k\big)^2\mathrm{d}y\mathrm{d}x \\ &\quad= 2C\sum_{i,j=1}^n\|\sigma_{ij}(u(s))-\sigma_{ij}(\bar u(s))\|^2_{\mathcal{L}_2(U;L^2(\mathcal{O}))}\le 2C_1C_\sigma^2\|u(s)-\bar u(s)\|^2_{L^2(\mathcal{O})}.\end{aligned}$$

Thus, by the Poincaré–Wirtinger inequality with constant $C_P>0$,

$$\begin{aligned}&e^{\eta t}\mathbb{E}H(u(t)|\bar u(t))-\mathbb{E}H(u(0)|\bar u(0)) \\ &\quad\le\bigg(C(\lambda)\eta+2C_1C_\sigma^2-\frac{\delta\lambda}{C_P^2}\bigg)\int_0^t e^{\eta s}\int_{\mathcal{O}}\|(u-\bar u)(s)\|^2_{L^2(\mathcal{O})}\mathrm{d}s.\end{aligned}$$

Without loss of generality, we may choose $C_1\ge 1/2$. We set $c_0^2:=\delta\lambda/(2C_1C_P^2)$, choose $0<C_\sigma<c_0$, and set $\eta:=(c_0^2-C_\sigma^2)/C(\lambda)$. Then $C(\lambda)\eta+2C_1C_\sigma^2-\delta\lambda/C_P^2=(1-2C_1)(c_0^2-C_\sigma^2)<0$, and we end the proof.

5. Numerical illustration

We discretize (1) for $n\ge 2$ species by a semi-implicit Euler–Maruyama scheme and centered finite differences in one space dimension. Let $\mathcal{O}=(0,1)$, $J\in\mathbb{N}$, $\Delta x=1/J$,

$\Delta t > 0$, $x_j = j\Delta x$ for $j = 0, \ldots, J$, and $t_k = k\Delta t$ for $k \in \mathbb{N}_0$. We approximate $u_i(x_j, t_k, \cdot)$ by u_{ij}^k, solving

$$u_{ij}^{k+1} = (I + \delta\Delta t A_\Delta)^{-1}\bigg(u_{ij}^k + \frac{\Delta t}{\Delta x}(F_{i,j+1/2}^k - F_{i,j-1/2}^k) + \sum_{j=1}^n \sigma_{ij}(u^k)\Delta W_j^k\bigg),$$

where the centered discrete fluxes are given by

$$F_{i,j+1/2}^k = \frac{1}{2\Delta x}(u_{i,j+1}^k + u_{ij}^k)\sum_{\ell=1}^n a_{i\ell}(u_{\ell,j+1}^k - u_{\ell,j}^k),$$

and we have set $u^k = (u_{ij}^k)_{i=1,2,\, j=0,\ldots,J}$. The matrix $A_\Delta \in \mathbb{R}^{(J+1)\times(J+1)}$ is associated to the discrete Laplacian with Neumann conditions, and $\Delta W_j^k = W_j(t_{k+1}) - W_j(t_k)$.

First, we choose the initial data

$$u_1^0(x) = 1_{[0,1/2]}(x), \quad u_2^0(x) = 10x^2\bigg(\frac{1}{2} - \frac{x}{3}\bigg) \quad \text{for } x \in [0,1],$$

the diffusion parameter $\delta = 1$, the coefficients $a_{11} = a_{22} = 2$, $a_{12} = a_{21} = 1$, and the stochastic diffusion $\sigma_{ij}(u) = 0.001\sqrt{1+u_i}\delta_{ij}$ for $i,j = 1,2$. Observe that the matrix (a_{ij}) is symmetric positive definite. The ℓ^2 errors of (u_1, u_2) versus time step size Δt (with $\Delta x = 0.02$ and 2^{11} samples) and space grid size Δx (with $\Delta t = 10^{-5}$ and 2^{11} samples) at time $t = 1$ are presented in Figure 1. For the reference solution, we have chosen $\Delta t = 5\cdot 10^{-5}$, $\Delta x = 0.02$ (left figure) and $\Delta t = 10^{-5}$ and $\Delta x = 1/128$ (right figure). As expected, we observe a half-order convergence in time and first-order convergence in space.

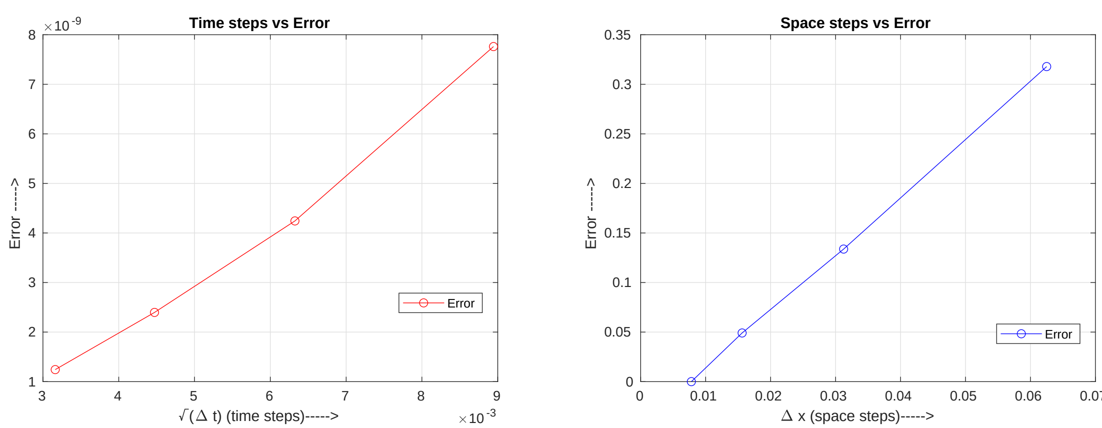

FIGURE 1. Discrete ℓ^2 error of (u_1, u_2) versus time step size (left) and space grid size (right). Observe that the x-axis of the left figure is scaled with $\sqrt{\Delta t}$.

Next, we present the long-time behavior of the discrete ℓ^2 norms of the densities in the three-species case. The parameters are $\delta = 1$, $a_{12} = a_{23} = a_{31} = 1$ and $a_{ij} = 0$ else, $\sigma_{ij} = 0.1\sqrt{1+u_i}\delta_{ij}$ for $i,j = 1,2,3$. Notice that the matrix (a_{ij}) does not satisfy the detailed-balance condition such that, strictly speaking, Theorem 2 is not applicable. The time step is $\Delta t = 10^{-4}$ and the grid size $\Delta x = 0.02$. Figure 2 shows the time evolution of the ℓ^2 norms of u_i in the deterministic case, while the right figure illustrates the dynamics of the ℓ^2 norms in the stochastic situation (with 2^{11} samples). We observe that the ℓ^2 norms, which are equivalent to the Rao entropy, converge (in expectation) as $t \to \infty$, and the behavior is similar in both cases.

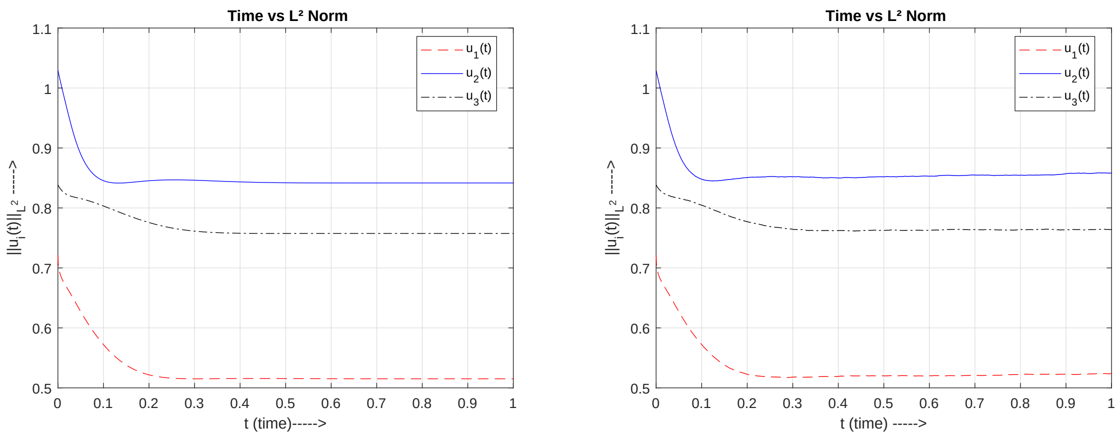

FIGURE 2. Long-time behavior of the ℓ^2 norms of u_i in the deterministic (left) and stochastic (right) case.

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